Swap Rate à la Stock: Bermudan Swaptions Made Easy

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Abstract
We show how Markovian projection, together with some clever parameter freezing, can be used to reduce a full-fledged local volatility interest rate model – such as the Cheyette model – to a "minimal" form in which the swap rate evolves essentially like a dividend-paying stock. Using a number of numerical examples, we compare such a minimal "poor man's" model to a full-fledged Cheyette local volatility model and the market benchmark Hull–White one-factor model. Numerical tests demonstrate that the "poor man's" model is in fact sufficient to price Bermudan interest rate swaptions. The main practical implication of this finding is that – once local volatility, dividend, and short rate parameters are properly stripped from the volatility surface and interest rate curve – one can readily use the widely popular equity derivatives software for pricing exotic interest rate options such as Bermudans.

Keywords
local volatility, Cheyette model, Bermudan options

MSC Subject Classification: 91B24, 91G20

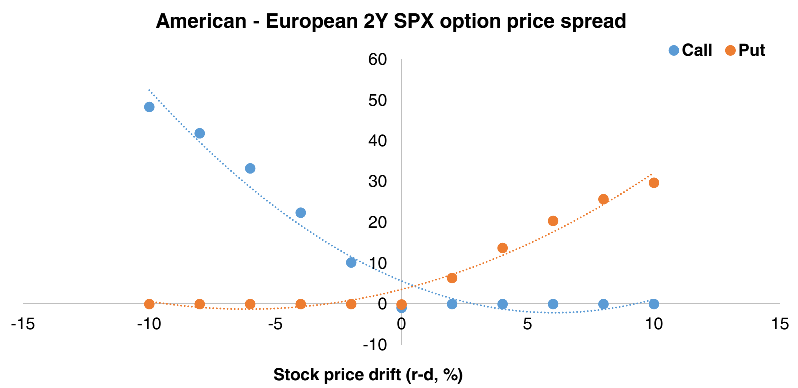
JEL Keywords: G12, G13

1 Introduction and motivation
Pricing interest rate derivatives – especially of the more exotic type – is often seen as more complicated than pricing derivatives on equity or commodity underlyings. After all, the latter deal with "tangible," self-contained assets. In contrast, in interest rate derivatives the underlyings are often not even, strictly speaking, proper assets (one cannot "buy" or "sell" an interest rate), and exist only as different bits of a general collection – i.e. the term structure or yield curve – making the conceptual challenge and computational effort considerably greater. This is perhaps most clearly seen in the case of products with path dependence or early exercise features, such as American or Bermudan options.

While a standard European option gives the right – but no obligation – to purchase a given underlying instrument at expiration for a pre-agreed price, American options allow buyers to exercise their right at any time before expiration. Similarly, Bermudans offer the possibility to exercise on any one of a specified set of dates before maturity. For a typical equity underlying, an American-style option gives the owner the right – but no obligation – to buy the underlying at a pre-agreed price at any time before expiration. Although there is no analytical solution for the price of an American equity option, the multiple-exercise feature can in fact be accurately and efficiently handled through least squares Monte Carlo or finite difference schemes, and can be shown to depend on the dividend yield and its relation to the short rate (Figure 1).

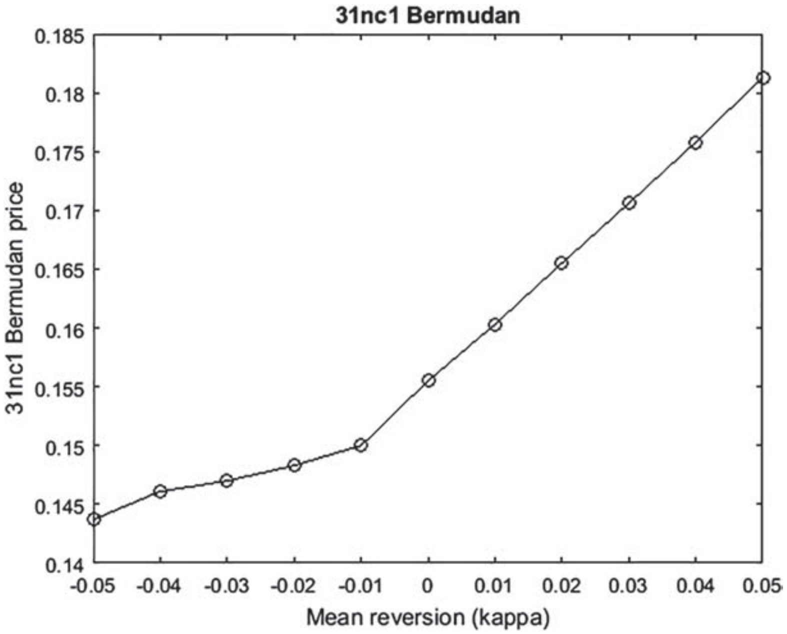
Contrast this with the problem of pricing a Bermudan swaption giving the right to enter into one of several fixed-for-floating interest rate swaps, each of which is observed on a different exerecise date and, technically, each is driven by a different forward process. Here, we no longer have an option on a single underlying, but rather a complicated best-of chooser option granting the holder the right to choose among several European options on swap rates with different fixing dates. Moreover, what determines the value of the early exercise premium is no longer the interdependence between the short rate and the dividend yield, but rather the mean reversion or correlation between the underlying forward swap rates (Figure 2).

Figure 1: The sensitivity of European and American options on S&P 500 to short rate and dividend.



Note: 2Y SPX options, local volatility model calibrated to market data as of April 27, 2018; ATM = 2678.81.

Figure 2: Sensitivity of a 30Y Bermudan swaption price to short rate mean reversion speed



Market benchmark model of Hull & White $dr(t) = (\theta(t) - \kappa(t)r(t))\,dt + \sigma(t)dW(t)$ calibrated to the same set of co-terminal swaptions each time with different κ. US data as of October 5, 2016.

Building again on the chooser option analogy, if the correlation among the payoffs is high, the option to exercise early is clearly not so valuable. Conversely, if the correlation is low or negative, then high payoffs on the early options are likely to be followed by low payoffs on the later ones and vice versa, so having the Bermudan exercise right improves the chances of making money in more scenarios. Unfortunately, accounting properly for the correlation among different swap rates requires a formal no-arbitrage model for the joint evolution of the entire term structure of interest rates. Or so it would seem at least...

In this paper we argue that a full-fledged term structure model is actually not necessary to price Bermudan swaptions. Instead, we show how a cleverly chosen set of approximations leads to an equity-like local volatility process for the swap rate, driven by its own "short rate" and "dividend yield" – both implied from the term structure of interest rates. The equity-like process effectively "synthesizes" the information on volatility and correlation contained in the core swap rates, spanned by a given Bermudan swaption. As a result, the latter can be priced as any American-style option on a dividend-paying stock, using standard equity derivatives software. While not exactly arbitrage free and limited in application to swap rate derivatives, such equity-like model can be seen as a "poor man's" approach that delivers very accurate approximations of Bermudan swaption prices at a considerably lower computational cost and effort than full-term structure models.

2 Notation and modeling framework

As usual with interest rate instruments, we start by introducing a uniformly spaced tenor structure:

$$0 = T_0 < T_1 < \cdots < T_N, \tag{2.1}$$

with $T_n = \delta n$. Typically, $\delta = 0.25$ – for quartely payments – or $\delta = 0.5$ for semiannual ones. Let $P(t, T)$ be the time t price of a zero-coupon discount bond paying 1 for sure at T. A **fixed-for-floating interest rate swap** (IRS) with unit notional, fixed rate (coupon) K, and a specified tenor structure $\mathcal{T} = \{T_n\}_{n=\alpha+1}^{\beta}$ is a contract whereby two parties exchange differently indexed cashflows over a pre-agreed time span. Specifically, on each date $T_n \in \mathcal{T}$, the fixed leg pays δK, whereas the floating leg pays the LIBOR rate given by

$$\frac{1 - P(T_{n-1}, T_n)}{\delta P(T_{n-1}, T_n)} \times \delta. \tag{2.2}$$

When the fixed leg is paid, the IRS is called a "payer," conversely the swap is called a "receiver." The **forward swap rate** $S_{\alpha,\beta}(t)$ corresponding to the tenor structure $\mathcal{T}$ is the rate in the fixed leg that sets it equal to the floating leg and hence makes the net present value of the transaction equal to zero:

$$S_{\alpha,\beta}(t) \equiv \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{n=\alpha+1}^{\beta} P(t, T_n)\delta}. \tag{2.3}$$

The portfolio of zero-coupon bonds in the denominator of (2.3) is the so-called annuity factor, for which we will often use a continuous-time definition, writing $N_{\alpha,\beta}(t) \equiv \int_{T_\alpha}^{T_\beta} P(t, u)du$ along with

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{N_{\alpha,\beta}(t)}. \tag{2.4}$$

A **European payer (receiver) swaption** with strike K, maturity T_α, and tenor $T_\beta - T_\alpha$ (henceforth referred to also as $T_\alpha \times (T_\beta - T_\alpha)$, or T_α-into-$(T_\beta - T_\alpha)$) is simply an option that gives the holder the right to enter at T_α into a payer (receiver) swap which matures at T_β and entitles to pay (receive) the fixed rate K in exchange for floating LIBOR rate on the tenor dates $\mathcal{T}$. Thus, the price of a European payer swaption with unit notional at T_α is given by

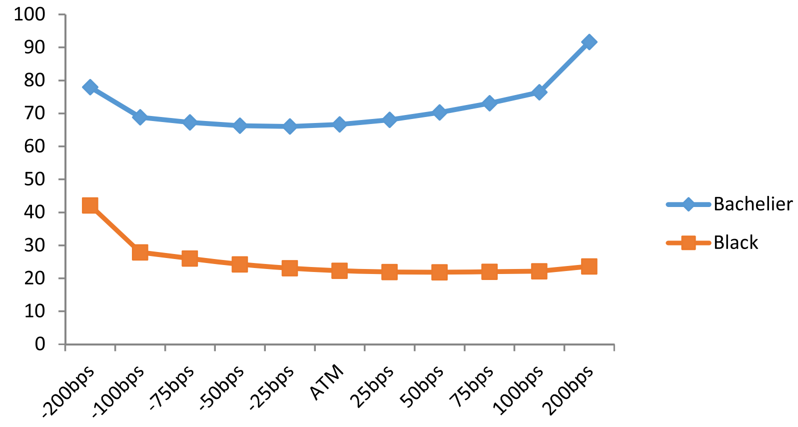
$$N_{\alpha,\beta}(T_\alpha)\left(S_{\alpha,\beta}(T_\alpha) - K\right)^+, \tag{2.5}$$

where $x^+ = \max(x, 0)$. Finally, a **Bermudan receiver (payer) swaption** is an option to enter at any time T_i, $i \in \{\alpha, \alpha+1, ..., \beta-1\}$, into a swap which terminates at T_β and gives the holder the right to receive (pay) a pre-determined fixed rate K in exchange for floating LIBOR. The period up to T_α is called the lockout or no-call period, and hence a Bermudan swaption with final exercise date $T_{\beta-1}$ and first exercise T_α is often called "T_β no-call T_α," or "$T_\beta$ncT_α." For example, a 11nc1 swaption with annually spaced exercise dates can be exercised at the beginning of any year, starting from year 1. By exercising the option, the holder enters a swap starting at the time of exercise (i.e. years 1, 2, 3, ..., 10) and ending at year 11.

By convention, European swaption prices are often expressed in terms of their Black implied volatilities, i.e. log-normal volatilities that – plugged into the Black swaption price formula – yield the market price. More recently, reflecting the popular recognition that interest rates in advanced economies can indeed go below zero, the Bachelier model has become the standard quotation mechanism.

Like most options markets, the swaption market exhibits a pronounced implied volatility skew – apparent in both Black and Bachelier conventions (Figure 3). This dependence of implied volatility on swaption strike is not consistent with the constant-volatility assumption underlying both Bachelier and Black approaches, calling for a more refined treatment.

Figure 3: Black and Bachelier implied volatilities for 1 × 10 USD swaption (in bp; as of April 27, 2018)



One of the most popular approaches for handling smiles – at least in equity and FX settings–has long been the so-called local volatility model developed by Derman and Kani (1994) and Dupire (1994), and later extended in multiple directions (see e.g. Carr and Madan, 1998; Carr et al., 2004; Carmona and Nadtochiy, 2009; Coleman et al., 2001; Durrleman, 2010; cf. also Gatheral, 2011 for a practical overview of volatility modeling). These models proposed a small departure from the Black–Scholes world by postulating that the instantaneous volatility of the underlying instrument be a deterministic function of its spot price and time. Such a framework retained the completeness of the Black–Scholes approach, thus allowing the option to be perfectly replicated by dynamically trading the underlying. Perhaps more importantly, Dupire showed that the local volatility function could be derived in a non-parametric way from quoted vanilla option prices, which by construction ensured a perfect fit to the market smile and hence allowed us to price exotics consistently with vanilla options. Dupire's famous local volatility equation featured – as one of the inputs – a partial derivative of option price with respect to expiration (a "calendar spread"). While not problematic for equity underlyings, the derivative of option price with respect to maturity is not properly defined for interest rate underlyings,[1] which proved to be a major hurdle in applying the local volatility approach in the interest rate space. In a recent contribution, however, we have tried to present a unified approach to local volatility modeling, one that encompasses all asset classes, and applies equally to interest rate underlyings. We present the thrust of the argument for completeness, adapting it to the case at hand.

As a first step, we need to introduce a new spot process S, defined as the value of a forward swap rate (2.3), with fixed termination date T, rolling with time t into successive fixing dates (for convenience, we opt for continuous notation):

$$S(t) \equiv \frac{1 - P(t,T)}{\int_t^T P(t,s)ds}, \quad t < T. \tag{2.6}$$

We do not need to specify formally the dynamics of $S(t)$, other than by saying that it is an Itô process with diffusion $\sigma(t)$ satisfying the usual regularity conditions. The instantaneous forward rate $f(t,T)$ with maturity T contracted at t is defined by

$$f(t,T) \equiv -\frac{\partial \ln P(t,T)}{\partial T} \iff P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \tag{2.7}$$

while the instantaneous spot rate $r(t)$ – i.e. the short rate – capturing the locally risk-free return from a continuously compounded money market account $B(t) \equiv \exp\left\{\int_0^t r(s)ds\right\}$ can be formalized as

$$r(t) \equiv f(t,t). \tag{2.8}$$

We also let $N(t) = \int_t^T P(t,s)ds$ and for notational convenience denote $Q(t) = \int_t^T P(0,s)ds$ and $D(t) = \frac{N(t)}{B(t)}$. We may now rewrite (2.5) in our new notation, expressing the time-zero value of an option with strike K and maturity t on a swap terminating at T as

$$C(t,K) \quad = \mathbb{E}^{\mathbb{Q}}\left[(S(t)-K)^+ D(t)\right]. \tag{2.9}$$

Note that since $S(\cdot)$ is now defined as a spot process, $C(t,K)$ is properly defined as a function of maturity time t. We can apply Tanaka's formula to the payoff to get

$$d(S(t)-K)^+ = \mathbb{1}_{\{S(t)>K\}} dS(t) + \frac{1}{2}\delta(S(t)-K)d\langle S(t)\rangle_t. \tag{2.10}$$

Thus, after some algebra:

$$d\left(D(t)(S(t)-K)^+\right) = \frac{1}{2}D(t)\delta(S(t)-K)\sigma^2(t)dt$$
$$+ \mathbb{1}_{\{S(t)>K\}}\left(d\left(D(t)S(t)\right) - KdD(t)\right).$$

Taking the expected value of both sides ultimately yields (see Gatarek and Jabłecki, 2019 for details of the proof)

$$\frac{\partial C(t,K)}{\partial t} = \frac{1}{2}\frac{\partial^2 C(t,K)}{\partial K^2}\mathbb{E}^t\left[\sigma^2(t)|S(t)=K\right]$$
$$-\int_K^\infty \frac{\partial^2 C(t,x)}{\partial x^2}\left\{\mathbb{E}^t\left[\frac{r(t)}{N(t)}\middle| S(t)=x\right] + K\mathbb{E}^t\left[\frac{1}{N(t)}\middle| S(t)=x\right]\right\}dx, \tag{2.11}$$

where $\mathbb{E}^t[\cdot]$ is the expectation under a t-forward measure defined for any ξ by

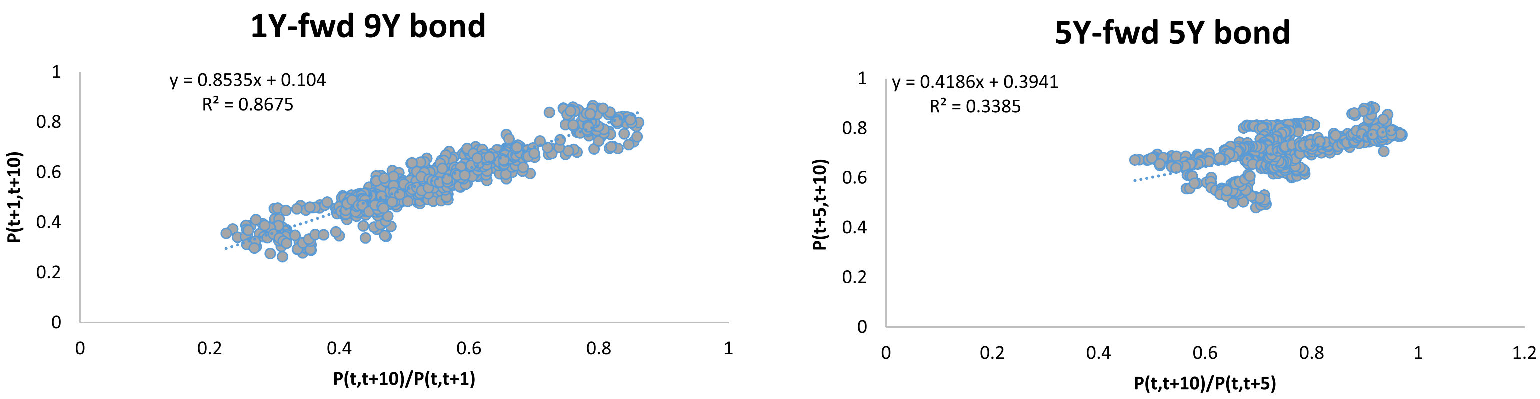
$$\mathbb{E}^t(\xi) \equiv \frac{\mathbb{E}^{\mathbb{Q}}[D(t)\xi]}{\mathbb{E}^{\mathbb{Q}}[D(t)]}. \tag{2.12}$$

The result above is clearly unwieldy because of the conditional expectations in the integrand. Hence, Gatarek and Jabłecki (2019) propose to "localize" the respective drift and diffusion terms in processes S and D by introducing a term structure model, borrowed from Cheyette (1992), in which yields and all derivable quantities – such as swap rates – are driven by a single state variable. This then makes it possible to extract the local volatility function $\mathbb{E}^t\left[\sigma^2(t)|S(t)=K\right]$ directly from market prices, map it onto the short rate volatility in the Cheyette model, and price swaptions in a local volatility framework through Monte Carlo or finite differences. Here, we consider a somewhat different approach which–at the cost of some rigor and formalism–delivers a substantial improvement in simplicity and intuitive appeal.

3 Swap rate à la stock

The main idea is to naively "localize" the expressions under expectations in (2.11) through a clever approximation, thus circumventing the need for the introduction of a formal yield curve model. To achieve this, we resort to a standard practice of freezing the yield curve, as done by Brace et al. (1997). This is a fairly standard

Figure 4: Regression fit of the 1-year-ahead 9-year zero-coupon bond price and the 5-year-ahead 5-year zero-coupon bond price against their time *t* "frozen" forecasts



technique, often used by practitioners and academics alike (see e.g. Beveridge and Joshi, 2014; Chen and Sandmann, 2012; or Grzelak and Oosterlee, 2012 for some recent examples). Concretely, we assume that

$$P(t,T) \approx \frac{P(0,T)}{P(0,t)}. \tag{3.1}$$

The approximation in (3.1) is actually not as restrictive as it might seem. To verify this, we regress the 1-year-ahead 9-year zero-coupon bond price $P(t+1,t+10)$ and the 5-year-ahead 5-year zero-coupon bond price $P(t+5,t+10)$ against their time t "frozen" forecasts calculated as $P(t,t+10)/P(t,t+1)$ and $P(t,t+10)/P(t,t+5)$, respectively. We use US Treasury yield curve data for the period June 1961 through March 2018. In both cases the fit is reasonably good (with correlation 93 and 58 percent and RMSE 10 and 13 percent of the bond price), although it clearly deteriorates somewhat as we move further out in time (Figure 4).

It follows from (3.1) that

$$P(0,t)\int_t^T P(t,s)ds \approx \int_t^T P(0,s)ds, \tag{3.2}$$

so that

$$\frac{r(t)}{N(t)} = \frac{r(t)S(t)}{1-P(t,T)} \approx \frac{f(0,t)P(0,t)}{P(0,t)-P(0,T)}S(t) \tag{3.3}$$

and

$$\frac{1}{N(t)} \approx \frac{P(0,t)}{Q(t)}. \tag{3.4}$$

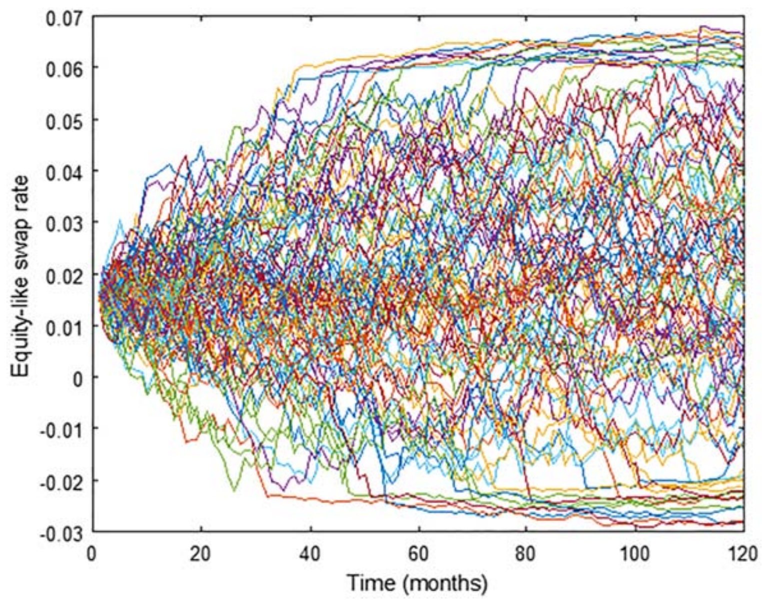
Set $p(t) \equiv \frac{f(0,t)P(0,t)}{P(0,t)-P(0,T)}$ and $q(t) \equiv -\frac{Q'(t)}{Q(t)}$. Under these approximations, the integrand in (2.11) simplifies greatly and we obtain

$$\frac{\partial C(t,K)}{\partial t} - \frac{1}{2}\frac{\partial^2 C(t,K)}{\partial K^2}\sigma^2(t,K) \approx K\frac{\partial C(t,K)}{\partial K}\left(p(t)-q(t)\right) - p(t)C(t,K). \tag{3.5}$$

Differentiating both sides of (3.5) yields the Fokker–Planck equation

$$\frac{\partial C^3(t,K)}{\partial t \partial K^2} - \frac{1}{2}\frac{\partial^2}{\partial K^2}\left(\frac{\partial^2 C(t,K)}{\partial K^2}\sigma^2(t,K)\right) = -\frac{\partial}{\partial K}\left(\frac{\partial^2 C(t,K)}{\partial K^2}\left\{p(t)-Kq(t)\right\}\right) + \frac{\partial^2 C(t,K)}{\partial K^2}q(t,K). \tag{3.6}$$

Figure 5: Sample paths of the equity-like swap rate $\widetilde{S}(t)$



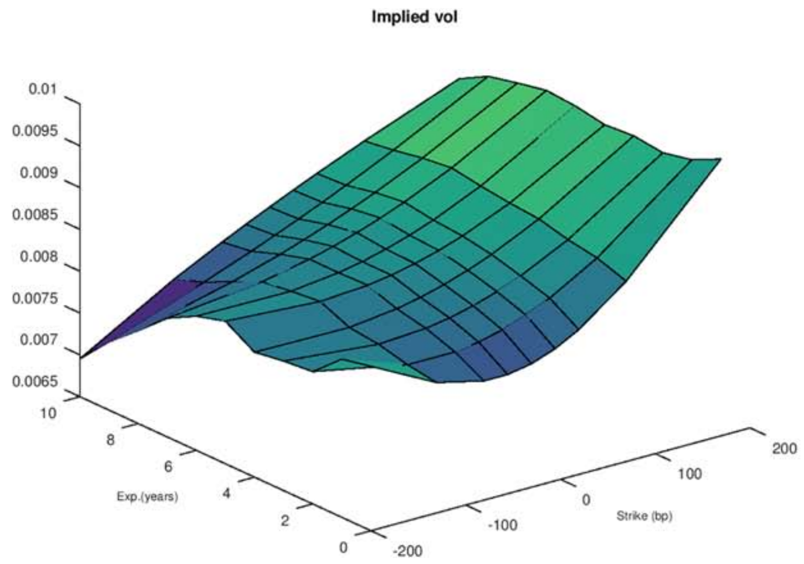
Note: Paths generated under the assumption that $T = 11$ years and $\sigma\left(t,\widetilde{S}(t)\right)$ is flat at 0.009.

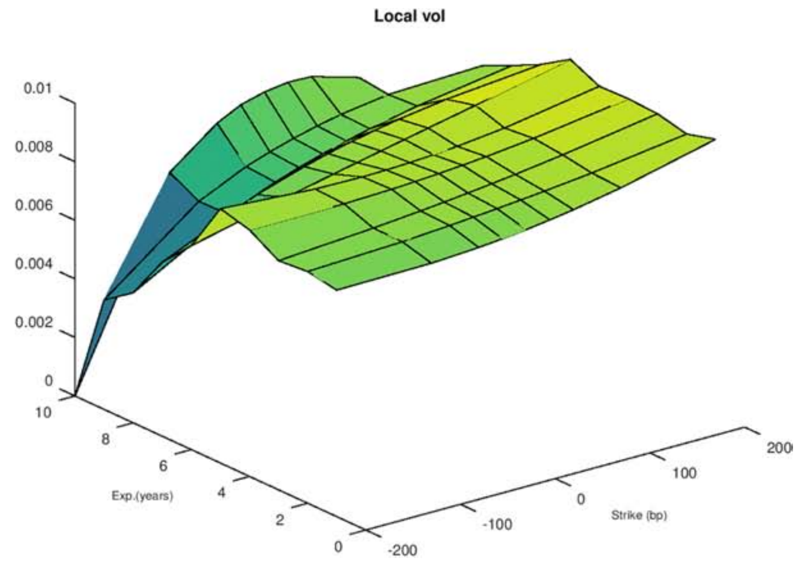
Now, it can be shown that the unique solution to (3.6) is the transition density generated by the following diffusion:

$$d\widetilde{S} = \sigma\left(t,\widetilde{S}(t)\right)d\widetilde{W}(t) + \widetilde{S}(t)\left(q(t)-p(t)\right)dt. \tag{3.7}$$

The process $\widetilde{S}(t)$ is quite interesting: its instantaneous volatility is given by the local volatility function of the rolling swap rate (2.6) and its drift is driven by the spread $q(t)-p(t)$. Note that this is very similar to the situation we encounter in the equity world where the risk-neutral drift of a share of stock features the risk-free rate and the stock's dividend yield. Thus, we have effectively constructed equity-like dynamics for the swap rate process $\widetilde{S}(t)$ with its own "short rate" $q(t)$ and "dividend yield" $p(t)$ (Figure 5).

Figure 6: Co-terminal normal implied volatility surface and corresponding stripped local volatility surface





Note: Terminal swap maturity is fixed at 11 years; data as of October 5, 2016.

Building on this analogy further, we consider a new discount process $Q(t)$ with dynamics

$$dQ(t) = -Q(t)q(t)dt, \tag{3.8}$$

which follows directly from (3.4), since

$$q(t) = \frac{P(0,t)}{Q(t)} = -\frac{dQ(t)}{dt}\frac{1}{Q(t)}.$$

Using the Feynman–Kac theorem we can express the solution of the partial differential equation (PDE) in (3.5) as the expected value of the underlying function's terminal payoff. The payoff itself is a function of our equity-like process whose drift and diffusion coefficients are implicitly defined as the coefficients of the PDE. Thus, the time-zero price of an option $\widetilde{C}$ on $\widetilde{S}$ with maturity t and strike K is given by

$$\mathbb{E}\left[Q(t)\left(\widetilde{S}(t)-K\right)^{+}\right] \tag{3.9}$$

and needs to satisfy the general PDE

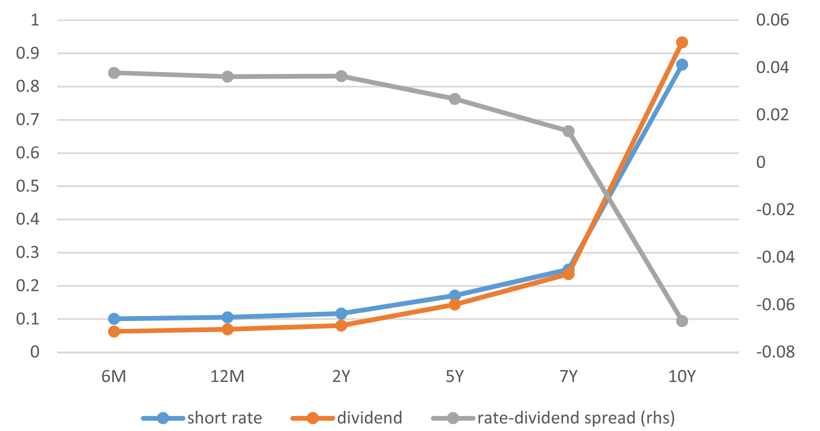
$$\frac{\partial\widetilde{C}(t,K)}{\partial t} - \frac{1}{2}\frac{\partial^2\widetilde{C}(t,K)}{\partial K^2}\sigma^2(t,K) = K\frac{\partial\widetilde{C}(t,K)}{\partial K}\left(p(t)-q(t)\right) - p(t)\widetilde{C}(t,K), \tag{3.10}$$

which itself is an approximation of the general equation (2.11). Hence, we conclude that

$$\begin{aligned}\widetilde{C}(t,K) &= \mathbb{E}\left[Q(t)\left(\widetilde{S}(t)-K\right)^{+}\right] = Q(t)\mathbb{E}\left[\left(\widetilde{S}(t)-K\right)^{+}\right]\\ &\approx \mathbb{E}\left[D(t)\left(S(t)-K\right)^{+}\right] = C(t,K),\end{aligned} \tag{3.11}$$

or equivalently, that $\mathbb{E}\left[Q(t)\left(\widetilde{S}(t)-K\right)^{+}\right] \approx \mathbb{E}^t\left[Q(t)\left(S(t)-K\right)^{+}\right]$. The above conclusion can greatly simplify the process of pricing swaptions, reducing it to the calculation of a risk-neutral expectation of $\widetilde{S}(t)$, with no need for a full-fledged term structure model or an estimation of the swap level. In fact, (3.11) can be handled using a classic equity option Monte Carlo or finite difference pricer, which should make it particularly useful for practitioners. Naturally, the swaption price derived using (3.11) is just an approximation of the swaption price derived using a full-term structure model (such as Cheyette), but it turns out to be a very accurate one. We demonstrate this point in numerical examples below.

Figure 7: Stripped "risk-free" rate and "dividend" yield of the equity-like swap rate process (3.7)

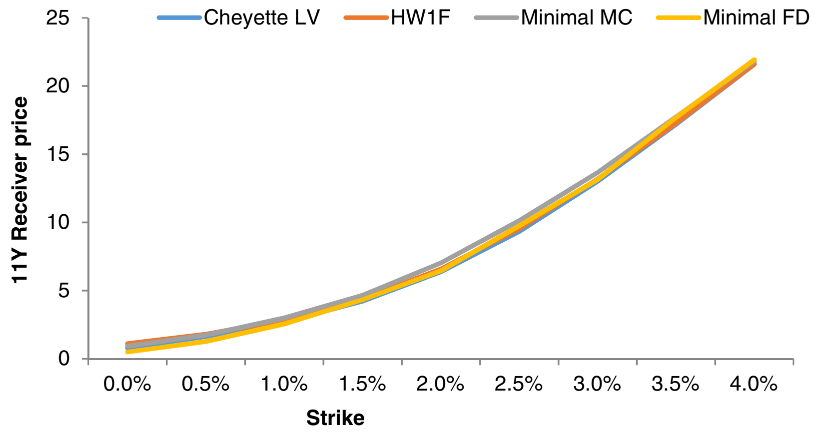


Note: $p(t) = \frac{f(0,t)P(0,t)}{P(0,t)-P(0,T)}$ and $q(t) = \frac{P(0,t)}{\int_t^T P(0,s)ds}$, $T = 11$. Values stripped from yield curve data as of October 5, 2016.

4 Bermudan swaption pricing

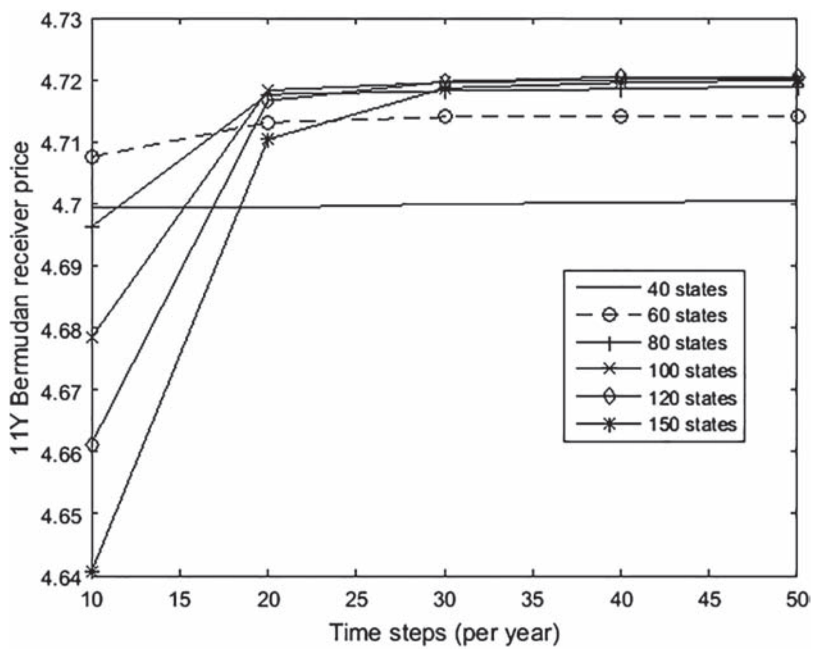
Ultimately, the proof of the pudding is in the eating. Thus, we now test our equity-like swap rate model on a sample Bermudan swaption structure, using different numerical techniques, and comparing the results against the full local volatility Cheyette model and the standard Hull–White model. Recall that, given a

Figure 8: Prices and Monte Carlo standard errors (where appropriate) of a 11nc1 Bermudan receiver swaption with 100 notional.



Note: "MC" stands for least squares Monte Carlo; "FD" denotes Crank–Nicholson finite difference scheme. The models are all calibrated to the same set of co-terminal swaptions: $1 \times 10, 2 \times 9, 3 \times 8, \ldots, 10 \times 1$ using USD market data as of October 5, 2016; local volatility surface used is given in Figure 6.

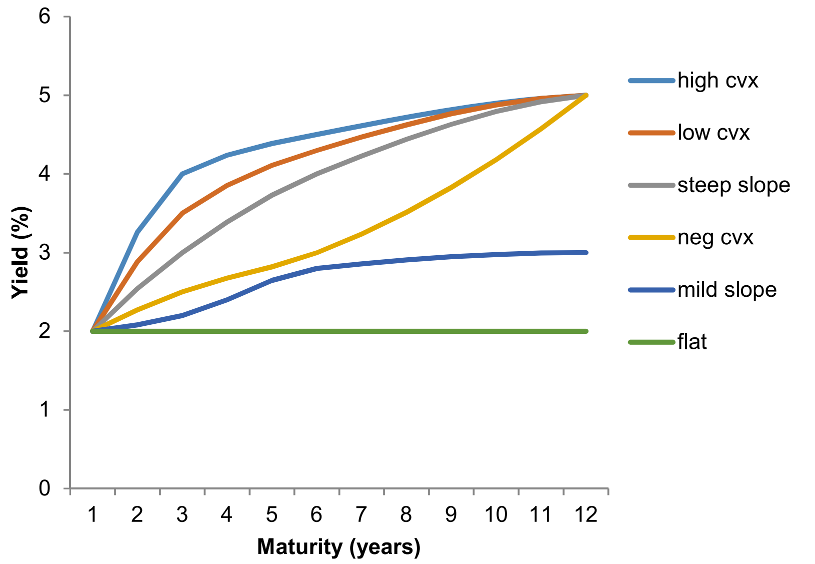
Figure 9: Convergence profiles for a finite difference implementation of the equity-like "minimal model" (11Y Bermudan swaption, ATM = 0.0163)



tenor structure $\mathcal{T} = \{T_n\}_{n=\alpha}^{\beta}$, the price of "$T_\beta$ no-call T_α" Bermudan receiver swaption $\mathbf{RBS}_{\alpha,\beta}(t,K)$ has time-t value given by

$$\mathbf{RBS}_{\alpha,\beta}(t,K) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left((K - S_{\tau,\beta}(\tau))^+ D(\tau)\right). \tag{4.1}$$

Figure 10: Stylized yield curve shapes scenarios



But we already know from (3.11) that $\mathbb{E}\left[Q(t)\left(K - \widetilde{S}(t)\right)^+\right] \approx \mathbb{E}\left[D(t)(K - S(t))^+\right]$, which suggests that

$$\mathbf{RBS}_{\alpha,\beta}(t,K) \approx \sup_{\tau \in \mathcal{T}} \mathbb{E}\left((K - \widetilde{S}_{\tau,\beta}(\tau))^+ Q(\tau)\right) = \widetilde{\mathbf{RBS}}_{\alpha,\beta}(t,K). \tag{4.2}$$

Concretely, we now set $T_\alpha = 1$ and $T_\beta = 11$ for a 11nc1 Bermudan swaption which we price using USD interest rate curve and volatility data as of October 5, 2016. As a first step, we extract the local volatility surface from the quoted prices of co-terminal swaptions $1 \times 10, 2 \times 9, 3 \times 8, \ldots, 10 \times 1$ using the rearranged PDE (3.5):[2]

$$\sigma(t,K)^2 \approx 2\frac{\partial_t C(t,K) - \left(q(t) - p(t)\right) K \partial_K C(t,K) + p(t)C(t,K)}{\partial^2_{KK} C(t,K)}. \tag{4.3}$$

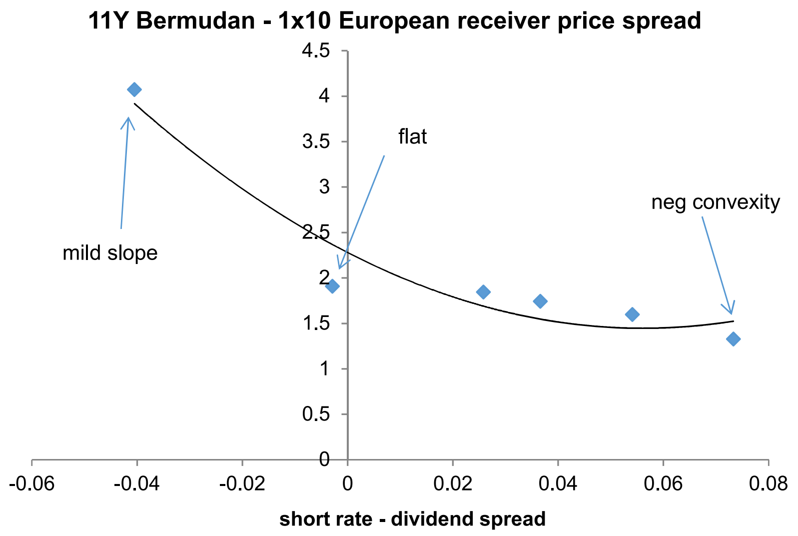
Figure 6 shows the implied volatility surface of co-terminal swaptions with terminal fixed at 11 years and the corresponding local volatility, while Figure 7 shows the generic risk-free rate and dividend yield. Note how the two are related and rise in tandem as t approaches the final maturity.

To estimate $\widetilde{\mathbf{RBS}}_{\alpha,\beta}(t,K)$ we use two numerical techniques: least-squares Monte Carlo (LSMC) and the Crank–Nicholson finite difference scheme (CN). For the LSMC we choose simply the underlying "swap rate" along with its second and third powers, and a constant, in the continuation value regression. The Monte Carlo uses 12 time steps per year and 20,000 paths, and the entire routine is repeated 100 times which allows for the calculation of the approximate simulation standard error. The CN scheme uses the following mesh parameters:

$$S_{ini} = 0.0163, \quad S_0 = 0, \quad S_{max} = 2S_0$$
$$dS = 0.0001 \implies M = 326$$
$$dt = 0.02 \implies N = 50.$$

We benchmark these results against the market standard Hull–White one-factor model implemented in a tree-based approach as well as a local volatility Cheyette model with mean reversion parameter $\kappa = 0.1\%$ simulated using LSMC.

Figure 11: 11Y Bermudan–1 × 10 European receiver swaption price spread (exercise premium)



Note: Short rate and dividend rates calculated as averages over the tenor structure.

The results for a range of strikes are shown in Figure 8. The prices produced by the four models are generally closely aligned, but what is particularly important is how close the minimal model – with an evolution of only a single process –is to the Cheyette model which features a full-fledged, no-arbitrage term structure dynamics. The minimal model lends itself to a straightforward implementation via a finite difference scheme which guarantees faster convergence as compared to the Cheyette Monte Carlo implementation. Figure 9 shows convergence profiles for the Crank–Nicholson scheme under different grid set-ups. As for standard equity options, convergence of the finite difference method for the minimal model is very fast in terms of both δt and δS – in fact all price estimates are within 0.04 percent of the number obtained using the highest grid resolution.

5 Discussion

We have presented above a simple trick that allows us to price Bermudan swaptions essentially as if they were options on a dividend-paying stock. The practical significance of this contribution is that, while the approach is not strictly arbitrage free and unfortunately can be directly applied only to swap rate derivatives, it offers a potentially attractive solution – especially in cases when the development of a full-fledged interest rate model is too time or resource consuming. Perhaps the most attractive feature of the model is that it makes it possible to transpose some of the well-established rules of thumb from the world of equities to interest rate space. While this is still a work in progress, we show below a stylized example demonstrating how the exercise premium in a Bermudan swaption is linked to the spread between the implied “short rate” and the “dividend,” which in turn can be linked to the observed yield curve shapes. Specifically, we consider five yield curve regimes and a benchmark case of a flat term structure, shown in Figure 10.

For each of the scenarios we price an 11Y Bermudan receiver swaption and the corresponding 1 × 10 European swaption. As shown in Figure 11, the difference between the two – i.e. the exercise premium – varies with the rolling swap rate drift, captured by the short rate–dividend spread implied by each yield curve scenario. Thus, our model provides an interesting analogy to the case of equities (Figure 1), and may lead to useful rules of thumb for gauging the size of exercise premium, depending on the specific yield curve view.

From a theoretical standpoint, we hope to have countered the widely held belief that the pricing of Bermudan swaptions requires a formal term structure model. This may be counterintuitive. After all, it is well known that what determines the value of the early exercise premium in a Bermudan structure is the correlation between the underlying forward swap rates – which at first glance does not appear to be captured by our equity-like swap rate model. In fact, however, information about rate correlations and volatilities is captured by the local volatility and, to a lesser extent, also the “short rate” $q(t)$ and “dividend” $p(t)$. To see this, consider the forward swap rates into which our “T_β no-call T_α” Bermudan receiver allows exercising: $S_{\alpha,\beta}(t), S_{\alpha+1,\beta}(t), \ldots, S_{\beta-1,\beta}(t)$. Suppose we were to use a proper multi-factor yield curve model (say, LMM) so that the swap rates would evolve according to the following Itô processes:

$$
\begin{aligned}
dS_{\alpha,\beta}(t) &= \mu_\alpha(t)dt + \sigma_\alpha dW_\alpha(t)\\
dS_{\alpha+1,\beta}(t) &= \mu_{\alpha+1}(t)dt + \sigma_{\alpha+1} dW_{\alpha+1}(t)\\
&\ldots\\
dS_{\beta-1,\beta}(t) &= \mu_{\beta-1}(t)dt + \sigma_{\beta-1} dW_{\beta-1}(t)
\end{aligned}
$$

with $dW_i dW_j = \rho_{ij}(t)dt$. Then, as shown by Rebonato, the terminal correlation $\mathrm{corr}(S_{i,\beta}(T_i), S_{j,\beta}(T_j))$ – which is the key value driver for our Bermudan –could be roughly approximated by

$$
\frac{\int_0^{T_i} \sigma_i(t)\sigma_j(t)\rho_{i,j}(t)dt}{\sqrt{\int_0^{T_i} \sigma_i^2(t)dt}\sqrt{\int_0^{T_j} \sigma_j^2(t)dt}}. \tag{5.1}
$$

Now, (5.1) clearly depends not only on $\rho_{i,j}(t)$ – which we may be accused of missing through our choice of a single equity-like process for the Bermudan's underlying – but it is also driven by the instantaneous volatilities $\sigma_i(t), \sigma_j(t)$. The latter, however, are picked up by the local volatility function of our equity-like swap rate, as $\sigma(t, K)$ is determined from the implied volatilities of the forward swap rates $S_{\alpha,\beta}(t), S_{\alpha+1,\beta}(t), \ldots, S_{\beta-1,\beta}(t)$.

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Endnotes

1. Indeed, there is typically only a single option on a given interest rate product –e.g. there is only one option on an interest rate swap initiating in, say, 5 years and maturing in 15 years, namely the 5y-into-10y swaption. A swaption on the fixed-tenor 10-year swap with a different maturity, say 1 year, will be written on a different underlying, driven by a different forward process.
2. To be precise, we convert swaption prices into Bachelier implied volatilities and then perform the stripping, which seems to produce much more stable results.

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