Interest Rate Volatility

III. Working with SABR

Andrew Lesniewski
Baruch College and Posnania Inc

First Baruch Volatility Workshop
New York

June 16 - 18, 2015

Outline

Arbitrage free approach

- The arbitrage free approach to SABR [5] replaces the explicit asymptotic expressions discussed in Presentation II with an efficient numerical solution of the model.
- The probability density function:

$$\begin{aligned} & p(t, x, y; T, F, \Sigma)\, dF\, d\Sigma \\ & \quad = \mathrm{Prob}(F < F(T) < F + dF, \Sigma < \sigma(T) < \Sigma + d\Sigma \mid F(t) = x, \sigma(t) = y) \end{aligned} \tag{1}$$

satisfies the forward Kolmogorov equation:

$$\frac{\partial}{\partial T}\, p = \frac{1}{2}\, \frac{\partial^2}{\partial F^2} \big(\Sigma^2 C(F)^2 p\big) + \rho\alpha\, \frac{\partial^2}{\partial F \partial \Sigma} \big(\Sigma^2 C(F) p\big) + \frac{1}{2}\, \alpha^2\, \frac{\partial^2}{\partial \Sigma^2} \big(\Sigma^2 p\big), \tag{2}$$

with the initial condition:

$$p(t, x, y; t, F, \Sigma) = \delta(F - x)\delta(\Sigma - y). \tag{3}$$

Arbitrage free approach

- We have the following probability conservation laws:

$$\begin{aligned}\int_0^\infty \frac{\partial^2}{\partial F \partial \Sigma}\left(\Sigma^2 C(F)^2 p\right) d\Sigma &= \frac{\partial}{\partial F}\left(\Sigma^2 C(F)^2 p\right)\Big|_0^\infty \\ &= 0, \\ \int_0^\infty \frac{\partial^2}{\partial \Sigma \partial \Sigma}\left(\Sigma^2 p\right) d\sigma &= \frac{\partial}{\partial \Sigma}\left(\Sigma^2 p\right)\Big|_0^\infty \\ &= 0,\end{aligned} \tag{4}$$

- Introduce now the moments:

$$Q^{(k)}(t, x, y; T, F) = \int_0^\infty \Sigma^k p(t, x, y; T, F, \Sigma)\, d\Sigma, \tag{5}$$

  for $k = 0, 1, \ldots$. Clearly, $Q^{(0)}(t, x, y; T, F)$ is the terminal probability of F, given the state (x, y) at time t.
- In the following, we will suppress the explicit dependence on (t, x, y) of $Q^{(k)}$.

Effective forward equation

- Integrating the forward Kolmogorov equation over all Σ's and using the probability conservation laws (4) yields the following equation:

$$\frac{\partial}{\partial T} Q^{(0)} = \frac{1}{2} \frac{\partial^2}{\partial F^2} \big(C(F)^2 Q^{(2)}\big). \tag{6}$$

  The time evolution of the marginal PDF $Q^{(0)}$ depends thus on the second moment $Q^{(2)}$.

- Now, each of the moments $Q^{(k)}$ satisfies the backward Kolmogorov equation:

$$\begin{gathered}\frac{\partial}{\partial t} Q^{(k)} + \frac{1}{2} y^2 C(x)^2 \frac{\partial^2}{\partial x^2} Q^{(k)} + \rho\alpha y \frac{\partial^2}{\partial x \partial y} Q^{(k)} + \frac{1}{2}\alpha^2 y^2 \frac{\partial^2}{\partial y^2} Q^{(k)} = 0, \\ Q^{(k)}(T, x, y; T, F) = y^k \delta(F - x).\end{gathered} \tag{7}$$

- Rather than finding an explicit solution to (7), we seek to express $Q^{(2)}$ in terms of $Q^{(0)}$, in order to close the forward equation (6).

Effective forward equation

- A detailed analysis using asymptotic analysis of the the backward Kolmogorov equation for $Q^{(0)}$ and $Q^{(2)}$show that:

$$\begin{aligned} Q^{(2)}(T,F) &= y^2(1+2\rho\zeta+\zeta^2)\, e^{\rho\alpha y\Gamma(T-t)} Q^{(0)}(T,F)\big(1+O(\varepsilon^3)\big) \\ &= y^2 I(\zeta)^2\, e^{\rho\alpha y\Gamma(T-t)} Q^{(0)}(T,F)\big(1+O(\varepsilon^3)\big), \end{aligned}$$

where

$$\begin{aligned} \zeta &= \frac{\alpha}{y}\int_x^F \frac{du}{C(u)}\,, \\ I(\zeta) &= \sqrt{1+2\rho\zeta+\zeta^2}\,, \\ \Gamma &= \frac{C(F)-C(x)}{F-x}\,. \end{aligned}$$

- The marginal PDF $Q^{(0)}(T,F)$ satisfies thus the *effective forward equation*:

$$\frac{\partial}{\partial T}\, Q^{(0)} = \frac{1}{2}\,\frac{\partial^2}{\partial F^2}\,\big(y^2 I(\zeta)^2\, e^{\rho\alpha y\Gamma(T-t)} C(F)^2 Q^{(0)}\big). \tag{8}$$

- The approximation above is accurate through $O(\varepsilon^2)$, which is the same accuracy as the original SABR analysis.

Option prices

- To price an option we thus proceed in the following steps.
- We solve numerically the effective forward equation:

$$\frac{\partial}{\partial T} Q^{(0)} = \frac{1}{2} \frac{\partial^2}{\partial F^2} \big(y^2 I(\zeta)^2 \, e^{\rho\alpha y \Gamma(T-t)} C(F)^2 Q^{(0)}\big), \tag{9}$$

with the initial condition:

$$Q^{(0)}(0, F) = \delta(F - F_0), \text{ at } T = 0. \tag{10}$$

- We assume that $0 < F < F_{\max}$, where $F_{\max}$ is a suitably chosen maximum value of the forward (say 10%).
- We assume absorbing (Dirichlet) boundary conditions so that $F(t)$ is a martingale:

$$\begin{aligned} Q^{(0)} &= 0, \text{ at } F = 0, \\ Q^{(0)} &= 0, \text{ at } F = F_{\max}. \end{aligned}$$

Numerical solution

- The reduced problem is one dimensional.
  (i) Its solution is implemented using the moment preserving Crank-Nicolson scheme.
  (ii) Its run time is virtually instantaneous.
- Furthermore, the method
  (i) guarantees that probability is exactly preserved, and that $F(t)$ is a martingale:

$$\begin{aligned} \int_0^\infty p(T, F)\, dF &= 1, \\ \int_0^\infty F p(T, F)\, dF &= F_0; \end{aligned} \tag{11}$$

  (ii) the maximum principle for parabolic equations guarantees that

$$p(t, F) \geq 0, \text{ for all } F. \tag{12}$$

Numerical solution

- Option prices are given by the integrals:

$$\begin{aligned} P^{\text{call}} &= \mathcal{N}(0) \int_K^\infty (F-K) Q^{(0)}(T,F)\, dF, \\ P^{\text{put}} &= \mathcal{N}(0) \int_0^K (K-F) Q^{(0)}(T,F)\, dF, \end{aligned} \tag{13}$$

  which are calculated numerically.
- The PDF is independent of the strike and can be used for pricing options of all strikes.
- The numerical solution is an arbitrage free model.

Boundary layer

- Arbitrage free approach yields nearly the same values as the explicit SABR formulas $\sigma_{\mathrm{n}}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$, except for low strikes and forwards.
- Using asymptotic methods to solve the effective forward equation leads to the same explicit formulas for σ_{n} as in the original analysis, unless the forward or strike is near zero.
- Explicit formulas for σ_{n} do not hold in a boundary layer around zero.
- Boundary layer occurs where a significant fraction of the paths get absorbed at 0 before option expiration.

Boundary layer effects

- At the money vols decrease linearly for small rates (Figure 1).
- As F_0 decreases, an increasing percentage of the paths reach the boundary prior to expiration, which reduces the ATM volatility. This creates a “knee” in the graph.

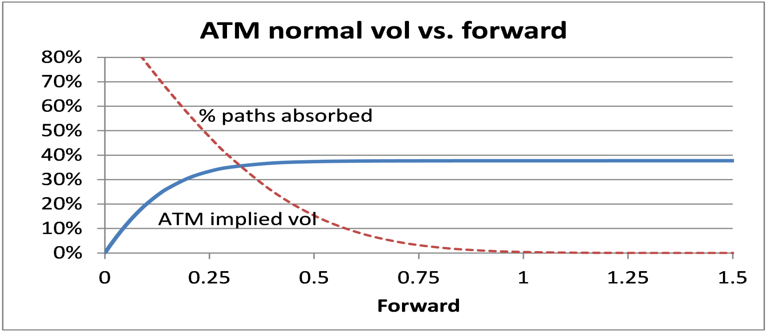

Figure: 1. ATM implied vol for small rates

Boundary layer effects

- Figure 2 shows the smiles $\sigma_{\mathrm{n}}(K)$ obtained for different values of F_0, using the same SABR parameters.

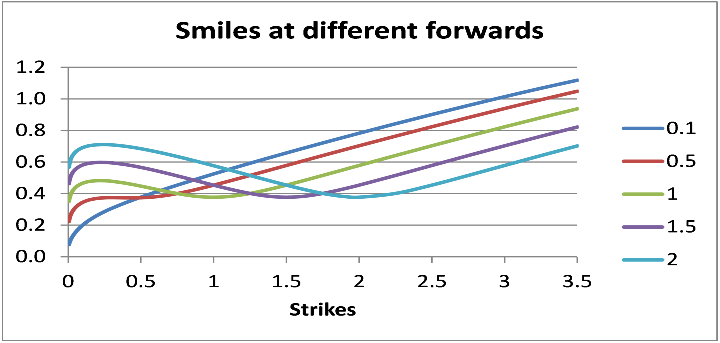

Figure: 2. Smiles for different values of the forward

Boundary layer effects

- The knee is often attributed to market switching from normal to log normal behavior in very low rate environments.
- This is incorrect as, in fact, the decline in volatility is caused solely by the boundary layer.
- This phenomenon has its roots in the fact that the explicit implied volatility formulas are used to calibrate the SABR model. Calibrating the explicit formulas to observed smiles can lead to relatively high values of β and/or ρ for low forward rates.
- Since high values of β and ρ increase the volatilities for high strikes, this can create mispricing for instruments, which are sensitive to high strikes such as CMS caps / floors and swaps.
- Historical analysis shows that for higher forwards, the ATM normal volatilities are reasonably constant; for low forwards, they decrease linearly with the rate

Historical market data

- Figure 3 compares the historical data to the implied volatility from SABR with $\sigma_0 = 0.65\%$, $\alpha = 0.75$, $\beta = 0.25$, and $\rho = 0$.

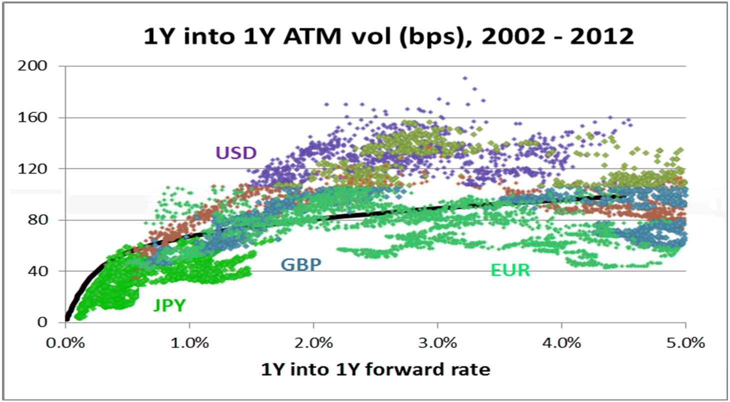

Figure: 3. Historic swaption vols for 2002 through 2012

Calibrating the SABR model

- σ_0 controls the at-the-money vol, α controls the smile, but both ρ and β control the skew.
- Figure 4 shows SABR calibrated to same market data with β chosen to be 0, $1/2$, and 1.

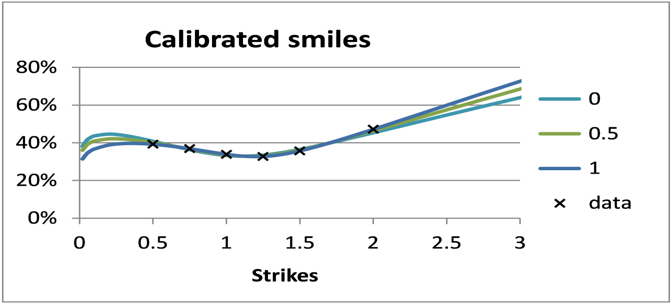

Figure: 4. ATM implied vol for small rates

Calibrating the SABR model

- The calibrated parameters used in Figure 4 are summarized in Table 1 below.

σ	31.8%	32.9%	35.1%
β	0	0.5	1
ρ	-18.3%	-45.5%	-64.4%
α	0.777	0.867	0.985

Table: 1. Calibrated SABR parameters corresponding to various choices of β

- Although tails are somewhat different, all three sets of parameters fit the actual market data well within market noise.
- As already mentioned in Presentation II, ρ can largely compensate for β.

Term structure modeling

- One of the challenges in modeling interest rates is the existence of a term structure of interest rates embodied in the shape of the forward curve. Fixed income instruments typically depend on a segment of the forward curve rather than a single point.
- Pricing such instruments requires thus a model describing a stochastic time evolution of the entire forward curve.
- There exists a large number of term structure models based on different choices of state variables parameterizing the curve, number of dynamic factors, volatility smile characteristics, etc. We describe two approaches to term structure modeling:
  - (i) *Short rate models*, in which the stochastic state variable is taken to be the instantaneous spot rate. Historically, these were the earliest successful term structure models. We shall focus on the Hull-White model and its stochastic volatility extensions.
  - (ii) *HJM style models*, in which the stochastic state variable is the entire forward curve. We shall focus on the LMM model and its stochastic volatility extension LMM-SABR, which are descendants of the HJM approach.

Short rate models

- Short rates models use the instantaneous spot rate $r(t)$ as the basic state variable.
- In the LIBOR / OIS framework, the short rate is defined as $r(t) = f(t, t)$, where $f(t, s)$ denotes the instantaneous discount (OIS) rate.
- The instantaneous index rate (LIBOR) $l(t)$ is given by $r(t) + b(t)$, where $b(t)$ is the instantaneous LIBOR / OIS basis.
- The stochastic dynamics of the short rate $r(t)$ is driven by a number of random factors, usually one, two, or three, which are modeled as Brownian motions. Depending on the number of these stochastic drivers, we refer to the model as one-, two- or three-factor.
- The stochastic differential equations specifying the dynamics are typically stated under the spot measure.

Short rate models

- In the one-factor case the dynamics has the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{14}$$

  where μ and σ are suitably chosen drift and diffusion coefficients, and W is the Brownian motion driving the process.
- Various choices of the coefficients μ and σ lead to different dynamics of the instantaneous rate.
- In a *multi-factor model* the rate $r(t)$ is represented as the sum of a deterministic component and several stochastic components, each of which describes the evolution of a stochastic factor. The factors are specified so that the combined dynamics captures closely observed interest rate curve behavior.

One-factor Hull-White model

- The one factor Hull-White model is given by the following SDE:

$$dr(t) = \Big(\frac{d\mu(t)}{dt} + \lambda(\mu(t) - r(t))\Big)dt + \sigma(t)\, dW(t)\,. \tag{15}$$

Here $\mu(t)$ is the time dependent deterministic long term mean, and $\sigma(t)$ is the deterministic instantaneous volatility function. We assume that

$$\mu(0) = r_0. \tag{16}$$

- Solving (15) (using the method of variation of constants) yields

$$r(t) = \mu(t) + \int_0^t e^{-\lambda(t-u)}\sigma(u)\, dW(u)\,, \tag{17}$$

and thus

$$\begin{aligned} \mathrm{E}^{\mathrm{Q}}[r(t)] &= \mu(t)\,, \\ \mathrm{Var}[r(t)] &= \int_0^t e^{-2\lambda(t-u)}\sigma(u)^2\, du. \end{aligned} \tag{18}$$

One-factor Hull-White model

- Note that (17) implies that

$$r(t) = \mu(t) + (r(s) - \mu(s))e^{-\lambda(t-s)} + \int_s^t e^{-\lambda(t-u)}\sigma(u)\, dW(u)\,, \tag{19}$$

  for any $s < t$.
- The instantaneous 3 month LIBOR rate $l(t)$ is given by

$$l(t) = r(t) + b(t)\,, \tag{20}$$

  where $b(t)$ is the basis between the instantaneous LIBOR and OIS rates.
- As usual, for simplicity of exposition we assume that the basis curve is given by a deterministic function rather than a stochastic process.

Multi-factor Hull-White model

- In the multi-factor Hull-White model, the instantaneous rate is represented as the sum of
  (i) the deterministic function $\mu(t)$, and
  (ii) K stochastic state variables $X_j(t)$ $j = 1, \ldots, K$. Typically, $K = 2$.
- In other words,
$$r(t) = \mu(t) + X_1(t) + \ldots + X_K(t). \tag{21}$$
  A natural interpretation of these variables is that $X_1(t)$ controls the levels of the rates, while $X_2(t)$ controls the steepness of the forward curve.
- We assume the stochastic dynamics for each of the factors X_j:
$$dX_j(t) = -\lambda_j X_j(t)\, dt + \sigma_j(t)\, dW_j(t), \tag{22}$$
  where $\sigma_j(t)$ is the deterministic instantaneous volatility of X_j, and λ_j is its mean reversion speed.
- The Brownian motions are correlated,
$$\mathrm{E}[dW_i(t)\, dW_i(t)] = \rho_{ij}\, dt. \tag{23}$$
- In the two-factor case, the correlation coefficient ρ_{12} is typically a large negative number ($\rho \sim -0.9$) reflecting the fact that steepening curve moves tend to correlate negatively with parallel moves.

The zero coupon bond in the Hull-White model

- The key to all pricing is the coupon bond $P(t,T)$. It is given by the expected value of the stochastic discount factor,

$$P(t,T)=\mathrm{E}_t^{\mathrm{Q}}\left[e^{-\int_t^T r(u)du}\right], \tag{24}$$

where the subscript t indicates conditioning on $\mathscr{F}_t$.
- Within the Hull-White model this expected value can be computed in closed form!
- Let us consider the one-factor case. We proceed as follows:

$$\begin{aligned}\mathrm{E}_t^{\mathrm{Q}}[e^{-\int_t^T r(u)du}] &= \mathrm{E}_t^{\mathrm{Q}}[e^{-\int_t^T(\mu(u)+e^{-\lambda(u-t)}(r(t)-\mu(t))+\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s))du}]\\ &= e^{-\int_t^T\mu(u)du-h_\lambda(T-t)(r(t)-\mu(t))}\,\mathrm{E}_t^{\mathrm{Q}}[e^{-\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)dW(s)du}],\end{aligned}$$

where

$$h_\lambda(t)=\frac{1-e^{-\lambda t}}{\lambda}\,. \tag{25}$$

- Integrating by parts, we transform the double integral in the exponent into a single integral

$$\int_t^T\int_t^u e^{-\lambda(u-s)}\sigma(s)\;dW(s)\;du=\int_t^T h_\lambda(T-s)\,\sigma(s)\,dW(s)\,.$$

The zero coupon bond in the Hull-White model

- Finally, using the fact that

$$\mathrm{E}_t\left[e^{\int_t^T \varphi(s)dW(s)}\right] = e^{\frac{1}{2}\int_t^T \varphi(s)^2 ds},$$

we obtain the following expression for the price of a zero coupon bond:

$$P(t,T) = A(t,T)e^{-h_\lambda(T-t)r(t)}, \tag{26}$$

where

$$A(t,T) = e^{-\int_t^T \mu(s)ds + \mu(t)h_\lambda(T-t) + \frac{1}{2}\int_t^T h_\lambda(T-s)^2\sigma(s)^2 ds}. \tag{27}$$

- Generalizing (26) to the multi-factor case is straightforward:

$$P(t,T) = A(t,T)e^{-\sum_j h_{\lambda_j}(T-t)X_j(t)}, \tag{28}$$

where,

$$A(t,T) = e^{-\int_t^T \mu(s)ds + \frac{1}{2}\sum_{i,j}\int_t^T \rho_{ij}h_{\lambda_i}(T-s)h_{\lambda_j}(T-s)\sigma_i(s)\sigma_j(s)ds}. \tag{29}$$

Calibration of the Hull-White model

- A term structure model has to be *calibrated* to the market before it can be used for valuation purposes.
- All the free parameters of the model should be assigned values, so that the model reprices exactly (or close enough) the prices of a selected set of liquid vanilla instruments.
- In the case of the Hull-White model, this amounts to:
  (i) Matching the current discount curve.
  (ii) Matching the volatilities of selected options.

Calibration of the Hull-White model

- These two tasks have to be performed simultaneously. Note that today's value (in the one-factor model) of the discount factor is

$$P(0,T) = e^{-\int_0^T \mu(s)ds + \frac{1}{2}\int_0^T h_\lambda(T-s)^2 \sigma(s)^2 ds}. \tag{30}$$

- This implies that

$$-\frac{\partial \log P(0,T)}{\partial T} = \mu(T) - \int_0^T e^{-\lambda(T-s)} h_\lambda(T-s)\sigma(s)^2 \, ds, \tag{31}$$

and so

$$\mu(t) = f(0,t) + \int_0^t e^{-\lambda(t-s)} h_\lambda(t-s)\sigma(s)^2 \, ds. \tag{32}$$

- As a result, the curve data ($\mu(t)$) are entangled with the dynamic model data (λ and $\sigma(t)$), and they require joined calibration. This phenomenon is typical of all short rate models.

Calibration of the Hull-White model

- It is impossible to calibrate the Hull-White model in such a way that the prices of all caps / floors and swaptions for all expirations, strikes and underlying tenors are matched.
- This is a consequence of:
  - (i) the volatility dynamics of the Hull-White model (normal, which implies that its intrinsic smile is inconsistent with the market smile),
  - (i) the paucity of model parameters available for calibration.
- Commonly used calibration strategies are:
  - (i) *Global optimization*, suitable for a portfolio.
  - (ii) Deal specific *local calibration* or *autocalibration*, suitable for an individual instrument.
- Global optimization consists in selecting the parameters σ_j so as to minimize the objective function

$$\mathcal{L}(\sigma) = \frac{1}{2} \sum_{\text{all instruments}} \left(\sigma_{\mathrm{n}}(\sigma) - \overline{\sigma}_{\mathrm{n}}\right)^2, \tag{33}$$

  where $\overline{\sigma}_{\mathrm{n}}$ and $\sigma_{\mathrm{n}}(\sigma)$ are the market and model prices of all calibration instruments, respectively.

Local calibration

- Local calibration consists in selecting a set of instruments (swaptions or caps / floors) whose risk characteristics match the risk characteristics of a particular trade. This methodology goes back to [4] and [3].
- For example, in order to model a Bermudan swaption (to be discussed later in the course), one often selects *co-terminal swaptions* of the same strike (not necessarily at the money) as calibrating instruments.
- Co-terminal swaptions are defined as swaptions whose underlying swaps have the same final maturities, e.g. 1Y$\to$ 10Y, 2$\to$ 9,. . ., 10 $\to$ 1.
- Calibration to co-terminal swaptions is close to exact.
- In addition to the co-terminal swaptions, other instruments are used to calibrate the mean reversion speed(s).

Local calibration

- The advantages of an auto-calibrated short rate model are:
  - (i) The calibrating instruments (OTM swaptions) are repriced exactly to the market, even though they are typically far from the money.
  - (ii) Its calibration and run times are fast, making it very suitable for trading desk usage.
- On the other hand, the risk sensitivities of an instrument are calculated based on the model's internal (i.e. normal) smile dynamics. These risk sensitivities are incompatible with the market risk of vanilla options (such as calculated by SABR) and among each other.
- At the portfolio level, this may lead to:
  - (i) Inaccurate risk aggregation among various instruments.
  - (ii) Discrepancy between the securities portfolio and the hedging portfolio.

Stochastic volatility Hull-White model

- An alternative to a locally calibrated short rate model is a short rate model that has a built in stochastic volatility dynamics.
- As above, we let $r(t)$ and $l(t)$ denote the instantaneous discount and index rates, respectively. We assume that these rates evolve around the time-dependent deterministic functions $\mu(t)$ and $\mu(t) + b(t)$, but are driven by a common finite dimensional diffusion processes $X(t) = (X_1(t), \ldots, X_K(t))$:

$$\begin{aligned} r(t) &= \mu(t) + \sum_{1 \le j \le K} X_j(t), \\ l(t) &= r(t) + b(t). \end{aligned} \tag{34}$$

- We assume that each $X_j(t)$ is a mean reverting diffusion driven by a Brownian motion $W_j(t)$, with mean zero:

$$\begin{aligned} &dX_j(t) = -\lambda_j X_j(t)\,dt + \sigma_j(t)\,v_j(t)\,dW_j(t), \\ &X_j(0) = 0. \end{aligned} \tag{35}$$

- Here, λ_j is the speed of mean reversion of factor j. The function $\sigma_j(t)$ is the deterministic component of instantaneous volatility of X_j, and the process $v_j(t)$ is its stochastic component.

Stochastic volatility Hull-White model

- We assume that $v_j(t)$ follows the lognormal process:

$$\begin{aligned} dv_j(t) &= \alpha_j(t)\, v_j(t)\, dZ_j(t)\,, \\ v_j(0) &= 1. \end{aligned} \tag{36}$$

- The correlations between the Brownian motions are given by

$$\begin{aligned} dW_i(t)dW_j(t) &= \rho_{ij}dt, \\ dZ_i(t)dW_j(t) &= r_{ij}dt, \\ dZ_i(t)dZ_j(t) &= \eta_{ij}dt, \end{aligned} \tag{37}$$

where the block correlation matrix

$$\Pi = \begin{bmatrix} \rho & r \\ r^{\mathrm{T}} & \eta \end{bmatrix} \tag{38}$$

is positive definite.

Stochastic volatility Hull-White model

- The choice of $v_j(0) = 1$ in (36) is no loss of generality, as the value $v_j(0)$ can be multiplicatively absorbed in the deterministic instantaneous function $\sigma_j(t)$.
- Equation (36) has a closed form solution:

$$\begin{aligned} v_j(t) &= v_j(t_0) \exp\Big(\int_{t_0}^{t} \alpha_j(u)\, dZ_j(u) - \frac{1}{2} \int_{t_0}^{t} \alpha_j^2(u)\, du \Big) \\ &\triangleq v_j(t_0) \mathcal{E}_j(t|t_0). \end{aligned} \tag{39}$$

- This implies that equation (35) has the following solution:

$$X_j(t) = X_j(t_0) e^{-\lambda_j(t-t_0)} + v_j(t_0) \int_{t_0}^{t} e^{-\lambda_j(t-s)} \sigma_j(s)\, \mathcal{E}_j(s|t_0) dW_j(s)\,, \tag{40}$$

  for $t \geq t_0$.
- Recall that the short rates $r(t)$ and $l(t)$ are sums of the X_j's and corresponding deterministic functions.

Remark

- A simpler form of the specification could be considered. Namely, we could assume that there is only one factor driving the stochastic volatility of the short rate.
- Specifically,

$$\begin{aligned} dX_j(t) &= -\lambda_j X_j(t)\,dt + \sigma_j(t)\,v(t)\,dW_j(t)\,, \\ dv(t) &= \alpha(t)v(t)dZ(t), \end{aligned} \tag{41}$$

where

$$dW_j(t)\,dZ(t) = r_j dt, \tag{42}$$

and $X_j(0) = 0$, $v(0) = 1$.

Price of a zero coupon bond

- Let $P(t,T)$ denote the risk neutral price of zero coupon bond defined as:

$$P(t,T) = \mathsf{E}_t^{\mathsf{Q}}\big[e^{-\int_t^T r(s)ds}\big]. \tag{43}$$

- We find that

$$\begin{aligned} P(t,T) =& e^{-\int_t^T \mu(s)ds - \sum_j h_{\lambda_j}(T-t)X_j(t)} \\ & \times \mathsf{E}_t^{\mathsf{Q}}\big[e^{-\sum_j v_j(t)\int_t^T h_{\lambda_j}(T-s)\sigma_j(s)\mathcal{E}_j(s|t)dW_j(s)}\big]. \end{aligned} \tag{44}$$

- The integral in the exponent inside the expectation involves integration of $\mathcal{E}_j(s|t)$ with respect to the Brownian motion W_j. Since $\mathcal{E}_j(s|t)$ is a lognormal process, the expectation cannot be calculated in closed form (as was the case for the classic Hull-White model).

Price of a zero coupon bond

- Let us introduce the notation:

$$E(t, T) = \mathrm{E}^{\mathrm{Q}}\left[e^{-\sum_j v_j(t) \int_t^T h_{\lambda_j}(T-s)\sigma_j(s)\mathcal{E}_j(s|t)dW_j(s)}\right], \tag{45}$$

so that

$$P(t, T) = e^{-\int_t^T \mu(s)ds + \mu(t)\sum_j h_{\lambda_j}(T-t) - \sum_j X_j(s)h_{\lambda_j}(T-t)} E(t, T). \tag{46}$$

Notice that $\partial/\partial T \log E(0, T)$ is a convexity term that depends on both the deterministic and stochastic components of volatility.

- As a consequence, the initial curve can be expressed in the following way in terms of E:

$$f(0, t) = \mu(t) + \frac{\partial}{\partial t} \log E(0, t). \tag{47}$$

- This formula can be made practical, after an approximation to $\log E(0, t)$ is derived [6].

References

- Andersen, L., and Piterbarg, V.: *Interest Rate Modeling*, Vol. 2, Atlantic Financial Press (2010).
- Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).
- Hagan, P.: Methodology for callable swaps and Bermudan “exercise into” swaptions, working paper (2000).
- Hagan, P., and Woodward, D.: Markov interest rate models, *Appl. Math. Finance*, **6**, 233 - 260 (1999).
- Hagan, P., Kumar, D., Lesniewski, A., and Woodward, D.: Arbitrage Free SABR, *Wilmott Magazine*, **January** (2014).
- Lesniewski, A., Sun, H., and Wu, Q.: Short rate models with stochastic volatility, working paper (2015).