Salomon Brothers

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Overview of Forward Rate Analysis

Understanding the Yield Curve: Part 1

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INTRODUCTION

In recent years, advances have been made in the theoretical and the empirical analysis of the term structure of interest rates. However, such analysis is often very quantitative, and it rarely emphasizes practical investment applications. There appears to be a need to bridge the gap between theory and practice and to set up an accessible framework for sophisticated yield curve analysis. This report serves as an overview of a forthcoming series of reports that will examine the theme *Understanding the Yield Curve*. After briefly describing the computation of par, spot and forward rates, it presents a framework for interpreting the forward rates by identifying their main influences and finally, it develops practical tools for using forward rate analysis in active bond portfolio management. Subsequent reports will discuss these topics in detail.[1]

The three main influences on the Treasury yield curve shape are: (1) the market's expectations of future rate changes; (2) bond risk premia (expected return differentials across bonds of different maturities); and (3) convexity bias. Conceptually, it is easy to divide the yield curve (or the term structure of forward rates) into these three components. It is much harder to interpret real-world yield curve shapes, but the potential benefits are substantial. For example, investors often wonder whether the curve's steepness reflects the market's expectations of rising rates or positive risk premia. The answer to this question determines whether a duration extension increases expected returns. It also shows whether we can view forward rates as the market's expectations of future spot rates. In addition, our analysis will describe how the market's curve reshaping expectations and volatility expectations influence the shape of today's yield curve. These expectations determine the cost of enhancing a portfolio's convexity via a duration-neutral yield curve trade.

Forward rate analysis also can be valuable in direct applications. Forward rates may be used as break-even rates with which subjective rate forecasts are compared or as relative value tools to identify attractive yield curve sectors. Subsequent reports will analyze many aspects of yield curve trades, such as barbell-bullet trades, and present empirical evidence about their historical behavior.

COMPUTATION OF PAR, SPOT AND FORWARD RATES

At the outset, it is useful to review the concepts "yield to maturity," "par yield," "spot rate," and "forward rate" to ensure that we are using our terms consistently. Appendix A is a reference that describes the notation and definitions of the main concepts used throughout the series *Understanding the Yield Curve*. Our analysis focuses on government bonds that have known cash flows (no default risk, no embedded options). *Yield to maturity* is the single discount rate that equates the present value of a bond's cash flows to its market price. *A yield curve* is a graph of bond yields against their maturities. (Alternatively, bond yields may be plotted against their durations, as we do in many figures in this report.) The best-known yield curve is the on-the-run Treasury curve. On-the-run bonds are the most recently issued government bonds at each maturity sector.

[1] This overview will contain few references to earlier studies, but later reports in this series will provide a guide to academic and practitioner literature for interested readers.

Because these bonds are always issued with price near par (100), the on-the-run curve often resembles the *par yield* curve, which is a curve constructed for theoretical bonds whose prices equal par.

While the yield to maturity is a convenient summary measure of a bond's expected return — and therefore a popular tool in relative value analysis — the use of a single rate to discount multiple cash flows can be problematic unless the yield curve is flat. First, all cash flows of a given bond are discounted at the same rate, even if the yield curve slope suggests that different discount rates are appropriate for different cash flow dates. Second, the assumed reinvestment rate of a cash flow paid at a given date can vary across bonds because it depends on the yield of the bond to which the cash flow is attached. This report will show how to analyze the yield curve using simpler building blocks — single cash flows and one-period discount rates — than the yield to maturity, an *average* discount rate of multiple cash flows with various maturities.

A coupon bond can be viewed as a bundle of zero-coupon bonds (zeros). Thus, it can be unbundled into a set of zeros, which can be valued separately. These zeros then can be rebundled into a more complex bond, whose price should equal the sum of the component prices.[2] The *spot rate* is the discount rate of a single future cash flow such as a zero. Equation (1) shows the simple relation between an n-year zero's price P_n and the annualized n-year spot rate s_n.

$$P_n = \frac{100}{(1 + s_n)^n} \tag{1}$$

A single cash flow is easy to analyze, but its discount rate can be unbundled even further to one-period rates. That is, a multiyear spot rate can be decomposed into a product of *one-year forward rates*, the simplest building blocks in a term structure of interest rates. A given term structure of spot rates implies a specific term structure of forward rates. For example, if the m-year and n-year spot rates are known, the annualized forward rate between maturities m and n, $f_{m,n}$, is easily computed from Equation (2).

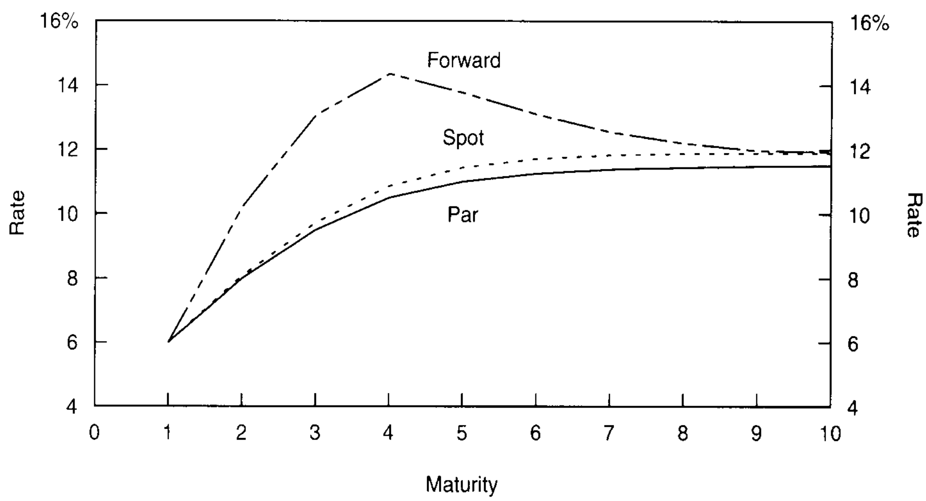
$$(1 + f_{m,n})^{n-m} = \frac{(1 + s_n)^n}{(1 + s_m)^m} \tag{2}$$

The forward rate is the interest rate for a loan between any two dates in the future, contracted today. Any forward rate can be "locked in" today by buying one unit of the n-year zero at price $P_n = 100/(1+s_n)^n$ and by shortselling P_n/P_m units of the m-year zero at price $P_m = 100/(1+s_m)^m$. (Such a weighting requires no net investment today because both the cash inflow and the cash outflow amount to P_n.) The one-year forward rate ($f_{n-1,n}$ such as $f_{1,2}$, $f_{2,3}$, $f_{3,4}$, ...) represents a special case of Equation (2) in which m = n-1. The spot rate represents another special case in which m = 0; thus, $s_n = f_{0,n}$.

[2] Arbitrage activities ensure that a bond's present value is similar when its cash flows are discounted using the marketwide spot rates as when its cash flows are discounted using the bond's own yield to maturity. However, some deviations are possible because of transaction costs and other market imperfections. In other words, the term structure of spot rates gives a consistent set of discount rates for all government bonds, but all bonds' market prices are not exactly consistent with these discount rates. Individual bonds may be rich or cheap relative to the curve because of bond-specific liquidity, coupon, tax, or supply effects. For example, the Salomon Brothers Government Bond Strategy Group reports daily each bond's spread off the estimated Treasury Model curve. (Because cheapness appears to persist over time, many investors prefer to use the Model spread relative to its own history as a relative value indicator.)

To summarize, a par rate is used to discount a set of cash flows (those of a par bond) to today, a spot rate is used to discount a single future cash flow to today and a forward rate is used to discount a single future cash flow to another (nearer) future date. The par yield curve, the spot rate curve and the forward rate curve contain the same information about today's term structure of interest rates[3] — if one set of rates is known, it is easy to compute the other sets.[4] Figure 1 shows a hypothetical example of the three curves. In Appendix B, we show how the spot and forward rates were computed based on the par yields.

Figure 1. Par, Spot and One-Year Forward Rate Curves



In this example, the par and spot curves are monotonically upward sloping, while the forward rate curve[5] first slopes upward and then inverts (because of the flattening of the spot curve). The spot curve lies above the par curve, and the forward rate curve lies above the spot curve. This is always the case if the spot curve is upward sloping. If it is inverted, the ordering is reversed: The par curve is highest and the forward curve lowest. Thus, loose characterizations of one curve (for example, steeply upward-sloping, flat, inverted, humped) are generally applicable to the other curves. However, the three curves are identical only if they are horizontal. In other cases, the forward rate curve magnifies any variation in the slope of the spot curve. One-year forward rates measure the *marginal* reward for lengthening the maturity of the investment by one year, while the spot rates measure an investment's *average* reward from today to maturity n. Therefore, spot rates are (geometric) averages of one or more forward

[3] These curves can be computed directly by interpolating between on-the-run bond yields (approximate par curve) or between zero yields (spot curve). Because these assets have special liquidity characteristics, these curves may not be representative of the broad Treasury market. Therefore, the par, spot or forward rate curve is typically estimated using a broad universe of coupon Treasury bond prices. Many different "curve fitting" techniques exist, but a common goal is to fit the prices well with a reasonably shaped curve. This report does not focus on yield curve estimation but on the interpretation and practical uses of the curve once it has been estimated.

[4] Further, one can use today's spot rates and Equation (2) to back out implied spot curves for any future date and implied future paths for the spot rate of any maturity. It is important to distinguish the implied spot curve one year forward ($f_{1,2}$, $f_{1,3}$, $f_{1,4}$, ...), a special case of Equation (2) in which m = 1, from the constant maturity one-year forward rate curve ($f_{1,2}$, $f_{2,3}$, $f_{3,4}$, ...). Today's spot curve can be subtracted from the former curve to derive the yield changes implied by the forwards. (This terminology is somewhat misleading because these "implied" forward curves/paths do not reflect only the market's expectations of future rates.)

[5] Note that all one-year forward rates actually have a one-year maturity even though, in the x-axis of Figure 1, each forward rate's maturity refers to the final maturity. For example, the one-year forward rate between n-1 and n ($f_{n-1,n}$) matures n years from today.

rates. Similarly, par rates are averages of one or more spot rates; thus, par curves have the flattest shape of the three curves. In Appendix C, we discuss further the relationship between spot and forward rate curves.

It is useful to view forward rates as **break-even rates.** The implied spot rates one year forward ($f_{1,2}$, $f_{1,3}$, $f_{1,4}$, ...) are, by construction, equal to such future spot rates that would make all government bonds earn the same return over the next year as the (riskless) one-year zero. For example, the holding-period return of today's two-year zero (whose rate today is s_2) will depend on its selling rate (as a one-year zero) in one year's time. The implied one-year spot rate one year forward ($f_{1,2}$) is computed as the selling rate that would make the two-year zero's return (the left-hand side of Equation (3)) equal to the one-year spot rate (the right-hand side of Equation (3))[6]. Formally, Equation (3) is derived from Equation (2) by setting m = 1 and n = 2 and rearranging.

$$\frac{(1 + s_2)^2}{1 + f_{1,2}} = 1 + s_1 \tag{3}$$

Consider an example using numbers from Figure 1, in which the one-year spot rate (s_1) equals 6% and the two-year spot rate (s_2) equals 8.08%. Plugging these spot rates into Equation (3), we find that the implied one-year spot rate one year forward ($f_{1,2}$) equals 10.20%. If this implied forward rate is exactly realized one year hence, today's two-year zero will be worth 100/1.1020 = 90.74 next year. Today, this zero is worth $100/1.0808^2 = 85.61$; thus, its return over the next year would be 90.74/85.61-1 = 6%, exactly the same as today's one-year spot rate. Thus, 10.20% is the break-even level of future one-year spot rate. In other words, the one-year rate has to increase by more than 420 basis points (10.20%-6.00%) before the two-year zero underperforms the one-year zero over the next year. If the one-year rate increases, but by less than 420 basis points, the capital loss of the two-year zero will not fully offset its initial yield advantage over the one-year zero.

More generally, **if the yield changes implied by the forward rates are subsequently realized, all government bonds, regardless of maturity, earn the same holding-period return**. In addition, all self-financed positions of government bonds (such as long a barbell versus short a bullet) earn a return of 0%; that is, they break even. In contrast, **if the yield curve remains unchanged over a year, each n-year zero earns the corresponding one-year forward rate $f_{n-1,n}$.** This can be seen from Equation (2) when m = n-1; $1+f_{n-1,n}$ equals $(1+s_n)^n/(1+s_{n-1})^{n-1}$, which is the holding-period return from buying an n-year zero at rate s_n, and selling it one year later at rate s_{n-1}. Thus, the one-year forward rate equals a zero's horizon return for an unchanged yield curve (see Appendix C for details).

[6] An alternative interpretation is also possible. Instead of viewing $f_{1,2}$ as the break-even *selling* rate of the two-year zero in one year's time, we can view it as the break-even *reinvestment* rate of the one-year zero over the second year. In the first case, we equate the uncertain one-year return of the two-year zero with the known return of a (horizon-matching) one-year zero. In the second case, we equate the uncertain two-year return of a roll-over-one-year-zeros strategy with the known return of a two-year zero.

In this section, we describe some economic forces that influence the term structure of forward rates or, more generally, the yield curve shape. The three main influences are **the market's rate expectations**, **the bond risk premia** (expected return differentials across bonds) and the so-called **convexity bias**. In fact, these three components fully determine the yield curve; we will show in later reports that the difference between each one-year forward rate and the one-year spot rate is approximately equal to the sum of an expected spot rate change, a bond risk premium and the convexity bias. We first discuss how each component influences the curve shape, and then we analyze their combined impact.

Rate Expectations

It is clear that the market's expectation of future rate changes is an important determinant of the yield curve shape. For example, a steeply upward-sloping curve may indicate that the market expects near-term Fed tightening or rising inflation. However, it may be too restrictive to assume that the yield differences across bonds with different maturities *only* reflect the market's rate expectations. The well-known pure expectations hypothesis has such an extreme implication. **The pure expectations hypothesis asserts that all government bonds have the same near-term expected return** (as the nominally riskless short-term bond) because the return-seeking activity of risk-neutral traders removes all expected return differentials across bonds. **Near-term expected returns are equalized if all bonds that have higher yields than the short-term rate are expected to suffer capital losses that offset their yield advantage**. When the market expects an increase in bond yields, the current term structure becomes upward-sloping so that any long-term bond's yield advantage and expected capital loss (due to the expected yield increase) exactly offset each other. In other words, if investors expect that their long-term bond investments will lose value because of an increase in interest rates, they will require a higher initial yield as a compensation for duration extension. Conversely, expectations of yield declines and capital gains will lower current long-term bond yields below the short-term rate, making the term structure inverted.

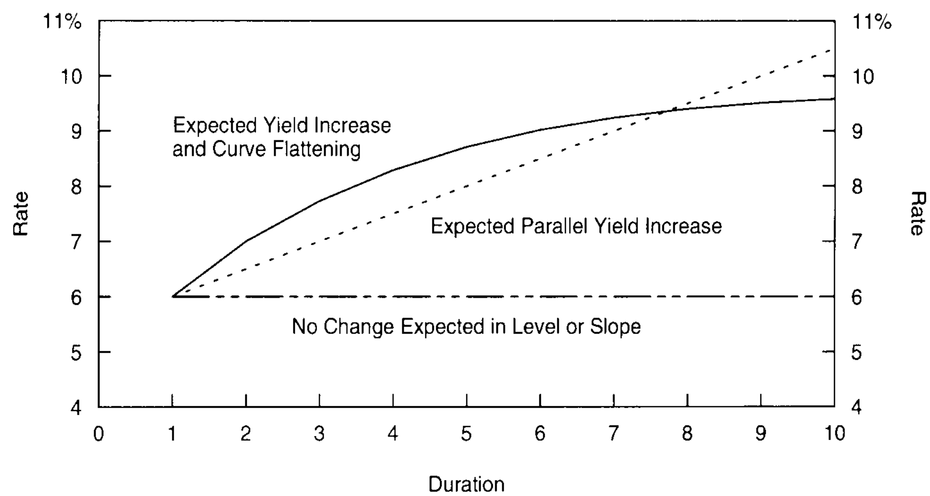
The same logic — that positive (negative) initial yield spreads offset expected capital losses (gains) to equate near-term expected returns — also holds for combinations of bonds, including duration-neutral yield curve positions. One example is a trade that benefits from the flattening of the yield curve between two- and ten-year maturities: selling a unit of the two-year bond, buying a duration-weighted amount of the ten-year bond and putting the remaining proceeds from the sale to "cash" (very short-term bonds). Given the typical concave yield curve shape (as a function of duration), such a curve flattening position earns a negative carry.[7] The trade will be profitable only if the curve flattens enough to offset the impact of the negative carry. Implied forward rates indicate how much flattening (narrowing of the two- to ten-year spread) is needed for the trade to break even.

In the same way as the market's expectations regarding the future level of rates influence the *steepness* of today's yield curve, the market's expectations regarding the future steepness of the yield curve influence the

[7] A concave shape means that the (upward-sloping) yield curve is steeper at the front end than at the long end. The yield loss of moving from the two-year bond to cash produces a greater yield loss than the yield gain achieved by moving from the two-year bond to the ten-year bond. Thus, the yield earned from the combination of cash and tens is lower than the foregone yield from twos.

curvature of today's yield curve. If the market expects more curve flattening, the negative carry of the flattening trades needs to increase (to offset the expected capital gains), making today's yield curve more concave (curved). Figure 2 illustrates these points. This figure plots coupon bonds' yields against their durations or, equivalently, zeros' yields against their maturities, given various rate expectations. Ignoring the bond risk premia and convexity bias, if the market expects no change in the level or slope of the curve, today's yield curve will be horizontal. **If the market expects a parallel rise in rates** over the next year but no reshaping, **today's yield curve will be linearly increasing** (as a function of duration). **If the market expects rising rates *and* a flattening curve, today's yield curve will be increasing and concave** (as a function of duration).[8]

Figure 2.Yield Curves Given the Market's Various Rate Expectations



Bond Risk Premium

A key assumption in the pure expectations hypothesis is that all government bonds, regardless of maturity, have the same expected return. In contrast, many theories and empirical evidence suggest that expected returns vary across bonds. We define the *bond risk premium* as a longer-term bond's expected one-period return in excess of the one-period bond's riskless return. A positive bond risk premium would tend to make the yield curve slope upward. However, various theories disagree about the sign (+/-), the determinants and the constancy (over time) of the bond risk premium. The classic liquidity premium hypothesis argues that most investors dislike short-term fluctuations in asset prices; these investors will hold long-term bonds only if they offer a positive risk premium as a compensation for their greater return volatility. Also some modern asset pricing theories suggest that the bond risk premium should increase with a bond's duration, its return volatility or its covariance with market wealth. In contrast, the preferred habitat hypothesis argues that the risk premium may decrease with duration; long-duration liability holders may perceive the long-term bond as the riskless asset and require higher expected returns for holding short-term assets. While academic analysis focuses on risk-related premia, market practitioners often emphasize other factors that cause expected return differentials across the yield curve. These include

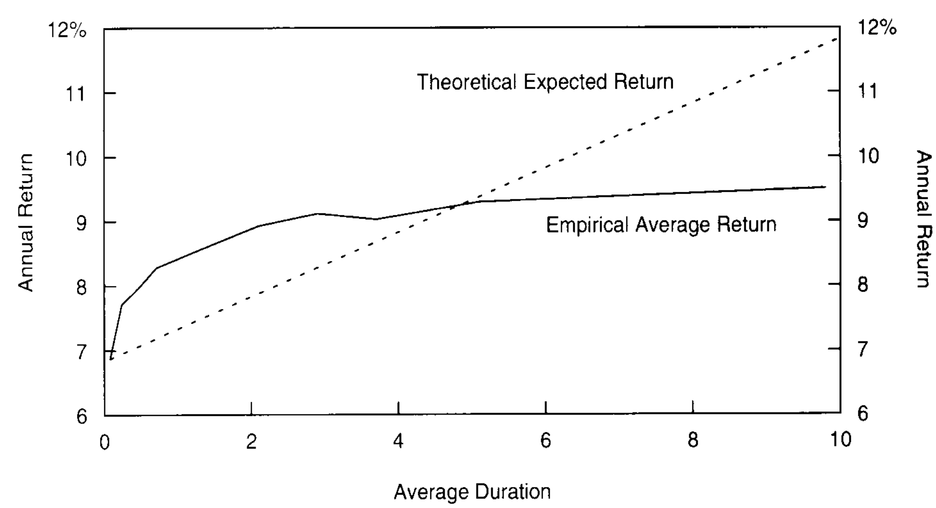
[8] Part 2 of this series, *Market's Rate Expectations and Forward Rates*, discusses these issues in detail.

liquidity differences between market sectors, institutional restrictions and supply and demand effects. **We use the term "bond risk premium" broadly to encompass all expected return differentials across bonds,** including those caused by factors unrelated to risk.

Historical data on U.S. Treasury bonds provide evidence about the empirical behavior of the bond risk premium. For example, the fact that the Treasury yield curve has been upward sloping nearly 90% of the time in recent decades may reflect the impact of positive bond risk premia. Historical average returns provide more direct evidence about expected returns across maturities than do historical yields. Even though weekly and monthly fluctuations in bond returns are mostly unexpected, the impact of unexpected yield rises and declines should wash out over a long sample period. Therefore, the *historical average returns* of various maturity sectors should reflect the *long-run expected returns.*

Figure 3 shows the empirical average return curve as a function of average duration and contrasts it to *one* theoretical expected return curve, one that increases linearly with duration. The theoretical bond risk premia are measured in Figure 3 by the difference between the annualized expected returns at various duration points and the annualized return of the riskless one-month bill (the leftmost point on the curve). Similarly, the empirical bond risk premia are measured by the historical average bond returns at various durations in excess of the one-month bill.[9] **Historical experience suggests that the bond risk premia are not linear in duration, but that they increase steeply with duration in the front end of the curve and much more slowly after two years.** The concave shape may reflect the demand for long-term bonds from pension funds and other long-duration liability holders.

Figure 3. Theoretical and Empirical Bond Risk Premia



[9] The empirical bond risk premia are computed based on monthly returns of various maturity-subsector portfolios of Treasury bills or bonds between 1970 and 1994. This period does not have an obvious bearish or bullish bias because long-term yields were at roughly similar level in the end of 1994 as they were in the beginning of 1970. Figure 3 plots arithmetic average annual returns on average durations. The geometric average returns would be a bit lower, and the curve would be essentially flat after two years.

Figure 3 may give us the best empirical estimates of the long-run average bond risk premia at various durations. However, **empirical studies also suggest that the bond risk premia are not constant but vary over time**. That is, it is possible to identify in advance periods when the near-term bond risk premia are abnormally high or low. These premia tend to be high after poor economic conditions and low after strong economic conditions. A possible explanation for such countercyclic variation in the bond risk premium is that investors become more risk averse when their wealth is relatively low, and they demand larger compensation for holding risky assets such as long-term bonds.[10]

Convexity Bias

The third influence on the yield curve — the convexity bias — is probably the least well known. Different bonds have different convexity characteristics, and the convexity differences across maturities can give rise to (offsetting) yield differences. In particular, long-term zeros exhibit very high convexity (see top panel of Figure 4), which tends to depress their yields. Convexity bias refers to the impact these convexity differences have on the yield curve shape.

Convexity is closely related to the nonlinearity in the bond price-yield relationship. All noncallable bonds exhibit positive convexity; their prices rise more for a given yield decline than they fall for a similar yield increase. All else being equal, positive convexity is a desirable characteristic because it increases a bond's return (relative to return in the absence of convexity) whether yields go up or down — as long as they move somewhere. Because positive convexity can only improve a bond's performance (for a given yield), more convex bonds tend to have lower yields than less convex bonds with the same duration.[11] In other words, investors tend to demand less yield if they have the prospect of improving their returns as a result of convexity. Investors are primarily interested in expected returns, and these high-convexity bonds can offer a given expected return at a lower yield level.

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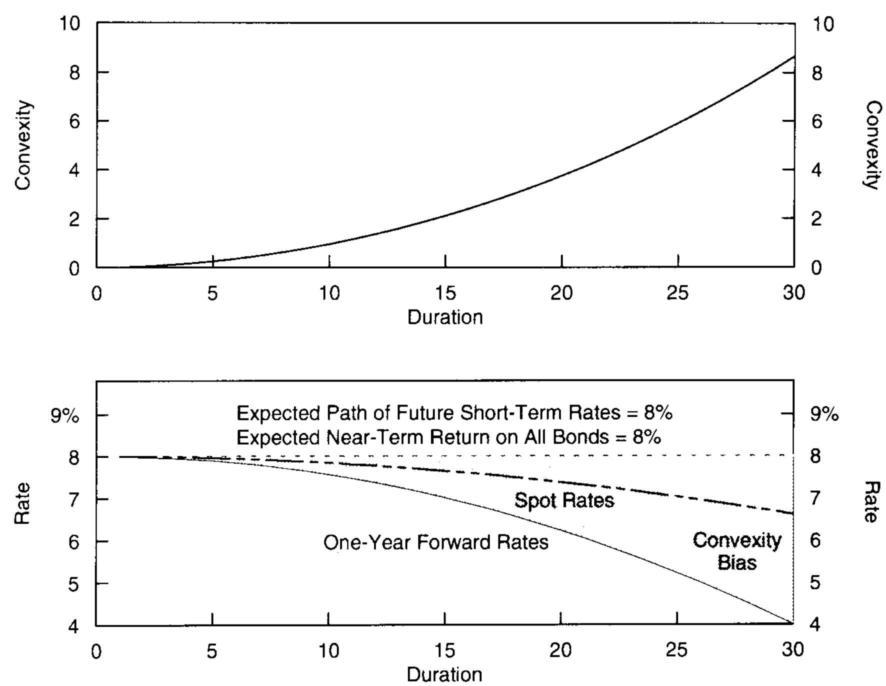
[10] Parts 3 and 4 of this series describe the empirical behavior of the bond risk premia. *Does Duration Extension Enhance Long-Term Expected Returns?* focuses on the long-run average return differentials across bonds with different maturities. *Forecasting U.S. Bond Returns* focuses on the near-term expected return differentials across bonds and on the time-variation in the bond risk premia.

[11] The degree of convexity varies across bonds, mainly depending on their option characteristics and durations. Embedded short options decrease convexity. For bonds without embedded options, convexity increases roughly as a square of duration (see Figure 4, top panel). There also are convexity differences between bonds that have the same duration. A barbell position (with very dispersed cash flows) exhibits more convexity than a duration-matched bullet bond. The reason is that a yield rise reduces the relative weight of the barbell's longer cash flows (because their present values decline more than those of the shorter cash flows), shortening the barbell's duration. The inverse relation between duration and yield level increases a barbell's convexity, limiting its losses when yields rise and enhancing its gains when yields decline. Of all bonds with the same duration, a zero has the smallest convexity because its cash flows are not dispersed; thus, its Macaulay duration does not vary with the yield level.

Figure 4. Convexity and the Yield Curve



Note: Volatility of annual yield changes is assumed to be 100 basis points. Thus, the convexity bias is -0.5 * the zero's convexity * 1.

The lower panel of Figure 4 illustrates the pure impact of convexity on the curve shape by plotting the spot rate curve and the curve of one-year forward rates when all bonds have the same expected return (8%) and the short-term rates are expected to remain at the current level. With no bond risk premia and no expected rate changes, one might expect these curves to be horizontal at 8%. Instead, they slope down at an increasing pace because **lower yields are needed to offset the convexity advantage of longer-duration bonds** and thereby to equate the near-term expected returns across bonds.[12] Short-term bonds have little convexity; therefore, there is little convexity bias at the front end of the yield curve, but convexity can have a dramatic impact on the curve shape at very long durations. Convexity bias can be one of the main reasons for the typical concave yield curve shape (that is, for the tendency of the curve to flatten or invert at long durations).

The value of convexity increases with the magnitude of yield changes. Therefore, **increasing volatility should make the overall yield curve shape more concave (curved)** and widen the spreads between more and less convex bonds (duration-matched coupon bonds versus zeros and barbells versus bullets).[13]

[12] Convexity bias is closely related to the distinction between different versions of the pure expectations hypothesis. Above, we referred to *the* pure expectations hypothesis. In fact, alternative versions of this hypothesis exist that are not exactly consistent with each other. The local expectations hypothesis (LEH) assumes that "all bonds earn the same expected return over the next period" while the unbiased expectations hypothesis (UEH) assumes that "forward rates equal expected spot rates." In Figure 4 (lower panel), the LEH is assumed to hold; thus, UEH is not exactly true. The expected future short rates are flat at 8% even though the curve of one-year forward rates is inverted. In yield terms, the difference between the LEH and the UEH is the convexity bias.

[13] Part 5 of this series, *Convexity Bias and the Yield Curve*, discusses these topics in more detail.

Putting the Pieces Together

Of course, all three forces influence bond yields simultaneously, making the task of interpreting the overall yield curve shape quite difficult. **A steeply upward-sloping curve can reflect either the market's expectations of rising rates or high required risk premia. A strongly humped curve (that is, high curvature) can reflect the market's expectations of either curve flattening or high volatility** (which makes convexity more valuable), or even the concave shape of the risk premium curve.

In theory, the yield curve can be neatly decomposed into expectations, risk premia and convexity bias. **In reality, exact decomposition is not possible** because the three components vary over time and are not directly observable but must be estimated.[14] Even though an exact decomposition is not possible, the analysis in this and subsequent reports should give investors a framework for interpreting various yield curve shapes. These reports will characterize the behavior of rate expectations, risk premia and convexity bias; show how they influence the curve; and evaluate the magnitude of their impact using historical data. Furthermore, our survey of earlier literature and our new empirical work will evaluate which theories and market myths are correct (consistent with data) and which are false. The main conclusions are as follows:

- We often hear that **"forward rates show the market's expectations of future rates."** However, **this statement is only true if no bond risk premia exist and the convexity bias is very small.**[15] If the goal is to infer expected short-term rates one or two years ahead, the convexity bias is so small that it can be ignored. In contrast, our empirical analysis shows that the bond risk premia are important at short maturities. Therefore, if the forward rates are used to infer the market's near-term rate expectations, some measures of bond risk premia should be subtracted from the forwards, or the estimate of the market's rate expectations will be strongly upward biased.

- The traditional term structure theories assume a zero risk premium (pure expectations hypothesis) or a nonzero but *constant* risk premium (liquidity premium hypothesis, preferred habitat hypothesis) which is inconsistent with historical data. According to the pure expectations hypothesis, an upward-sloping curve should predict increases in long-term rates, so that a capital loss offsets the long-term bonds' yield advantage. However, empirical evidence shows that, on average, small declines in long-term rates, which *augment* the long-term bonds' yield advantage, follow upward-sloping curves. The steeper the yield curve is, the higher the expected bond risk premia. **This finding clearly violates the pure expectations hypothesis and supports hypotheses about time-varying risk premia**.

- Modern term structure models make less restrictive assumptions than the traditional theories above. Yet, many popular one-factor models assume that bonds with the same duration earn the same expected return. Such an assumption implies that duration-neutral positions with more or less convexity earn the same expected return (because a yield disadvantage

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[14] In later papers, we will show how interest rate expectations can be measured using survey data, how bond risk premia can be estimated using historical return data and how the convexity bias can be inferred using option prices. Alternatively, all three components can be estimated from the yield curve if one is willing to impose the structure of some term structure model (with its possibly unrealistic assumptions).

[15] A related assertion claims that if near-term expected returns were not equal across bonds, it would imply the existence of *riskless arbitrage opportunities*. This assertion is erroneous. It is true that if forward contracts were traded assets, arbitrage forces would require their pricing to be consistent with zero prices according to Equation (2). However, the arbitrage argument says nothing about the economic determinants of the zero prices themselves, such as rate expectations or risk premia. The bond market's performance in 1994 shows that buying long-term bonds is not riskless even if they have higher expected returns than short-term bonds.

exactly offsets any convexity advantage). However, if the market values very highly the insurance characteristics of positively convex positions, **more convex positions may earn lower expected returns**. Our analysis of the empirical performance of duration-neutral barbell-bullet trades will show that, in the long run, barbells tend to marginally underperform bullets.

USING FORWARD RATE ANALYSIS IN YIELD CURVE TRADES

Recall that if the pure expectations hypothesis holds, all bond positions have the same near-term expected return. In particular, an upward-sloping yield curve reflects expectations of rising rates and capital losses, and convexity is priced so that a yield disadvantage exactly offsets the convexity advantage. In such a world, yields do not reflect value, no trades have favorable odds and active management can add value only if an investor has truly superior forecasting ability. Fortunately, **the real world is not quite like this theoretical world**. Empirical evidence (presented in parts 2-4 of this series of reports) shows that expected returns do vary across bonds. The main reason is probably that most investors exhibit risk aversion and preferences for other asset characteristics; moreover, investor behavior may not always be fully rational. Therefore, yields reflect value and certain relative value trades have favorable odds.

The previous section provided a framework for thinking about the term structure shapes. In this section, we describe practical applications — different ways to use forward rates in yield curve trades. The first approach requires strong subjective rate views and faith in one's forecasting ability.

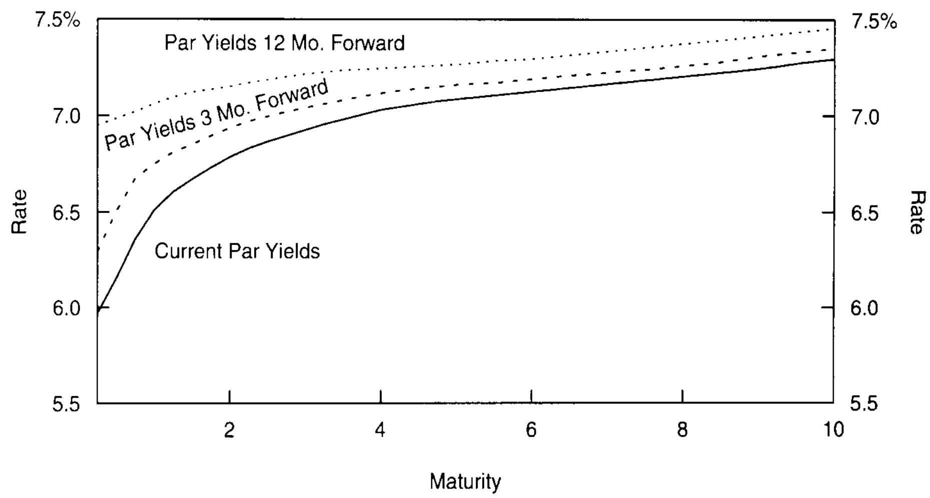
Forwards as Break-Even Rates for Active Yield Curve Views

The forward rates show a path of break-even future rates and spreads. **This path provides a clear yardstick for an active portfolio manager's subjective yield curve scenarios and yield path forecasts**. It incorporates directly the impact of carry on the profitability of the trade. For example, a manager should take a bearish portfolio position only if he expects rates to rise by more than what the forwards imply. However, if he expects rates to rise by less than what the forwards imply (that is, by less than what is needed to offset the positive carry), he should take a bullish portfolio position. If the manager's forecast is correct, the position will be profitable. In contrast, managers who take bearish portfolio positions whenever they expect bond yields to rise — ignoring the forwards — may find that their positions lose money, because of the negative carry, even though their rate forecasts are correct.

One positive aspect about the role of forward rates as break-even rates is that it does not depend on assumptions regarding expectations, risk premia or convexity bias. The rules are simple. If forward rates are realized, all positions earn the same return. If yields rise by more than the forwards imply, bearish positions are profitable and bullish positions lose money. If yields rise by less than the forwards imply, the opposite is true. Similar statements hold for any yield spreads and related positions, such as curve-flattening positions.

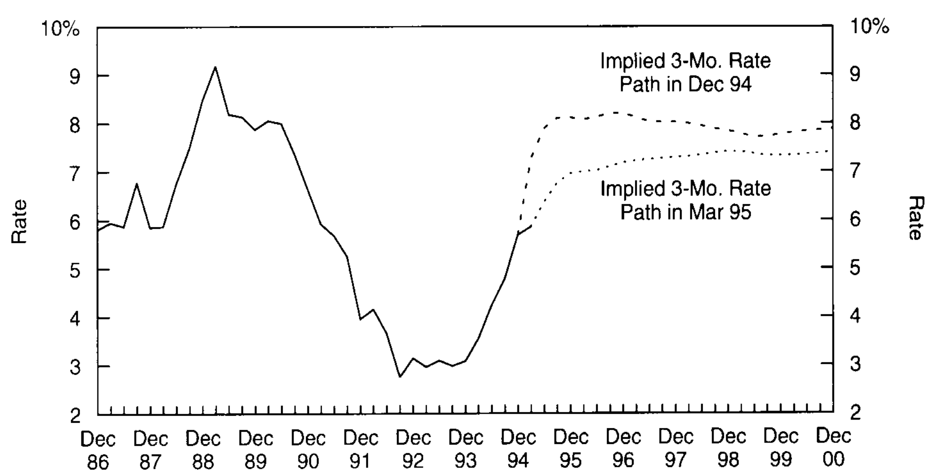
Figure 5 shows the U.S. par yield curve and the implied par curves three months forward and 12 months forward based on the Salomon Brothers Treasury Model as of March 31, 1995. If we believe that forward rates only reflect the market's rate expectations, a comparison of these curves tells us that the market expects rates to rise and the curve to flatten over the next year. Alternatively, the implied yield rise may reflect a bond risk premium and the implied curve flattening may reflect the value of convexity. Either way, the forward yield curves reflect the break-even levels between profits and losses.

Figure 5. Current and Forward Par Yield Curves, as of 31 Mar 95



The information in the forward rate structure can be expressed in several ways. Figure 5 is useful for an investor who wants to contrast his subjective view of the future yield curve with an objective break-even curve at some future horizon. Another graph may be more useful for an investor who wants to see the break-even future path of any constant-maturity rate (instead of the whole curve) and contrast it with his own forecast, which may be based on a macroeconomic forecast or on the subjective view about the speed of Fed tightening. As an example, Figure 6 shows such a break-even path of future three-month rates at the end of March 1995.[16] To add perspective, the graph also contains the historical path of the three-month rate over the past eight years and the break-even path of the future three-month rates at the end of 1994 when the Treasury market sentiment was much more bearish.

Figure 6. Historical Three-Month Rates and Implied Forward Three-Month Rate Path, as of 30 Dec 94 and 31 Mar 95

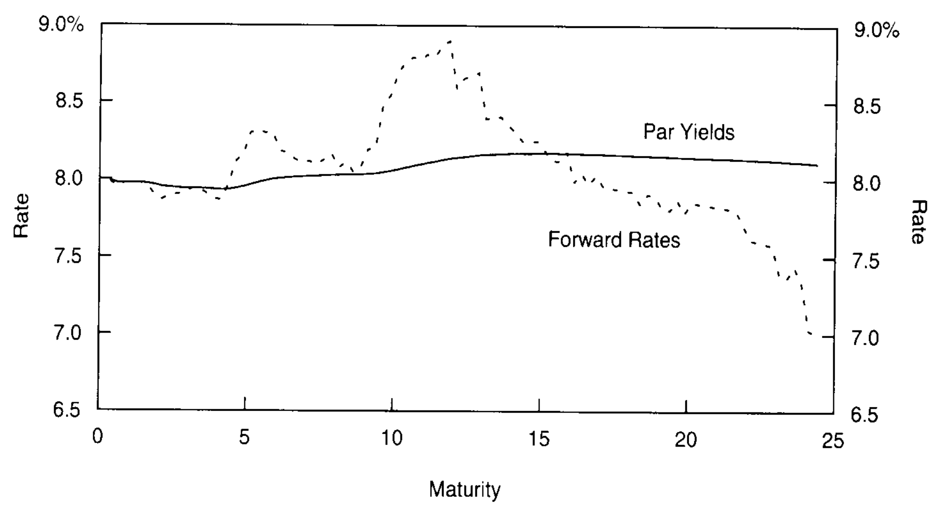


[16] Note that the first point in each implied forward par curve in Figure 5 is the implied forward three-month rate at a given future date. Therefore, the forward path in Figure 6 can be constructed by tracing through the three-month points in the three curves of Figure 5 and through similar curves at other horizons. Because Figure 6 depicts a rate path over time, the x-axis is calendar years and not maturity.

Forwards as Indicators of Cheap Maturity Sectors
The other ways to use forwards require less subjective judgment than the first one. As a simple example, the forward rate curve can be used to identify cheap maturity sectors visually. Abnormally high forward rates are more visible than high spot or par rates because the latter are averages of forward rates.

Figure 7 shows one real-world example from the beginning of this decade when the par yield curve was extremely flat (although forwards may be equally useful when the par curve is not flat). Even though the par yield curve was almost horizontal (all par yields were within 25 basis points), the range of three-month annualized forward rates was almost 200 basis points because the forward rate curve magnifies the cheapness/richness of different maturity sectors. High forward rates identify the six-year sector and the 12-year sector as cheap, and low forward rates identify the four-year sector and the nine-year sector as expensive.[17]

Figure 7. Par Yields and Three-Month Forward Rates, as of 2 Jan 90



Once an investor has identified a sector with abnormally high forward rates (for example, between nine and 12 years), he can exploit the cheapness of this sector by buying a bond that matures at the end of the period (12 years) and by selling a bond that matures at the beginning of the period (nine years). If equal market values of these bonds are bought and sold, the position is exposed to a general increase in rates and a steepening yield curve. More elaborate trades can be constructed (for example, by selling both the nine-year and 15-year bonds against the 12-year bonds with appropriate weights) to retain level and slope neutrality. To the extent that bumps and kinks in the forward curve reflect temporary local cheapness, the trade will earn capital gains when the forward curve becomes flatter and the cheap sector richens (in addition to the higher yield and rolldown the position earns). In the example of Figure 7, such "richening" actually did happen over the next three months.

[17] Forward rates are also very low at the long maturities, but this characteristic probably reflects the convexity bias. Forward rates are downward-biased estimates of expected returns because they ignore the convexity advantage which is especially large at long maturities.

Forwards as Relative Value Tools for Yield Curve Trades

Above, forwards are used quite loosely to identify cheap maturity sectors. A more formal way to use forwards is to construct quantitative cheapness indicators for duration-neutral flattening trades, such as barbell-bullet trades. We first introduce some concepts with an example of a market-directional trade.

When the yield curve is upward sloping, long-term bonds' yield advantage over the riskless short-term bond provides a cushion against rising yields. In a sense, duration extensions are "cheap" when the yield curve is very steep and the cushion (positive carry) is large. These trades only lose money if capital losses caused by rising rates offset the initial yield advantage. Moreover, the longer-term bonds' rolling yield advantages[18] over the short-term bond are even larger than their yield advantages. **The one-year forward rate ($f_{n-1,n}$) is, by construction, equal to the n-year zero's rolling yield** (see Appendix C). Thus, it is a direct measure of the n-year zero's rolling yield advantage. (Another forward-related measure, the change in the n-1 year spot rate implied by the forwards ($f_{1,n}$ -s_{n-1}), tells how much the yield curve has to shift to offset this advantage and to equate the holding-period returns of the n-year zero and the one-year zero.)

Because one-period forward rates measure zeros' near-term expected returns, they can be viewed as indicators of cheap maturity sectors. The use of such cheapness indicators does not require any subjective interest rate view. Instead, it requires a belief, motivated by history, that an unchanged yield curve is a good base case scenario.[19] If this is true, long-term bonds have higher (lower) near-term expected returns than short-term bonds when the forward rate curve is upward sloping (downward sloping). In the long run, a strategy that adjusts the portfolio duration dynamically based on the curve shape should earn a higher average return than constant-duration strategies.[20]

Similar analysis holds for curve-flattening trades. Recall that when the yield curve is concave as a function of duration, any duration-neutral flattening trade earns a negative carry. Higher concavity (curvature) in the yield curve indicates less attractive terms for a flattening trade (larger negative carry) and more "implied flattening" by the forwards (which is needed to offset the negative carry). Therefore, **the amount of spread change implied by the forwards is a useful cheapness indicator for yield curve trades** at different parts of the curve. If the implied change is historically wide, the trade is expensive, and vice versa.

Figure 8 shows an example of a recent situation in which the flattening trades were extremely expensive. At the end of December 1994, the three-month to two-year sector of the Treasury curve was very steep (a spread of 200 basis points) and the two- to 30-year sector was quite flat (a spread of 20 basis points). The high curvature indicated strong flattening expectations — forwards implied an inversion of the two- to 30-year spread by March — or high expected volatility (high value of convexity).

---

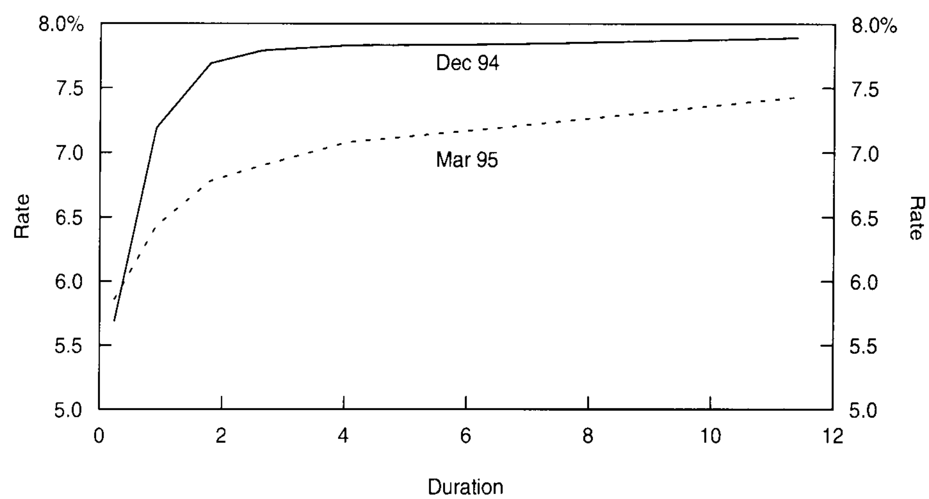
[18] As bonds age, they roll down the upward-sloping yield curve and earn some rolldown return (capital gain due to this yield change) if the yield curve remains unchanged. A bond's rolling yield, or horizon return, includes the yield and the rolldown return given a scenario of no change in the yield curve.

[19] The one-period forward rate can proxy for the near-term expected return — albeit with a downward-bias because it ignores the value of convexity — *if* the current yield curve is not expected to change. Empirical studies show that the assumption of an unchanged curve is more realistic than the assumption that forward rates reflect expected future yields. Historically, current spot rates predict future spot rates better than current forward rates do because the yield changes implied by the forwards have not been realized, on average.

[20] Part 4 of this series, *Forecasting U.S. Bond Returns*, evaluates the historical performance of dynamic strategies that exploit the predictability of long-term bonds' near-term returns. The dynamic strategies have consistently outperformed static strategies that do not actively adjust the portfolio duration.

The barbell (of the 30-year bond and three-month bill) over the duration-matched two-year bullet would become profitable only if the curve flattened even more than the forwards implied or if a sudden increase in volatility occurred. Purely on yield grounds, the two-year bullet (a steepening position) appeared cheap in an absolute comparison (across bonds) and in a historical comparison (over time). With the benefit of hindsight, we know that the cheapness indicator gave a correct signal. The two-year bullet outperformed various duration-matched barbell positions substantially over the next quarter as it earned large capital gains in addition to its high initial yield and rolldown advantage. By the end of March, the front end of the curve had flattened by 108 basis points and the long end had steepened by 45 basis points. Figure 8 illustrates the decline in curvature by plotting the Treasury on-the-run yield curves (as a function of duration) on December 30 and on March 31. In later reports, we will show how to use forward rate analysis to evaluate opportunities like this.

Figure 8. On-the-Run Yield Curves, as of 30 Dec 94 and 31 Mar 95



APPENDIX A. NOTATION AND DEFINITIONS USED IN THE SERIES UNDERSTANDING THE YIELD CURVE

P	Market price of a bond.
P_n	Market price of an n-year zero.
C	Coupon rate (in percentage; other rates are expressed as a decimal).
y	Annualized yield to maturity (YTM) of a bond.
n	Time to maturity of a bond (in years).
s_n	Annualized n-year spot rate; the discount rate of an n-year zero.
s^*_{n-1}	Annualized n-1 year spot rate next period; superscript * denotes next period's (year's) value.
Δs_{n-1}	Realized change in the n-1 year spot rate between today and next period (= $s^*_{n-1} - s_{n-1}$)
$f_{m,n}$	Annualized forward rate between maturities m and n.
$f_{n-1,n}$	One-year forward rate between maturities (n-1) and n; also the n-year zero's rolling yield.
$f_{1,n}$	Annualized forward rate between maturities 1 and n; also called the implied n-1 year spot rate one year forward.
Δf_{n-1}	Implied change in the n-1 year spot rate between today and next period (= $f_{1,n} - s_{n-1}$); also called the break-even yield change (over the next period) implied by the forwards.
Δfz_n	Implied change in the yield of an n-year zero, a specific bond, over the next period (= $f_{1,n} - s_n$).
FSP	Forward-spot premium ($FSP_n = f_{n-1,n} - s_1$).
h_n	Realized holding-period return of an n-year zero over one period (year).
Rolling Yield	A bond's horizon return given a scenario of unchanged yield curve; sum of yield and rolldown return.
Bond Risk Premium (BRP)	Expected return of a long-term bond over the next period (year) in excess of the riskless one-period bond; for the n-year zero, $BRP_n = E(h_n - s_1)$.
Realized BRP	Realized one-year holding-period return of a long-term bond in excess of the one-year bond; also called excess bond return; realized $BRP_n = h_n - s_1$.
Persistence Factor (PF)	Slope coefficient in a regression of the annual realized BRP_n on FSP_n.
Term Spread	Yield difference between a long-term bond and a short-term bond; for the n-year zero, $= s_n - s_1$.
Real Yield	Difference between a long-term bond yield and a proxy for expected inflation; our proxy is the recently published year-on-year consumer price inflation rate.
Inverse Wealth	Ratio of exponentially weighted past wealth to the current wealth; we proxy wealth W by the stock market level; $= (W_{t-1} + 0.9*W_{t-2} + 0.9^2 *W_{t-3} + ...)*0.1/W_t$
Duration (Dur)	Measure of a bond price's interest rate sensitivity; Dur = $-(dP/dy) * (1/P)$
Convexity (Cx)	Measure of the nonlinearity in a bond's P/y -relation; Cx = $(d^2P/dy^2) * (1/P)$
Convexity Bias (CB)	Impact of convexity on the curve of one-year forward rates; $CB_n = -0.5 * Cx_n * (\text{Volatility of } \Delta s_n)^2$

Figure 9. Par, Spot and One-Year Forward Rates

Maturity	Par Rate	Spot Rate	Forward Rate
1	6.00%	6.00%	6.00%
2	8.00	8.08	10.20
3	9.50	9.72	13.07
4	10.50	10.86	14.36
5	11.00	11.44	13.77
6	11.25	11.71	13.10
7	11.38	11.83	12.55
8	11.44	11.88	12.20
9	11.48	11.89	11.97
10	11.50	11.89	11.93

APPENDIX C. RELATIONS BETWEEN SPOT RATES, FORWARD RATES, ROLLING YIELDS, AND BOND RETURNS

Investors often want to make quick "back-of-the-envelope" calculations with spot rates, forward rates and bond returns. In this appendix, we discuss some simple relations between these variables, beginning with a useful approximate relation between spot rates and one-year forward rates. These relations are discussed in more detail in the appendix of *Market's Rate Expectations and Forward Rates*. Equation (2) showed exactly how the forward rate between years m and n is related to m- and n-year spot rates. Equation (8) shows the same relation in an approximate but simpler form; this equation ignores nonlinear effects such as the convexity bias. The relation is exact if spot rates and forward rates are continuously compounded.

$$f_{m,n} \approx \frac{ns_n - ms_m}{n - m} \qquad (8)$$

For one-year forward rates (m = n-1), Equation (8) can be simplified to

$$f_{n-1,n} \approx s_n + (n-1) * (s_n - s_{n-1}). \qquad (9)$$

Equation (9) shows that the forward rate is equal to an n-year zero's one-year horizon return given an unchanged yield curve scenario: a sum of the initial yield and the rolldown return (the zero's duration at horizon (n-1) multiplied by the amount the zero rolls down the yield curve as it ages). This horizon return is often called the rolling yield. Thus, the one-year forward rates proxy for near-term expected returns at different parts of the yield curve if the yield curve is expected to remain unchanged. We can gain intuition about the equality of the one-year forward rate and the rolling yield by examining the n-year zero's realized holding-period return h_n over the next year, in Equation (10). The zero earns its initial yield s_n plus a capital gain/loss, which is approximated by the product of the zero's year-end duration and its realized yield change.

$$h_n \approx s_n + (n-1) * (s_n - s^*_{n-1}) \qquad (10)$$

A simple example illustrates how spot rates and forward rates are computed on a coupon date when the par curve is known (and coupon payments and compounding frequency are annual). The basis of the procedure is the fact that a bond's price will be the same, the sum of the present values of its cash flows, whether it is priced via yield to maturity (Equation (4)) or via the spot rate curve (Equation (5)):

$$P = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C+100}{(1+y)^n} \tag{4}$$

$$P = \frac{C}{1+s_1} + \frac{C}{(1+s_2)^2} + \ldots + \frac{C+100}{(1+s_n)^n} \tag{5}$$

where P is the bond price, C is the coupon rate (in percentage), y is the annual yield to maturity (expressed as a decimal), s is the annual spot rate (expressed as a decimal), and n is the time to maturity (in years).

We only show the computation for the first two years, which have par rates of 6% and 8%. For the first year, par, spot, and forward rates are equal (6%). Longer spot rates are solved recursively using known values of the par bond's price and cash flows and the previously solved spot rates. Every par bond's price is 100 (par) by construction; thus, its yield (the par rate) equals its coupon rate. Because the two-year par bond's market price (100) and cash flows (8 and 108) are known, as is the one-year spot rate (6%), it is easy to solve for the two-year spot rate as the only unknown in the following equation:

$$100 = \frac{C}{1+s_1} + \frac{C}{(1+s_2)^2} = \frac{8}{1.06} + \frac{108}{(1+s_2)^2}\,. \tag{6}$$

A little manipulation shows that the solution for s_2 is 8.08%. Equation (6) also can be used to compute par rates when only spot rates are known. If the spot rates are known, the coupon rate C, which equals the par rate, is the only unknown in Equation (6).

The forward rate between one and two years is computed using Equation (3) and the known one-year and two-year spot rates.

$$(1 + f_{1,2}) = \frac{(1+s_2)^2}{1+s_1} = \frac{(1.0808)^2}{1.06} = 1.1020 \tag{7}$$

The solution for $f_{1,2}$ is 10.20%. The other spot rates and one-year forward rates ($f_{2,3}$, $f_{3,4}$, etc.) in Figure 9 are computed in the same way. These numbers are shown graphically in Figure 1.

where s^*_{n-1} is the n-1 year spot rate next year. If the yield curve follows a random walk, the best forecast for s^*_{n-1} is (today's) s_{n-1}. Therefore, the n-year zero's *expected* holding period return equals the one-year forward rate in Equation (9). The key question is whether it is more reasonable to assume that the current spot rates are the optimal forecasts of future spot rates than to assume that forwards are the optimal forecasts. We present later empirical evidence which shows that the "random walk" forecast of an unchanged yield curve is more accurate than the forecast implied by the forwards.

Equation (9) shows that the (one-year) forward rate curve lies above the spot curve as long as the latter is upward sloping (and the rolldown return is positive). Conversely, if the spot curve is inverted, the rolldown return is negative, and the forward rate curve lies below the spot curve. If the spot curve is first rising and then declining, the forward rate curve crosses it from above at its peak. Finally, the forward rate curve can become downward sloping even when the spot curve is upward sloping, if the spot curve's slope is first steep and then flattens (reducing the rolldown return). The calculations below illustrate this point and show that the approximation is good — within a few basis points from the correct values (10.20-13.07-14.36-13.77) in Figure 9:

$f_{1,2} \approx 8.08 + 1 * (8.08 - 6.00) = 8.08 + 2.08 = 10.16;$

$f_{2,3} \approx 9.72 + 2 * (9.72 - 8.08) = 9.72 + 3.28 = 13.00;$

$f_{3,4} \approx 10.86 + 3 * (10.86 - 9.72) = 10.86 + 3.42 = 14.28;$ and

$f_{4,5} \approx 11.44 + 4 * (11.44 - 10.86) = 11.44 + 2.32 = 13.76.$

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Market's Rate Expectations and Forward Rates

Understanding the Yield Curve: Part 2

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FIGURES

INTRODUCTION

Our recent report *Overview of Forward Rate Analysis* introduced a series on the theme *Understanding the Yield Curve*. It argued that three main forces determine the term structure of forward rates: the market's rate expectations; required bond risk premia; and the convexity bias. Separate reports discuss each of these forces. This report focuses on the impact of the market's rate expectations on the yield curve shape.

The impact of rate expectations on today's yield curve shape is best isolated by assuming that the pure expectations hypothesis holds. According to this hypothesis, all government bonds have the same near-term expected return (that is, all bond risk premia are zero). If the near-term expected returns are equal across maturities, initial yield differences must offset any expected capital gains or losses that are caused by the market's rate expectations. For example, **if the market expects rates to rise and the long-term bonds to suffer capital losses, these bonds must have an initial yield advantage over the one-period bond** (to offset the expected capital losses). **Therefore, expectations of rising rates tend to make today's yield curve upward sloping.** Conversely, expectations of declining future rates tend to make today's yield curve inverted. In a similar way, the market's expectations of future curve flattening or steepening influence the curvature of today's yield curve.

We emphasize the distinction between the statements "the forwards imply rising spot rates" and "the market expects rising spot rates." The first statement is related to the forward rates' role as break-even levels of future spot rates. By construction, the spot rate changes that the forwards imply for the next period are such rate changes that would make all government bonds earn the same one-period return. **Whenever the spot rate curve is upward sloping, the forwards imply rising rates. That is, rising rates are needed to offset long-term bonds' yield advantage. However, it does not necessarily follow that the market expects rising rates**. An upward-sloping spot rate curve also may reflect higher near-term required returns for long-term bonds than for the riskless one-period bond (so-called positive bond risk premia). The changes in spot rates that the forwards imply would be approximately equal to the expected spot rate changes *only if* the restrictive pure expectations hypothesis were true.

In this report, we also present empirical evidence about rate expectations and conclude that the pure expectations hypothesis is, in many ways, at odds with historical experience. We show that **forward rates are poor predictors of future spot rates. In fact, long-term rates tend to move *away* from the direction implied by the forward rates.** We also contrast the expectations "implied" in the forward rate structure to the expectations revealed in explicit surveys among bond market participants. The comparison suggests that forward rates are *upward-biased* measures of the market's rate expectations because the market appears to require higher expected returns for long-term bonds than for short-term bonds.

ALGEBRAIC RELATIONS BETWEEN SPOT AND FORWARD RATES

This section reviews the relations between spot rates and forward rates and describes forward rates' role as break-even rates. The discussion in this section overlaps — but expands on — the discussion in *Overview of Forward Rate Analysis*. Readers familiar with the basic concepts in forward rate analysis may wish to move directly to the section "The Expectations Hypothesis and the Yield Curve."

Computation of Forward Rates

A **spot rate** is the discount rate of a single future cash flow such as a zero-coupon bond (zero). A coupon bond can be viewed and valued as a bundle of zeros. Given the price P_n of an n-year zero, the annualized n-year spot rate s_n can be computed as in Equation (1):[1]

$$P_n = \frac{100}{(1 + s_n)^n} . \tag{1}$$

A **forward rate** is the interest rate for a loan between any two future dates, contracted today. The rate may be explicit in the price of a traded forward contract or it may be implicit in today's spot rate curve. A zero's discount rate (a multiyear spot rate) can be decomposed into a product of one-year forward rates. Thus, the spot rate is a geometric average of one-year forward rates:[2]

$$(1 + s_n)^n = (1+ f_{0,1})(1 + f_{1,2})(1 + f_{2,3}) \ldots (1 + f_{n-1,n}) , \tag{2}$$

where $f_{n-1,n}$ is the one-year forward rate between maturities n-1 and n, and $f_{0,1} = s_1$. If only one spot rate is known, all forward rates cannot be computed. However, if spot rates are known for each maturity, a given term structure of spot rates implies a specific term structure of forward rates. For example, if the m-year and n-year spot rates are known, $f_{m,n}$ (the annualized n-m year rate m years forward) can be computed as in Equation (3):

$$(1 + f_{m,n})^{n-m} = \frac{(1 + s_n)^n}{(1 + s_m)^m} . \tag{3}$$

Figure 1 presents a hypothetical spot rate curve and two series derived using Equation (3): the implied forward path of the constant-maturity one-year rate at various future dates and the implied spot curve at one future date, one year hence.[3] It is important to **distinguish the curve of one-year forward rates** in column B from the **implied spot rate curve one year forward** in column C. Unfortunately the terminology varies because both are sometimes called the forward curves. To clarify the relations between these curves, Figure 2 shows the future years covered by ten spot rates (s_1 to s_{10}), ten one-year forward rates ($f_{0,1}$ to $f_{9,10}$) and nine implied spot rates one year forward ($f_{1,2}$ to $f_{1,10}$). The three curves are shown graphically in Figures 3 and 4.

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[1] In practice, the spot rates are rarely inferred from the zero-coupon bond (STRIPS) prices because STRIPS differ from the more common Treasury coupon bonds in terms of liquidity and tax treatment. More often, the spot curve is estimated from the prices of all or most Treasury coupon bonds. Once the spot curve is known, it is a matter of algebra to construct the **par yield** curve which many investors prefer to follow. Alternatively, if the par curve is known, it is easy to construct the spot curve. For details, see *Using STRIPS in a Treasury Portfolio*, Janet Showers, Salomon Brothers Inc, August 1992, and *Overview of Forward Rate Analysis*, Salomon Brothers Inc, May 1995.

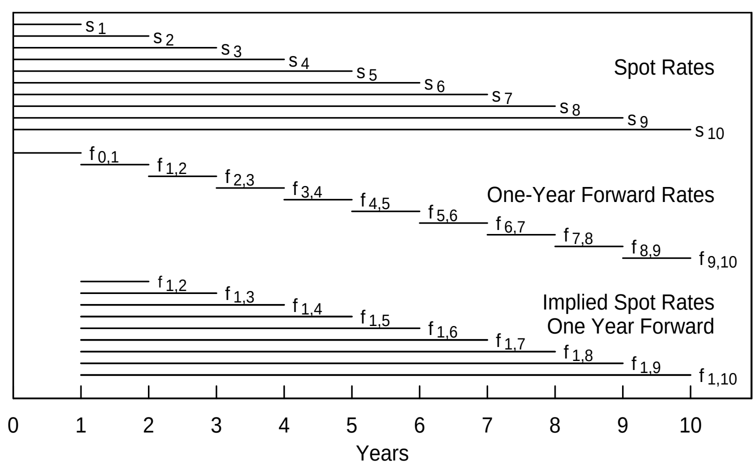
[2] To simplify notation, this report focuses on spot rates rather than on par yields. Moreover, all analysis is presented using one-year forward rates and annual compounding frequency. In practice, semiannual compounding and three-month forward rates are more popular, but the equations would have to include various annualization terms.

[3] The forward path is computed by fixing m = n-1 and letting m vary from 0 to 9. The implied spot curve one year forward is computed by fixing m = 1 and letting n vary from 2 to 10. In a similar way, we can construct the forward path of any constant-maturity rate or the implied spot curve for any future horizon.

Figure 1. Spot Curve, Curve of One-Year Forward Rates and Implied Spot Curve One Year Forward

A Spot Rate Today		B One-Year Forward Rate		C Implied Spot Rate One Year Forward		D = C - A Implied Change in the Spot Rate	
s_1	6.00%	$f_{0,1}$	6.00%	$f_{1,2}$	8.01%	Δf_1	2.01%
s_2	7.00	$f_{1,2}$	8.01	$f_{1,3}$	8.64	Δf_2	1.64
s_3	7.75	$f_{2,3}$	9.27	$f_{1,4}$	9.09	Δf_3	1.34
s_4	8.31	$f_{3,4}$	10.02	$f_{1,5}$	9.43	Δf_4	1.12
s_5	8.73	$f_{4,5}$	10.44	$f_{1,6}$	9.67	Δf_5	0.94
s_6	9.05	$f_{5,6}$	10.65	$f_{1,7}$	9.85	Δf_6	0.80
s_7	9.29	$f_{6,7}$	10.72	$f_{1,8}$	9.97	Δf_7	0.68
s_8	9.47	$f_{7,8}$	10.72	$f_{1,9}$	10.06	Δf_8	0.59
s_9	9.60	$f_{8,9}$	10.67	$f_{1,10}$	10.12	Δf_9	0.52
s_{10}	9.70	$f_{9,10}$	10.60				

Figure 2. Future Years Covered by Each Rate



Forwards as Break-Even Rates

The numbers in column D in Figure 1 — the difference between the implied spot curve one year forward and today's spot curve — show that "the forwards imply rising rates." How should this statement be interpreted? It does not necessarily mean that the market expects rising rates. Instead, the forwards tell how much the spot curve needs to change over the next year to make all bonds earn the same holding-period return. Recall that the holding-period return is a sum of a bond's initial yield and its capital gains or losses caused by yield changes. For example, if today's spot curve is upward sloping, longer-term bonds have a yield advantage over the one-period bond. To equate holding-period returns across bonds, longer bonds have to suffer capital losses that offset their initial yield advantage. Forwards show exactly how much long-term rates have to increase to cause such capital losses. Stated in terms of rate levels instead of rate changes, **the implied spot rates one year forward** (column C in Figure 1) **are such future spot rates that would make all government bonds earn the same holding-period return over the next year.** (Moreover, this same return must be the return of the one-year zero because it is already known today.) This break-even relation follows from Equation (3) by setting m = 1 and rearranging:

$$\frac{(1 + s_n)^n}{(1 + f_{1,n})^{n-1}} = 1 + s_1 \ . \qquad (4)$$

The left-hand side of Equation (4) is the return of buying an n-year zero at rate s_n today and selling it a year later at rate $f_{1,n}$. The right-hand side is the riskless return of the one-year zero. Thus, $f_{1,n}$ is the selling rate at which the n-year zero's holding-period return equals the return of the riskless asset. A numerical example illustrates the computation of the break-even rate $f_{1,2}$. The numbers are from Figure 1: a 6% one-year spot rate and a 7% two-year spot rate. Over the next year, the return of a one-year zero is known to be 6%, while the holding-period return of a two-year zero depends on its selling price at the end of the year — when its remaining maturity is one year. So, the question is: "What should the one-year spot rate be one year hence to make the longer zero's holding-period return 6%?" A little math shows that the answer is 8.01%. At this selling rate, the longer zero's price would rise from 87.34 [= $100/(1.07^2)$] to 92.58 [= 100/1.0801], earning it 6% [= 92.58/87.34 - 1]. Thus, the implied one-year spot rate one year forward, $f_{1,2}$ = 8.01%, is the level of future one-year rate that would make investors *ex post* indifferent to holding either of the two zeros.

It is worth reiterating that the forwards imply distinct yield changes for actual bonds and for constant-maturity rates. **The forwards tell how much *the yield of a given security* — a longer-term bond — needs to change to offset an initial yield spread over the short-term rate. Alternatively, the forwards tell how much *a given point on the spot curve* has to shift to equate holding-period returns across bonds.** In the example above, the two-year zero's yield had to rise by 1.01% and the constant-maturity one-year spot rate had to rise by 2.01%. The 1% difference is due to the so-called "rolling down the yield curve" effect, which we discuss shortly. However, first, we provide a rule-of-thumb relation between initial yield spreads and break-even yield changes — and some economic intuition about this relation. We compute the implied two-year spot rate one year forward ($f_{1,3}$) by equating the holding-period returns of the three-year zero and the one-year zero over the next year. (Recall that the value of *the two-year spot rate* after one year determines the holding-period return of what is today a *three-year zero*.) The left-hand side of Equation (5) is the approximate holding-period return of the three-year zero[4] (given a selling rate $f_{1,3}$), and the right-hand side is the holding-period return of the one-year zero.

$$s_3 - Dur_2 * (f_{1,3} - s_3) \approx s_1 \ . \qquad (5)$$

Note that $f_{1,3} - s_3$ is not the three-year zero's actual subsequent yield change but the break-even yield change implied by the forward rates today. Rearranging Equation (5) gives a rule-of-thumb that **the break-even yield change for the three-year zero equals the yield spread divided by the bond's duration** (at horizon): $f_{1,3} - s_3 \approx (s_3 - s_1)/Dur_2$. The following

[4] The holding-period return of a three-year zero over the next year is equal to its initial yield (s_3) plus the capital gains or losses caused by any yield change. In this example, we use linear approximation of the capital gains (percentage price change equals minus duration times yield change), ignoring convexity effects. For this reason, the forward rate $f_{1,3}$ that we compute will be 1.5 basis points lower than that in Figure 1 (which is computed without any approximation error).

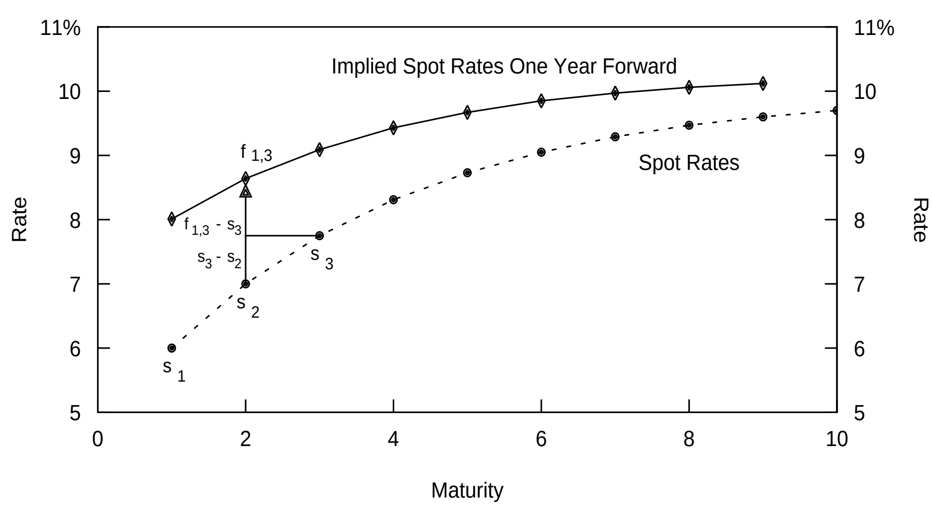
observation may be helpful. A large yield spread means that a purchase of a three-year zero financed by the sale of a one-year zero has a large positive "carry." The profit of this position is the sum of the yield carry (s_3 - s_1) and the capital gains or losses caused by the longer zero's yield changes. The position is bullish — it profits from falling rates and suffers from rising rates — but the positive carry provides a *cushion* against rising rates. The trade will lose money only if the two-year spot rate rises above $f_{1,3}$ in one year.

The break-even yield change $f_{1,3}$ - s_3 shows how much the three-year zero's yield can *rise* before its carry advantage is offset. However, if an upward-sloping yield curve remains unchanged, the zero's yield will *fall* as it rolls down the curve (from s_3 to s_2). The capital gains from this rolldown tendency provide the three-year zero with an additional cushion against rising rates. **The yield advantage component and the rolldown component can be added up to answer the question, "How much should the constant-maturity two-year spot rate change in the next year to make the three-year zero and the one-year zero earn the same return?"** The answer is the implied, or break-even, change in the two-year spot rate over the next year (Δf_2 in column D in Figure 1):

$$\Delta f_2 = (f_{1,3} - s_3) + (s_3 - s_2) = f_{1,3} - s_2 \,. \tag{6}$$

If only the spot rates in Figure 1 are known (s_1 = 6%, s_2 = 7%, s_3 = 7.75%), the yield advantage component is 0.875% [= (7.75 - 6.00)/2] and the rolldown component is 0.75% [= 7.75 - 7.00]. Thus, Δf_2 = 1.625% [= 0.875 + 0.75] and $f_{1,3}$ = 8.625% [= 7.00 + 1.625]. In words, Δf_2 is the difference between the implied two-year spot rate one year forward and the two-year spot rate today. Figure 3 illustrates graphically how this break-even yield change is decomposed into **the break-even yield change needed to offset the carry** ($f_{1,3}$ - s_3) and **the rolldown yield change** (s_3 - s_2).[5]

Figure 3. Spot Curve and Implied Spot Curve One Year Forward



[5] The x-axis in Figure 3 (and in subsequent figures) is denoted as maturity, but it could as well be denoted as Macaulay duration because for zeros, duration equals maturity. (This report does not emphasize the distinction between Macaulay duration and modified duration.)

An interesting relation exists between the curve of one-year forward rates and the implied spot curve one year forward (columns B and C in Figure 1). The steeper the former curve is, the higher the latter curve must be. A steep curve of one-year forward rates reflects a large rolling yield advantage for long-term bonds over the one-year bond. (We show in the Appendix that the one-year forward rate between maturities n-1 and n is equal to the n-year zero's rolling yield, that is, its one-year horizon return if the yield curve does not change. Thus, the one-year forward rate $f_{n-1,n}$ is equal to the spot rate s_n plus the rolldown return.) The larger this rolling yield advantage is, the larger the yield increase required to offset it is, and the higher the implied spot curve one year forward is (see the Appendix for details).

Break-Even Yield Changes for Curve-Flattening Positions

We can extend the above analysis to more complex yield curve positions. If the implied spot rates one year forward are realized, all self-financed positions of government bonds will break even (earn a return of 0%). To break even, any position with a positive carry will have to suffer capital losses; the forwards show how large spread changes over the next period would cause capital losses that offset the positive carry. Conversely, any position with a negative carry will have to earn capital gains to break even; thus, forwards imply spread changes that cause such capital gains. The following example clarifies this point.

Consider an investor who has a strong view, with a one-year horizon, that the spot curve will flatten between two- and four-year maturities. He could implement this curve flattening view by selling a three-year zero and by buying with the sale proceeds equal market values of a five-year zero and a one-year zero (which represents cash at horizon).[6] Such a "barbell-bullet" trade is duration-neutral; thus, the position is not sensitive to parallel changes in interest rates, but it profits from the curve flattening. In a typical yield curve environment, this trade earns a negative carry. That is, when the spot curve is concave (steeper slope in the front end than in the long end), the yield loss of moving from the three-year zero to the one-year zero will be greater than the yield gain of moving from the three-year zero to the five-year zero. For example, in Figure 1 the negative carry is 39 basis points (0.5 * (6.00 - 7.75) + 0.5 * (8.73 - 7.75) = -0.88 + 0.49 = -0.39). **For the trade to make money, capital gains caused by future flattening of the spot curve must offset the negative carry**.

The implied spot curve one year forward indicates the future level of the two- to four-year spread at which the trade — like any bond position with no net investment — exactly breaks even. This is the sense in which the forward rates "imply" flattening of the spot curve. **More curvature in today's spot curve (a lower one-year rate or five-year rate for a given three-year rate) indicates less attractive terms for a flattening trade (a larger negative carry) and more implied flattening by the forwards** (which is needed to offset the negative carry). If today's spot curve were linear and not at all curved, the flattening trade would give up no yield, and, consequently, the forwards would imply no flattening of the spot curve. The break-even change in the two- to four-year spread would be zero. Finally, if today's spot curve were convex, the (barbell-bullet) flattening trade would actually pick up yield, and the forwards would imply steepening of the spot curve.

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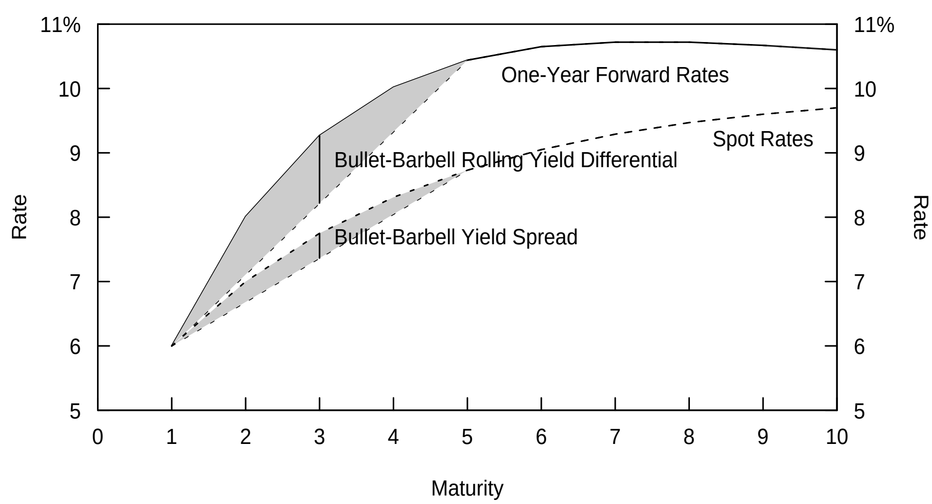
[6] It may seem puzzling that a view about two- to four-year spread is implemented by trading three- and five-year zeros. The reason is that, in this example, the investor has a view about the spread change *in one year's time* and the zeros' maturities shorten by one year over the horizon. If the investor had a view with an immediate horizon, he would implement it by selling twos and buying fours and cash.

Note that the break-even change in the (constant-maturity) two- to four-year yield spread depends not only on the barbell's and the bullet's initial yields, but also on their rolldown tendencies. When the spot curve is concave, a rolldown return advantage augments the bullet's yield advantage. (The three-year zero rolls down a steeper part of the curve than the five-year zero, and the one-year zero earns no rolldown return because it has zero duration at horizon.) Thus, the rolling yield curve — that is, the curve of one-year forward rates — is even more concave than the spot curve. The amount of implied flattening (which is needed to offset the negative yield carry and the difference in rolldown returns) really depends on the curvature of the rolling yield curve.

Figure 4 illustrates a typical spot curve and the corresponding curve of one-year forward rates (columns A and B in Figure 1). To evaluate the degree of curvature in these curves, we draw a straight line between a pair of zeros with different maturities (durations). Each point on these lines represents a barbell portfolio of the two zeros. The curvature is measured by the vertical distance between a barbell portfolio and a duration-matched bullet bond. The curvature of the spot curve reflects the bullet-barbell yield spread, while the curvature of the curve of one-year forward rates reflects the bullet-barbell rolling yield differential. The larger the vertical distance between the barbell and the duration-matched bullet is, the more "expensive" the barbell is in the sense that larger curve flattening is needed for the trade to break even. For example, the initial rolling yield differential of 105 basis points at the three-year point (0.5 * 6.00 + 0.5 * 10.44 - 9.27 = -1.05) will only be offset if the two- to four-year spread narrows by 52 basis points in one year. The break-even spread change can be computed from column D in Figure 1: 112 - 164 = -52 basis points.

Figure 4. Measuring the Degree of Curvature



How Do Rate Expectations Influence Today's Yield Curve Shape?
It is widely agreed that the market's rate expectations have a strong influence on the yield curve shape.[7] It is much more controversial to argue that such expectations are the only determinant of the yield curve shape. However, this is roughly what the **pure expectations hypothesis (PEH)** claims. This hypothesis **assumes that all government bonds, regardless of their maturity, have the same near-term expected return.** The motivation is that the market prices of bonds are set by risk-neutral traders, whose activity eliminates any expected return differentials across bonds.

If all government bonds have the same near-term expected return, any yield differences across bonds must imply expectations of future rate changes (so that expected capital gains or losses offset the impact of initial yield differences). For example, if investors expect rates to rise and long-term bonds to lose value, they require higher initial yields for long-term bonds than for short-term bonds, making today's yield curve upward sloping. This kind of break-even argument is similar to the one used in the previous section, except that now the *expected* (as opposed to realized) returns are being equalized across bonds.

Figure 5 illustrates how different types of expectations influence today's spot curve (if there are no expected return differences across bonds and if the convexity bias is ignored). Expectations of unchanged future rates lead to a horizontal spot curve, rising rate expectations (for example, a parallel shift of 100 basis points over one year) lead to a linearly upward-sloping spot curve and curve-flattening expectations lead to a concave curve shape. A break-even type of argument can motivate each situation:

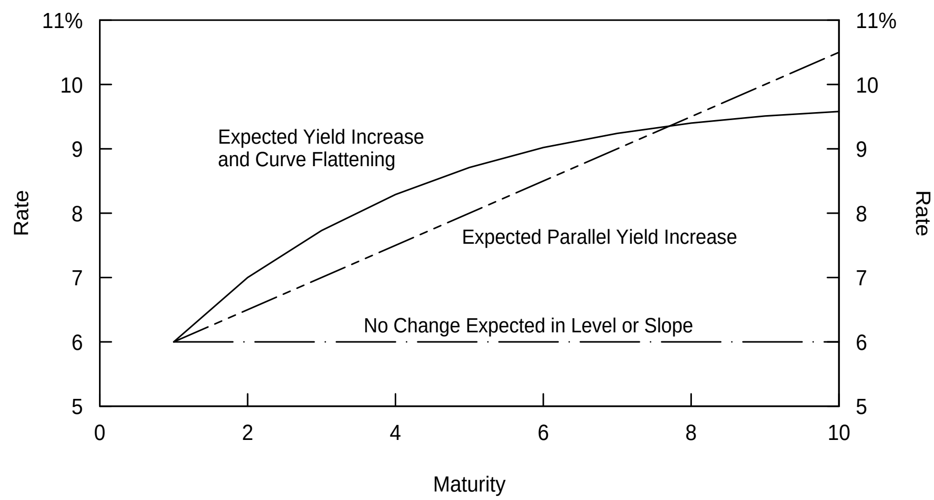
• If the market expects *no rate changes*, today's spot curve is flat because no expected gains or losses need to be offset by an initial yield spread.

• If the market expects rates to *rise in a parallel fashion*, longer-term bonds are expected to earn greater capital losses than shorter-term bonds. An initial yield advantage must offset these expected losses. Because the expected capital losses are proportional to duration, the yield advantage is also proportional to duration. Therefore, today's spot curve is linearly upward sloping. In a similar way, expectations of declining future rates make today's spot curve inverted.

[7] This report takes the market's rate expectations as given, except that this footnote briefly discusses *the economic determinants of these expectations*. An old Wall Street adage — that the central bank determines the level of short-term rates while the market's inflation expectations drive the long-term rates — is an oversimplification, but probably captures the main determinants of rate behavior. (i) Market participants are likely to have clearer expectations regarding the near-term behavior of short rates than about more distant events. These expectations are closely related to the market's view on the economic conditions and on the direction of monetary policy. If the market expects an increase in economic growth rate and in credit demand, it often expects both the real rates and the inflation to increase, and vice versa. Moreover, central banks often try to influence inflation rates and economic growth by raising short-term (nominal and real) rates when fast economic growth and capacity constraints are risking an inflation pickup and decreasing short rates when the economy is in a recession. The bond market appears to incorporate expectations of such countercyclic monetary policy into the term structure. Thus, the market's expectations about short-term rate changes may be reasonably flat when the monetary policy is inactive and declining (rising) in periods when the central bank has begun to actively ease (tighten) monetary policy. Central banks often adjust short rates gradually, in trends; thus, the first easing (tightening) moves may create market expectations of further rate declines (increases). However, expectations also may have a mean-reverting component. If the market perceives certain rate levels to be "normal," it may expect rate changes even if the monetary policy is inactive but short rates are exceptionally high (1981-82) or low (1992-93). (ii) The market's distant rate expectations probably reflect mainly its long-term inflation views, which are influenced by factors such as the country's inflation history, the size of the government debt and budget deficits (government's incentive to "monetize" debt), the strength of anti-inflationary forces (central bank independence, discipline from financial markets, political clout of long-term savers versus debt holders), and the exchange rate policy. The market's expectations about the real rates in the long run probably move quite slowly, based on its perception of the future investment-saving balance.

- If the market expects the *curve to flatten* in the future, barbells and other curve-flattening positions are expected to earn capital gains. An initial negative carry must offset these expected capital gains. Therefore, today's spot curve is concave, and barbell portfolios have lower yields and rolling yields than duration-matched bullet bonds. In a similar way, expectations of future curve steepening tend to make today's spot curve convex, and barbells have higher yields than bullets.

Figure 5. Spot Curves Given the Market's Various Rate Expectations



Expectations versus Risk Premia

We emphasize that the PEH is nothing more than a hypothesis. **Much empirical evidence shows that the extreme assumption of equal expected returns across bonds is false.** Thus, it is unreasonable to assume that an upward-sloping yield curve reflects only expectations of rising rates. It is at least as reasonable to assume that such a shape reflects only the premium that investors require for holding long-term bonds. (We ignore the convexity bias until the next subsection.) In this light, the numbers in Figure 1 (s_1 = 6%, s_2 = 7%, $f_{1,2}$ = 8.01%) can be interpreted in two ways.

- According to the PEH, the one- and two-year zeros have the same expected return over the next year. The return of the shorter zero is known to be 6%. The one-year return of the longer zero will be 6% only if its yield rises to 8.01% (thereby causing capital losses that offset its initial yield advantage). Thus, $f_{1,2}$ reflects the expected level of the one-year spot rate one year hence.

- According to the risk premium hypothesis, $f_{1,2}$ reflects only the longer zero's one-year expected return and no expected rate changes. Recall that one-year forward rates measure the zeros' one-year expected returns given no change in the yield curve. If the spot curve is unchanged in a year, the longer zero earns the initial 7% yield plus a 1.01% rolldown return when its yield declines to 6%.

In both cases, the longer zero will earn the same return over a two-year period (14.49%); this return is known from its 7% annual yield today. However, in the first case the zero is expected to earn 6% in year one and 8.01% in year two, while in the second case, it is expected to earn 8.01% in year one and 6% in year two.

Let us put this example in a broader context. We show in the Appendix (Equation (13)) that, as a linear approximation, the yield change implied by the forwards can be split to an expected change in the n-1 year spot rate over the next year, $E(\Delta s_{n-1})$, and a bond risk premium (that is, the expected return of an n-year bond over the next year in excess of the riskless one-year rate, BRP_n):

$$f_{1,n} - s_{n-1} \approx E(\Delta s_{n-1}) + BRP_n/(n-1) \qquad (7)$$

Equation (7) helps in contrasting different assumptions about the yield curve behavior. For better understanding, one can think of $f_{1,n} - s_{n-1}$ loosely as one measure of the yield curve steepness. Thus, the equation says that the curve steepness reflects market's future rate expectations, or expected return differentials across bonds, or some combination. The two cases above make two polar assumptions:

- The PEH assumes that BRP = 0. Thus, all government bonds have the same near-term expected return as the riskless asset, and forwards reflect only the market's expectations of future rate changes.
- The risk premium hypothesis assumes that $E(\Delta s_{n-1}) = 0$. Thus, forwards reflect only the near-term expected return differentials across bonds.

In reality, of course, neither polar assumption is correct; the truth lies somewhere between.[8] Fortunately, the interpretation of forward rates as break-even rates is valid whether forward rates reflect the market's expectations of future rates, risk premia, or both. We will present some empirical evidence later that indicates that **if one of the two polar assumptions has to be chosen, the risk premium hypothesis is the more realistic one.**

Alternative Versions of the Expectations Hypothesis and the Convexity Bias

The bond risk premium is not the only reason that the forward rates are not equal to expected spot rates. Another reason is the so-called convexity bias. The PEH is often identified with two statements: "All bonds have the same near-term expected return" and "forward rates are optimal (unbiased) forecasts of future spot rates." It turns out that these two statements are not exactly consistent with each other. That is, **two distinct versions of the PEH exist**; the local expectations hypothesis is associated with the first statement and the unbiased expectations hypothesis is associated with the second statement. The difference between these hypotheses is related to convexity, that is, the nonlinearity in a bond's price-yield curve.

We only try to give here some intuition about the convexity bias. Consider a situation in which the spot curve and the implied forward curves are flat at the 6% level. Because there are no yield differences across bonds and no rolldown, will the expected returns be equal across bonds? No, they will not, because some bonds are more convex than others. Positive convexity can only increase expected return *for a given yield*; thus, the longer-duration bonds, which exhibit greater convexity, will have higher expected

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[8] It is sometimes asserted that the PEH must hold because the existence of any near-term expected return differentials across bonds would imply arbitrage opportunities. This statement is only true in a theoretical risk-neutral world (which is often used in derivatives pricing); it is *not true in the real world.* Even though arbitrage arguments determine the levels of forward rates — these must be consistent with spot rates according to Equation (3) — these arguments do not say whether forward rates reflect rate expectations or required bond risk premia. Asness presents a lucid discussion on this complex issue in "OAS Models, Expected Returns, and a Steep Yield Curve" in the *Journal of Portfolio Management,* Summer 1993.

returns. Therefore, even though forward rates equal expected spot rates in this example, all bonds will not have the same expected return. We refer to the impact of convexity on the yield curve shape as the convexity bias.[9]

To conclude, in this section we have described how the market's rate expectations influence the shape of the yield curve, but we also emphasized that expectations are not the only determinants of the curve. The statement "forward rates show the market's expectations of future spot rates" is valid only if the bond risk premia and the convexity bias can be ignored. Ordinarily they cannot. In general, it is difficult to say whether the yield curve's upward slope reflects rising rate expectations or positive bond risk premia. It is equally difficult to say how much of the curvature in the yield curve reflects the market's flattening expectations and how much of it reflects the convexity bias. We will discuss these issues in the next section.

EMPIRICAL EVIDENCE ABOUT RATE EXPECTATIONS AND FORWARD RATES

Do forwards reflect the market's rate expectations, required risk premia or both? Are forward rates or current spot rates better forecasts of future spot rates? Are expected bond returns equal across maturities, as the PEH asserts? In this final section, we address these questions and evaluate empirically what we really know about forward rates and the market's expectations. (We ignore nonlinear effects such as convexity bias and, thus, make no distinction between the different versions of the PEH.)

Because expectations are not observable, academic researchers have studied these questions using two different methods. Many authors examine the forward rates' ability to predict actual subsequent rate changes and required bond risk premia. Others take a more direct approach and use surveys of interest rate forecasts to proxy for the market's rate expectations.

Forwards' Ability to Forecast Future Rate Changes and Risk Premia

We first examine forwards' ability to forecast future spot rate changes and realized future bond risk premia. The underlying idea is that the market's expectations are rational, and any forecast errors are "noise" that should wash out during the sample period. If the PEH holds, forwards are optimal predictors of future spot rates. (By "optimality," we mean here unbiasedness. The forwards do not need to be very accurate predictors, but they should not contain systematic forecast biases.) However, if the risk premium hypothesis holds, forwards are optimal predictors of near-term expected bond returns, and the current spot curve is the optimal forecast for future spot curves. The implications of the two extreme hypotheses are summarized in Figure 6.

9 Because short-term bonds have very small convexity, the convexity bias is only a few basis points at maturities of less than three years. Thus, it is reasonable to ignore the bias, as an approximation, when analyzing only short-term bonds. For longer-term bonds, the rolling yields are clearly downward-biased estimates of expected returns because they ignore convexity's significantly positive impact. This impact is probably the main explanation for the typically concave (humped) shape of the long end of the spot curve. We discuss these topics in more detail in a forthcoming report *Convexity Bias and the Yield Curve.*

Figure 7 reports the correlation of the forward-spot premium ($f_{n-1,n}$ - s_1 or FSP_n; see Equation (16) in the Appendix), first, with the subsequent change in the n-1 year spot rate Δs_{n-1} over the next month and, second, with the subsequent realized bond risk premium (realized BRP_n or the monthly holding-period return of an n-year zero in excess of the one-month rate). For better understanding, one can view FSP_n as another measure of the yield curve steepness. We compute these correlations for six maturities (three- and six-month bills and estimated two-, three-, four-, and five-year zeros) using monthly Treasury market data from the 1970-94 period.

Figure 6. The Implications of Two Hypotheses About the Yield Curve Behavior

	Pure Expectations Hypothesis	Risk Premium Hypothesis
What Is the Information in Forward Rates?	Market's Rate Expectations	Required Risk Premia
What Future Events Should Forward Rates Forecast?	Future Rate Changes	Near-Term Return Differentials Across Bonds
What Is the Best Forecast of an n-Year Zero's One-Period Expected Return?	One-Period Riskless Rate	One-Period Forward Rate ($f_{n-1,n}$), that is, the Zero's Rolling Yield)
What Is the Best Forecast of Next Period's Spot Curve?	Implied Spot Curve One Year Forward	Current Spot Curve
CORR(FSP_n, Δs_{n-1})	Positive	0
CORR(FSP_n, Realized BRP_n)	0	Positive

Figure 7. Evaluating Forward Rates' Ability to Predict Monthly Rate Changes and Risk Premia

	3 Month	6 Month	2 Year	3 Year	4 Year	5 Year
CORR(FSP_n, Δs_{n-1})	0.12	0.05	-0.05	-0.07	-0.11	-0.08
CORR(FSP_n, Realized BRP_n)	0.37	0.22	0.17	0.17	0.19	0.15

Source: Center of Research for Security Prices at the University of Chicago and Salomon Brothers Inc.

The first row in Figure 7 provides the main finding. **The forward-spot premia are negatively correlated with future changes in long-term rates. That is, when the yield curve is upward sloping, long-term rates do not tend to increase, as the PEH says they should, to offset their initial yield advantage over short-term bonds. Instead, long-term rates tend to decline, causing capital gains that *augment* the long-term bonds' yield advantage.**[10] Thus, it is not surprising that the forward-spot premia are positively correlated with future bond risk premia (the second row). In the front end of the curve, forwards tend to at least predict the rate direction correctly. Therefore, the optimal forecast of a future spot rate is a weighted average of the current spot rate and the implied spot rate one period forward. In the long end, forwards appear to be inverse indicators of future rate changes. Overall, Figure 7 suggests that the yield curve steepness tends to reflect more near-term expected return differentials across bonds than the market's rate expectations. These findings are clearly inconsistent with the pure expectations hypothesis, but they may be explained by time-varying bond risk premia.

[10] The finding that forwards predict the "wrong" sign for long-term rate changes is not new; it was noted in many academic studies in the 1980s (see "Literature Guide"). Moreover, Frederick Macaulay, who pioneered the concept of duration, remarked on this pattern already in 1938. We must caution, however, that the estimated negative relation is not very strong. For the long-term bonds, the negative correlation coefficients (the first line in Figure 7) are only about one standard deviation away from zero. The estimated correlation coefficients between FSP_n and realized BRP_n (the second line in Figure 7) are more significant, about two standard deviations above zero.

We can relate our empirical analysis to the so-called persistence factors (PFs) that reflect the expectation that spot rates will remain at their current levels.[11] The two polar assumptions that we presented earlier — the PEH and the risk premium hypothesis — are associated with PFs 0 and 1. A zero value means that today's spot curve is expected to give way to the implied spot curve one period forward. A unit value means that today's spot curve is expected to remain unchanged (that is, to *persist*). Thus, if PF equals zero, forwards are optimal (unbiased) forecasts of future spot rates, and if PF equals one, current spot rates are the optimal forecasts. The PF can be estimated empirically by the slope coefficient in a regression of the annualized realized bond risk premium on the forward-spot premium (see Equation (16) in the Appendix). The value of the PF tells by how much an asset's near-term expected return increases for a given increase in the forward-spot premium. We find that for maturities beyond one year, for which forwards tend to give the wrong signal about future rate changes, the estimated PFs are greater than one (1.4 - 2.3). For maturities less than one year, for which forwards at least predict the rate direction correctly, the estimated PFs are smaller than one (about 0.8). These findings are consistent with the signs of the correlations in Figure 7.

Survey Evidence

A more direct approach is to use surveys of bond market analysts' interest rate forecasts to proxy for the market's rate expectations. The survey forecasts can be compared with forward rates; any difference should reflect a required risk premium.

We use the *Wall Street Journal*'s semiannual survey of Wall Street economists and analysts conducted 27 times between December 1981 and December 1994. In each survey, the participants predict the three-month bill rate and the 30-year bond yield at the end of the next June (December). We compute the expected rate change by subtracting the mid-December (June) rate from the survey median.[12] We then compare these survey forecasts with the yield changes that the forwards imply and with actual subsequent yield changes over each 6.5-month period. The forward yield changes are computed based on the on-the-run Treasury yield curve in mid-December (June). Figure 8 shows the main results separately for the six-month change in the bill rate and the bond rate and their spread.

[11] Persistence factors were discussed in *Global Fixed-Income Investments: The Persistence Effect*, Martin Leibowitz, Lawrence Bader and Stanley Kogelman, Salomon Brothers Inc, February 1994.

[12] There are two main criticisms regarding the use of survey data. First, one can argue that the analyst forecasts are not "representative" of the true market expectations. Second, we do not know exactly *when* the rate forecasts were made; thus, the expected rate change is measured with error if we subtract a wrong base (beginning) yield from the forecast. We use midmonth rates as the base yields because, according to the *Journal*, most survey responses are returned around the 15th. It is not clear that either criticism should bias the forecasts systematically one way or the other and, thus, have any significant impact on the average numbers over time.

Figure 8. Accuracy and Bias of Implied Rate Predictions from Forwards and from a Wall Street Journal Survey

A. Average Size of the Forecast Error (Mean Absolute Deviation)

	Forward	Survey	No Change
Δ 3-Month	1.21	1.03	1.03
Δ 30-Year	0.88	1.00	0.86
Δ Spread (30 Year-3 Month)	0.91	0.75	0.81

B. Correlations Between Rate Changes

	Forward/Survey	Forward/Actual	Survey/Actual
Δ 3-Month	0.43	0.03	0.26
Δ 30-Year	0.23	-0.01	-0.22
Δ Spread (30 Year-3 Month)	0.21	0.33	0.28

C. Average (Expected) Rate Change Over Six Months

	Forward	Survey	Actual
Δ 3-Month	+0.80	-0.26	-0.25
Δ 30-Year	+0.10	-0.14	-0.24
Δ Spread (30 Year-3 Month)	-0.70	+0.12	+0.01

Panel A examines the accuracy of the forwards, the surveys and the no-change predictions by showing the average magnitude of the difference between the predicted yield change and the subsequent yield change. (Note that the no-change prediction is consistent with the risk premium hypothesis and that its forecast error is simply the actual yield change.) Panel A shows that all forecasts are quite poor; average errors are of the same order of magnitude as the average yield changes, roughly 100 basis points in six months. Another observation is that the forwards have somewhat larger forecast errors than the no-change predictions, suggesting that **the current spot curve predicts future spot curves better than the implied forward curve does**. Given the small number of observations, these differences are not statistically significant. A more serious problem regarding the forwards' predictive ability is that long-term rates tend to move away from the forwards, not toward them — recall the evidence in Figure 7. However, the negative correlation is quite small in this sample (-0.01), and the expert forecasters have been even worse predictors of long-rate changes (the correlation is -0.22 in panel B). Panel C is the most interesting part of Figure 8. It reveals a **systematic bias in the forward rates**; during the 13-year period, the forwards typically implied rising short-term and long-term rates and a flattening yield curve, unlike the survey forecasts and the actual rate changes.

Why are the forwards' implied rate predictions higher than survey forecasts and actual rate changes? A possible explanation is that the market requires a positive risk premium for holding the long-term bond. In fact, we can use the survey forecasts of future rates to compute a direct estimate of the average bond risk premium. The difference between the yield change implied by the forwards and the expected future yield change is approximately proportional to the risk premium. The 106 [80 - (-26)]-basis-point difference for the three-month bill translates to an average annualized risk premium of 53 basis points.[13] A similar calculation shows

[13] The computation is based on Equation (13) in the Appendix, adjusted for the six-month holding period. The forwards implied, on average, 106 basis points higher yield increases than the surveys. Ignoring convexity, we can approximate the impact of a 106-basis-point yield increase on bond returns by multiplying 106 with the bond's duration at horizon (for a three-month bill, "n-1" = 0.25). ($BRP_n \approx (\Delta f_{0.25} - \Delta s_{0.25}) * 0.25 = (80 - (-26))*0.25 = 26.5$ basis points.) This 26.5-basis-point average risk premium is the average difference between the expected six-month holding-period return of a nine-month bill and a riskless six-month bill. After annualization, we have an estimated 53-basis-point annual risk premium for a strategy of rolling over nine-month bills every six months.

that the 24 [10 - (-14)]-basis-point difference for the 30-year bond translates to an average annualized risk premium of 480 basis points. These findings suggest that positive bond risk premia exist.

Figure 9 shows **how consistently the forward prediction of the three-month rate exceeds the survey forecast**. Both series move together with the actual three-month bill rate (which is typically between the two series). As explained above, the difference between the two series is proportional to a bond risk premium. Figure 9 shows that this **risk premium is not constant over time**. Its time-variation appears economically sensible in that the premium is exceptionally high around the 1982 recession and after the 1987 stock market crash when a recession was widely expected. The premium was quite low in 1993, despite the yield curve steepness, and it rose substantially in 1994.[14]

Figure 9. Forward- and Survey-Expected Three-Month Treasury Bill Rate Six Months Ahead

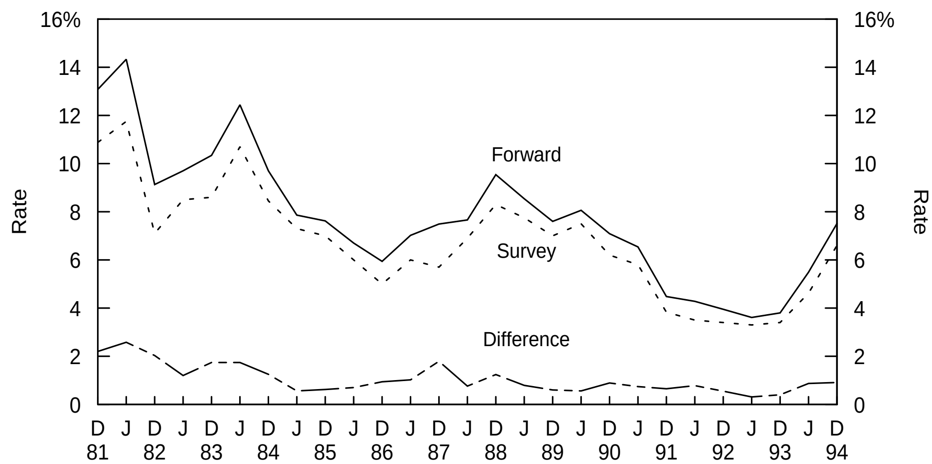
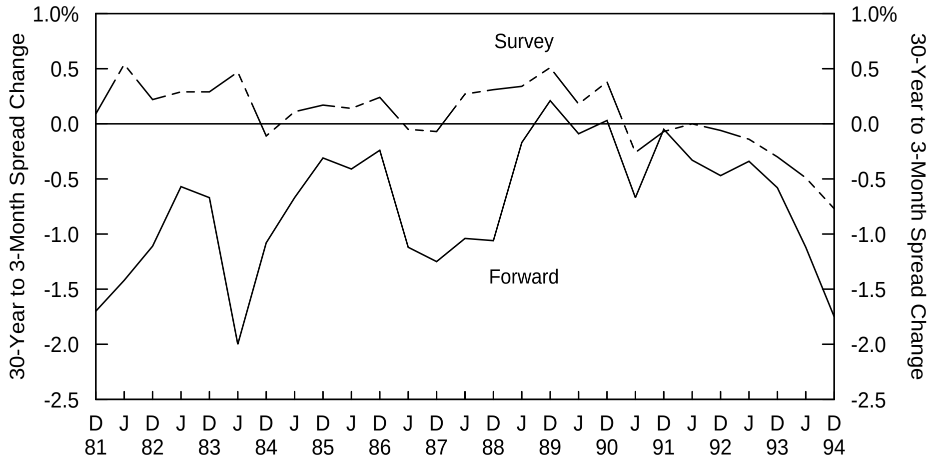


Figure 10 plots the implied forward *changes* in the three-month to 30-year yield *spread* and the corresponding survey forecasts. Forwards have quite consistently implied yield curve flattening, while the survey respondents were, until recently, often predicting curve steepening. We noted earlier that the typical concave yield curve shape causes forwards to imply curve flattening, and we conjectured that most of this implied flattening probably reflects the value of convexity rather than actual flattening expectations. Figure 10 is consistent with this conjecture. The difference between the two series in Figure 10 should, in fact, be a reasonable measure of the market price of convexity.

[14] We discuss the empirical behavior of the bond risk premium further in forthcoming reports *Does Duration Extension Enhance Long-Term Expected Returns?* and *Forecasting U.S. Bond Returns.*

Figure 10. Forward- and Survey-Expected Yield Curve Steepening Six Months Ahead



Investment Implications of the Empirical Findings

Overall, the empirical evidence is much more consistent with the risk premium hypothesis than with the pure expectations hypothesis. Forwards tell us more about near-term expected return differentials across bonds than about future rate changes. If either today's spot curve or the implied spot curve one period forward must be used as a predictor of the next period's spot curve, the evidence supports the use of the former as the neutral base case. These findings have obvious investment implications; **rolling yield differentials between bonds or the corresponding break-even yield changes provide potentially useful relative value indicators for duration-extension trades.** Intuitively, because the rate changes that forwards imply are not realized, on average, bonds with high rolling yields tend to keep their yield and rolldown advantage. Similarly, break-even changes of yield spreads may be good relative value indicators for duration-neutral yield curve positions such as barbells versus bullets. In later reports, we will evaluate the historical performance of these indicators and traditional cheapness indicators such as yield spreads.

More generally, the empirical failure of the pure expectations hypothesis is good news for active managers. If all bond positions always had the same near-term expected returns, it would be extremely difficult to add value. Therefore, our findings provide some **empirical justification for yield-seeking active investment strategies**.

This Appendix shows how forward rates are related to expected spot rate changes and expected holding-period returns and how one-year forward rates are related to implied spot rates one year forward. These relations are *linear approximations* that ignore nonlinear effects such as the convexity bias. Equation (3) shows how the annualized forward rate between maturities m and n is related to m- and n-year spot rates. By taking a first-order approximation of both sides of this equation and rearranging, we get a nice linear relation:

$$f_{m,n} \approx \frac{ns_n - ms_m}{n - m} . \tag{8}$$

To compute the implied spot curve one year forward, we set m = 1:

$$f_{1,n} \approx \frac{ns_n - s_1}{n-1} . \tag{9}$$

Equation (10) shows the n-year zero-coupon bond's holding period return over the next period (h_n). The zero earns its initial yield, s_n, plus a capital gain which is approximated by the product of the zero's duration at horizon (n-1) and its yield change.

$$h_n \approx s_n + (n-1) * (s_n - s^*_{n-1}) , \tag{10}$$

where s^* is the next period's rate (at which the bond is sold). Now we substitute Equation (10) into Equation (9), noting that $ns_n = s_n + (n-1) * s_n$:

$$f_{1,n} \approx \frac{h_n + (n-1) * s^*_{n-1} - s_1}{n-1} = s^*_{n-1} + \frac{h_n - s_1}{n-1} . \tag{11}$$

We call $h_n - s_1$ (the one-year holding-period return in excess of the short-term rate) the realized risk premium of the n-year zero. Because the equality in Equation (11) holds for realized returns, it also should hold in expectations if they are rational. Thus,

$$f_{1,n} \approx E(s^*_{n-1}) + \frac{BRP_n}{n-1} , \tag{12}$$

where BRP_n (the bond risk premium) is the expected holding-period return of an n-year zero in excess of the riskless one-year rate or $E(h_n - s_1)$. Subtracting s_{n-1} from both sides of Equation (12) gives the break-even change in the n-1 year spot rate:

$$\Delta f_{n-1} \equiv f_{1,n} - s_{n-1} \approx E(\Delta s_{n-1}) + \frac{BRP_n}{n-1}\,, \tag{13}$$

where $\Delta s_{n-1} = s^*_{n-1} - s_{n-1}$. (Note that Δs_{n-1} denotes actual rate change over time, whereas Δf_{n-1} is the difference between two rates that are known today.) Equation (13) states that the break-even change in the n-1 year spot rate, implied by the forwards, is equal to the sum of the expected change in the n-1 year spot rate over the next year and the bond risk premium of the n-year zero (divided by n-1). The information in the forward rates reflects either expected future yield changes, or expected bond risk premia, or some combination of the two.

Similarly, we can compute an approximation of the one-year forward rates (m = n-1 in Equation (8)):

$$f_{n-1,n} \approx ns_n - (n-1) * s_{n-1} = s_n + (n-1) * (s_n - s_{n-1})\,. \tag{14}$$

Now we substitute Equation (10) into Equation (14):

$$f_{n-1,n} \approx (n-1) * (\Delta s_{n-1}) + h_n\,. \tag{15}$$

If the yield curve remains unchanged, the first term in the right-hand side of Equation (15) equals zero and $f_{n-1,n} = h_n$. In other words, the one-year forward rate is equal to a zero's holding-period return given an unchanged yield curve, or its rolling yield. Equation (14) shows that such return is equal to a sum of the initial yield and the rolldown return (the zero's duration at horizon (n-1) multiplied by the amount that the zero rolls down the yield curve as it ages).

We can subtract s_1 from both sides of Equation (15) to get the forward-spot premium ($f_{n-1,n} - s_1$ or FSP_n) and realized risk premium ($h_n - s_1$), and then take expectations:

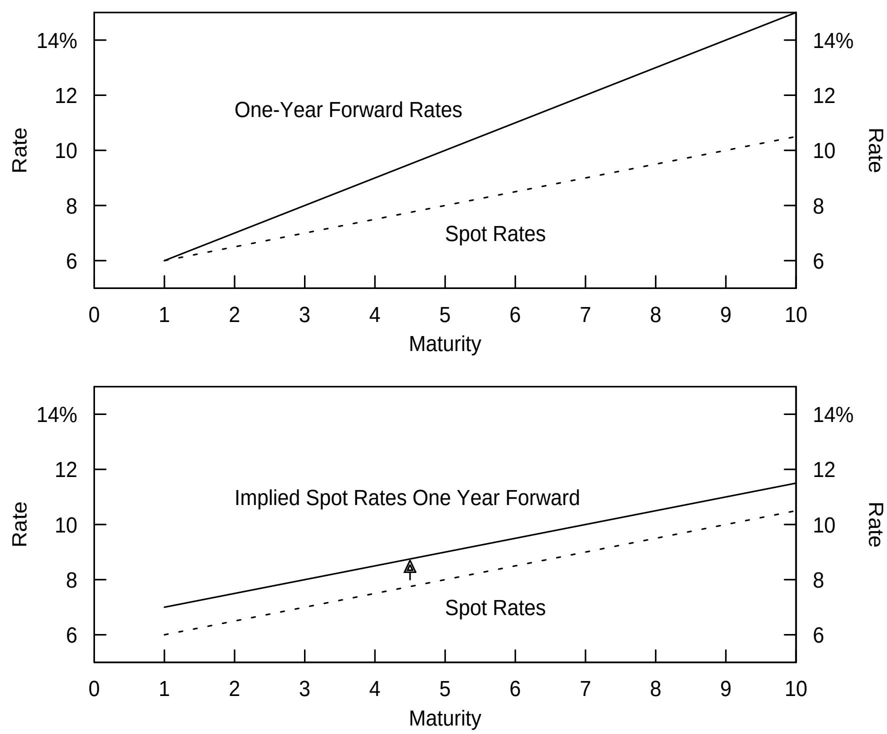
$$FSP_n \equiv f_{n-1,n} - s_1 \approx (n-1) * E(\Delta s_{n-1}) + BRP_n\,. \tag{16}$$

A comparison of Equations (13) and (16) shows that **the forward-spot premium is proportional to the break-even yield change**:[15] $FSP_n = (n-1) * \Delta f_{n-1}$. **Intuitively, FSP_n measures the rolling yield advantage of**

[15] Moreover, both measures are closely related to other measures of yield curve steepness. Equations (14) and (16) show that the forward-spot premium can be written as the sum of the *term spread* (or the yield difference between a long rate and a short rate, s_n-s_1) and the rolldown return (n-1) * (s_n-s_{n-1}). If the spot curve is linear, the forward-spot premium will be exactly twice the term spread. Even if the curve is not linear, the forward-spot premium and the term spread are very highly correlated. Finally, many of the relations developed in this Appendix for zeros also hold approximately for coupon bonds if we substitute their durations for maturities (n), their end-of-horizon durations for n-1, and their yields for spot rates (s_n).

the n-year zero over the riskless one-year zero, while Δf_{n-1} shows how large change in the n-1 year spot rate is needed to offset such a rolling yield advantage. The ratio of proportionality is n-1 because the product of duration-at-horizon and the break-even yield change gives a capital loss that is as large as the rolling yield advantage. Figure 11 illustrates this relation. The top panel shows a spot curve and the corresponding curve of one-year forward rates (rolling yields) which increase linearly with maturity. If the spot curve remains unchanged over the next year, the ten-year zero earns its annual yield 10.50% as well as a rolldown return of 4.50% (nine years end-of-horizon duration times 50 basis points rolldown yield change). Thus, its holding-period return is 15% if the spot curve does not change over the next year — as shown by the right-most point on the curve of one-year forward rates. The ten-year zero has a 9% rolling yield advantage over the one-year zero. The lower panel shows the implied spot curve one year forward (together with the same spot curve as in the top panel); that is, the break-even levels of future spot rates that would offset the longer-term bonds' rolling yield advantage over the one-year zero. The nine-year spot rate needs to increase by 100 basis points to cause a 9% capital loss for today's ten-year zero (whose maturity is nine years after a year). Similar calculations for various-maturity zeros show that a parallel increase of 100 basis points would make all government bonds earn the same 6% holding-period return as the one-year zero. (If convexity effects are taken into account, the break-even yield shift would not be exactly parallel.)

Figure 11. Link Between Zeros' Rolling Yields and Break-Even Spot Rate Changes



The link between initial rolling yields and the break-even changes in the spot rates is straightforward because the relationship is mathematical. If today's spot curve and the rolling yield curve were flatter than in the top panel in Figure 11, a smaller increase in the spot curve would be needed to offset the long-term bonds' rolling yield advantage over the one-year zero, and the forwards would imply smaller rate increases. If today's spot curve were inverted, long-term bonds would even have lower yields and rolling yields than the one-year zero, and the forwards would imply a rate decline to offset the long-term bonds' rolling yield disadvantage. If today's spot curve were upward sloping and concave (the "typical" shape; see Figure 4, which corresponds to the top panel in Figure 11), the forwards would imply rising rates and flattening curve to offset the rolling yield advantage of long-term bonds and steepening positions (bullets versus barbells; see Figure 3, which corresponds to the lower panel in Figure 11).

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Does Duration Extension Enhance Long-Term Expected Returns?

Understanding the Yield Curve: Part 3

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FIGURES

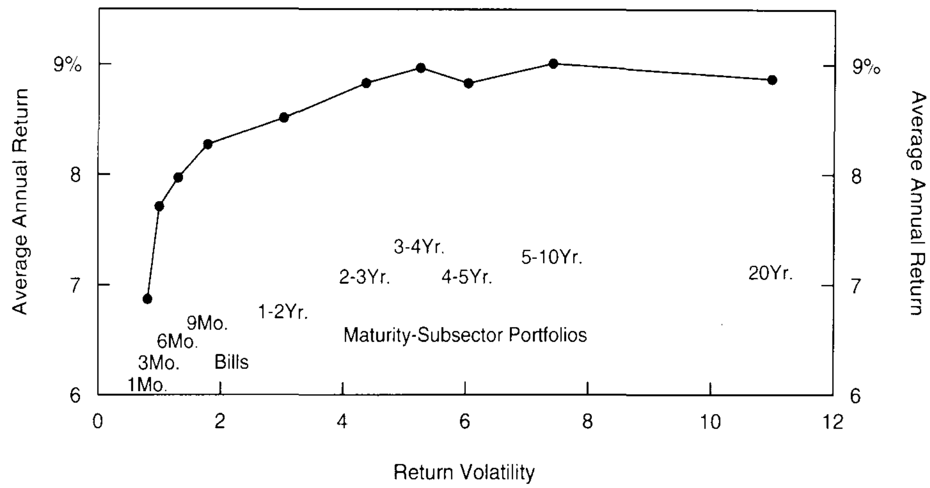
INTRODUCTION

Consider the situation of an investor — such as a central bank, a commercial bank, an insurance company, or a pension fund sponsor — that has **to choose the neutral benchmark duration for its U.S. dollar portfolio. This choice depends on the long-run reward-risk trade-off offered in the U.S. bond market** (as well as on the investor's investment horizon and risk tolerance) and not on any tactical interest rate views. Three directors of the investing institution meet to discuss their combined knowledge about the long-run bond risk premium. One director argues that the typical upward slope of the yield curve is evidence of a positive risk premium. Another director points out that the curve shape might reflect expectations of rising rates instead of a risk premium. It is better to look directly at historical return data, he argues, and presents the others some data that show how average returns over the past decade increased strongly with duration. The third director recalls that over a very long period (1926-94) long-term bonds earned only somewhat higher average returns than one-month bills and lower average returns than intermediate-term bonds. These findings are hard to reconcile until the directors realize that the recent sample reflects findings from a disinflationary period that was exceptionally favorable for long-term bonds. In contrast, the poor returns of long-term bonds in the longer sample partly reflect the yield rise over the decades. What should the directors conclude?

The goal of this paper is to help investors assess whether duration extension is rewarded in the long run. We present extensive empirical evidence mainly from the U.S. Treasury bond market over the past 25 years. All findings about historical returns depend on the interest rate trend in the sample period, but we alleviate concerns about sample-specific results by studying a period without a strong trend. Further, by examining the historical returns over many subperiods, across markets and from several perspectives, we can give as conclusive answers about long-run *expected* returns as possible.

The main conclusion is that **duration extension does increase expected returns at the front end of the curve** — the one-year bill earns about a 150 basis point higher annual return than the one-month bill. The slope of the average return curve flattens gradually, and **for durations longer than two years, no conclusive evidence exists of rising expected returns** (see Figure 1, which we explain in detail further in the report). Subperiod analysis shows that the average return differentials at short durations are quite stable, suggesting that the shortest Treasury bills are quite inefficient investments. In contrast, the relative performance of intermediate-term and long-term bonds varies with the interest rate trend (bull and bear markets).

Figure 1. Return-Risk Trade-Off in the U.S. Treasury Market, 1970-94



This report is the third part in a series titled *Understanding the Yield Curve*, and it focuses on the long-run expected return differentials across bonds with different maturities. **We refer to a long-term bond's expected holding-period return in excess of the short-term riskless rate as the bond risk premium.** (We discuss this terminology at some length in the Appendix.) The bond risk premium is an important determinant of the yield curve shape, but it is not the only determinant. Parts 2 (*Market's Rate Expectations and Forward Rates*) and 5 (*Convexity Bias and the Yield Curve*) in the series describe how the market's rate expectations and convexity bias influence the curve shape. Moreover, the risk premium may not be constant; thus, the long-run average of realized excess bond returns may not be the best forecast of the near-term bond risk premium. Part 4 in the series (*Forecasting U.S. Bond Returns*) discusses the evidence about the time-varying risk premium and its investment implications.

WHAT DO THEORIES TELL US ABOUT THE BOND RISK PREMIUM?

Various theories make very different predictions about the bond risk premium. These theories suggest many possible determinants of the bond risk premium; they tell us something about its likely sign, shape across maturities and constancy over time; but they tell us very little about its likely magnitude.

Our brief survey discusses six alternative theories. We begin with three classic term structure hypotheses. (i) The pure expectations hypothesis implies that no bond risk premium exists. That is, the influence of risk-neutral arbitrageurs drives all government bonds' expected returns to equal the short-term riskless rate. (ii) According to the liquidity (or risk) premium hypothesis, long-term bonds earn a positive risk premium as a compensation for their return volatility. Underlying this hypothesis is the idea that most investors dislike short-term fluctuations in returns.[1]

[1] In other words, they are risk averse and have a short investment horizon. An alternative and more subtle argument states that most investors have a vague investment horizon. If the horizon is so uncertain that it does not guide an investor's decision making and if he is more averse to price risk than to reinvestment risk, he is likely to bias the portfolio toward a short duration. Public accountability makes many investors more averse to price risk than to reinvestment risk. Erring toward a too-short duration exposes an investor "only" to reinvestment risk, which is akin to an opportunity cost. Erring toward a too-long duration exposes an investor to price risk, which is visible and, if realized, is more likely to cause a public outcry.

(iii) The preferred habitat hypothesis states that expected returns may increase or decrease with duration. Many pension funds and life insurance companies view the long-term bond as less risky than the short-term asset because it better matches the average duration of their liabilities. These investors, which we refer to as long-horizon investors, would accept a lower yield for the long-term bond than for the short-term asset. Even if horizons and subjective risk preferences vary across investors, each asset has only *one* market price. For this reason, the risk premium offered by the market will depend on "the market's investment horizon" and, therefore, on the relative importance of short-horizon and long-horizon investors. Casual empiricism suggests that the long-horizon investors still represent a minority: thus, the risk premium should increase with duration.[2] However, the risk premium offered by the market may be lower than that required by the short-horizon investors.

Modern asset pricing theories relate risk premium to amount of risk and price of risk rather than to investment horizons and the relative importance of different investor groups. (iv) In many one-factor term structure models, a bond's risk premium is proportional to its return volatility. In partial equilibrium models, bonds are viewed in isolation and volatility is the relevant risk measure. These models ignore the correlations between bond returns and other assets or other economic variables. (v) In the Capital Asset Pricing Model, any asset's risk depends on its sensitivity to the aggregate wealth. This is often measured by an asset's stock market sensitivity (that is, its beta or the asset's relative volatility multiplied with its correlation with the stock market). An asset's risk premium is the product of its beta and the market risk premium, which in turn depends on the market volatility and the market's risk aversion level. Given that long-term bond returns tend to be positively correlated with the stock market return, their betas (and the bond risk premia) are positive. In fact, bonds' estimated return volatilities and betas are approximately proportional to their durations. Thus, **many theories imply that the bond risk premium should increase linearly with duration.**[3]

The most complex theories allow risks and rewards to be time-varying instead of constant, and they allow multiple factors that reflect fundamental economic risks. (vi) The intuition behind all general equilibrium models is that assets that perform poorly in "bad times"[4] should earn a positive risk premium. In contrast, assets which perform well in "bad times" are accepted for very low yields. To the extent that long-term bonds are a good hedge against recessions, they might even earn a negative risk premium. This may have been the case during the Great Depression of the 1930s, but it certainly has not been the case in the post-World War II period. Bonds performed extremely poorly during the inflationary recessions of 1973-75 and 1980-82. Thus, the spirit of the general equilibrium models suggests that long-term bonds should earn a positive risk premium.

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[2] In a sense, the long-horizon investors are fortunate to be in the minority among market participants; they earn a positive risk premium even though they might accept a lower yield for long-term bonds. Andre Perold and William Sharpe show that an investment strategy's long-run profitability is inversely related to its popularity in the marketplace: see "Dynamic Strategies for Asset Allocation," *Financial Analysts Journal*, January-February 1989.

[3] However, these models specify a linear relation between expected returns and return volatility (or beta). A linear relation between expected returns and duration only follows if yields are equally volatile across the curve (because a bond's return volatility is approximately equal to its duration times the volatility of the yield changes). Empirically, however, the short-term rates tend to be more volatile than the long-term rates, making the return volatility increase by less than one-for-one with duration. Because return volatilities are somewhat concave as a function of duration, also expected returns (and bond risk premia) should be somewhat concave as a function of duration.

[4] In these models, bad times are associated with a high marginal utility of a dollar. Intuitively, a dollar is more valuable when you are hungry and poor. For the economy as a whole, periods of high marginal utility ("bad times") may coincide with recessions.

Many bond market participants feel that the expected return differentials across bonds mostly reflect bonds' characteristics that are not related to the risk characteristics on which the modern theories focus. For example, less liquid bonds earn higher expected returns, as evidenced by the positive yield spreads between duration-matched short-term Treasury coupon bonds and Treasury bills, between off-the-run and on-the-run bonds, and between the illiquid 20-year sector and the liquid 10- and 30-year sectors.[5] Unpopular assets, such as recent poor performers, may earn higher returns because holding them exposes portfolio managers to a "career risk." Temporary supply and demand imbalances also can cause expected return differentials across the curve sectors. In general, most asset-pricing theories ignore such technical factors, institutional constraints and any supply effects. In this report, **the term "risk premium" encompasses all expected return differentials across bonds,** whether risk-related or other factors cause them.

To summarize this survey, many theories suggest that the long-term bonds are riskier than short-term bonds and that investors can earn a *positive* risk premium for bearing this risk. Some models specify that expected returns are *linear* in duration or in return volatility. According to the various models, several factors can influence the slope of the expected return curve. For example, the slope may increase with bond market volatility, stock-bond correlation, the market's risk aversion level, the relative wealth of short-horizon investors (versus long-horizon investors), and the relative supply of government bonds across the curve. In the rest of this report, we examine empirically whether (and by how much) expected returns increase with duration and whether this relation, if it exists, is linear. It is more difficult to explain which factors cause the documented expected return differentials.

EMPIRICAL EVIDENCE ABOUT THE BOND RISK PREMIUM IN THE TREASURY MARKET

Estimating the Risk Premium from Historical Yield or Return Data

The (expected) bond risk premium is not directly observable. However, one can use historical yield or return data to estimate the average risk premium. We will use both approaches, but first we discuss their underlying assumptions and the pitfalls in their use. We also discuss these topics and the terminology in the Appendix.

Average yield curve shapes may help us estimate the average bond risk premium. The term spreads (that is, yield differentials between long-term bonds and short-term bonds) contain information about required bond risk premia, but they also reflect the market's expectations of future rate changes. It is notoriously difficult to disentangle these components. Conceptually, they can be isolated in the extreme versions of the pure expectations hypothesis and the liquidity premium hypothesis. According to the pure expectations hypothesis, an upward-sloping yield curve only reflects expectations of future rate changes; there are no risk premia.[6] The

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[5] However, more liquid bonds have two advantages over less liquid bonds that may offset their lower yield and expected cheapening (as they lose their liquidity premium). First, liquid bonds are more often "special" in the repo market; thus, they offer a financing advantage. Second, their smaller bid-ask spread can be viewed as an option to trade at small transaction costs.

[6] The expected rise in a long-term bond's yield will cause a capital loss that exactly offsets the bond's initial yield advantage over the short-term bond. The capital loss equals the product of the bond's expected yield rise and its duration — if we ignore the convexity bias (which we discuss in other parts of this series).

liquidity premium hypothesis makes the opposite claim: An upward-sloping yield curve reflects only required risk premia and no rate expectations. In reality, the shape of the yield curve probably reflects both rate expectations and risk premia.

The average term spread may be a good measure of the long-run average bond risk premium if the expected yield changes average to zero in the sample period. This requirement is often violated in short sample periods. For example, if the market has persistently expected rising rates during the sample, the average yield curve shape exaggerates the risk premium.

It is more direct to study return data. Historical average return differences are often used to estimate the expected risk premium. Even this approach contains implicit assumptions. By definition, any realized return can be split into an expected part and an unexpected part. Similarly, realized excess return can be split into the bond risk premium and the unexpected excess return. For a given day's or month's realized return of a risky asset, the unexpected part dominates. Yet, when many observations are averaged over time, the positive and negative unexpected parts begin to offset each other. Thus, a long-run average reflects the expected part more than the unexpected part. However, the historical average of realized excess returns is a good measure of the long-run expected risk premium only if the unexpected parts exactly wash out.[7] This is more likely to happen if the sample period is long and does not contain an excessively bearish or bullish bias (yield trend).

In other words, this approach is valid if the market's yield forecasts are correct, on average, during the sample period, so that the average unexpected yield changes are zero. The disinflation of recent years has surprised the bond markets positively, causing a realized risk premium that exaggerates the expected premium. (Many firms' databases begin in the early 1980s, near the peak yield levels, which may have given bond market participants a too optimistic view about expected bond returns.) Much longer sample periods suffer from the opposite problem, because of the persistent inflation surprises since the 1950s, which have caused capital losses to bondholders. It is not reasonable to assume that the market correctly anticipated the increase in long-term rates from the 3% levels in the 1950s.

This discussion illustrates how empirical evidence about historical average returns can vary dramatically across samples even when long sample periods are used. Period specificity is a problem that sophisticated econometric techniques cannot overcome. In this report, we focus on a neutral sample period, chosen so that the beginning and ending yield levels are not far apart. (Of course, it is possible that the expected and unexpected rate changes are large but offsetting, even when the realized rate changes average to zero.)

[7] Even if the historical average risk premium is the optimal predictor of the long-run future risk premium, it is not the optimal predictor of the *near-term* risk premium unless the risk premium is *constant* over time. However, many recent studies show that the bond risk premium varies over time. At the end of this report, we present simple evidence that illustrates the time-variation in the risk premium.

Data Description

We analyze average yields and returns of strategies that concentrate portfolio holdings in a certain maturity sector of the U.S. Treasury market. We also offer some additional evidence from other U.S. bond market sectors and from international government bond markets.

The main analysis covers the past quarter century (1970-94). We chose this period for three reasons:

• **Length.** 300 monthly observations reduce the problem of period-specific findings.

• **Relevance.** Lengthening the sample period makes sense only if the world has not changed so much that old data are irrelevant. This quarter century has been a period of fiat money (that is, money backed only by the government's promise), floating exchange rates, volatile inflation, and large budget deficits. However, some may argue that bond markets have changed so dramatically with globalization, deregulation, securitization, and technological change, that the 1970s data are not relevant. If we eliminated the 1970s data, we would be left with a biased sample that covers only the disinflationary 1980s and 1990s.

• **Neutrality.** Net yield changes (declines in the short-term rates and increases in the long-term rates) were small between January 1, 1970, and December 31, 1994. Thus, a sample-specific yield trend does not excessively influence the historical average returns during this period.[8]

Because this report studies the behavior of bond markets over a longer period, we need to analyze portfolios whose characteristics do not change too much over time, such as constant-maturity or maturity-subsector portfolios. Therefore, we use yield and return series whose underlying assets are rebalanced monthly. The ten yield series include one-month, three-month, six-month, nine-month, and 12-month Treasury bill series constructed by the Center of Research for Security Prices (CRSP) at the University of Chicago[9], Salomon Brothers's "on-the-run" two-year, three-year, five-year, and ten-year Treasury bond series, and Ibbotson Associates's 20-year Treasury bond index. The ten return series include four Treasury bill portfolios (one-month, three-month, six-month, and nine-month) and five maturity-subsector Treasury bond portfolios (one to two years, two to three years, three to four years, four to five years, and five to ten years) from CRSP, and the 20-year Treasury bond index from Ibbotson Associates. The 20-year bond is the longest that we study because 30-year bonds were not issued regularly before 1977.

Evidence From the Treasury Yield Curve Shapes

Figure 2 displays the path of the short-term rate and the long-term rate during the sample period. The time series have a distinct inverse "V" shape. In the first half, both rates increased dramatically; in the second half, they declined equally dramatically. In the first half, the yield curve frequently was inverted; through most of the second half, the curve was steeply upward sloping.

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[8] The earliest reasonable starting year would be 1952 in order to exclude a period of regulated long-term rates from the sample. The 1952-94 sample period would be somewhat less relevant and, because of the rising rates, much less neutral than the 1970-94 period.

[9] The CRSP series have been updated with Salomon Brothers data for 1994.

Figure 2. Yield Levels, 1970-94

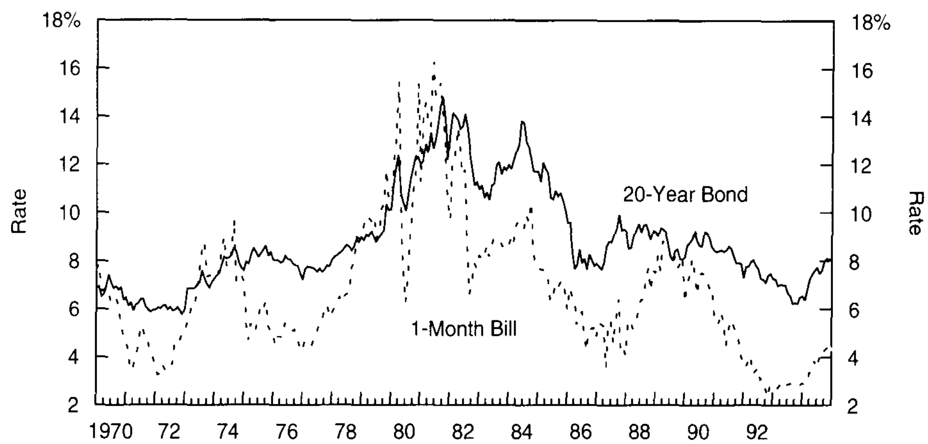


Figure 3 reports average yields (semiannually compounded) and yield spreads over the shortest rate, as well as the annualized standard deviations of monthly yield changes. The main conclusions are as follows:

- Average yields are increasing across the curve. An upward-sloping curve shape probably reflects a positive bond risk premium, but perhaps also rising rate expectations. Such expectations may have been rational even if they were not realized, given the inflation fears in a world of fiat money and large budget deficits.
- The curve is concave in maturity (as well as in duration), that is, yields increase at a decreasing rate as a function of maturity. Potential explanations for this shape include the demand for long-term bonds from the long-horizon investors and the convexity advantage of long-term bonds.
- The term structure of yield volatilities is inverted, likely reflecting mean reversion in short rates.[10] This observation implies that return volatility does not increase quite one-for-one with duration. For this reason, we present the risk-reward trade-off in Figure 1 by plotting average bond returns on return volatilities, not on durations.

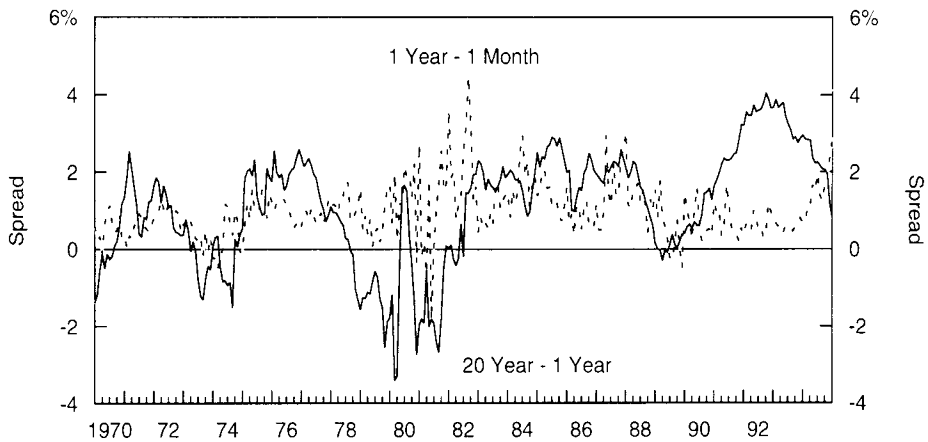
Figure 3. Treasury Instrument Yields, 1970-94

	Bills					On-the-Runs				Ibbotson
	1 Mo.	3 Mo.	6 Mo.	9 Mo.	12 Mo.	2 Year	3 Year	5 Year	10 Year	20 Year
Average Yield	6.75%	7.21%	7.56%	7.66%	7.73%	8.04%	8.18%	8.44%	8.63%	8.85%
Volatility of Yield Changes	3.04	2.40	2.42	2.41	2.37	2.07	1.87	1.71	1.41	1.31
Term Spread Over One-Month Rate	0.00	0.46	0.81	0.92	0.98	1.30	1.44	1.70	1.89	2.11

[10] It is widely known that interest rate volatility was exceptionally high between 1979 and 1982, when the Federal Reserve did not target the short-term rate behavior. Over the past decade, volatilities have been lower and the term structure of volatility has been flatter than in Figure 3. For the 1985-94 period, the volatilities of all maturity rates between three months and 20 years are 1.1%-1.3% (110-130 basis points), peaking at intermediate maturities.

Figure 4 displays the term spreads at the short end and at the long end of the curve. The shorter spread has been much more consistently positive. This may be an indication of the persistence of a positive bond risk premium at short maturities. We will next examine return data to study this issue in more detail.

Figure 4. Yield Spreads, 1970-94



Evidence From Government Bond Returns

As explained before, historical bond returns offer more direct evidence about the bond risk premium than historical bond yields do. Figure 5 shows the annual arithmetic and geometric means (averages) and other statistics for the ten return series described above. Most of the analysis in this report focuses on geometric mean returns rather than on the arithmetic means. The geometric mean reflects the multiperiod compound return that various strategies would have accumulated over the sample. The arithmetic mean exaggerates the historical performance, but it may be a better measure of expected return.[11]

The arithmetic mean return curve increases almost monotonically, while the geometric mean return curve is quite flat after two years. There appears to be a positive bond risk premium, but mainly at the front end of the curve: roughly 150 basis points between one-month and one-year durations and an additional 50 basis points between one- and two-year durations. Beyond two years, it is unclear whether duration extension increases expected returns at all.[12] The pattern of Sharpe ratios confirms that the reward to volatility decreases with maturity.[13]

[11] The arithmetic mean (AM) and geometric mean (GM) are computed using the following equations:

$$AM = (h_1+h_2 + ... + h_N)/N$$

$$GM = [(1 + h_1) * (1 + h_2) * ... * (1 + h_N)]^{1/N} - 1.$$

where h are one-period holding-period returns and N is the sample size. The geometric mean is less than or equal to the arithmetic mean, and the difference increases with the return volatility. The geometric mean is the correct number to use in historical analysis. It is harder to say which number is relevant when describing the future prospects of a given strategy. The arithmetic mean is the mathematically correct measure of expected return, while the geometric mean better represents a typical outcome (median). For further discussion, see "What Practitioners Need to Know about Future Value," Kritzman, *Financial Analysts Journal*, May-June 1994.

[12] In the earlier academic analysis of the average bond risk premium, long-term bonds perform even more poorly. Fama (1984) finds that over the 1953-82 period, average returns peak at the 12- to 18-month maturity. Fama's sample period was, however, clearly inflationary and thus "bearish"; as explained above, the 1970-94 period is more neutral.

[13] Incidentally, the t-statistics of the excess returns are five times larger than the Sharpe ratios (given a sample of 300 months); thus, most bonds have statistically significant positive excess returns.

Figure 5. Treasury Maturity Subsector Annual Returns and Other Statistics, 1970-94

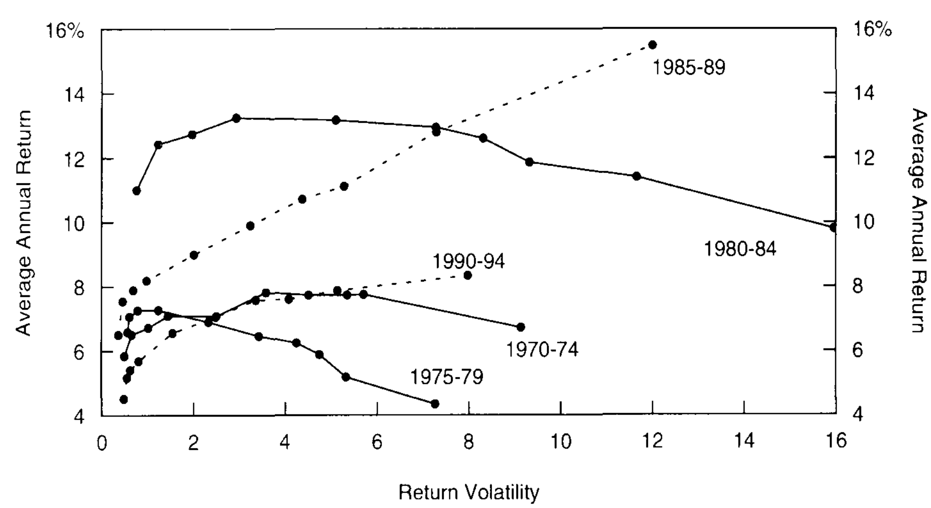
	1 Mo.	3 Mo.	6 Mo.	9 Mo.	1-2 Yr.	2-3 Yr.	3-4 Yr.	4-5 Yr.	5-10 Yr.	20 Yr.
Arithmetic Mean	6.87%	7.71%	7.98%	8.28%	8.56%	8.91%	9.10%	9.01%	9.28%	9.51%
Geometric Mean	6.87	7.71	7.97	8.27	8.52	8.81	8.95	8.82	8.98	8.87
Geom. Premium	0.00	0.84	1.11	1.40	1.65	1.94	2.09	1.95	2.12	2.00
Volatility	0.81	1.00	1.31	1.79	3.02	4.36	5.26	6.03	7.43	10.98
Avg. Duration	0.08	0.24	0.48	0.71	1.3	2.1	2.9	3.7	5.1	9.8
Sharpe Ratio	NA	1.92	1.10	0.87	0.55	0.45	0.40	0.34	0.31	0.22

NA Not applicable. Note: The Geom. premium is the annualized geometric mean return of a bond portfolio in excess of the one-month rate. Volatility is the annualized standard deviation of a bond portfolio's monthly returns. The Sharpe ratio is the annualized mean-to-volatility ratio of a bond portfolio's excess return.

Figure 1 shows the ex-post risk-reward trade-off in the bond market (based on data from Figure 5) by plotting the geometric mean returns on their return volatilities. Recall that many theories predict that expected returns increase linearly with return volatility or with duration. The pattern in Figure 1 contradicts these predictions; average returns are concave in return volatility. The explanation that many market participants would offer is related to the old preferred habitat hypothesis. **The expected returns of the long-term bonds are "pulled down" by the demand from long-horizon investors,** such as pension funds, which perceive the long-term bond as the least risky asset because it best matches the average duration of their liabilities. However, these long-horizon investors are a minority in the marketplace; thus, they do not pull the expected return of the long-term bonds quite as low as that of the short-term bonds.

Even if the sample period is well chosen, the findings are still period-specific unless the expected bond risk premium is very stable. We try to alleviate the problem of period specificity by conducting extensive subperiod analysis to search for patterns that hold across periods. Figure 6 shows separate reward-risk curves (similar to Figure 1) for five five-year subperiods. The bond markets were bearish or neutral in the first three subperiods and bullish (trend declines in long yields) in the last two

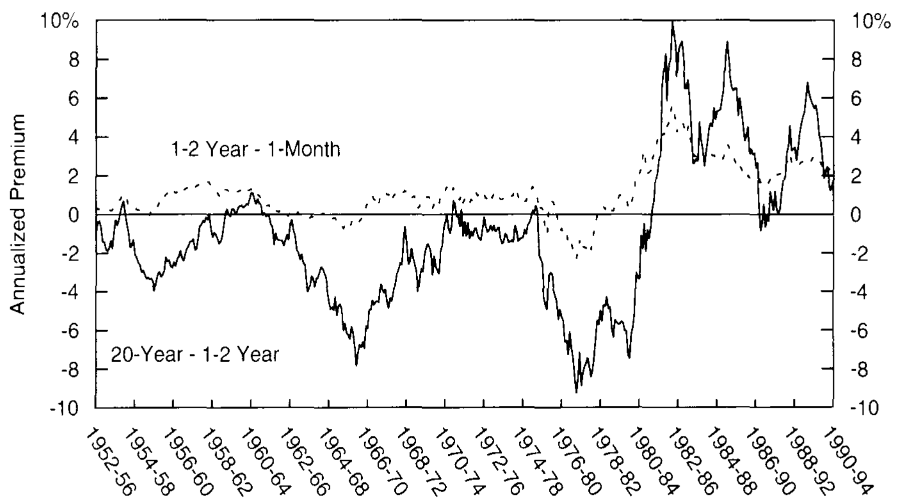
Figure 6. Return-Risk Trade-Off in the U.S. Treasury Market in Five Subperiods Between 1970 and 1994



subperiods. One striking pattern in Figure 6 is that **average returns increase monotonically from the one-month bill to the nine-month bill in all five subperiods**. This pattern provides further evidence regarding a persistent positive risk premium at the front end of the yield curve.[14]

We further study the stability of the bond risk premium over time by plotting in Figure 7 a moving average of the past 60 months' excess bond returns at the front end of the curve (one- to two-year bonds minus one-month bill) and at the long end of the curve (20-year bond minus one- to two-year bonds). We include in this figure the rolling premium already from the 1950s and 1960s to illustrate how bearish the bond market environment was before our main sample period. Again, **the premium at the front end is almost always positive.** In contrast, the premium at the long end is very often negative. In fact, the performance of the 20-year bond is surprisingly consistently bad until the mid-1980s. Only very recent samples support the claim that long-term bonds offer higher returns than intermediate-term bonds. These findings reflect the powerful impact that the slow and systematic changes in inflation rates have had on long-term bond returns.

Figure 7. Rolling 60-Month Return Premium, 1957-94



We turn to one more way to study the bond risk premium. We estimate the probability of earning a positive bond risk premium in a short period. We also evaluate the marginal benefit from duration extension by estimating the probability of earning a higher holding-period return than the previous-maturity asset. The intuition behind this analysis is the following. If bond returns are symmetrically distributed and no risk premium exists, the outcome of a duration extension is like a coin toss. There is a 50% probability of gain and a 50% probability of loss. If a positive risk premium exists, long-term bonds will outperform short-term bonds more frequently than half of the time.[15]

[14] It is also worth noting that return volatility peaked in the early 1980s even though bond durations were at their lowest (because of high yield levels). Thus, the increased yield volatility more than offset the risk-reducing impact of higher yields on bond durations.

[15] An alternative explanation is that returns are not symmetrically distributed. Even if long-term bonds outperform short-term bonds 60% of the time, it is conceivable that the negative returns of long-term bonds are rare but severe, leading to the same average returns as for the short-term bond.

The first panel of Figure 8 shows that the yield curve has been upward sloping in the bill market about 95% of the sample and somewhat less frequently at longer maturities. The second and third panels show how frequently each asset outperforms the previous-maturity asset and the one-month bill at monthly and annual horizons. Our comments focus on the third panel, because many investors are concerned about the performance of different strategies at an annual horizon. Again we see that there is a consistent positive risk premium in the bill market. For example, a strategy of rolling over three-month bills outperforms a strategy of rolling over one-month bills 99% of the time, and a strategy of rolling over six-month bills outperforms a strategy of rolling over three-month bills 67% of the time. **At the longer end, the reward for a marginal duration increase approaches a coin toss.** However, the four- to five-year maturity sector is the only area in which a marginal duration increase makes underperformance more likely.

Figure 8. Frequency of Upward-Sloping Yield Curve or Return Curve, 1970-94

Frequency of an Asset's *Monthly Yield* Exceeding the Monthly Yield of	1 Mo.	3 Mo.	6 Mo.	9 Mo.	12 Mo.	2 Yr.	3 Yr.	5 Yr.	10 Yr.	20 Yr.
Previous Maturity	NA	0.94	0.96	0.78	0.74	0.81	0.79	0.75	0.72	0.78
One-Month Bill	NA	0.94	0.98	0.97	0.95	0.90	0.88	0.88	0.84	0.85

Frequency of an Asset's *Monthly Return* Exceeding the Monthly Return of	1 Mo.	3 Mo.	6 Mo.	9 Mo.	1-2 Yr.	2-3 Yr.	3-4 Yr.	4-5 Yr.	5-10 Yr.	20 Yr.
Previous Maturity	NA	0.81	0.58	0.57	0.54	0.52	0.53	0.47	0.48	0.48
One-Month Bill	NA	0.81	0.68	0.66	0.58	0.56	0.55	0.56	0.56	0.51

Frequency of an Asset's *Annual Return* Exceeding the Annual Return of	1 Mo.	3 Mo.	6 Mo.	9 Mo.	1-2 Yr.	2-3 Yr.	3-4 Yr.	4-5 Yr.	5-10 Yr.	20 Yr.
Previous Maturity	NA	0.99	0.67	0.67	0.56	0.55	0.53	0.44	0.53	0.51
One-Month Bill	NA	0.99	0.88	0.82	0.69	0.64	0.61	0.59	0.57	0.56

NA Not applicable.

EVIDENCE FROM OTHER MARKETS

In this section, we examine whether the yield and return patterns documented above are specific to the U.S. Treasury markets. We extend our historical analysis to the U.S. non-government debt markets and to the government debt markets outside the United States. All yields in this section are expressed in the semiannual compounding frequency, and all returns are geometric averages. Figure 9 shows the average yields for various money market instruments. The last column shows that all private-sector yield curves are much flatter than the Treasury bill curve. In fact, the average return curves would be even steeper for Treasuries because they tend to roll down the steeper bill curve and earn larger rolldown returns in addition to their yields.

Figure 9. Average Yield Curve Steepness in Public- and Private-Issuer Money Markets, 1970-94

	1 Mo.	3 Mo.	6 Mo.	Spread (6 Mo.-1 Mo.)
Treasury Bill	6.75%	7.21%	7.56%	0.81%
Certificate of Deposit	7.68	7.81	7.97	0.29
Commercial Paper	7.87	8.01	8.14	0.27
Eurodeposit	8.23	8.39	8.57	0.34

From another perspective, Figure 9 shows that the credit spreads are wider at a one-month maturity than at a six-month maturity. Fama (1986) already has noted such inversion of the term structure of money market credit spreads. This shape can be contrasted with the typical upward-sloping credit spread curve in the corporate bond market beyond one year [see Litterman and Iben (1991) and Iwanowski and Chandra (1995)]. Only one spread is available at shorter and longer maturities than one year: Treasuries versus Eurodeposits. Figure 10 confirms that, between 1985 and 1994, the term structure of this spread typically had a "V" shape. One investment implication is that it often makes sense to take a large share of the desired credit exposure at short maturities.

The wide spread between one-month bill and other assets is difficult to explain as a rational credit spread. More likely, it **reflects some investors' return-insensitive demand for the ultimate safe asset.** The narrowing of the Eurodeposit-Treasury bill spread in recent years may indicate that such demand for safety "at any cost" is shrinking. (The spread at one-month maturity averaged more than 160 basis points both in the 1970s and in the 1980s, but only 73 basis points in the 1990s.)

Figure 10. Average Yields in Treasury and Eurodeposit Curves, 1985-94

	1 Mo.	3 Mo.	6 Mo.	9 Mo.	12 Mo.	2 Yr.	3 Yr.	5 Yr.
Treasury	5.43%	5.94%	6.17%	6.31%	6.40%	6.96%	7.20%	7.62%
Eurodeposit	6.49	6.58	6.68	6.81	6.95	7.46	7.84	8.30
Credit Spread	1.06	0.64	0.51	0.50	0.54	0.51	0.64	0.68

Figures 11 and 12 offer further evidence of the risk premium from other bond markets. We compare yields and returns in the one- to three-year maturity subsector and the seven- to ten-year maturity subsector of each market. Data availability restricts the analysis to the past decade. Figure 11 shows that the reward for duration extension in the corporate bond market is somewhat lower than in the Treasury market. However, this conclusion is subject to several reservations: (1) the duration difference between the short and long maturity subsector is smaller in the corporate bond market than in the Treasury market; (2) the yields in Figure 11 ignore the impact of the bonds' option features (negative convexity); and (3) both the yield spreads and the return premia may be biased because different sectors have different industry structures.[16]

Figure 11. Average Yield Spread and Return Premium in Various U.S. Bond Market Sectors, 1985-94

	Yield			Return		
	1-3 Yr.	7-10 Yr.	Spread	1-3 Yr.	7-10 Yr.	Premium
Treasury	6.93%	8.04%	1.11%	8.00%	10.52%	2.52%
Agency	7.20	8.53	1.33	8.19	10.47	2.28
AAA/AA	7.79	8.75	0.96	8.64	10.41	1.77
A	8.09	9.04	0.95	8.81	10.48	1.67
BBB	8.62	9.77	1.15	9.03	10.85	1.81

Source: Salomon Brothers's Broad Investment Grade Index.

Figure 12 shows local currency yields and returns in eight countries' government bond markets. **Yield spreads and return premia are positive almost everywhere, but lower than in the United States.** In most countries, the average return premium is higher than the average yield spread; the capital gains caused by long-term bonds' yield decline between 1985 and 1994 augment the premium. Clearly, the past decade offered a favorable environment for bondholders, except in Germany and the

[16] In general, more creditworthy borrowers are able to issue longer-term debt. In the U.S. corporate bond market, the (relatively safe) public utilities are important issuers of long-term debt, while the (more risky) financial companies typically issue short-term debt. These issuance patterns flatten the term structure of aggregate corporate credit spreads.

Netherlands.[17] Unfortunately, few government bond markets outside the United States are liquid at very short durations; thus, we cannot study whether the return curves in countries other than the United States have the concave shape of the average return curve in Figure 1.[18]

Figure 12. Average Yield Spread and Return Premium in International Government Bond Markets, 1985-94

	Yield 1-3 Yr.	Yield 7-10 Yr.	Spread	Return 1-3 Yr.	Return 7-10 Yr.	Premium
United States	6.93%	8.04%	1.11%	8.00%	10.52%	2.52%
Canada	9.01	9.42	0.41	9.59	10.94	1.36
Japan	4.83	5.38	0.55	5.48	7.12	1.63
Australia	11.28	11.65	0.37	12.16	13.82	1.65
Britain	9.58	9.82	0.24	10.08	11.53	1.45
France	8.43	8.67	0.24	9.30	10.97	1.67
Netherlands	6.95	7.15	0.21	7.03	6.98	-0.05
Germany	6.47	7.03	0.56	6.65	6.70	0.06

Source: Salomon Brothers's World Government Bond Index.

CONCLUSIONS AND EXTENSIONS

What Is the Best Estimate of the Long-Run Bond Risk Premium Today?

Any statements about the expected risk premium are partly subjective because expectations are not directly observable. Thus, caution is warranted when interpreting the empirical findings. However, we can draw some general conclusions. **The U.S. Treasury market does reward duration risk, but expected returns do not increase linearly with duration** (or even with return volatility). The reward for duration extension is high at the front end of the Treasury curve (almost 200 basis points from the one-month to the two-year duration), but after two years, the expected return curve appears quite flat.

We argue that the numbers in Figure 5 are our best estimates of the long-run bond risk premium in the U.S. Treasury market. If we can take these numbers at face value, yield curve analysts can subtract each maturity's risk premium from today's yield curve and, after adjusting for the rolldown effect and the convexity bias, infer the market's expectations of future rates. However, this approach is not valid if the risk premium varies over time.

While expected returns do not always increase with duration, short-run return volatility always does. This finding has important implications for fixed-income investors. **If an investor has a short investment horizon** and he is averse to the short-run fluctuations in bond returns, **he has little incentive to extend the long-run benchmark duration beyond two years.** Of course, long-duration bonds are good investments for investors who have long-duration liabilities or an otherwise long investment horizon. In addition, long-duration bonds may be excellent tactical investments if an investor can identify in advance periods of declining interest rates or if the yield curve is abnormally steep beyond the two-year maturity.

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[17] Analysis of average returns is notoriously sensitive to the chosen sample period. The period specificity is illustrated well by the fact that an extensive historical study by Bisignano (1987) identified Germany as the country with the highest reward for maturity extension. Bisignano used bond market data between the 1960s and mid-1980s. Rising rates caused by the German reunification have now pushed the former star performer to near the bottom of the ladder.

[18] One-month Eurodeposit rates are available for all eight countries, however. The average annual returns from rolling over these deposits are 6.68%, 8.92%, 5.19%, 12.34%, 10.98%, 9.44%, 6.99%, and 6.66%, respectively. Thus, the average premium of the one- to three-year government bond sector over the one-month Eurodeposits was negative in four of the eight countries. The average return curves in other countries than the United States appear to have different shapes than Figure 1, but we stress that ten years is quite short period for this type of analysis and that the comparison is contaminated by the use of default-risky data. Further analysis is clearly needed.

Another major finding is that **the shortest Treasury bills appear to be systematically overpriced.** In particular, the one-month bill has offered quite consistently a 100 basis point lower return than the more liquid three-month bill or other high-quality one-month papers in the money market.[19] Substituting longer bills or other money market instruments for the one-month bills in a portfolio may well provide the best reward-to-risk ratio in all capital markets.

Will the Bond Risk Premium Be Different in the Future?

We conclude with some observations about the stability of these risk premium estimates. Realistically, the long-run bond risk premium will change over time. It probably has changed quite a bit during the past 40 years. Figure 13 shows that many plausible measures of long-term bonds' riskiness[20] were low until the mid-1960s and then rose systematically for 15 years. However, this fact is only known with the benefit of hindsight. Surely, the bond investors of the 1950s and 1960s were not demanding as high a risk premium as today's bond investors are. Part of the U.S. bond market's poor performance in the 1960s and 1970s probably reflects the reassessment of the market's riskiness, which increased the required risk premium and thus (initially) led to higher yields and lower bond prices. Now that this major reassessment is over, bondholders can "enjoy" the higher expected returns. In fact, opposite forces may have helped the bond markets in the past decade. Inflation rates have declined and bond volatility has subsided. In addition to the reduced risk, structural changes may be lowering the long-run bond risk premium that the market offers, such as:

- The increasing importance of long-horizon investors who perceive the long-term assets as safe;
- Strengthening anti-inflationary tendencies such as central bank independence and the discipline imposed by financial markets;
- Risk-reduction caused by greater international diversification; and
- Improving liquidity.

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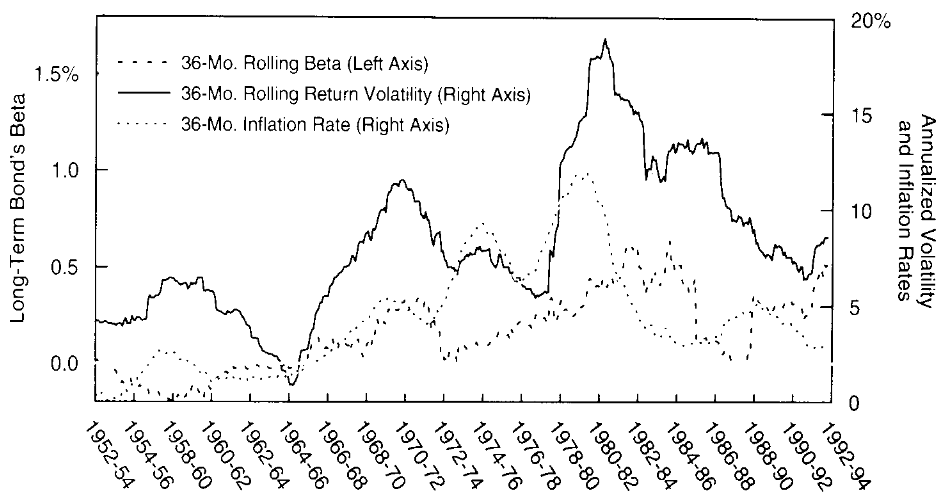
[19] The wide credit spreads at the front end imply that it is not easy to exploit the positive risk premium. A simple strategy of purchasing leveraged two-year notes will lose a large part of its profits when the borrowing is done at a private-issuer rate and not at a Treasury bill rate. Because most arbitrageurs must borrow at the private-issuer rate, they cannot eliminate the overpricing of short-term Treasury bills; only the holders of the expensive bills or the government can do it (by selling or by issuing more bills).

[20] The figure shows the 20-year bond's annualized return volatility and its sensitivity (beta) to U.S. stock market returns as well as the recent 36 months' annualized inflation rate. Many market participants think that bond risk (and not just losses from bond holdings) increases with the inflation level because inflation uncertainty appears to increase with the inflation level.

Figure 13. Reevaluating Long-Term Bond Riskiness, 1955-94



The historical average risk premium is the optimal forecast of the future risk premium only if the required risk premium is constant over time. However, the above discussion shows that we expect the long-run risk premium to vary slowly when there are *structural* changes. In addition, the bond risk premium appears to fluctuate in a (short-run) *cyclical* fashion. As an introduction to the time-variation in expected returns, we offer a simple analysis in Figure 14.

Figure 14. Average Return of the 20-Year Treasury Bond in Months that Begin with an Inverted, Mildly Upward-Sloping or Steep Yield Curve, 1970-94

Spread (20 Yr.-1 Mo.)	No. of Months	Annualized Return
<0bp	45	-2.57%
0-300bp	148	9.41
>300bp	107	12.46

bp Basis points.

The central question is whether we can identify, *ex ante*, periods when the near-term bond risk premium is particularly high or low. The most natural predictor is the steepness of the yield curve. Figure 14 shows that the curve shape has been able to distinguish good and bad times to invest in long-term bonds. **Steep curves tend to be followed by abnormally high returns, and inverted curves tend to be followed by negative returns.** These patterns have obvious investment implications, suggesting that strategies that adjust duration dynamically can produce superior long-run returns. We discuss the time-variation in the bond risk premium extensively in other papers (see "Literature Guide").

APPENDIX: BOND RISK PREMIUM TERMINOLOGY

We discuss bond yields, returns, and risk premia from many different perspectives in our series of reports on the theme *Understanding the Yield Curve*. In this Appendix, we describe and motivate some key concepts and the terminology used throughout the series. We begin with a definition: ***The bond risk premium* is the expected holding-period return of a long-term bond in excess of the riskless return of the one-period bond.**

Why the *name* "bond risk premium"? Based on many academic theories, expected return differentials across bonds compensate for risk differentials across bonds. Nevertheless, we use the term "bond risk premium" broadly to include any expected return differential over the riskless rate, whether it is caused by risk or by factors unrelated to risk. The term "bond risk premium" has many synonyms: interest rate risk premium; term premium; liquidity premium; and the more neutral "expected excess bond return."

Why *return*? Most investors are primarily interested in an investment's expected return, *as opposed to its yield*. For this reason, our analysis focuses on expected return differentials across bonds. Yield spreads do reflect these expected return differentials, but they also are influenced by other factors, such as the market's expectations about future rates. Furthermore, yields of different bonds are directly comparable only under restrictive conditions.

Why *excess* return? It is useful to decompose any bond's holding-period return to the *riskless return*[21] over the holding period (reward for time), which is known in advance and common to all bonds, and to the *excess return* over the riskless rate (reward for risk or for bond's other characteristics), which is uncertain and may be specific to each bond. (Sometimes the excess bond returns are low even though bond returns are quite high, for example, when inflation and short-term rates are very high.)

Why *expected* excess return? Realized returns have an *expected* part and an *unexpected* part. Active investors must try to earn high realized excess returns by capturing high expected excess returns, even though a large part of the realized excess returns is unexpected.[22]

Which *holding-period* return? In our theoretical analysis, we use *annual* holding periods because it simplifies the notation (because yields are expressed as percent per annum). In our empirical analysis, we focus on *monthly* holding periods, and we examine the excess returns of long-term bonds over the nominally riskless one-month rate.

How is the bond risk premium *estimated*? The answer to this question depends on the stability of the risk premium. If the risk premium is *constant* over time, a historical average return differential between the long-term bond and the riskless short-term bond is the best estimate of the future bond risk premium.[23] (Over a long sample period, the unexpected parts of the monthly returns should wash out, leaving only the expected

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[21] We measure the riskless return by the return of the Treasury bill that matures at the end of the horizon (holding period). This return is nominally riskless because the bill's holding-period return is known from its price today and its known maturity value (100). Treasury issues are perceived to be default-free but they have some purchasing power (or inflation) risk.

[22] We sometimes add the redundant word "expected" before bond risk premium to emphasize the distinction between the expected bond risk premium and the realized bond risk premium (or equivalently, between the expected excess return and the realized excess return). We may also use the term "required return" instead of expected return, because the latter term may have a misleading optimistic connotation: in reality, expected bond returns are more likely to be high in bad times when investors *require* a high risk premium for holding risky assets.

[23] A further question is whether we should use an arithmetic or a geometric average of the monthly returns, or perhaps an arithmetic average of the continuously compounded returns.

return differential.) However, if the bond risk premium *varies over time*, we should use the information in the current yield curve and in other variables that describe current economic conditions to find out whether the near-term bond risk premium is abnormally high or low.[24] In Figure 14 of this report, we use the term spread as a crude measure of the information in the yield curve. A better measure would include the impact of the so-called rolldown return. The rolling yield differential between a long-term bond and the riskless rate is a proxy for the bond risk premium under the scenario of no change in the yield curve, but even this measure ignores the impact of convexity on expected returns. Finally, we can combine the information in the yield curve and in other predictor variables to develop an optimal forecast for the near-term bond risk premium. The other reports in this series discuss these topics in detail.

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[24] A historical average of excess bond returns may still be an excellent forecast for the long-run expected excess bond return (relevant for strategic investment decisions) but not the optimal forecast for the near-term excess bond return. The near-term and the long-run forecasts are equal only if the bond risk premium is constant over time.

On the Historical Performance of the U.S. Treasury Bonds with Various Maturities

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Ibbotson Associates, *Stocks, Bonds, Bills, and Inflation 1993 Yearbook*, 1994.

Tyson, "Estimation of the Term Premium," in *Fixed-Income Investment: Recent Research*, ed. Ho, 1995.

On the Historical Behavior of the U.S. Non-Government Money and Bond Market Behavior

Fama, "Term Premiums and Default Premiums in Money Markets," *Journal of Financial Economics*, 1986.

Litterman and Iben, "Corporate Bond Valuation and the Term Structure of Credit Spreads," *Journal of Portfolio Management*, 1991.

Iwanowski and Chandra, "*How Do Corporate Spread Curves Move Over Time?*," Salomon Brothers Inc, July 1995.

On the Historical Performance of the Non-U.S. Government Bond Markets

Bisignano, *"A Study of Efficiency and Volatility in Government Securities Markets,"* Bank for International Settlements, 1987.

Siegel and Kaplan, "Stocks, Bonds, Bills, and Inflation around the World," in *Managing Institutional Assets*, ed. Fabozzi, 1990.

On the Time-Variation in Expected Bond Returns

Ilmanen, "Time-Varying Expected Returns in International Bond Markets," *Journal of Finance*, 1995.

Ilmanen, "*Forecasting U.S. Bond Returns*," Salomon Brothers Inc, 1995.

Ilmanen, "When Do Bond Markets Reward Investors for Interest Rate Risk?," *Journal of Portfolio Management* (forthcoming), 1996.

Salomon Brothers

Forecasting U.S. Bond Returns

Understanding the Yield Curve: Part 4

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INTRODUCTION

It is extremely difficult to forecast bond market fluctuations. However, recent research shows that these fluctuations are not fully unpredictable: **It is possible to identify in advance periods when the reward for duration extension is likely to be abnormally high or abnormally low.** In this report, we first describe a few variables that have the ability to predict near-term bond market performance. We then show how to combine the information that these predictors contain into a single forecast and, further, into implementable investment strategies. Finally, we backtest the historical performance of these strategies in a realistic "out-of-sample" setting.

This report is the fourth part of a series titled *Understanding the Yield Curve*. Historical analysis included in Part 3, *Does Duration Extension Enhance Long-Term Expected Returns?*, showed that intermediate- and long-term bonds earn higher average returns than short-term bonds. This evidence suggests that the long-run bond risk premium is positive.[1] If the risk premium is constant over time, the long-run average risk premium is also our best forecast for the near-term bond market performance. However, we showed in Part 3 that steeply upward-sloping yield curves tend to precede high excess bond returns and inverted yield curves tend to precede negative excess bond returns. It follows then that **the risk premium is not constant** and that **the current shape of the yield curve provides valuable information about the time-varying bond risk premium**. In this report, we show that **other variables can enhance the yield curve's ability to forecast the near-term excess bond return**. The predictability of the excess bond return has important implications for investors who are willing to use so-called tactical asset allocation strategies. Based on our extensive historical analysis, **strategies that adjust the portfolio duration dynamically using the signals from the predictor variables would have earned substantially higher long-run returns than did static strategies** that do not actively adjust the portfolio duration.

REASON FOR THE CHANGING RISK PREMIUM: A CYCLE OF FEAR AND GREED

Earlier empirical research shows that the expected excess returns of stocks and long-term bonds vary over the business cycle; they tend to be high at the end of the recession and low at the end of the expansion. Are there any intuitive reasons that expected excess returns should vary with economic conditions?

If the expected returns reflect rational risk premia, they change over time as the amount of risk in assets or the market price of risk varies over time. The risk premium that bond market participants expect should increase if the risk increases (because of higher interest rate volatility, higher covariance between bonds and stocks, greater inflation uncertainty, etc.). In addition, the required risk premium should increase if the market participants' aggregate risk aversion level increases. **We propose that investors are more risk averse (afraid) when their current wealth is**

[1] We define the *bond risk premium* as the near-term (say, one-month) expected return of a long-term government bond in excess of the return of the near-term riskless asset (say, the one-month Treasury bill). The *long-run* bond risk premium is the long-run average expected return of a long-duration strategy in excess of the long-run average expected return of a short-duration strategy. It may differ from the (near-term) bond risk premium if the latter is abnormally high or low. Our definition of bond risk premium encompasses any expected return differential between a long-term bond and the near-term riskless bond, whether it is actually related to risk or to some technical factor. For this reason, we often call the bond risk premium, more neutrally, the "expected excess bond return."

low relative to their past wealth. The higher risk aversion makes them demand higher risk premia (larger compensation for holding risky assets) near business cycle troughs. Conversely, higher wealth near business cycle peaks makes investors less risk averse (more complacent and greedy); therefore, they bid down the required risk premia (by bidding up the asset prices). Thus, **the observed time-variation in risk premia may be explained by the old Wall Street adage about the cycle of fear and greed**.

ARE EXCESS BOND RETURNS PREDICTABLE?

Which Variables Forecast the Excess Bond Return?

As mentioned above, measures of yield curve steepness have some ability to predict the subsequent excess bond return. In the appendix to Part 2 of this series, *Market's Rate Expectations and Forward Rates*, we showed that a steep yield curve may reflect a high required risk premium or the market's expectations of rising rates. If the second term is assumed to be zero (the current yield curve is the market's best forecast for future yield curves), then the curve steepness is a good proxy for the bond risk premium. We measure the curve steepness by the term spread (the difference between a long-term rate and a short-term rate).

Conveniently, we can use the **term spread as an overall proxy for the bond risk premium** even if we do not know what causes the expected return differentials across bonds. For this reason, the term spread will be our first predictor variable. Yet, if we are trying to forecast bond returns, why restrict ourselves to just one predictor? It is likely that the term spread is sometimes influenced by the market's rate expectations, making the previous assumption unrealistic. Because the rate expectations are unobservable, we cannot know how much "noise" they introduce to our risk premium proxy. Thus, we do not know to what extent a given shape of the curve reflects the required bond risk premium and to what extent it reflects the market's rate expectations. Using other predictor variables together with the term spread should help us *filter out* the noise and give us a better signal about the future risk premium.

The filter variables should be correlated with the risk premium. Based on our hypothesis of wealth-dependent risk aversion, we **combine the information in the term spread and in the stock market's recent performance**. The inverse of the recent stock market performance is our proxy for the (unobservable) aggregate level of risk aversion. If a high term spread coincides with a depressed stock market, the curve steepness is less likely to reflect rising rate expectations (because monetary policy tightening and inflation threat are less likely in this environment) and more likely to reflect high required risk premia (because low stock prices may reflect high required returns on risky assets, or even cause them via wealth-dependent risk aversion). We measure the recent stock market performance by "inverse wealth," a weighted average of past returns, where more distant observations have lower weights.

As a third predictor, we will examine the **real bond yield**, which is sometimes used as the overall proxy for the bond risk premium instead of the term spread. This measure incorporates the inflation rate into the forecasting model. Our final predictor, **momentum**, is a dummy variable that simulates a simple moving average trading rule to exploit the persistence (positive autocorrelation) in bond returns.[2] This strategy tries to capture large trending moves in the bond market. To reduce trading when the yields are oscillating within a narrow trading range and, thereby, to avoid "whipsaw" losses from buying at low yields and selling at high yields, we impose a neutral trading range in which no position is held. Somewhat arbitrarily, we use a six-month moving average window and a ten-basis-point neutral trading range. Thus, the rule is to take a long (short) position in the bond market when the long-term bond yield declines (increases) to more than five basis points below (above) its six-month moving average; such a break-out from a trading range is attributed to positive (negative) momentum in the bond market. If the bond yield returns to (stays within) the ten-basis-point range around its six-month average, the rule is to take (retain) a neutral position. The dummy variable takes value 1, -1 or 0 if the strategy is long, short or neutral, respectively.

Our empirical analysis will confirm that the term spread can forecast future excess bond returns, but combining the information contained in several predictor variables improves these forecasts further. Linear regression is the most common way to combine the information in several variables.[3] We will run a multiple regression of the realized excess bond return on the term spread, the real bond yield and measures of recent stock and bond market performance, and we will use the fitted value from this regression as an estimate of the (expected) bond risk premium.

Correlations Between Predictor Variables and Subsequent Excess Bond Return

We will examine **the predictability of the monthly excess return of a 20-year Treasury bond over the one-month bill rate between January 1965 and July 1995.**[4] We focus on the four predictor variables described in Figure 1: the term spread; the real bond yield; inverse wealth; and momentum.

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[2] Moving average trading rules are perhaps the most popular trend-following strategies among traders. Such strategies are profitable if the market moves more in trends than sideways within a trading range. Even though academic research in the 1960s and 1970s found that common technical trading rules do not consistently outperform buy-and-hold strategies, more recent studies have shown that some trend-following strategies are profitable, especially in the foreign exchange market. See, for example, "Technical Trading: When It Works and When It Doesn't," William Silber, *Journal of Derivatives*, Spring 1994. Momentum indicators are often viewed as indicators of the market sentiment. An alternative interpretation, which is consistent with economic theories with rational behavior, is that the trends in bond markets reflect slow declines (increases) in the bond risk premium that coincide with bull (bear) markets.

[3] The Equity Portfolio Analysis Group at Salomon Brothers has developed more sophisticated ways to combine the information in several predictive variables. See, for example, *Salomon Brothers Global Quantitative Strategy: Anatomy of the Global Allocation Model*, Eric Sorensen et al., Salomon Brothers Inc, September 1991, and "When Is a Tree a Hedge?," Joe Mezrich, *Financial Analysts Journal*, November-December 1994.

[4] We forecast the *excess* return rather than the return for three reasons: (1) The former is a proxy for the realized risk premium of a long-term bond (because any asset's return can be viewed as the sum of the riskless return and a realized risk premium); (2) it corresponds to a return on a self-financed position; and (3) it is harder to predict (because we subtract the riskless return which is known at the time of forecasting). Anyway, this choice hardly affects the predictability findings because the correlation between returns and excess returns is 0.997. (We also could forecast the long-term rate changes, whose correlation with excess bond returns is -0.97.) Finally, we examine a *20-year* bond because it has a long historical return series; however, the main findings of this report are similar if we examine a shorter history of a ten-year or a 30-year bond instead. This similarity is not surprising because the returns of all long-term government bonds are highly correlated.

Figure 1. Description of the Predictor Variables and the Predicted Variable

Variable	Definition	Data Source
Term Spread	Difference between the estimated five-year spot rate and the three-month spot rate.	Center of Research for Security Prices at the University of Chicago. Salomon Brothers since 1994.
Real Yield	Difference between the estimated five-year spot rate and the most recently published yearly consumer price inflation rate.	Center of Research for Security Prices at the University of Chicago. Salomon Brothers since 1994.
Inverse Wealth	Ratio of the exponentially weighted past stock market level to the current stock market level (W_t). Formally, = $(W_{t-1} + 0.9*W_{t-2} + 0.9^2*W_{t-3} + ...)*0.1/W_t$.	Ibbotson Associates — Standard and Poor's 500 total return index.
Momentum	A dummy variable which takes value 1 if the bond yield is more than five basis points below its six-month average, -1 if the bond yield is more than five basis points above its six-month average, and 0 otherwise.	Ibbotson Associates — yield of a long-term government bond with an approximate maturity of 20 years.
Excess Bond Return	Monthly return of a long-term Treasury bond in excess of the nominally riskless return of a one-month Treasury bill. Also called Realized Bond Risk Premium.	Ibbotson Associates — total return index of a long-term government bond with an approximate maturity of 20 years.

Note: All rates and returns are compounded continuously.

Figure 2 shows the correlations between the excess bond return and various predictor variables.[5] The conventional view that risk premia cannot be forecast using available information implies that all these correlations should be very close to zero. This conventional view is partly based on the finding that some obvious predictor candidates have limited forecasting ability. For example, the first three columns in Figure 2 show that a bond's yield level, its lagged monthly return, and its past volatility (measured by the 12-month rolling standard deviation of monthly excess returns) all have low correlations with next month's excess bond return (0.03-0.11). In contrast, **the predictors that we have identified above — the term spread, the real yield, inverse wealth, and momentum — have correlations with the subsequent excess bond return between 0.09 and 0.21**. Note that our momentum variable, which is based on a moving average strategy, has somewhat better forecasting ability than a simple lagged return (which could be used as an alternative proxy for the market's momentum). Finally, **combining the information in these four predictors gives even more accurate return predictions, with a correlation of 0.32.**[6] Steep yield curves, high real yields, depressed stock markets, and rallying bond markets are all positive indicators of subsequent bond market performance.

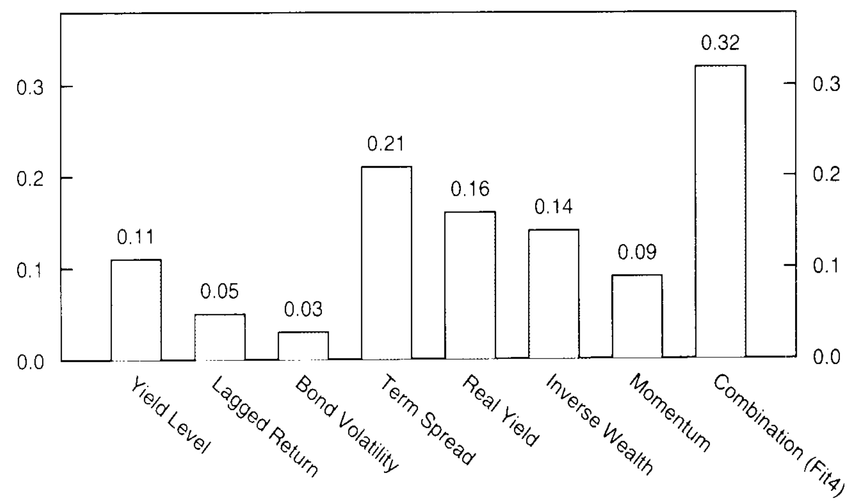
Our predictor variables are financial market data. Many bond market participants are more used to forecasting market movements based on available "fundamental" macroeconomic data. Previous empirical research (see the Literature Guide) suggests, however, that **financial market variables are better predictors of asset returns than macroeconomic variables** such as production growth rates, perhaps because the latter are less accurately measured and less timely. While market-based variables are forward looking (partly reflecting the market's expectations about future

[5] A correlation coefficient measures how closely two series move together. Its possible values range from -1 to 1, where 1 indicates a perfect positive correlation, -1 indicates a perfect negative correlation and 0 indicates the lack of any correlation. The square of the correlation coefficient, so-called R^2, measures what part of the variability in a regression's dependent variable (say, excess bond return) the variation in the independent variables explains (or predicts).

[6] The correlations may appear low to those readers that are used to examining contemporaneous relations between bond returns and other variables. It is easy to *explain* a large part of the fluctuations in monthly excess bond returns, but it is very difficult to *forecast* those returns. Because most of the realized monthly excess returns are unexpected, even the optimal predictor will have a low correlation with the realized return. In fact, if the bond risk premium is constant over time, we should expect to see zero correlations between the excess bond return and its predictors, which are known at the beginning of the forecasting month.

economic developments), contemporaneous macroeconomic data describe past events, and with a publication lag. Another finding worth noting from previous studies is the low correlation between various risk measures (such as volatility in Figure 2) and future bond returns: periods of high risk do not seem to provide bondholders with high near-term expected returns.

Figure 2. Correlation of Various Predictors with Subsequent Monthly Excess Bond Return, 1965-95



Correlations are not the only way to show that our predictors can discriminate between good and bad times to hold long-term bonds. In Figure 3, we examine the average monthly returns in subsamples that are based on the beginning-of-month values of the term spread and inverse wealth. **The annualized average excess return is -12.4% in months that begin with an inverted curve and 2.6% in months that begin with an upward-sloping curve** (87% of the time). This finding is consistent with the hypothesis of wealth-dependent risk aversion above. Periods of steep yield curves and high risk premia tend to coincide with cyclical troughs (high risk aversion), while periods of flat or inverted yield curves and low risk premia tend to coincide with cyclical peaks (low risk aversion). **Future bond returns also tend to be higher when inverse wealth is high (the stock market is depressed) than when it is low. Combining the information in these two predictors sharpens our return predictions further. The average excess bond return is higher when an upward-sloping yield curve coincides with a depressed stock market (12%) than when it coincides with a strong stock market (1%).** In the latter case, the curve steepness is more likely to reflect the market's expectations about rising rates than about the required bond risk premium.

Figure 3. Excess Bond Return in Subsamples, 1965-95

Months Begin With	Term Spread > 0	Term Spread < 0
Average	2.63%	-12.40%
No. of Months of Total (%)	87	13

Months Begin With	Inverse Wealth > 1	Inverse Wealth < 1
Average	6.24%	-0.36%
No. of Months of Total (%)	16	84

Months Begin With	Term Sp. > 0 and Inv. Wealth > 1	Term Sp. > 0 and Inv. Wealth < 1	Term Sp. < 0 and Inv. Wealth > 1	Term Sp. < 0 and Inv. Wealth < 1
Average	11.95%	1.23%	-9.76%	-13.60%
No. of Months of Total (%)	12	75	4	9

Note: **Average** is the annualized average of a 20-year bond's monthly excess return in each subsample.

The inverse relation between stock market level and subsequent bond returns may be interpreted in many ways. We proposed earlier that declining wealth level makes investors more risk averse and increases the risk premium that they require for holding risky assets. Alternatively, the relation may be caused by lagged portfolio flows. Poor recent stock market performance can make investors shift money to bonds, either because these are less risky or because investors extrapolate and expect the poor stock market performance to continue. More generally, the time-variation in expected returns may reflect rational factors (time-varying risk or risk aversion level) or irrational factors (such as swings in market sentiment or an underreaction of long-term rate expectations to current inflation shocks).

Another way to think about the patterns on the next-to-last row in Figure 3 is that when *both* bond and stock markets appear to be "cheap" (the term spread is high and inverse wealth is high), investors can rely more on these cheapness indicators and expect high future returns for risky assets. Conversely, when *both* bond and stock market indicators signal "richness" (the term spread is negative and inverse wealth is low), investors can more confidently expect low future returns. In this light, our three first predictors may be viewed as "value" indicators that tend to give buysignals when asset markets are weak. These predictors are complemented by the fourth, momentum, which gives a buysignal when bond prices are trending higher.

Figure 4 shows the regression results for our whole sample period. All four predictors are statistically significantly related to subsequent excess bond returns. The regression coefficients show that the expected excess bond returns are high when the yield curve is steeply upward sloping, the real yield is high, the stock market is depressed, and the bond market has positive momentum. Together, the four predictors capture 10% of the monthly variation in excess bond returns. The fact that 90% of the return variation is unpredictable tells us that even if strategies that exploit these patterns are profitable, they certainly will not be riskless.

Figure 4. Results from Regressing Excess Bond Return on Four Predictors, 1965-95

	Coefficient	T-Statistic
Constant	-10.15	-4.98
Term Spread	0.37	2.24
Real Yield	0.20	2.89
Inverse Wealth	9.85	4.81
Momentum	0.34	2.02
R^2		10.3%

Out-of-Sample Estimation of Return Predictions

The regression splits each month's excess bond return to a fitted part and a residual. The fitted part can be viewed as the expected excess bond return and the residual as the unexpected excess bond return. Because the current value of each predictor is known, we can compute the current forecast for the near-term excess bond return by using the following equation:

Expected Excess Bond Return =
-10.15 + 0.37 * Term Spread + 0.20 * Real Yield + 9.85 * Inverse Wealth + 0.34 * Momentum

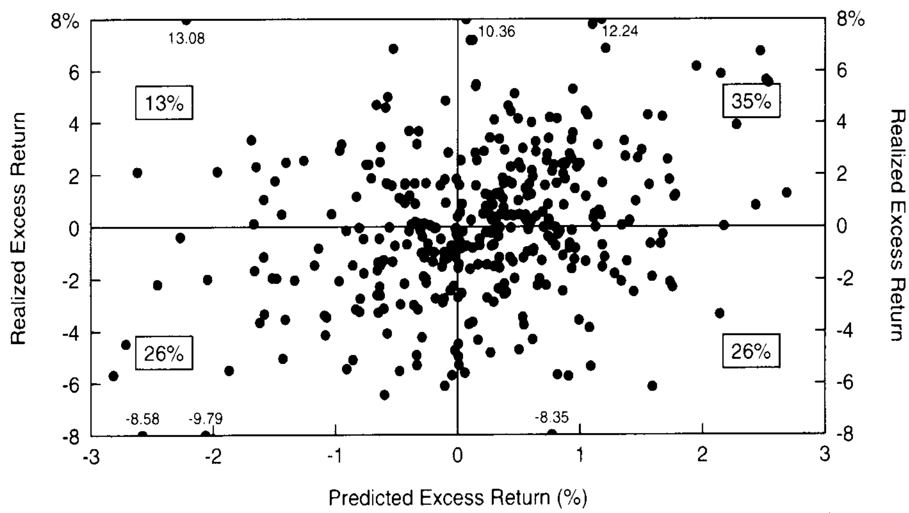
We are only using available information to make this forecast; we combine current values of the predictors with the historical estimates of the regression coefficients. For this reason, we can call this an *out-of-sample* forecast, as opposed to an *in-sample* forecast. As an example of an in-sample forecast, we could combine the predictor values at the beginning of January 1965 with the above regression coefficients and treat the fitted value as the expected excess bond return for January 1965. In doing so, we would be peeking into the future and assuming that investors already knew the above regression coefficients in 1965. In reality, these coefficients were estimated in 1995 using data from 1965 to 1995. The use of in-sample forecasts adds an element of hindsight to the analysis, which leads, at best, to an exaggerated view of return predictability and, at worst, to totally spurious findings. Many investors find the use of in-sample forecasts unrealistic and unappealing.

In general, investors are well advised to be concerned about the potential impact of data-snooping bias when faced with any "exciting" new empirical findings. Data snooping refers to the fact that many investors and researchers are intensively searching for profitable regularities in the financial market data. The bias means that some apparently significant findings are likely to be period-specific and spurious. **We try to guard against such data-snooping bias by conducting out-of-sample analysis, in which the return predictions are made each month using only data that are available at the time of forecasting.** When we make the forecast of the monthly U.S. excess bond return for January 1965, we run a regression using all historical data from January 1955 to December 1964. The forecast (the fitted part of the regression) combines the estimated regression coefficients with the values of the predictors at the end of December 1964. To make the forecast for February 1965, we run another regression which uses data from January 1955 to January 1965. We run these monthly rolling regressions with an expanding historical sample until July 1995. This process gives us a series of monthly out-of-sample excess bond return forecasts.[7]

[7] This approach still leaves one element of hindsight to the analysis: The predictors may have been chosen based on their historical fit. It may be unrealistic to assume that bond analysts chose to focus on this particular set of predictors (out of many alternative sets) a long time ago. We have three answers to such criticism: (1) We can motivate our predictors with economic reasoning; (2) the relation between the predictors and future bond returns is reasonably stable. Data analysis could have alerted investors of these relations a long time ago; and (3) the ultimate test is the predictors' performance with "new" data. We will present some evidence from the 1990s, after the predictive relations were identified.

We begin the evaluation of the out-of-sample forecasts' predictive ability by showing in Figure 5 a scatter plot of realized monthly excess bond returns on the out-of-sample predictions. To enhance visual clarity, we have trimmed the range of the y-axis to (-8%, +8%) and marked six exceptional observations on the borders. If the forecasts tend to have correct signs (realized excess returns are positive when they were predicted to be positive and negative when they were predicted to be negative), most observations will lie in the upper-right quadrant or in the lower-left quadrant. Without any forecasting ability, all quadrants should contain 25% of the observations. Figure 5 shows that the forecasts have the correct sign in 61% (= 35 + 26) of the months. These odds are better than 50-50, but clearly the forecasts are not infallible.

Figure 5. Realized Excess Return versus Predicted Excess Return, 1965-95



The relation between the predicted and the realized excess returns in Figure 5 may not appear very impressive, reflecting the fact that most of the short-term fluctuations in excess bond returns are unpredictable. Perhaps the long-term fluctuations are more predictable; averaging many monthly returns will smooth the return series and may increase the share of the predictable returns. Our unpublished analysis shows that in a scatter plot of the subsequent 12-month realized excess bond returns on the predicted excess returns, 84% of the observations are in the upper-right quadrant or in the lower-left quadrant. Moreover, the correlation between the out-of-sample predictions and the subsequent 12-month excess returns is 0.57, much larger than the 0.26 correlation between these predictions and the subsequent monthly excess returns.

Figure 6 displays a time series plot of the monthly predicted excess returns and the subsequent 12-month average excess returns. We also plot the time series of each predictor variable in Figure 7, to better identify the sources of fluctuations in expected excess bond returns. **Figure 6 shows that the predictions track the movements in the realized bond returns reasonably well.** Both series in Figure 6 were low in the 1960s and 1970s and exceptionally high in the 1982-85 period, reflecting slow-moving changes in the real yield and in the term spread. Aside from these broad movements, both series exhibit apparent business cycle patterns: **The predicted returns tend to increase during cyclical contractions such as those in 1970, 1974 and 1982.** Figure 7 shows that these increases in expected returns as well as those in the aftermath of the 1987 stock market crash (when a recession was widely expected) and in the most recent recession (1990 Gulf War) coincide with inverse wealth spikes, that is, with poor stock market performance. These patterns are consistent with our hypothesis that wealth-dependent risk aversion causes the required bond risk premium to vary over time with economic conditions. However, the negative excess return forecasts in the 1960s, 1973-74, 1979-80, and 1989 are difficult to interpret; most bond market participants think that the bond risk premium is always positive. Finally, **it is worth noting that the forecasting model is currently quite bearish. The excess bond return predictions have been negative since the end of May 1995**, mainly because of the strong stock market (low inverse wealth) and the relatively flat yield curve (low term spread).[8]

Figure 6. Predicted Excess Bond Return and Subsequent Realized 12-Month Excess Bond Return, 1965-95

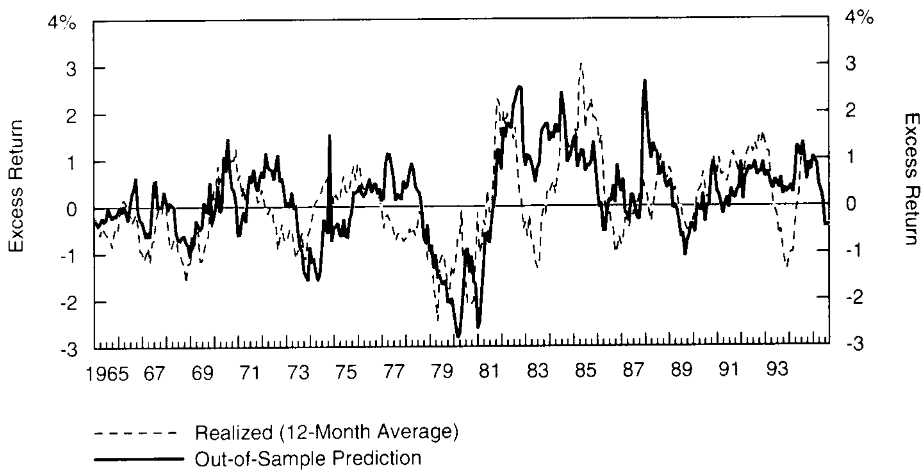
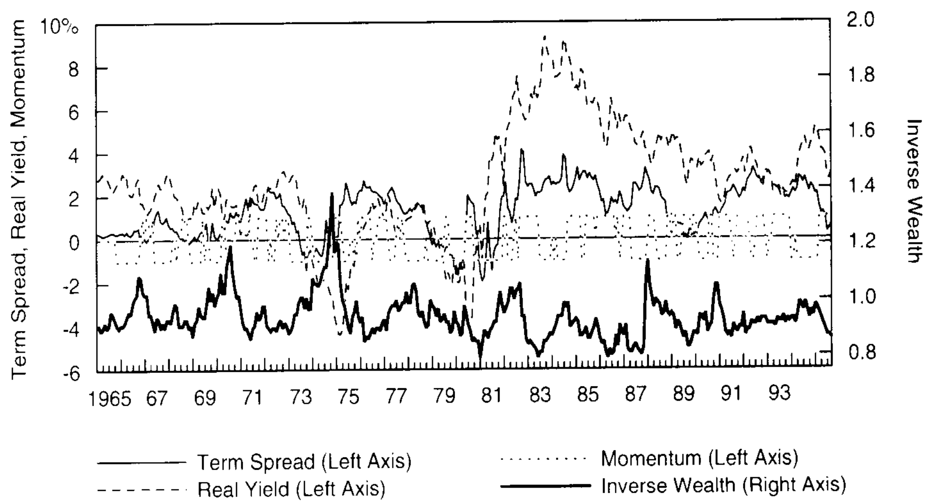


Figure 7. Historical Levels of the Predictor Variables, 1965-95



[8] We reemphasize that the model only produces estimates and that these estimates capture, at best, 10% of the future variation in excess bond returns. Thus, if the *unexpected* economic news in the coming months lowers the market's rate expectations, the bond market may perform well in spite of the low risk premium. The model just signals that the expected return cushion in favor of the long-term bonds now is abnormally low, making these bonds vulnerable to bad news and to a rising risk premium.

Exploiting Return Predictability by Using Dynamic Investment Strategies

Even if the predictor variables appear to have some ability to forecast excess bond returns in an out-of-sample setting, should investors care about these findings? **For portfolio managers, the key question is whether investment strategies that exploit the return predictability produce economically significant profits.** In this section, we first describe the implementation of such dynamic investment strategies and then present extensive analysis of their historical performance. In particular, we compare their historical returns to the returns of static strategies that have a constant portfolio composition regardless of economic conditions. One goal is to show that the information in the yield curve and in other predictors could have been used to enhance long-run returns. Another goal is to provide a tool kit to evaluate the future profitability of any forecasting strategy and a set of critical questions (economic reason for success, stability of success, sensitivity to transaction costs and to risk adjustment) that an investor should ask when faced with backtest evidence of an apparently attractive investment strategy.

The static strategies are called "always-bond" and "bond-cash combination." The former strategy involves always holding a 20-year Treasury bond, while the latter involves always holding 50% of the portfolio in cash (one-month Treasury bill) and 50% in the 20-year Treasury bond, with monthly rebalancing. The dynamic strategies adjust the allocation of the portfolio between cash and the 20-year Treasury bond each month, based on the predicted value of next month's excess bond return. The two dynamic strategies are called "scaled" and "1/0." The 1/0 strategy is simpler: It involves holding one unit of the 20-year bond when its predicted excess return is positive and zero when it is negative (thus, holding cash). This approach ignores information about the magnitude of the predicted excess return. In contrast, the scaled strategy involves buying more long-term bonds the larger the predicted excess return is. Specifically, investors should buy or short-sell the long-term bond in proportion to the size of the predicted excess return.[9]

Strategy returns are expressed in excess of the one-month bill return. For investors who have investable funds, each strategy's total return would be approximately equal to its excess return plus the one-month bill's return (which is the same number for all of the strategies). For arbitrage traders who only hold self-financed positions, the reported excess return can be interpreted as the total return of their "zero-net-investment" position — to the extent that they can finance their positions using the one-month bill rate.

Figure 8 shows, for each strategy, the annualized average excess return, the volatility of excess returns as well as the Sharpe ratio. Note that a cash portfolio earns zero excess return by definition; it is equivalent to holding cash financed with cash. Therefore, the 50-50 bond-cash combination has exactly half of the excess return and volatility of the always-bond strategy. **The static strategies yielded only insignificant excess returns over the sample period. In other words, long-term bonds and short-term bonds earned quite similar average returns. The dynamic strategies**

[9] An example illustrates how the scaled strategy works. If the predicted bond risk premium (BRP) over the next month is 0, the scaled strategy involves buying no long-term bonds, just cash. If the BRP is 1%, the strategy involves buying one unit of the long-term bond, no cash. If the BRP is 2%, the strategy involves buying two units of the long-term bond by using leverage (borrowing cash). If the BRP is -1%, the strategy involves short-selling one unit of the long-term bond and investing the sale proceeds in cash. Because the scaled strategy often involves either leveraging or short-selling, it is much riskier than the 1/0 strategy.

performed much better. The scaled strategy earned almost a 9% average annual excess return while the 1/0 strategy earned about half of that. Also the rewards to volatility (Sharpe ratios) of the dynamic strategies are much larger than those of the static strategies.[10]

Figure 8. Performance of Self-Financed Dynamic and Static Investment Strategies, 1965-95

Dynamic Strategies	
Scaled Strategy	
Average Excess Return	**8.64%**
Volatility	12.80
Sharpe Ratio	0.68
1/0 Strategy	
Average Excess Return	**4.16%**
Volatility	7.92
Sharpe Ratio	0.53
Static Strategies	
Always-Bond	
Average Excess Return	**0.67%**
Volatility	10.42
Sharpe Ratio	0.06
Bond-Cash Combination	
Average Excess Return	**0.33%**
Volatility	5.21
Sharpe Ratio	0.06

Note: **Average excess return** is the annualized average excess return of each strategy over the one-month bill. **Volatility** is the annualized standard deviation of the excess return series. The **Sharpe ratio** is the ratio of the (annualized) average excess return to volatility.

We can compare the performance of the dynamic strategies in Figure 8 with the performance of a dynamic strategy that uses only the information in the term spread. The scaled strategy would have earned 3.87% per annum and the 1/0 strategy 2.96% per annum if the out-of-sample forecasts had been based on the term spread alone. Comparison with the average returns in Figure 8 (8.64% and 4.16%) indicates that the marginal value of the other predictors has been substantial. It is also worth noting that the scaled strategy would have earned 11.15% per annum and the 1/0 strategy 4.94% per annum if the predictions had been based on the in-sample estimates from the regression of excess bond returns on the four predictors. The difference between the performance of the in-sample and the out-of-sample forecasts may reflect the data-snooping bias.

Stability of the Predictive Relations

The analysis above shows, first, that over the past 30 years, our predictors have been able to forecast near-term bond returns and, second, that strategies that exploit such predictability have earned economically meaningful profits. In this section, we examine the stability of these findings over time. If the predictive ability and exceptional performance arise from a couple of extreme observations, we become skeptical about the reliability of these findings. However, if the observed relations are consistent across subperiods, we think that they are less likely to be spurious. Thus, we become more comfortable in expecting that the historical experience (good predictive ability and the dynamic strategies' exceptional performance) will be repeated in the future. We will study three types of evidence: Rolling correlations; subperiod analysis of average returns; and cumulative performance of various investment strategies.

[10] Note that the scaling intensity used in the scaled strategy is arbitrary. More aggressive scaling factors would lead to higher average returns and higher volatilities. Fortunately, the Sharpe ratios do not depend on the scaling factor. Figure 8 shows that the scaled strategy has the highest Sharpe ratios.

Figure 9 shows that estimated rolling 60-month correlations between the predictors and the subsequent bond return are not constant, but they are positive in most subperiods. In the 1990s, the real yield and momentum have had little forecasting ability, but both the term spread and inverse wealth have had predictive correlations near 20%. The combined predictor tends to have better forecasting ability than any of the individual predictors. Similar subperiod analysis shows that the frequency of correctly predicting the sign (+/-) of the next month's excess bond return is reasonably stable and near 60%.

Figure 9. Rolling 60-Month Correlation of Various Predictors with Subsequent Excess Bond Return, 1965-95

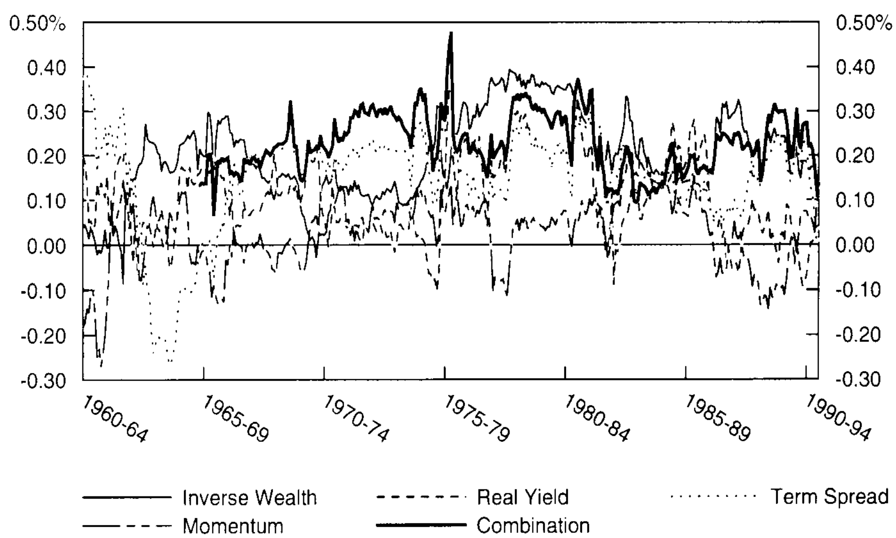


Figure 10 reports the statistics from Figure 8 for three decade-long subsamples and for the 1990s subperiod. It is encouraging to see that the observed patterns are stable across decade-long subsamples. In particular, both dynamic strategies outperform the bond-cash combination strategy by at least 200 basis points in all subperiods.

Figure 10. Subperiod Performance of Various Investment Strategies, 1965-95

	1965-74	1975-84	1985-94	1990-95
Dynamic Strategies				
Scaled Strategy				
Average Excess Return	**4.09%**	**15.29%**	**6.40%**	**5.30%**
Volatility	5.65	20.30	7.32	4.80
Sharpe Ratio	0.72	0.75	0.87	1.10
1/0 Strategy				
Average Excess Return	**0.89%**	**3.20%**	**7.27%**	**6.73%**
Volatility	5.56	8.92	8.66	7.79
Sharpe Ratio	0.16	0.36	0.84	0.86
Static Strategies				
Always-Bond				
Average Excess Return	**-3.13%**	**-1.68%**	**5.62%**	**5.25%**
Volatility	8.35	12.36	10.05	8.28
Sharpe Ratio	-0.38	-0.14	0.56	0.63
Bond-Cash Combination				
Average Excess Return	**-1.57%**	**-0.84%**	**2.81%**	**2.72%**
Volatility	4.18	6.18	5.03	4.14
Sharpe Ratio	-0.38	-0.14	0.56	0.63

The most informative way to display the stability of a predictive relation is to plot the cumulative wealth of an investment strategy that exploits the predictive relation. Such a graph shows how the profits from the strategy grow over time. Note that the cumulative wealth of an ideal perfect-foresight strategy would never decline; moreover, it should also be rising faster than the cumulative wealth of any competing strategies. Alternatively, we can plot the relative performance of two investment strategies, and again, the line representing a perfect-foresight strategy should always be rising (or flat if it matches the performance of the other strategies).

Figure 11 shows the cumulative wealth growth of both dynamic strategies and the always-bond strategy (plotted on a log-scale where constant percentage growth produces a straight line). Because the lines cumulate each strategy's monthly returns in excess of cash (the one-month bill), we also can interpret these lines as relative performance versus cash. Figure 12 measures the relative performance of the two dynamic strategies versus a more realistic benchmark, a 50-50 combination of cash and the long-term bond. These graphs show that **the dynamic strategies have had a consistent ability to outperform the static strategies.** The scaled strategy earned very high returns in the late 1970s and early 1980s by short-selling the long-term bond during the bear market. During the subsequent bull market, the dynamic strategies have earned similar returns as the static bondholding strategy. This result must be viewed as satisfactory, because this bull market has been exceptionally strong and long, making long-term bond returns a difficult target to beat. Figures 11 and 12 show that the dynamic strategies never underperformed the benchmark static strategies for an extended period. And what about the recent experience? The dynamic strategies have outperformed the static strategies in the 1990-95 period — see Figure 10 — but last year (1994) both dynamic strategies underperformed the cash-bond combination because they remained in long-term bonds throughout a period of rising rates.

Figure 11. Cumulative Wealth Growth from Three Self-Financed Strategies, 1965-95

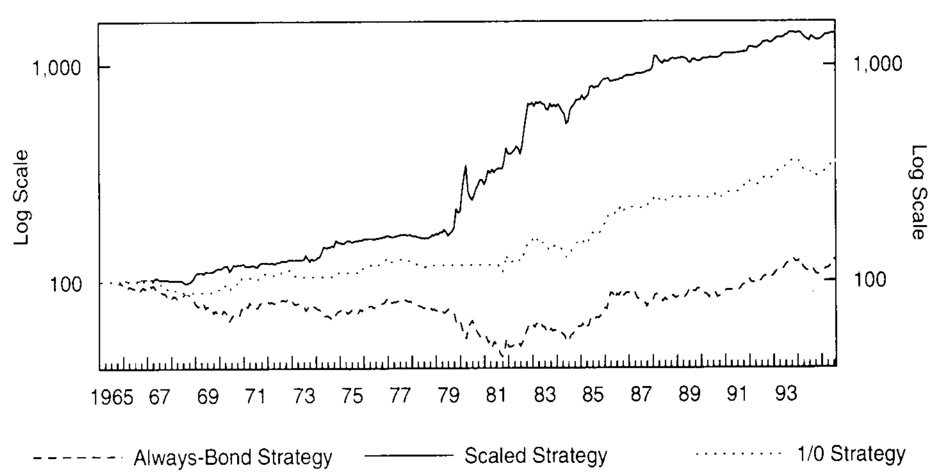
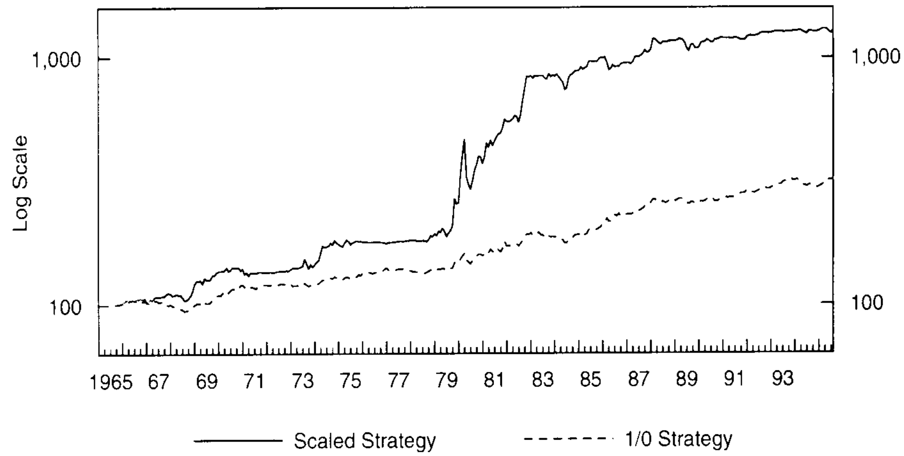


Figure 12. Dynamic Strategies' Relative Performance versus Bond-Cash Combination, 1965-95



Other Critical Considerations

The backtest results suggest that bond investors could enhance their performance substantially by exploiting the forecasting ability of the term spread, the real yield, inverse wealth, and momentum. However, historical success does not guarantee future success. We stress that any reported findings of apparently profitable investment strategies should be subjected to a set of critical questions. We already addressed the important **concern about data-snooping bias** — we used out-of-sample forecasts, we restricted the predictors to economically well-motivated variables, and we ensured that the observed findings are relatively stable across subperiods. Other reservations include **the sensitivity of the findings to transaction costs and to risk adjustment.**

Transaction costs will reduce the profitability of any investment strategy. However, government bonds have such small transaction costs for institutional investors that their impact on the reported returns should be small. In particular, the results of the 1/0 strategy are hardly affected because this strategy involves very infrequent trading — on average 1.5 trades per year. The scaled strategy is more transaction intensive, and it also involves short-selling. Thus, the reported results are somewhat exaggerated.

The dynamic strategies offer higher returns than the static strategies, but they excel even more when the comparison is made between risk-adjusted returns. First, if risk is measured by the volatility of returns, the Sharpe ratios in Figure 8 provide a risk-adjusted comparison. **The volatility of the scaled strategy is higher than that of the static bond strategy, but its reward-to-volatility ratio is more than ten times higher. The volatility of the 1/0 strategy is lower and the average return higher than that of the static always-bond strategy.** Second, if investors are concerned with downside risk, the dynamic strategies will look even better. The historical success of these strategies partly reflects their ability to avoid long-term bonds during bear markets. For example, we can infer from Figure 5 that the 1/0 strategy underperformed cash in only 26% of the months in the sample (outperforming it 35% of the time and matching its performance 39% of the time when the predicted excess return was negative and the strategy involved holding cash). However, if the return predictability reflects a time-varying risk premium, it is possible that the abnormally high returns of the dynamic strategies reflect only a fair compensation for taking up additional risk at times when either the amount of risk or risk aversion is abnormally high.

In spite of the apparent attractiveness of the dynamic strategies, few investors have tried to systematically exploit the predictability of bond returns. For those investors who venture to do that, this fact is good news. The profit opportunities are not likely to be "arbitraged away" any time soon. One major reason is that these strategies are not riskless arbitrages — they involve a lot of short-term risk because the forecasts are wrong 40% of the time. Nonetheless, 60-40 odds are attractive in competitive financial markets. **Therefore, what could make investors forego the exceptionally favorable odds that the dynamic strategies offer?** Here are some possible explanations:

- Many investors prefer the more subjective interest rate forecasting approach even if its track record is rarely good. Other investors believe that market fluctuations cannot be predicted; thus, they do not want to take any market-directional positions. Such investors would attribute our predictability findings to data snooping or to events that the market was expecting but that were not realized during the sample period.

- The potential losses from such strategies may loom larger than the potential gains. The short-term risk in the dynamic strategies may expose portfolio managers to substantial career risk[11] even if the strategies are likely to outperform in the long run. Moreover, the losses may have a tendency to occur at especially unpleasant times.[12] The dynamic strategies' high expected return may be a reward for such discomforts.

Even if some investors find the strategy too mechanical or too risky to be used systematically, they may want to use it *selectively*. For example, they may want to use the strategy only when the signal is very strong. What do historical data say about such an approach? The fact that the scaled strategy outperforms the 1/0 strategy suggests that the magnitude of the predicted excess return contains valuable information beyond the sign. Figure 13 studies whether the return predictions become more reliable when the forecast deviates much from zero. It also reports the average returns at different levels of predicted excess returns. We can see that the frequency of correct-sign forecasts is only weakly related to the absolute value of the forecast. The average excess returns show clearer patterns: large negative values when the predictions are very negative and large positive values when the predictions are very positive.

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[11] First, the dynamic strategy can make the portfolio differ significantly from most peer portfolios. Given the frequent performance evaluations, short-term underperformance relative to a peer group often implies serious career risk. Therefore, portfolio managers may avoid the dynamic strategies because "it is better to lose conventionally than to gain unconventionally." Second, the dynamic strategy works quite slowly. Many traders prefer making many trades with fast outcomes to making one trade with a slow outcome. The former approach allows better diversification (across a large number of trades within a given period) — and smaller career risk — than the latter approach.

[12] For example, the dynamic strategies might systematically underperform during recessions when many investors find it particularly difficult to tolerate losses. However, our empirical analysis shows that the dynamic strategies tend to perform particularly well in "bad times." During cyclical recessions, as defined by the National Bureau of Economic Research, the always-bond strategy earned an annualized average excess return of 6.55%, while the scaled strategy earned 19.75% and the 1/0 strategy earned 9.08%. Similarly, both dynamic strategies tended to outperform the static strategies near business cycle troughs, in months when excess bond returns were negative and following years of exceptionally poor bond market performance.

Figure 13. Impact of the Forecast Signal's Strength on the Return Predictability, 1965-95

	f < -1	-1 < f < -0.5	-0.5 < f < 0	0 < f < 0.5	0.5 < f < 1	f > 1
Frequency of Correct-Sign Predictions	65%	63%	65%	53%	57%	67%
Average Excess Return	-15.68	-3.80	-8.57	3.92	4.15	15.66
No. of Months of Total (%)	9	11	19	25	21	15

Note: **f** is the beginning-of-month out-of-sample forecast of the 20-year bond's monthly excess return, expressed in percent per month. **Average excess return** is the annualized average of the 20-year bond's monthly excess returns within each subsample, in percent.

Impact of Investment Horizon

We suggested above that the dynamic strategies may involve an unacceptably high risk of short-term underperformance (see footnote 11). However, the long-run performance of these strategies should make them very appealing for investors who can afford to take a long investment horizon. In this section, we focus on the impact of investment horizon on the attractiveness of the dynamic strategies. The crucial question we address is: **How long a horizon is long enough for investors to be confident that these strategies outperform cash and/or bonds?** Recall that the out-of-sample forecasts of next *month's* excess bond return have a correct sign in 61% of the months in the sample (see Figure 5). Increasing the investment horizon from one month makes the dynamic strategies look better and better. For example, Figure 14 shows that the scaled strategy outperformed the always-bond strategy in 23 calendar years out of 30 in this sample, and the 1/0 strategy underperformed the always-bond strategy in only two calendar years (and matched it in ten other years when it kept holding the long-term bond).

Figure 14. Annual Excess Returns of Three Self-Financed Strategies, 1965-94

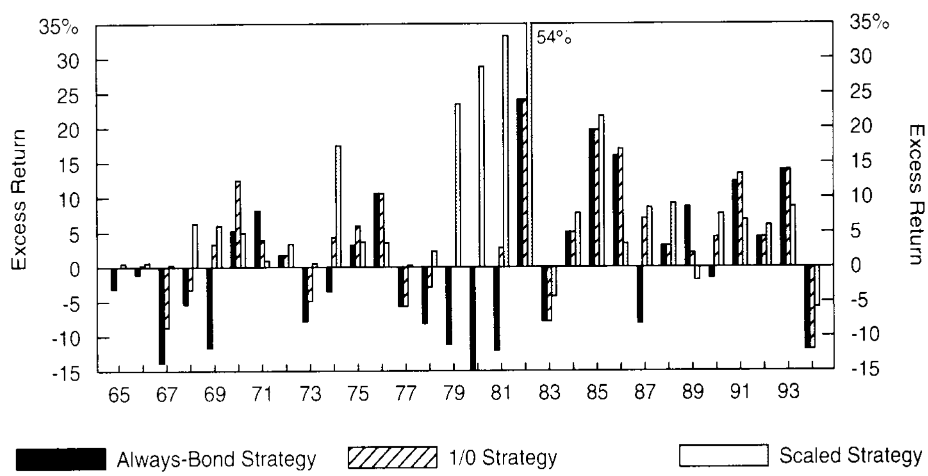


Figure 15 shows — for various horizons — the frequency at which the dynamic strategies outperformed or matched cash, the long-term bond or both. The longer horizon numbers are based on overlapping monthly data.[13]

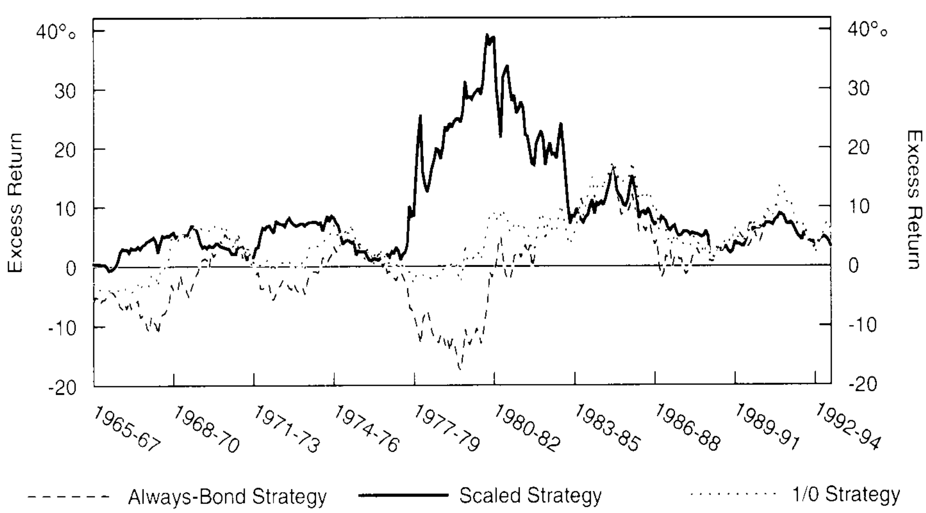
[13] For example, the evaluation at the five-year horizon compares the holding-period returns of various investment strategies between January 1965 and December 1969, February 1965 and January 1970, March 1965 and February 1970, ... until August 1990 and July 1995. There are a total of 307 overlapping five-year periods. (Recall that each strategy may involve monthly rebalancing: the dynamic strategies use the predictions to adjust the portfolio, the bond-cash combination is rebalanced to 50/50 shares and even the always-bond strategy requires occasional rebalancing so that the portfolio maturity does not deviate too much from 20 years.) The last row in Figure 15 reports how frequently — that is, in what part of the 307 five-year periods — the dynamic strategies beat or matched the performance of cash, bond or both.

Focusing on the toughest comparison, the dynamic strategies outperformed **both** bonds and cash in roughly 60% of the one-year periods in the sample. For the three-year horizon, the frequency increases to about 80% of the sample, and for the five-year horizon to 92%. In a way, the dynamic strategies would have provided a free outperformance option for long-term investors. Figure 16 illustrates this point by showing the rolling 36-month excess return for each strategy. If the dynamic strategies outperform both cash and bonds, their excess returns should lie above the excess returns of cash (zero line) and the always-bond strategy. This is roughly what we see in the graph. We conclude that although historical analysis provides no guarantee about the future and although backtest results are rarely achieved in real-world investments, the odds in favor of the dynamic strategies appear excellent for three- to five-year investment horizons.

Figure 15. Impact of the Horizon Length to the Strategy's Success Rate, 1965-95

	Scaled Strategy			1/0 Strategy		
Horizon	Beat/ Match Cash	Beat/ Match Bond	Beat/ Match Both	Beat/ Match Cash	Beat/ Match Bond	Beat/ Match Both
Month	61%	56%	35%	74%	86%	61%
Quarter	68	61	46	71	85	58
Year	84	69	58	73	87	64
Three Years	99	86	85	82	91	73
Five Years	100	92	92	94	97	92

Figure 16. Rolling 36-Month Excess Returns of Three Self-Financed Strategies, 1968-95



In this report, we have shown that long-term bond returns are predictable. A set of four predictors — yield curve steepness, real bond yield, recent stock market performance, and bond market momentum — is able to forecast 10% of the monthly variation in long-term bonds' excess returns. Thus, when making inferences about the yield curve behavior, bond market analysts should not assume that the bond risk premium is constant over time. **The bond risk premium may be small, on average, but we can identify in advance periods when it is abnormally large or small. A forecasting model gives us an estimate of the near-term bond risk premium — but even the best models' estimates are subject to various errors. Nevertheless, such models can be valuable tools for long-term investors.** We find that dynamic investment strategies that exploit the bond return predictability have consistently outperformed static investment strategies over long investment horizons.

There are **many ways to implement** investment strategies that exploit return predictability. This report presents two dynamic strategies (scaled and 1/0) that shift funds between cash and a long-term bond based on the sign and the magnitude of the return prediction. One alternative way to implement the strategy is through active duration management using on-the-run Treasury bonds or bond futures. An investor could modify his portfolio duration dynamically based on the magnitude of the return prediction. The range of durations would depend on the investor's risk aversion level and on his confidence in the proposed strategy. Figure 17 gives an example of how various investor types (aggressive, moderate, conservative) with a neutral benchmark duration of four years could vary their portfolio's target duration with economic conditions. Of course, more conservative implementation would reduce the potential for return enhancement.

Figure 17. Implementing the Strategy: From Return Predictions to Target Portfolio Durations

Predicted Excess Bond Return	Target Duration for an Aggressive Investor	Target Duration for a Moderate Investor	Target Duration for a Conservative Investor
Negative	-2	0	3.0
Low	1	2	3.5
Average	4	4	4.0
High	7	6	4.5
Very High	10	8	5.0

The analysis in this report focuses on the predictability of the excess return of a 20-year U.S. Treasury bond using four predictor variables. Obviously, **the analysis could be extended in various directions:**

- One might improve the forecasts by using **a broader set of predictors** or by combining their information in a more sophisticated way than a simple linear regression. However, our small set of predictors may have more robust forecasting ability in an out-of-sample setting than a broad predictor set would. We have not found other predictors that consistently and significantly improve the forecasting ability of our four-predictor model.

- One can examine the return predictability over **a shorter investment horizon** than one month. The predictors we use above may be too slow-moving for short-term traders. They often prefer to trade either on their fundamental views or on momentum and overreaction effects (price trends and reversals) or on other technical factors (supply effects and portfolio flows). It might be a good idea to subject even these trading approaches to the performance evaluation proposed in this report.

• One can examine the return predictability of **other government bonds**. We show elsewhere that our predictors can also forecast the excess returns of shorter-maturity bills and bonds in the United States and that similar variables forecast international bond returns (see the Literature Guide).

• One can examine **the predictability of the relative performance of various bond market sectors** and markets (changes in the yield spreads across maturities, across market sectors and across countries). In this report, we combine the information in the term spread and other predictors to improve our forecasts of excess bond returns. In a similar way, we could combine the information in mean-reverting yield spreads and in other predictors to develop better relative value indicators. **The tools presented in this report also can be used to evaluate the forecasting performance of various relative value indicators.**

LITERATURE GUIDE

Extensive literature exists discussing the predictability of bond, stock, and currency returns and the dynamic (or "tactical asset allocation") strategies that try to exploit the return predictability. We focus here on studies about bond return predictability.

On the Predictability of U.S. Bond Returns and on the Concept of Time-Varying Risk Premium

Keim and Stambaugh, "Predicting Returns in the Bond and Stock Markets," *Journal of Financial Economics*, 1986.

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Marcus, "An Equilibrium Theory of Excess Volatility and Mean Reversion in Stock Market Prices," National Bureau of Economic Research working paper #3106, 1989.

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Salomon Brothers

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Convexity Bias and the Yield Curve

Understanding the Yield Curve: Part 5

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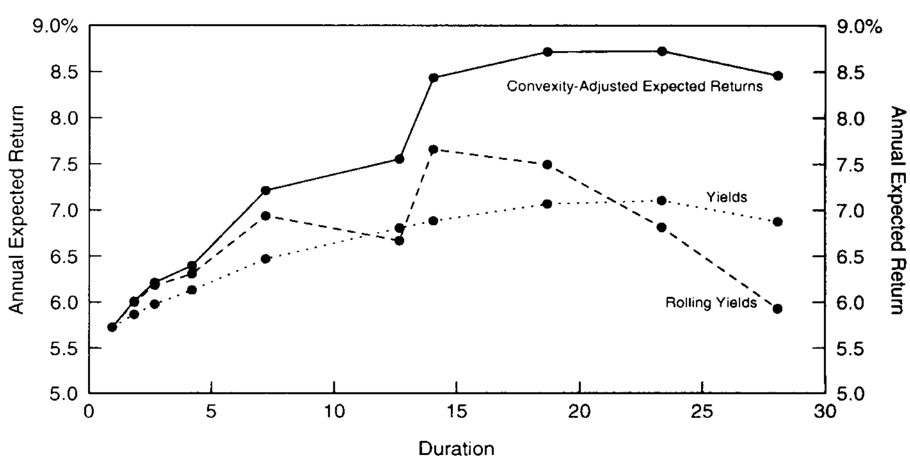
FIGURES

Few fixed-income assets' values are linearly related to interest rate levels; most bonds' price-yield curves exhibit positive or negative convexity. Market participants have long known that positive convexity can enhance a bond portfolio's performance. Therefore, convexity differentials across bonds have a significant effect on the yield curve's shape and on bond returns. This report describes these effects and presents empirical evidence of their importance in the U.S. Treasury market.

For a given level of expected returns, many investors are willing to accept lower yields for more convex bond positions. Long-term bonds are much more convex than short-term bonds because convexity increases very quickly as a function of duration. Because of the value of convexity, long-term bonds can have lower yields than short-term bonds and yet, offer the same near-term expected returns. **Thus, the convexity differentials across bonds tend to make the Treasury yield curve inverted or "humped."** We refer to the impact of such convexity differentials on the yield curve shape as the convexity bias. Our historical analysis shows that the bias is small at the front end of the curve, but it can be quite large at the long end.

Convexity bias can also be viewed from another perspective — the value of convexity as a part of the expected bond return. Widely used relative value tools in the Treasury market, such as yield to maturity and rolling yield, assign no value to convexity. **In this report, we show how yield-based expected return measures can be adjusted to include the value of convexity.** The value of convexity depends crucially on the yield volatility level; the larger the yield shift, the more beneficial positive convexity is. In contrast, the rolling yield is a bond's expected holding-period return given *one* scenario, an unchanged yield curve. Thus, the rolling yield implicitly assumes zero volatility and ignores the value of convexity, making it a downward-biased measure of near-term expected bond return. To counteract this problem, we can simply add up the two sources of expected return. **A bond's convexity-adjusted expected return is equal to the sum of its rolling yield and the value of convexity.** Figure 1 shows that, at long durations, the convexity-adjusted expected returns can be substantially different from the yield-based expected returns. (We describe the construction of this figure further in the report.)

Figure 1. Three Alternative Expected Return Curves, as of 1 Sep 95



Note: Each curve is constructed by connecting ten individual bonds' yields, rolling yields or convexity-adjusted expected returns. The first six points on each curve represent par bonds of 1- to 30-year maturities and the last four points represent zero-coupon bonds of 15- to 30-year maturities, estimated from the Salomon Brothers Treasury Model curve.

In the section "Basics of Convexity," we define convexity, describe how it varies across bonds and discuss the relation between volatility and the value of convexity. We then examine convexity's impact on the yield curve shape and on expected returns and explain why we advocate the use of convexity-adjusted expected returns in the evaluation of duration-neutral barbell-bullet trades. Finally, we present historical evidence about convexity's impact on realized long-term bond returns and on the performance of a barbell-bullet trade.

While this report focuses on convexity's impact on the yield curve (and on bond returns), we stress that the convexity bias is not the only determinant of the yield curve shape. Positive bond risk premia tend to offset the negative impact of convexity, making the yield curve slope upward, at least at short durations. Moreover, the market's expectations about future rate changes can make the yield curve take any shape. This report is the fifth part of a series titled *Understanding the Yield Curve;* earlier reports in this series describe how the market's rate expectations and the required bond risk premia influence the curve shape.

BASICS OF CONVEXITY[1]

What Is Convexity and How Does It Vary Across Treasury Bonds?

Convexity refers to the curvature (nonlinearity) in a bond's price-yield curve. All noncallable bonds exhibit varying degrees of positive convexity. When a price-yield curve is positively convex, a bond's price rises more for a given yield decline than it falls for a similar yield increase. It is often stated that positive convexity can only improve a bond portfolio's performance. Figure 2, which shows the price-yield curve of a 30-year zero, illustrates in what sense this statement is true: A linear approximation of a positively convex curve always lies below the curve. That is, a duration-based approximation of a bond's price change for a given yield change will always understate the bond price. The error is small for small yield changes but large for large yield changes. We can approximate the true price-yield curve much better by adding a quadratic (convexity) term to the linear approximation. Thus, a bond's percentage price change ($100 * \Delta P/P$) for a given yield change is:[2]

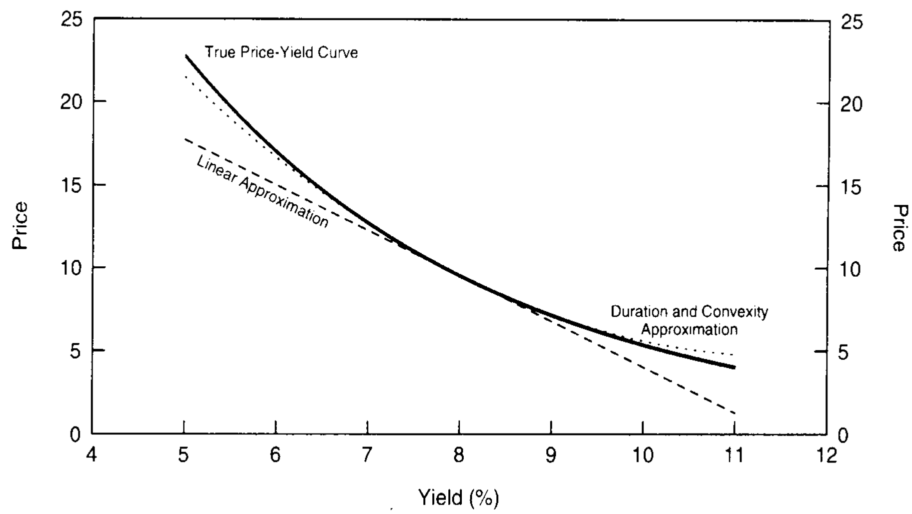
$$100 * \Delta P/P \approx - \text{duration} * \Delta y + 0.5 * \text{convexity} * (\Delta y)^2 \qquad (1)$$

where duration = $-(100/P) * (dP/dy)$, convexity = $(100/P) * (dP^2/dy^2)$, Δy is the yield change, and yields are expressed in percentage terms.

[1] This section provides a brief overview of convexity. Readers who are not familiar with this concept may want to read first a text with a more extensive discussion, such as Klotz (1985) or Tuckman (1995).

[2] Equation (1) is based on a two-term Taylor series expansion of a bond's price as a function of its yield, divided by the price. The Taylor series can be used to approximate the bond price with any desired level of accuracy. A duration-based approximation is based on a one-term Taylor series expansion; it only uses the first derivative of the price function (dP/dy). The two-term Taylor series expansion also uses the second derivative (d2P/dy2) but ignores higher-order terms. In Equation (1), the word "convexity" is used narrowly for the difference between the two-term approximation and the linear approximation, but the word is sometimes used more broadly for the whole difference between the true price-yield curve and the linear approximation. Given the price-yield curves of Treasury bonds and typical yield volatilities in the Treasury market, the two-term approximation in Equation (1) is quite accurate. As an "eyeball test," we note that Figure 2 shows the most nonlinear price-yield curve among noncallable Treasury bonds and yet, the two-term approximation is visually indistinguishable from the true price-yield curve within a 300-basis-point yield range.

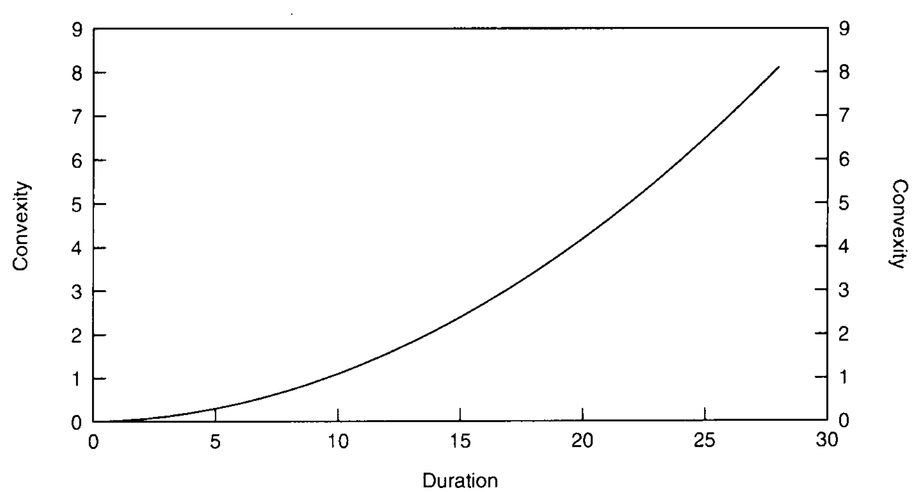
Figure 2. Price-Yield Curve of a 30-Year Zero



In general, **the most important determinants of bond convexity are the option features** attached to bonds. Bonds with embedded short options often exhibit negative convexity. The negative convexity arises because the borrower's call or prepayment option effectively caps the bond's price appreciation potential when yields decline. However, this report does not analyze bonds with option features. **For noncallable bonds, convexity depends on duration and on the dispersion of cash flows** (see Appendix A for details).

Figure 3 shows the convexity of zero-coupon bonds as a function of (modified) duration. Convexity not only increases with duration, but it increases at a rising speed. For zeros, a good rule of thumb is that

Figure 3. Convexity of Zeros as a Function of Duration



convexity equals the square of duration (divided by 100).[3] **Convexity also increases with the dispersion of cash flows.** A barbell portfolio of a short-term zero and a long-term zero has more dispersed cash flows than a duration-matched bullet intermediate-term zero. **Of all bonds with the same duration, a zero has the smallest convexity because it has no cash flow dispersion.** As discussed in Appendix A, a coupon bond's or a portfolio's convexity can be viewed as the sum of a duration-matched zero's convexity and the additional convexity caused by cash flow dispersion.

Volatility and the Value of Convexity

Convexity is valuable because of a basic characteristic of positively convex price-yield curves that we alluded to earlier: A given yield decline raises the bond price more than a yield increase of equal magnitude reduces it. Even if investors know nothing about the direction of rates, they can expect gains to be larger than losses because of the nonlinearity of the price-yield curve. Figure 2 illustrated that convexity has little impact on the bond price if the yield shift is small, but a big impact if the yield shift is large. The more convex the bond and the larger the absolute magnitude of the yield shift, the greater the realized value of convexity is. We do not know in advance how large the realized yield shift will be, but we can measure its expected magnitude with a volatility forecast.[4] **If we expect high near-term yield volatility, we expect a high value of convexity.**

The value of convexity is a nebulous concept; it may be hard for investors to see how higher volatility can increase expected returns. We try to make the concept more concrete and intuitive with the following example. Figure 4 compares the expected value of a 30-year zero in a world of certainty and in a world of uncertainty. In a world of certainty, investors know that a bond's yield will remain unchanged at 8%; thus, there is no volatility and convexity has no value. In the second case, we introduce uncertainty in the simplest possible way: The bond's yield either moves to 10% or to 6% immediately, with equal probability. That is, investors do not know in which direction the rates are moving (on average, they expect no change), but they do know that the rates will shift up or down by 200 basis points. Note that the two possible final bond prices (y = 10%, P = \$5.40 and y = 6%, P = \$17.00) are higher than those implied by a linear approximation. The expected bond price is an average of the two possible final prices: E(P) = 0.5 * \$5.40 + 0.5 * \$17.00 = \$11.20. This expected price is higher than the price given no yield change (y = 8%, P = \$9.50). The \$1.70 price difference reflects the expected value of convexity; the bond's expected price is \$1.70 higher if volatility is 200 basis points than if volatility is 0 basis points. Thus, higher volatility enhances the (expected) performance of positively convex positions.[5]

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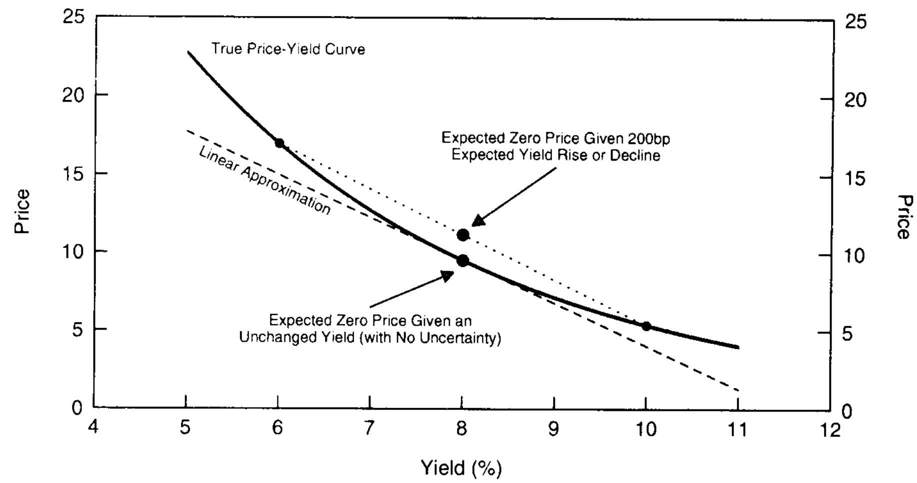
[3] The convexity of a given security can be quoted in many ways, depending, in part, on the way that yields are quoted. If yields are *expressed in percent* (200 basis points = 2%), as in Equation (1), the convexity of a long zero with a duration of 15 is quoted as roughly 2.25 (= 15^2 / 100). However, if yields are *expressed in decimals* (200 basis points = 0.02), the same bond's convexity is quoted as 225 (= 15^2). We decided to use the former method of expressing yields and quoting convexity because it is more common in practice. (For careful readers, we point out that in Appendix A of *Overview of Forward Rate Analysis*, titled "Notation and Definitions Used in the Series Understanding the Yield Curve," we expressed yields in decimals and, thus, used the other quotation method for duration and convexity.) Fortunately, the quotation method does not influence convexity's impact on bond returns. The convexity impact of a 200-basis-point yield change on the long zero's return is approximately 0.5 * convexity * $(\Delta y_{percent})^2$ = 0.5 * 2.25 * 2^2 = 4.5%. We get the same result if the yield change is expressed in percent and convexity is scaled correctly: 0.5 * (100 * convexity) * $(\Delta y_{decimal})^2$ = 0.5 * 225 * 0.02^2 = 0.045 or 4.5%.

[4] Equation (1) shows that the impact of convexity on percentage price changes can be approximated by 0.5 * convexity * $(\Delta y)^2$. The expected value of convexity is, therefore, 0.5 * convexity * $E(\Delta y)^2$. Appendix B shows that $E(\Delta y)^2$ is roughly equal to the squared volatility of basis-point yield changes, $(Vol(\Delta y))^2$.

[5] This example suggests that scenario analysis is one way to incorporate the value of convexity to expected returns. If we compare the average expected bond price from two rate scenarios (+/-2%) to the expected price given one scenario, the difference will be positive for positively convex bonds (if the scenarios are not biased). In reality, more than two possible rate scenarios exist, but the same intuition holds: the expected value of convexity depends on volatility (also if this is computed from 500 yield curve scenarios instead of two).

Figure 4. Value of Convexity in the Price-Yield Curve of a 30-Year Zero



bp Basis points.

The impact of volatility is very clear in the spread behavior between positively and negatively convex bonds (noncallable government bonds versus callable bonds or mortgage-backed securities). It is more subtle in the spread behavior within the government bond market where all bonds exhibit positive convexity. When volatility is high, the yield curve tends to be more humped and is more likely to be inverted at the long end, widening the spreads between duration-matched barbells and bullets and between duration-matched coupon bonds and zeros.

CONVEXITY, YIELD CURVE AND EXPECTED RETURNS

Convexity Bias: The Impact of Convexity on the Curve Shape

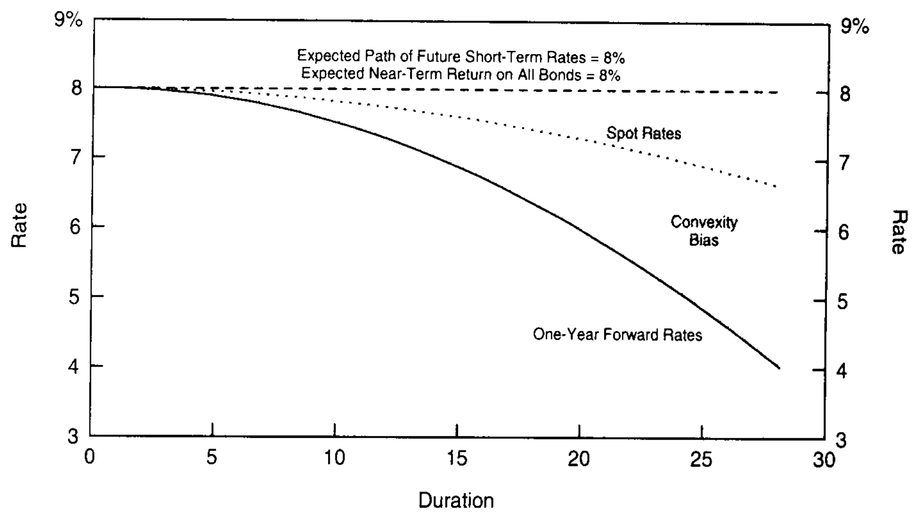
We have demonstrated that positive convexity is a valuable property for a fixed-income asset and that different-maturity bonds exhibit large convexity differences. Now we will show that these convexity differences give rise to offsetting yield differences across maturities. Investors tend to demand less yield for more convex positions because they have the prospect of enhancing their returns as a result of convexity. In particular, Figure 3 showed that long-term bonds exhibit very high convexity. Because of their high convexity, these bonds can offer lower yields than a short-term bond and still offer the same near-term expected returns.

We isolate the impact of convexity on the yield curve shape, or the convexity bias[6], by presenting a hypothetical situation where the other influences on the curve shape are neutral. Specifically, we assume that all bonds have the same expected return (8%) and that the market expects the short-term rates to remain at the current (8%) level, and we examine the behavior of the spot curve and the curve of one-year forward rates. With no bond risk premia and no expected rate changes, one might expect these curves to be horizontal at 8%. Instead, Figure 5 shows that **they slope**

[6] Our use of the term "convexity bias" is slightly different from its use in a recent article "A Question of Bias," by Burghardt and Hoskins, *Risk*, March 1995. In that article, convexity bias refers to the difference between the forward price and the futures price in the Eurodollar market. This bias also reflects varying degrees of curvature in the price-yield curves of different fixed-income assets; the mark-to-market system makes the future's price-yield curve linear, while the forward price is a convex function of yield.

down at an increasing pace because lower yields are needed to offset the convexity advantage of long-duration bonds (and thus to equate the near-term expected returns across bonds). Note the symmetry between the curve shapes in Figures 3 and 5.

Figure 5. Pure Impact of Convexity on the Yield Curve Shape



Note: Convexity bias is the difference between the curve of one-year forward rates and the expected return curve. Formally, Convexity bias ≈ -0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of a one-year horizon. The assumed yield volatility is 100 basis points per annum for all bonds; that is $\mathrm{Vol}(\Delta y) = 1\%$.

Where did the numbers in Figure 5 come from? Unlike the real world, where the spot rates are the easiest to observe, in this example, we take the expected returns as given and work our way back to forward rates and then to spot rates. Given our assumption that the market has no directional views about the yield curve, each zero earns the near-term expected return from the rolling yield[7] and from convexity:[8]

Convexity-adjusted expected return = rolling yield + value of convexity, (2)

where value of convexity ≈ 0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$.

Using our assumption that all bonds have convexity-adjusted expected return of 8% and using some volatility assumption (which determines the value of convexity), we can back out the rolling yields for various-maturity zeros from Equation (2). Our volatility assumption of 100 basis points means roughly that we expect all rates to move 100 basis points (up or down) from their current level over the next year. For example, if the convexity of a long zero is 2.25 (see footnote 3), the value of convexity is

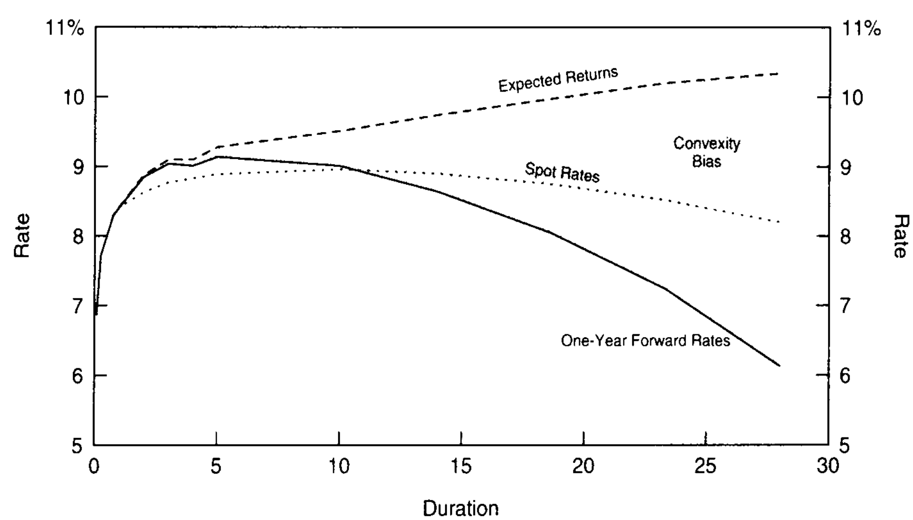
[7] The rolling yield is a bond's holding-period return given an unchanged yield curve. If a downward-sloping yield curve remains unchanged, long-term bonds earn their initial yields and negative rolldown returns (because they "roll up the curve" as their maturities shorten). An n-year zero-coupon bond's rolling yield over the next year is equal to the one-year forward rate between n-1 and n. For details, see *Market's Rate Expectations and Forward Rates*, Salomon Brothers Inc, June 1995.

[8] Here is an intuitive "proof." A bond's expected holding-period return can be split into *a part that reflects an unchanged yield curve* (the rolling yield) and *a part that reflects expected changes in the yield curve*. The second part can be approximated by taking expectations of Equation (1). If we expect the yield curve to remain unchanged, as a base case, but allow for positive volatility, the duration impact will be zero, leaving only the value of convexity. (Some modifications are needed because Equation (1) holds instantaneously for constant-maturity rates, while the actual bond price changes occur over a horizon.)

approximately 0.5 * 2.25 * 1^2 = 1.125%. The zero's rolling yield is 6.875% but its annualized near-term expected return is 8%, by assumption. For coupon bonds, which have smaller convexities, the value of convexity is much smaller. The final step in constructing Figure 5 is to compute the spot curve from the curve of one-year forward rates (the rolling yield curve).

Convexity bias is simply the inverse of the value of convexity, or -0.5 * convexity * $(\text{Vol}(\Delta y))^2$. Figure 5 shows that the convexity bias, by itself, tends to make the yield curve inverted, especially at long durations. However, actual yield curves rarely invert as they do in this hypothetical example, in which we assumed, in particular, that all bonds across the curve have the same near-term expected return and the same basis-point yield volatility. We now relax each of these two assumptions, one at a time. First, convexity is not the only influence on the curve shape. The typical historical yield curve shape is upward sloping, probably reflecting positive bond risk premia (the fact that investors require higher expected returns for long-term bonds than for short-term bonds). At the front end of the curve, the convexity bias is so small that it does not offset the impact of positive bond risk premia. At the long end, the convexity bias can be so large that the yield curve becomes inverted in spite of positive risk premia. Figure 6 shows that **in the presence of positive risk premia, convexity bias tends to make the yield curve humped rather than inverted**. In this figure, we use historical average returns of various maturity subsectors to proxy for expected returns.

Figure 6. Impact of Convexity with Positive Bond Risk Premia

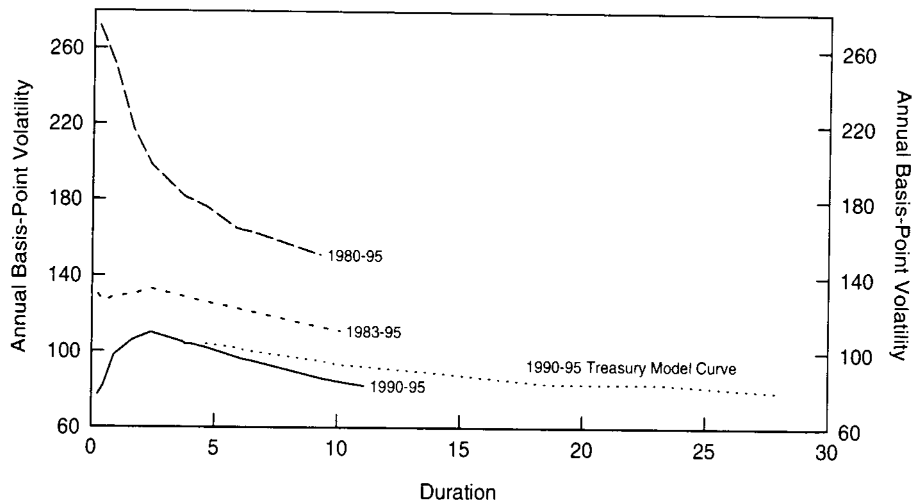


Note: The figure is constructed in the same way as Figure 5 except that all bonds' expected returns are not 8% but are based on the (arithmetic) mean realized returns of Treasury bond maturity-subsectors between 1970 and 1994. The curve is extrapolated between ten- and 30-year durations because of a lack of data. The curve of one-year forward rates is computed by adding the convexity bias from Figure 5 to the expected return curve. The spot curve is computed from the curve of one-year forward rates.

As explained earlier, the value of convexity increases with yield volatility. **Thus far we have assumed that yield volatility is equally high across the curve. Figure 7 shows that historically, the term structure of volatility has often been inverted** — long-term rates have been less volatile than short-term rates. Therefore, the value of convexity does not increase quite as a square of duration even though convexity itself does.

However, **the value of convexity does increase quite quickly with duration even when the volatility term structure is taken into account;** its inversion only dampens the rate of increase (see Figure 8).

Figure 7. Historical Term Structure of (Basis-Point) Yield Volatility

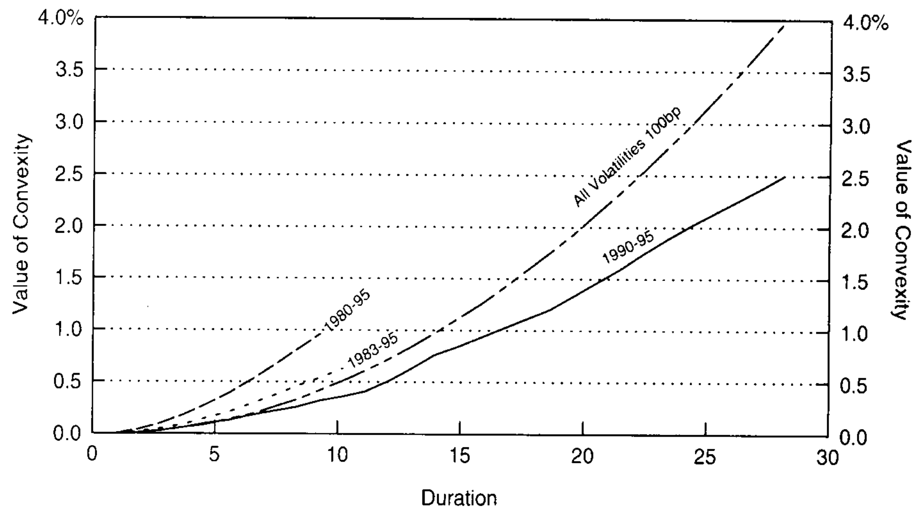


Note: Historical yield volatility is the annualized standard deviation of weekly basis-point yield changes. Yield volatilities are computed for several on-the-run Treasury bill and bond series (and plotted on their average durations) over three sample periods. In addition, yield volatilities are computed for the five-, ten-, 15-, 20-, 25-, and 30-year points on the Salomon Brothers Treasury Model spot curve over the January 1990-August 1995 period. Yields are compounded annually and the yield volatilities are expressed in basis points.

The levels and shapes of the volatility term structures are very different in Figure 7, depending on the sample period. In the 1980s — and especially at the beginning of the decade — yield volatilities were very high and the term structure of volatility was inverted. In the 1990s, volatilities have been lower and the term structure of volatility has been flat or humped. It is difficult to choose the appropriate sample period for computing the yield volatility and Figure 8 shows that this choice will have a significant impact on the estimated value of convexity. Our view is that the relevant choice is between the 1983-95 and the 1990-95 sample periods because we do not expect to see again the volatility levels experienced in 1979-82 — at least not without clear warning signs. This period coincided with a different monetary policy regime in which the Federal Reserve targeted the money supply and tolerated much higher yield volatility than after October 1982.[9]

[9] Whenever the period 1979-82 is included in a historical sample, the estimated volatilities will be much higher, the term structures of volatility will be more inverted and basis-point yield volatilities will appear to be more "level-dependent" than if the sample period begins after 1982. In many countries outside the United States, the inversion and the level-dependency also have been apparent features of the volatility structure recently. These features seem to become stronger if the central bank subordinates the short-term rates to be tools for some other monetary policy goal, such as money supply (United States 1979-82) or currency stability (for example, countries in the European Monetary System). Figure 7 also illustrates interesting findings about the term structure of volatility in the 1990s. The shape is humped, not inverted, because the intermediate-term yields have been more volatile than either the short-term or long-term yields. Moreover, yield volatility is not just a function of duration; it also depends on a bond's cash flow distribution. *For a given duration*, zeros have exhibited greater yield volatility than coupon bonds. This pattern probably reflects the coupon bonds' diversification benefits (unlike zeros, these bonds have cash flows in many parts of the yield curve that are imperfectly correlated) as well as the humped shape of the volatility structure.

Figure 8. Value of Convexity Given Various Volatility Structures



bp Basis points.
Note: Value of Convexity ≈ 0.5 * convexity * $(\text{Vol}(\Delta y))^2$, expressed in percent per annum, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of a one-year horizon. Yield volatilities are based on Figure 7. Because we only have spot rate data for the 1990s, we cannot compute long zeros' value of convexity for the two longer samples.

Instead of sample-specific historical volatilities, we could use implied volatilities from current option prices (based on the cap-curve, options on various futures contracts, OTC options on individual on-the-run bonds) to compute the (expected) value of convexity. The main reason that we have not done this is that such implied volatilities are not available for all maturities. In addition, it is not clear from empirical evidence that implied volatilities predict future yield volatilities any better than historical volatilities do.

In Appendix B, we describe the various volatility measures used in this report and discuss the relations between them. In particular, we emphasize that the option prices are typically quoted in relative yield volatilities ($\text{Vol}(\Delta y/y)$) rather than in the basis-point volatilities ($\text{Vol}(\Delta y)$) that we use. For example, a 13% implied volatility quote has to be multiplied by the yield level, say 7%, to get the basis-point volatility (91 basis points = 0.91% = 13% * 7%).

The Impact of Convexity on Expected Bond Returns

Figure 8 shows that positive convexity can be quite valuable, especially in a high-volatility environment. However, yield-based measures of expected bond return assign no value to convexity. For example, the rolling yield is a bond's holding-period return given *one* scenario (an unchanged yield curve), essentially assuming no rate uncertainty. **Because volatility can only be positive, the rolling yield is a downward-biased measure of expected return** for bonds with positive convexity.[10] Fortunately, it is possible to add the impact of rate uncertainty (the expected value of convexity) to rolling yields. Equation (2) showed that if the base case expectation is an unchanged yield curve, **a bond's near-term expected**

[10] This point is most easily seen by considering a horizontal yield curve. All bonds have same yields and rolling yields, but their expected returns are not the same. Long-term bonds are more convex than short-term bonds; thus, they have higher near-term expected returns.

return is simply the sum of the rolling yield and the value of convexity.[11] This relation holds approximately for coupon bonds as well as for zeros.

In Figure 9, we calculate three expected return measures (yield, rolling yield, convexity-adjusted expected return) and the value of convexity on September 1, 1995 for six Treasury par bonds and four long-duration zeros (estimated from the Salomon Brothers Treasury Model curve which represents off-the-run bonds). In addition, we describe two barbell positions that can be compared with duration-matched bullets. Figure 1 showed graphically the three alternative expected return curves as a function of duration.

Figure 9. Expected One-Year Returns on Various Bonds as of 1 Sep 95

	(Modified) Duration	Convexity	Historical Vol(Δy)	(Annual) Yield	Rolling Yield	Value of Convexity	Conv.-Adjusted Expected Return
Par Bonds							
1 Year	0.95	0.02	0.98%	5.73%	5.73%	**0.00%**	5.73%
2 Year	1.84	0.05	1.06	5.87	6.00	**0.01**	6.01
3 Year	2.67	0.10	1.10	5.98	6.18	**0.03**	6.21
5 Year	4.20	0.23	1.04	6.13	6.31	**0.09**	6.40
10 Year	7.20	0.67	0.95	6.47	6.94	**0.27**	7.21
30 Year	12.66	2.57	0.82	6.81	6.67	**0.88**	7.55
Long Zeros							
15 Year	14.03	2.10	0.89%	6.88%	7.66%	**0.78%**	8.44%
20 Year	18.68	3.66	0.83	7.07	7.49	**1.22**	8.71
25 Year	23.34	5.67	0.83	7.11	6.81	**1.91**	8.72
30 Year	28.07	8.14	0.79	6.88	5.93	**2.53**	8.46
Par Barbells							
1 Year and 10 Year	4.19	0.36	0.95%	6.11%	6.35%	**0.14%**	6.50%
1 Year and 30 Year	7.18	1.38	0.82	6.30	6.23	**0.47**	6.70

Note: Convexity-adjusted expected return = rolling yield + value of convexity, where rolling yield = yield + rolldown return and where value of convexity = 0.5 * convexity * $(\text{Vol}(\Delta y))^2$, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of the one-year horizon. Historical volatilities are the annualized standard deviations of weekly basis-point yield changes between January 1990 and August 1995. All measures use annually compounded yields and are expressed in percentage terms. The first (second) barbell is a combination of the one-year par bond and the ten-year (30-year) par bond, duration-matched to the end-of-horizon duration of the five-year (ten-year) par bond; thus, the current durations are not exactly matched. All other measures for the barbells are market value-weighted averages, but the barbell's yield volatility is market value * duration-weighted.

We use maturity-specific historical volatilities from the 1990-95 period to proxy for expected volatility, and we use a one-year horizon. These choices give one illustration of the ideas developed in this report; we stress that it is possible to use other volatility measures or other horizon. In particular, Figure 7 shows that the volatility estimates would be much higher if we extended our sample period to the 1980s. (The par bonds' yield volatilities are similar to those of the on-the-run bonds in Figure 7.) For a given yield curve, these higher volatility estimates could more than double the estimated value of convexity and, thus, increase the convexity-adjusted expected returns. Using a one-year horizon makes the notation easier because the value of convexity is expressed in annualized terms as are yields and volatilities. If we used a three-month horizon, all three expected return measures and the value of convexity would be roughly one fourth of the numbers in Figure 9. For example, if a 30-year par bond's convexity is 2.57 and the annual volatility is 82 basis points, the quarterly volatility is approximately 41 basis points ($82/\sqrt{4}$), and the quarterly value of convexity is $0.5 * 2.57 * 0.41^2 = 0.22\%$ ($\approx 0.88\%/4$), or 22 basis points.

[11] Our empirical analysis in *Market's Rate Expectations and Forward Rates* indicates that it is reasonable to take today's yield curve as the base forecast for the future yield curve. Therefore, the rolling yield can proxy for a bond's near-term expected return (assuming zero volatility). Other hypotheses about the yield curve behavior would lead to other expected return proxies than the rolling yield, but the value of convexity could be added to any such proxy. For example, if the implied forward curve were the best forecast for the future yield curve, the near-term expected return of each bond would be the sum of the near-term riskless rate and the (bond-specific) value of convexity. Or, if investors have strong subjective expectations about curve-reshaping, the impact of such expectations can be easily added to the convexity-adjusted expected returns — as a third term on the right-hand side of Equation (2).

Figures 1 and 9 show that **the convexity adjustment has little impact at short durations because short-term bonds exhibit little convexity**. Even for the longest coupon bond, the annual impact is 88 basis points. **In contrast, for the longest zeros, the value of convexity is very large** both as an absolute number (253 basis points) and as a proportion of their expected return (30% = 2.53/8.46). More generally, the value of convexity can partly explain the rolling yield curve's typical concave (humped) shape, but even the convexity-adjusted expected return curve inverts after 25 years. The longest-maturity zeros appear to have genuinely low expected returns, perhaps reflecting their liquidity advantage and financing advantage.

One advantage of this analysis is that it gives an **improved view of the overall reward-risk trade-off in the government bond market**. Until the 1970s, fixed-income investors evaluated this reward - risk trade-off by plotting bond yields on their maturities. Eventually investors learned that the rolling yield measures near-term expected return better than yield and that duration measures risk better than maturity.[12] In the mid-1980s, investors became familiar with the concept of convexity (see Literature Guide), although few have incorporated it formally into their expected return measures. However, convexity-adjusted expected returns are even better expected return measures than rolling yields — and the adjustment is reasonably simple. To move all the way to mean-variance analysis, as advocated by the modern portfolio theory, we should adjust bond durations by their yield volatilities; then, Figure 1 would plot bonds' expected returns on their return volatilities. Of course, convexity-adjusted expected returns are not perfect; for example, if investors can predict yield curve reshapings consistently, they can construct even better expected return measures.

In addition, our analysis helps investors to interpret varying yield curve shapes, and more directly, it gives them tools to evaluate relative value trades between duration-matched barbells and bullets and between duration-matched coupon bonds and zeros. This is the topic of the next subsection.

Applications to Barbell-Bullet Analysis

A barbell-bullet trade involves the sale of an intermediate bullet bond and the purchase of a barbell portfolio of a short-term bond and a long-term bond. Often the trade is weighted so that it is cash-neutral and duration-neutral; that is, one unit of the intermediate bond is sold, a duration-weighted amount of the long bond is bought and the remaining proceeds from the sale are put into "cash" (a short-term bond that matures at the end of horizon). For simplicity, we will only study such barbells in this report. In Appendix A, we explain that a barbell portfolio has a convexity advantage over a duration-matched bullet because the barbell's duration varies more (inversely) with the yield level. Figure 3 provides another illustration of the convexity difference between barbells and bullets. If we draw a straight line between any two points on the zeros' convexity-duration curve, each point on this line corresponds to a barbell portfolio (with varying weights of the long-term and the short-term zero). The convexity of this barbell is the market-value-weighted average of the component bonds' convexities. Because the connecting straight line always

[12] *Total Return Management*, Martin L. Leibowitz, Salomon Brothers Inc, 1979, and *Understanding Duration and Volatility*, Salomon Brothers Inc, September 1985, among other papers, made the concepts of rolling yield and duration widely known among bond investors.

lies above the zeros' convexity-duration curve, the barbell's convexity is always higher than that of a duration-matched bullet. Furthermore, the maximum convexity pick-up for any duration occurs when we connect the shortest and longest zeros.

In a similar way, we can connect any two points in Figure 5 and find that the rolling yield of any barbell is below the rolling yield of a duration-matched bullet. More generally, the rolling yield curve (as well as the yield curve) almost always has a concave shape as a function of duration; that is, the curve increases at a decreasing rate or decreases at an increasing rate. Therefore, a rolling yield disadvantage tends to offset the convexity advantage of a barbell-bullet trade. **If an investor wants to evaluate the relative cheapness of a barbell-bullet trade, he needs to compare two numbers, the rolling yield give-up and the convexity pick-up. The advantage of the convexity-adjusted expected return is that it provides a single number to measure the attractiveness of these trades.** For example, the ones-30s barbell in Figure 9 has a 71-basis-point rolling yield give-up relative to the ten-year bullet (= 6.23%-6.94%), but how does this give-up compare with the convexity pick-up (1.38 versus 0.67)? The numbers in the last column show that the barbell still has a 51-basis-point give-up (= 6.70%-7.21%) when measured in terms of convexity-adjusted expected returns and given our volatility forecasts. Incidentally, the shorter barbell in Figure 9 even picks up rolling yield over the duration-matched five-year bullet; this exceptional situation reflects the convex shape in parts of the rolling yield curve in Figure 1.

The performance of a duration-neutral barbell-bullet trade depends on curve reshaping, on parallel curve shifts and on the initial yields: (1) The trade profits from curve flattening and loses from curve steepening (between the two longer bonds); (2) the trade is constructed to be neutral to small parallel curve shifts, but the barbell profits from large shifts in either direction because of its convexity advantage; and (3) the initial rolling yield give-up is greater the more curved (concave) the yield curve is. Such a shape may be caused by the market's expectations of curve flattening or of high volatility, either of which would generate capital gains for the trade in the future.

Typical barbell-bullet trades are more curve flattening trades than convexity trades. The following break-even analysis illustrates this point. Consider the long barbell-bullet trade in Figure 9. It consists of selling a ten-year par bond (rolling yield 6.94%) and buying a barbell of the 30-year par bond (rolling yield 6.67%) and the one-year bond (rolling yield 5.73%), with a one-year investment horizon. Thus, at the end of horizon, the components will be a nine-year bond, a 29-year bond and cash. The constraints that the trade is duration-neutral and cash-neutral require weights 0.53 and 0.47 for the long bond and the short bond. Given the duration-neutral weighting of the barbell, the rolling yield give-up is 71 basis points (= 0.53 * 6.67% + 0.47 * 5.73% - 6.94%). We isolate the flattening and convexity effects in the trade by asking two questions:

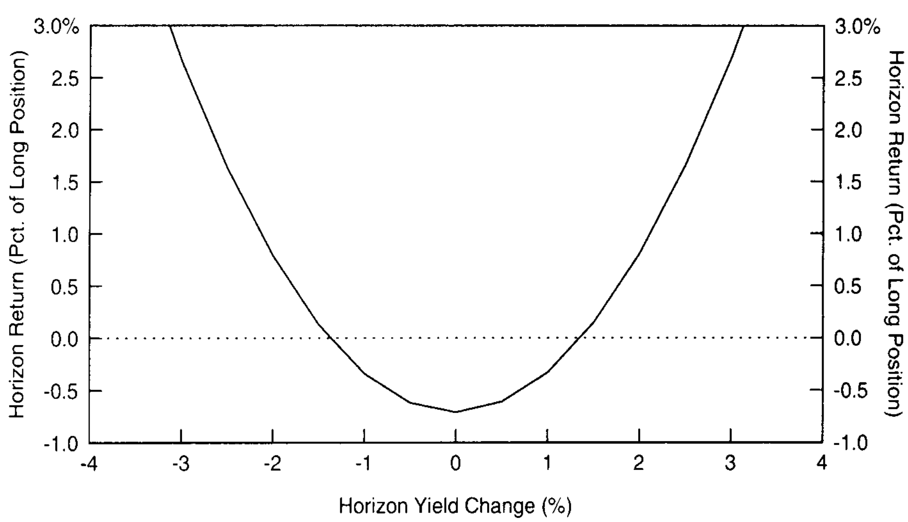
- How much would the yield spread between tens and 30s (or more exactly, between nines and 29s at the end of horizon) have to narrow to offset this give-up, if no parallel shifts occur?

- How large must the parallel shifts be to make the convexity advantage offset this give-up, if no curve reshaping occurs?

A little math shows that the necessary break-even changes are an 11-basis-point spread narrowing (curve flattening) and a 138-basis-point parallel shift. Historical experience suggests that the former event is more plausible than the latter: Over the past 15 years, the tens-30s spread narrowed by at least 11 basis points in a year 30% of the time, while the ten-year yield level shifted by more than 138 basis points in a year only 17% of the time. Thus, **it is more likely that a given rolling yield disadvantage is offset via curve flattening than via the barbell's convexity advantage**. However, the relative roles of curve-reshaping and convexity vary across different barbell-bullet trades. The reshaping effects are clearly more important at shorter durations (between most coupon bonds), while convexity can be more important at longer durations (between very long zeros). It follows that the time-variation in the rolling yield spread between barbell and bullet coupon bonds— or in the yield curve curvature below the ten-year duration — depends more on the market's changing expectations about future curve flattening/steepening than on its changing volatility expectations.

The convexity aspect of the previous example illustrates **the similarity between a barbell-bullet trade and a purchase of a long option straddle** (a purchase of a call and a put with the same strike price and exercise date). Figure 10 shows the almost U-shaped pattern that is familiar from option analysis. The rolling-yield disadvantage corresponds to the long call and put positions' initial cost (premium), which large market movements in either direction would offset. The trade would only be profitable if the yield level increased or declined by at least 138 basis points, assuming

Figure 10. The Payoff Profile of a Barbell-Bullet Trade, Assuming Parallel Yield Shifts



parallel yield shifts. If the yield curve does not move at all from the initial level, the maximum loss (71 basis points) occurs. Of course, Figure 10 ignores the substantial curve-reshaping risk in this trade.[13]

Another way to measure the cheapness of the barbell-bullet trade is to compute its implied yield volatility and compare it with the implied volatility in option markets. We can back out an implied volatility number for each barbell-bullet trade based on the observable rolling yield spread and convexity difference, if we assume that the duration-matched barbell and bullet earn the same expected returns and that the rolling yield spread reflects only the value of convexity — and no curve-flattening expectations.[14] In that case, high curvature (concavity) in the yield curve and high bullet-barbell rolling yield spreads indicate high implied volatility. In contrast, if the yield curve is a convex function of duration, barbells pick up yield *and* convexity and the implied volatility is negative — typically an indication of the market's strong expectations about near-term curve steepening.

HISTORICAL EVIDENCE ABOUT CONVEXITY AND BOND RETURNS

The intuition behind convexity-adjusted expected returns is that if investors care about expected return rather than yield, they will rationally accept lower yields and rolling yields from more convex bonds. In this sense, convexity is priced: It influences bond yields. However, **a more subtle question is whether convexity also influences expected returns** that are not directly observable. It is possible that the rolling yield disadvantage exactly offsets convexity advantage so that two bond positions with the same duration but different convexities have the same near-term expected return. It is also possible that convexity is such a desirable characteristic — because of the insurance-type payoff pattern — that the market (investors in the aggregate) accepts lower expected returns for more convex bonds. Finally, it is possible that current-income seekers dominate the marketplace, leading to a price premium (lower expected returns) for higher-yielding, less convex bonds. The jury is still out on this question. The evidence from historical bond returns that we present below suggests that more convex positions earn somewhat lower returns in the long run than less convex positions.

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[13] The barbell-bullet trade that we analyze over a static one-year horizon is comparable to a strategy of buying and holding a straddle. Readers familiar with options know that the profitability of this strategy depends solely on the starting and ending yield levels and not on the yield path during the horizon. Option traders may use this strategy if they expect yields to end up far away from the current levels. It is useful to contrast this option strategy with another option strategy: buying a delta-hedged straddle and rebalancing the position dynamically throughout the horizon. The profitability of this strategy depends on the level of volatility (yield path) during the horizon and not on the ending yield level. Option traders initiate this strategy (and "go long volatility") when they think that the current implied volatility is "too low." If the realized volatility turns out to be higher than the initial implied volatility, the trade makes money from profitable rebalancing trades — even if the ending yield is the same as the starting yield. These two option positions are analogous to two types of barbell-bullet strategies. In the first type (our example), the barbell and the bullet are duration-matched to horizon and no rebalancing occurs. In the second type, the trade is duration-matched instantaneously and the match is rebalanced frequently. The appropriate strategy in a given situation depends on several factors, including the following: (1) whether the investor has a particular view about the likely horizon yields (for example, "far away from the current level") or about the implied volatility during the horizon; (2) whether the investor tolerates some duration drift (because in the first case, the duration would drift during the year) or has a strict duration target; and (3) whether the investor expects rates to be mean-reverting (in which case he may want to rebalance and lock in the convexity gains after significant rate movements).

[14] The assumption of no curve-flattening expectations is realistic when describing the long-run average behavior of the yield curve, but may be unrealistic at times, especially if the Fed has recently begun easing or tightening. Because the performance of the barbell-bullet trade depends more on the curve reshaping than on convexity effects, curvature (the rolling yield spread between a barbell and a bullet) provides very noisy implied volatility estimates. Thus, it might be more useful to try to extract the market's curve-flattening expectations from the curvature by subtracting the value of convexity (based on, say, the implied volatility from option prices) from the rolling yield spread.

In this final section, we examine the historical performance of a long-term bond position and of a wide barbell-bullet position between January 1980 and December 1994, focusing on the impact of convexity on realized returns. The first strategy involves always investing in the on-the-run 30-year Treasury bond; this strategy is long convexity by holding a long-duration bond. The second strategy involves rolling over a fives-thirties flattening trade each month. Specifically, we sell short the on-the-run five-year Treasury bond each month and buy a barbell of the 30-year bond and one-month bill. The trade is duration-matched to horizon; that is, the weight of the 30-year bond in the barbell is such that the barbell and the bullet have the same expected duration at the end of the month. A little algebra shows that the weight is the ratio of the five-year bond's duration to the 30-year bond's duration (at horizon). Although the trade is cash-neutral and duration-neutral, it is long convexity because a barbell is more convex than a bullet.

We first show some summary statistics of various bond positions in Figure 11 but focus on the last two columns. **The bullet has roughly a 100-basis-point higher average return and average yield than the duration-matched barbell.**[15] Thus, the barbell's convexity pick-up (0.69 versus 0.19) and the impact of yield curve reshaping do not offset its initial yield give-up. However, the barbell does have clearly lower return volatility than the bullet, reflecting the lower yield volatility of the 30-year bond than the five-year bond.

Figure 11. Description of Various On-the-Run Bond Positions, 1980-94

	1-Month	30-Year	5-Year "Bullet"	"Barbell"
Average Return	7.21%	10.34%	9.75%	8.69%
Volatility of Return	0.91	12.84	7.02	5.43
Average Yield	7.34	9.61	9.20	8.22
Volatility of Yield Change	3.46	1.44	1.88	1.44
Average Duration	0.08	9.87	3.91	3.91
Average Convexity	0.00	1.79	0.19	0.69

Note: Average returns are simply annualized by *12 and volatilities by * $\sqrt{12}$. The barbell is a combination of the one-month bill and the 30-year bond, duration-matched each month to the end-of-month duration of the five-year bond. All other measures for the barbell are market value-weighted averages, but the barbell's yield volatility is market value * duration-weighted.

We can decompose any bond's holding-period return into four parts: the yield impact; the duration impact; the convexity impact; and a residual term. Recall from Equation (1) that duration and convexity effects can approximate a bond's instantaneous return well. Over time, a bond also earns some income from coupons or from price accrual; we estimate this income from a bond's yield. Thus, we approximate a bond's holding-period return by Equation (3).[16] The difference between the actual return and its three-term approximation is the residual term; if the approximation is good, the residual should be relatively small. We split the

[15] The bullet's outperformance is consistent with the finding in Part 3 of this series, *Does Duration Extension Enhance Long-Term Expected Returns?*, that historical average returns do not increase linearly with duration. Instead, the average return curve is concave, indicating that the intermediate-term bonds earn higher average returns than duration-matched pairs of short-term bonds and long-term bonds.

[16] Why is the first term on the right-hand side of Equation (3) yield and not rolling yield? Equation (3) is the correct way to approximate a bond's holding-period return when we study actual bond-specific yield changes (which can be viewed as the sum of the rolldown yield changes and the changes in constant-maturity rates). In this case, the rolldown return is a part of the duration and convexity impact. Alternatively, if we studied in Equation (3) the changes in constant-maturity rates (which do not include the rolldown yield change), we should include the rolldown return into the first term on the right-hand side; it would be rolling yield instead of yield.

30-year bond's monthly returns to four components and describe the average behavior and volatility of each component in the top panel of Figure 12.[17]

$$\text{Return} \approx \text{yield impact} - \text{duration} * \Delta y + 0.5 * \text{convexity} * (\Delta y)^2. \qquad (3)$$

The return volatility numbers in the top panel of Figure 12 show that in any given month, the duration impact largely drives the long bond's return — it is the source behind 99% of the monthly return fluctuations. However, yield increases and decreases tend to offset each other over time, having little impact on long-term average returns.[18] Over our 15-year sample period, the long bond's average return reflects more the average yield (91%) and less the convexity (14%) and duration (-5%) effects. The residual term has a small mean and volatility, indicating that the approximation in Equation (3) works well. Subperiod analysis shows that over three-year horizons, the duration effect can still have a significant positive or negative impact — the 1983-85 and 1989-91 subperiods were clearly bull markets and the three other subperiods were bear markets. In contrast, the yield and convexity effects are always positive (by construction). The convexity impact was largest in the early 1980s when yield volatility was very high. **During the whole sample, the annualized convexity impact was 148 basis points. In the 1990s, it was about half of that.**

Similarly, we can split the five-year bullet's and the duration-matched barbell's monthly returns into four components based on Equation (3). The lower panel of Figure 12 describes the average behavior and volatility of their difference, which can be viewed as a duration-matched and cash-neutral barbell-bullet trade. Again, the volatility numbers show that most of the monthly fluctuations (99%) come from the duration impact. The trade is duration-neutral; thus, the duration impact refers to the capital gains or losses caused by curve reshaping. That is, although $Dur_{Barbell} = Dur_{Bullet}$, the duration impacts of the barbell and the bullet differ unless the yield changes are parallel ($-Dur_{Barbell} * \Delta y_{Barbell} \neq -Dur_{Bullet} * \Delta y_{Bullet}$). Over the whole sample, these effects tend to cancel out, and the average return depends largely (90%) on initial yields. The barbell has a 105-basis-point lower average annual return than the bullet, mainly because of its yield disadvantage (-95 basis points) and partly due to losses caused by the curve steepening (-36 basis points); these are only partly offset by the barbell's convexity advantage (30 basis points). **In four out of five subperiods, the bullet outperformed the barbell, suggesting that a barbell's convexity advantage is rarely sufficient to offset the negative**

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[17] The percentage contributions of average returns in Figure 12 add up to 100% because we use an approximate method of annualizing monthly returns (multiplying by 12). In contrast, the percentage contributions of volatilities do not add up to 100% because volatilities are not additive (whether annualized or not).

[18] A careful reader may find it puzzling that the average duration impact on bond returns is negative over a sample period when the bond yields declined, on average. There are two explanations. First, the duration impact is a product of duration and yield changes, and it turns out that yield declines (from high yield levels) tended to coincide with relatively short durations, while yield increases (from low yield levels) tended to coincide with long durations. Thus, yield increases are "weighted" more heavily than yield declines. Second, historical yield changes that are based on a time series of on-the-run yield levels can be misleading because they ignore the impact of changing on-the-run bonds. For example, if a new bond is issued on August 15, the on-the-run yield change from July 31 to August 31 compares the yields of different bonds, the old one and the new one. Typically, the old bond loses some of its liquidity premium; thus, its end-of-month yield tends to be higher than that of the new bond — a pattern hidden in the on-the-run yield level series. For the analysis in Figure 12, we create a clean series of yield changes that always compares the beginning- and end-of-month yields of one bond. The average monthly yield change in the clean series is one basis point higher than in the unadjusted series.

carry over a multiyear period.[19] In addition, the impact of curve-reshaping is larger, in absolute magnitude, than the convexity impact in each subperiod. Again, the residual has a small mean and volatility; thus, the approximation in Equation (3) appears to work well.

Figure 12. Decomposing Returns to Yield, Duration and Convexity Effects

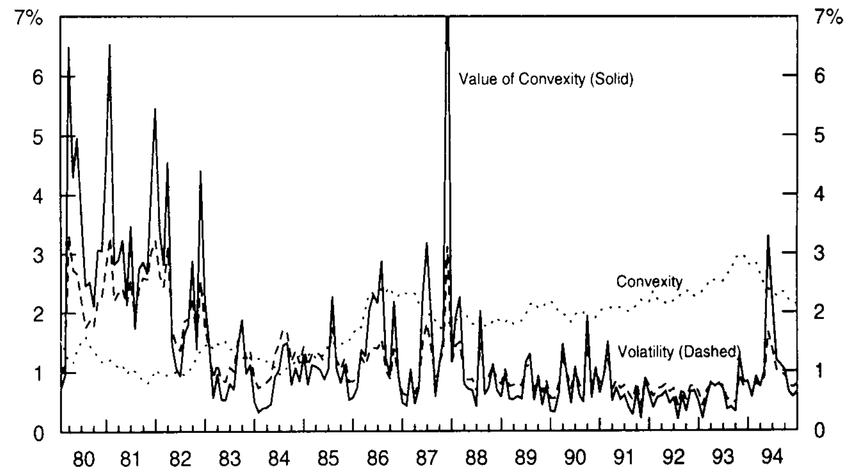
1980-94	Total Return	Yield Impact	Duration Impact	Convexity Impact	Residual
30-Year Bond's Monthly Returns					
Average Return	10.34%	9.41%	-0.49%	**1.48%**	-0.06%
Volatility of Return	12.84	0.61	12.66	0.61	0.11
Pct. of Average Return	100	**91**	-5	14	-1
Pct. of Volatility of Return	100	5	**99**	5	1
Subperiod Average Returns					
1980-82	11.19%	12.21%	-3.70%	**2.83%**	-0.15%
1983-85	14.83	11.22	2.24	**1.44**	-0.07
1986-88	8.16	8.27	-1.73	**1.65**	-0.03
1989-91	13.57	8.26	4.56	**0.80**	-0.04
1992-94	3.94	7.10	-3.81	**0.67**	-0.02
Barbell-Bullet Trade's Monthly Returns					
Average Return	**-1.05%**	**-0.95%**	**-0.36%**	**0.30%**	**-0.04%**
Volatility of Return	2.96	0.19	2.93	0.18	0.09
Pct. of Average Return	100	**90**	34	-28	4
Pct. of Volatility of Return	100	6	**99**	6	3
Subperiod Average Returns					
1980-82	**-1.44%**	-0.66%	-1.10%	**0.42%**	-0.09%
1983-85	**-1.86**	-1.21	-0.94	**0.39**	-0.10
1986-88	**0.58**	-0.97	1.12	**0.42**	0.01
1989-91	**-2.39**	-0.65	-1.90	**0.17**	-0.01
1992-94	**-0.14**	-1.25	1.02	**0.10**	0.00

Note: For the bond, the returns and their components are raw returns. For the barbell-bullet, these figures are return differences between the barbell and the duration-matched five-year bullet. Averages of total returns and their components are simply annualized by *12 and expressed in percent; volatilities are annualized by * $\sqrt{12}$. Yield impact is the return from yield (where the barbell's yield is market value-weighted). Duration impact is -Duration-at-horizon * Δy, where the yield change for the barbell is market value * duration-weighted. Convexity impact is 0.5 * convexity-at-horizon * $(\Delta y)^2$. Residual is the difference between the total return and the three components (yield impact, duration impact and convexity impact).

Figure 12 describes the impact of convexity, and two other effects, on realized bond returns. While characterization of past returns is sometimes useful, most investors are more interested in the future impact of convexity. If volatility and convexity were constant, we could use the historical average convexity impact to proxy for the expected value of convexity. However, volatility and convexity vary over time. Figure 13 shows the behavior of convexity, the rolling 20-day historical volatility and the (expected) value of convexity of the 30-year bond between 1980 and 1994. (Recent historical volatility is often used as an estimate for near-term future volatility.) Convexity has increased as yields declined, but the volatility level has declined even more except for spikes after the 1987 stock market crash and after the Fed's tightening in spring 1994. **In the early 1980s, convexity was worth several hundred basis points for the 30-year bond — while more recently, the value of convexity has rarely exceeded 100 basis points.** Such variation implies that any estimates of the value of convexity are as good as the underlying estimates of future volatility. Therefore, when computing convexity-adjusted expected returns, investors should use the information in the current yield curve combined with their best forecasts of the near-term yield volatility.

[19] One should not generalize these findings about wide barbells to narrower barbells. The yield curve exhibits less curvature in the intermediate sector than between the extreme front end and long end. For example, a barbell-bullet trade from fives to twos and tens tends to have a much smaller yield give-up than the trade from fives to cash and thirties — and a smaller convexity pick-up.

Figure 13. Convexity and Volatility of the 30-Year Bond Over Time



Note: Volatility (Vol (Δy)) is the annualized 20-day historical volatility of the 30-year on-the-run bond's basis-point-yield changes. Convexity is the same bond's convexity. Value of convexity ≈ 0.5 * convexity * $(\text{Vol}(\Delta y))^2$.

APPENDIX A. HOW DOES CONVEXITY VARY ACROSS NONCALLABLE TREASURY BONDS?

For bonds with known cash flows, **convexity depends on the bond's duration and on the dispersion of the bond's cash flows.** The longer the duration, the higher the convexity (for a given cash flow dispersion), and the more dispersed the cash flows, the higher the convexity (for a given duration). In this subsection, we discuss the algebra and the intuition behind these relations. We begin by analyzing zero-coupon bonds.

The price of an n-year zero is

$$P = \frac{100}{(1 + y/100)^n} \tag{4}$$

where P is the bond's price, y is its annually compounded yield, expressed in percent, and n is its maturity. Taking the derivative of price with respect to yield reveals that

$$\frac{dP}{dy} = \frac{-n}{(1 + y/100)^{n+1}} = \frac{-n * (P/100)}{1 + y/100}\,. \tag{5}$$

The second equality holds because $1/(1 + y/100)^n = P/100$, based on Equation (4). Multiplying both sides of Equation (5) by (-100/P) gives the definition of (modified) duration:

$$\text{Dur} = -\frac{100}{P} * \frac{dP}{dy} = \frac{n}{1 + y/100}. \tag{6}$$

For zeros, maturity (n) equals Macaulay duration (T). Thus, Equation (6) confirms the familiar relation between modified duration and Macaulay duration: Dur = T/(1 + y/100), given annual compounding.

Taking the second derivative of price with respect to yield reveals that

$$\frac{d^2P}{dy^2} = \frac{-n * (-n-1)}{100 * (1 + y/100)^{n+2}} = \frac{(n^2 + n) * (P/100)}{100 * (1 + y/100)^2}. \tag{7}$$

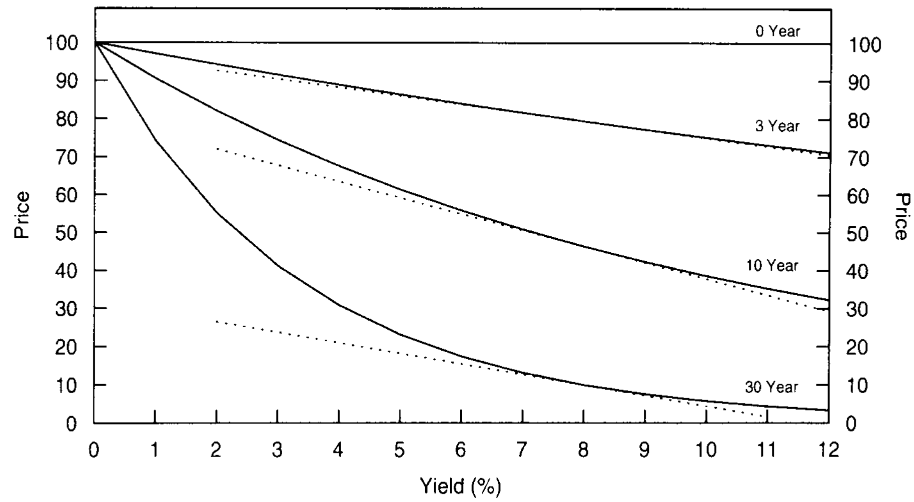
Multiplying both sides by (100/P) gives the definition of convexity (Cx):

$$\text{Cx} = \frac{100}{P} * \frac{d^2P^2}{dy^2} = \frac{n^2 + n}{100 * (1+ y/100)^2}. \tag{8}$$

Expressed in terms of Macaulay duration or modified duration, a zero's convexity is $(T^2 + T)/[100 * (1 + y/100)^2] = [\text{Dur}^2 + \text{Dur}/(1+y/100)]/100$. For long-term bonds, the square of duration is much larger than duration — thus, the **rule of thumb that the convexity of zeros increases as a square of duration** divided by 100. For example, for a zero with modified duration of 20 and yield of 8%, convexity is approximately 4.0 ($= 20^2/100 \approx (20^2 + 20/1.08)/100 = 4.18$).

The relation between the convexity and duration of zeros, illustrated in Figure 3, is simply a mathematical fact. With Figure 14 we try to offer some intuition as to *why* long-term bonds have much *more nonlinear* (convex) price-yield curves than short-term bonds. This figure shows price as a function of yield for various-maturity zeros. All curves are downward sloping but not linear. However large the discounting term $(1 + y/100)^n$ is, prices cannot become negative as long as $y > 0$. Intuitively, high convexity (that is, a large change in the slope of the price-yield curve) is needed to keep bond prices positive if the price-yield curve is initially very steep. Otherwise the linear approximation of the long bond's price-yield curve would hit zero very fast (at a yield of 11% for a 30-year zero in Figure 14 versus at a yield of 43% for a three-year zero).

Figure 14. Price-Yield Curves of Zeros with Various Maturities and Their Linear Approximations



For a given duration, convexity increases with the dispersion of cash flows. A barbell portfolio of a short-term zero and a long-term zero has more dispersed cash flows than a duration-matched bullet intermediate-term zero. The bullet, in fact, has no cash flow dispersion. **The barbell exhibits more convexity because of the inverse relation between yield level and portfolio duration.** A given yield rise reduces the present value of the longer cash flow more than it reduces that of the shorter cash flow, and the decline in the longer cash flow's relative weight shortens the barbell's duration, limiting losses if yields rise further. (Recall that the Macaulay duration of a portfolio is the *present-value-weighted* average duration of its constituent cash flows.) Of all bonds with the same duration, a zero has the smallest convexity because it has no cash flow dispersion. Thus, its Macaulay duration does not vary with the yield level.

In fact, **a coupon bond's or a portfolio's convexity can be viewed as a sum of a duration-matched zero's convexity and additional convexity caused by cash flow dispersion**. That is, the convexity of a bond portfolio with a Macaulay duration T is:

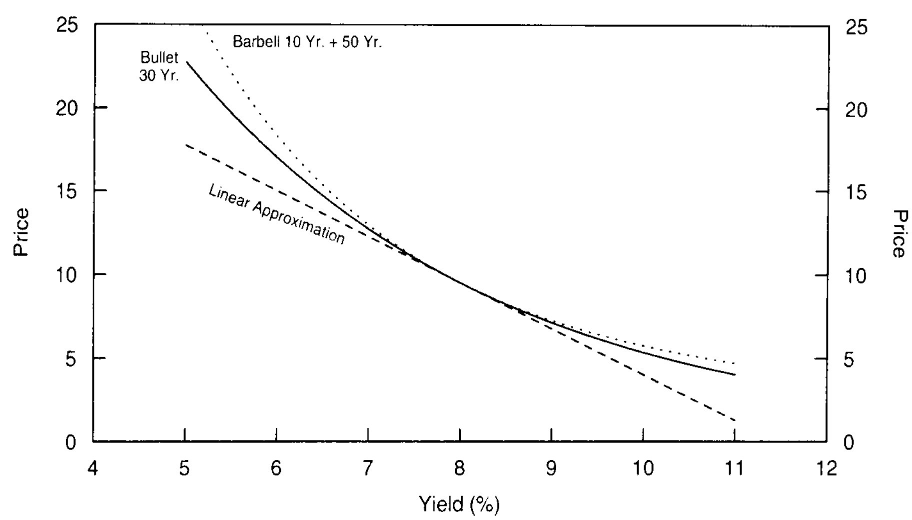
$$Cx = \frac{T^2 + T}{100 * (1 + y/100)^2} + \frac{\text{Dispersion}}{100 * (1 + y/100)^2}. \qquad (9)$$

where the first term on the right-hand side equals a duration-matched zero's convexity (see Equation (8)) and "dispersion" is the standard deviation of the maturities of the portfolio's cash flows about their present-value-weighted average (the Macaulay duration).[20]

[20] Stan Kogelman derived Equation (9) in "Dispersion: An Important Component of Convexity and Performance," an unpublished research piece, Salomon Brothers Inc. 1986.

Figure 15 illustrates the convexity difference between a bullet (a 30-year zero) and a duration-matched barbell portfolio of ten-year and 50-year zeros. We use such an extreme example and a hypothetical 50-year bond only to make the difference in the two price-yield curve shapes visually discernible. If the yield curve is flat at 8% and can undergo only parallel yield shifts, the barbell will, at worst, match the bullet's performance (if yields stay at 8%) and, at best, outperform the bullet substantially (if yields shift up or down by a large amount). Clearly, high positive convexity is a valuable characteristic. In fact, because it is valuable, the situation in Figure 15 is unrealistic. If the flat curve/parallel shifts assumption were literally true, investors could earn riskless arbitrage profits by being long the barbell and short the bullet. In reality, market prices adjust so that the yield curve is typically concave rather than flat (that is, the barbell has a lower yield than the bullet), and nonparallel shifts such as curve steepening can make the bullet outperform the barbell.

Figure 15. Price-Yield Curves of a Barbell and a Bullet with the Same Duration (30 Years)



APPENDIX B. RELATIONS BETWEEN VARIOUS VOLATILITY MEASURES

Equation (1) shows that $0.5 * Cx * (\Delta y)^2$ approximates the impact of convexity on a bond's percentage price changes. Thus, the expected value of convexity $\approx 0.5 * Cx * E(\Delta y)^2$. Now we discuss **relations between $E(\Delta y)^2$ and some volatility measures.** The variance of basis-point yield changes is defined as

$$\mathrm{Var}(\Delta y) = E(\Delta y - E(\Delta y))^2. \qquad (10)$$

Because yield changes are mostly unpredictable, it is reasonable to assume that $E(\Delta y) \approx 0$. Therefore, $\mathrm{Var}(\Delta y) \approx E(\Delta y - 0)^2 = E(\Delta y)^2$. The volatility of yield changes ($\mathrm{Vol}(\Delta y)$) is often measured by standard deviation — the square root of variance. Thus,

$$\text{Value of convexity} \approx 0.5 * Cx * \mathrm{Var}(\Delta y) = 0.5 * Cx * (\mathrm{Vol}(\Delta y))^2. \qquad (11)$$

As long as $E(\Delta y) \approx 0$, volatility is roughly proportional to the expected absolute magnitude of the yield change, $E(|\Delta y|)$. Note that it makes sense to assume that $E(|\Delta y|)$ is positive even when $E(\Delta y) = 0$. Even if an investor thinks that the current yield curve is the best forecast for next year's yield curve, he can think that the curve is likely to move up or down by, say, 100 basis points from the current level over the next year. In fact, it would be extreme to assume that $E(|\Delta y|) = 0$; this assumption would imply zero volatility (no rate uncertainty).

Next we show that **for zero-coupon bonds the value of convexity is proportional to the variance of returns**. Both yields and returns are expressed in percent. Short-term fluctuations in bonds' holding-period returns (h) mostly reflect the duration impact ($-Dur * \Delta y$) because the yield and convexity impacts are either so stable or so small that they contribute little to the return variance (see Equation (3) and Figure 12). Therefore,

$$Var(h) \approx Var(-Dur * \Delta y) = Dur^2 * Var(\Delta y) \approx 100 * Cx * Var(\Delta y). \qquad (12)$$

The relation $Cx \approx Dur^2/100$ is explained below Equation (8). A comparison of Equations (11) and (12) shows that the value of convexity for zeros is approximately equal to the variance of returns divided by 200. Interestingly, also the difference between an arithmetic mean and a geometric mean is approximately equal to the variance of returns divided by 200.[21] It appears that a duration extension enhances convexity and increases the (arithmetic) expected return, but the ensuing increase in volatility drags down the geometric mean and offsets the convexity advantage.

Equation (12) illustrates the relation between a bond's return volatility and yield volatility. We finish by stressing **the distinction between the volatility of basis-point yield changes Vol(Δy) and the volatility of relative yield changes Vol(Δy/y)**. The volatility quotes in option markets and in Bloomberg or Yield Book typically refer to Vol(Δy/y), while our analysis focuses on Vol(Δy).

$$Vol(h) \approx Dur * Vol(\Delta y) \approx Dur * Vol(\Delta y/y) * y. \qquad (13)$$

In Figure 7, we use the historical basis-point yield volatility to proxy for the expected basis-point yield volatility. Alternatively, we could compute the historical relative yield volatility and multiply it by the current yield level. The latter approach would be appropriate if the relative yield volatility is believed to be constant over time, making the basis-point yield volatility vary one-for-one with the yield level. Empirically, this has not been the case in the United States since 1982 (see footnote 9).

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[21] The arithmetic mean (AM) and geometric mean (GM) are computed using the following equations: $AM = (h_1 + h_2 + ... + h_K) / K$ and $GM = ([(1 + h_1/100) * (1 + h_2/100) * ... * (1 + h_K/100)]^{1/K} - 1) * 100$, where h_k is the one-period holding-period return at time k, and K is the sample size. It can be shown that $GM = AM - Var(h)/200$.

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A Framework for Analyzing Yield Curve Trades

Understanding the Yield Curve: Part 6

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INTRODUCTION

In Part 1 of this series, *Overview of Forward Rate Analysis*, we argued that **the shape of the yield curve depends on three factors: the market's rate expectations; the required bond risk premia; and the convexity bias.** After examining these determinants in detail in Parts 2-5, we now return to the "big picture" to show how we can decompose the forward rate curve into these three determinants. Even though we cannot directly observe these determinants, the decomposition can clarify our thinking about the yield curve.

Our analysis also produces direct applications — it provides **a systematic framework for relative value analysis of noncallable government bonds.** Analogous to the decomposition of forward rates, **the total expected return of any government bond position can be viewed as the sum of a few simple building blocks: (1) the yield income; (2) the rolldown return; (3) the value of convexity; and (4) the duration impact of the rate view.** A fifth term, the financing advantage, should be added for bonds that trade "special" in the repo market.

The following observations motivate this decomposition. A bond's near-term expected return is a sum of its horizon return given an unchanged yield curve and its expected return from expected changes in the yield curve. The first item, the horizon return, is also called the rolling yield because it is a sum of the bond's yield income and the rolldown return (the capital gain that the bond earns because its yield declines as its maturity shortens and it "rolls down" an upward-sloping yield curve). The second item, the expected return from expected changes in the yield curve, can be approximated by duration and convexity effects. The duration impact is zero if the yield curve is expected to remain unchanged, but it may be the main source of expected return if the rate predictions are based on a subjective market view or on a quantitative forecasting model. The value of convexity is always positive and depends on the bond's convexity and on the perceived level of yield volatility.

We argue that both prospective and historical **relative value analysis should focus on near-term expected return differentials across bond positions** instead of on yield spreads. The former measures **take into account all sources of expected return. Moreover, they provide a consistent framework for evaluating all types of government bond positions.** We also show, with practical examples, how various expected return measures are computed and how our framework for relative value analysis is related to the better-known scenario analysis.

FORWARD RATES AND THEIR DETERMINANTS

How Do the Main Determinants Influence the Yield Curve Shape?

We first describe how the market's rate expectations, the required bond risk premia[1], and the convexity bias influence the term structure of spot and forward rates. The market's expectations regarding the future interest rate behavior are probably the most important influences on today's term structure. **Expectations for parallel increases in yields tend to make today's term structure linearly upward sloping, and expectations for**

[1] The bond risk premium is defined as a bond's expected (near-term) holding-period return in excess of the riskless short rate. Historical experience suggests that long-term bonds command some risk premium because of their greater perceived riskiness. However, our term "bond risk premium" also covers required return differentials across bonds that are caused by other factors than risk, such as liquidity differences, supply effects or market sentiment.

falling yields tend to make today's term structure inverted. Expectations for future curve flattening induce today's spot and forward rate curves to be concave (functions of maturity), and expectations for future curve steepening induce today's spot and forward rate curves to be convex.[2] These are the facts, but what is the intuition behind these relationships?

The traditional intuition is based on the *pure expectations hypothesis*. In the absence of risk premia and convexity bias, a long rate is a weighted average of the expected short rates over the life of the long bond. If the short rates are expected to rise, the expected average future short rate (that is, the long rate) is higher than the current short rate, making today's term structure upward sloping. A similar logic explains why expectations of falling rates make today's term structure inverted. However, this logic gives few insights about the relation between the market's expectations regarding future curve reshaping and the curvature of today's term structure.

Another perspective to the pure expectations hypothesis may provide a better intuition. The absence of risk premia means that all bonds, independent of maturity, have the same near-term expected return. Recall that a bond's holding-period return equals the sum of the initial yield and the capital gains/losses that yield changes cause. Therefore, **if all bonds are to have the same expected return, initial yield differentials across bonds must offset any expected capital gains/losses**. Similarly, each bond *portfolio* with expected capital gains must have a yield disadvantage relative to the riskless asset. If investors expect the long bonds to gain value because of a decline in interest rates, they accept a lower initial yield for long bonds than for short bonds, making today's spot and forward rate curves inverted. Conversely, if investors expect the long bonds to lose value because of an increase in interest rates, they demand a higher initial yield for long bonds than for short bonds, making today's spot and forward rate curves upward sloping. Similarly, if investors expect the curve-flattening positions to earn capital gains because of future curve flattening, they accept a lower initial yield for these positions. In such a case, barbells would have lower yields than duration-matched bullets (to equate their near-term expected returns), making today's spot and forward rate curves concave. A converse logic links the market's curve-steepening expectations to convex spot and forward rate curves.

The above analysis presumes that all bond positions have the same near-term expected returns. **In reality, investors require higher returns for holding long bonds than short bonds.** Many models that acknowledge bond risk premia assume that they increase linearly with duration (or with return volatility) and that they are constant over time. Parts 3 and 4 of this series showed that empirical evidence contradicts both assumptions. Historical average returns increase substantially with duration at the front end of the curve but only marginally after the two-year duration. Thus, **the bond risk premia make the term structure upward-sloping and concave**, on average. Moreover, it is possible to forecast when the required bond risk premia are abnormally high or low. Thus, **the time-variation in the bond risk premia can cause significant variation in the shape of the term structure**.

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[2] A concave (but upward-sloping) curve has a steeper slope at short maturities than at long maturities; thus, a line connecting two points on the curve is always below the curve. A convex (but upward-sloping) curve has a steeper slope at long maturities than at short maturities; thus, a line connecting two points on the curve is always above the curve.

Convexity bias refers to the impact that the nonlinearity of a bond's price-yield curve has on the shape of the term structure. This impact is very small at the front end but can be quite significant at very long durations. A positively convex price-yield curve has the property that a given yield decline raises the bond price more than a yield increase of equal magnitude reduces it. All else equal, this property makes a high-convexity bond more valuable than a low-convexity bond, especially if the volatility is high. It follows that investors tend to accept a lower initial yield for a more convex bond because they have the prospect of enhancing their returns as a result of convexity. Because a long bond exhibits much greater convexity than a short bond, it can have a lower yield and yet, offer the same near-term expected return. Thus, **in the absence of bond risk premia, the convexity bias would make the term structure inverted. In the presence of positive bond risk premia, the convexity bias tends to make the term structure humped** — because the negative effect of convexity bias overtakes the positive effect of bond risk premia only at long durations. An increase in the interest rate volatility makes the bias stronger and, thus, tends to make the term structure more humped.

The three determinants influence the shape of the term structure simultaneously, making it difficult to distinguish their individual effects. One central theme in this series has been that **the shape of the term structure does not only reflect the market's rate expectations**. Forward rates are good measures of the market's rate expectations only if the bond risk premia and the convexity bias can be ignored. This is hardly the case, even though a large portion of the short-term variation in the shape of the curve probably reflects the market's changing expectations about the future level and shape of the curve. The steepness of the curve on a given day depends mainly on the market's view regarding the rate direction, but in the long run, the impact of positive and negative rate expectations largely washes out. Therefore, the average upward slope of the yield curve is mainly attributable to positive bond risk premia. The curvature of the term structure may reflect all three components. On a given day, the spot rate curve is especially concave (humped) if market participants have strong expectations of future curve flattening or of high future volatility. In the long run, the reshaping expectations should wash out, and the average concave shape of the term structure reflects the concavity of the risk premium curve and the convexity bias.

Decomposing Forward Rates Into Their Main Determinants

Conceptually, each one-period forward rate can be decomposed to three parts: the impact of rate expectations; the bond risk premium; and the convexity bias. So far, this statement is just an assertion. In this subsection, we show intuitively why this relationship holds between the forward rates and their three determinants. We provide a more formal derivation in Appendix A (where we take into account the fact that the analysis is not instantaneous but yield changes occur over a discrete horizon, during which invested capital grows). In Appendix B, we tie some loose strings together by summarizing various statements about the forward rates and by clarifying the relations between these statements.

Figure 1 shows how the yield change of an n-year zero-coupon bond over one period (dashed arrow) can be split to the rolldown yield change and the one-period change in an n-1 year constant-maturity spot rate s_{n-1} ($\Delta s_{n-1} = s^*_{n-1} - s_{n-1}$) (two solid arrows).[3] A zero-coupon bond's price

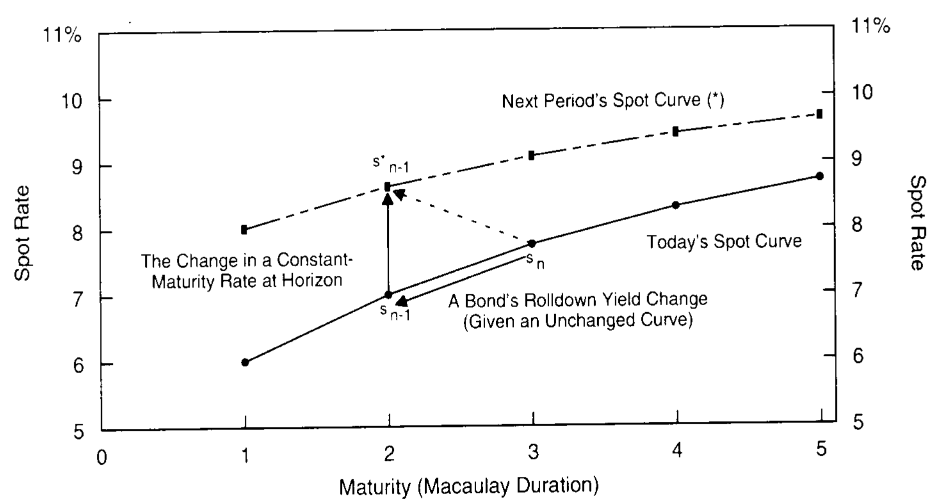
[3] All rates and returns in this report are expressed in percentage terms (200 basis points = 2%), whereas in the equations in Parts 1 and 2 of this series they were expressed in decimal terms (200 basis points = 0.02).

can be split in a similar way (see Appendix A). Thus, an n-year zero's holding-period return over the next period h_n is:

$$\begin{aligned} h_n &= \text{return if the curve is unchanged} && + [\text{return from the curve changes}] \\ &= \text{rolling yield} && + [\text{percentage price change (at horizon)}] \\ &\approx \text{(one-period) forward rate} && + [-\text{duration}*(\Delta s_{n-1}) + 0.5*\text{convexity}*(\Delta s_{n-1})^2]. \end{aligned} \tag{1}$$

Equation (1) is based on the following relations. First, a bond's one-period horizon return given an unchanged yield curve is called the rolling yield. A zero-coupon bond's rolling yield equals the one-period forward rate ($f_{n-1,n}$). For example, if the four-year (five-year) constant-maturity rate remains unchanged at 9.5% (10%) over the next year, a five-year zero bought today at 10% can be sold next year at 9.5%, as a four-year zero; then the bond's horizon return is $1.10^5/1.095^4 - 1 = 0.1202 = 12.02\%$, which is the one-year forward rate between four- and five-year maturities (see Equation (12) in Appendix B). The second source of a zero's holding-period return, the price change caused by the yield curve shift, is approximated very well by duration and convexity effects for all but extremely large yield curve shifts.

Figure 1. Splitting a Zero-Coupon Bond's One-Period Yield Change Into Two Parts



It is more interesting to relate the forward rates to expected returns and expected rate changes than to the realized ones. We take expectations of both sides of Equation (1), split the bond's expected holding-period return into the short rate and the bond risk premium, and recall that $E(\Delta s_{n-1})^2 \approx (\mathrm{Vol}(\Delta s_{n-1}))^2$. Then we can rearrange the equation to express the one-period forward rate as a sum of the other terms:

$$\textbf{Forward rate} \approx \textbf{short rate} + \textbf{duration}*E(\Delta s_{n-1}) + \textbf{bond risk premium} + \textbf{convexity bias}, \tag{2}$$

where bond risk premium $= E(h_n - s_1)$ and convexity bias $\approx -0.5*\text{convexity}*(\mathrm{Vol}(\Delta s_{n-1}))^2$.

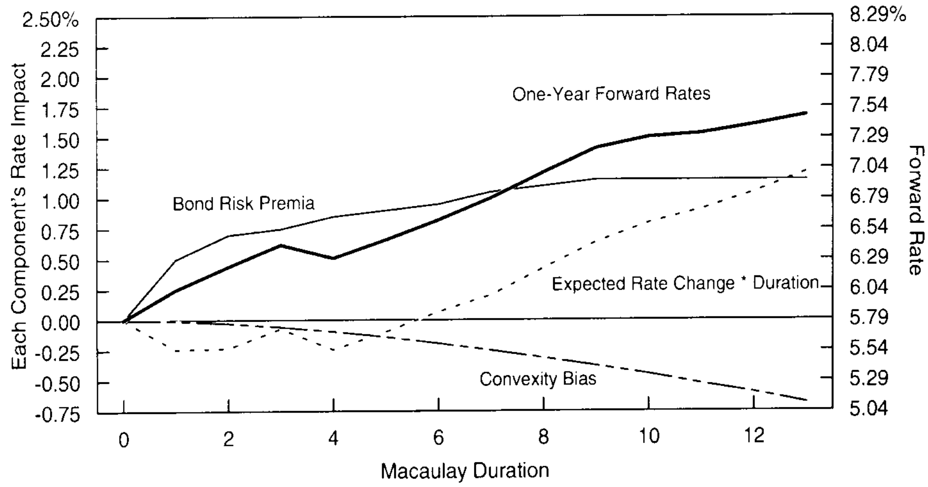
If we move the short rate to the left-hand side of the equation, we decompose the "forward-spot premium" ($f_{n-1,n} - s_1$) into a rate expectation term, a risk premium term and a convexity term (see Equation (11) in Appendix A). We interpret the expectations in Equation (2) as the market's rate and volatility expectations and as the expected risk premium that the market requires for holding long-term bonds. The market's expectations are weighted averages of individual market participants' expectations.

Some readers may wonder why our analysis deals with forward rates and not with the more familiar par and spot rates. The reason is the simplicity of the one-period forward rates. A one-period forward rate is the most basic unit in term structure analysis, the discount rate of one cash flow over one period. A spot rate is the average discount rate of one cash flow over many periods, whereas a par rate is the average discount rate of many cash flows — those of a par bond — over many periods. All the averaging makes the decomposition messier for the spot rates and the par rates than it is for the one-period forward rate in Equation (2). However, because the spot and the par rates are complex averages of the one-period forward rates, they too can be conceptually decomposed into the three main determinants.

Because the approximate decomposition in Equation (2) is derived mathematically without making specific economic assumptions, it is true in general. In reality, however, it is hard to make this decomposition because the components are not observable and because they vary over time. Further assumptions or proxies are needed for such a decomposition. In Figure 2, we use historical average returns to compute the bond risk premia and historical volatilities to compute the convexity bias — together with the observable market forward rates (as of September 26, 1995) — and back out the only unknown term in Equation (2): the expected spot rate change times duration. We also could divide this term by duration to infer the market's rate expectations. The rate expectations that we back out in Figure 2 suggest that the market expects small declines in short rates and small increases in long rates.

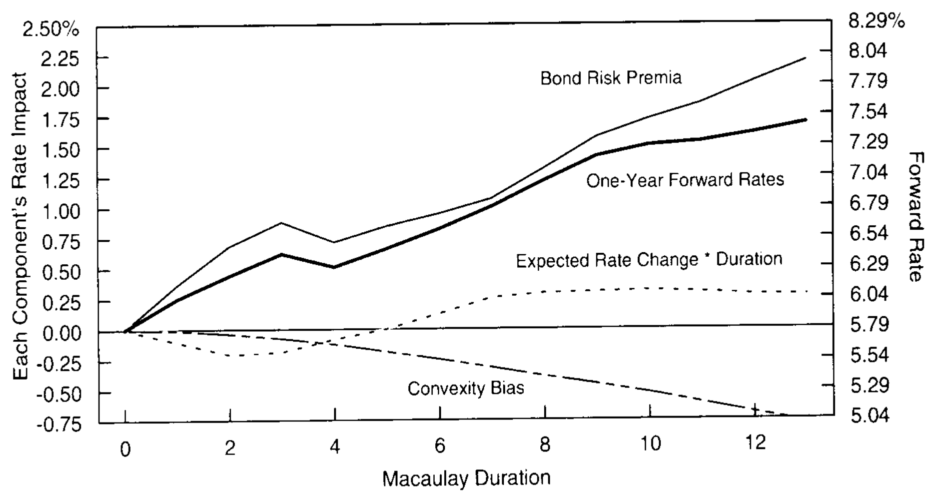
If bond risk premia vary over time, the use of historical average risk premia may be misleading. As an alternative, we can use survey data or rate predictions based on a quantitative forecasting model to proxy for the market's rate expectations. In Figure 3, we use the consensus interest rate forecasts from the *Blue Chip Financial Forecast* to proxy for the market's rate expectations. In addition, we use implied volatilities from option prices to compute the convexity bias. These components can be used together with the one-year forward rates to back out estimates of the unobservable bond risk premia.

Figure 2. Decomposing Forward Rates Into Their Components, Using Historical Average Risk Premia and Volatilities



Note: The one-year forward rates are based on the Salomon Brothers Treasury Model curve on September 26, 1995. The bond risk premia are based on the historical arithmetic average returns of various maturity-subsector portfolios between 1970-94, expressed in excess of the riskless one-year return. The convexity bias is based on the historical volatilities of various on-the-run bonds' weekly yield changes between 1990 and 1995. The rate expectation term for each duration is then backed out as the difference — one-year forward rate - one-year spot rate - bond risk premium - convexity bias.

Figure 3. Decomposing Forward Rates Into Their Components, Using Survey Rate Expectations and Implied Volatilities



Note: The one-year forward rates are based on the Salomon Brothers Treasury Model curve on September 26, 1995. The market's rate expectations are proxied by the consensus interest rate forecasts from the *Blue Chip Financial Forecast* (October 1, 1995 issue, a survey conducted among industry economists and analysts on September 26-27, 1995). The convexity bias is based on the implied volatilities from the prices of Salomon Brothers' OTC options for various on-the-run bonds on September 26. The bond risk premium for each duration is then backed out as the difference — one-year forward rate - one-year spot rate - expected rate change * duration - convexity bias.

A comparison of Figures 2 and 3 shows that the two decompositions look similar up to the seven-year duration, but quite different beyond that point. The similarity of the convexity bias components in these two figures suggests that the use of historical or implied volatilities makes little difference, at least in this case. It is also clear that the Blue Chip survey predicted small declines in the short rates and small increases in the long rates, just as the inferred rate expectations in Figure 2. However, the predicted increases in long rates were smaller in this survey (where the largest increase was four basis points) than the inferred forecasts of Figure 2 (where the largest increase was eight basis points). Because the forward rate curve is the same in both figures, the smaller predicted rate increases lead to higher bond risk premia in Figure 3 than in Figure 2.[4]

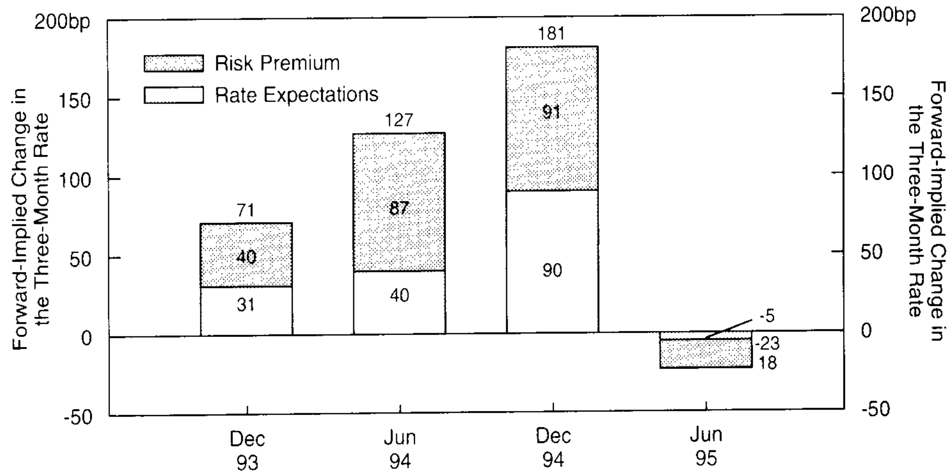
Figures 2 and 3 are snapshots of the forward rates and their components on one date. A comparison of similar decompositions over time would provide insights into the relative variability of each component. In Figure 4, we try to illustrate the impact of changing rate expectations and risk premia on the steepness of the U.S. Treasury bill curve. The figure shows that on four recent dates the forwards always implied larger increases in the three-month rate than the market expected, based on surveys of bond market analysts. The difference is proportional to the required bond risk premium of longer bills over shorter bills (because bills exhibit negligible convexity, its impact can be ignored). Not surprisingly, this difference is always positive; moreover, it varies over time.

The time-variation in the bond risk premium in Figure 4 appears economically reasonable. In December 1993, after a long bull market, market participants were complacent, neither expecting much higher rates nor demanding much compensation for duration extension. After the Fed began to tighten the monetary policy, the market expected further rate increases. In addition, the increase in volatility and in risk aversion levels (perhaps caused by own losses as well as the well-publicized losses of other investors) increased the required bond risk premia. By the end of 1994, the market was extremely bearish, expecting almost 100 basis points higher three-month rates in six months. However, the forwards were implying almost 200 basis points higher rates — the difference reflects an abnormally large risk premium. In 1995, the bond market rallied strongly as the market's expectations for further Fed tightening receded and turned into expectations of easing policy. However, a large part of this rally was caused by the collapse in the required bond risk premium, perhaps reflecting lower inflation uncertainty and higher wealth that reduced the market's risk perceptions and risk aversion. In general, **the time-variation in required returns appears to have contributed as much to the changing slope of the yield curve as has the time-variation in the**

[4] We hasten to point out that these calculations are quite imprecise, especially at long durations. Even an error of a couple of basis points in our proxy for the market's rate expectation will have a large impact on any long bond's expected return (and thus on the estimated bond risk premium), because the expected yield change is scaled up by duration. Such sensitivity reduces the usefulness of this decomposition at long durations.

market's interest rate expectations.[5] Finally, we note that the time-variation in the estimated bond risk premia has been very market-directional over the past two years; this may not always be the case.

Figure 4. Forward Implied Yield Changes versus Survey-Expected Yield Changes in the Treasury Bill Market, 1993-95



Note: Forward-implied yield change is the difference between the implied three-month rate six months forward ($f_{0.5,\,0.75}$) and the current three-month rate ($s_{0.25}$) (see the data below). Survey-expected yield change is the difference between the expected three-month rate six months ahead (E(s0.25)) and the current three-month rate, where the market's rate expectation is proxied by the mean in the *Wall Street Journal*'s semiannual survey. Part 2 of this series discusses the survey data and their shortcomings. It also shows that the difference between the forward-implied yield change and the survey-expected yield change is proportional to the bond risk premium.

	$s_{0.25}$	$f_{0.5,\,0.75}$	$E(s_{0.25})$
Dec 93	3.09%	3.80%	3.40%
Jun 94	4.22	5.49	4.62
Dec 94	5.70	7.51	6.60
Jun 95	5.63	5.58	5.40

DECOMPOSING EXPECTED RETURNS OF BOND POSITIONS

Five Alternative Expected Return Measures

Our framework for decomposing the yield curve also provides a framework for systematic relative value analysis of government bonds with known cash flows. We can evaluate all bond positions' expected returns comprehensively, yet with simple and intuitive building blocks. We emphasize that relative value analysis should be based on near-term expected return differentials, not on yield spreads, which are only one part

[5] In the earlier parts of this series, we provide further evidence of the importance of time-varying risk premia. Why do so many market participants and analysts think that the rate expectations are *much* more important determinants of the yield curve steepness than are the bond risk premia, in spite of the contradicting empirical evidence? We offer one possible explanation: Individual market participants have their individual rate views and individual required risk premia. (Few investors think explicitly in terms of such premia, but they will extend duration only if they expect longer bonds to outperform shorter bonds by a margin sufficient to offset their greater risks.) However, what matters for the yield curve is the *market's aggregate* rate view and aggregate required risk premia. Both vary over time as individual rate views and risk perceptions and preferences change (and as the composition of market participants changes). We stress that the changing individual rate views may have smaller aggregate effects than the changing risk perceptions and preferences even if individual rate views are more volatile than individual risk perceptions and preferences. This effect can occur if the changes in risk perceptions and preferences are much *more highly correlated across individuals* than are changes in rate views. For example, when volatility is high, most market participants are likely to demand abnormally high bond risk premia even if they have widely different views about the future rate direction. Perhaps most market participants focus on the fact that (their) individual rate views vary more over time than do their risk perceptions and preferences, ignoring the fact that market rates are driven by the aggregate effects, for which correlations across individuals matter a lot.

of them. That is, **total return investors should care more about expected returns than about yields.** Thus, our approach brings fixed-income investors closer to mean-variance analysis in which various positions are evaluated based on the trade-off between their expected return and return volatility.

Equation (1) shows that a zero's holding-period return is a sum of its return given an unchanged yield curve and its return caused by the changes in the yield curve. The return given an unchanged yield curve is called the rolling yield because it is a sum of the zero's yield and the rolldown return. The return caused by changes in the yield curve can be approximated well by duration and convexity effects. Taking expectations of Equation (1) and splitting the rolling yield into yield income and rolldown return, the near-term expected return of a zero is:

$$\begin{aligned}\textbf{Expected return} \approx\ & \textbf{Yield income} \\ & + \textbf{Rolldown return} \\ & + \textbf{Value of convexity} \\ & + \textbf{Expected capital gain/loss from the rate "view"}\end{aligned} \tag{3}$$

For details, see Equation (9) in Appendix A or the notes below Figure 5. A similar relation holds approximately for coupon bonds, and we will describe the three-month expected return of some on-the-run Treasury bonds as the sum of the four components in the right-hand side of Equation (3).[6]

This framework is especially useful when evaluating positions of two or more government bonds, such as duration-neutral barbells versus bullets. We first compute expected return separately for each component and then compute the portfolio's expected return by taking a market-value weighted average of all the components' expected returns.

It may be helpful to show step by step how the expected return measures are improved, starting from simple yields and moving toward more comprehensive measures:

• A bond's **yield income** includes coupon income, accrued interest and the accretion/amortization of price toward par value. Yield to maturity is the correct return measure if all interim cash flows can be reinvested at the yield and the bond can be sold at its purchasing yield.[7] Yield ignores the rolldown return the bond earns if the yield curve stays unchanged.

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[6] However, certain modifications are needed when Equation (3) is used to describe the expected returns of coupon bonds rather than those of zeros — and the approximation will be somewhat worse. We use each bond's rolling yield to measure the horizon return given an unchanged yield curve; this measure no longer equals the one-period forward rate. We also use the end-of-horizon duration and convexity as well as the change in the constant-maturity rate of a constant-coupon curve at horizon, and we adjust the duration and convexity effects for the fact that the bond's value increases to (1+ rolling yield/100) by the end of the horizon. Besides the approximation error of ignoring higher-order terms than duration and convexity effects, another source of error exists for coupon bonds: The reinvestment rate assumptions vary across bonds. Recall that the calculation of the yield to maturity implicitly assumes that all cash flows are reinvested at the bond's yield to maturity. This fact may lead to exaggerated estimates of yield income for long-term bonds if the yield curve is upward sloping, a problem common to all expected return measures that use the concept of yield to maturity. Even though our approach of using bond-specific yields does not ensure internal consistency of the reinvestment rate assumptions across bonds, any inconsistencies should have a relatively small impact on the overall level of bonds' expected returns.

[7] The yield to maturity of a single cash flow is unambiguous, whereas the yield of a portfolio of multiple cash flows is a more controversial measure. The duration-times-market-value weighted yield is a good proxy for a portfolio's true yield to maturity (internal rate of return). However, a portfolio's market-value weighted yield may be a better estimate of the portfolio's *likely yield income over a short horizon* (its near-term expected return) than is its yield to maturity. The yield to maturity weighs longer cash flows more heavily and is more influenced by the built-in reinvestment rate assumptions. We will return to this topic in a future report.

- **Rolling yield** is a better expected return proxy if an unchanged curve is a reasonable base case. Yet, it ignores the value of convexity and, thus, implicitly assumes no rate uncertainty. Thus, the rolling yield measures expected return if no curve change and no volatility is expected.

- Combining the rolling yield with the value of convexity improves the expected return measure further. In Part 5 of this series, we showed that a bond's **convexity-adjusted expected return** equals the sum of the rolling yield and the value of convexity. This measure recognizes the impact of rate uncertainty but implies that no change is expected in the yield curve. Empirical evidence described in Part 2 suggests that an unchanged yield curve is often a reasonable base "view."

- If investors want, they can replace the prediction of an unchanged curve with some other rate (or spread) "view." One possibility is to use survey-based information of the market's current rate forecasts; such an approach may be useful for backing out the market's required return for each bond. Alternatively, investors may ignore the market view and input either their own rate views or an economist's subjective rate forecasts or rate predictions from some quantitative model. For example, the predictors identified in Part 4 of this series can be used to forecast the changes in the long rates. The impact of any rate view is approximated by the expected yield change scaled by duration (see Equation (10) in Appendix A), which may be added to the convexity-adjusted expected return. The sum gives us the "**expected return with a view**" — the four-term expected return measure in Equation (3). However, this equation is a perfect description of expected returns only for bonds that lie on the fitted curve. Thus, the relative value measures above ignore "local" or bond-specific richness or cheapness relative to the curve.

- Many technical factors can make a specific bond "locally" rich or cheap (relative to adjacent-maturity bonds), or they can make a whole maturity sector rich or cheap relative to the fitted curve. Such factors include supply effects (temporary price pressure on a sector caused by new issuance), demand effects (maturity limitations or preferences of important market participants — for example, the richness of quarter-end bills), liquidity effects (lower transaction costs for on-the-runs versus off-the-runs, for 30-year bonds versus 25-year bonds, for Treasury bills versus duration-matched coupon bonds, etc.), coupon effects (motivated by tax benefits, accounting rules, etc.), and above all, the financing effects (the "special" repo income that is common for on-the-runs).[8] Fortunately, it is easy to **add to the four-term expected return measures the financing advantage and two local cheapness measures —** the spread off the fitted curve and the expected cheapening toward the fitted curve. The five-term expected return measures are comprehensive measures of **total expected returns —** ignoring small approximation errors, they incorporate all sources of expected return for noncallable government bonds.[9]

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[8] Whether such local cheapness effects appear as deviations from a fitted yield curve or as "wiggles" or "kinks" in the fitted curve depends on the curve-estimation technique. Recall that all curve-estimation techniques try to fit bond prices well while keeping the curve reasonably shaped. If the goodness of fit is heavily weighted, all bonds have small or no deviations from the fitted curve. However, a close fit may lead to "unreasonably" jagged forward rate curves. Based on Equation (2), the forward rate curve should be smooth, rather than jagged, because maturity-specific expectations of rate or volatility behavior are hard to justify and because arbitrageurs presumably are quick to exploit any abnormally large expected return differentials between adjacent-maturity bonds.

[9] In our analysis, we include the local effects into the expected bond returns separately as a fifth term. As an alternative, we could include the financing advantage (repo income) and the spread off the curve in the yield income, and we could include the expected cheapening in the rolldown return. "Rich" bonds, such as the on-the-runs, are unlikely to roll down the fitted curve if the overall curve remains unchanged. More likely, they will lose their relative richness eventually. It may be reasonable to assume that an on-the-run bond's yield advantage and *expected* cheapening exactly offset its *expected* financing advantage.

As a numerical illustration, Figure 5 shows the various expected return measures for three bonds (the three-month Treasury bill and the three-year and ten-year on-the-run Treasury notes) and for the barbell combination of the three-month bill and the ten-year bond. In this example, we use as much market-based data as possible: for example, implied volatilities, not historical, to estimate the value of convexity, and the "view" (rate predictions) based on survey evidence of the market's rate expectations, not on a quantitative forecasting model. All the numbers are based on the market prices as of September 26, 1995.

Figure 5. Three-Month Expected Return Measures and Their Components, as of 26 Sep 95

Maturity	0.25	3	10	Barbell
Yield Income	**1.349%**	**1.474%**	**1.568%**	**1.425%**
+ Rolldown Return	*0.000*	*0.065*	*0.108*	*0.038*
= Rolling Yield	**1.349**	**1.537**	**1.676**	**1.463**
+ Value of Convexity	*0.000*	*0.014*	*0.082*	*0.029*
= Convexity-Adj. Expected Return	**1.349**	**1.551**	**1.758**	**1.492**
+ Duration Impact of the "View"	*0.000*	*-0.056*	*-0.284*	*-0.099*
= Expected Return with a View	**1.349**	**1.495**	**1.474**	**1.393**
+ Total Local Rich/Cheap Effect	*-0.015*	*-0.039*	*0.025*	*-0.002*
= Total Expected Return	**1.334**	**1.456**	**1.499**	**1.391**
Background Information				
Par Yield	5.507	6.011	6.408	NA
Rolldown Yield Change	NA	-0.026	-0.015	NA
Duration now	0.245	2.708	7.300	2.706
Duration at Horizon	0.000	2.500	7.168	2.500
Convexity now	0.002	0.090	0.669	0.235
Convexity at Horizon	0.000	0.077	0.643	0.224
Yield Volatility	NA	0.598	0.502	NA
Yield Change "View"	-0.046	0.022	0.039	NA
On-the-Run Yield	5.446	5.978	6.261	NA
Financing Advantage	0.000	0.038	0.463	NA
Spread to the Par Curve	-0.015	-0.008	-0.037	NA
Expected Cheapening Return	0.000	-0.068	-0.401	NA

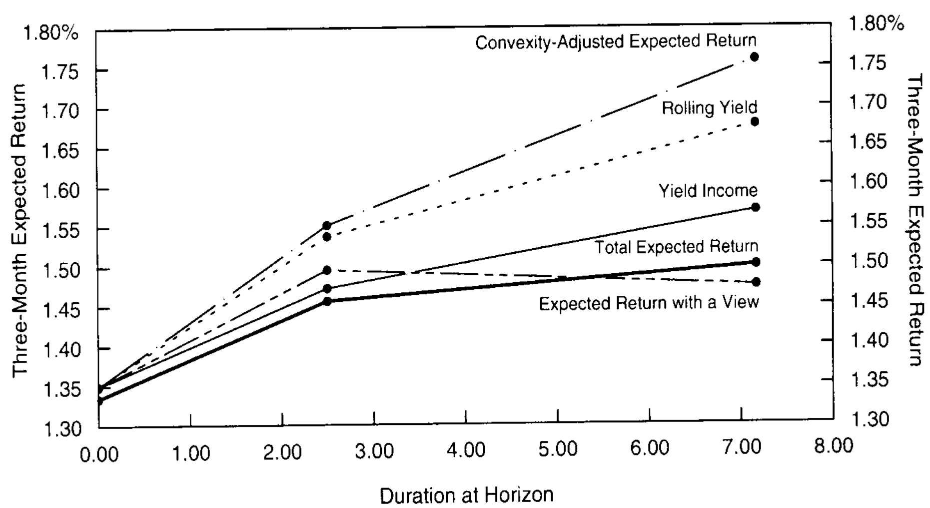
NA Not available.

Note: Barbell is a combination of 0.53 units of the ten-year par bond and 0.47 units of the three-month bill; these weights duration-match the barbell with the three-year par bond bullet. Yield income is the return that a par bond earns over three months if it can be sold at its yield and if any cash flows are reinvested at the yield. The yields are semiannually compounded and based on the Salomon Brothers Treasury Model's par yield curve. Rolldown return is the capital gain that a bond earns from the rolldown yield change. Rolling yield is a bond's horizon return given an unchanged yield curve. Value of convexity is approximated by 0.5*convexity at horizon*(yield volatility)^2*(1 + rolling yield/100), where yield volatility is the basis-point yield volatility over a three-month horizon. The latter is computed by multiplying the on-the-run bond's relative yield volatility — implied volatility based on the price of a three-month OTC option written on this bond — by its yield level and dividing by two (for deannualization). For the three-year bond, we interpolate between the implied volatilities of on-the-run twos and fives. Duration impact of the "view" is (-duration at horizon)*(expected change in a constant-maturity rate over the next three months)*(1 + rolling yield/100). In this example, the "view" reflects the market's yield curve expectations, as measured by the Blue Chip consensus forecasts (that are based on a survey of professional economists and market analysts conducted on September 26-27, 1995). The "expected return with a view" measures the expected return for a hypothetical par bond that lies exactly on the Model curve, ignoring any local cheapness or financing advantage of actual bonds. We can add to this four-term measure a fifth component called the total local rich/cheap effect. It is the sum of three additional sources of return for specific bonds: (1) the financing advantage (the difference between the three-month term repo rate for general collateral and the three-month special term repo rate for the on-the-run bond, divided by four for deannualization); (2) the spread between the on-the-run bond yield and the Model par yield, divided by four for deannualization; and (3) the bond's expected cheapening as it loses the richness associated with the on-the-run status (estimated by the Salomon Brothers Government Bond Strategy Group).

The top panel of Figure 5 shows **how nicely the different components of expected returns can be added to each other. Moreover, the barbell's expected return measures are simply the market-value weighted averages of its components' expected returns.** In this case, the yield income, the rolldown return and the value of convexity are all higher for the longer bonds. In contrast, the duration impact of the market's rate view is negative, because the Blue Chip survey suggested that the market expected small increases in the three-year and the ten-year rates over the next quarter. The local rich/cheap effect is negative for the shorter instruments but positive for the ten-year note; the reason is that the negative yield spread and the expected cheapening of the ten-year note are not sufficient to offset the ten-year note's high repo market advantage. According to all five expected return measures, the barbell has a lower expected return than the duration-matched bullet.

Figure 6 shows the five different expected return curves plotted on the three bonds' durations. In this case, the simplest expected return measure (yield income) and the most comprehensive measure (total expected return) happen to be quite similar. In general, the relative importance of the five components may be dramatically different from that in Figure 5 where the yield income dominates. The longer the asset's duration and the shorter the investment horizon, the greater is the relative importance of the duration impact. It is worth noting that realized returns can be decomposed in the same way as the expected returns and that the duration impact typically dominates the realized returns even more.[10]

Figure 6. Expected Returns of a Three-Month Bill, a Three-Year Bond and a Ten-Year Bond, as of 26 Sep 95



The total expected returns, if estimated carefully, should produce the most useful signals for relative value analysis because they include all sources of expected returns. Yield spreads may be useful signals, but they are only a part of the picture. Therefore, **we advocate the monitoring of broader expected return measures relative to their history as cheapness indicators** — just as yield spreads are often monitored relative to their history.

The components of expected returns discussed above are not new. However, few investors have combined these components into an integrated framework and based their historical analysis on broad expected return measures. **An additional useful feature of this framework is that all types of government bond trades can be evaluated consistently within it: the portfolio duration decision (market-directional view); the maturity-sector positioning and barbell-bullet decision (curve-reshaping view); and the individual issue selection (local cheapness view).** With small modifications, the framework can be extended to include the

[10] Realized returns can be split into an expected part and an unexpected part, and both parts can be decomposed further. Equation (3) describes the decomposition of the expected part, while the unexpected part can be split into duration and convexity effects. This type of return attribution can have a useful role in risk management and performance evaluation, but these two activities are not our focus in this report.

cross-country analysis of currency-hedged government bond positions. Other possible future extensions include the analysis of foreign exchange exposure and the analysis of spread positions between government bonds and other fixed-income assets.

We note some reservations. Even if two investors use the same general framework and the same type of expected return measure, they may come up with **different numbers because of different data sources and different estimation techniques.**

- **The whole analysis can be made with any raw material; we emphasize the importance of good-quality inputs. Various candidates for the raw material include on-the-run and off -the-run government bonds, STRIPS, Eurodeposits, swaps, and Eurodeposit futures**. [This multitude of course opens the possibility of trading between these curves if we can assess how various characteristics (say, convexity) are priced in each curve.] The most common approach is first to estimate the spot curve (or discount function) using a broad universe of coupon government bonds as the raw material, and then to compute the forward rates and other relevant numbers. In European bond markets, the liquid swap curve (using cash Eurodeposits and swaps as the raw material) has gained more of a benchmark status. Of course, some credit and tax-related spread may exist between the swap curve and the government bond yield curve. Recently, yet another approach has become popular: Eurodeposit futures prices are used as the raw material. In this case, the forward rates are computed by adjusting for the convexity difference between a futures contract and a forward contract, and only then are spot rates computed from the forwards.

- **Some components of expected returns are easier to measure — and less debatable — than others.** The yield income is relatively unambiguous. The rolldown return and the local rich/cheap effects depend on the curve-fitting technique. The value of convexity depends on the volatility input and, thus, on the volatility estimation technique. The rate "view," the fourth term, can be based on various approaches, such as quantitative modeling or subjective forecasting that relies on fundamental or technical analysis. Even the quantitative approach is not purely objective because infinitely many alternative forecasting models and estimation techniques exist. Forecasting rate changes is of course the most difficult task as well as the one with greatest potential rewards and risks. Forecasting changes in yield spreads may be almost as difficult. **The short-term returns of most bond positions depend primarily on the duration impact (rate changes or spread changes)**. However, even if investors cannot predict rate changes, they may earn superior returns in the long run — and with less volatility — by systematically exploiting the more stable sources of expected return differentials across bonds: yields; rolldown returns; value of convexity; and local rich/cheap effects. More generally, while the total expected return differentials are, in theory, better relative value indicators than the yield spreads, in practice, measurement errors conceivably can make them so noisy that they give worse signals. Therefore, it is important to check with historical data that any supposedly superior relative value tools would have enhanced the investment performance at least in the past.

Link to Scenario Analysis

Many active investors base their investment decisions on subjective yield curve views, often with the help of scenario analysis. Our framework for relative value analysis is closely related to scenario analysis. It may be worthwhile to explore the linkages further.

An investor can perform the scenario analysis of government bonds in two steps. First, the investor specifies a few yield curve scenarios for a given horizon and computes the total return of his bond portfolio — or perhaps just a particular trade — under each scenario. **Second, the investor assigns subjective probabilities to the different scenarios and computes the probability-weighted expected return for his portfolio. Sometimes the second step is not completed** and investors only examine qualitatively the portfolio performance under each scenario. However, **we advocate performing this step because investors can gain valuable insights from it**. Specifically, the probability-weighted expected return is the "bottom-line" number a total return manager should care about. By assigning probabilities to scenarios, investors also can explicitly back out their implied views about the yield curve reshaping and about yield volatilities and correlations.

In scenario analysis, investors define the mean yield curve view and the volatility view implicitly by choosing a set of scenarios and by assigning them probabilities. In contrast, **our framework for relative value analysis involves explicitly specifying** one yield curve view (which corresponds to the probability-weighted mean yield curve scenario) and a volatility view (which corresponds to the dispersion of the yield curve scenarios). Either way, the yield curve view determines the duration impact and the volatility view determines the value of convexity (and these views together approximately define the expected yield distribution).

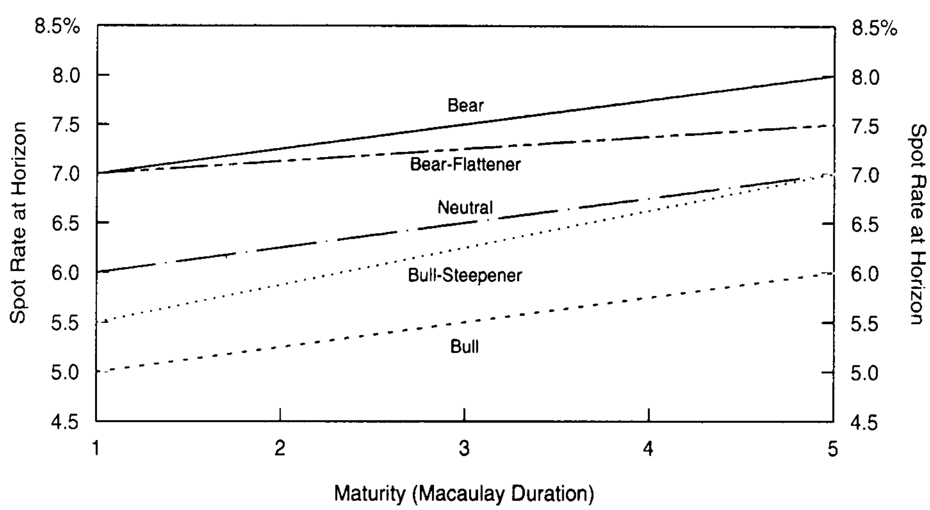
Figure 7 presents a portfolio that consists of five equally weighted zero-coupon bonds with maturities of one to five years and (annually compounded) yields between 6% and 7%. The portfolio's maturity — and its Macaulay duration — initially is three years. Over a one-year horizon, each zero's maturity shortens by one year. We specify five alternative yield curve scenarios over the horizon: parallel shifts of +100 basis points and -100 basis points; no change; a yield increase combined with a curve flattening; and a yield decline combined with a curve steepening (see Figure 8). We compute the one-year holding-period returns for each asset and for the portfolio under each scenario. In particular, the neutral scenario shows the rolling yield that each zero earns if the yield curve remains unchanged. We can evaluate each scenario separately. However, such analysis gives us limited insight — for example, the last column in Figure 7 shows just that bearish scenarios produce lower portfolio returns than bullish scenarios.

In contrast, if we assign probabilities to the scenarios, we can back out many numbers of potential interest. We begin with a simple example in which we only use the two first scenarios, parallel shifts of 100 basis points up or down. If we assign these scenarios equal probabilities (0.5), the expected return of the portfolio is 7.04% (= 0.5*5.02 + 0.5*9.06). On average, these scenarios have no view about curve changes; yet, this expected return is four basis points higher than the expected portfolio return given no change in the curve (that is, the 7% rolling yield computed in the neutral scenario). This difference reflects the value of convexity. If we only use one scenario, we implicitly assume zero volatility, which leads to downward-biased expected return estimates for positively convex bond positions. If we use the two first scenarios (bear and bull), we implicitly assume a 100-basis-point yield volatility; this assumption may or may not be reasonable, but it certainly is more reasonable than an assumption of no volatility. **This example highlights the importance of using multiple scenarios to recognize the value of convexity.** (The value is small here, however, because we focus on short-duration assets that have little convexity.)

Figure 7. Scenario Analysis and Expected Bond Returns

	Bond					Portfolio
Initial Maturity	**1**	**2**	**3**	**4**	**5**	3
Horizon Maturity	0	1	2	3	4	2
Initial Yield	6.00%	6.25%	6.50%	6.75%	7.00%	
Yield Change Scenarios						
(Of 1- to 5-Year Constant-Maturity Rates)						
Bear	1.00%	1.00%	1.00%	1.00%	1.00%	
Bull	-1.00	-1.00	-1.00	-1.00	-1.00	
Neutral	0.00	0.00	0.00	0.00	0.00	
Bear-Flattener	1.00	0.875	0.75	0.625	0.50	
Bull-Steepener	-0.50	-0.375	-0.25	-0.125	0.00	
One-Year Returns in Each Scenario						
Bear	6.00%	5.51%	5.02%	4.53%	4.05%	5.02%
Bull	6.00	7.51	9.04	10.59	12.15	9.06
Neutral	6.00	6.50	7.00	7.50	8.01	**7.00**
Bear-Flattener	6.00	5.51	5.26	5.26	5.51	5.51
Bull-Steepener	6.00	7.01	7.76	8.26	8.51	7.51
Assign **Equal Probability**						
(0.2) to Each Scenario and Back Out Various Statistics						
Mean Return	6.00%	6.41%	6.82%	7.23%	7.65%	**6.82%**
Vol. of Return	0.00	0.80	1.52	2.17	2.78	1.45
Mean Yield Change	0.10	0.10	0.10	0.10	0.10	
Vol. of Yield Change	0.80	0.76	0.72	0.69	0.66	

Figure 8. Various Yield Curve Scenarios



Now we return to the example with all five yield curve scenarios in Figure 8. As an illustration, we assign each scenario the same probability (p_i = 0.2). Then, it is easy to compute the portfolio's probability-weighted expected return:

$$E(h_p) = \sum_{i=1}^{5} p_i * h_i = 0.2 * (5.02 + 9.06 + 7.00 + 5.51 + 7.51) = 6.82 \qquad (4)$$

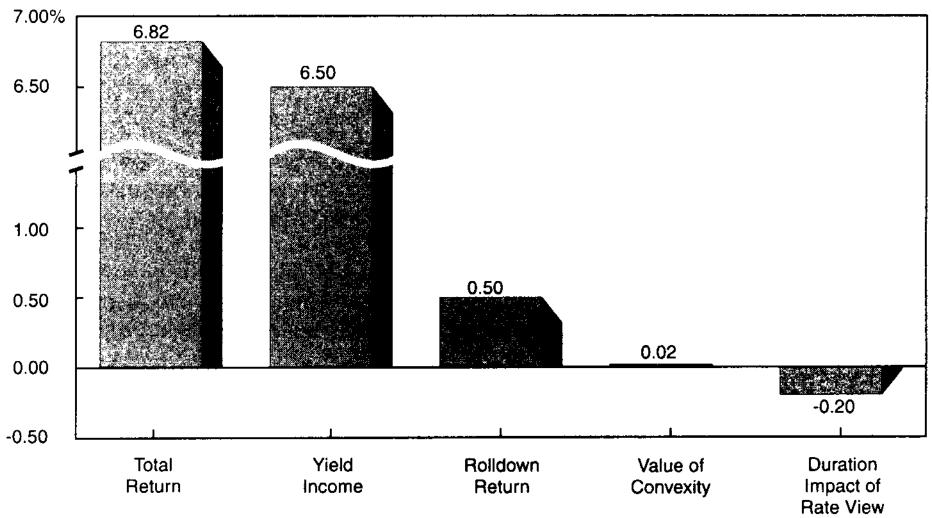
Given these probabilities, we can compute the expected return for each asset, and **it is possible to back out the implied yield curve views**. The lower panel in Figure 7 shows that the mean yield change across scenarios is +10 basis points for each rate (because the bear-flattener and the bull-steepener scenarios are not quite symmetric in magnitude in this example), **implying a mild bearish bias but no implied curve-steepness views**. In addition, we can back out the implied basis-point yield volatilities (or return volatilities) by measuring how much the yield change (or return) outcomes in each scenario deviate from the mean. These yield volatility levels are important determinants of the value of convexity. The last line in Figure 7 shows that the volatilities range from 80 to 66 basis points, **implying an inverted term structure of volatility**. Finally, we can compute implied correlations between various-maturity yield changes; the curve behavior across the five scenarios is so similar that all correlations are 0.92 or higher (not shown). Note that all correlations would equal 1.00 if only the first three scenarios were used; the imperfect correlations arise from the bear-flattener and the bull-steepener scenarios.

Whenever an investor uses scenario analysis, he should back out these implicit curve views, volatilities and correlations — and check that any biases are reasonable and consistent with his own views. Without assigning the probabilities to each scenario, this step cannot be completed; then, the investor may overlook hidden biases in his analysis, such as a biased curve view or a very high or low implicit volatility assumption which makes positive convexity positions appear too good or too bad. If investors use quantitative tools — such as scenario analysis, mean-variance optimization, or the approach outlined in this report — to evaluate expected returns, they should recognize the importance of their rate views in this process. Strong subjective views can make *any* particular position appear attractive. Therefore, investors should have the discipline and the ability to be fully aware of the views that are input into the quantitative tool.

In addition to the implied curve views, we can **back out the four components of expected returns** discussed above. In this example, we only analyze bonds that lie "on the curve" and thus can ignore the fifth component, the local rich/cheap effects. (1) We measure the **yield income** from the portfolio by a market-value weighted average yield of the five zeros, which is **6.50%** (see footnote 7). (2) Each asset's rolldown return is the difference between the horizon return given an unchanged yield curve and the yield income. Figure 7 shows that the horizon return for the portfolio is 7% in the neutral scenario; thus, the portfolio's (market-value weighted average) **rolldown return is 50 basis points** (= 7% - 6.5%). Note that the rolldown return is larger for longer bonds, reflecting the fact that the same rolldown yield change (25 basis points) produces larger capital gains for longer bonds. (3) The value of convexity for each zero can be approximated by 0.5*convexity at horizon*(basis-point yield volatility)^2*(1 + rolling yield/100). Using the implicit yield volatilities in Figure 7, this value varies between 0.6 and 4.5 basis points across bonds. The portfolio's **value of convexity** is a market-value weighted average of the bond-specific values of convexity, or roughly **two basis points**. (4) The **duration impact of the rate "view"** for each bond equals (-duration at horizon)*(expected yield change)*(1 + rolling yield/100). The last term is needed because each invested dollar grows to (1 + rolling yield/100) by the end of horizon when the repricing occurs. The core of the duration impact is the product of duration and expected yield change. The expected yield change refers to the change (over the investment horizon) in a constant-maturity rate of the bond's horizon maturity. In Figure 7, all rates are expected to increase by ten basis points, and the duration impact on specific bonds' returns varies between 0 and -40 basis points. The portfolio's duration impact is a market-value weighted average of bond-specific duration impacts, or about **-20 basis points.**[11]

Figure 9 shows that the four components add up to the total probability-weighted expected return of 6.82%. Decomposing expected returns into these components should help investors to better understand their own investment positions. For example, they can see what part of the expected return reflects static market conditions and what part reflects their subjective market view. Unless they are extremely confident about their market view, they can emphasize the part of expected return advantage that reflects static market conditions. In our example, the duration effect is small because the implied rate view is quite mild (ten basis points) and the one-year horizon is relatively long (the "slower" effects have time to accrue). With a shorter horizon and stronger rate views, the duration impact would easily dominate the other effects.

Figure 9. Decomposing the Total Expected Return into Four Components



[11] It is possible to compute the value of convexity and the duration impact in another way and end up with almost exactly the same numbers. If we compute the bond returns based on yield curve scenarios that are, on average, unbiased or "viewless" (that is, we subtract the ten-basis-point mean yield change from each scenario), the probability-weighted expected portfolio return is 7.02%. We can estimate the duration impact of a view by comparing the portfolio's probability-weighted expected return given the five scenarios *with the rate view* (6.82%) to the expected return given the *"viewless"* set of scenarios. The difference is again -20 basis points. Similarly, we can estimate the value of convexity by comparing the probability-weighted expected return of the portfolio under a "viewless" set of scenarios (7.02%) to the expected return under the neutral scenario that has the same average rate view — an unchanged yield curve — but assumes no volatility (7.00%). The difference, two basis points, measures the pure effect of the implicit volatility forecast without a bias from a yield curve view.

APPENDIX A. DECOMPOSING THE FORWARD RATE STRUCTURE INTO ITS MAIN DETERMINANTS

In this appendix, we show how the forward rate structure is related to the market's rate expectations, bond risk premia and convexity bias. In particular, the holding-period return of an n-year zero-coupon bond can be described as a sum of its horizon return given an unchanged yield curve and the end-of-horizon price change that is caused by a change in the n-1 year constant-maturity spot rate (Δs_{n-1}). The horizon return equals a one-year forward rate, and the end-of-horizon price change can be approximated by duration and convexity effects. These relations are used to decompose near-term expected bond returns and the one-period forward rates into simple building blocks. All rates and returns used in the following equations are compounded annually and expressed in percentage terms.

$$\frac{h_n}{100} = \frac{P^*_{n-1} - P_n}{P_n} = \frac{(P^*_{n-1} - P_{n-1}) + (P_{n-1} - P_n)}{P_n} \tag{5}$$

$$= \left(\frac{\Delta P_{n-1}}{P_{n-1}} * \frac{P_{n-1}}{P_n} \right) + \left(\frac{P_{n-1}}{P_n} - 1 \right),$$

where h_n is the one-period holding-period return of an n-year bond, P_n is its price (today), P^*_{n-1} is its price in the next period (when its maturity is n-1), and $\Delta P_{n-1} = P^*_{n-1} - P_{n-1}$. The second term on the right-hand side of Equation (5) is the bond's rolling yield (horizon return). The first term on the right-hand side of Equation (5) is the instantaneous percentage price change of an n-1 year zero, multiplied by an adjustment term P_{n-1}/P_n.[12]

Equation (6) shows that the zero's rolling yield (P_{n-1}/P_n - 1) equals, by construction, the one-year forward rate between n-1 and n. Moreover, the adjustment term equals one plus the forward rate.

$$1 + \frac{f_{n-1,n}}{100} = \frac{(1 + \frac{s_n}{100})^n}{(1 + \frac{s_{n-1}}{100})^{n-1}} = \frac{P_{n-1}}{P_n}\,. \tag{6}$$

Equation (7) shows the well-known result that the percentage price change ($\Delta P/P$) is closely approximated by the first two terms of a Taylor series expansion, duration and convexity effects.

$$100 * \frac{\Delta P}{P} \approx - Dur * (\Delta s) + 0.5 * Cx * (\Delta s)^2, \tag{7}$$

$$\text{where } Dur \equiv - \frac{dP}{ds} * \frac{100}{P} \text{ and } Cx \equiv \frac{d^2P}{d^2s} * \frac{100}{P}\,.$$

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[12] The adjustment term is needed because the bond's instantaneous price change occurs at the end of horizon, not today. The value of the bond position grows from one to P_{n-1}/P_n at the end of horizon if the yield curve is unchanged. The end-of-horizon value (P_{n-1}/P_n) would be subject to the yield shift at horizon.

Plugging Equations (6) and (7) into (5), we get:

$$h_n \approx f_{n-1,n} + (1 + \frac{f_{n-1,n}}{100})*[-Dur_{n-1}*(\Delta s_{n-1}) + 0.5*Cx_{n-1}*(s_{n-1})^2]. \qquad (8)$$

Even if the yield curve shifts occur during the horizon, **for performance calculation purposes the repricing takes place at the end of horizon**. This disparity causes various differences between the percentage price changes in Equations (7) and (8). First, the amount of capital that experiences the price change grows to $(1 + f_{n-1,n}/100)$ by the end of horizon. Second, the relevant yield change is the change in the n-1 year constant-maturity rate, not in the n-year zero's own yield (the difference is the rolldown yield change).[13] Third, the end-of-horizon (as opposed to the current) duration and convexity determine the price change.

The realized return can be split into an expected part and an unexpected part. Taking expectations of both sides of Equation (8) gives us the n-year zero's expected return over the next year:

$$E(h_n) \approx f_{n-1,n} + (1 + \frac{f_{n-1,n}}{100})*[-Dur_{n-1}*E(\Delta s_{n-1}) + 0.5*Cx_{n-1}*E(s_{n-1})^2]. \qquad (9)$$

Recall from Equation (6) that the one-period forward rate equals a zero's rolling yield, which can be split to yield and rolldown return components. In addition, the expected yield change squared is approximately equal to the variance of the yield change or the squared volatility, $E(\Delta s_{n-1})^2 \approx (Vol(\Delta s_{n-1}))^2$. This relation is exact if the expected yield change is zero. Thus, **the zero's near-term expected return can be written (approximately) as a sum of the yield income, the rolldown return, the value of convexity, and the expected capital gains from the rate "view"** (see Equation (3)).

We can interpret the expectations in Equation (9) to refer to the market's rate expectations. Mechanically, the forward rate structure and the market's rate expectations on the right-hand side of Equation (9) determine the near-term expected returns on the left-hand side. These expected returns should equal the required returns that the market demands for various bonds if the market's expectations are internally consistent. These required returns, in turn, depend on factors such as each bond's riskiness and the market's risk aversion level. Thus, it is more appropriate to think that **the market participants, in the aggregate, *set the bond market prices* to be such that given the forward rate structure and the consensus rate expectations, each bond is expected to earn its required return.**[14]

[13] If we used bonds' own yield changes in Equation (8), these yield changes would include the rolldown yield change. In this case, we should not use the forward rate (which includes the impact of the rolldown yield change on the return, in addition to the yield income) as the first term on the right-hand side of Equation (8). Instead, we would use the spot rate.

[14] Individual investors also can use Equation (9) but the interpretation is slightly different because most of their investments are so small that they do not influence the market rates; thus, they are "price-takers." Any individual investor can plug his subjective rate expectations into Equation (9) and back out the expected return given these expectations and the market-determined forward rates. These expected returns may differ from the required returns that the market demands; this discrepancy may prompt the investor to trade on his view.

Subtracting the one-period riskless rate (s_1) from both sides of Equation (9), we get:

$$E(h_n - s_1) \approx$$
$$(f_{n-1,n} - s_1) + (1 + \frac{f_{n-1,n}}{100})*[-Dur_{n-1}*E(\Delta s_{n-1}) + 0.5*Cx_{n-1}*(Vol(s_{n-1}))^2]. \tag{10}$$

We define the bond risk premium as $BRP_n \equiv E(h_n - s_1)$ and the forward-spot premium as $FSP_n \equiv f_{n-1,n} - s_1$. The forward-spot premium measures the steepness of the one-year forward rate curve (the difference between each point on the forward rate curve and the first point on that curve) and it is closely related to simpler measures of yield curve steepness. Rearranging Equation (10), we obtain:

$$FSP_n \approx$$
$$BRP_n + (1 + \frac{f_{n-1,n}}{100})*[-Dur_{n-1}*E(\Delta s_{n-1}) + 0.5*Cx_{n-1}*(Vol(s_{n-1}))^2]. \tag{11}$$

In other words, **the forward-spot premium is approximately equal to a sum of the bond risk premium, the impact of rate expectations** (expected capital gain/loss caused by the market's rate "view") **and the convexity bias** (expected capital gain caused by the rate uncertainty). Unfortunately, none of the three components is directly observable.

The analysis thus far has been very general, based on accounting identities and approximations, not on economic assumptions. Various term structure hypotheses and models differ in their assumptions. Certain simplifying assumptions lead to well-known hypotheses of the term structure behavior by making some terms in Equation (11) equal zero — although fully specified term structure models require even more specific assumptions. First, if constant-maturity rates follow *a random walk*, the forward-spot premium mainly reflects the bond risk premium, but also the convexity bias [$E(\Delta s_{n-1}) = 0 \Rightarrow FSP_n \approx BRP_n + CB_{n-1}$]. Second, if the *local expectations hypothesis* holds (all bonds have the same near-term expected return), the forward-spot premium mainly reflects the market's rate expectations, but also the convexity bias [$BRP_n = 0 \Rightarrow FSP_n \approx Dur_{n-1}*E(\Delta s_{n-1}) + CB_{n-1}$]. Third, if the *unbiased expectations hypothesis* holds, the forward-spot premium only reflects the market's rate expectations [$BRP_n + CB_{n-1} = 0 \Rightarrow FSP_n \approx Dur_{n-1}*E(\Delta s_{n-1})$)]. The last two cases illustrate the distinction between two versions of the pure expectations hypothesis.

APPENDIX B. RELATING VARIOUS STATEMENTS ABOUT FORWARD RATES TO EACH OTHER

In the series *Understanding the Yield Curve*, we make several statements about forward rates — describing, interpreting and decomposing them in various ways. The multitude of these statements may be confusing; therefore, we now try to clarify the relationships between them.

We refer to the spot curve and the forward curves on a given date as if they were unambiguous. In reality, **different analysts can produce somewhat different estimates of the spot curve on a given date if they use different curve-fitting techniques or different underlying data** (asset universe or pricing source). We acknowledge the importance of these issues — having good raw material is important to any kind of yield curve analysis — but in our reports we ignore these differences. We take the estimated spot curve as given and focus on showing how to interpret and use the information in this curve.

In contrast, the relations between various depictions of the term structure of interest rates (par, spot and forward rate curves) are unambiguous. In particular, **once a spot curve has been estimated, any forward rate can be mathematically computed by using Equation (12):**

$$(1 + \frac{f_{m,n}}{100})^{n-m} = \frac{(1 + \frac{s_n}{100})^n}{(1 + \frac{s_m}{100})^m}, \qquad (12)$$

where $f_{m,n}$ is the annualized n-m year interest rate m years forward and s_n and s_m are the annualized n-year and m-year spot rates, expressed in percent. Thus, **a one-to-one mapping exists between forward rates and *current* spot rates.** The statement "the forwards imply rising rates" is equivalent to saying that "the spot curve is upward sloping," and the statement "the forwards imply curve flattening" is equivalent to saying that "the spot curve is concave." Moreover, an unambiguous mapping exists between various types of forward curves, such as the implied spot curve one year forward ($f_{1,n}$) and the curve of constant-maturity one-year forward rates ($f_{n-1,n}$).

The forward rate can be the agreed interest rate on an **explicitly** traded contract, a loan between two future dates. More often, the forward rate is **implicitly** defined from today's spot curve based on Equation (12). However, arbitrage forces ensure that even the explicitly traded forward rates would equal the implied forward rates and, thus, be consistent with Equation (12). For example, the implied one-year spot rate four years forward (also called the one-year forward rate four years ahead, $f_{4,5}$) must be such that the equality $(1 + s_5/100)^5 = (1 + s_4/100)^4 *(1 + f_{4,5}/100)$ holds. If $f_{4,5}$ is higher than that, arbitrageurs can earn profits by short-selling the five-year zeros and buying the four-year zeros and the one-year forward contracts four years ahead, and vice versa. Such activity should make the equality hold within transaction costs.

Forward rates can be viewed in many ways: the arbitrage interpretation; the break-even interpretation; and the rolling yield interpretation. According to the arbitrage interpretation, implied forward rates are such rates that would ensure the absence of riskless arbitrage opportunities between spot contracts (zeros) and forward contracts if the latter were traded. According to the break-even interpretation of forward rates, implied forward rates are such *future* spot rates that would equate holding-period returns across bond positions. According to the rolling yield interpretation, the one-period forward rates show the one-period horizon returns that various zeros earn if the yield curve remains unchanged. Footnotes 15-17 show that each interpretation is useful for a certain purpose: active view-taking relative to the forwards (break-even); relative value analysis given no yield curve views (rolling yield); and valuation of derivatives (arbitrage).

All of these interpretations hold by construction (from Equation (12)). **Thus, they are not inconsistent with each other.** For example, the one-period forward rates can be interpreted and used in quite different ways. **The implied one-year spot rate four years forward ($f_{4,5}$) can be viewed as either the break-even one-year rate *four years into the future* or the rolling yield of a five-year zero *over the next year*.** Both interpretations follow from the equality $(1 + s_5/100)^5 = (1 + s_4/100)^4 *$ $(1 + f_{4,5}/100)$. This equation shows that the forward rate is the break-even one-year reinvestment rate that would equate the returns between two strategies (holding the five-year zero to maturity versus buying the four-year zero and reinvesting in the one-year zero when the four-year zero matures) over a five-year horizon. [Rewriting the equality as $(1 + s_4/100)^4 = (1 + s_5/100)^5/(1 + f_{4,5}/100)$ gives a slightly different viewpoint; the forward rate also is the break-even selling rate that would equate the returns between two strategies (holding the four-year zero to maturity versus buying the five-year zero and selling it after four years as a one-year zero) over a four-year horizon.] Finally, rewriting the equality as $1 + f_{4,5}/100 = (1 + s_5/100)^5/(1 + s_4/100)^4$ shows that the forward rate is the horizon return from buying a five-year zero at rate s_5 and selling it one year later, as a four-year zero, at rate s_4 (thus, the constant-maturity four-year rate is unchanged from today). In this series, we focus on the last (rolling yield) interpretation.

Interpreting the one-period forward rates as rolling yields enhances our understanding about the relation between the curve of one-year forward rates ($f_{0,1}, f_{1,2}, f_{2,3}, \ldots, f_{n-1,n}$) and the implied spot curve one year forward ($f_{1,2}$, $f_{1,3}, f_{1,4}, \ldots, f_{1,n}$). The latter "break-even" curve shows how much the spot curve needs to shift to cause capital gains/losses that exactly offset initial rolling yield differentials across zeros and, thereby, equalize the holding-period returns. Thus, a steeply upward-sloping curve of one-period forward rates requires, or "implies," a large offsetting increase in the spot curve over the horizon, while a flat curve of one-period forward rates only implies a small "break-even" shift in the spot curve.[15] A similar link exists for the rolling yield differential between a duration-neutral barbell versus bullet and the break-even yield spread change (curve flattening) that is needed to offset the bullet's rolling yield advantage. These examples provide insight as to why an upward-sloping spot curve implies rising rates and why a concave spot curve implies a flattening curve.

[15] Part 1 of this series describes one common way to use the break-even forward rates. Investors can compare their subjective views about the yield curve at some future date (or about the path of some constant-maturity rate over time) to the forward rates and directly determine whether bullish or bearish strategies are appropriate. If the rate changes that the forwards imply are realized, all bonds earn the riskless return because $(1 + s_n/100)^n/(1 + f_{1,n}/100)^{n-1} = (1 + s_1/100)$. If rates rise by more than that, long bonds underperform short bonds. If rates rise by less than that, long bonds outperform short bonds (because their capital losses do not quite offset their initial yield advantage).

Appendix A shows that forward rates can be conceptually decomposed into three main determinants (rate expectations, risk premia, convexity bias). **One might hope that the arbitrage, break-even or rolling yield interpretations could help us in backing out the relative roles of rate expectations, risk premia and convexity bias in a given day's forward rate structure.** However, such hope is in vain. The three interpretations hold quite generally because of their mathematical nature. Thus, they do not guide us in decomposing the forward rate structure.

Therefore, even when two analysts agree that today's forward rate structure is an approximate sum of three components, they may disagree about the relative roles of these components. We can try to address this question empirically. It is closely related to the question about the forward rates' ability to forecast future rate changes and future bond returns. Ignoring convexity bias, if the forwards primarily reflect rate expectations, they should be unbiased predictors of future spot rates (and they should tell little about future bond returns). However, if the forwards mainly reflect required bond risk premia, they should be unbiased predictors of future bond returns (and they should tell little about future rate changes). In Part 2 of this series, *Market's Rate Expectations and Forward Rates*, we present some empirical evidence indicating that the forward rates are better predictors of future bond returns than of future rate changes.[16,17]

Finally, our analysis does not reveal the fundamental economic determinants of the required risk premia or the market's rate expectations — nor does it tell us to what extent the nominal rate expectations reflect expected inflation and expected real rates. Macroeconomic news about economic growth, inflation rates, budget deficits, and so on, can influence both the required risk premia and the market's rate expectations. More work is clearly needed to improve our understanding about the mechanisms of these influences.

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[16] This evidence also suggests that the current spot curve is a better predictor of the next-period spot curve than is the implied spot curve one period forward. These findings imply that the rolling yields are reasonable proxies for the near-term expected bond returns — although even rolling yields capture a very small part of the short-term realized bond returns. Note that the poorer the forwards are in predicting future rate changes, the better they are in predicting bond returns — because then the implied rate changes that would offset initial yield advantages tend to occur more rarely. Note also that some investors may not care whether the forwards' ability to predict bond returns reflects rational risk premia or the market's inability to forecast rate changes; they want to earn any predictable profit irrespective of its reason.

[17] One common misconception is that the forward rates are used in the valuation of swaps, options and other derivative instruments *because* the forwards are good predictors of future spot rates. In fact, the forwards' ability to predict future spot rates has nothing to do with their usefulness in derivatives pricing. Unlike forecasting returns, the valuation of derivatives is based on arbitrage arguments. For example, traders can theoretically construct, by dynamic hedging, a riskless combination of a risky long-term bond and an option written on it. The price of the option should be such that the hedged position earns the riskless rate — otherwise a riskless arbitrage opportunity arises. The forward rates are central in this valuation because the traders, via their hedging activity, can lock in these rates for future periods. This arbitrage argument implies that the yield curve option pricing models should be calibrated to be consistent with the market forward rates *despite* the fact that the forwards are quite poor predictors of future spot rates.

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The Dynamics of the Shape of the Yield Curve: Empirical Evidence, Economic Interpretations and Theoretical Foundations

Understanding the Yield Curve: Part 7

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FIGURES

How can we interpret the shape (steepness and curvature) of the yield curve *on a given day*? And how does the yield curve evolve *over time*? In this report, we examine these two broad questions about the yield curve behavior. We have shown in earlier reports that the market's rate expectations, required bond risk premia and convexity bias determine the yield curve shape. Now we discuss various economic hypotheses and empirical evidence about the relative roles of these three determinants in influencing the curve steepness and curvature. We also discuss term structure models that describe the evolution of the yield curve over time and summarize relevant empirical evidence.

The key determinants of the curve steepness, or slope, are the market's rate expectations and the required bond risk premia. The pure expectations hypothesis assumes that all changes in steepness reflect the market's shifting rate expectations, while the risk premium hypothesis assumes that the changes in steepness only reflect changing bond risk premia. In reality, rate expectations *and* required risk premia influence the curve slope. Historical evidence suggests that **above-average bond returns, and not rising long rates, are likely to follow abnormally steep yield curves. Such evidence is inconsistent with the pure expectations hypothesis and may reflect time-varying bond risk premia.** Alternatively, the evidence may represent irrational investor behavior and the long rates' sluggish reaction to news about inflation or monetary policy.

The determinants of the yield curve's curvature have received less attention in earlier research. It appears that the curvature varies primarily with the market's curve reshaping expectations. Flattening expectations make the yield curve more concave (humped), and steepening expectations make it less concave or even convex (inversely humped). It seems unlikely, however, that the average concave shape of the yield curve results from systematic flattening expectations. More likely, it reflects the convexity bias and the apparent required return differential between barbells and bullets. If convexity bias were the only reason for the concave average yield curve shape, one would expect a barbell's convexity advantage to exactly offset a bullet's yield advantage, in which case duration-matched barbells and bullets would have the same expected returns. Historical evidence suggests otherwise: **In the long run, bullets have earned slightly higher returns than duration-matched barbells. That is, the risk premium curve appears to be concave rather than linear in duration.** We discuss plausible explanations for the fact that investors, in the aggregate, accept lower expected returns for barbells than for bullets: the barbell's lower return volatility (for the same duration); the tendency of a flattening position to outperform in a bearish environment; and the insurance characteristic of a positively convex position.

Turning to the second question, we describe some empirical characteristics of the yield curve behavior that are relevant for evaluating various term structure models. The models differ in their assumptions regarding the expected path of short rates (degree of mean reversion), the role of a risk premium, the behavior of the unexpected rate component (whether yield volatility varies over time, across maturities or with the rate level), and the number and identity of factors influencing interest rates. For example, the simple model of parallel yield curve shifts is consistent with no mean reversion in interest rates and with constant bond risk premia over time. Across bonds, the assumption of parallel shifts implies that the term structure of yield volatilities is flat and that rate shifts are perfectly correlated (and, thus, driven by one factor).

Empirical evidence suggests that short rates exhibit quite slow mean reversion, that required risk premia vary over time, that yield volatility varies over time (partly related to the yield level), that the term structure of basis-point yield volatilities is typically inverted or humped, and that rate changes are not perfectly correlated — but two or three factors can explain 95%-99% of the fluctuations in the yield curve.

In Appendix A, we survey the broad literature on term structure models and relate it to the framework described in this series. It turns out that many popular term structure models allow the decomposition of yields to a rate expectation component, a risk premium component and a convexity component. However, the term structure models are more consistent in their analysis of relations across bonds because they specify exactly how a small number of systematic factors influences the whole yield curve. In contrast, our approach analyzes expected returns, yields and yield volatilities separately for each bond. In Appendix B, we discuss the theoretical determinants of risk premia in multi-factor term structure models and in modern asset pricing models.

HOW SHOULD WE INTERPRET THE YIELD CURVE STEEPNESS?

The steepness of yield curve primarily reflects the market's rate expectations and required bond risk premia because the third determinant, convexity bias, is only important at the long end of the curve. A particularly steep yield curve may be a sign of prevalent expectations for rising rates, abnormally high bond risk premia, or some combination of the two. Conversely, an inverted yield curve may be a sign of expectations for declining rates, negative bond risk premia, or a combination of declining rate expectations and low bond risk premia.

We can map statements about the curve shape to statements about the forward rates. When the yield curve is upward sloping, longer bonds have a yield advantage over the risk-free short bond, and the forwards "imply" rising rates. The implied forward yield curves show the break-even levels of future yields that would exactly offset the longer bonds' yield advantage with capital losses and that would make all bonds earn the same holding-period return.

Because expectations are not observable, we do not know with certainty the relative roles of rate expectations and risk premia. **It may be useful to examine two extreme hypotheses that claim that the forwards reflect only the market's rate expectations or only the required risk premia.** If the pure expectations hypothesis holds, the forwards reflect the market's rate expectations, and the implied yield curve changes are likely to be realized (that is, rising rates tend to follow upward-sloping curves and declining rates tend to succeed inverted curves). In contrast, if the risk premium hypothesis holds, the implied yield curve changes are not likely to be realized, and higher-yielding bonds earn their rolling-yield advantages, on average (that is, high excess bond returns tend to follow upward-sloping curves and low excess bond returns tend to succeed inverted curves).

Empirical Evidence

To evaluate the above hypotheses, we compare implied forward yield changes (which are proportional to the steepness of the forward rate curve) **to subsequent average realizations of yield changes and excess bond returns.**[1] In Figure 1, we report (i) the average spot yield curve shape, (ii) the average of the yield changes that the forwards imply for various constant-maturity spot rates over a three-month horizon, (iii) the average of realized yield changes over the subsequent three-month horizon, (iv) the difference between (ii) and (iii), or the average "forecast error" of the forwards, and (v) the estimated correlation coefficient between the implied yield changes and the realized yield changes over three-month horizons. We use overlapping monthly data between January 1968 and December 1995 — deliberately selecting a long neutral period in which the beginning and ending yield curves are very similar.

Figure 1. Evaluating the Implied Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1968-95

	3 Mo.	6 Mo.	9 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.
Mean Spot Rate	7.04	7.37	7.47	7.57	7.86	8.00	8.12	8.25	8.32
Mean Implied Rate Change	0.65	0.32	0.27	0.23	0.14	0.12	0.11	0.08	0.07
Mean Realized Rate Change	0.003	0.001	0.000	0.000	0.001	0.001	0.001	0.002	0.002
Mean Forecast Error	0.65	0.32	0.27	0.23	0.14	0.12	0.11	0.08	0.07
Correlation Between Implied and Realized Rate Changes	-0.04	-0.08	-0.10	-0.08	-0.10	-0.13	-0.13	-0.12	-0.13

Notes: Data source for all figures is Salomon Brothers (although Figures 3 and 11 have additional sources). The spot yield curves are estimated based on Treasury on-the-run bill and bond data using a relatively simple interpolation technique. (Given that the use of such synthetic bond yields may induce some noise to our analysis, we have ensured that our main results also hold for yield curves and returns of actually traded bonds, such as on-the-run coupon bonds and maturity-subsector portfolios.) The implied rate change is the difference between the constant-maturity spot rate that the forwards imply in a three-month period and the current spot rate. The implied and realized spot rate changes are computed over a three-month horizon using (overlapping) monthly data. The forecast error is their difference.

Figure 1 shows that, **on average, the forwards imply rising rates**, especially at short maturities — simply because the yield curve tends to be upward sloping. However, the rate changes that would offset the yield advantage of longer bonds have not materialized, on average, leading to **positive forecast errors**. Our unpublished analysis shows that this conclusion holds over longer horizons than three months and over various subsamples, including flat and steep yield curve environments. The fact that the forwards tend to imply too high rate increases is probably caused by positive bond risk premia.

The last row in Figure 1 shows that the estimated correlations of the implied forward yield changes (or the steepness of the forward rate curve) with subsequent yield changes are negative. These estimates suggest that, **if anything, yields tend to move in the opposite direction than that which the forwards imply**. Intuitively, small declines in long rates have followed upward-sloping curves, on average, thus augmenting the yield advantage of longer bonds (rather than offsetting it). Conversely, small yield increases have succeeded inverted curves, on average. The big bull markets of the 1980s and 1990s occurred when the yield curve was upward sloping, while the big bear markets in the 1970s occurred when the curve was inverted. We stress, however, that the negative correlations in Figure 1 are quite weak; they are not statistically significant.[2]

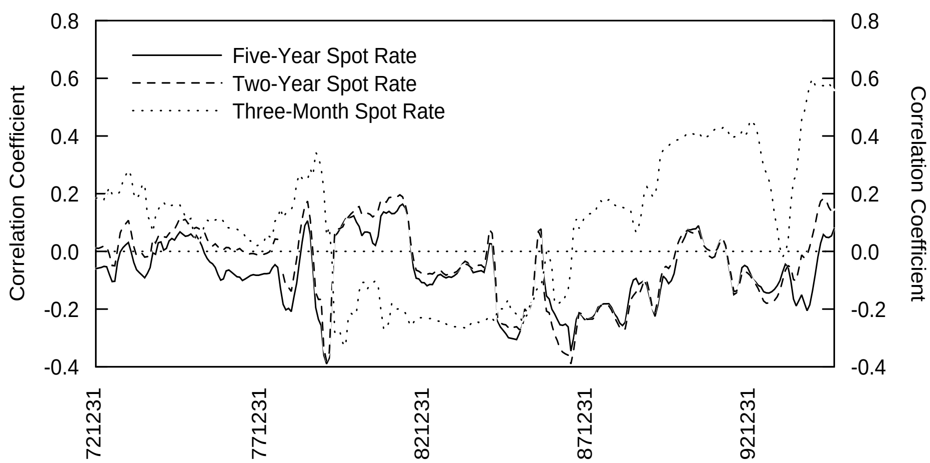
[1] Another way to get around the problem that the market's rate expectations are unobservable is to assume that a survey consensus view can proxy for these expectations. Comparing the forward rates with survey-based rate expectations indicates that changing rate expectations *and* changing bond risk premia induce changes in the curve steepness (see Figure 9 in Part 2 of this series and Figure 4 in Part 6).

[2] The deviations from the pure expectations hypothesis *are* statistically significant when we regress excess bond returns on the steepness of the forward rate curve. Moreover, as long as the correlations in Figure 1 are zero or below, long bonds tend to earn at least their rolling yields.

Many market participants believe that the bond risk premia are constant over time and that changes in the curve steepness, therefore, reflect shifts in the market's rate expectations. However, the empirical evidence in Figure 1 and in many earlier studies contradicts this conventional wisdom. **Historically, steep yield curves have been associated more with high subsequent excess bond returns than with ensuing bond yield increases**.[3]

One may argue that the historical evidence in Figure 1 is no longer relevant. Perhaps investors forecast yield movements better nowadays, partly because they can express their views more efficiently with easily tradable tools, such as the Eurodeposit futures. Some anecdotal evidence supports this view: Unlike the earlier yield curve inversions, the most recent inversions (1989 and 1995) were quickly followed by declining rates. **If market participants actually are becoming better forecasters, subperiod analysis should indicate that the implied forward rate changes have become better predictors of the subsequent rate changes**; that is, the rolling correlations between implied and realized rate changes should be higher in recent samples than earlier. In Figure 2, we plot such rolling correlations, demonstrating that **the estimated correlations have increased somewhat over the past decade**.

Figure 2. 60-Month Rolling Correlations Between the Implied Forward Rate Changes and Subsequent Spot Rate Changes, 1968-95



Notes: The Treasury spot yield curves are estimated based on on-the-run bill and bond data. The implied rate change is the difference between the constant-maturity spot rate that the forwards imply in a three-month period and the current spot rate. The implied and realized spot rate changes are computed over a three-month horizon using (overlapping) monthly data. The rolling correlations are based on the previous 60 months' data.

In Figure 3, we compare the forecasting ability of Eurodollar futures and Treasury bills/notes in the 1987-95 period. The average forecast errors are smaller in the Eurodeposit futures market than in the Treasury market, reflecting the flatter shape of the Eurodeposit spot curve (and perhaps the systematic "richness" of the shortest Treasury bills). In contrast, the correlations between implied and realized rate changes suggest that the Treasury forwards predict future rate changes slightly better than the

[3] Figure 7 in Part 2 shows that the forwards have predicted future excess bond returns better than they have anticipated future yield changes. Figures 2-4 in Part 4 show more general evidence of the forecastability of excess bond returns. In particular, combining yield curve information with other predictors can enhance the forecasts. The references in the cited reports provide formal evidence about the statistical significance of the predictability findings.

Eurodeposit futures do. A comparison with the correlations in Figure 1 (the long sample period) shows that the front-end Treasury forwards, in particular, have become much better predictors over time. For the three-month rates, this correlation rises from -0.04 to 0.45, while for the three-year rates, this correlation rises from -0.13 to 0.01. Thus, recent evidence is more consistent with the pure expectations hypothesis than the data in Figure 1, but these relations are so weak that it is too early to tell whether the underlying relation actually has changed. Anyway, even the recent correlations suggest that bonds longer than a year tend to earn their rolling yields.

Figure 3. Evaluating the Implied Eurodeposit and Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1987-95

	3 Mo.	6 Mo.	9 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.
Eurodeposits									
Mean Spot Rate	6.32	6.40	6.48	6.58	6.98	—	—	—	—
Mean Implied Rate Change	0.16	0.18	0.19	0.20	0.20				
Mean Realized Rate Change	-0.02	-0.02	-0.02	-0.02	-0.03				
Mean Forecast Error	0.18	0.20	0.21	0.22	0.23				
Correlation Between Implied and Realized Rate Changes	0.39	0.18	0.11	0.06	0.02				
Treasuries									
Mean Spot Rate	5.67	5.90	6.01	6.13	6.64	6.86	7.07	7.29	7.41
Mean Implied Rate Change	0.47	0.30	0.27	0.28	0.19	0.16	0.15	0.12	0.11
Mean Realized Rate Change	-0.01	-0.01	-0.01	-0.01	-0.02	-0.02	-0.03	-0.03	-0.03
Mean Forecast Error	0.47	0.30	0.28	0.29	0.21	0.18	0.18	0.14	0.14
Correlation Between Implied and Realized Rate Changes	0.45	0.32	0.28	0.17	0.04	0.01	-0.01	0.01	0.01

Notes: Data sources are Salomon Brothers and Chicago Mercantile Exchange. The Eurodeposit spot yield curves are estimated based on monthly Eurodeposit futures prices between 1987 and 1995. The Treasury spot yield curves are estimated based on on-the-run bill and bond data. (Note that the price-yield curve of Eurodeposit futures is linear; thus, the convexity bias does not influence the futures-based spot curve. However, convexity bias is worth only a couple of basis points for the two-year zeros.) For further details, see Figure 1.

Interpretations

The empirical evidence in Figure 1 is clearly inconsistent with the pure expectations hypothesis.[4] **One possible explanation is that curve steepness mainly reflects time-varying risk premia**, and this effect is variable enough to offset the otherwise positive relation between curve steepness and rate expectations. That is, if the market requires high risk premia, the current long rate will become higher and the curve steeper than what the rate expectations alone would imply — the yield of a long bond initially has to rise so high that it provides the required bond return by its high yield *and* by capital gain caused by its expected rate decline. In this case, rate expectations and risk premia are negatively related; the steep curve predicts high risk premia and declining long rates. This story could explain the steepening of the front end of the U.S. yield curve in spring 1994 (but not on many earlier occasions when policy tightening caused yield curve flattening).

The long-run average bond risk premia are positive (see Part 3 of this series and Figure 11 in this report) but the predictability evidence suggests that bond risk premia are time-varying rather than constant. **Why should required bond risk premia vary over time?** In general, an asset's risk premium reflects **the amount of risk and the market price of risk** (for

[4] However, some other evidence is more consistent with the expectations hypothesis than the short-run behavior of long rates. Namely, long rates often are reasonable estimates of the average level of the short rate over the life of the long bond (see John Campbell and Robert Shiller: "Yield Spreads and Interest Rate Movements: A Bird's Eye View," *Review of Economic Studies*, 1991).

details, see Appendix B). Both determinants can fluctuate over time and result in predictability. They may vary with the yield level (rate-level-dependent volatility) or market direction (asymmetric volatility or risk aversion) or with economic conditions. For example, **cyclical patterns in required bond returns may reflect wealth-dependent variation in the risk aversion level —** "the cycle of fear and greed."

Figure 4 shows the typical business cycle behavior of bond returns and yield curve steepness: **Bond returns are high and yield curves are steep near troughs, and bond returns are low and yield curves are flat/inverted near peaks.** These countercyclic patterns probably reflect the response of monetary policy to the economy's inflation dynamics, as well as time-varying risk premia (high risk aversion and required risk premia in "bad times" and vice versa). Figure 4 is constructed so that if bonds tend to earn their rolling yields, the two lines are perfectly aligned. However, the graph shows that bonds tend to earn additional capital gains (beyond rolling yields) from declining rates near cyclical troughs — and capital losses from rising rates near peaks. Thus, realized bond returns are related to the steepness of the yield curve and — in addition — to the level of economic activity.

Figure 4. Average Business Cycle Pattern of U.S. Realized Bond Risk Premium and Curve Steepness, 1968-95

Notes: Each line is constructed by computing the average value of a series in eight "stage of the business cycle" subsamples. Peak and trough subsamples refer to seven-month windows around the cyclical peaks and troughs, as defined by the National Bureau of Economic Research. In addition, each business cycle is split into three thirds of expansion and three thirds of contraction, and each month is assigned to one of these six subsamples. The spacing of subsamples in the x-axis is partially adjusted for the fact that expansions tend to last much longer than contractions. The forward-spot premium measures the steepness of the forward rate curve (the deannualized one-month rate 23 months forward minus the current one-month rate). The realized bond risk premium measures the monthly excess return of a synthetic two-year zero-coupon bond over a one-month bill. If the steepness of the forward rate curve is a one-for-one predictor of future excess returns, the two lines are perfectly aligned.

These empirical findings motivate the idea that the required bond risk premia vary over time with the steepness of the yield curve *and* with some other variables. In Part 4 of this series, we show that yield curve steepness indicators and real bond yields, combined with measures of recent stock and bond market performance, are **able to forecast up to 10% of the**

variation in monthly excess bond returns. That is, bond returns are partly forecastable. For quarterly or annual horizons, the predictable part is even larger.[5]

If market participants are rational, bond return predictability should reflect time-variation in the bond risk premia. Bond returns are predictably high when bonds command exceptionally high risk premia — either because bonds are particularly risky or because investors are exceptionally risk averse. Bond risk premia may also be high if increased supply of long bonds steepens the yield curve and increases the required bond returns. **An alternative interpretation is that systematic forecasting errors cause the predictability.** If forward rates really reflect the market's rate expectations (and no risk premia), these expectations are irrational. They tend to be too high when the yield curve is upward sloping and too low when the curve is inverted. The market appears to repeat costly mistakes that it could avoid simply by not trying to forecast rate shifts. Such irrational behavior is not consistent with market efficiency.

What kind of expectational errors would explain the observed patterns between yield curve shapes and subsequent bond returns? One explanation is a delayed reaction of the market's rate expectations to inflation news or to monetary policy actions. For example, if good inflation news reduces the current short-term rate but the expectations for future rates react sluggishly, the yield curve becomes upward-sloping, and subsequently the bond returns are high (as the impact of the good news is fully reflected in the rate expectations and in the long-term rates).[6]

Because expectations are not observable, we can never know to what extent the return predictability reflects time-varying bond risk premia and systematic forecast errors.[7] Academic researchers have tried to develop models that explain the predictability as rational variation in required returns. However, **yield volatility and other obvious risk measures seem to have little ability to predict future bond returns.** In contrast, the observed countercyclic patterns in expected returns suggest rational variation in the risk aversion level — although they also could reflect irrational changes in the market sentiment. Studies that use survey data to proxy for the market's expectations conclude that risk premia *and* irrational expectations contribute to the return predictability.

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[5] Our forecasting analysis focuses on excess return over the short rate, not the whole bond return. We do not discuss the time-variation in the short rate. The nominal short rate obviously reflects expected inflation and the required real short rate, both of which vary over time and across countries. From an international perspective, nominally riskless short-term rates in high-yielding countries may reflect expected depreciation and/or high required return (foreign exchange risk premium). In such countries, yield curves often are flat or inverted; investors earn a large compensation for holding the currency but little additional reward for duration extension.

[6] See Kenneth Froot's article "New Hope for the Expectations Hypothesis of the Term Structure of Interest Rates," *Journal of Finance*, 1989, and Werner DeBondt and Mary Bange's article "Inflation Forecast Errors and Time Variation in Term Premia," *Journal of Financial and Quantitative Analysis*, 1992.

[7] Other explanations to the apparent return predictability include "data mining" and "peso problem." Data mining or overfitting refers to situations in which excessive analysis of a data sample leads to spurious empirical findings. Peso problems refer to situations where investors *appear* to be making systematic forecast errors because the realized historical sample is not representative of the market's (rational) expectations. In the two decades between 1955 and 1975, Mexican interest rates were systematically higher than the U.S. interest rates although the peso-dollar exchange rate was stable. Because no devaluation occurred within this sample period, a statistician might infer that investors' expectations were irrational. This inference is based on the assumption that the ex post sample contains all the events that the market expects, with the correct frequency of occurrence. A more reasonable interpretation is that investors assigned a small probability to the devaluation of peso throughout this period. In fact, a large devaluation did occur in 1976, justifying the earlier investor concerns. Similar peso problems may occur in bond market analysis, for example, caused by unrealized fears of hyperinflation. That is, investors appear to be making systematic forecast errors when in fact investors are rational and the statistician is relying on benefit of hindsight. Similar problems occur when rational agents gradually learn about policy changes, and the statistician assumes that rational agents should know the eventual policy outcome during the sample period. However, while peso problems and learning could in principle induce some systematic forecast errors, it is not clear whether either phenomenon could cause exactly the type of systematic errors and return predictability that we observe.

Investment Implications

If expected bond returns vary over time, historical average returns contain less information about future returns than do indicators of the prevailing economic environment, such as the information in the current yield curve. In principle, the information in the forward rate structure is one of the central issues for fixed-income investors. **If the forwards** (adjusted for the convexity bias) **only reflect the market's rate expectations and if these expectations are unbiased** (they are realized, on average)**, then all government bond strategies would have the same near-term expected return. Yield-seeking activities (convergence trades and relative value trades) would be a waste of time and trading costs. Empirical evidence discussed above suggests that this is not the case**: Bond returns are partially predictable, and yield-seeking strategies are profitable in the long run.[8] However, it pays to use other predictors together with yields and to diversify across various positions, because the predictable part of bond returns is small and uncertain.

In practice, the key question is perhaps not whether the forwards reflect rate expectations or risk premia but whether actual return predictability exists and who should exploit it. No predictability exists if the forwards (adjusted for the convexity bias) reflect unbiased rate expectations. If predictability exists and is caused by expectations that are systematically wrong, everyone can exploit it. If predictability exists and is caused by rational variation in the bond risk premia, only some investors should take advantage of the opportunities to enhance long-run average returns; many others would find higher expected returns in "bad times" no more than a fair compensation for the greater risk or the higher risk aversion level. Only risk-neutral investors and atypical investors whose risk perception and risk tolerance does not vary synchronously with those of the market would want to exploit any profit opportunities — and these investors would not care whether rationally varying risk premia or the market's systematic forecast errors cause these opportunities.

HOW SHOULD WE INTERPRET THE YIELD CURVE CURVATURE?

The market's curve reshaping expectations, volatility expectations and expected return structure determine the curvature of the yield curve. Expectations for yield curve flattening imply expected profits for duration-neutral long-barbell versus short-bullet positions, tending to make the yield curve concave (thus, the yield disadvantage of these positions offsets their expected profits from the curve flattening). Expectations for higher volatility increase the value of convexity and the expected profits of these barbell-bullet positions, again inducing a concave yield curve shape. Finally, high required returns of intermediate bonds (bullets) relative to short and long bonds (barbells) makes the yield curve more concave. Conversely, expectations for yield curve steepening or for low volatility, together with bullets' low required returns, can even make the yield curve convex.

In this section, we analyze the yield curve curvature and **focus on two key questions: (1) How important are each of the three determinants in changing the curvature over time?; and (2) why is the long-run average shape of the yield curve concave?**

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[8] We provide empirical justification to a strategy that a naive investor would choose: Go for yield. A more sophisticated investor would say that this activity is wasteful because well-known theories — such as the pure expectations hypothesis in the bond market and the unbiased expectations hypothesis in the foreign exchange market — imply that positive yield spreads only reflect expectations of offsetting capital losses. Now we remind the sophisticated investor that these well-known theories tend to fail in practice.

Empirical Evidence

Some earlier studies suggest that the curvature of the yield curve is closely related to the market's volatility expectations, presumably due to the convexity bias. However, **our empirical analysis indicates that the curvature varies more with the market's curve-reshaping expectations than with the volatility expectations**. The broad curvature of the yield curve varies closely with the steepness of the curve, probably reflecting mean-reverting rate expectations.

Figure 5 plots the Treasury spot curve when the yield curve was at its steepest and at its most inverted in recent history and on a date when the curve was extremely flat. This graph suggests that historically low short rates have been associated with steep yield curves and high curvature (concave shape), while historically high short rates have been associated with inverted yield curves and negative curvature (convex shape).

Figure 5. Treasury Spot Yield Curves in Three Environments

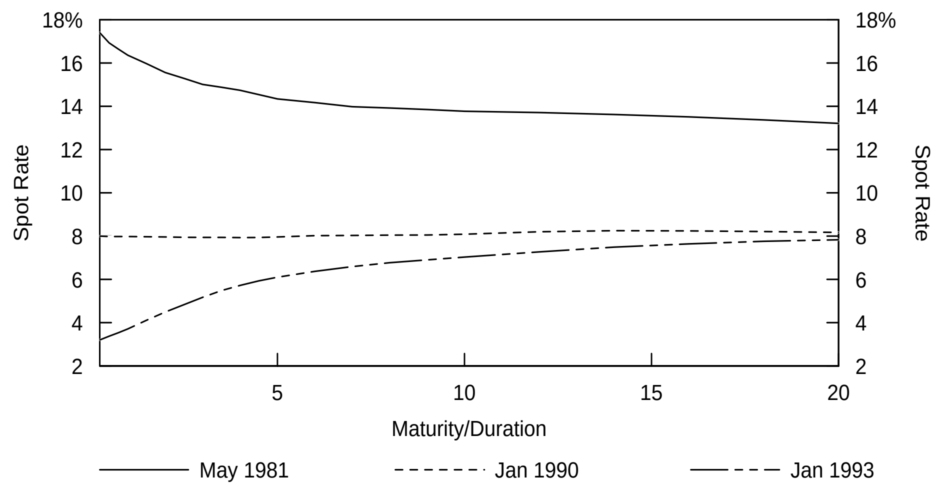


Figure 6. Correlation Matrix of Yield Curve Level, Steepness and Curvature, 1968-95

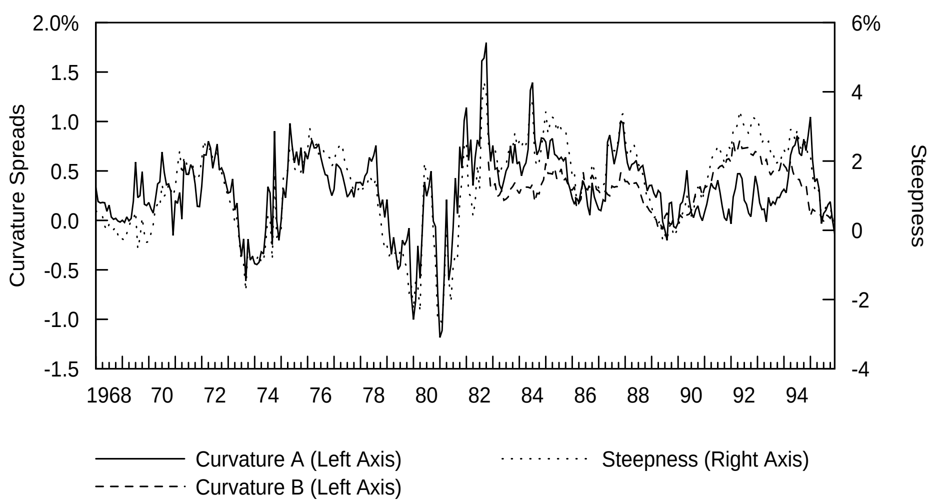
	3-Mo. Rate	6-Yr. Rate	Steepness	Curvature
3-Mo. Spot Rate	1.00			
6-Yr. Spot Rate	0.70	1.00		
Steepness	-0.43	-0.04	1.00	
Curvature	-0.20	0.10	0.79	1.00

Notes: The Treasury spot yield curves are estimated based on on-the-run bill and bond data (see Figure 1). The correlations are between the monthly changes in spot rates (or their spreads). Steepness refers to the yield spread between the six-year spot rate and the three-month spot rate. Curvature refers to the yield spread between a synthetic bullet (three-year zero) and a duration-matched barbell (0.5 * three-month zero + 0.5 * 5.75-year zero).

The correlation matrix of the monthly changes in yield levels, curve steepness and curvature in Figure 6 confirms these relations. **Steepness measures are negatively correlated with the short rate levels** (but almost uncorrelated with the long rate levels), reflecting the higher likelihood of bull steepeners and bear flatteners than bear steepeners and bull flatteners. However, we focus on **the high correlation (0.79) between the changes in the steepness and the changes in the curvature**. This relation has a nice economic logic. Our curvature measure can be viewed as the yield carry of a curve-steepening position, a duration-weighted bullet-barbell position (long a synthetic three-year zero and short equal amounts of a three-month zero and a 5.75-year zero). If market participants have mean-reverting rate expectations, they expect yield curves to revert to a certain average shape (slightly upward sloping) in the long run. Then, exceptionally steep curves are associated with expectations for subsequent curve flattening and for capital losses on steepening positions. Given the expected capital losses, these positions need to offer an initial yield pickup, which leads to a concave (humped) yield curve shape. Conversely, abnormally flat or inverted yield curves are associated with the market's expectations for subsequent curve steepening and for capital gains on steepening positions. Given the expected capital gains, these positions can offer an initial yield giveup, which induces a convex (inversely humped) yield curve.

Figure 7 illustrates **the close comovement between our curve steepness and curvature measures. The mean-reverting rate expectations described above are one possible explanation for this pattern.** Periods of steep yield curves (mid-1980s and early 1990s) are associated with high curvature and, thus, a large yield pickup for steepening positions, presumably to offset their expected losses as the yield curve flattens. In contrast, periods of flat or inverted curves (1979-81, 1989-90 and 1995) are associated with low curvature or even an inverse hump. Thus, barbells can pick up yield and convexity over duration-matched bullets, presumably to offset their expected losses when the yield curve is expected to steepen toward its normal shape.

Figure 7. Curvature and Steepness of the Treasury Curve, 1968-95

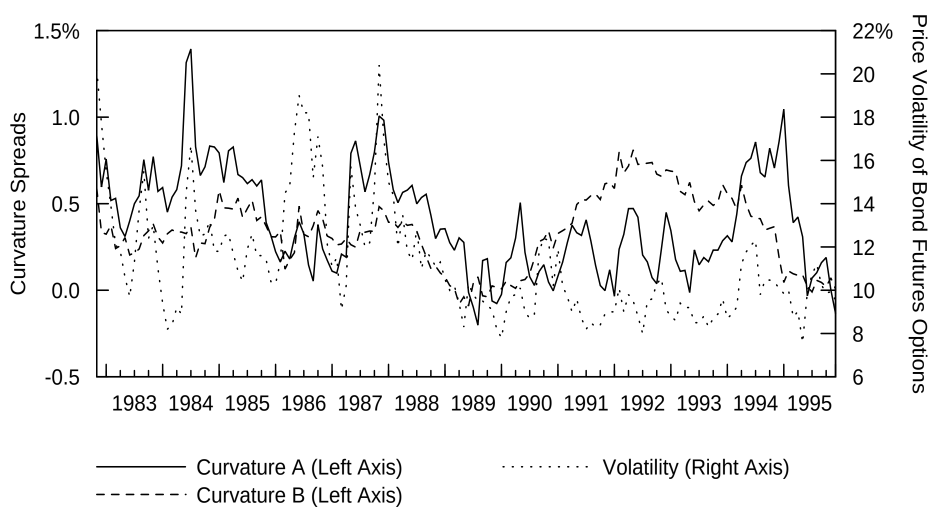


Notes: Curvature A refers to the yield spread between a long bullet (three-year zero) and a duration-matched short barbell (0.5 * three-month zero + 0.5 * 5.75-year zero). Curvature B refers to the yield spread between a bullet (ten-year on-the-run bond) and a duration-matched barbell (duration-weighted combination of two-year and 30-year on-the-run bonds); this series begins in 1982. Steepness refers to the yield spread between the six-year spot rate and the three-month spot rate.

The expectations for mean-reverting curve steepness influence the broad curvature of the yield curve. In addition, the curvature of the front end sometimes reflects the market's strong view about near-term monetary policy actions and their impact on the curve steepness. Historically, the Federal Reserve and other central banks have tried to smooth interest rate behavior by gradually adjusting the rates that they control. Such a rate-smoothing policy makes the central bank's actions partly predictable and induces a positive autocorrelation in short-term rate behavior. Thus, if the central bank has recently begun to ease (tighten) monetary policy, it is reasonable to expect the monetary easing (tightening) to continue and the curve to steepen (flatten).

In the earlier literature, the yield curve curvature has been mainly associated with the level of volatility. Litterman, Scheinkman and Weiss ("Volatility and the Yield Curve," *Journal of Fixed Income*, 1991) pointed out that higher volatility should make the yield curve more humped (because of convexity effects) and that a close relation appeared to exist between the yield curve curvature and the implied volatility in the Treasury bond futures options. However, **Figure 8 shows that the relation between curvature and volatility was close only during the sample period of the study (1984-88)**. Interestingly, no recessions occurred in the mid-1980s, the yield curve shifts were quite parallel and the flattening/steepening expectations were probably quite weak. The relation breaks down before and after the 1984-88 period — especially near recessions, when the Fed is active and the market may reasonably expect curve reshaping. For example, in 1981 yields were very volatile but the yield curve was convex (inversely humped); see Figures 5 and 13. It appears that the market's expectations for future curve reshaping are more important determinants of the yield curve curvature than are its volatility expectations (convexity bias). **The correlations of our curvature measures with the curve steepness are around 0.8 while those with the implied option volatility are around 0.1.** Therefore, it is not surprising that the implied volatility estimates that are based on the yield curve curvature are not closely related to the implied volatilities that are based on option prices. Using the yield curve shape to derive implied volatility can result in negative volatility estimates; this unreasonable outcome occurs in simple models when the expectations for curve steepening make the yield curve inversely humped (see Part 5 of this series).

Figure 8. Curvature and Volatility in the Treasury Market, 1982-95



Notes: Curvature A refers to the yield spread between a duration-matched long bullet (three-year zero) and short barbell (0.5 * three-month zero + 0.5 * 5.75-year zero). Curvature B refers to the yield spread between a bullet (ten-year on-the-run bond) and a duration-matched barbell (duration-weighted combination of two-year and 30-year on-the-run bonds). Volatility refers to the implied volatility of at-the-money options of the Treasury bond futures; these options began to trade in 1982.

Now we move to the second question "Why is the long-run average shape of the yield curve concave?" Figure 9 shows that the average par and spot curves have been concave over our 28-year sample period.[9] Recall that the concave shape means that the forwards have, on average, implied yield curve flattening (which would offset the intermediate bonds' initial yield advantage over duration-matched barbells). Figure 10 shows that, **on average, the implied flattening has not been matched by sufficient realized flattening.** Not surprisingly, flattenings and steepenings tend to wash out over time, whereas the concave spot curve shape has been quite persistent. In fact, a significant positive correlation exists between the implied and the realized curve flattening, but the average forecast errors in Figure 10 reveal a bias of too much implied flattening. This conclusion holds when we split the sample into shorter subperiods or into subsamples of a steep versus a flat yield curve environment or a rising-rate versus a falling-rate environment.

Figure 9. Average Yield Curve Shape, 1968-95

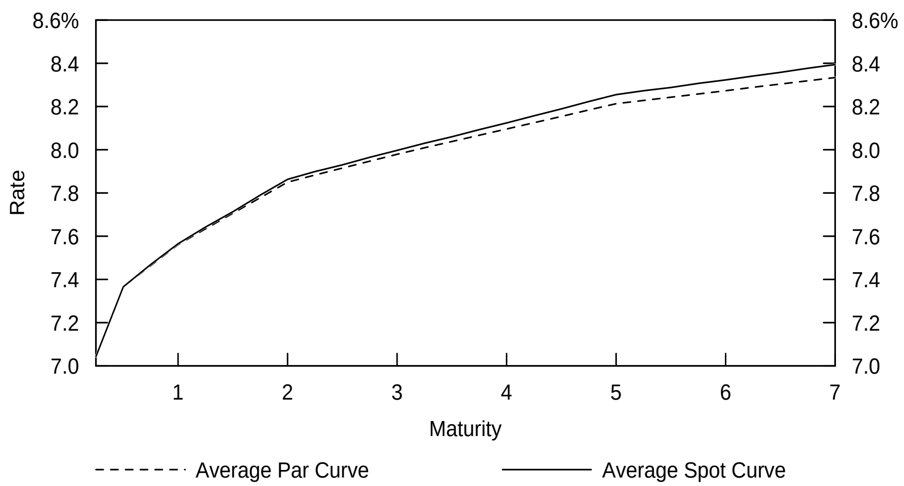


Figure 10. Evaluating the Implied Forward Yield Curve's Ability to Predict Actual Changes in the Spot Yield Curve's Steepness, 1968-95

	6 Mo.-3 Mo.	1 Yr.-6 Mo.	3 Yr.-1 Yr.	6 Yr.-3 Yr.	6 Yr.-3 Mo.
Mean Spread (Steepness)	0.33	0.19	0.44	0.32	1.28
Mean Implied Spread Change	-0.33	-0.09	-0.11	-0.05	-0.58
Mean Realized Spread Change	-0.002	-0.001	0.001	0.001	-0.001
Mean Forecast Error	-0.33	-0.09	-0.11	-0.05	-0.58
Correlation Between Implied and Realized Spread Changes	0.53	0.45	0.20	0.03	0.21

[9] Our discussion will focus on the concavity of the spot curve. Some authors have pointed out that the coupon bond yield curve tends to be concave (as we see in Figure 9) and have tried to explain this fact in the following way: If the spot curve were linearly upward-sloping and the par yields were linearly increasing in duration, the par curve would be a concave function of maturity because the par bonds' durations are concave in maturity. However, this is only a partial explanation to the par curve's concavity because Figure 9 shows that the average spot curve too is concave in maturity/duration.

Figure 10 shows that, on average, the capital gains caused by the curve flattening have not offset a barbell's yield disadvantage (relative to a duration-matched bullet). A more reasonable possibility is that the barbell's convexity advantage has offset its yield disadvantage. We can evaluate this possibility by examining the impact of convexity on realized returns over time. Empirical evidence suggests that the convexity advantage is not sufficient to offset the yield disadvantage (see Figure 12 in Part 5 of this series). Alternatively, **we can examine the shape of historical average returns because the realized returns should reflect the convexity advantage. This convexity effect is certainly a partial explanation for the typical yield curve shape — but it is the sole effect only if duration-matched barbells and bullets have the same expected returns.** Equivalently, if the required bond risk premium increases linearly with duration, the average returns of duration-matched barbells and bullets should be the same over a long neutral period (because the barbells' convexity advantage *exactly* offsets their yield disadvantage). The average return curve shape in Figure 1, Part 3 and the average barbell-bullet returns in Figure 11, Part 5 suggest that **bullets have somewhat higher long-run expected returns than duration-matched barbells**. We can also report the historical performance of synthetic zero positions over the 1968-95 period: The average annualized monthly return of a four-year zero is 9.14%, while the average returns of increasingly wide duration-matched barbells are progressively lower (3-year and 5-year 9.05%, 2-year and 6-year 9.00%, 1-year and 7-year 8.87%). **Overall, the typical concave shape of the yield curve likely reflects the convexity bias and the concave shape of the average bond risk premium curve** rather than systematic flattening expectations, given that the average flattening during the sample is zero.

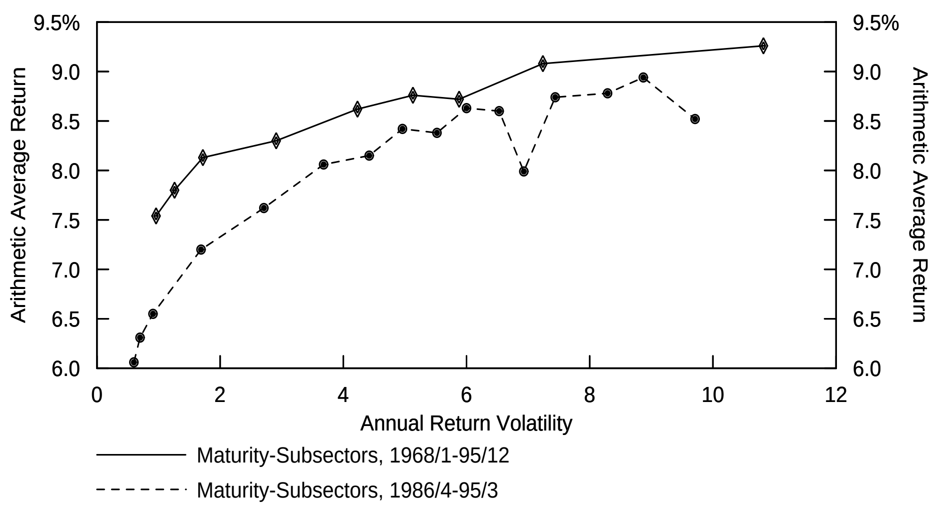
Interpretations

The impact of curve reshaping expectations and convexity bias on the yield curve shape are easy to understand, but the concave shape of the bond risk premium curve is more puzzling. In this subsection, we explore **why bullets should have a mild expected return advantage over duration-matched barbells**. **One likely answer is that duration is not the relevant risk measure.** However, we find that average returns are concave even in return volatility, suggesting a need for a multi-factor risk model. We first discuss various risk-based explanations in detail and then consider some alternative "technical" explanations for the observed average return patterns.

All one-factor term structure models imply that expected returns should increase linearly with the bond's sensitivity to the risk factor. Because these models assume that bond returns are perfectly correlated, expected returns should increase linearly with return volatility (whatever the risk factor is). However, bond durations are proportional to return volatilities only if all bonds have the same basis-point yield volatilities. **Perhaps the concave shape of the average return-duration curve is caused by (i) a linear relation between expected return and return volatility and (ii) a concave relation between return volatility and duration that, in turn, reflects an inverted or humped term structure of yield volatility** (see Figure 15). Intuitively, a concave relation between the actual return volatility and duration would make a barbell a more defensive (bearish) position than a duration-matched bullet. The return volatility of a barbell is simply a weighted average of its constituents' return volatilities (given the perfect correlation); thus, the barbell's volatility would be lower than that of a duration-matched bullet.

Figures 13 and 14 will demonstrate that **the empirical term structure of yield volatility has been inverted or humped most of the time. Thus, perhaps a barbell and a bullet with equal return volatilities (as opposed to equal durations) should have the same expected return. However, it turns out that the bullet's return advantage persists even when we plot average returns on historical return volatilities.** Figure 11 shows the historical average returns of various maturity-subsector portfolios of Treasury bonds as a function of return volatility. The average returns are based on two relatively neutral periods, January 1968 to December 1995 and April 1986 to March 1995. We still find that the average return curves have a somewhat concave shape. Note that we demonstrate the concave shape in a conservative way by graphing arithmetic average returns; the geometric average return curves would be even more concave.[10]

Figure 11. Average Treasury Maturity-Subsector Returns as a Function of Return Volatility



Notes: Data sources are Salomon Brothers, Center of Research for Security Prices at the University of Chicago and Ibbotson Associates. The curves show the annualized arithmetic averages of monthly returns of various Treasury bill and bond portfolios as a function of return volatility. The two curves differ in that we can split the Treasury market into narrower maturity-subsector buckets in the more recent sample. The first three points in each curve correspond to constant-maturity three-month, six-month, and nine-month bill portfolios. The next four points correspond to maturity-subsector portfolios of 1-2, 2-3, 3-4, and 4-5 year Treasuries. The last two points in the longer sample correspond to a 5-10 year bond portfolio and a 20-year bond portfolio. The last nine points in the shorter sample correspond to maturity-subsector portfolios of 5-6, 6-7, 7-8, 8-9, 9-10, 10-15, 15-20, 20-25, and 25-30 year Treasuries. Our return calculations ignore the on-the-run bonds' repo market advantage, partly explaining the low returns of the 9-10 year and the 25-30 year Treasury portfolios.

As explained above, one-factor term structure models assume that bond returns are perfectly correlated. One-factor asset pricing models are somewhat more general. They assume that realized bond returns are influenced by only one systematic risk factor but that they also contain a bond-specific residual risk component (which can make individual bond

[10] Here is another way of making our point: If short rates are more volatile than long rates, a duration-matched long-barbell versus short-bullet position would have a negative "empirical duration" or beta (rate level sensitivity). That is, even though the position has zero (traditional) duration, it tends to be profitable in a bearish environment (when curve flattening is more likely) and unprofitable in a bullish environment (when curve steepening is more likely). This negative beta property could explain the lower expected returns for barbells versus duration-matched bullets, if expected returns actually are linear in return volatility. However, the concave shapes of the average return curves in Figure 11 imply that even when barbells are weighted so that they have the same return volatility as bullets (and thus, the barbell-bullet position empirically has zero rate level sensitivity), they tend to have lower returns.

returns imperfectly correlated). Because the bond-specific risk is easily diversifiable, only systematic risk is rewarded in the marketplace. Therefore, expected returns are linear in the systematic part of return volatility. This distinction is not very important for government bonds because their bond-specific risk is so small. If we plot the average returns on systematic volatility only, the front end would be slightly less steep than in Figure 11 because a larger part of short bills' return volatility is asset-specific. Nonetheless, the overall shape of the average return curve would remain concave.

Convexity bias and the term structure of yield volatility explain the concave shape of the average yield curve partly, but a nonlinear expected return curve appears to be an additional reason. **Figure 11 suggests that expected returns are somewhat concave in return volatility. That is, long bonds have lower required returns than one-factor models imply. Some desirable property in the longer cash flows makes the market accept a lower expected excess return per unit of return volatility for them than for the intermediate cash flows. We need a second risk factor, besides the rate level risk, to explain this pattern.** Moreover, this pattern may teach us something about the nature of the second factor and about the likely sign of its risk premium. We will next discuss heuristically two popular candidates for the second factor — interest rate volatility and yield curve steepness. We further discuss the theoretical determinants of required risk premia in Appendix B.

Volatility as the second factor could explain the observed patterns if the market participants, in the aggregate, prefer insurance-type or "long-volatility" payoffs. Even nonoptionable government bonds have an optionlike characteristic because of the convex shape of their price-yield curves. As discussed in Part 5 of this series, the value of convexity increases with a bond's convexity and with the perceived level of yield volatility. If the volatility risk is not "priced" in expected returns (that is, if all "delta-neutral" option positions earn a zero risk premium), a yield disadvantage should exactly offset longer bonds' convexity advantage. However, the concave shape of the average return curve in Figure 11 suggests that positions that benefit from higher volatility have lower expected returns than positions that are adversely affected by higher volatility. Although the evidence is weak, we find the negative sign for the price of volatility risk intuitively appealing. **The Treasury market participants may be especially averse to losses in high-volatility states, or they may prefer insurance-type (skewed) payoffs so much that they accept lower long-run returns for them.**[11] Thus, the long bonds' low expected return could reflect the high value many investors assign to positive convexity. However, because short bonds exhibit little convexity, other factors are needed to explain the curvature at the front end of the yield curve.

Yield curve steepness as the second factor (or short rate and long rate as the two factors) could explain the observed patterns if curve-flattening positions tend to be profitable just when investors value them most. We do not think that the curve steepness is by itself a

[11] Some market participants prefer payoff patterns that provide them insurance. Other market participants prefer to sell insurance because it provides high current income. Based on the analysis of Andre Perold and William Sharpe ("Dynamic Strategies for Asset Allocation," *Financial Analysts Journal*, 1989), we argue that following the more popular strategy is likely to earn lower return (because the price of the strategy will be bid very high). It is likely that the Treasury market ordinarily contains more insurance seekers than income-seekers (insurance sellers), perhaps leading to a high price for insurance. However, the relative sizes of the two groups may vary over time. In good times, many investors reach for yield and don't care for insurance. In bad times, some of these investors want insurance — after the accident.

risk factor that investors worry about, but it may tend to coincide with a more fundamental factor. Recall that the concave average return curve suggests that self-financed curve-flattening positions have negative expected returns — because they are more sensitive to the long rates (with low reward for return volatility) than to the short/intermediate rates (with high reward for return volatility). **This negative risk premium can be justified theoretically if the flattening trades are especially good hedges against "bad times."** When asked what constitutes bad times, an academic's answer is a period of high marginal utility of profits, while a practitioner's reply probably is a deep recession or a bear market. The empirical evidence on this issue is mixed. It is clear that long bonds performed very well in deflationary recessions (the United States in the 1930s, Japan in the 1990s). However, they did not perform at all well in the stagflations of the 1970s when the predictable and realized excess bond returns were negative. Since the World War II, the U.S. long bond performance has been positively correlated with the stock market performance — although bonds turned out to be a good hedge during the stock market crash of October 1987. Turning now to flattening positions, these have not been good recession hedges either; the yield curves typically have been flat or inverted at the beginning of a recession and have steepened during it (see also Figure 4).[12] Nonetheless, flattening positions typically have been profitable in a rising rate environment; thus, they have been reasonable hedges against a bear market *for bonds*.

We conclude that risk factors that are related to volatility or curve steepness could perhaps explain the concave shape of the average return curve — but these are not the only possible explanations. "Technical" or "institutional" explanations include the value of liquidity (the ten-year note and the 30-year bond have greater liquidity and lower transaction costs than the 11-29 year bonds, and the on-the-run bonds can earn additional income when they are "special" in the repo market), institutional preferences (immunizing pension funds may accept lower yield for "riskless" long-horizon assets, institutionally constrained investors may demand the ultimate safety of one-month bills at any cost, fewer natural holders exist for intermediate bonds), and the segmentation of market participants (the typical short-end holders probably tolerate return volatility less well than do the typical long-end holders, which may lead to a higher reward for duration extension at the front end).[13]

Investment Implications

Bullets tend to outperform barbells in the long run, although not by much. It follows that as a long-run policy, it might be useful to bias the investment benchmarks and the core Treasury holdings toward intermediate bonds, given any duration. **In the short run, the relative performance of barbells and bullets varies substantially — and mainly with the yield curve reshaping.** Investors who try to "arbitrage" between the volatility implied in the curvature of the yield curve and the yield volatility implied in option prices will find it very difficult to neutralize the inherent curve shape exposure in these trades. An interesting task for future research is to

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[12] Another perspective may clarify our subtle point. Long bonds typically perform well in recessions, but leveraged extensions of intermediate bonds (that are duration-matched to long bonds) perform even better because their yields decline more. Thus, the recession-hedging argument cannot easily explain the long bonds' low expected returns relative to the intermediate bonds — unless various impediments to leveraging have made the long bonds the best realistic recession-hedging vehicles.

[13] Simple segmentation stories do not explain why arbitrageurs do not exploit the steep slope at the front end and the flatness beyond two years and thereby remove such opportunities. A partial explanation is that arbitrageurs cannot borrow at the Treasury bill rate; the higher funding cost limits their profit opportunities. These opportunities also are not riskless. In addition, while it is likely that supply and demand effects influence maturity-specific required returns and the yield curve shape in the short run, we would expect such effects to wash out in the long run.

study how well barbells' and bullets' relative short-run performance can be forecast using predictors such as the yield curve curvature (yield carry), yield volatility (value of convexity) and the expected mean reversion in the yield spread.

HOW DOES THE YIELD CURVE EVOLVE OVER TIME?

The framework used in the series *Understanding the Yield Curve* is very general; it is based on identities and approximations rather than on economic assumptions. As discussed in Appendix A, **many popular term structure models allow the decomposition of forward rates into a rate expectation component, a risk premium component, and a convexity bias component. However, various term structure models make different assumptions about the behavior of the yield curve over time.** Specifically, the models differ in their assumptions regarding the number and identity of factors influencing interest rates, the factors' expected behavior (the degree of mean reversion in short rates and the role of a risk premium) and the factors' unexpected behavior (for example, the dependency of yield volatility on the yield level). **In this section, we describe some empirical characteristics of the yield curve behavior that are relevant for evaluating the realism of various term structure models.**[14] In Appendix A, we survey other aspects of the term structure modelling literature. Our literature references are listed after the appendices; until then we refer to these articles by author's name.

The simple model of only parallel shifts in the spot curve makes extremely restrictive and unreasonable assumptions — for example, it does not preclude negative interest rates.[15] In fact, it is equivalent to the Vasicek (1977) model with no mean reversion. All one-factor models imply that rate changes are perfectly correlated across bonds. The parallel shift assumption requires, in addition, that the basis-point yield volatilities are equal across bonds. Other one-factor models may imply other (deterministic) relations between the yield changes across the curve, such as multiplicative shifts or greater volatility of short rates than of long rates. Multi-factor models are needed to explain the observed imperfect correlations across bonds — as well as the nonlinear shape of expected bond returns as a function of return volatility that was discussed above.

Time-Series Evidence

In our brief survey of empirical evidence, we find it useful to first focus on the time-series implications of various models and then on their cross-sectional implications. **We begin by examining the expected part of yield changes, or the degree of mean reversion in interest rate levels and spreads.** If interest rates follow a random walk, the current interest rate is the best forecast for future rates — that is, changes in rates are unpredictable. In this case, the correlation of (say) a monthly change in a rate with the beginning-of-month rate level or with the previous month's rate change should be zero. If interest rates do not follow a random walk, these correlations need not equal zero. In particular, if rates are

[14] We provide empirical evidence on the historical behavior of nominal interest rates. This evidence is not directly relevant for evaluating term structure models in some important situations. First, when term structure models are used to value derivatives in an arbitrage-free framework, these models make assumptions concerning the risk-neutral probability distribution of interest rates, not concerning the real-world distribution. Second, equilibrium term structure models often describe the behavior of real interest rates, not nominal rates.

[15] Moreover, a model with parallel shifts would offer riskless arbitrage opportunities if the yield curves were flat. Duration-matched long-barbell versus short-bullet positions with positive convexity could only be profitable (or break even) because there would be no yield giveup or any possibility of capital losses caused by the curve steepening. However, the parallel shift model would not offer riskless arbitrage opportunities if the spot curves were concave (humped) because the barbell-bullet positions' yield giveup could more than offset their convexity advantage.

mean-reverting, the slope coefficient in a regression of rate changes on rate levels should be negative. That is, falling rates should follow abnormally high rates and rising rates should succeed abnormally low rates.

Figure 12 shows that **interest rates do not exhibit much mean reversion over short horizons**. The slope coefficients of yield changes on yield levels are negative, consistent with mean reversion, but they are not quite statistically significant. Yield curve steepness measures are more mean-reverting than yield levels. Mean reversion is more apparent at the annual horizon than at the monthly horizon, consistent with the idea that mean reversion is slow. In fact, yield changes seem to exhibit some trending tendency in the short run (the autocorrelation between the monthly yield changes are positive), until a "rubber-band effect" begins to pull yields back when they get too far from the perceived long-run mean. Such a long-run mean probably reflects the market's views on sustainable real rate and inflation levels as well as a perception that a hyperinflation is unlikely and that negative nominal interest rates are ruled out (in the presence of cash currency). If we focus on the evidence from the 1990s (not shown), the main results are similar to those in Figure 12, but short rates are more predictable (more mean-reverting and more highly autocorrelated) than long rates, probably reflecting the Fed's rate-smoothing behavior.

Figure 12. Mean Reversion and Autocorrelation of U.S. Yield Levels and Curve Steepness, 1968-95

	3 Mo.		2 Yr.		30 Yr.		2 Yr.-3 Mo.		30 Yr.-2 Yr.	
	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.
Mean-Reversion										
Coefficient	-0.03	-0.26	-0.02	-0.26	-0.01	-0.20	-0.13	-0.74	-0.06	-0.42
t-statistic	(-1.26)	(-1.75)	(-1.20)	(-1.74)	(-1.10)	(-1.36)	(-3.40)	(-2.49)	(-2.19)	(-2.70)
R^2	2%	12%	1%	13%	1%	10%	7%	37%	3%	22%
First Autocorrelation										
Coefficient	0.10	0.24	0.17	-0.06	0.15	-0.10	-0.12	-0.21	0.08	0.10
t-statistic	(0.87)	(1.26)	(2.46)	(-0.39)	(2.08)	(-0.46)	(-1.39)	(-1.48)	(1.08)	(0.56)
R^2	1%	6%	3%	1%	2%	1%	1%	4%	1%	1%

Notes: These numbers are based on the on-the-run yields of a three-month bill, a two-year note, and a 30-year (or the longest available) bond. We use 335 monthly observations and 27 annual observations. The mean-reversion coefficient is the slope coefficient in a regression of each yield change on its beginning-of-period level. The first-order autocorrelation coefficient is the slope coefficient in a regression of each yield change on the previous period's yield change. The (robust) t-statistics measure the statistical significance and the (unadjusted) R^2 values measure the explanatory power in the regression.

Moving to the unexpected part of yield changes, we analyze the behavior of (basis-point) yield volatility over time. In an influential study, Chan, Karolyi, Longstaff, and Sanders (1992) show that various specifications of common one-factor term structure models differ in two respects: the degree of mean reversion and the level-dependency of yield volatility. **Empirically, they find insignificant mean reversion and significantly level-dependent volatility — more than a one-for one relation.**[16] Moreover, they find that the evaluation of various one-factor models' realism depends crucially on the volatility assumption; models that best fit U.S. data have a level-sensitivity coefficient of 1.5. According to these models, future yield volatility depends on the current rate level and nothing else: High yields predict high volatility. Another class of models — so-called GARCH models — stipulate that future yield volatility

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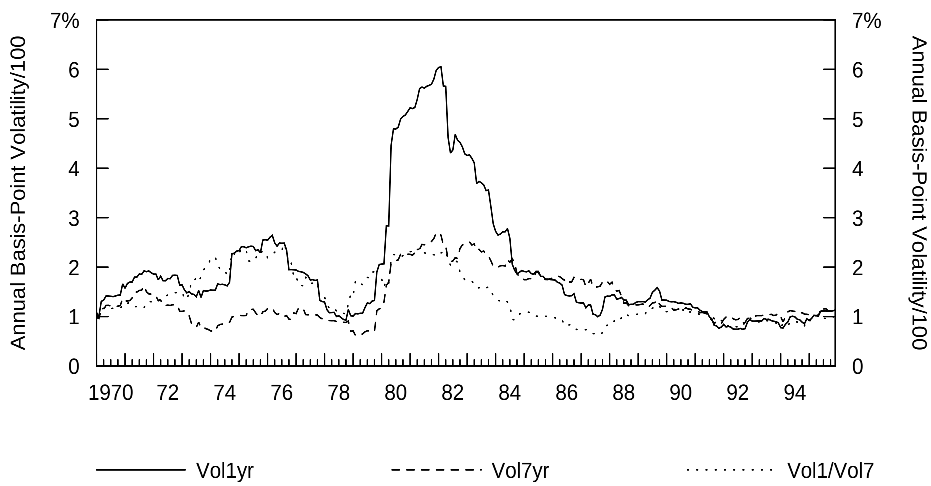
[16] As shown in Equation (13) in Appendix A, the short rate volatility in many term structure models can be expressed as proportional to r^γ where is the coefficient of volatility's sensitivity on the rate level. For example, in the Vasicek model (additive or normal rate process), $\gamma = 0$, while in the Cox-Ingersoll-Ross model (square root process), $\gamma = 0.5$. The Black-Derman-Toy model (multiplicative or lognormal rate process) is not directly comparable but $\gamma \approx 1$. If $\gamma = 0$, the basis-point yield volatility [Vol(Δy)] does not vary with the yield level. If $\gamma = 1$, the basis-point yield volatility varies one-for-one with the yield level — and the relative yield volatility [Vol(Δy/y)] is independent of the yield level (see Equation (13) in Part 5 of this series).

depends on the past volatility: High recent volatility and large recent shocks (squared yield changes) predict high volatility. Brenner, Harjes and Kroner (1996) show that empirically **the most successful models assume that yield volatility depends on the yield level *and* on past volatility**. With GARCH effects, the level-sensitivity coefficient drops to approximately 0.5. Finally, all of these studies include the exceptional period 1979-82 which dominates the results (see Figure 13). In this period, yields rose to unprecedented levels — but the increase in yield volatility was even more extraordinary. **Since 1983, the U.S. yield volatility has varied much less closely with the rate level.**[17]

Figure 13. 24-Month Rolling Spot Rate Volatilities in the United States



Notes: The graphs plot the annualized volatilities of the monthly basis-point changes in one-year and seven-year spot rates (and their ratio) over 24-month subperiods.

A few words about the required bond risk premia. In all one-factor models, the bond risk premium is a product of the market price of risk, which is assumed to be constant, and the amount of risk in a bond. Risk is proportional to return volatility, roughly a product of duration and yield volatility. Thus, models that assume rate-level-dependent yield volatility imply that the bond risk premia vary directly with the yield level. Empirical evidence indicates that the bond risk premia are not constant — but they also do not vary closely with either the yield level or yield volatility (see Figure 2 in Part 4). Instead, the market price of risk appears to vary with economic conditions, as discussed above Figure 4. One point upon which theory and empirical evidence agree is the sign of the market price of risk. Our finding that the bond risk premia increase with return volatility is consistent with a negative market price of interest rate risk. (Negative market price of risk and negative bond price sensitivity to interest rate changes together produce positive bond risk premia.) Many

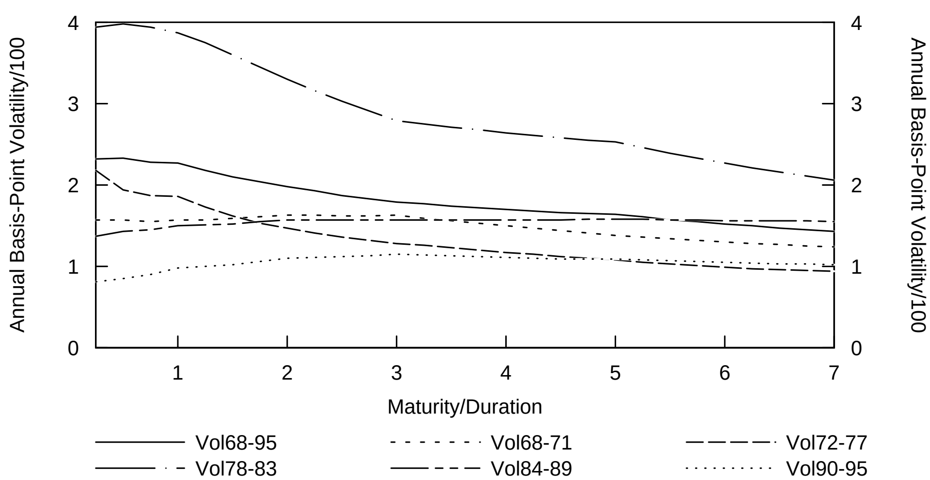
[17] When we estimate the coefficient (yield volatility's sensitivity to the rate level — see Equation (13) in Appendix A) using daily changes of the three-month Treasury bill rate, we find that the coefficient falls from 1.44 between 1977-94 to 0.71 between 1983-94. Moreover, when we reestimate the coefficient in a model that accounts for simple GARCH effects, it falls to 0.37 and 0.17, suggesting little level-dependency. (The GARCH coefficient on the past variance is 0.87 and 0.95 in the two samples, and the GARCH coefficient on the previous squared yield change is 0.02 and 0.03.) GARCH refers to "generalized autoregressive conditional heteroscedasticity," or more simply, time-varying volatility. GARCH models or other stochastic volatility models are one way to explain the fact that the actual distribution of interest rate changes have fatter tails than the normal distribution (that is, that the normal distribution underestimates the actual frequency of extreme events).

theoretical models, including the Cox-Ingersoll-Ross model, imply that the market price of interest rate risk is negative as long as changing interest rates covary negatively with the changing market wealth level.

Cross-Sectional Evidence

We first discuss the shape of the term structure of yield volatilities and its implications for bond risk measures and later describe the correlations across various parts of the yield curve. **The term structure of basis-point yield volatilities in Figure 14 is steeply inverted when we use a long historical sample period. Theoretical models suggest that the inversion in the volatility structure is mainly due to mean-reverting rate expectations** (see Appendix A). Intuitively, if long rates are perceived as averages of expected future short rates, temporary fluctuations in the short rates would have a lesser impact on the long rates. The observation that the term structure of volatility inverts quite slowly is consistent with expectations for very slow mean reversion. **In fact, after the 1979-82 period, the term structure of volatility has been reasonably flat** — as evidenced by the ratio of short rate volatility to long rate volatility in Figure 13. The subperiod evidence in Figure 14 confirms that the term structure of volatility has recently been humped rather than inverted. The upward slope at the front end of the volatility structure may reflect the Fed's smoothing (anchoring) of very short rates while the one- to three-year rates vary more freely with the market's rate expectations and with the changing bond risk premia.

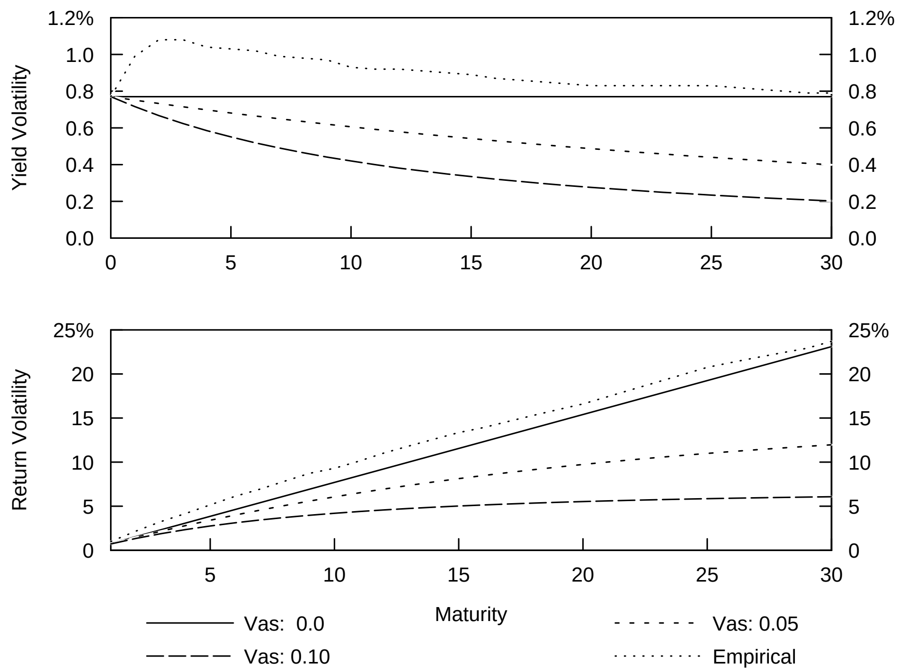
Figure 14. Term Structure of Spot Rate Volatilities in the United States



Notes: Each volatility term structure plots the annualized volatilities of the monthly basis-point changes in various-maturity spot rates over a given sample period. For example, the three-month rate's volatility was 2.32% (232 basis points) for the full sample.

The nonflat shape of the term structure of yield volatility has important implications on the relative riskiness of various bond positions. The traditional duration is an appropriate risk measure only if the yield volatility structure is flat. We pointed out earlier that inverted or humped yield volatility structures would make the return volatility curve a concave function of duration. Figure 15 shows examples of flat, humped and inverted yield volatility structures (upper panel) — and the corresponding return volatility structures (lower panel). The humped volatility structure reflects empirical yield volatilities in the 1990s, while the flat and inverted volatility structures are based on the Vasicek model with mean reversion coefficients of 0.00, 0.05, and 0.10. The model's short-rate volatility is calibrated to match that of the three-month rate in the 1990s (77 basis points or 0.77%). It is clear from this figure that the **traditional duration exaggerates the relative riskiness of long bonds** whenever the term structure of yield volatility is inverted or humped. Moreover, the relative riskiness will be quite misleading if the assumed volatility structure is inverted (as in the long sample period in Figure 14) while the actual volatility structure is flat or humped (as in the 1990s).

Figure 15. Basis-Point Yield Volatilities and Return Volatilities for Various Models



Notes: This figure shows four term structures of annualized basis-point spot yield volatilities and the corresponding return volatilities. The empirical yield volatilities are based on weekly Treasury market data between 1990-95, while the three other volatility structures are based on distinct specifications of the Vasicek model — one with no mean reversion (k=0.0 in Equation (9) in Appendix A, or the world of parallel shifts), another with mild mean reversion (k=0.05) and a third one with stronger mean reversion (k=0.10). The return volatilities are computed for each zero-coupon bond by the product of basis-point yield volatility and duration. These return volatilities are proportional to the factor sensitivities in one-factor models.

Historical analysis shows that **correlations of yield changes across the Treasury yield curve are not perfect but are typically very high** beyond the money market sector (0.82-0.98 for the monthly changes of the two- to 30-year on-the-run bonds between 1968-95) and reasonably high even for the most distant points, the three-month bills and 30-year bonds (0.57). Thus, the evidence is not consistent with a one-factor model, but it appears that **two or three systematic factors can explain 95%-99% of the fluctuations in the yield curve** (see Garbade (1986), Litterman and Scheinkman (1991), Ilmanen (1992)). Based on the patterns of sensitivities to each factor across bonds of different maturities, **the three most important factors are often interpreted as the level, slope and curvature factors**.[18]

[18] Principal components analysis is used to extract from the data first the systematic factor that explains as much of the common variation in yields as possible, then a second factor that explains as much as possible of the remaining variation, and so on. These statistically derived factors are not directly observable — but we can gain insight into each factor by examining the pattern of various bonds' sensitivities to it. These factors are not exactly equivalent to the actual shifts in the level, slope and curvature. For example, the level factor is not exactly parallel, as its shape typically depends on the term structure of yield volatility. In addition, the statistically derived factors are uncorrelated, by construction, whereas Figure 6 shows that the actual shifts in the yield curve level, slope and curvature are not uncorrelated.

A vast literature exists on quantitative modelling of the term structure of interest rates. Because of the large number of these models and the fact that the use of stochastic calculus is needed to derive these models, many investors view them as inaccessible and not useful for their day-to-day portfolio management. However, investors use these models extensively in the pricing and hedging of fixed-income derivative instruments and, implicitly, when they consider such measures as option-adjusted spreads or the delivery option in Treasury bond futures. Furthermore, these models can provide useful insights into the relationships between the expected returns of bonds of different maturities and their time-series properties.

It is important that investors understand the assumptions and implications of these models to choose the appropriate model for the particular objective at hand (such as valuation, hedging or forecasting) and that the features of the chosen model are consistent with the investor's beliefs about the market. Although the models are developed through the use of stochastic calculus, it is not necessary that the investor have a complete understanding of these techniques to derive some insight from the models. One goal of this section is to make these models accessible to the fixed-income investor by relating them to risk concepts with which he is familiar, such as duration, convexity and volatility.

Equation (1) in Part 5 of this series gives the expression of the percentage change in a bond's price ($\Delta P/P$) as a function of changes in its own yield (Δy):

$$100 * \Delta P/P \approx - \text{Duration} * \Delta y + 0.5 * \text{Convexity} * (\Delta y)^2 \; . \qquad (1)$$

This expression, which is derived from the Taylor series expansion of the price-yield formula, is a perfectly valid linkage of changes in a bond's own yield to returns and expected returns through traditional bond risk measures such as duration and convexity.

One problem with this approach is that every bond's return is expressed as a function of its *own* yield. This expression says nothing about the relationship between the return of a particular bond and the returns of other bonds. Therefore, it may have limited usefulness for hedging and relative valuation purposes. One must impose some simplifying assumptions to make these equations valid for cross-sectional comparisons. In particular, more specific assumptions are needed for the valuation of derivative instruments and uncertain cash flows. Of course, the marginal value of more sophisticated term structure models depends on the empirical accuracy of their specification and calibration.

Factor Model Approach

Term structure models typically start with a simple assumption that the prices of all bonds can be expressed as a function of time and a small number of factors. For ease of explanation, the analysis is often restricted to default-free bonds and their derivatives. **We first discuss**

[19] This Appendix is an abbreviated version of Iwanowski (1996), an unpublished research piece that is available upon request. In this survey, we mention several term structure models; a complete reference list can be found after the appendices.

one-factor models which assume that one factor (F_t)[20] drives the changes in all bond prices and the dynamics of the factor is given by the following stochastic differential equation:

$$\frac{dF}{F} = m(F,t)dt + s(F,t)dz \tag{2}$$

where F can be any stochastic factor such as the yield on a particular bond or the real growth rate of an economy, dt is the passage of a small (instantaneous) time interval, and dz is Brownian motion (a random process that is normally distributed with a mean of 0 and a standard deviation of $\sqrt{dt}$). The letter "d" in front of a variable can be viewed as shorthand for "change in". Equation (2) is an expression for the percentage change of the factor which is split into expected and unexpected parts. The "drift" term m(F,t)dt is the expected percentage change in the factor (over a very short interval dt). This expectation can change as the factor level changes or as time passes. In the unexpected part, s(F,t) is the volatility of the factor (also dependent on the factor level and on time) and dz is Brownian motion. For now, we leave the expression of the factors as general, but various one-factor models differ by the specifications of F, m(F,t) and s(F,t).

Let the price at time t of a zero-coupon bond which pays $1 at time T be expressed as $P_i(F,t,T)$. Because F is the only stochastic component of P_i, Ito's Lemma — roughly, the stochastic calculus equivalent of taking a derivative — gives the following expression for the dynamics of the bond price:

$$\frac{dP_i\,(F,t,T)}{P_i} = \mu_i dt + \sigma_i dz \tag{3}$$

$$\text{where } \mu_i = \frac{\partial P_i}{\partial t}\frac{1}{P_i} + \frac{\partial P_i}{\partial F}\frac{1}{P_i}\,m(F,t)F + \frac{1}{2}\frac{\partial^2 P_i}{\partial F^2}\frac{1}{P_i}\,s(F,t)^2\,F^2$$

$$\text{and } \sigma_i = \frac{\partial P_i}{\partial F}\frac{1}{P_i}\,s(F,t)F.$$

In this framework, Ito's Lemma gives us an expression for the percentage change in price of the bond over the time dt for a given realization of F at time t. μ_i is the expected percentage change in the price (drift) of bond i over the period dt and σ_i is the volatility of bond i.

The unexpected part of the bond return depends on the bond's "duration" with respect to the factor (its factor sensitivity)[21] and the unexpected factor realization. The return volatility of bond i (σ_i) is the product of its factor sensitivity and the volatility of the factor.

Equation (3) shows that the decomposition of expected returns in Part 6 of this series is very general. The expected part of the bond return over dt is given by the expected percentage price change μ_i because zero-coupon bonds do not earn coupon income. Consider the three components of the expected return:

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[20] The subscript refers to the realization of factor F at time t. For convenience, we subsequently drop this subscript.

[21] At this point, we refer to "duration" in quotes to signify that this is a duration with respect to the factor and not necessarily the traditional modified Macaulay duration.

(1) The first term is the change in price due to the passage of time. Because our bonds are zero-coupon bonds, this change (accretion) will always be positive and represents a "rolling yield" component;

(2) The second term is the expected change in the factor (mF) multiplied by the sensitivity of the bond's price to changes in the factor. This price sensitivity is like "duration" with respect to the relevant factor; and

(3) The third term comprises of the second derivative of the price with respect to changes in the factor and the variance of the factor. The second derivative is like "convexity" with respect to the factor.

Suppose we specify the factor F to be the yield on bond i (y_i). Then, the expected change in price of bond i over the short time period (dt) is given by the familiar equation that we developed in the previous parts of this series:

$$E\left(\frac{dP_i(y,t,T)}{P_i}\right) = \mu_i dt = \frac{\partial P_i}{\partial t}\frac{1}{P_i} + \frac{\partial P_i}{\partial y_i}\frac{1}{P_i}E(\Delta y_i) + \frac{1}{2}\frac{\partial^2 P_i}{\partial y_i^2}\frac{1}{P_i}\text{variance }(\Delta y_i) \tag{4}$$

$$= \text{Rolling Yield}_i - \text{Duration}_i * E(\Delta y_i) + \frac{1}{2}\text{Convexity}_i * \text{variance }(\Delta y_i)$$

where Δy_i is the change in the yield of bond i. We can also use Equation (4) to similarly link the factor model approach to the decompositions of forward rates made in the previous parts of this series. It can be shown (for "time-homogeneous" models) that the instantaneous forward rate T periods ahead equals the rolling yield component. Therefore, we rewrite Equation (4) in terms of the forward rate as follows:

$$f_{T,T+dt} = \frac{\partial P_i}{\partial t}\frac{1}{P_i} = E\left(\frac{dP_i}{P_i}\right) - \frac{\partial P_i}{\partial y_i}\frac{1}{P_i}E(\Delta y_i) - \frac{1}{2}\frac{\partial^2 P_i}{\partial y_i^2}\frac{1}{P_i}\text{variance }(\Delta y_i) \tag{5}$$

$$= \text{Expected Return}_i + \text{Duration}_i * E(\Delta y_i) - \frac{1}{2}\text{Convexity}_i * \text{variance }(\Delta y_i)$$

The expected return term can be further decomposed into the risk-free short rate and the risk premium for bond i. Thus, forward rates can be decomposed into the rate expectation term (drift), a risk premium term and a convexity bias (or a Jensen's inequality) term. Other term structure models contain analogous but more complex terms.

Unfortunately, by defining the one relevant factor to be the bond's own yield, Equation (4) only holds for bond i. For any other bond j, the chain rule in calculus tells us that

$$-\frac{\partial P_j}{\partial y_i}\frac{1}{P_i} = -\frac{\partial P_j}{\partial y_j}\frac{\partial y_j}{\partial y_i}\frac{1}{P_i} \text{ which equals Duration}_j \text{ only if } \frac{\partial y_j}{\partial y_i} = 1. \tag{6}$$

Therefore, **in a one-factor world where y_i represents the relevant factor, Equation (4) only holds for bonds other than bond i if all shifts of the yield curve are parallel**. While this observation suggests that more sophisticated term structure models are needed for derivatives valuation, it does not deem useless the framework developed in this series. In particular, this framework is valuable in applications such as interpreting yield curve shapes and forecasting the relative performance of various government bond positions. Such forecasts are not restricted to parallel curve shifts if we predict separately each bond's yield change (or if we predict a few points in the curve and interpolate between them). The problem with using maturity-specific yield and volatility forecasts is that the consistency of the forecasts across bonds and the absence of arbitrage opportunities are not explicitly guaranteed.

Arbitrage-Free Restriction

For the time being, we return to the world where the factor, F, is unspecified and the change in price (return) of any bond i is given by Equation (3). How should bonds be priced relative to each other? The first term of Equation (3) ($\mu_i dt$) is deterministic — that is, we know today what the value of this component will be at the end of time t+dt. However, the value of second term ($\sigma_i dz$) is unknown until the end of time t+dt. In fact, in this one-factor framework, this is the only unknown component of any bond's returns. **If we can form a portfolio whereby we eliminate all exposure to the one stochastic factor, then the return on the portfolio is known with certainty. If the return is known with certainty, then it must earn the riskless rate r or else arbitrage opportunities would exist.**

It also follows that **in our one-factor world, the ratio of expected excess return over the return volatility must be equal for any two bonds to prevent arbitrage opportunities**. This relation must hold for all bonds or portfolios of bonds, and in Equation (7) below is the value of this ratio, often known as the "market price of the factor risk".

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda \tag{7}$$

where r is the riskless short rate.

Solutions of the Term Structure Models for Bond Prices

Combining Equations (3) and (7) leads to the following differential equation:

$$\frac{\partial P_1}{\partial t} + \frac{\partial P_1}{\partial F}(mF - \lambda s) + \frac{1}{2}\frac{\partial^2 P_1}{\partial F^2} s^2 F^2 = rP_1. \tag{8}$$

This differential equation is solved to obtain bond prices and derivatives of bonds. Virtually all of the existing one-factor term-structure models are developed in this framework.[22] The next step is to impose a set of boundary conditions specific to the instrument that is being priced and then

[22] This is also the framework in which the Black-Scholes model to price equity options is developed.

solve the differential equation for P(F, t, T). One boundary condition for zero-coupon bonds is that the price of the bond at maturity is equal to par (P(F, T, T) = 100). Another example of a boundary condition is that the value of a European call option on bonds, at the expiration of the option, is given by C(t,T,K) = max[P(F,t,T)-K,0]. Various term structure models differ in the definition of the relevant factor and the specification of its dynamics. Specifically, the one-factor models differ from each other in how the variable F and the functions m(F,t) (factor drift) and s(F,t) (factor volatility) are specified. Different specifications lead to distinct solutions of Equation (8) and distinct implications for bond prices and yields. In the rest of this section, we will analyze one such specification to give the reader an intuitive interpretation of these models, and then we qualitatively discuss the trade-offs between various popular models.

One Example: The Vasicek Model

Many of the existing term-structure models begin by specifying the one stochastic factor that affects all bond returns as the riskless interest rate (r) on an investment that matures at the end of dt. One of the earliest such model developed by Vasicek (1977) took this approach and specified the dynamics of the short rate as follows:

$$dr = k(l - r)dt + sdz. \tag{9}$$

This fits in the framework of Equation (2) if F is defined to be r and m(r,t) = (k(l-r))/r and s(r,t)=s/r. The second term indicates that the **short rate is normally distributed with a constant volatility of s which does not depend on the current level of r**. The basis-point yield volatility is the same regardless of whether the short rate is equal to 5% or 20%. The drift term requires some interpretation. **In the Vasicek model, the short rate follows a *mean-reverting* process.** This means that there is some long-term mean level toward which the short rate tends to move. If the current short rate is high relative to this long-term level, the expected change in the short-rate is negative. Of course, even if the *expected* change over the next period is negative, we do not know for sure that the *actual* change will be negative because of the stochastic component. In Equation (9), l is the long-term level of the short rate and k is the speed of mean-reversion. If k=0, there is no mean-reversion of the short rate. If k is large, the short rate reverts to its long-term level quite quickly and the stochastic component will be small relative to the mean reversion component.

This specification falls into a class of models known as the "affine" yield class. "Affine" essentially means that all continuously compounded spot rates are linear in the short rate. Many of the popular one-factor models belong to this class. For the affine term structure models, the solution of Equation (8) for zero-coupon bond prices is of the following form:

$$P(r,t,T) = e^{A(t,T) - B(t,T)r} \tag{10}$$

Typically, A(r,t,T) and B(r,t,T) are functions of the various parameters describing the interest rate dynamics such as k, l, s, and λ. It is easy to show that the "duration" of the zero-coupon bond with respect to the short rate equals B(t,T). How does this duration measure differ from our traditional definition of duration with respect to a bond's own yield? For example, in the Vasicek model, the solution for B(r,t,T) is given by the following:

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k} . \tag{11}$$

Therefore, the duration measure with respect to changes in the short rate is a function of the speed of mean-reversion parameter, k. As this parameter approaches 0, the duration of a bond with respect to changes in the short rate approaches the traditional duration measure with respect to changes in the bond's own yield. **Without mean reversion, the Vasicek model implies parallel yield shifts, and Equation (4) holds. However, as the mean reversion speed gets larger, long bonds' prices are only slightly more sensitive to changes in the short rate than are intermediate bonds' prices because the impact of longer (traditional) duration is partly offset by the decay in yield volatility** (see Figure 15). With mean reversion, long rates are less volatile than short rates. In this case, the traditional duration measure would overstate the relative riskiness of long bonds.

Comparisons of Various Models

Most of the one-factor term structure models that have evolved over the past 20 years are remarkably similar in the sense that they all essentially were derived in the framework that we described above. However, dissatisfaction with certain aspects of the existing technologies have motivated researchers in the industry and in academia to continue to develop new versions of term structure models. Four issues that have motivated the model-builders are:

- Consistency of factor dynamics with empirical observations;
- Ability to fit the current term structure and volatility structure;
- Computational efficiency; and
- Adequacy of one factor to satisfactorily describe the term structure dynamics.

Differing Factor Specifications. Some of the one-factor models differ by the definition of the one common factor. However, the vast majority of the models assume that the factor is the short rate and the models differ by the specification of the dynamics of the factor.

For example, the mean-reverting normally distributed process for the short rate that is used to derive the Vasicek model (Equation (9)) leads to features that many users find problematic. Specifically, nominal interest rates can become negative and the basis-point volatility of the short rate is not affected by the current level of interest rates. The Cox-Ingersoll-Ross model (CIR) is based on the following specification of the short rate which precludes negative interest rates and allows for level-dependent volatility:

$$dr = k(l - r)dt + s\sqrt{r}\,dz. \tag{12}$$

Because this model is a member of the affine yield class, the solution of the model is of the form shown in Equation (10). The function B(t, T), which represents the "duration" of the zero-coupon bond price with respect to changes in the short rate, is a complex function of the parameters k, l, s, and λ. As in the Vasicek model, when the mean-reversion parameter is non-zero, the durations of long bonds with respect to changes in the short rate are significantly lower than the traditional duration.

Chan, Karolyi, Longstaff and Sanders (CKLS, 1992) empirically compare the various models by noting that most of the one-factor models developed in the 1970s and 1980s are quite similar in that they define the one factor to be the short rate, r, and their dynamics are described by the following equation:

$$dr = k(l - r)dt + sr^{\gamma}dz. \tag{13}$$

The differences between the models are in their specification of k and γ. For example, the Vasicek model has a non-zero k and $\gamma = 0$. CIR also has a non-zero k and $\gamma = 0.5$. We discuss the findings of CKLS and subsequent researchers in the section "How Does the Yield Curve Evolve Over Time?".

Fitting the Current Yield Curve and Volatility Structure. One of the problems that practitioners have with the early term structure models such as the original Vasicek and CIR models is that the parameters of the short-rate dynamics (k, l, s) and the market price of risk, λ, must be estimated using historical data or by minimizing the pricing errors of the current universe of bonds. **Nothing ensures that the market prices of a set of benchmark bonds matches the model prices.** Therefore, a user of the model must conclude that either the benchmarks are "rich" or "cheap" or that the model is misspecified. Practitioners who must price derivatives from the model typically are not comfortable assuming that the market prices the benchmark Treasury bonds incorrectly.

In 1986, Ho and Lee introduced a model that addressed this concern by specifying that the "risk-neutral" drift of the spot rate is a function of time. This addition **allows the user to calibrate the model in such a way that a set of benchmark bonds are correctly priced** without making assumptions regarding the market price of risk. Subsequently developed models address some shortcomings in the process implied by the Ho-Lee model (possibility of negative interest rates) or fit more market information (term structure of implied volatilities). Such models include Black-Derman-Toy, Black-Karasinski, Hull-White, and Heath-Jarrow-Morton. These models have become known as the "arbitrage-free" models, as opposed to the earlier "equilibrium" models. Our brief discussion does not do justice to these models; interested readers are referred to surveys by Ho (1994) and Duffie (1995).

These arbitrage-free models represent the current "state of the art" for pricing and hedging fixed-income derivative instruments. One theoretical problem with these models is that they are time-inconsistent. The models are calibrated to fit the market data and then bonds and derivatives are priced with the implicit assumption that the parameters of the stochastic process remain as specified. However, as soon as the market changes, the model needs to be recalibrated, thereby violating the implicit assumption (see Dybvig (1995)). In reality, most practitioners find this inconsistency a small price to pay for the ability to calibrate the model to market prices.

Computational Efficiency. Some of the issues in choosing a model involve computational efficiency. For example, some of the models have the feature that the price of bonds and many derivatives on bonds have a closed-form solution, but others must be solved numerically by techniques such as Monte Carlo methods and finite differences. Because such techniques can be employed quite quickly, most practitioners do not feel that a closed-form solution is necessary. However, a closed-form solution allows a better understanding of the model and the sensitivities of the price to the various input variables.

Many practitioners and researchers prefer the Heath-Jarrow-Morton model, which specifies the entire term structure as the underlying factor, because it provides the user with the most degrees of freedom in calibrating the model. However, the major shortcoming of this model is that, when implemented on a lattice (or tree) structure, the nodes of the lattice do not recombine. Therefore, the number of nodes grows exponentially as the number of time steps increase, rendering the time to obtain a price unacceptably long for many applications. Much of the recent research has been devoted to approximating this model to make it more computationally efficient.

Extensions to Multi-Factor Models. Empirical analysis by Litterman and Scheinkman (1991), among others, shows that two or three factors can explain most of the cross-sectional differences in Treasury bond returns. A glance at the imperfect correlations between bond returns provides even simpler evidence of the insufficiency of a one-factor model. Yet, while multi-factor models, by definition, explain more of the dynamics of the term structure than a one-factor model, the cost of the additional complexity and computational time can be significant. In assessing whether a one-, two- or three-factor model is appropriate, the tradeoff is the efficiency gained in pricing and hedging because of the additional factors against these costs. For certain applications in the fixed-income markets, a one-factor model is adequate. For a systematic and detailed comparison of one-factor models vs. two-factor models, see Canabarro (1995).

The general framework in which a multi-factor term structure model is derived is similar to the one-factor model with the n factors specified in a similar manner as in Equation (2):

$$\frac{dF_j}{F_j} = m_j(F_j,t)dt + s_j(F_j,t)dz_j, \tag{14}$$

where $j = 1, ..., n$ and the dz_j's can be correlated with correlations given by $\rho_{jk}(F,t)$.

For example, the Cox-Ingersoll-Ross model can be extended into a multi-factor model. To keep the analysis tractable, most term structure models define a small number of factors ($n = 2$ or 3). Some examples in the literature include the Brennan-Schwartz model, which specifies the two factors as a long and a short rate, the Brown-Schaefer model, which specifies the two factors as a long rate and the yield curve steepness, the Longstaff-Schwartz model, which specifies the two factors as a short rate and the volatility of the short rate, and the Duffie-Kan model, which specifies the factors as the yields on n bonds. A multi-factor version of Ito's Lemma provides the following expression for the return of bonds in the multi-factor world:

$$\frac{dP_i\ (F,t,T)}{P_i} = \mu_i dt + \sigma_i dz, \tag{15}$$

$$\text{where } \mu_i = \frac{\partial P_i}{\partial t}\frac{1}{P_i} + \sum_{j=1}^{n} \frac{\partial P_i}{\partial F_j}\frac{1}{P_i}\, m_j(F,t)F_j$$

$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 P_i}{\partial F_j \partial F_k}\frac{1}{P_i}\, s_j(F,t)s_k(F,t)\rho_{jk}(F,t)F_jF_k$$

$$\text{and } \sigma_i = \sum_{j=1}^{n} \frac{\partial P_i}{\partial F_j}\frac{1}{P_i}\, s_jF_j\ .$$

While this expression may appear onerous, it is really a restatement of Equation (3). Qualitatively, Equation (15) simply states that the return on a bond can be decomposed in the multi-factor world as follows:

Return on bond i = expected return on bond i + unexpected return on bond i

where expected return on bond i =

return on bond i due to the passage of time (rolling yield)

- the sum of the "durations" with respect to each factor * the expected realization of the factor

+ the value of all the convexity and cross-convexity terms,

and where unexpected return = the sum of the durations with respect to each factor * the realization of the factors.

In this Appendix, we **link the return decomposition in Equation (15) to the broader asset pricing literature** in modern finance, emphasizing the determination of bond risk premia. While term structure models focus on the expected returns and risks of only default-free bonds, asset pricing models analyze the expected returns and risks of all assets (stocks, bonds, cash, currencies, real estate, etc.).

The traditional explanation for positive bond risk premia is that long bonds should offer higher returns (than short bonds) because their returns are more volatile.[23] However, a central theme in modern asset pricing models is that **an asset's riskiness does not depend on its return volatility but on its sensitivity to (or covariation with) systematic risk factors**. Part of each asset's return volatility may be nonsystematic or asset-specific. Recall that the realized return is a sum of expected return and unexpected return. Unexpected return depends (i) on an asset's sensitivity to systematic risk factors and actual realizations of those risk factors and (ii) on asset-specific residual risk. Expected return depends only on the first term because the second term can be diversified away. That is, the market does not reward investors for assuming diversifiable risk. Note that the term structure models assume that only systematic factors influence bond returns. This approach is justifiable by the empirical observation that the asset-specific component is a much smaller part of a government bond's return than a corporate bond's or a common stock's return.

The best-known asset pricing model, the Capital Asset Pricing Model **(CAPM), posits that any asset's expected return is a sum of the risk-free rate and the asset's required risk premium**. **This risk premium depends on each asset's sensitivity to the overall market movements and on the market price of risk.** The overall market is often proxied by the stock market (although a broader measure is probably more appropriate when analyzing bonds). Then, each asset's risk depends on its sensitivity to stock market fluctuations (beta). Intuitively, high-beta assets that accentuate the volatility of diversified portfolios should offer higher expected returns, while negative-beta assets that reduce portfolio volatility can offer low expected returns. The market price of risk is common to all investors and depends on the market's overall volatility and on the *aggregate* risk aversion level. Note that **in a world of parallel yield curve shifts and positive correlation between stocks and bonds, all bonds would have positive betas — and these would be proportional to the traditional duration measures.** This is one explanation for the observed positive bond risk premia.

In the CAPM, the market risk is the only systematic risk factor. In reality, investors face many different sources of risk. Multi-factor asset pricing models can be viewed as generalized versions of the CAPM. **All these models state that each asset's expected return depends on the risk-free rate (reward for time) and on the asset's required risk premium (reward for taking various risks). The latter, in turn, depends on the asset's sensitivities ("durations") to systematic risk factors and on these factors' market prices of risk**. These market prices

[23] One problem with this explanation is that short positions in long-term bonds are equally volatile as long positions in them; yet, the former earn a negative risk premium. Stated differently, why would borrowers issue long-term debt that costs more and is more volatile than short-term debt? The classic liquidity premium hypothesis offered the following "institutional" answer: Most investors prefer to lend short (to avoid price volatility) while most borrowers prefer to borrow long (to fix the cost of a long-term project or to ensure continuity of funding). However, we focus above on the explanations that modern finance offers.

of risk may vary across factors; investors are not indifferent to the source of return volatility. An example of undesirable volatility is a factor that makes portfolios perform poorly at times when it hurts investors the most (that is, when so-called marginal utility of profits/losses is high). Such a factor would command a positive risk premium; investors would only hold assets that covary closely with this factor if they are sufficiently rewarded. Conversely, investors are willing to accept a low risk premium for a factor that makes portfolios perform well in bad times. Thus, if long bonds were good recession hedges, they could even command a negative risk premium (lower required return than the risk-free rate).

The multi-factor framework provides a natural explanation for why assets' expected returns may not be linear in return volatility. One can show that expected returns are concave in return volatility if two factors with different market prices of risk influence the yield curve — and the factor with a lower market price of risk has a relatively greater influence on the long rates. That is, **if long bonds are highly sensitive to the factor with a low market price of risk and less sensitive to the factor with a high market price of risk, they may exhibit high return volatility and low expected returns** (per unit of return volatility).

What kind of systematic factors should be included in a multi-factor model? By definition, systematic factors are factors that influence many assets' returns. Two plausible candidates for the fundamental factors that drive asset markets are a real output growth factor (that influences all assets but the stock market in particular) and an inflation factor (that influences nominal bonds in particular). The expected excess return of each asset would be a sum of two products: (i) the asset's sensitivity to the growth factor * the market price of risk for the growth factor and (ii) the asset's sensitivity to the inflation factor * the market price of risk for the inflation factor. However, these macroeconomic factors cannot be measured accurately; moreover, asset returns depend on the market's expectations rather than on past observations. Partly for these reasons, the term structure models tend to use yield-based factors plausibly — as proxies for the fundamental economic determinants of bond returns.

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