The Swaption Cube

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We infer conditional swap rate moments model independently from swaption cubes. Conditional volatility and skewness exhibit systematic variation across swap maturities and option expiries (conditional kurtosis less so), with conditional skewness sometimes changing sign. Conditional skewness displays some relation to the level and volatility of swap rates but is most consistently related to the conditional correlation between swap rates and swap rate variances. From realized excess returns on synthetic variance and skewness swap contracts, we infer that variance and (to a lesser extent) skewness risk premia are negative and time varying. For the most part, results hold true in both the USD and EUR markets and in both precrisis and crisis subsamples. We design and estimate a dynamic term structure model that captures much of the dynamics of conditional swap rate moments. (*JEL* E43, G12, G13)

A vast body of term structure literature has investigated the dynamics of interest rates and risk premia (see, e.g., the survey by Dai and Singleton 2003). However, despite many advances, we still know surprisingly little about the dynamics of higher-order moments of interest rates and their associated risk premia. Such knowledge is critical for the trading, pricing, hedging, and risk management of interest rate derivatives, particularly those that are in or out of the money. In this paper, we use a new database of options on interest rate swaps, so-called swaptions, to provide insights on higher-order interest rate moments. Our database is uniquely suited for this purpose, because it is the first to provide extensive information about swaptions that are not at the money.

Specifically, our paper has two related objectives: First, using a model-independent approach, we characterize the variation in swap rate moments,

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their linkages, and their associated risk premia. Second, we investigate the extent to which a parsimonious dynamic term structure model is able to match the model-independent results.

The data are provided by the largest interdealer broker in the interest rate derivatives market and records prices of swaptions along three dimensions: the maturities of the underlying swaps, the expiries of the options, and the option strikes. This three-dimensional grid of prices is known, among market participants, as the *swaption cube*. The range along all three dimensions is wide with swap maturities from 2 years to 30 years, option expiries from 1 month to 10 years, and strike intervals up to 800 basis points (bps). The data cover a period from 2001 to 2010, spanning two recessions, the financial crisis, and periods of inflation and deflation fears. Moreover, the data contain both USD- and EUR-denominated swaptions. As such, it is arguably the most comprehensive data set on interest rate derivatives available.

The model-independent analysis is based on the following insight: for a given swap maturity and option expiry, the conditional moments (under the probability measure appropriate for pricing swaptions) of the underlying swap rate at a time horizon equal to the option expiry can be obtained by suitably integrating over prices of swaptions with different strikes. For instance, suppose we consider the 1-year option on the 10-year swap, then the strike-dimension of swaption prices provides the conditional moments of the 10-year swap rate at a 1-year horizon. This way we obtain time series of conditional swap rate moments for each combination of swap maturity and option expiry in the database.

First, we study the moments individually. Conditional volatility (the square-root of conditional variance, annualized, and measured in basis points) is, on average, a hump-shaped function of swap maturity at short horizons, a decreasing function of swap maturity at longer horizons, a hump-shaped function of horizon for the shortest swap maturity, and a decreasing function of horizon for longer swap maturities. In the USD market, the sample means of conditional volatility vary between 70 bps (for the 30-year swap rate at a 10-year horizon) and 124 bps (for the 2-year swap rate at a 2-year horizon). Conditional volatility varies significantly over time; for instance, conditional volatility of the USD 10-year swap rate at a 1-year horizon varies between 67 bps (on June 6, 2007) and 177 bps (on November 19, 2008). Moreover, conditional volatility displays a high degree of commonality across swap maturity and horizon with the first two factors from a principal component analysis (PCA) of weekly changes in volatility explaining 84% and 9%, respectively, of the variation in the USD market.

Conditional skewness is, on average, positive for most combinations of swap maturity and horizon, a decreasing function of swap maturity, and a concave and mostly increasing function of horizon. In the USD market, the sample means of conditional skewness vary between -0.04 (for the 30-year swap rate at a 1-month horizon) and 0.43 (for the 2-year swap rate at a 5-year horizon).

The fact that implied swap rate distributions remain highly skewed at long horizons is suggestive of skewness arising from nonzero correlation between increments to swap rates and swap rate variances. In particular, with a highly persistent variance factor (under the pricing measure), skewness may peak at very long horizons.[1] Conditional skewness varies significantly over time to the extent that the sign sometimes changes. For example, conditional skewness of the USD 10-year swap rate at a 1-year horizon varies between -0.38 (on October 31, 2007) and 0.87 (on April 29, 2009). This time-series pattern of conditional skewness is suggestive of the correlation between swap rates and swap rate variances being time varying with occasional sign changes. Variation in conditional skewness is systematic across swap maturity and horizon with the first two factors from a PCA of weekly changes in skewness explaining 85% and 7%, respectively, of the variation in the USD market.

Conditional kurtosis is always in excess of three, indicating that the swap rate distributions have heavier tails than the normal distribution. Compared with conditional volatility and skewness, conditional kurtosis appears more stable over time. For example, conditional kurtosis of the USD 10-year swap rate at a 1-year horizon fluctuates between 3.22 and 3.67. Conditional kurtosis also exhibits less systematic variation across swap maturity and horizon than is the case for conditional volatility and skewness. The first two factors from a PCA of weekly changes in kurtosis explain only 42% and 11%, respectively, of the variation in the USD market. For this reason, we mainly focus on volatility and skewness in the paper.

Second, we investigate the linkages among the term structure, volatility, and skewness by studying the unconditional correlations between principal components. We find a positive correlation between the first volatility factor and the first interest rate factor (the level factor of the term structure), consistent with conditional skewness being positive, on average, for most combinations of swap maturity and option expiry. We also observe a positive correlation between the first volatility factor and the third interest rate factor (the curvature factor of the term structure), consistent with the convexity effect in the term structure. For skewness, we find a negative correlation between the first skewness factor and the first interest rate factor. We also observe a negative (positive) correlation between the first skewness factor and the first volatility factor in the USD (EUR) market.

While the dynamics of conditional skewness provide indirect evidence of stochastic correlation between swap rates and swap rate variances, we also look for direct evidence. For each combination of swap maturity and option expiry, we estimate the conditional correlation between swap rates and swap rate variances using dynamic conditional correlation models. Conditional

[1] Alternatively, skewness may arise from independent increments to swap rates that have a common asymmetric distribution. However, in this case, skewness decays rapidly like the reciprocal of the square root of the horizon, which is inconsistent with the observed pattern.

correlations vary significantly over time with occasional sign changes. Taking the 10-year swap maturity and the 1-year option expiry in the USD market as an example, the conditional correlation varies between -0.65 and 0.82. Conditional correlations display a high degree of commonality across swap maturity and option expiry with the first two factors from a PCA of weekly changes in correlations explaining 73% and 8%, respectively, of the variation in the USD market. Regressing the first skewness factor on the first correlation factor confirms that variation in the conditional correlation is a significant determinant of variation in conditional skewness. This holds true controlling for the level and volatility of swap rates.

Third, we study variance and skewness risk premia. We follow the approach of Carr and Wu (2009) and estimate these premia from average excess returns on synthetic variance and skewness swaps. These contracts pay the difference between the realized interest rate variance or skewness and a predetermined rate. We design the payoffs such that the contracts can be valued model independently, even when the underlying interest rate exhibits jumps and when the floating legs of the variance and skewness swaps are computed with discrete sampling. We consider one-month variance and skewness swaps with underlying interest rate maturities corresponding to those in the swaption cube. Variance and (to a lesser extent) skewness risk premia are negative. In the USD market, annualized Sharpe ratios corrected for return-autocorrelation lie between -0.55 and -0.69 for variance swaps and between -0.21 and -0.46 for skewness swaps, depending on the underlying interest rate maturity. For comparison, annualized Sharpe ratios lie between 0.48 and 0.65 for interest rate swaps, although these values are likely inflated by the downward trend in interest rates over the sample period. The fact that investors appear willing to accept negative average excess returns on contracts that hedge against unexpected increases in the variance and skewness of interest rates suggests that investors dislike higher variance and skewness. Because an increase in interest rate skewness corresponds to a decrease in bond return skewness, the skewness results are consistent with Arditti (1967) and Scott and Horvath (1980), who show that, under very general assumptions, investors have a preference for positive skewness in returns. In contrast to recent evidence from equity indexes, excess returns on variance and skewness swaps are not highly correlated. Finally, predictive regressions indicate that variance and skewness risk premia are time varying, and that variance (skewness) risk premia become more negative, when the level of variance (skewness) increases.

Most results hold true across both markets. In the EUR market, conditional volatility is generally lower and conditional skewness is generally lower at short horizons and higher at long horizons. If anything, conditional skewness is more highly time varying in the EUR market, and mean excess returns on variance and skewness swaps are generally more negative. In some cases, we also divide the sample period into a precrisis sample and a crisis sample and show that results, for the most part, hold true across subsamples. The robustness across markets and subsamples suggests that our findings are not likely to be spurious.

The second main contribution of the paper is the design and estimation of a parsimonious dynamic term structure model capable of matching many of the stylized facts obtained from the model-independent analysis. Among the key features of the model are hump-shaped innovations to the term structure of instantaneous forward rates (to match the average "volatility surface"), stochastic mean-reverting variance (to match variation in conditional volatility), nonzero correlations between innovations to the term structure and variance (to match the average "skewness surface"), stochastic correlations (to match variation in conditional skewness), and multiple term-structure and variance factors (to match the multifactor nature of moment dynamics). The model is highly tractable with swaptions priced via a fast and accurate Fourier-based pricing formula. It is estimated by maximum likelihood on the entire data set consisting of swap term structures and swaption cubes. The model captures a large fraction of the variation in swap rate moments, explaining 85% (73%) of the variation in conditional volatility (skewness), when averaged across swap maturities and option expiries. In particular, it is able to match the occasional switch in the sign of conditional skewness. Estimated market price of risk parameters indicate that variance and skewness risk premia are negative in both markets, corroborating the model-independent estimates.

Our paper differs from the existing literature on interest rate derivatives in terms of both data and methodology. In terms of data, virtually all existing studies of the swaption market use at-the-money (ATM) swaption data.[2] In contrast, we exploit the information in the entire swaption cube to obtain insights into higher-order swap rate moments. Other papers study the related market for interest rate caps/floors, using implied volatility smile data.[3] The information in the cap/floor smile data and the swaption cube data are complementary. However, the information in the swaption cube is more detailed in that caps/floors provide insights on the dynamics of 3-month LIBOR rates, whereas swaptions provide insights on the dynamics of swap rates along the wide maturity spectrum from 2 years to 30 years. Also, from a practical perspective, the swaption market constitutes a significantly larger fraction of the total interest rate derivatives market.

In terms of methodology, most papers in the interest rate derivatives literature focus on the pricing and/or hedging performance of models. In contrast, we perform an extensive model-independent analysis of the data, providing robust

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[2] Papers that use ATM swaption data include Longstaff, Santa-Clara, and Schwartz (2001a, 2001b), Heidari and Wu (2003), Driessen, Klaassen, and Melenberg (2003), Fan, Gupta, and Ritchken (2003), de Jong, Driessen, and Pelsser (2004), Han (2007), Joslin (2007), Duarte (2008), Trolle and Schwartz (2009), and Carr, Gabaix, and Wu (2009).

[3] Such data are readily available via Bloomberg and other data providers. Papers that use cap/floor implied volatility smile data include Gupta and Subrahmanyam (2005), Li and Zhao (2006, 2009), Jarrow, Li, and Zhao (2007), Deuskar, Gupta, and Subrahmanyam (2008), and Trolle and Schwartz (2009).

stylized facts about higher-order swap rate moments. There are additional benefits to the swaption cube data, when used in conjunction with a model-independent analysis. First, inferring conditional swap rate moments from swaption data has numerical advantages to inferring conditional LIBOR rate moments from caps data. The latter requires an initial stripping of caplets (the individual LIBOR options) from caps, which is nontrivial, typically requires inputs from options on Eurodollar futures, and invariably introduces some model dependency into an otherwise model-independent analysis. Second, the shortest cap/floor expiry is one year, whereas the shortest swaption expiry is one month. This allows us to study nonoverlapping returns on short-term synthetic variance and skewness swaps to quantify variance and skewness risk premia.[4]

The paper most closely related to ours is Li and Zhao (2009), who study the probability density functions (PDFs) of future LIBOR rates implied from caps data and link their shapes to the MBS market. PDFs of future swap rates can be implied from swaption cube data, but our approach instead is to summarize the PDFs by their moments and study how these vary and interact and how exposures to moments are compensated. In recent work, Leippold and Stromberg (2012) develop an extended LIBOR market model, which they fit to implied volatility smiles on both caps and swaptions. Their focus is on the model-implied relative valuation of caps and swaptions during the financial crisis.

The rest of the paper is organized as follows: Section 1 describes the swaption cube data. Section 2 analyzes the variation in swap rate moments, their linkages, and their associated risk premia. Section 3 describes and evaluates a dynamic term structure model. Section 4 concludes. Several appendices, including an online appendix, contain derivations and additional analyses.

1. The Swaption Market

1.1 The swaption cube data

A standard European swaption is an option to enter into a fixed versus floating forward starting interest rate swap at a predetermined rate on the fixed leg. A receiver swaption gives the right to enter a swap, receiving the fixed leg and paying the floating leg, whereas a payer swaption gives the right to enter a swap, paying the fixed leg and receiving the floating leg. For instance, a two-year into ten-year, 5% payer swaption is the option to pay a fixed rate of 5% on a ten-year swap, starting two years from today.

[4] Our paper also differs in terms of model design. Early models of the swaption market assumed a constant covariance matrix of interest rates (see, e.g., Longstaff, Santa-Clara, and Schwartz 2001a, 2001b; Driessen, Klaassen, and Melenberg 2003; Fan, Gupta, and Ritchken 2003; de Jong, Driessen, and Pelsser 2004), although the parameters of the covariance matrix were often recalibrated over time. Subsequent papers introduced models with stochastic variance (see, e.g., Han 2007; Joslin 2007; Trolle and Schwartz 2009; Carr, Gabaix, and Wu 2009) or allowed the covariance matrix to depend on prepayment activity in the MBS market (Duarte 2008). A key feature of our model is stochastic correlation between interest rates and variance in order to generate stochastic skewness in swap rates.

The swaption cube is an object that shows how swaption prices vary along three dimensions: the maturities of the underlying swaps (the swap tenors), the expiries of the options, and the option strikes. In the swaption cube data provided to us, prices are quoted for five different swap tenors (2, 5, 10, 20, and 30 years), eight different option expiries (1, 3, 6, and 9 months and 1, 2, 5, and 10 years), and up to fifteen degrees of moneyness defined as strike minus the forward swap rate (± 400, ± 300, ± 200, ± 150, ± 100, ± 50, ± 25, and 0 bps). Price quotes are for at- or out-of-the-money swaptions; that is, receiver swaptions, when the strike is less than the forward swap rate, and payer swaptions, when the strike is greater than the forward swap rate.

Swaptions trade over the counter (OTC) and typically via interdealer brokers, who act as intermediaries; they facilitate price discovery and transparency by communicating dealer interests and transactions, enhance liquidity, and allow financial institutions anonymity in terms of their trading activities. Our swaption cube data are from ICAP plc., which is the largest interdealer broker in the interest rate derivatives market, and as such provides the most accurate quotations. In addition to swaption prices, ICAP also provided the underlying forward swap rates, as well as spot swap rates with maturities up to 40 years. Swaption prices and swap rates reflect aggregates of dealer midquotes between 16:00 and 16:30 GMT.

We consider swaptions denominated in both USD and EUR, which are by far the most liquid markets (see below). The data are available at a daily frequency; however, most of the analyses are conducted using weekly data sampled on Wednesdays. For the USD market, the data are from December 19, 2001 to January 27, 2010 (419 weeks), whereas for the EUR market, the data are from June 6, 2001 to January 27, 2010 (449 weeks). We discard obvious mistakes in the quotations.[5] Also, according to market sources, quotes for extremely deep out-of-the-money swaptions are less reliable, and so we only consider options for which the price is greater than USD (EUR) 100 in the case of a swap notional of USD (EUR) 1,000,000. At a weekly frequency, we are left with a total of 172,658 quotes in the USD market and 172,500 quotes in the EUR market.

Although swaptions are quoted in terms of prices, it is often more convenient to represent prices in terms of implied volatilities, either log-normal or normal.[6] Most market participants think in terms of normal implied volatilities, as these are more uniform across the swaption grid and more stable over time than are log-normal implied volatilities. Therefore, in this paper, implied volatilities always refer to the normal type, unless otherwise stated.

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[5] Anomalous quotes were identified by fitting a flexible curve to each of the implied volatility smiles. Quotes for which the normal implied volatilities (see definition below) deviated more than 20% from the fitted values were identified as potentially anomalous. Upon closer examination, some of these quotes were discarded. Quotes corresponding to negative strikes were also discarded.

[6] The log-normal (or percentage) implied volatility is the volatility parameter that, plugged into the log-normal (or Black 1976) pricing formula, matches a given price. The normal (or absolute or basis point) implied volatility is the volatility parameter that, plugged into the normal pricing formula, matches a given price.

1.2 Liquidity

Given its OTC nature, the swaption market is relatively opaque. However, since 2010, the TriOptima trade repository, which records all trades for fourteen of the largest OTC derivatives dealers, has contributed to increased transparency. For instance, as of July 1, 2011, TriOptima reports 213,457 outstanding swaption contracts with a combined notional value of 35.026 trillion USD equivalent. While this covers all currencies, the majority of trades are in USD and EUR.

Using data from the same set of firms, the International Swaps and Derivatives Association (ISDA) recently reported the volume of swaption trades during the period April to June, 2011. The average daily volume amounted to 988 trades with a combined notional of USD 151 billion. This understates the true trading volume as it does not cover trades in which none of the fourteen dealers is a counterparty. To put these numbers into perspective, ISDA also reported the average daily trading volume of caps/floors, which amounted to a combined notional of USD 19 billion. Moreover, data from the Chicago Mercantile Exchange show that, during the same period, the average daily trading volume of all options on Treasury futures amounted to a combined notional of USD 29 billion. As such, even taking the large number of quotes in the swaption cube into account, it appears that liquidity in the swaption market compares favorably with liquidity in other segments of the interest rate options market.

In terms of the distribution of liquidity across swap maturities and option expiries, TriOptima reports statistics on outstanding swaption contracts in the following seven swaption maturity (swap maturity plus option expiry) buckets: 0–2 years, 2–5 years, 5–10 years, 10–15 years, 15–20 years, 20–30 years, and 30+ years. As of July 1, 2011, the distribution was 5.6% (13.3%), 16.8% (23.6%), 20.8% (21.8%), 25.0% (21.6%), 9.6% (7.9%), 13.0% (7.2%), and 9.3% (4.6%) in terms of the number of contracts (notional values), showing that even ultralong swaptions trade frequently. In terms of the distribution of liquidity across moneyness, there is no publicly available data. ATM swaptions are most liquid, although market sources indicate that there is significant trading activity in out-of-the-money (OTM) swaptions. Part of the reason is that embedded options in callable bonds and structured products, which are often hedged via swaptions, are typically out of the money. At the end of Section 2.2, we use indicative bid-ask spreads from major investment banks to gauge the impact of liquidity on our results.

2. A Model-Independent Analysis of the Swaption Cube

In this section, we analyze the swaption cube from a model-independent perspective. For a given swap maturity and option expiry, the conditional moments (under the appropriate pricing measure) of the underlying swap rate at a time horizon equal to the option expiry can be inferred from the implied volatility smile. We characterize the variation in swap rate moments, their linkages, and their associated risk premia.

In principle, using the insight from Breeden and Litzenberger (1978), we could infer the entire conditional densities of future swap rates, rather than just the conditional moments. Beber and Brandt (2006) and Li and Zhao (2009) study such option-implied densities of Treasury futures prices and LIBOR rates, respectively. In practice, however, it is a nontrivial matter to obtain the conditional density from a finite number of option prices, and results may be quite sensitive to the choice of numerical scheme. In contrast, conditional moments can be recovered in a robust fashion. Nevertheless, for illustrative purposes, we display estimated probability densities in the Online Appendix.

2.1 Inferring conditional moments of swap rates

Consider a fixed versus floating interest rate swap for the period T_m to T_n with a fixed rate of K. At every time T_j in a prespecified set of dates $T_{m+1},...,T_n$, the fixed leg pays $\tau_{j-1}K$ and the floating leg pays $\tau_{j-1}L(T_j,T_{j-1})$, where τ_{j-1} is the year-fraction between times T_{j-1} and T_j and $L(T_j,T_{j-1})$ is the τ_{j-1}-maturity LIBOR rate set at T_{j-1}.[7] From the perspective of the fixed-rate payer, the value of the swap at time $t<T_m$ (assuming a notional of one) is given by

$$V_{m,n}(t)=\sum_{j=m+1}^{n}\tau_{j-1}E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_j}r(s)ds}\left(L(T_j,T_{j-1})-K\right)\right], \tag{1}$$

where $r(t)$ is the short rate and $\mathbb{Q}$ denotes the risk-neutral measure. Assuming that $r(t)$ incorporates the same credit and liquidity risk as LIBOR, the above formula simplifies considerably to

$$V_{m,n}(t)=P(t,T_m)-P(t,T_n)-K\,A_{m,n}(t), \tag{2}$$

where

$$A_{m,n}(t)=\sum_{j=m+1}^{n}\tau_{j-1}P(t,T_j) \tag{3}$$

is the swap annuity factor, and $P(t,T)$ denotes the time-t price of a zero-coupon bond maturing at time T. In reality, interdealer contracts are virtually always collateralized, and cash flows should, in principle, be discounted using the rate that is paid on collateral, making swap (and swaption) valuation significantly more involved (see, e.g., Johannes and Sundaresan 2007; Filipovic and Trolle 2013). In the interest of simplicity and in line with most studies, we use the traditional approach to valuing swaps (and swaptions).

[7] For expositional ease, we assume that the fixed and floating leg payments occur at the same frequency, although this is typically not the case. In the final valuation equations, it is only the frequency of the fixed-leg payments that matter.

The time-t forward swap rate, $S_{m,n}(t)$, is the rate on the fixed leg that makes the present value of the swap equal to zero and is given by

$$S_{m,n}(t)=\frac{P(t,T_m)-P(t,T_n)}{A_{m,n}(t)}. \tag{4}$$

The forward swap rate becomes the spot swap rate at time T_m.

A payer swaption is an option to enter into an interest rate swap, paying the fixed leg at a predetermined rate and receiving the floating leg. Let $\mathcal{P}_{m,n}(t,K)$ denote the time-t value of a European payer swaption expiring at T_m with strike K on a swap for the period T_m to T_n. At expiration, the swaption has a payoff of[8]

$$\begin{aligned} V_{m,n}(T_m)^+ &= (1-P(T_m,T_n)-K\,A_{m,n}(T_m))^+ \\ &= A_{m,n}(T_m)(S_{m,n}(T_m)-K)^+. \end{aligned} \tag{5}$$

At time $t<T_m$, its price is given by

$$\begin{aligned} \mathcal{P}_{m,n}(t,K) &= E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_m} r(s)ds} A_{m,n}(T_m)(S_{m,n}(T_m)-K)^+\right] \\ &= A_{m,n}(t)E_t^{\mathbb{A}}\left[(S_{m,n}(T_m)-K)^+\right], \end{aligned} \tag{6}$$

where $\mathbb{A}$ denotes the annuity measure associated with using $A_{m,n}(t)$ as numeraire (see, e.g., Jamshidian 1997). Note that the annuity measure changes with m and n; to lighten notation, we have suppressed this dependence. The corresponding receiver swaption is denoted by $\mathcal{R}_{m,n}(t,K)$, and has a time-t price of

$$\mathcal{R}_{m,n}(t,K)=A_{m,n}(t)E_t^{\mathbb{A}}\left[(K-S_{m,n}(T_m))^+\right]. \tag{7}$$

From (6) and (7) it is apparent that a receiver swaption can be viewed as a put option on a swap rate, whereas a payer swaption can be viewed as a call option on a swap rate.

Using the insights from Bakshi and Madan (2000), Carr and Madan (2001), and Bakshi, Kapadia, and Madan (2003), it follows that for any fixed Z, we can write any twice continuously differentiable function of $S_{m,n}(T_m)$, $g(S_{m,n}(T_m))$, as

$$\begin{aligned} g(S_{m,n}(T_m)) &= g(Z)+g'(Z)(S_{m,n}(T_m)-Z)+\int_Z^\infty g''(K)(S_{m,n}(T_m)-K)^+dK \\ &\quad +\int_0^Z g''(K)(K-S_{m,n}(T_m))^+dK. \end{aligned} \tag{8}$$

[8] EUR swaptions are typically cash-settled and have a payoff given by

$$\left(S_{m,n}(T_m)-K\right)^+ \sum_{j=m+1}^{n} \tau_{j-1}\frac{1}{(1+S_{m,n}(T_m))^{\tau_{m,j}}},$$

where $\tau_{m,j}$ is the year-fraction between times T_m and T_j. The advantage of using this formula, rather than (5) is that counterparties only have to agree upon a single swap rate, rather than a complete set of discount factors, to compute the cash settlement value. In practice, the difference between the two payoff formulas is very small, and in the paper we also use (5) for EUR swaptions (see, e.g., Andersen and Piterbarg 2010 for further details).

Taking expectations under the annuity measure and setting $Z=S_{m,n}(t)$, we obtain an expression in terms of prices of OTM receiver and payer swaptions

$$E_t^{\mathbb{A}}\left[g(S_{m,n}(T_m))\right]=g(S_{m,n}(t))+\frac{1}{A_{m,n}(t)}\left(\int_{S_{m,n}(t)}^{\infty}g''(K)\mathcal{P}_{m,n}(t,K)dK\right.$$
$$\left.+\int_0^{S_{m,n}(t)}g''(K)\mathcal{R}_{m,n}(t,K)dK\right). \tag{9}$$

We can use this result to calculate conditional moments of the future swap rate at a time horizon equal to the expiry of the option. First, by construction of the annuity measure, the conditional mean of the future swap rate is simply the current forward swap rate:

$$\mu_t\equiv E_t^{\mathbb{A}}\left[S_{m,n}(T_m)\right]=S_{m,n}(t). \tag{10}$$

Then, using (9), we get the following expressions for conditional variance, skewness, and kurtosis of the future swap rate:

$$\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)=E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m)-\mu_t\right)^2\right]=$$
$$\frac{2}{A_{m,n}(t)}\left(\int_{S_{m,n}(t)}^{\infty}\mathcal{P}_{m,n}(t,K)dK+\int_0^{S_{m,n}(t)}\mathcal{R}_{m,n}(t,K)dK\right) \tag{11}$$

$$\mathrm{Skew}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)=\frac{E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m)-\mu_t\right)^3\right]}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)^{3/2}}=$$
$$\frac{\frac{6}{A_{m,n}(t)}\left(\int_{S_{m,n}(t)}^{\infty}\left(K-S_{m,n}(t)\right)\mathcal{P}_{m,n}(t,K)dK+\int_0^{S_{m,n}(t)}\left(K-S_{m,n}(t)\right)\mathcal{R}_{m,n}(t,K)dK\right)}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)^{3/2}} \tag{12}$$

$$\mathrm{Kurt}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)=\frac{E_t^{\mathbb{A}}\left[\left(S_{m,n}(T_m)-\mu_t\right)^4\right]}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)^{2}}=$$
$$\frac{\frac{12}{A_{m,n}(t)}\left(\int_{S_{m,n}(t)}^{\infty}\left(K-S_{m,n}(t)\right)^2\mathcal{P}_{m,n}(t,K)dK+\int_0^{S_{m,n}(t)}\left(K-S_{m,n}(t)\right)^2\mathcal{R}_{m,n}(t,K)dK\right)}{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)^{2}}. \tag{13}$$

As discussed in Section 1.1, swaptions are only available for a finite set of strikes, while these formulas presume the existence of a continuum of strikes. Swaption prices corresponding to the required strikes in the scheme used for numerical integration are obtained by, first, linearly interpolating between the available normal implied volatilities, and then converting from implied volatilities to prices. For strikes below the lowest available strike, we use the implied volatility at the lowest strike. Similarly, for strikes above the highest available strike, we use the implied volatility at the highest strike. With the exception of swaptions with 10-year option expiries in times of high volatility, the approximation error caused by the extrapolation of implied volatilities is very small, because swaption prices are very low in the regions of strikes in which extrapolation is necessary.

2.2 Variation in swap rate moments

We investigate how conditional swap rate moments vary in the cross-section (with option expiry and swap maturity) and over time. Clearly, when comparing moments for different swap maturities and at different option expiries, one should keep in mind that these values are computed under different measures. However, a simulation exercise based on the estimated dynamic term structure model in Section 3, shows that moments computed under the common risk-neutral measure and the (swap maturity, option expiry)-specific annuity measures are quite similar. Therefore, the cross-sectional variation in moments does contain useful information for model design.

2.2.1 Volatility. Table 1 displays results for the conditional volatility (the square-root of the conditional variance, annualized, and measured in basis points, that is, $\sqrt{\mathrm{Var}_t^{\mathbb{A}}(S_{m,n}(T_m))/(T_m-t)}\times 10{,}000$) of the future swap rate for different swap maturities and at different horizons, corresponding to the available option expiries (we leave out the 9-month option expiry from the table). It reports the sample means and, in parentheses, the sample standard deviations. It also reports the sample means of the slopes of the volatility surface along both the swap maturity and horizon dimensions, with Newey and West (1987) t-statistics in brackets. On average, conditional volatility is a hump-shaped function of swap maturity at short horizons and a decreasing function of swap maturity at longer horizons, where the spreads between conditional volatility of the 30-year and 2-year swap rates are highly statistically significant. Similarly, on average, conditional volatility is a hump-shaped function of horizon for the shortest swap maturity, and a decreasing function of horizon for longer swap maturities, where the spreads between conditional volatility at the 10-year and 1-month horizons are statistically significant (less so in the EUR market). These results hold true in both markets. In the USD market, the sample means of conditional volatility vary between 70 bps (for the 30-year swap rate at a 10-year horizon) and 124 bps (for the 2-year swap rate at a 2-year horizon). In the EUR market, volatility is generally lower.

To understand the implications for model design, we note that the conditional variance of the future swap rate at a given horizon reflects the conditional expectation (under the pricing measure) of the realized variance of the forward swap rate over that horizon. Specifically, consider a partition of the interval $[t,T_m]$ into N equally spaced subintervals

$$t=t_0<t_1<\cdots<t_N=T_m, \tag{14}$$

Table 1
Volatility of future swap rates

Tenor	Horizon							
	1 mo	3 mo	6 mo	1 yr	2 yr	5 yr	10 yr	Slope
Panel A: USD market								
2 yr	110.1	112.0	114.4	120.7	123.6	116.9	98.2	−11.9
	(35.5)	(31.5)	(27.9)	(26.6)	(24.6)	(18.0)	(12.0)	[−1.35]
5 yr	122.4	122.2	121.8	121.3	120.4	111.7	93.2	−29.2***
	(37.4)	(33.5)	(29.4)	(26.3)	(23.2)	(16.5)	(10.4)	[−3.10]
10 yr	115.9	115.8	115.4	114.8	113.3	104.7	86.8	−29.1***
	(36.6)	(32.7)	(28.7)	(25.0)	(22.0)	(15.0)	(9.2)	[−3.19]
20 yr	106.9	105.7	104.0	101.8	99.5	89.9	74.1	−32.8***
	(36.1)	(31.4)	(26.7)	(21.8)	(18.6)	(12.3)	(7.9)	[−3.71]
30 yr	103.5	101.7	99.9	97.5	95.1	85.4	69.6	−33.9***
	(37.6)	(31.5)	(26.3)	(20.9)	(17.1)	(10.8)	(6.3)	[−3.62]
Slope	−6.6	−10.3	−14.5**	−23.2***	−28.5***	−31.5***	−28.6***	—
	[−0.88]	[−1.55]	[−2.38]	[−4.71]	[−6.50]	[−7.37]	[−8.46]	
Panel B: EUR market								
2 yr	80.5	81.1	81.6	80.8	80.7	78.0	72.2	−8.4
	(29.3)	(24.7)	(20.3)	(15.3)	(12.9)	(8.6)	(6.3)	[−1.13]
5 yr	83.5	82.4	80.9	78.6	77.1	74.2	69.1	−14.4**
	(25.3)	(21.2)	(17.1)	(13.4)	(11.5)	(8.5)	(7.2)	[−2.41]
10 yr	74.7	74.7	74.4	73.4	73.5	72.0	67.1	−7.5
	(23.2)	(20.6)	(17.6)	(15.0)	(13.6)	(9.9)	(7.6)	[−1.50]
20 yr	72.3	72.1	71.3	70.1	69.4	67.3	61.9	−10.5
	(30.1)	(26.5)	(21.8)	(18.0)	(15.4)	(10.9)	(8.2)	[−1.64]
30 yr	71.7	71.3	70.4	68.7	67.9	65.3	59.8	−11.9
	(36.9)	(32.6)	(26.7)	(20.9)	(17.6)	(12.9)	(9.3)	[−1.61]
Slope	−8.8	−9.8*	−11.2**	−12.1***	−12.8***	−12.7***	−12.4***	—
	[−1.58]	[−1.82]	[−2.27]	[−2.72]	[−3.44]	[−4.01]	[−6.35]	

This table shows summary statistics of conditional volatility of future swap rates. Conditional volatility is the square-root of conditional variance, annualized, and measured in basis points; that is $\sqrt{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)/(T_m-t)}\times 10{,}000$. For each combination of swap maturity (tenor) and horizon, the table reports the sample mean of conditional volatility, with the sample standard deviation of conditional volatility reported in parenthesis. For each swap maturity, the column labeled "Slope" reports the sample mean of the difference between conditional volatility at the 10-year and 1-month horizons, with associated t-statistics reported in brackets. For each horizon, the row labeled "Slope" reports the sample mean of the difference between conditional volatility of the 30-year and 2-year swap rates, with associated t-statistics reported in brackets. t-statistics are corrected for heteroscedasticity and serial correlation up to 52 lags (i.e., one year) using the approach of Newey and West (1987). Conditional volatility is computed under the annuity measure $\mathbb{A}$. In the USD market, the sample period is from December 19, 2001 to January 27, 2010 (419 weekly observations). In the EUR market, the sample period is from June 6, 2001 to January 27, 2010 (449 weekly observations). *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

and let $\Delta S(t_i) = S_{m,n}(t_i) - S_{m,n}(t_{i-1})$ denote changes in the forward swap rate. We have that

$$\mathrm{Var}_t^{\mathbb{A}}(S_{m,n}(T_m)) = E_t^{\mathbb{A}}\left[(S_{m,n}(T_m)-S_{m,n}(t))^2\right] = E_t^{\mathbb{A}}\left[\left(\sum_{i=1}^{N}\Delta S(t_i)\right)^2\right]$$
$$= E_t^{\mathbb{A}}\left[\sum_{i=1}^{N}(\Delta S(t_i))^2\right], \tag{15}$$

where the last equality follows from $S_{m,n}(t)$ being a martingale under $\mathbb{A}$. Within a Heath, Jarrow, and Morton (1992) framework, the pattern in Table 1 is suggestive of innovations to the term structure of instantaneous forward rates

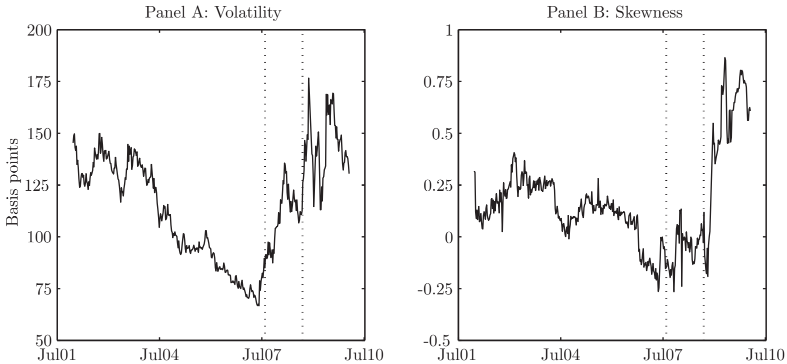

Figure 1
Volatility and skewness of the USD 10-year swap rate at a 1-year horizon
Panel A (Panel B) shows the time series of conditional volatility (skewness) of the USD 10-year swap rate at a 1-year horizon. Conditional volatility is the square-root of conditional variance, annualized, and measured in basis points; that is, $\sqrt{\mathrm{Var}_t^{\mathbb{A}}\left(S_{m,n}(T_m)\right)/(T_m-t)}\times 10{,}000$. Conditional volatility and skewness are computed under the annuity measure $\mathbb{A}$. The vertical dotted lines mark the beginning of the financial crisis on August 9, 2007, and the Lehman Brothers bankruptcy filing on September 15, 2008. The sample period is from December 19, 2001 to January 27, 2010 (419 weekly observations).

exhibiting a hump shape. In that case, starting from a short swap maturity, $T_n - T_m$, and a short horizon, $T_m - t$, the instantaneous variance (and the annualized expected realized variance from t to T_m) of a forward swap rate initially increases, but eventually decreases, with swap maturity and/or horizon.

Conditional volatility exhibits significant variation over time. For instance, Panel A in Figure 1 displays the conditional volatility of the 10-year swap rate at a 1-year horizon in the USD market, which varies between 67 bps (on June 6, 2007) and 177 bps (on November 19, 2008) during the sample period. Table 1 shows that for a given swap maturity, the variation in conditional volatility decreases with option expiry. This is suggestive of the instantaneous variance of forward swap rates being stochastic and mean reverting.

To investigate commonality in volatility dynamics across swap maturities and option expiries, we perform a PCA of weekly changes in conditional volatilities. We observe large common variation in conditional volatility; for instance, in the USD market, the first two PCs explain 84% and 9% of the variation, respectively. Panels A and B in Figure 2 display their loadings on the volatility surface. For both PCs, it holds that the impact along the swap maturity dimension is relatively flat (albeit with a slight hump), whereas the impact along the option expiry dimension is more distinct. The first PC impacts all volatilities positively, with the effect decreasing in option expiry. The second PC has a positive impact on volatility at short option expiries and a negative impact on volatility at medium-to-long option expiries. Therefore, it mainly affects the slope of the term structure of volatility along the option expiry dimension. The factor loadings look similar in the EUR market. Roughly similar results are obtained by Heidari and Wu (2003) and Carr, Gabaix, and Wu (2009) in their

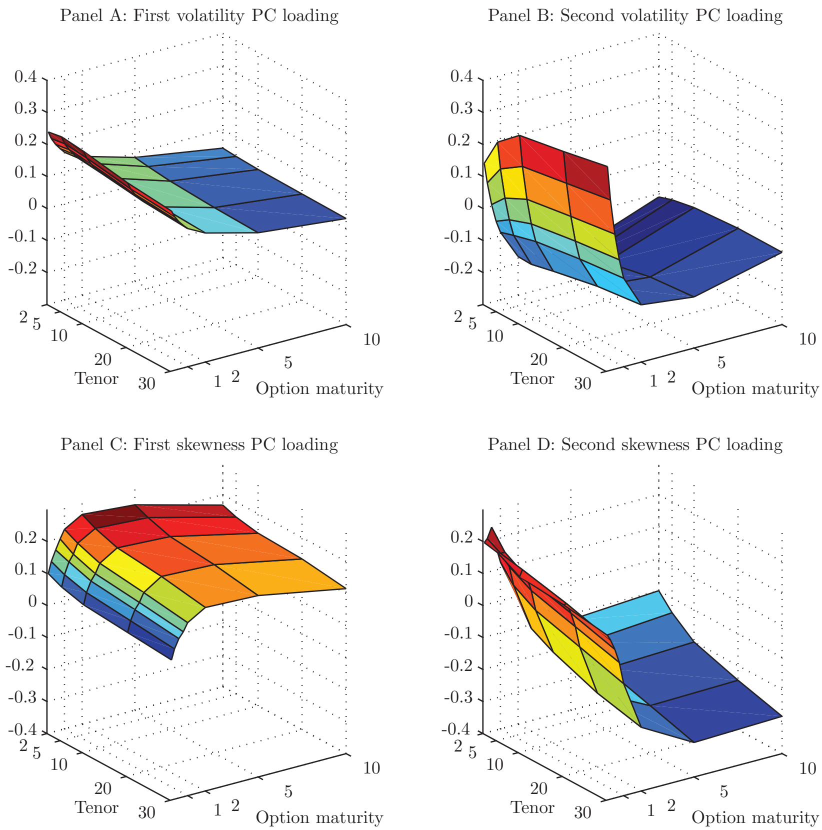

Figure 2
Factor loadings on volatility and skewness surface in the USD market
Panel A (Panel B) shows the loading of the first (second) principal component of volatility on the volatility surface. These loadings are constructed from the eigenvectors corresponding to the two largest eigenvalues of the covariance matrix of weekly changes in conditional volatility. Panel C (Panel D) shows the loading of the first (second) principal component of skewness on the skewness surface. These loadings are constructed from the eigenvectors corresponding to the two largest eigenvalues of the covariance matrix of weekly changes in conditional skewness. Conditional volatility and skewness are computed under the annuity measure $\mathbb{A}$. The sample period is from December 19, 2001 to January 27, 2010 (418 weekly observations).

principal component analyses of ATM log-normal implied volatility surfaces. Taken together, these results indicate the existence of multiple variance factors, one of which is highly persistent (under the pricing measure).[9]

2.2.2 Skewness. Table 2 displays results for the conditional skewness of the future swap rate for different swap maturities and at different horizons.

9 Another way to see the multifactor nature of volatility dynamics is via the autocorrelation of the volatility time series. For both markets it holds that the autocorrelation is fairly uniform along the swap maturity dimension but increases along the option expiry dimension, again pointing to the existence of both a transitory and persistent component of volatility.

Table 2
Skewness of future swap rates

Tenor	Horizon							
	1 mo	3 mo	6 mo	1 yr	2 yr	5 yr	10 yr	Slope
Panel A: USD market								
2 yr	0.00	0.20	0.24	0.27	0.30	0.43	0.42	0.42***
	(0.30)	(0.36)	(0.43)	(0.42)	(0.41)	(0.33)	(0.29)	[7.09]
5 yr	0.01	0.17	0.19	0.21	0.23	0.33	0.37	0.36***
	(0.20)	(0.23)	(0.28)	(0.28)	(0.34)	(0.35)	(0.30)	[6.73]
10 yr	−0.00	0.15	0.16	0.16	0.18	0.29	0.32	0.33
	(0.17)	(0.18)	(0.22)	(0.23)	(0.30)	(0.34)	(0.31)	[6.44]
20 yr	−0.03	0.12	0.13	0.13	0.18	0.27	0.30	0.33***
	(0.13)	(0.13)	(0.16)	(0.19)	(0.25)	(0.32)	(0.36)	[4.65]
30 yr	−0.04	0.10	0.12	0.12	0.15	0.22	0.27	0.31***
	(0.14)	(0.13)	(0.15)	(0.17)	(0.23)	(0.30)	(0.33)	[4.42]
Slope	−0.04	−0.10	−0.12	−0.15*	−0.15**	−0.21***	−0.15***	—
	[−0.85]	[−1.31]	[−1.19]	[−1.72]	[−2.57]	[−11.47]	[−5.70]	
Panel B: EUR market								
2 yr	0.00	0.17	0.28	0.37	0.47	0.60	0.63	0.63***
	(0.15)	(0.21)	(0.27)	(0.29)	(0.35)	(0.33)	(0.38)	[7.33]
5 yr	−0.15	−0.03	0.04	0.13	0.22	0.40	0.46	0.61***
	(0.38)	(0.41)	(0.40)	(0.35)	(0.30)	(0.27)	(0.25)	[5.27]
10 yr	−0.21	−0.11	−0.06	0.02	0.12	0.29	0.36	0.57***
	(0.37)	(0.39)	(0.36)	(0.30)	(0.25)	(0.25)	(0.25)	[8.16]
20 yr	−0.29	−0.16	−0.11	−0.01	0.05	0.24	0.35	0.64***
	(0.28)	(0.27)	(0.24)	(0.22)	(0.22)	(0.24)	(0.27)	[9.08]
30 yr	−0.32	−0.18	−0.13	−0.04	0.00	0.22	0.33	0.65***
	(0.29)	(0.27)	(0.25)	(0.24)	(0.25)	(0.28)	(0.34)	[8.49]
Slope	−0.32***	−0.35***	−0.41***	−0.41***	−0.47***	−0.39***	−0.30***	—
	[−4.33]	[−3.98]	[−4.16]	[−4.46]	[−5.44]	[−6.44]	[−2.73]	

This table shows summary statistics of conditional skewness of future swap rates. For each combination of swap maturity (tenor) and horizon, the table reports the sample mean of conditional skewness, with the sample standard deviation of conditional skewness reported in parenthesis. For each swap maturity, the column labeled “Slope” reports the sample mean of the difference between conditional skewness at the 10-year and 1-month horizons, with associated t-statistics reported in brackets. For each horizon, the row labeled “Slope” reports the sample mean of the difference between conditional skewness of the 30-year and 2-year swap rates, with associated t-statistics reported in brackets. t-statistics are corrected for heteroscedasticity and serial correlation up to 52 lags (i.e., one year) using the approach of Newey and West (1987). Conditional skewness is computed under the annuity measure $\mathbb{A}$. In the USD market, the sample period is from December 19, 2001 to January 27, 2010 (419 weekly observations). In the EUR market, the sample period is from June 6, 2001 to January 27, 2010 (449 weekly observations). *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

Like the previous table, it reports the sample means and, in parentheses, the sample standard deviations. It also reports the sample means of the slopes of the skewness surface along both the swap maturity and horizon dimensions, with Newey and West (1987) t-statistics in brackets. Conditional skewness is positive, on average, for most combinations of swap maturity and option expiry, reflecting that OTM payer swaptions tend to be more expensive than equivalently OTM receiver swaptions. At any horizon, average conditional skewness decreases with swap maturity, and the spreads between conditional skewness of the 30-year and 2-year swap rates are highly statistically significant at longer horizons (all horizons in the EUR market). For a given swap maturity, average conditional skewness is a concave and mostly increasing function of horizon, with the spreads between conditional skewness at the 10-year and 1-month horizons being highly statistically significant for all swap maturities.

These results hold true in both markets. In the USD market, the sample means of conditional skewness vary between -0.04 (for the 30-year swap rate at a 1-month horizon) and 0.43 (for the 2-year swap rate at a 5-year horizon). In the EUR market, skewness is generally lower at short horizons and higher at long horizons.

To understand the implications for model design, we consider again the partition (14), write $V(t) = \mathrm{Var}_t^{\mathbb{A}}(S_{m,n}(T_m))$ for notational ease, and let $\Delta V(t_i) = V(t_i) - V(t_{i-1})$ denote changes in the conditional variance of the future swap rate. Applying Proposition 1 in Neuberger (2012) to our setting, where $S_{m,n}(t)$ is a martingale under $\mathbb{A}$, we have

$$\begin{aligned} E_t^{\mathbb{A}}\left[(S_{m,n}(T_m) - \mu_t)^3\right] &= E_t^{\mathbb{A}}\left[(S_{m,n}(T_m) - S_{m,n}(t))^3\right] \\ &= E_t^{\mathbb{A}}\left[\sum_{i=1}^{N}\left((\Delta S(t_i))^3 + 3\Delta S(t_i)\Delta V(t_i)\right)\right]. \end{aligned} \tag{16}$$

This shows that the third moment of the future swap rate at a given horizon arises from two sources. The first source is the conditional expectation of the realized third moment of the forward swap rate. The second source is the conditional expectation of the realized covariation between the forward swap rate and the conditional variance of the future swap rate. Scaling by $V(t)^{3/2}$, we get

$$\mathrm{Skew}_t^{\mathbb{A}}(S_{m,n}(T_m)) = \frac{1}{V(t)^{3/2}} E_t^{\mathbb{A}}\left[\sum_{i=1}^{N}\left((\Delta S(t_i))^3 + 3\Delta S(t_i)\Delta V(t_i)\right)\right]. \tag{17}$$

Consequently, one approach to generating skewness is assuming that increments to swap rates are independent and identically distributed (iid.) with a common asymmetric distribution. However, in this case skewness decays rapidly like the reciprocal of the square root of the horizon, which is clearly inconsistent with Table 2.[10] Another approach to generating skewness is assuming that increments to swap rates are correlated with increments to the swap rate variances. In a simple setting with a single swap rate factor and a single (stationary) variance factor, skewness is a hump-shaped function of the horizon, with the sign depending on the correlation between rates and variances (see, e.g., Das and Sundaram 1999 for an analysis of the term structure of conditional moments in stochastic volatility models). In a more realistic setting with multiple factors driving swap rates and swap rate variances, much more intricate skewness patterns are possible. If the variance factors are stationary, skewness eventually decreases with the horizon. However, in the presence of a highly persistent variance factor (under the pricing measure), skewness may peak at very long horizons, similar to what is observed in Table 2.

[10] The decay in skewness follows from the central limit theorem, which requires the increments to have finite second moments. Carr and Wu (2003) develop a class of models in which this assumption is violated and skewness does not decay with the horizon despite increments being iid.

Like conditional volatility, conditional skewness also exhibits significant variation over time. The variation is such that the sign of conditional skewness sometimes changes. For example, Panel B in Figure 1 displays the conditional skewness of the 10-year swap rate at a 1-year horizon in the USD market, which varies between -0.38 (on October 31, 2007) and 0.87 (on April 29, 2009) through the sample period. Table 2 shows that for most combinations of swap maturity and option expiry, conditional skewness is even more volatile in the EUR market than in the USD market. These time-series dynamics are suggestive of swap rates and swap rate variances exhibiting time-varying correlation with occasional sign changes. We return to this issue in Section 2.3.

A PCA of weekly changes in conditional skewness across swap maturities and option expiries reveals large common variation in conditional skewness. For instance, in the USD market, the first two PCs explain 85% and 7% of the variation, respectively. Panels C and D in Figure 2 display their loadings on the skewness surface. As for volatility, the impact along the swap maturity dimension is relatively flat (slightly decreasing), whereas the more interesting action is along the option expiry dimension. The first PC affects all skewness coefficients positively, with a distinct hump shape along the option expiry dimension. The second PC has a positive impact on skewness at short option expiries and a negative impact on skewness at medium to long option expiries. Therefore, it mainly affects the slope of the term structure of skewness along the option expiry dimension. Roughly similar factor loadings are observed in the EUR market.

2.2.3 Kurtosis. Because of space constraints, we only briefly summarize the results for the conditional kurtosis of future swap rates. Conditional kurtosis is always in excess of three, indicating that the swap rate distributions have heavier tails than the normal distribution. For a given swap maturity, the term structure of conditional kurtosis is hump shaped with a peak between 5 and 10 years, again suggestive of the existence of a highly persistent variance factor. Conditional kurtosis appears to be more stable over time than is conditional volatility and skewness. For example, the conditional kurtosis of the USD 10-year swap rate at a 1-year horizon varies between 3.22 and 3.67 through the sample period. Furthermore, a PCA of weekly changes in conditional kurtosis across swap maturities and option expiries shows that variation in conditional kurtosis is less systematic than is the case for conditional volatility and skewness. For instance, in the USD market, the first two PCs only explain 42% and 11% of the variation, respectively. For this reason, in the rest of the paper, we will mainly focus on volatility and skewness.

2.2.4 Liquidity issues. A natural question is how sensitive our results are to liquidity issues in the swaption market. Taking the 1-year option on the 10-year swap rate in the USD market as an example, we polled major investment banks for indicative bid-ask spreads in 2010. Typical bid-ask spreads for ATM, 50 bp

OTM, and 100 bp OTM swaptions were 1.5%, 2%, and 5%, respectively, of the midprice. We recompute conditional volatility, skewness, and kurtosis of the 10-year swap rate at a 1-year horizon in the USD market using, respectively, bid prices and ask prices. In doing so, we assume that bid-ask spreads are 1.5% for strikes up to 25 bp OTM, 2% for strikes between 25 bp and 75 bp OTM, and 5% for strikes of more than 75 bp OTM. We apply these bid-ask spreads to the entire period, although it is likely that bid-ask spreads were wider at the height of the financial crises, and tighter in the middle of the decade.

For conditional volatility and skewness, the impact of bid-ask spreads is small. The average conditional volatility computed from bid, mid, and ask prices is 114.0, 114.8, and 115.7 bps, respectively, and the average conditional skewness computed from bid, mid, and ask prices is 0.1592, 0.1595, and 0.1598, respectively. Although OTM swaptions with larger bid-ask spreads receive more weight in the computation of skewness than in the computation of volatility, OTM receiver and payer swaptions enter with opposite signs in the skewness formula, resulting in an overall small impact of bid-ask spreads. For conditional kurtosis, the impact of bid-ask spreads is somewhat larger, though still economically small, with the average conditional kurtosis computed from bid, mid, and ask prices being equal to 3.436, 3.424, and 3.411, respectively.

2.3 Linkages among the term structure, volatility, and skewness

We now consider the linkages among the term structure, volatility, and skewness. To make the analysis concise, we base it on principal components that parsimoniously describe the complex moment structures. Before proceeding, we briefly discuss the PCs of the term structure. As expected, interest rates exhibit significant common variation; for instance, a PCA applied to weekly swap rate changes in the USD market shows that the first three PCs explain 91%, 6%, and 2% of the variation, respectively. The three PCs act as level, slope, and curvature shocks to the term structure.

In the following, we let PC_i^{IR}, PC_i^{V}, and PC_i^{S} denote the i'th PC of weekly changes in interest rates, volatilities, and skewness, respectively. Table 3 shows unconditional correlations among the first three interest rate PCs, the first two volatility PCs, and the first two skewness PCs (we include the third interest rate PC, because theory suggests a relation between the curvature of the term structure and volatility).[11] We display results for the full sample period, for a precrisis subsample using data to August 8, 2007, and for a crisis subsample using data after August 08, 2007. To increase readability, we have highlighted in bold those correlations that are greater than or equal to 0.20 in absolute values.

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11 By construction, PC_1^{IR}, PC_2^{IR}, and PC_3^{IR} are unconditionally uncorrelated. The same applies to PC_1^{V} and PC_2^{V}, as well as to PC_1^{S} and PC_2^{S}. Table 3 displays contemporaneous correlations between PCs, and we have also computed correlations at leads and lags up to ten weeks, but find only few instances of significant lead-lag relationships.

Table 3
Correlations between swap rates, volatility, and skewness

	Full sample				Precrisis				Crisis			
	PC_1^V	PC_2^V	PC_1^S	PC_2^S	PC_1^V	PC_2^V	PC_1^S	PC_2^S	PC_1^V	PC_2^V	PC_1^S	PC_2^S
Panel A: USD market												
PC_1^{IR}	**0.29**	−0.06	**−0.21**	0.08	**0.30**	0.06	**−0.37**	−0.18	**0.30**	−0.14	−0.11	−0.12
PC_2^{IR}	0.10	0.06	0.07	0.19	0.18	−0.05	−0.09	**0.28**	0.04	0.08	0.12	0.10
PC_3^{IR}	**0.23**	−0.00	−0.17	−0.15	**0.25**	**−0.23**	0.03	−0.20	**0.26**	0.02	**−0.22**	−0.04
PC_1^V	1	0	**−0.22**	−0.18	1	0	0.06	0.02	1	0	**−0.32**	−0.12
PC_2^V	0	1	0.12	−0.16	0	1	0.10	0.08	0	1	0.13	−0.12
Panel B: EUR market												
PC_1^{IR}	0.15	−0.01	**−0.40**	0.11	**0.23**	0.10	−0.16	**−0.22**	0.10	−0.10	**−0.63**	0.03
PC_2^{IR}	−0.13	−0.15	−0.13	0.16	0.05	−0.03	0.09	−0.17	**−0.23**	−0.19	**−0.21**	**−0.23**
PC_3^{IR}	**0.38**	−0.08	0.11	0.05	0.16	0.10	0.08	0.05	**0.51**	−0.09	0.07	0.12
PC_1^V	1	0	**0.20**	−0.01	1	0	0.14	0.08	1	0	**0.25**	0.18
PC_2^V	0	1	0.12	−0.02	0	1	−0.01	−0.15	0	1	0.17	−0.04

This table shows unconditional correlations between principal components (PCs) of swap rates, volatilities, and skewness. PC_1^{IR}, PC_2^{IR}, and PC_3^{IR} denote the first three PCs of weekly changes in swap rates across swap maturities. PC_1^V and PC_2^V denote the first two PCs of weekly changes in conditional volatilities across the swap maturity-option expiry grid. PC_1^S and PC_2^S denote the first two PCs of weekly changes in conditional skewness across the swap maturity-option expiry grid. Conditional volatility and skewness are computed under the annuity measure $\mathbb{A}$. In the USD market, the full sample period is from December 19, 2001 to January 27, 2010 (418 observations). In the EUR market, the full sample period is from June 6, 2001 to January 27, 2010 (448 observations). The precrisis period covers data until August 8, 2007 (290 and 320 observations in the USD and EUR markets, respectively), and the crisis period covers data after August 8, 2007 (128 observations in both markets). Correlations greater than or equal to 0.20 in absolute values are highlighted in bold.

2.3.1 Volatility. For volatility, we observe a positive correlation between the first volatility PC and the first interest rate PC (the level factor of the term structure). This result holds true in all sample periods and both currencies, with the relation being strongest in the USD market. For example, for the full sample period in the USD market, the correlation is 0.29. The positive level-volatility relation is consistent with papers showing a level-dependence in short-rate volatility (see, e.g., Andersen and Lund 1997; Ball and Torous 1999). It is also consistent with the observation that conditional skewness is positive, on average, for most combinations of swap maturity and option expiry (Table 2).

We also find a positive correlation between the first volatility PC and the third interest rate PC (the curvature factor of the term structure). Again, this holds true across all sample periods and both currencies. For instance, for the full sample period in the USD market, the correlation is 0.23. If anything, the relation is stronger in the EUR market, particularly in the crisis period. The positive curvature-volatility relation is a feature of most term structure models as explained in, for example, Brown and Schaefer (1994), and is also consistent with the empirical findings in Litterman, Scheinkman, and Weiss (1991) in the case of the Treasury market.

We only find a small positive correlation between the first volatility PC and the second interest rate PC (the slope factor of the term structure). Furthermore, in the case of the second volatility PC, we do not find a systematic relation with the term structure in either of the markets. That is, variation in the slope of the term structure of volatility along the option expiry dimension appears largely unrelated to variation in the shape of the swap term structure.

2.3.2 Skewness. Concerning the relation between skewness and the term structure, we observe a negative correlation between the first skewness PC and the first interest rate PC. That is, an increase in the level of swap rates tends to be associated with a general decrease in the skewness of future swap rates. This result holds true in all sample periods and both currencies. For instance, for the full sample period in the USD market, the correlation is -0.21. The relation is stronger in the EUR market, particularly in the crisis period. Concerning the relation between skewness and volatility, the results are less clear. In the USD market, we observe a negative correlation between the first skewness PC and the first volatility PC, whereas in the EUR market we observe a positive correlation. For the second skewness PC, we do not find a systematic relation with the term structure or volatility in either of the markets, suggesting that this component of skewness evolves relatively autonomously.[12]

As noted in Section 2.2.2, the dynamics of conditional skewness provide indirect evidence of stochastic correlations between swap rates and swap rate variances. We now look for direct evidence of stochastic correlations and investigate if there is a link between variation in skewness and variation in correlations. Of course, skewness reflects correlation expectations under the pricing measure, and time-varying skewness risk premia will weaken the link between skewness and observed correlations. We leave a discussion of skewness risk premia for Section 2.4 and proceed here in three steps. First, for each combination of swap maturity and option expiry, we infer the conditional correlation between daily increments to the forward swap rate, $\Delta S(t)$, and the conditional variance of the future swap rate, $\Delta V(t)$, using the dynamic conditional correlation (DCC) GARCH model of Engle (2002). Recent applications of this framework include Baele, Bekaert, and Inghelbrecht (2010) and Bali and Engle (2010), and details are provided in the Online Appendix. Conditional correlations vary significantly over time with occasional sign changes. Taking the 10-year swap maturity and the 1-year option expiry in the USD market as an example, the conditional correlation varies between -0.65 and 0.82 over the full sample period with an unconditional correlation of 0.25.

Second, we perform a PCA of weekly changes in conditional correlations across swap maturities and option expiries. This reveals large common variation

12 In addition to correlations between principal components, we also compute correlations between changes in forward swap rates, volatilities, and skewness for each combination of swap maturity and option expiry. Taking the full sample period in the USD market as an example, the forward rate-volatility correlation ranges from 0.16 to 0.37, the forward rate-skewness correlation ranges from -0.34 to 0.03, and the volatility-skewness correlation ranges from -0.33 to 0.08. This is consistent with the correlations between PC_1^{IR}, PC_1^{V}, and PC_1^{S} in Table 3.

Table 4
Explaining variation in skewness

	Full sample			Precrisis			Crisis		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
Panel A: USD market									
PC_1^{IR}	−0.08*		−0.11***	−0.13***		−0.14***	−0.02		−0.07
	[−1.80]		[−2.65]	[−7.31]		[−8.35]	[−0.16]		[−0.80]
PC_1^V	−0.07		−0.07	0.06**		0.05*	−0.24*		−0.18*
	[−1.24]		[−1.17]	[2.54]		[1.84]	[−1.93]		[−1.69]
PC_1^C		0.12**	0.14***		0.07***	0.08***		0.28*	0.26*
		[2.24]	[2.73]		[3.32]	[4.43]		[1.75]	[1.73]
R^2	0.06	0.06	0.14	0.16	0.04	0.23	0.09	0.12	0.19
Panel B: EUR market									
PC_1^{IR}	−0.12***		−0.12***	−0.02		−0.04**	−0.20***		−0.19***
	[−5.03]		[−5.12]	[−1.08]		[−2.14]	[−4.03]		[−3.66]
PC_1^V	0.06***		0.06***	0.04*		0.03	0.08**		0.09**
	[3.25]		[2.98]	[1.85]		[1.35]	[2.23]		[2.15]
PC_1^C		0.09***	0.09***		0.10***	0.10***		0.10**	0.06**
		[3.61]	[4.53]		[3.35]	[3.67]		[2.44]	[1.97]
R^2	0.21	0.09	0.30	0.02	0.11	0.14	0.45	0.09	0.48

This table shows results from regressing PC_1^S (the first principal component [PC] of weekly changes in conditional skewness across the swap maturity-option expiry grid) on PC_1^{IR} (the first PC of weekly changes in swap rates across swap maturities), PC_1^V (the first PC of weekly changes in conditional volatilities across the swap maturity-option expiry grid), and PC_1^C (the first PC of weekly changes in conditional correlations between forward swap rates and variances across the swap maturity-option expiry grid). PC_1^{IR}, PC_1^V, and PC_1^C are normalized to have unit standard deviation. t-statistics are reported in brackets and are corrected for heteroscedasticity and serial correlation up to two lags using the approach of Newey and West (1987). In the USD market, the full sample period is from December 19, 2001 to January 27, 2010 (418 observations). In the EUR market, the full sample period is from June 6, 2001 to January 27, 2010 (448 observations). The precrisis period covers data until August 8, 2007 (290 and 320 observations in the USD and EUR markets, respectively), and the crisis period covers data after August 8, 2007 (128 observations in both markets). *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

in conditional correlations for all sample periods and both currencies. For instance, for the full sample period in the USD market, the first two PCs explain 73% and 8% of the variation, respectively.

Third, we run versions of the following regression

$$PC_{1,t}^S = \beta_0 + \beta_1 PC_{1,t}^{IR} + \beta_2 PC_{1,t}^V + \beta_3 PC_{1,t}^C + \epsilon_t, \tag{18}$$

where PC_1^C denotes the first PC of weekly changes in correlations, and PC_1^{IR}, PC_1^V, and PC_1^S were defined previously. Table 4 shows the results, with t-statistics corrected for heteroscedasticity and serial correlation in brackets. In the univariate regressions involving only PC_1^C, its coefficient is positive and statistically significant. This holds true in all sample periods and both currencies. In the multivariate regressions, the coefficient on PC_1^C remains positive and statistically significant. The signs of the coefficients on PC_1^{IR} and PC_1^V are consistent with the correlations reported in Table 3 but are only statistically significant in certain samples.[13] We conclude that while conditional

[13] We confirm the results in Table 4 by running the regressions for each combination of swap maturity and option expiry instead of using principal components. That is, regressing changes in conditional skewness on changes

skewness displays some relation to the level and volatility of swap rates, it is most consistently related to the conditional correlation between swap rates and swap rate variances.

2.4 Variance and skewness risk premia

The pricing of variance risk has received considerable attention in the equity literature. Here, we quantify variance and skewness risk premia in the interest rate swap market using realized excess returns on synthetic variance and skewness swap contracts. This approach is similar to that taken in Carr and Wu (2009) to quantify variance risk premia for equity indexes and individual stocks, although the contract designs differ.

2.4.1 Variance and skewness swap contracts. We consider again the partition (14) and use the notation in which $\Delta S(t_i)$ and $\Delta V(t_i)$ denote changes in the forward swap rate and the conditional variance of the future swap rate, respectively.

In our variance swap design, the payoff at time T_m is given by

$$(\mathrm{RVar}(t,T_m)-K)\times A_{m,n}(T_m), \tag{19}$$

where

$$\mathrm{RVar}(t,T_m)=\sum_{i=1}^{N}(\Delta S(t_i))^2 \tag{20}$$

is the realized variance of $S_{m,n}(t)$ over the period t to T_m and K is a constant. The multiplication by the annuity factor is for tractability purposes and is due to Mele and Obayashi (2012), who propose a related variance swap design. The time-t variance swap rate, $\mathcal{V}(t,T_m)$, is defined as the value of K that makes the variance swap have zero initial value. That is, $\mathcal{V}(t,T_m)$ solves

$$E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_m} r(s)ds}(\mathrm{RVar}(t,T_m)-\mathcal{V}(t,T_m))\times A_{m,n}(T_m)\right]=0, \tag{21}$$

which, after changing measure from $\mathbb{Q}$ to $\mathbb{A}$ and using (15), implies that

$$\mathcal{V}(t,T_m)=\mathrm{Var}_t^{\mathbb{A}}(S_{m,n}(T_m)). \tag{22}$$

In our skewness swap design, the payoff at time T_m is given by

$$(\mathrm{RSkew}(t,T_m)-K)\times A_{m,n}(T_m), \tag{23}$$

in the forward swap rate, the conditional volatility, and the conditional correlation between the forward swap rate and the conditional variance of the future swap rate. Taking the multivariate regression estimated on the full sample period in the USD market as an example, the coefficient on the conditional rate-variance correlation is statistically significant at the 5% level in 37 out of 40 regressions.

where

$$\text{RSkew}(t,T_m)=\frac{1}{V(t)^{3/2}}\sum_{i=1}^{N}\left((\Delta S(t_i))^3+3\Delta S(t_i)\Delta V(t_i)\right) \tag{24}$$

is the realized skewness over the period t to T_m, K is a constant, and the multiplication by the annuity factor is again for tractability purposes. The time-t skewness swap rate, $\mathcal{S}(t,T_m)$, is defined as the value of K that makes the skewness swap have zero initial value. That is, $\mathcal{S}(t,T_m)$ solves

$$E_t^{\mathbb{Q}}\left[e^{-\int_t^{T_m} r(s)ds}(\text{RSkew}(t,T_m)-\mathcal{S}(t,T_m))\times A_{m,n}(T_m)\right]=0, \tag{25}$$

which, after changing measure from $\mathbb{Q}$ to $\mathbb{A}$ and using (17), implies that

$$\mathcal{S}(t,T_m)=\text{Skew}_t^{\mathbb{A}}(S_{m,n}(T_m)). \tag{26}$$

Note that the model-independent valuation of the variance and skewness swaps hold true regardless of whether the underlying interest rate exhibits jumps and despite the floating legs of the variance and skewness swaps being computed by sampling interest rates (and variances) discretely. It only relies on $S_{m,n}(t)$ being a martingale under $\mathbb{A}$ and, in the case of the skewness swap, $V(t)$ being observable. This robustness is in contrast to the variance swap designs based on log returns typically used in the equity market (see, e.g., the discussion in Carr, Lee, and Wu 2012).

2.4.2 Implementation. Long positions (receive realized, pay fixed) in variance and skewness swaps are initiated on the last business day of each month using closing prices (we have verified that our results hold true regardless of the day of the month that the positions are initiated). Realized variance and skewness are computed from daily data. Because variance and skewness swaps, like regular interest rate swaps, are zero net present value investments, we follow Duarte, Longstaff, and Yu (2007) in assuming that there is an initial amount of capital, X, required to enter the contracts. In this case, the excess returns on variance and skewness swaps from t to T_m are

$$(\text{RVar}(t,T_m)-\mathcal{V}(t,T_m))\times A_{m,n}(T_m)\big/X \tag{27}$$

and

$$(\text{RSkew}(t,T_m)-\mathcal{S}(t,T_m))\times A_{m,n}(T_m)\big/X, \tag{28}$$

respectively. As in Duarte, Longstaff, and Yu (2007), we adjust the amount of initial capital to achieve a 10% unconditional annualized volatility of each strategy's realized excess returns.[14]

[14] The choice of 10% is inconsequential for our conclusions. A more sophisticated approach would adjust X over time to achieve a certain conditional excess return volatility. However, because that would reduce the sample size by the length of the initial window used to estimate the conditional volatility model, we do not pursue that approach. Carr and Wu (2009) compute excess variance swap returns by assuming that the required amount of capital equals the variance swap rate. For skewness swaps, this is not a practical return definition, because the skewness swap rate could be zero.

In principle, we could consider variance and skewness swap maturities corresponding to each of the option expiries in the swaption cube. However, to mitigate the problem of overlapping observations, we only consider one-month variance and skewness swaps. In this case, the return time series consist of nonoverlapping monthly excess returns. We consider maturities of the underlying interest rate swaps corresponding to each of the five maturities in the swaption cube (2, 5, 10, 20, and 30 years). For both variance and skewness swaps, we also consider an equally weighted portfolio (in terms of notional) of the five individual swap strategies.

Recall that $S_{m,n}(t)$ is the forward swap rate for the period T_m to T_n. To compute the floating leg payments in the variance and skewness swaps, we need to monitor this rate daily over the interval $[t, T_m]$. That is, we track the forward swap rate until it eventually becomes the spot swap rate at time T_m. Because the data set only contains information on spot swap rates and forward swap rates with forward starting times corresponding to the option expiries, we interpolate with a spline to get the forward swap rates at the required horizons.

Similarly, to compute the floating leg payments in the skewness swap, we need to monitor $V(t)$ daily over the interval $[t, T_m]$. That is, we track the conditional variance until time T_m, when it becomes zero. Again, we only have conditional variances at horizons corresponding to the option expiries, and we interpolate with a spline to get the conditional variances at the required horizons.

2.4.3 Pricing of variance risk. Table 5 shows summary statistics of the performance of variance swaps. Mean excess returns on variance swaps are negative for all underlying interest rate swap maturities, reflecting that interest rate variance implied from swaptions, on average, exceeds subsequent realized interest rate variance. In general, excess returns are more negative in the EUR market than in the USD market. Mean excess returns are an inverse hump-shaped function of the underlying interest rate swap maturity, being most negative at the 5-year maturity. This mirrors the hump-shaped function of average conditional variances in the swap maturity dimension observed in Table 1 and suggests a positive relationship between the level of variance and the absolute magnitude of the variance risk premium in the cross-section. t-statistics adjusted for heteroscedasticity and autocorrelation show that most of the mean excess returns are statistically significant, particularly for intermediate interest rate swap maturities.

Excess returns display positive excess kurtosis, but no consistent sign on skewness (positive in the USD market and negative in the EUR market). Therefore, within the sample, the variance risk premium does not appear to be a compensation for crash risk. Excess returns exhibit very little autocorrelation in the USD market, but positive autocorrelation in the EUR market.

Annualized Sharpe ratios corrected for return-autocorrelation using the approach of Lo (2002) lie between -0.55 and -0.69 in the USD market and

Table 5
Pricing of variance risk

Tenor	Mean	SD	Skew	Kurt	Auto	SR
Panel A: USD market						
2 yr	−0.44*	2.89	3.69	16.21	−0.15	−0.60*
	[−1.72]					[−1.69]
5 yr	−0.66**	2.89	1.70	8.61	0.01	−0.69**
	[−2.24]					[−2.19]
10 yr	−0.58*	2.89	0.86	6.12	0.06	−0.64*
	[−1.91]					[−1.86]
20 yr	−0.50*	2.89	0.01	9.98	−0.03	−0.64*
	[−1.79]					[−1.75]
30 yr	−0.42	2.89	0.80	11.06	0.02	−0.55
	[−1.47]					[−1.44]
EW	−0.49*	2.71	0.33	8.75	0.02	−0.63*
	[−1.78]					[−1.74]
Panel B: EUR market						
2 yr	−0.88**	2.89	−1.21	6.00	0.45	−0.60**
	[−2.40]					[−2.29]
5 yr	−1.53***	2.89	−2.06	8.27	0.62	−0.81***
	[−3.86]					[−3.40]
10 yr	−1.17***	2.89	−2.35	11.25	0.29	−0.73***
	[−3.52]					[−3.31]
20 yr	−0.71**	2.89	−2.53	12.69	0.24	−0.55*
	[−2.05]					[−1.95]
30 yr	−0.50	2.89	−3.08	14.57	0.24	−0.42
	[−1.43]					[−1.39]
EW	−0.71**	2.71	−3.44	14.59	0.27	−0.55**
	[−2.15]					[−2.07]

This table shows summary statistics of nonoverlapping monthly excess returns on one-month variance swaps. The swap contracts are initiated on the last business day of each month using closing prices. The maturities of the underlying interest rate swaps are 2, 5, 10, 20, and 30 years, respectively. Returns are computed from the perspective of the counterpart who receives the floating leg. We assume that an initial amount of capital is required to enter each of the variance swap contracts and adjust this amount to achieve a 10% annualized excess return volatility. "EW" denotes an equally weighted (based on notional amounts) portfolio of the five individual variance swaps. "Mean," "SD," "Skew," "Kurt," and "Auto" denote the sample mean, standard deviation, skewness, excess kurtosis, and first-order autocorrelation of excess returns, respectively. "SR" denotes the annualized Sharpe ratio corrected for return-autocorrelation using the approach of Lo (2002). t-statistics are reported in brackets and are corrected for heteroscedasticity and serial correlation up to two lags using the approach of Newey and West (1987). In the USD market, the time series consist of 97 monthly returns from January 2002 to January 2010. In the EUR market, the time series consist of 103 monthly returns from July 2001 to January 2010. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

-0.42 and -0.81 in the EUR market, and most are statistically significant. The equally weighted portfolios produce annualized Sharpe ratios of -0.63 and -0.55 in the USD and EUR markets, respectively. Similar, though somewhat model dependent, results are obtained by Duarte, Longstaff, and Yu (2007), who report annualized Sharpe ratios between 0.47 and 0.82 on a strategy of selling interest rate volatility via delta-hedged caps. Results in Joslin (2007) and Mueller, Vedolin, and Yen (2013) are also indicative of a negative interest rate variance risk premium. Practitioners have long realized the benefits of selling volatility either directly or indirectly via the prepayment options embedded in mortgage-backed securities (see, e.g., Goodman and Ho 1997).

Figure 3 illustrates the performance of the variance swap strategy in the USD market, in the case in which the underlying interest rate swap has a 10-year

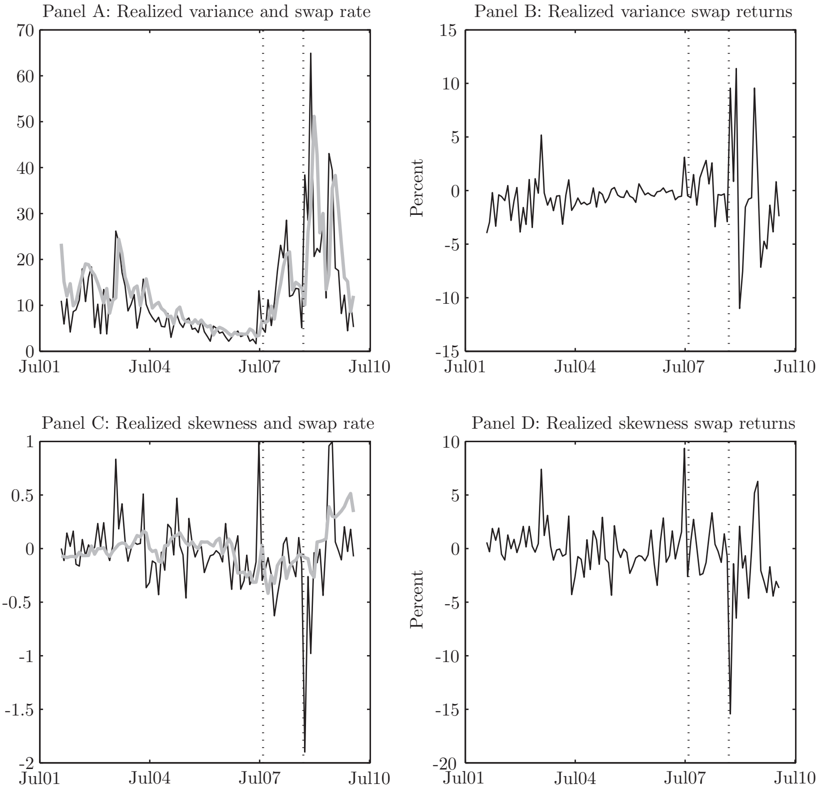

Figure 3
Excess returns on variance and skewness swaps on the USD 10-year swap rate
Panel A shows the variance swap rate ($\mathcal{V}(t,T_m)$, thick gray line) and the realized variance (RVar(t,T_m), thin black line) for nonoverlapping one-month variance swaps, for which the underlying interest rate swap has a 10-year maturity. Both $\mathcal{V}(t,T_m)$ and RVar(t,T_m) are multiplied by 10^6. Panel B shows the corresponding monthly realized excess returns. Panel C shows the skewness swap rate ($\mathcal{S}(t,T_m)$, thick gray line) and the realized skewness (RSkew(t,T_m), thin black line) for nonoverlapping one-month skewness swaps, for which the underlying interest rate swap has a 10-year maturity. Panel D shows the corresponding monthly realized excess returns. The swap contracts are initiated on the last business day of each month using closing prices and are held to maturity. The swap contracts are capitalized to achieve an annualized excess return volatility of 10%. The vertical dotted lines mark the beginning of the financial crisis on August 9, 2007 and the Lehman Brothers bankruptcy filing on September 15, 2008. The time series consist of 97 monthly observations from January 2002 to January 2010.

maturity. Panel A shows the variance swap rate ($\mathcal{V}(t,T_m)$, thick gray line) and the realized variance (RVar(t,T_m), thin black line), and Panel B displays the realized excess returns. Excess returns exhibit a significant amount of heteroscedasticity, with return volatility being particularly elevated at the height of the financial crisis.

The fact that investors appear willing to accept negative average excess returns on contracts that hedge against unexpected increases in variance suggests that investors dislike higher interest rate variance (and, therefore, also higher bond return variance).

Table 6
Pricing of skewness risk

Tenor	Mean	SD	Skew	Kurt	Auto	Corr	SR
Panel A: USD market							
2 yr	−0.38	2.89	−0.63	17.12	0.06	−0.30	−0.46
	[−1.29]						[−1.26]
5 yr	−0.44	2.89	−0.48	11.74	0.17	−0.07	−0.44
	[−1.49]						[−1.46]
10 yr	−0.32	2.89	−0.90	10.85	0.14	−0.03	−0.31
	[−1.08]						[−1.05]
20 yr	−0.21	2.89	−1.09	11.12	0.18	−0.15	−0.23
	[−0.72]						[−0.70]
30 yr	−0.20	2.89	−1.85	13.52	0.17	−0.27	−0.21
	[−0.70]						[−0.67]
EW	−0.26	2.73	−1.43	12.97	0.16	−0.19	−0.30
	[−0.95]						[−0.94]
Panel B: EUR market							
2 yr	−0.58*	2.89	0.38	2.97	0.36	−0.17	−0.43
	[−1.67]						[−1.62]
5 yr	−0.66*	2.89	0.27	3.14	0.20	−0.28	−0.46*
	[−1.91]						[−1.85]
10 yr	−0.56	2.89	0.58	4.63	0.21	−0.32	−0.38
	[−1.61]						[−1.56]
20 yr	−0.36	2.89	0.08	4.09	0.20	−0.10	−0.25
	[−1.05]						[−1.02]
30 yr	−0.16	2.89	−1.04	6.40	0.29	−0.15	−0.12
	[−0.45]						[−0.42]
EW	−0.34	2.41	0.15	4.81	0.17	−0.19	−0.33
	[−1.19]						[−1.16]

This table shows summary statistics of nonoverlapping monthly excess returns on one-month skewness swaps. The swap contracts are initiated on the last business day of each month using closing prices. The maturities of the underlying interest rate swaps are 2, 5, 10, 20, and 30 years, respectively. Returns are computed from the perspective of the counterpart who receives the floating leg. We assume that an initial amount of capital is required to enter each of the skewness swap contracts and adjust this amount to achieve a 10% annualized excess return volatility. "EW" denotes an equally weighted (based on notional amounts) portfolio of the five individual skewness swaps. "Mean," "SD," "Skew," "Kurt," and "Auto" denote the sample mean, standard deviation, skewness, excess kurtosis, and first-order autocorrelation of excess returns, respectively. "Corr" denotes the correlation between excess returns on skewness swaps and the corresponding variance swaps. "SR" denotes the annualized Sharpe ratio corrected for return-autocorrelation using the approach of Lo (2002). t-statistics are reported in brackets and are corrected for heteroscedasticity and serial correlation up to two lags using the approach of Newey and West (1987). In the USD market, the time series consist of 97 monthly returns from January 2002 to January 2010. In the EUR market, the time series consist of 103 monthly returns from July 2001 to January 2010. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

2.4.4 Pricing of skewness risk. Table 6 shows summary statistics of the performance of skewness swaps. Mean excess returns on skewness swaps are negative for all underlying interest rate swap maturities, reflecting that interest rate skewness implied from swaptions, on average, exceeds subsequent realized interest rate skewness. Again, excess returns tend to be more negative in the EUR market than in the USD market, although in both markets the mean excess returns on skewness swaps are less negative than those on variance swaps. Once we control for heteroscedasticity and the autocorrelation, few of the mean excess returns are statistically significant.

In terms of risk characteristics, excess returns display positive excess kurtosis, but opposite sign on skewness across the two markets (negative in the USD market and positive in the EUR market), and some positive serial dependence, particularly in the EUR market. We also report the correlation between excess returns on skewness swaps and the corresponding variance swaps. The correlations are relatively small (and negative), again indicating that variance and skewness are fairly distinct risk factors.

Annualized Sharpe ratios corrected for return-autocorrelation lie between -0.21 and -0.46 in the USD market and -0.12 and -0.46 in the EUR market, with the equally weighted portfolios producing annualized Sharpe ratios of -0.30 and -0.33 in the two markets, respectively. Only one of the Sharpe ratios is statistically significant.

Figure 3 also illustrates the performance of the skewness swap strategy in the USD market, in the case in which the underlying interest rate swap has a 10-year maturity. Panel C shows the skewness swap rate ($\mathcal{S}(t, T_m)$, thick gray line) and the realized skewness (RSkew(t, T_m), thin black line), and Panel D displays the realized excess returns. It appears that skewness swap returns are less heteroscedastic than variance swap returns.

Even if the skewness risk premium is smaller than the variance risk premium, the fact that investors appear willing to accept negative average excess returns on contracts that hedge against unexpected increases in skewness suggests that investors dislike higher interest rate skewness. Because an increase in interest rate skewness corresponds to a decrease in bond return skewness, this finding is consistent with Arditti (1967) and Scott and Horvath (1980), who show that, under very general assumptions, investors have a preference for positive skewness in returns.

2.4.5 Pricing of interest rate risk. To put the compensation for variance and skewness risk into perspective, we also analyze the compensation for directional interest rate exposure. Details are in the Online Appendix; we summarize the results here. We consider a strategy of entering into interest rate swaps, receiving the fixed leg and paying the floating leg, on the last business day of each month and holding the positions over the following month. The risk profile of such a strategy is similar to a long position in a par coupon bond and benefits from a decrease in interest rates. Mean excess returns are positive and statistically significant for short to intermediate swap maturities. An equally weighted portfolio (in terms of notional) of the individual interest rate swaps generates annualized Sharpe ratios (corrected for return-autocorrelation) of 0.57 and 0.61 in the USD and EUR markets, respectively. Although this is similar in magnitude to the Sharpe ratios on variance swaps, the Sharpe ratios on interest rate swaps are inflated by the downward trend in interest rates over the sample period. In contrast, there is no clear trend in realized variance and skewness over the sample period. As such, the compensation for variance risk appears to be greater than the compensation for interest rate risk.

2.4.6 Variation in variance and skewness risk premia. To investigate time variation in variance and skewness risk premia, we follow Carr and Wu (2009)

Table 7
Predictive regressions

	Variance			Skewness		
	β_0	β_1	R^2	β_0	β_1	R^2
Panel A: USD market						
2 yr	2.77*	0.65***	0.28	−0.08	0.54***	0.07
	[1.79]	[−3.27]		[−1.61]	[−3.10]	
5 yr	2.15	0.72***	0.42	−0.05	0.67**	0.12
	[1.53]	[−3.53]		[−1.51]	[−2.37]	
10 yr	2.21*	0.70***	0.39	−0.04	0.68	0.10
	[1.87]	[−4.54]		[−1.06]	[−1.47]	
20 yr	2.63**	0.63***	0.34	−0.04	0.58	0.03
	[2.16]	[−5.01]		[−1.05]	[−1.23]	
30 yr	2.41**	0.65***	0.33	−0.04	0.55	0.03
	[2.10]	[−4.14]		[−1.15]	[−1.58]	
Panel B: EUR market						
2 yr	2.16***	0.44***	0.31	−0.04	0.30***	0.04
	[2.62]	[−6.05]		[−1.01]	[−3.28]	
5 yr	1.67**	0.47***	0.44	−0.05*	0.24***	0.03
	[2.54]	[−6.19]		[−1.93]	[−3.74]	
10 yr	0.79*	0.62***	0.49	−0.07***	0.38***	0.05
	[1.87]	[−4.46]		[−3.61]	[−4.07]	
20 yr	1.14***	0.55***	0.37	−0.18***	0.15***	0.03
	[2.60]	[−4.29]		[−4.12]	[−5.08]	
30 yr	1.51***	0.51***	0.36	−0.18***	0.22***	0.03
	[3.15]	[−4.86]		[−3.56]	[−5.04]	

This table shows results for predictive regressions. The results under "Variance" are for the regression

$$\mathrm{RVar}(t,T_m)=\beta_0+\beta_1\mathcal{V}(t,T_m)+\epsilon, \tag{30}$$

where $\mathrm{RVar}(t,T_m)$ denotes the one-month realized variance given in (20), and $\mathcal{V}(t,T_m)$ denotes the one-month variance swap rate given in (22). The results under "Skewness" are for the regression

$$\mathrm{RSkew}(t,T_m)=\beta_0+\beta_1\mathcal{S}(t,T_m)+\epsilon, \tag{31}$$

where $\mathrm{RSkew}(t,T_m)$ denotes the one-month realized skewness given in (24), and $\mathcal{S}(t,T_m)$ denotes the one-month skewness swap rate given in (26). The swap contracts are initiated on the last business day of each month using closing prices. The maturities of the underlying interest rate swaps are 2, 5, 10, 20, and 30 years, respectively. t-statistics under the null hypotheses of $\beta_0=0$ and $\beta_1=1$ are reported in brackets and are corrected for heteroscedasticity and serial correlation up to 12 lags using the approach of Newey and West (1987). In the USD market, the time series consist of 97 monthly returns from January 2002 to January 2010. In the EUR market, the time series consist of 103 monthly returns from July 2001 to January 2010. *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.

and run the following predictive regressions

$$\mathrm{RVar}(t,T_m)=\beta_0+\beta_1\mathcal{V}(t,T_m)+\epsilon \tag{29}$$

and

$$\mathrm{RSkew}(t,T_m)=\beta_0+\beta_1\mathcal{S}(t,T_m)+\epsilon. \tag{32}$$

Under the null hypotheses of zero variance and skewness risk premia, we have $\beta_0=0$ and $\beta_1=1$. Table 7 reports the results of the predictive regressions, with Newey and West (1987) t-statistics under the null hypotheses of $\beta_0=0$ and $\beta_1=1$ reported in brackets. In both markets, for all underlying interest rate swap maturities, and in both the variance and skewness regressions, we obtain positive estimates of β_1. This implies that the variance (skewness) swap rate contains information about future realized variance (skewness). At the same time, in the variance regressions, all estimates of β_1 are significantly less than one. In the skewness regressions, estimates of β_1 are significantly less than one in the EUR market as well as in the USD market for short underlying interest rate swap maturities.[15] This suggests that variance and skewness risk premia are time varying. In particular, it suggests that variance (skewness) risk premia become more negative, when the level of variance (skewness) increases. In the Online Appendix, we show that variance and skewness risk premia are not explained by exposure to standard risk factors such as market, value, and momentum.

2.5 Summary

To summarize, the main findings are the following: conditional volatility is (1) when averaged over time, a hump-shaped function of swap maturity at short horizons, a decreasing function of swap maturity at longer horizons, a hump-shaped function of horizon for the shortest swap maturity, and a decreasing function of horizon for longer swap maturities, (2) highly time varying with a high degree of commonality across swap maturity and option expiry, (3) positively correlated with the level and curvature of the term structure, and (4) associated with a significant negative risk premium, the size of which is correlated with the level of variance.

Conditional skewness is (1) when averaged over time, mostly positive, decreasing with swap maturity, and a concave and mostly increasing function of option expiry, (2) highly time varying to the extent that the sign sometimes changes and exhibits a high degree of commonality across swap maturity and option expiry, (3) somewhat related to the level and volatility of swap rates but most consistently positively related to the conditional correlation between swap rates and swap rate variances, and (4) associated with a negative risk premium that is less significant than the variance risk premium but is correlated with the level of skewness.

It is instructive to compare these characteristics with what is known about the dynamics of volatility and skewness in other asset markets. Volatility is obviously stochastic in virtually every market. The interest rate variance risk premium appears less negative than typical (precrisis) estimates of the variance risk premium for equity indexes (see, e.g., Bakshi and Kapadia 2003; Driessen, Maenhout, and Vilkov 2009; Carr and Wu 2009; Egloff, Leippold, and Wu 2010), but more negative than typical estimates of the variance risk premium

15 To control for the errors-in-variable problem that arises from the synthetic variance and skewness swap rates being measured with error, we also estimate the predictive regressions with the state-space approach used by Carr and Wu (2009). Although this increases the β_1-estimates, most remain significantly less than one. These results are available upon request.

for individual equities (see Carr and Wu 2009). Several papers find that the variance risk premium in the equity market varies with the level of variance (see, e.g., Carr and Wu 2009; Wu 2011).

That implied swap rate distributions remain highly nonnormal at long horizons is a feature shared with implied distributions of equity index returns (see, e.g., Carr and Wu 2003; Foresi and Wu 2005) and FX returns (see Carr and Wu 2007). Implied skewness of equity index returns does exhibit some variation over time but is consistently negative. In contrast, implied skewness of FX returns exhibits significant variation over time and occasionally changes sign (see Carr and Wu 2007), similar to what we observe for the implied skewness of future swap rates. Several papers find that implied skewness of equity index returns is more negative than realized skewness, pointing to a skewness risk premium. However, in contrast to Kozhan, Neuberger, and Schneider (2013), who find that excess returns on variance and skewness swaps are very highly correlated in the case of equity indexes, we observe a low correlation in the interest rate swap market.

3. A Dynamic Term Structure Model for Swap Rates

Guided by the model-independent analysis, we now design a parsimonious and tractable dynamic term structure model. We estimate the model and evaluate the extent to which it matches the dynamics of conditional swap rate moments.

3.1 The model

We have already hinted at key requirements for a successful model design: (1) hump-shaped innovations to the term structure of instantaneous forward rates (to match the average volatility surface), (2) stochastic mean-reverting variance (to match variation in conditional volatility), (3) nonzero correlations between innovations to the term structure and variance (to match the average skewness surface), (4) stochastic correlations (to match variation in conditional skewness), and (5) multiple term-structure and variance factors (to match the multifactor nature of moment dynamics).

We first set up the model under the risk-neutral measure and then find its dynamics under the physical and annuity measures.

3.1.1 Dynamics under the risk-neutral measure. Let $P(t,T)$ denote the time-t price of a zero-coupon bond maturing at time T. We assume the following general specification for the dynamics of zero-coupon bond prices

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sum_{i=1}^{N} \sigma_{P,i}(t,T)\left(\sqrt{v_1(t)}dW_i^{\mathbb{Q}}(t) + \sqrt{v_2(t)}d\overline{W}_i^{\mathbb{Q}}(t)\right), \quad (33)$$

$$dv_1(t) = (\eta(t) - \kappa v_1(t))dt + \sqrt{v_1(t)}dZ^{\mathbb{Q}}(t), \quad (34)$$

$$dv_2(t) = (\eta(t) - \kappa v_2(t))dt + \sqrt{v_2(t)}d\overline{Z}^{\mathbb{Q}}(t), \tag{35}$$

$$d\eta(t) = \left(\overline{\eta} - \kappa_\eta \eta(t)\right)dt + \sigma_\eta \sqrt{\eta(t)}d\widetilde{Z}^{\mathbb{Q}}(t), \tag{36}$$

where $W_i^{\mathbb{Q}}(t)$ and $\overline{W}_i^{\mathbb{Q}}(t)$, $i=1,...,N$, as well as $Z^{\mathbb{Q}}(t)$, $\overline{Z}^{\mathbb{Q}}(t)$, and $\widetilde{Z}^{\mathbb{Q}}(t)$ denote Wiener processes under the risk-neutral measure. We allow for correlations between $Z^{\mathbb{Q}}(t)$ and $W_i^{\mathbb{Q}}(t)$, $i=1,...,N$, with correlations denoted by ρ_i. Similarly, we allow for correlations between $\overline{Z}^{\mathbb{Q}}(t)$ and $\overline{W}_i^{\mathbb{Q}}(t)$, $i=1,...,N$, and denote these correlations by $\overline{\rho}_i$. This is the most general correlation structure that preserves the tractability of the model (we require that $\sum_{i=1}^N \rho_i^2 \leq 1$ and $\sum_{i=1}^N \overline{\rho}_i^2 \leq 1$). For parsimony, we have imposed that the first two variance factors, $v_1(t)$ and $v_2(t)$, have the same mean-reversion level and speed, and only differ in terms of their correlations with the term structure. The third variance factor, $\eta(t)$, captures variation in the mean-reversion level of $v_1(t)$ and $v_2(t)$. It is immediately clear that the model matches requirements (2), (3), and (5). In Sections 3.1.4 and 3.1.5, we show that it also has the ability to match requirements (4) and (1), respectively.

Applying Ito's lemma to Equation (4) gives the dynamics of the forward swap rate under $\mathbb{Q}$

$$dS_{m,n}(t) = \left(-\sum_{i=1}^N \sigma_{S,i}(t,T_m,T_n)\sigma_{A,i}(t,T_m,T_n)(v_1(t)+v_2(t))\right)dt + \sum_{i=1}^N \sigma_{S,i}(t,T_m,T_n)\left(\sqrt{v_1(t)}dW_i^{\mathbb{Q}}(t) + \sqrt{v_2(t)}d\overline{W}_i^{\mathbb{Q}}(t)\right), \tag{37}$$

where

$$\sigma_{S,i}(t,T_m,T_n) = \sum_{j=m}^n \zeta_j(t)\sigma_{P,i}(t,T_j) \tag{38}$$

$$\sigma_{A,i}(t,T_m,T_n) = \sum_{j=m+1}^n \chi_j(t)\sigma_{P,i}(t,T_j), \tag{39}$$

and $\zeta_j(t)$ and $\chi_j(t)$ are (stochastic) weights that are given in Appendix A.

3.1.2 Dynamics under the physical measure. The dynamics under the physical probability measure $\mathbb{P}$ are obtained by specifying the market prices of risk that link the Wiener processes under $\mathbb{Q}$ and $\mathbb{P}$. We apply the following standard specifications

$$dW_i^{\mathbb{P}}(t) = dW_i^{\mathbb{Q}}(t) - \lambda_i\sqrt{v_1(t)}dt,\ d\overline{W}_i^{\mathbb{P}}(t) = d\overline{W}_i^{\mathbb{Q}}(t) - \overline{\lambda}_i\sqrt{v_2(t)}dt,\ i=1,...,N \tag{40}$$

and

$$dZ^{\mathbb{P}}(t)=dZ^{\mathbb{Q}}(t)-\nu\sqrt{v_1(t)}dt,\ d\overline{Z}^{\mathbb{P}}(t)=d\overline{Z}^{\mathbb{Q}}(t)-\overline{\nu}\sqrt{v_2(t)}dt,$$

$$d\widetilde{Z}^{\mathbb{P}}(t)=d\widetilde{Z}^{\mathbb{Q}}(t)-\widetilde{\nu}\sqrt{\eta(t)}dt. \tag{41}$$

This leads to the following dynamics of the forward swap rate under $\mathbb{P}$

$$dS_{m,n}(t)=\left(-\sum_{i=1}^{N}\sigma_{S,i}(t,T_m,T_n)((\sigma_{A,i}(t,T_m,T_n)+\lambda_i)v_1(t)\right.$$
$$\left.+(\sigma_{A,i}(t,T_m,T_n)+\overline{\lambda}_i)v_2(t))\right)dt$$
$$+\sum_{i=1}^{N}\sigma_{S,i}(t,T_m,T_n)\left(\sqrt{v_1(t)}dW_i^{\mathbb{P}}(t)+\sqrt{v_2(t)}d\overline{W}_i^{\mathbb{P}}(t)\right), \tag{42}$$

where

$$dv_1(t)=\left(\eta(t)-\kappa_1^{\mathbb{P}}v_1(t)\right)dt+\sqrt{v_1(t)}dZ^{\mathbb{P}}(t), \tag{43}$$

$$dv_2(t)=\left(\eta(t)-\kappa_2^{\mathbb{P}}v_2(t)\right)dt+\sqrt{v_2(t)}d\overline{Z}^{\mathbb{P}}(t), \tag{44}$$

$$d\eta(t)=\left(\overline{\eta}-\kappa_\eta^{\mathbb{P}}\eta(t)\right)dt+\sigma_\eta\sqrt{\eta(t)}d\widetilde{Z}^{\mathbb{P}}(t), \tag{45}$$

and $\kappa_1^{\mathbb{P}}=\kappa-\nu$, $\kappa_2^{\mathbb{P}}=\kappa-\overline{\nu}$, and $\kappa_\eta^{\mathbb{P}}=\kappa_\eta-\sigma_\eta\widetilde{\nu}$.

3.1.3 Dynamics under the annuity measure. As discussed in Section 2.1, for pricing swaptions it is convenient to work under the annuity measure, $\mathbb{A}$. Straightforward calculations give the following dynamics of the forward swap rate under $\mathbb{A}$

$$dS_{m,n}(t)=\sum_{i=1}^{N}\sigma_{S,i}(t,T_m,T_n)\left(\sqrt{v_1(t)}dW_i^{\mathbb{A}}(t)+\sqrt{v_2(t)}d\overline{W}_i^{\mathbb{A}}(t)\right), \tag{46}$$

where

$$dv_1(t)=\left(\eta(t)-\kappa_1^{\mathbb{A}}v_1(t)\right)dt+\sqrt{v_1(t)}dZ^{\mathbb{A}}(t), \tag{47}$$

$$dv_2(t)=\left(\eta(t)-\kappa_2^{\mathbb{A}}v_2(t)\right)dt+\sqrt{v_2(t)}d\overline{Z}^{\mathbb{A}}(t), \tag{48}$$

$$d\eta(t)=\left(\overline{\eta}-\kappa_\eta\eta(t)\right)dt+\sigma_\eta\sqrt{\eta(t)}d\widetilde{Z}^{\mathbb{A}}(t), \tag{49}$$

and $\kappa_1^{\mathbb{A}}=\kappa-\sum_{i=1}^{N}\rho_i\sigma_{A,i}(t,T_m,T_n)$ and $\kappa_2^{\mathbb{A}}=\kappa-\sum_{i=1}^{N}\overline{\rho}_i\sigma_{A,i}(t,T_m,T_n)$. The dynamics of $S_{m,n}(t)$ under $\mathbb{A}$ are not entirely affine, because the weights $\zeta_j(t)$ and $\chi_j(t)$ in (38) and (39) are stochastic. By "freezing" $\zeta_j(t)$ and $\chi_j(t)$ at their initial values, we obtain affine dynamics of $S_{m,n}(t)$ which leads to a fast and accurate Fourier-based pricing formula for swaptions given in Appendix A. An efficient pricing formula is critical for our empirical analysis, where the model is estimated on all swaptions across time, swap maturity, option expiry, and strike.

3.1.4 Stochastic skewness. That $v_1(t)$ and $v_2(t)$ differ in terms of their correlation with the term structure allows the model to capture variation in conditional swap rate skewness, including the switch in the sign of skewness. To see this, note that the instantaneous variance of the forward swap rate is given by

$$\frac{1}{dt}Var(dS_{m,n}(t))=\left(\sum_{i=1}^{N}\sigma_{S,i}^2\right)(v_1(t)+v_2(t)). \tag{50}$$

This implies that the instantaneous correlation between innovations to the forward swap rate and its variance is given by

$$\Omega_{m,n}(t)=\sum_{i=1}^{N}\frac{\sigma_{S,i}}{\sqrt{\sum_{i=1}^{N}\sigma_{S,i}^2}}((1-w(t))\rho_i+w(t)\overline{\rho}_i), \tag{51}$$

where

$$w(t)=\frac{v_2(t)}{v_1(t)+v_2(t)}. \tag{52}$$

The instantaneous correlation is the sum of weighted averages of ρ_i and $\overline{\rho}_i$. The weight on $\overline{\rho}_i$ is given by the ratio of $v_2(t)$ to the sum of $v_1(t)$ and $v_2(t)$ and obviously lies between zero and one. The implication is that $\Omega_{m,n}(t)$ is stochastic and may switch sign if ρ_i and $\overline{\rho}_i$ have opposite signs. Consequently, conditional swap rate skewness is also stochastic and subject to possible sign change.[16]

3.1.5 Level, slope, and curvature shocks. We allow for three term structure factors by setting $N=3$. For the bond price volatility functions in (33), we note that we can equally well specify the volatility functions for instantaneous forward rates, because the two are related by $\sigma_{P,i}(t,T)=-\int_t^T\sigma_{f,i}(t,u)du$. As it is generally easier to relate to interest rate volatility than to bond price volatility, we show both. The specifications we use are

$$\begin{array}{lllllll}
\sigma_{f,1}(t,T) & = & \alpha_1 e^{-\xi(T-t)} & \Rightarrow & \sigma_{P,1}(t,T) & = & \frac{\alpha_1}{\xi}\left(e^{-\xi(T-t)}-1\right)\\
\sigma_{f,2}(t,T) & = & \alpha_2 e^{-\gamma(T-t)} & \Rightarrow & \sigma_{P,2}(t,T) & = & \frac{\alpha_2}{\gamma}\left(e^{-\gamma(T-t)}-1\right)\\
\sigma_{f,3}(t,T) & = & \alpha_3(T-t)e^{-\gamma(T-t)} & \Rightarrow & \sigma_{P,3}(t,T) & = & \frac{\alpha_3}{\gamma^2}\left(e^{-\gamma(T-t)}(\gamma(T-t)+1)-1\right).
\end{array} \tag{53}$$

The second and third are the "slope" and "curvature" factor loadings proposed by Nelson and Siegel (1987), whereas the first becomes their "level" factor

16 Carr and Wu (2007) rely on a similar mechanism for generating stochastic skewness in FX returns. We also note the similarity with the structural BEGE framework of Bekaert and Engstrom (2010), which features two state-variables generating either positive or negative skewness in consumption growth and returns. Skewness is stochastic and may change sign, depending on the relative magnitudes of the two state variables. However, because there are only two fundamental shocks within their framework, it implies a tight link between the term structure, variance, and skewness if applied to a fixed income setting.

loading in the limit $\xi \to 0$. These factor loadings are popular in the term structure literature as they parsimoniously capture the predominant shocks to the term structure and allow for hump-shaped innovations to the term structure of instantaneous forward rates.

Modifying the first factor loading relative to Nelson and Siegel (1987) allows us to express the dynamics of the term structure in terms of a finite-dimensional affine state vector. Specifically, we show in Appendix B that $P(t,T)$ is given by

$$P(t,T)=\frac{P(0,T)}{P(0,t)}\exp\left\{\sum_{i=1}^{3} B_{x_i}(T-t)x_i(t)+\sum_{i=1}^{5} B_{\phi_i}(T-t)\phi_i(t)\right\}, \qquad (54)$$

where $B_{x_i}(T-t)=\sigma_{P,i}(t,T)$, and $B_{\phi_i}(T-t)$ as well as the $\mathbb{Q}$- and $\mathbb{P}$-dynamics of $x(t)\equiv(x_1(t),...,x_3(t))$ and $\phi(t)\equiv(\phi_1(t),...,\phi_5(t))$ are given in that appendix. $\phi(t)$ consists of "auxiliary," locally deterministic, state variables that ensure a low-dimensional Markovian representation of the term structure. $x(t)$ and $\phi(t)$, along with $v_1(t)$, $v_2(t)$, and $\eta(t)$, jointly constitute an affine state vector under both $\mathbb{Q}$ and $\mathbb{P}$. This facilitates model estimation using well-established techniques from the vast body of literature on affine models.

Even with the parsimonious market price of risk specifications outlined in Section 3.1.2, there are six market price of risk parameters associated with the term structure, which is excessive given the length of the sample period. To reduce this number, we follow Cochrane and Piazzesi (2008) in assuming that bond risk premia only arise from exposure to shocks in the "level" factor (we refer to their paper for an in-depth discussion of the evidence supporting this assumption). In other words, we impose $\lambda_2=\overline{\lambda}_2=\lambda_3=\overline{\lambda}_3=0$.

3.2 Estimation results

3.2.1 Maximum likelihood estimates. We estimate the model using weekly data on all available swap rates and swaptions. The estimation approach is quasi-maximum likelihood in conjunction with Kalman filtering. Details are provided in Appendix C. Table 8 displays parameter estimates with asymptotic standard errors in parentheses. Consider first the parameters of the term structure factors. The ξ-parameter is close to zero, which for all practical purposes makes the first term structure factor act like the level factor in Nelson and Siegel (1987). The magnitude of the γ-parameter means that the curvature factor has most impact on instantaneous forward rates with intermediate maturities (around five years in the USD market). The magnitudes of the α-parameters imply that both level, slope, and curvature shocks are important. The φ-parameter, which equals the infinite-maturity forward rate as explained in Appendix C, has a reasonable magnitude (4.78% in the USD market).

Next, consider the parameters of the variance factors. The correlation parameters ρ_i and $\overline{\rho}_i$ have opposite signs for all i, which is consistent with time-varying conditional skewness that may switch sign. When $v_2(t)$ is low

Table 8
Maximum likelihood estimates

	α_1	α_2	α_3	ξ	γ	φ
USD	0.0067	0.0060	0.0052	0.0009	0.2192	0.0478
	(0.0005)	(0.0008)	(0.0005)	(0.0002)	(0.0211)	(0.0078)
EUR	0.0098	0.0094	0.0028	0.0010	0.1403	0.0447
	(0.0008)	(0.0011)	(0.0004)	(0.0002)	(0.0201)	(0.0075)
	ρ_1	$\overline{\rho}_1$	ρ_2	$\overline{\rho}_2$	ρ_3	$\overline{\rho}_3$
USD	−0.3443	0.5773	−0.2085	0.4677	−0.3685	0.4014
	(0.0503)	(0.0438)	(0.0434)	(0.0721)	(0.0555)	(0.089)
EUR	−0.2416	0.5919	−0.2689	0.4071	−0.2302	0.5868
	(0.0409)	(0.0599)	(0.0632)	(0.0677)	(0.0407)	(0.0511)
	κ	κ_η	σ_η	$\overline{\eta}$	σ_{rates}	$\sigma_{options}$
USD	1.2399	0.3450	0.6287	0.1925	6.2676	4.0239
	(0.0846)	(0.0697)	(0.0601)	(0.0305)	(0.5101)	(0.4291)
EUR	1.5714	0.2533	0.5733	0.0978	5.4903	3.8990
	(0.2801)	(0.0441)	(0.0559)	(0.0107)	(0.7154)	(0.4074)
	λ_1	$\overline{\lambda}_1$	ν	$\overline{\nu}$	$\widetilde{\nu}$	
USD	−0.1695	−0.3765	−0.1215	−0.5510	−0.1042	
	(0.1135)	(0.1691)	(0.1184)	(0.2058)	(0.0970)	
EUR	−0.1600	−0.3094	−0.1498	−0.5696	−0.1848	
	(0.0937)	(0.1562)	(0.1240)	(0.2301)	(0.1140)	

The table shows maximum likelihood estimates of model parameters, with White (1982) robust standard errors reported in parentheses. σ_{rates} denotes the standard deviation of swap rate measurement errors, and $\sigma_{options}$ denotes the standard deviation of scaled swaption price measurement errors. Both σ_{rates} and $\sigma_{options}$ are measured in basis points. The log-likelihood values are -239,803 and -210,591 in the USD and EUR markets, respectively. In the USD market, the sample period is from December 19, 2001 to January 27, 2010 (419 weekly observations). In the EUR market, the sample period is from June 6, 2001 to January 27, 2010 (449 weekly observations).

relative to $v_1(t)$ (i.e., when $w(t)$ is close to zero), future swap rates are skewed towards lower values, whereas when $v_2(t)$ is high relative to $v_1(t)$ (i.e., when $w(t)$ is close to one), future swap rates are skewed towards higher values. Note also that $v_1(t)$ and $v_2(t)$ have faster risk-neutral mean-reversion than $\eta(t)$, suggesting that the former capture more transitory shocks to volatility, whereas the latter captures more persistent shocks to volatility.

3.2.2 Model fit. From the filtered state variables, we compute the fitted values of swap rates as well as swaption prices and implied volatilities. We also compute conditional swap rate moments using the fitted swaption prices and the same interpolation/extrapolation scheme outlined in Section 2.1. The fit to the term structure is comparable to that of other three-factor models, with a root-mean-squared-error (RMSE) for swap rates of 6.27 and 7.09 bps in the USD and EUR market, respectively. The overall fit to swaptions is good, with a RMSE for implied volatilities of 3.53 and 2.74 bps in the USD and EUR market, respectively. In general, the fit to conditional volatility is better in the EUR market (a RMSE for volatility of 1.52 bps versus 3.31 bps in the USD market), whereas the fit to conditional skewness is better in the USD market (a RMSE for skewness of 0.09 versus 0.17 in the EUR market). The Online Appendix contains more information about RMSEs.

To illustrate how model performance varies across the different dimensions of the data, we compute for each combination of swap maturity and option

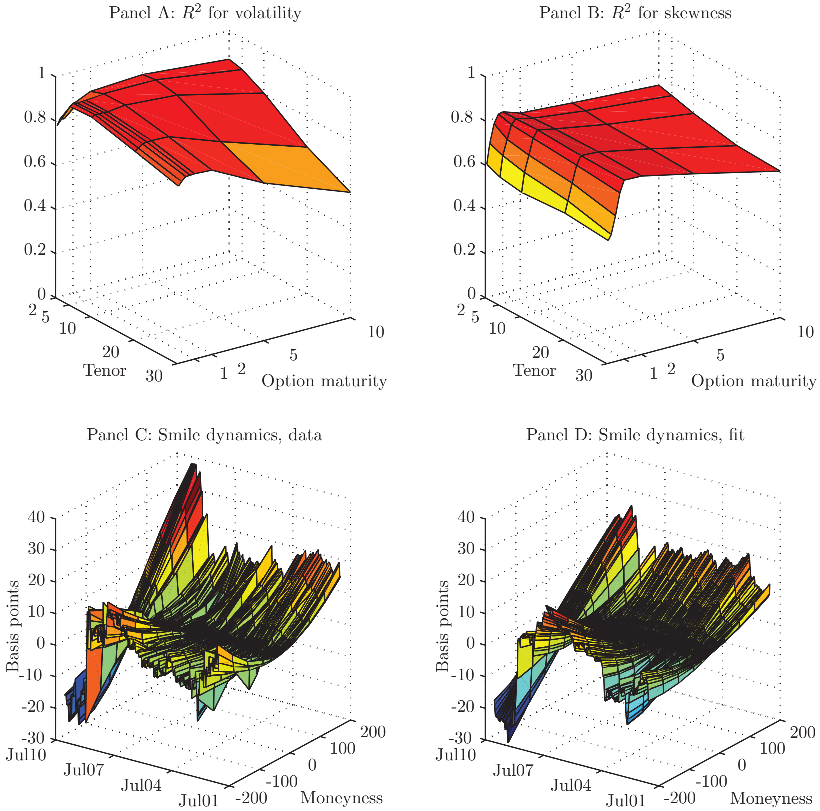

Figure 4
Fit to volatility and skewness in the USD market
For each combination of swap maturity (tenor) and option expiry, Panel A (Panel B) shows the fraction of variation in conditional volatility (skewness) that is explained by the model. Conditional volatility and skewness is under the annuity measure $\mathbb{A}$. Panel C (Panel D) shows the actual (fitted) time series of the USD implied volatility smile of the 1-year option on the 10-year swap rate. The smiles are the differences between the normal implied volatilities for different strikes and the ATM normal implied volatility, and the units are basis points. The sample period is from December 19, 2001 to January 27, 2010 (419 weekly observations).

expiry the fraction of variation in conditional volatility and skewness that is explained by the model. We focus on the USD market, but results are similar in the EUR market. Panel A in Figure 4 displays the results for conditional volatility. The model explains 85% of the variation in conditional volatility, on average. However, the performance deteriorates as we move towards the edges of the swap maturity-option expiry grid. In particular, the model does not generate sufficient variation in conditional volatility for very long swap maturities at very long option expiries. To remedy this, one would likely need to augment the model with another variance factor with very high risk-neutral persistence.

Panel B in Figure 4 displays the results for conditional skewness. The model explains 73% of the variation in conditional skewness, on average. The fit is good for option expiries of one year and more, but the fit deteriorates at very short option expiries. To generate sufficient variation in conditional skewness over very short horizons, one would likely need to add jump components to the model, possibly along the lines of Carr and Wu (2007).

To visualize how the fit to conditional skewness translates into the fit to the implied volatility smile, Figure 4 shows actual (in Panel C) and fitted (in Panel D) time series of the USD implied volatility smile of the 1-year option on the 10-year swap rate (one out of the 40 time series of swaption smiles in the data set). Evidently, the model matches the smile dynamics well, although it does not quite capture the significant increase in the skewness of the smile towards the end of the sample period.

3.2.3 Model-implied variance and skewness risk premia. Consider the market price of risk parameters in Table 8. ν, $\overline{\nu}$, and $\widetilde{\nu}$ are negative in both markets, and several are statistically significant. This implies that the variance factors have lower long-run means under $\mathbb{P}$ than under $\mathbb{Q}$, consistent with a negative variance risk premium. Furthermore, in both markets, $\overline{\nu}$ is more negative than ν, which implies that $v_2(t)$ has a lower long-run mean than $v_1(t)$ under $\mathbb{P}$. Because both factors have the same long-run mean under $\mathbb{Q}$ and because skewness is determined by the relative magnitudes of $v_2(t)$ and $v_1(t)$, this is consistent with a negative skewness risk premium.

To investigate if the magnitudes of the model-implied risk premia are consistent with the model-independent estimates reported earlier, we perform a simulation exercise. For each currency, we simulate long time series of swap rates and swaption prices. We then repeat the research design of Section 2.4 on the simulated data, computing monthly excess returns on variance and skewness swaps. Achieving the target excess return volatility requires more leverage than in the real data, and the mean excess returns and Sharpe ratios are somewhat less negative. Taking the USD market as an example, the annualized Sharpe ratio for the equally weighted portfolio of volatility (skewness) swaps is -0.42 (-0.17) in the simulated data versus -0.63 (-0.30) in the real data.

4. Conclusion

In this paper, we use a comprehensive database of interdealer quotes to conduct the first empirical analysis of the swaption cube. We infer conditional swap rate moments model independently. We study the time-series and cross-sectional variation in conditional swap rate moments and show that conditional volatility and skewness exhibit systematic variation across swap maturities and option expiries (conditional kurtosis less so), with conditional skewness sometimes changing sign. Conditional skewness displays some relation to the level and volatility of swap rates but is most consistently related to the conditional correlation between swap rates and swap rate variances. We quantify variance and skewness risk premia from realized excess returns on synthetic variance and skewness swap contracts and show that variance and (to a lesser extent) skewness risk premia are negative and time varying. For the most part, results hold true in both the USD and EUR markets and in both precrisis and crisis subsamples. Finally, we design and estimate a parsimonious and tractable dynamic term structure model, which captures much of the dynamics of conditional swap rate moments.

Appendix A. Swaption Pricing

A.1. The weights in (38) and (39)

The weights $\zeta_j(t)$ in (38) are given by

$$\zeta_m(t)=\frac{P(t,T_m)}{A_{m,n}(t)}, \tag{A1}$$

$$\zeta_j(t)=-\tau_{j-1}S_{m,n}(t)\frac{P(t,T_j)}{A_{m,n}(t)},\ j=m+1,...,n-1, \tag{A2}$$

$$\zeta_n(t)=-(1+\tau_{n-1}S_{m,n}(t))\frac{P(t,T_n)}{A_{m,n}(t)}, \tag{A3}$$

whereas the weights $\chi_j(t)$ in (39) are given by

$$\chi_j(t)=\frac{\tau_{j-1}P(t,T_j)}{A_{m,n}(t)},\ j=m+1,...,n. \tag{A4}$$

A.2. Fourier-based pricing formula for swaptions

The dynamics of the forward swap rate under $\mathbb{A}$ are not entirely affine because of the stochastic weights $\zeta_j(t)$ and $\chi_j(t)$. However, these are low variance martingales under $\mathbb{A}$, and following much of the literature on LIBOR market models, we may "freeze" these at their initial values to obtain a truly affine model, in which case swaptions can be priced quasi-analytically. As is the case for LIBOR market models, the resulting biases in swaption prices are also very small in our context. First, we find the characteristic function of $S_{m,n}(T_m)$ given by

$$\psi_{m,n}(u,t)=E_t^{\mathbb{A}}\left[e^{\mathtt{i}uS_{m,n}(T_m)}\right], \tag{A5}$$

where $\mathtt{i}=\sqrt{-1}$. This has an exponentially affine solution as demonstrated in the following proposition:

Proposition 1. (A5) is given by

$$\psi_{m,n}(u,t)=e^{M(T_m-t)+N_1(T_m-t)v_1(t)+N_2(T_m-t)v_2(t)+N_3(T_m-t)\eta(t)+\mathtt{i}uS_{m,n}(t)}, \tag{A6}$$

where $M(\tau)$, $N_1(\tau)$, $N_2(\tau)$, and $N_3(\tau)$ solve the following system of ODEs

$$\frac{dM(\tau)}{d\tau}=N_3(\tau)\overline{\eta} \tag{A7}$$

$$\frac{dN_1(\tau)}{d\tau}=\left(-\kappa_1^{\mathbb{A}}+\mathrm{i}u\sum_{i=1}^{N}\rho_i\sigma_{S,i}(t,T_m,T_n)\right)N_1(\tau)+\frac{1}{2}N_1(\tau)^2-\frac{1}{2}u^2\sum_{i=1}^{N}\sigma_{S,i}(t,T_m,T_n)^2 \tag{A8}$$

$$\frac{dN_2(\tau)}{d\tau}=\left(-\kappa_2^{\mathbb{A}}+\mathrm{i}u\sum_{i=1}^{N}\overline{\rho}_i\sigma_{S,i}(t,T_m,T_n)\right)N_2(\tau)+\frac{1}{2}N_2(\tau)^2-\frac{1}{2}u^2\sum_{i=1}^{N}\sigma_{S,i}(t,T_m,T_n)^2 \tag{A9}$$

$$\frac{dN_3(\tau)}{d\tau}=N_1(\tau)+N_2(\tau)-\kappa_\eta N_3(\tau)+\frac{1}{2}N_3(\tau)^2\sigma_\eta^2 \tag{A10}$$

subject to the boundary conditions $M(0)=N_1(0)=N_2(0)=N_3(0)=0$.

Proof. See Online Appendix. ∎

Next, we follow the general approach of Carr and Madan (1999) and Lee (2004) to price swaptions. The idea is that the Fourier transform of the modified swaption price,

$$\widehat{\mathcal{P}}_{m,n}(t,K)=e^{\alpha K}\mathcal{P}_{m,n}(t,K), \tag{A11}$$

can be expressed in terms of the characteristic function of $S_{m,n}(T_m)$ (the control parameter α must be chosen to ensure that the modified swaption price is square integrable, which is a sufficient condition for its Fourier transform to exist). The swaption price is then obtained by applying the Fourier inversion theorem. The result is given in the following proposition:

Proposition 2. The time-t price of a European payer swaption expiring at T_m with strike K on a swap for the period T_m to T_n, $\mathcal{P}_{m,n}(t,K)$, is given by

$$\mathcal{P}_{m,n}(t,K)=A_{m,n}(t)\frac{e^{-\alpha K}}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{e^{-\mathrm{i}uK}\psi_{m,n}(u-\mathrm{i}\alpha,t)}{(\alpha+\mathrm{i}u)^2}\right]du. \tag{A12}$$

Proof. See Online Appendix. ∎

Appendix B. The Affine Term Structure Specification

Although the model is based on the Heath, Jarrow, and Morton (1992) framework, it can be represented as an affine model with a finite-dimensional state vector.

Proposition 3. The time-t instantaneous forward interest rate for borrowing and lending at time T, $f(t,T)$, is given by

$$f(t,T)=f(0,T)+\sum_{i=1}^{3}\mathcal{B}_{x_i}(T-t)x_i(t)+\sum_{i=1}^{5}\mathcal{B}_{\phi_i}(T-t)\phi_i(t), \tag{B1}$$

where

$$\begin{aligned}
\mathcal{B}_{x_1}(\tau)&=\alpha_1 e^{-\xi\tau}\\
\mathcal{B}_{x_2}(\tau)&=\alpha_2 e^{-\gamma\tau}\\
\mathcal{B}_{x_3}(\tau)&=\alpha_3 \tau e^{-\gamma\tau}\\
\mathcal{B}_{\phi_1}(\tau)&=-\frac{\alpha_1^2}{\xi}e^{-2\xi\tau}\\
\mathcal{B}_{\phi_2}(\tau)&=-\left(\frac{\alpha_2^2}{\gamma}+\left(\tau+\gamma\tau^2\right)\frac{\alpha_3^2}{\gamma^2}\right)e^{-2\gamma\tau}\\
\mathcal{B}_{\phi_3}(\tau)&=\alpha_3 e^{-\gamma\tau}\\
\mathcal{B}_{\phi_4}(\tau)&=-(1+2\gamma\tau)\frac{\alpha_3^2}{\gamma^2}e^{-2\gamma\tau}\\
\mathcal{B}_{\phi_5}(\tau)&=-\frac{\alpha_3^2}{\gamma}e^{-2\gamma\tau},
\end{aligned} \tag{B2}$$

and the $\mathbb{Q}$-dynamics of $x_i(t)$ and $\phi_i(t)$ are given by

$$\begin{aligned}
dx_1(t)&=\left(\frac{\alpha_1}{\xi}(v_1(t)+v_2(t))-\xi x_1(t)\right)dt+\sqrt{v_1(t)}dW_1^{\mathbb{Q}}(t)+\sqrt{v_2(t)}d\overline{W}_1^{\mathbb{Q}}(t)\\
dx_2(t)&=\left(\frac{\alpha_2}{\gamma}(v_1(t)+v_2(t))-\gamma x_2(t)\right)dt+\sqrt{v_1(t)}dW_2^{\mathbb{Q}}(t)+\sqrt{v_2(t)}d\overline{W}_2^{\mathbb{Q}}(t)\\
dx_3(t)&=\left(\frac{\alpha_3}{\gamma^2}(v_1(t)+v_2(t))-\gamma x_3(t)\right)dt+\sqrt{v_1(t)}dW_3^{\mathbb{Q}}(t)+\sqrt{v_2(t)}d\overline{W}_3^{\mathbb{Q}}(t)\\
d\phi_1(t)&=(v_1(t)+v_2(t)-2\xi\phi_1(t))dt\\
d\phi_2(t)&=(v_1(t)+v_2(t)-2\gamma\phi_2(t))dt\\
d\phi_3(t)&=(x_3(t)-\gamma\phi_3(t))dt\\
d\phi_4(t)&=(\phi_2(t)-2\gamma\phi_4(t))dt\\
d\phi_5(t)&=(2\phi_4(t)-2\gamma\phi_5(t))dt,
\end{aligned} \tag{B3}$$

with initial conditions $x_i(t)=0$ and $\phi_i(t)=0$.

Proof. See Online Appendix. ■

It follows from Proposition 3 that the time-t price of a zero-coupon bond maturing at time T, $P(t,T)$, is given by

$$\begin{aligned}
P(t,T)&\equiv\exp\left\{-\int_t^T f(t,u)du\right\}\\
&=\frac{P(0,T)}{P(0,t)}\exp\left\{\sum_{i=1}^{3}B_{x_i}(T-t)x_i(t)+\sum_{i=1}^{5}B_{\phi_i}(T-t)\phi_i(t)\right\},
\end{aligned} \tag{B4}$$

where

$$
\begin{aligned}
B_{x_1}(\tau) &= \frac{\alpha_1}{\xi}\left(e^{-\xi\tau}-1\right)\\
B_{x_2}(\tau) &= \frac{\alpha_2}{\gamma}\left(e^{-\gamma\tau}-1\right)\\
B_{x_3}(\tau) &= \frac{\alpha_3}{\gamma^2}\left(e^{-\gamma\tau}(\gamma\tau+1)-1\right)\\
B_{\phi_1}(\tau) &= \frac{\alpha_1^2}{2\xi^2}\left(1-e^{-2\xi\tau}\right)\\
B_{\phi_2}(\tau) &= \frac{\alpha_2^2}{2\gamma^2}\left(1-e^{-2\gamma\tau}\right)+\frac{\alpha_3^2}{2\gamma^4}\left(1-e^{-2\gamma\tau}\left(\gamma^2\tau^2+2\gamma\tau+1\right)\right)\\
B_{\phi_3}(\tau) &= \frac{\alpha_3}{\gamma}\left(e^{-\gamma\tau}-1\right)\\
B_{\phi_4}(\tau) &= \frac{\alpha_3^2}{\gamma^3}\left(1-e^{-2\gamma\tau}(\gamma\tau+1)\right)\\
B_{\phi_5}(\tau) &= \frac{\alpha_3^2}{2\gamma^2}\left(1-e^{-2\gamma\tau}\right).
\end{aligned}
\tag{B5}
$$

For the estimation, we need the dynamics of $x(t)\equiv(x_1(t),...,x_3(t))$ and $\phi(t)\equiv(\phi_1(t),...,\phi_5(t))$ under the physical probability measure $\mathbb{P}$. Given the market price of risk specification in (40), the $\mathbb{P}$-dynamics of $x(t)$ are

$$
\begin{aligned}
dx_1(t) &= \left(\frac{\alpha_1+\lambda_1\xi}{\xi}v_1(t)+\frac{\alpha_1+\overline{\lambda}_1\xi}{\xi}v_2(t)-\xi x_1(t)\right)dt+\sqrt{v_1(t)}dW_1^{\mathbb{P}}(t)+\sqrt{v_2(t)}d\overline{W}_1^{\mathbb{P}}(t)\\
dx_2(t) &= \left(\frac{\alpha_2+\lambda_2\gamma}{\gamma}v_1(t)+\frac{\alpha_2+\overline{\lambda}_2\gamma}{\gamma}v_2(t)-\gamma x_2(t)\right)dt+\sqrt{v_1(t)}dW_2^{\mathbb{P}}(t)+\sqrt{v_2(t)}d\overline{W}_2^{\mathbb{P}}(t)\\
dx_3(t) &= \left(\frac{\alpha_3+\lambda_3\gamma^2}{\gamma^2}v_1(t)+\frac{\alpha_3+\overline{\lambda}_3\gamma^2}{\gamma^2}v_2(t)-\gamma x_3(t)\right)dt+\sqrt{v_1(t)}dW_3^{\mathbb{P}}(t)+\sqrt{v_2(t)}d\overline{W}_3^{\mathbb{P}}(t),
\end{aligned}
$$

whereas the dynamics of $\phi(t)$ are invariant to the measure change.

Appendix C. Quasi-Maximum Likelihood Estimation

We cast the model in state space form, which consists of a measurement equation and a transition equation. The measurement equation describes the relation between the state variables and the observable swaption prices and swap rates, whereas the transition equation describes the discrete-time dynamics of the state variables.

The vector of state variables, $X(t)$, is given by

$$X(t)=(x(t),\phi(t),v_1(t),v_2(t),\eta(t))'. \tag{C1}$$

Because $X(t)$ follows an affine diffusion process under $\mathbb{P}$, the conditional mean and variance of its transition density can be computed even if the density itself is unknown. We approximate the transition density with a Gaussian density with identical first and second moments, in which case the transition equation is of the form

$$X(t)=\Phi_0+\Phi_X X(t-1)+w(t),\quad w(t)\sim N(0,Q(t)). \tag{C2}$$

The measurement equation is given by

$$Z(t) = h(X(t)) + u(t), \quad u(t) \sim N(0, \Omega), \tag{C3}$$

where $Z(t)$ is a vector consisting of all swaption prices available in the time-t swaption cube, as well as the term structure of spot swap rates; h is the pricing function relating swaption prices and swap rates to $X(t)$; and $u(t)$ is a vector of iid. Gaussian pricing errors with covariance matrix Ω.

Swap rates are related to $x(t)$ and $\phi(t)$ through (3), (4), and (54). In principle, the model is time-inhomogeneous and fits the initial term structure at time 0 by construction. For the purpose of estimation, it is more convenient to work with the model's time-homogeneous counterpart. To this end, we assume that the initial forward rate curve is flat and equal to a constant φ, which amounts to setting $\frac{P(0,T)}{P(0,t)} = \exp(-\varphi(T-t))$ in (54). φ is an additional parameter of the model to be estimated (see also de Jong and Santa-Clara 1999; Trolle and Schwartz 2009). One can show that φ equals the infinite-maturity forward rate.

Swaption prices are related to $v_1(t)$, $v_2(t)$, and $\eta(t)$ via the formulas in Propositions 1 and 2 in Appendix A. Note that we price swaptions based on the actual forward swap rates and annuity factors observed in the market, making swaption prices independent of $x(t)$ and $\phi(t)$. This has two advantages: First, an imperfect fit to the swap term structure is not reflected in swaption prices, which in turn provides a cleaner estimate of the volatility processes. Second, the resulting separation of the state vector makes the likelihood optimization more stable.[17]

To reduce the number of parameters in Ω, we assume that the measurement errors are cross-sectionally uncorrelated (that is, Ω is diagonal), that one variance, σ^2_{rates}, applies to all pricing errors for swap rates, and that another variance, $\sigma^2_{swaption}$, applies to all pricing errors for scaled swaption prices.

Because of the nonlinearities in the relation between observations and state variables, we apply the nonlinear unscented Kalman filter.[18] Details are provided in the Online Appendix. The Kalman filter produces one-step-ahead forecasts for Z_t, $\hat{Z}_{t|t-1}$, and the corresponding error covariance matrices, $F_{t|t-1}$, from which we construct the log-likelihood function as

$$\mathcal{L}(\Theta) = -\frac{1}{2}\log 2\pi \sum_{t=1}^{T} n_t - \frac{1}{2}\sum_{t=1}^{T}\log|F_{t|t-1}| - \frac{1}{2}\sum_{t=1}^{T}(Z_t - \hat{Z}_{t|t-1})' F^{-1}_{t|t-1}(Z_t - \hat{Z}_{t|t-1}), \tag{C4}$$

where Θ is the vector of parameters; T is the number of observation dates; and n_t is the dimension of Z_t. The maximum likelihood estimator, $\hat{\Theta}$, is then

$$\hat{\Theta} = \arg\max_{\Theta} \mathcal{L}(\Theta). \tag{C5}$$

Note that the estimation procedure is quasi-maximum likelihood because we approximate the true transition density of X_t with a Gaussian.

---

17 Ideally, we would like to fit the model directly to normal implied volatilities, which are more stable than are prices (or log-normal implied volatilities) along the swap maturity, option expiry, strike, and time-series dimensions. This is not practical, however, because computing implied volatilities from prices requires a numerical inversion for each swaption, which would add an extra layer of complexity to the estimation procedure. Instead, we fit the model to option prices scaled by their normal vegas (i.e., the sensitivities of swaption prices to variations in volatility within the normal pricing model). This essentially converts swaption pricing errors in terms of prices into swaption pricing errors in terms of normal implied volatilities, via a linear approximation. Similar vega-scaling has been used by Carr and Wu (2007), Bakshi, Carr, and Wu (2008), and Trolle and Schwartz (2009), among others.

18 Leippold and Wu (2007) appear to be the first to apply the unscented Kalman filter to the estimation of dynamic term structure models. Christoffersen, Jacobs, Karoui, and Mimouni (2009) perform an extensive Monte Carlo experiment, which shows that the unscented Kalman filter significantly outperforms the extended Kalman filter in the context of estimating dynamic term structure models with swap rates. Their results most likely carry over to our context, in which swaptions are also used in the estimation.

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