Advanced Statistics

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Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer

- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber

- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability and Statistical Inference, Ninth Edition,** Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman
- **Probability Demystified**, Allan G. Bluman
- **Practical Statistics for Data Scientists: 50 Essential Concepts,** Peter Bruce and Andrew Bruce
- **Schaum's Outline of Probability,** Second Edition, Seymour Lipschutz, Marc Lipson
- **Python for Probability, Statistics, and Machine Learning,** José Unpingco

References

- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer
- Elementary Statistics, Tenth Edition, Mario F. Triola

These notes contain material from the above resources.

Basic Concepts of Regression

- In some cases, **two variables** are related in a **deterministic way**, meaning that given **a value for one variable**, the value of the **other variable** is **automatically determined** without any **error**.

- For example, the **total cost y** of an item with a list price of **x** and a **sales tax of 5%** can be found by using the deterministic equation $y = 1.05x$. If an item is priced at **\$100**, its total cost is **\$105**.

Probabilistic Models

- In **probabilistic models**, meaning that one variable is **not determined** completely by the **other variable**.

- For example, a **child's height** is not determined completely by the **height of the father (or mother).**

- **Sir Francis Galton (1822–1911)** studied the phenomenon of heredity and showed that when **tall or short couples have children**, the heights of those children **tend to regress, or revert** to the more typical **mean height** for people of the **same gender**.

Notations

- The **regression equation** expresses a relationship between ***x*** (called the **explanatory variable**, or **predictor variable**, or **independent variable**) $\hat{y}$ and (called the **response variable**, or **dependent variable**).

- The typical equation of a straight line ***y = mx + b*** is expressed in the form $\hat{y}$ = $\boldsymbol{b_0} + \boldsymbol{b_1}\mathbf{x}$ or $\hat{y}$ = $\boldsymbol{a}$ + $\mathbf{bx}$, where $\boldsymbol{b_0}$ **or a** is the ***y*-intercept** and $\boldsymbol{b_1}$ or **b** is the **slope**.

- The given notation shows that b_0 and b_1 are **sample statistics** used to estimate the population parameters β_0 and β_1.

- We will use **paired sample data** to **estimate the regression equation**. Using only sample data, we can't find the **exact values** of the population parameters β_0 and β_1, but we can use the sample data to estimate them with b_0 and b_1

Requirements

1. The sample of **paired (*x*, *y*) data** is a ***random* sample** of **quantitative data**.

2. Visual examination of the **scatterplot shows** that the **points** approximate **a straight-line pattern**.

3. Any **outliers** must be **removed** if they are known to be errors. Consider the effects of any outliers that are not known errors.

Requirements

Note: Requirements 2 and 3 above are simplified attempts at checking these formal requirements for regression analysis:

- For each **fixed value of *x***, the **corresponding values of *y*** have a distribution that is **bell-shaped**.
- For the **different fixed values of *x***, the distributions of the corresponding ***y*-values all have the same variance**.
- For the different fixed values of *x*, the distributions of the corresponding *y*-values have **means that lie along the same straight line**.
- The *y* values are independent.

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Requirements

- Results are **not seriously affected** if departures from **normal distributions** and equal variances are not too extreme.

Definitions

Given a collection of paired sample data, the **regression equation**

$\hat{y} = \boldsymbol{b_0} + \boldsymbol{b_1}\mathbf{x}$

algebraically describes the relationship **between the two variables**. The graph of the **regression equation** is called the **regression line** (or ***line of best fit***, or ***least-squares line***).

Notation for Regression Equation

	Population Parameter	Sample Statistic
y-intercept of regression equation	β_0	b_0
Slope of regression equation	β_1	b_1
Equation of the regression line	$Y = \beta_0 + \beta_1 x$	$\hat{y} = b_0 + b_1 x$
Finding the slope b_1 and y-intercept b_0 in the regression equation $\hat{y} = b_0 + b_1 x$		

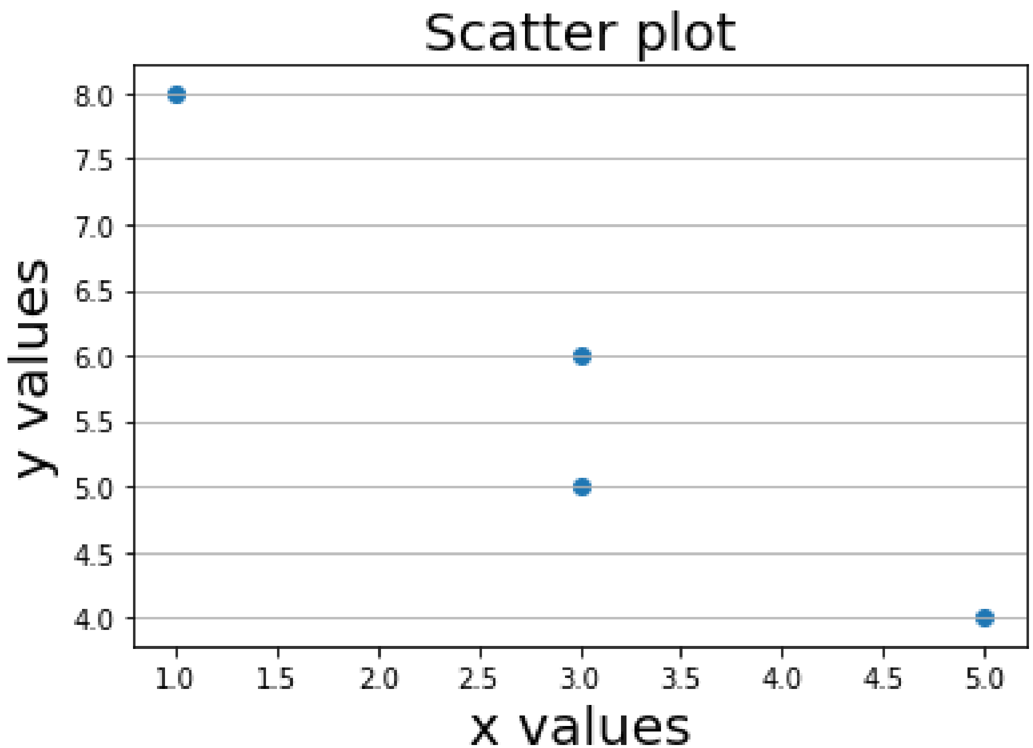
Slope	$b_1 = \dfrac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}$
***y*-intercept:**	$b_0 = \bar{y} - b_1\bar{x}$ or $b_0 = \dfrac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{n(\sum x^2) - (\sum x)^2}$

Example Finding the Regression Equation

Use the given sample data to find the regression equation.

x	3	1	3	5
y	5	8	6	4

REQUIREMENT The data are a simple random sample. The accompanying Python-generated scatterplot shows a pattern of points that does appear to be a straight-line pattern. There are no outliers. We can proceed to find the slope and intercept of the regression line.



x	y	xy	x^2	y^2
3	5	15	9	25
1	8	8	1	64
3	6	18	9	36
5	4	20	25	16
$\sum x$ = 12	$\sum y$ = 23	$\sum xy$ = 61	$\sum x^2$ = 44	$\sum y^2$ = 141

$$\mathbf{b_1 = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2}}$$

$$\mathbf{b_1} = \frac{4(61) - (12)(23)}{4(44) - (12)^2} = \frac{-32}{32} = -1$$

$$\bar{x} = \frac{12}{4} = 3$$

$$\bar{y} = \frac{23}{4} = 5.75$$

$$\mathbf{b_0 = \bar{y} - b_1\bar{x}}$$

$$b_0 = 5.75 - (-1)(3)$$

$$\mathbf{b_0} = 8.75$$

```python
import numpy as np
from scipy.stats import linregress

# Given data points
x = np.array([3, 1, 3, 5])  # Independent variable
y = np.array([5, 8, 6, 4])  # Dependent variable

# Calculate the slope and intercept using linregress
slope, intercept, r_value, p_value, std_err = linregress(x, y)

# Formulate the regression equation
regression_equation = f"y = {intercept:.2f} + {slope:.2f}x"

# Output results
print("Slope:", slope)
print("Intercept:", intercept)
print("Regression Equation:", regression_equation)
```

Explanation

np.array: Converts lists to numpy arrays for easy manipulation.

linregress: Calculates the slope, intercept, and other regression statistics.

Print Statements: Display the results, including the slope, intercept, and formatted regression equation.

This code will output:

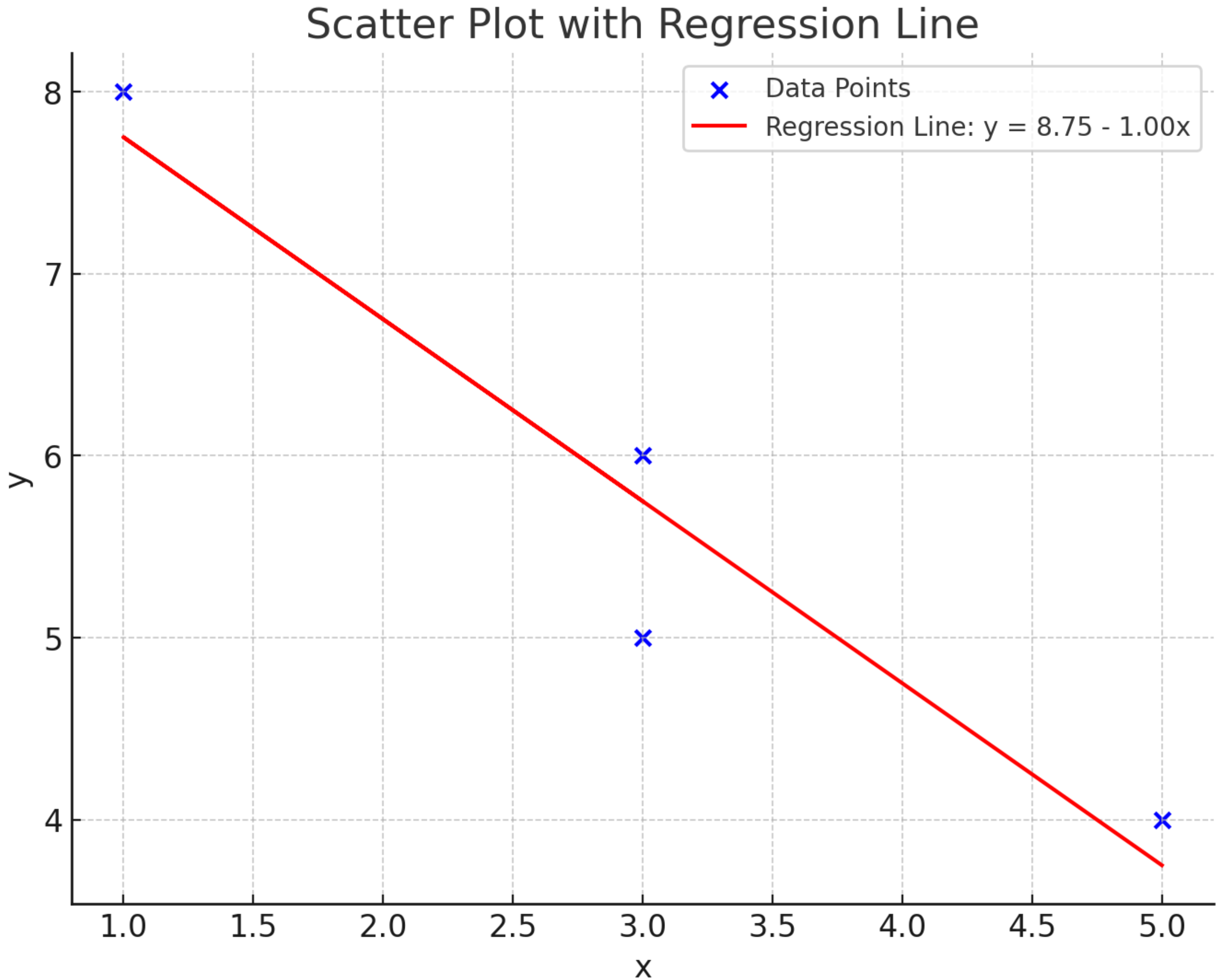
Slope: -1.0

Intercept: 8.75

Regression Equation: y = 8.75 –1.00x

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import linregress

# Given data points
x = np.array([3, 1, 3, 5])  # Independent variable
y = np.array([5, 8, 6, 4])  # Dependent variable

# Calculate the slope and intercept using linregress
slope, intercept, r_value, p_value, std_err =
linregress(x, y)

# Generate predicted y values based on the regression line
y_pred = intercept + slope * x
```

```python
# Create scatter plot of the original data points
plt.figure(figsize=(8, 6))
plt.scatter(x, y, color='blue', label="Data Points")

# Plot the regression line
plt.plot(x, y_pred, color='red', label=f"Regression
Line: y = {intercept:.2f} - {abs(slope):.2f}x")

# Add titles and labels
plt.title("Scatter Plot with Regression Line")
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.grid(True)

# Show the plot
plt.show()
```


Scatter Plot with Regression Line
Data Points
Regression Line: y = 8.75 - 1.00x
8
7
6
5
4
y
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
x

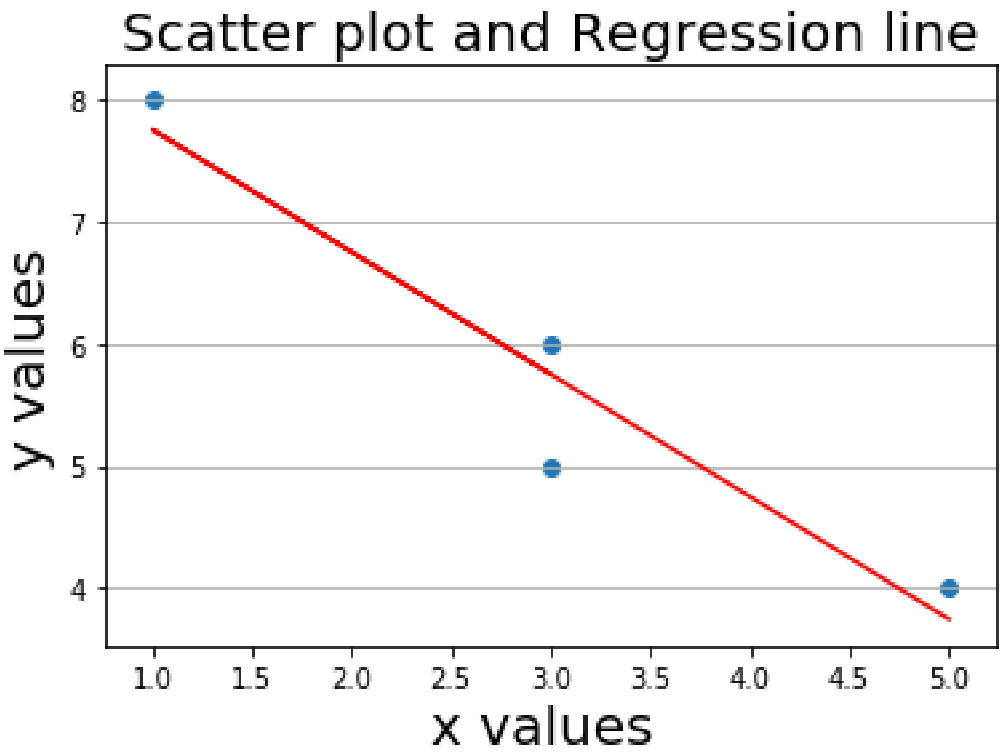
- Knowing the slope ***b_1*** and ***y*-intercept *b_0***, we can now express the **estimated equation** of the **regression line** as

$$\hat{y} = \mathbf{b_0 + b_1 x}$$

$$\hat{y} = \mathbf{8.75 - 1x}$$

- We should realize that this equation is an ***estimate*** of the **true regression equation $Y = \beta_0 + \beta_1 x$.** This estimate is based on one **particular set of sample data**, but another sample drawn from the same population would probably lead to a slightly different equation.

Scatter plot and Regression line



Using the Regression Equation for Predictions

- **Regression equations** are often useful for ***predicting*** the **value of one variable,** given **some particular value of the other variable**.

- If the **regression line fits** the data quite well, then it makes sense to use its **equation for predictions**, provided that we don’t go beyond the scope of the available values.

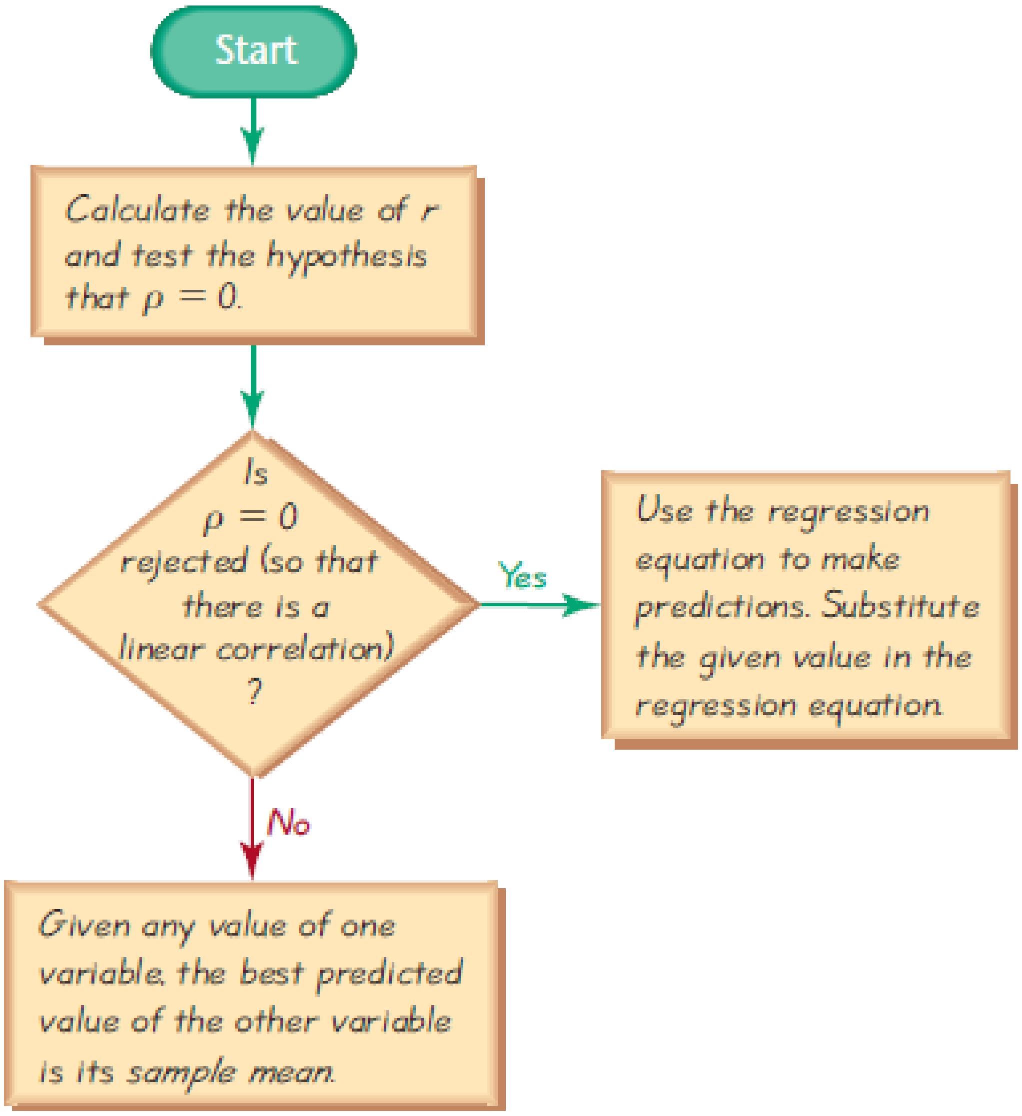
Using the Regression Equation for Predictions

In **predicting a value** of ***y*** **based** on some given value of ***x*** . . .

1. If there is ***not*** **a linear correlation**, the best predicted ***y*-value** is $\bar{y}$.

2. If there is a **linear correlation**, the **best predicted *y*-value** is found by **substituting the *x*-value** into the **regression equation**.

Procedure for Predicting



Guidelines for Using the Regression Equation

1. If there is **no linear correlation**, don’t use the **regression equation** to **make predictions**.

2. When using the **regression equation for predictions**, stay within the **scope of the available sample data**. If you find a regression equation that **relates women’s heights** and **shoe sizes**, it’s absurd to predict the **shoe size** of a woman who is **10 ft tall**.

Guidelines for Using the Regression Equation

3.A **regression equation** based **on old data** is not **necessarily valid now**. The regression equation relating **used-car prices** and **ages of cars** is no longer usable

if it's based on data from the 1990s.

4.Don't **make predictions** about **a population** that is different **from the population** from which the **sample data were drawn**. If we collect sample data from men and develop a regression equation **relating age and TV remote-control usage**, the results don't necessarily apply to women. If we use state averages to develop a regression equation relating SAT math scores and SAT verbal scores, the results don't necessarily apply to individuals.