Equivalent Black Volatilities*

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Abstract

We consider European calls and puts on an asset whose forward price $F(t)$ obeys

$$dF(t) = \alpha(t)A(F)dW(t)$$

under the forward measure. By using singular perturbation techniques, we obtain explicit algebraic formulas for the implied volatility σ_B in terms of today's forward price $F_0 \equiv F(0)$, the strike K of the option, and the time to expiry t_{ex}. The price of any call or put can then be calculated simply by substituting this implied volatility into Black's formula.

For example, for a power law (constant elasticity of variance) model

$$dF(t) = aF^\beta dW(t)$$

we obtain

$$\sigma_B = \frac{a}{f_{av}^{1-\beta}}\left\{1 + \frac{(1-\beta)(2+\beta)}{24}\left(\frac{F_0 - K}{f_{av}}\right)^2 + \frac{(1-\beta)^2}{24}\frac{a^2 t_{ex}}{f_{av}^{2-2\beta}} + \dots\right\}$$

where $f_{av} = \frac{1}{2}(F_0 + K)$.

Our formula for the implied volatility is not exact. However, we show that the error is insignificant, rarely approaching $\frac{1}{1000}$ of the time value of the option. We also present more accurate (albeit more complicated) formulas which can be used for the implied volatility.

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Market prices of many European options are quoted in terms of their Black-Scholes volatility σ_B. To obtain the cash price, let t_{ex} be the option's exercise date, let t_s be the option's settlement date, and define $F(t)$ to be the forward price of the asset (for a contract maturing at the settlement date t_s) as seen at date t. Then the cash price of calls and puts is given by Black's formula [1, 6]

$$V_{call} = D(0,t_s)\{F_0\mathcal{N}(d_1) - K\mathcal{N}(d_2)\} \tag{1.1a}$$

$$V_{put} = D(0,t_s)\{K\mathcal{N}(-d_2) - F_0\mathcal{N}(-d_1)\} \tag{1.1b}$$

with

$$d_{1,2} = \frac{\log(F_0/K) \pm \frac{1}{2}\sigma_B^2 t_{ex}}{\sigma_B\sqrt{t_{ex}}}. \tag{1.1c}$$

Here $F_0 \equiv F(0)$ is today's forward value of the asset, $D(0,t_s)$ is today's discount factor to the settlement date, and σ_B is the quoted volatility of the option.

The derivation of Black's formula presumes that the forward prices $F(t)$ are distributed log normally about today's forward price F_0,

$$dF = \sigma_B F\,dW, \quad F(0) = F_0. \tag{1.2}$$

However, prices are rarely distributed log normally. Consequently market volatilities σ_B usually vary with the strike K and time-to-exercise t_{ex}.

Changing σ_B with the strike and exercise date essentially means that a *different* model is being used for each strike and expiry. This presents several difficulties when managing large books of options. First, it is not clear that the delta and vega risks calculated at a given strike are consistent with the same risks calculated at other strikes, which interjects uncertainty into the consolidation and hedging of risks across strikes. Second, if σ_B varies with the strike K, it seems likely that σ_B also varies systematically as the forward price F changes [3, 4]. Any vega risk arising from the *systematic* change of σ_B with F could be hedged more properly (and inexpensively) as delta risk. Finally, it is difficult to know which volatility to use in pricing exotic options.

An alternative approach is to use a single model which correctly prices options at different strikes and exercise dates without "adjustment." Commonly these models are of the form [3, 4]

$$dF = \alpha(t)A(F)dW, \quad F(0) = F_0 \tag{1.3}$$

under the forward measure, with call and put prices given by the expected values

$$V_{call} = D(0,t_s)E\{[F(t_{ex}) - K]^+ \mid F(0) = F_0\} \tag{1.4a}$$

$$V_{put} = D(0,t_s)E\{[K - F(t_{ex})]^+ \mid F(0) = F_0\} \tag{1.4b}$$

in this measure.

In Appendix A we use singular perturbation techniques [9] to analyze these models and find explicit expressions for the values of European calls and puts. These formulas are then used to obtain the implied volatilities σ_B of the options. For both calls and puts, we find that the implied volatility is

$$\begin{aligned}\sigma_B = \frac{a|A(f_{av})|}{f_{av}}\Big\{1 &+ \tfrac{1}{24}\Big[\frac{A''}{A} - 2\Big(\frac{A'}{A}\Big)^2 + \frac{2}{f_{av}^2}\Big](F_0-K)^2 \\ &+ \tfrac{1}{24}\Big[2\frac{A''}{A} - \Big(\frac{A'}{A}\Big)^2 + \frac{1}{f_{av}^2}\Big]a^2A^2(f_{av})t_{ex} + \dots\Big\}.\end{aligned} \tag{1.5a}$$

Here A and its derivatives are to be evaluated midway between today's forward price and the strike, at

$$f_{av} = \tfrac{1}{2}(F_0 + K), \tag{1.5b}$$

and a is the "sum-of-squares average" of $\alpha(t)$,

$$a = \Big(\frac{1}{t_{ex}} \int_0^{t_{ex}} \alpha^2(t')dt'\Big)^{1/2}. \tag{1.5c}$$

For example consider a power law model

$$dF = \alpha(t) F^{\beta} dW. \tag{1.6}$$

This model is just the constant elasticity of variance (CEV) model [2] written in terms of the forward price $F(t)$. For this model the implied volatility is

$$\sigma_B = \frac{a}{f_{av}^{1-\beta}} \Big\{ 1 + \tfrac{1}{24}(1-\beta)(2+\beta)\Big(\frac{F_0 - K}{f_{av}}\Big)^2 + \tfrac{1}{24}(1-\beta)^2 \frac{a^2 t_{ex}}{f_{av}^{2-2\beta}} + \dots \Big\}, \tag{1.7}$$

where f_{av} and a are given by (1.5b) and (1.5c) as before.

The "equivalent vol" formula (1.5) provides a very quick and very simple method for pricing calls and puts. Instead of using lattice, PDE, or Monte Carlo methods to compute the value of the option, one simply uses (1.5) to obtain the implied volatility σ_B, and then substitutes this σ_B into Black's formula (1.1).

Option hedges can also be obtained by writing (1.5) as

$$\sigma_B \equiv \sigma_B(F_0, K, t_{ex}) \tag{1.8a}$$

and then writing the option price as

$$V = V_{Black}(F_0, K, t_{ex}, \sigma_B(F_0, K, t_{ex})), \tag{1.8b}$$

where V_{Black} is the call or put formula in (1.1). Differentiating (1.8b) with respect to F_0 yields

$$\frac{\partial V}{\partial F_0} = \frac{\partial V_{Black}}{\partial F_0} + \frac{\partial \sigma_B}{\partial F_0}\frac{\partial V_{Black}}{\partial \sigma_B}. \tag{1.9}$$

Thus, the delta risk of the option is composed of *two* terms: the usual delta risk from Black's formula, plus a term proportional to the Black vega. This last term arises from the systematic change in the implied volatility σ_B of the option caused by changes in the forward price F.

The equivalent vol formula (1.5) is not exact. To test its accuracy, we compared the option values obtained using the equivalent vol formula (1.5) against the "exact" option prices obtained by numerical computation for different $A(F)$. We found that the equivalent vol formula is surprisingly accurate, usually as accurate as an excellent tree or PDE valuation, and much more accurate than Monte Carlo solutions. To demonstrate its accuracy, consider the power law model (1.6). For power law models, we found that the worst errors occur when $\beta = 0$, which is to be expected since $\beta = 0$ is "furthest" from the log normal Black model. Accordingly, in figures 1-3 we show the error between the exact option price and the price obtained using the equivalent vol formula (1.7) when $\beta = 0$. Shown is the error as a function of the strike K for 1 year, 2 year, 5 year, and 10 year options. Figure 1 exhibits the error as the difference between the exact implied vol and the equivalent vol obtained from (1.7); figure 2 exhibits the error as a fraction of the forward price F_0; and figure 3 exhibits the error as a fraction of the option's time value.

The singular perturbation analysis in Appendix A can be carried out to arbitrarily high order. In Appendix B we state a more accurate equivalent vol formula obtained by carrying out the analysis through $O(\epsilon^4)$. Figure 4 shows the error made using this more accurate formula. Although this formula is clearly much more accurate than (1.5), we believe that (1.5) is precise enough to make this improvement superfluous in most cases.

Singular perturbation techniques can be used to solve many related pricing problems. For example, these techniques can be used to solve intrinsically time-dependent models

$$dF = A(t, F)dW \tag{1.10}$$

to obtain accurate equivalent vol formulas for calls and puts; to analyze one- and two-factor term-structure models to obtain accurate equivalent vol formulas for swaptions, floors, and caps; and to obtain quick and accurate pricing of many exotics.

A Singular Perturbation Expansion

Consider a European call with expiration date t_{ex}, settlement date t_s, and strike K. As before, let $F(t)$ be the stochastic process for the forward price as seen at date t. We are assuming that

$$dF = \alpha(t)A(F)dW \tag{A.1}$$

under the forward measure. Under this measure, the value of the option at date t is $V(t, F(t))$, where the function $V(t, f)$ is given by the expected value [7, 8]

$$V(t, f) = D(t, t_s)E\big\{[F(t_{ex}) - K]^+ \mid F(t) = f\big\}. \tag{A.2}$$

Here $D(t, t_s)$ is the discount factor to the settlement date t_s at date t.

To simplify the problem, let us strip the discount factor from V,

$$V(t, f) \equiv D(t, t_s)Q(t, f). \tag{A.3}$$

Then $Q(t, f)$ is defined by

$$Q(t, f) = E\big\{[F(t_{ex}) - K]^+ \mid F(t) = f\big\}, \tag{A.4a}$$

and the expectation is over the probability distribution generated by the process

$$dF(t) = \alpha(t)A(F)dW(t). \tag{A.4b}$$

Therefore $Q(t, f)$ satisfies the backward Kolmogorov equation [10]

$$Q_t + \tfrac{1}{2}\alpha^2(t)A^2(f)Q_{ff} = 0, \quad t < t_{ex} \tag{A.5a}$$

with the final condition

$$Q = [f - K]^+ \quad \text{at } t = t_{ex}. \tag{A.5b}$$

A.1 Scaling and asymptotic solution

To obtain the option value, we first scale equtions (A.5a) and (A.5b) appropriately, and then use singular perturbation methods to solve the scaled problem. To obtain the equivalent vol, we analyze Black's model to determine the volatility σ_B which would yield the same value of the option.

To scale the equations appropriately, define

$$\epsilon \equiv A(K) << 1, \tag{A.6a}$$

and the new variables

$$\tau(t) = \int_t^{t_e} a^2(t')dt', \quad x = \frac{1}{\epsilon}(f - K), \quad \tilde{Q}(\tau, x) = \frac{1}{\epsilon}Q(t, f). \tag{A.6b}$$

In terms of the new variables, (A.5) becomes

$$\tilde{Q}_\tau - \tfrac{1}{2}\frac{A^2(K+\epsilon x)}{A^2(K)}\tilde{Q}_{xx} = 0 \quad \text{for } \tau > 0 \tag{A.7a}$$

with

$$\tilde{Q} = x^+ \quad \text{at } \tau = 0. \tag{A.7b}$$

After solving (A.7) for $\tilde{Q}(\tau, x)$, the option value will be given by

$$V(t, f) = D(t, t_s)A(K)\tilde{Q}\big(\tau(t), \tfrac{f-K}{A(K)}\big). \tag{A.8}$$

By expanding

$$A(K + \epsilon x) = A(K)\big\{1 + \epsilon\nu_1 x + \tfrac{1}{2}\epsilon^2\nu_2 x^2 + \ldots\big\} \tag{A.9a}$$

where

$$\nu_1 = A'(K)/A(K), \quad \nu_2 = A''(K)/A(K), \quad \ldots, \tag{A.9b}$$

problem (A.7) becomes

$$\tilde{Q}_\tau - \tfrac{1}{2}\tilde{Q}_{xx} = \epsilon\nu_1 x\tilde{Q}_{xx} + \tfrac{1}{2}\epsilon^2(\nu_2 + \nu_1^2)x^2\tilde{Q}_{xx} + \ldots, \quad \tau > 0 \tag{A.10a}$$

with

$$\tilde{Q} = x^+ \quad \text{at } \tau = 0. \tag{A.10b}$$

We can solve (A.10) by expanding $\tilde{Q}$ as

$$\tilde{Q}(\tau, x) = Q^0(\tau, x) + \epsilon Q^1(\tau, x) + \epsilon^2 Q^2(\tau, x) + \ldots. \tag{A.11}$$

Substituting (A.11) into (A.10) and equating like powers of ϵ yields a hierarchy of problems [9]. At leading order, we obtain

$$Q^0_\tau - \tfrac{1}{2}Q^0_{xx} = 0 \quad \text{for } \tau > 0, \tag{A.12a}$$

$$Q^0 = x^+ \quad \text{at } \tau = 0. \tag{A.12b}$$

At order ϵ, we have

(A.13a) $$Q^1_\tau - \tfrac{1}{2} Q^1_{xx} = \nu_1 x Q^0_{xx} \quad \text{for } \tau > 0,$$

(A.13b) $$Q^1 = 0 \quad \text{at } \tau = 0.$$

And at order ϵ^2,

(A.14a) $$Q^2_\tau - \tfrac{1}{2} Q^2_{xx} = \nu_1 x Q^1_{xx} + \tfrac{1}{2}(\nu_2 + \nu_1^2) x^2 Q^0_{xx} \quad \text{for } \tau > 0,$$

(A.14b) $$Q^2 = 0 \quad \text{at } \tau = 0,$$

and so on.

The solution of (A.12a) yields

(A.15a) $$Q^0(\tau, x) \equiv G(\tau, x) = x\left(\tfrac{x}{\sqrt{\tau}}\right) + \sqrt{\tfrac{\tau}{2\pi}} e^{-x^2/2\tau},$$

as can be verified by direct substitution. Note that $G(\tau, x)$ is essentially the value of a call option using a normal (as opposed to a log normal) model for the forward price. For later convenience, we note that

(A.15b) $$G_\tau = \tfrac{1}{2} \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}, \quad G_x = \left(\tfrac{x}{\sqrt{\tau}}\right),$$

(A.15c) $$G_{\tau\tau} = \frac{x^2 - \tau}{4\tau^2} \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}, \quad G_{x\tau} = -\frac{x}{2\tau} \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}, \quad G_{xx} = \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}},$$

(A.15d) $$G_{\tau\tau\tau} = \frac{x^4 - 6x^2\tau + 3\tau^2}{8\tau^4} \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}.$$

At $O(\epsilon)$, we substitute $Q^0 \equiv G(\tau, x)$ into (A.13) and use $G_{xx} = 2G_\tau$ and (A.15c). This yields

(A.16a) $$Q^1_\tau - \tfrac{1}{2} Q^1_{xx} = \nu_1 x G_{xx} = -2\nu_1 \tau G_{x\tau} \quad \text{for } \tau > 0$$

(A.16b) $$Q^1 = 0 \quad \text{at } \tau = 0.$$

The solution of (A.16) is

(A.17) $$Q^1 = -\nu_1 \tau^2 G_{x\tau} \equiv \nu_1 \tau x G_\tau,$$

as can be verified by direct substitution.

At $O(\epsilon^2)$, we substitute $Q^0 = G$ and $Q^1 = \nu_1 \tau x G_\tau$ into (A.14). This yields

(A.18a) $$Q^2_\tau - \tfrac{1}{2} Q^2_{xx} = \left(\nu_1^2 \frac{x^4 - 2\tau x^2}{2\tau} + \tfrac{1}{2}\nu_2 x^2\right) \frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}} \quad \text{for } \tau > 0$$

(A.18b) $$Q^2 = 0 \quad \text{at } \tau = 0.$$

The solution of (A.18) is

(A.19) $$Q^2 = \nu_1^2\big[\tau^4 G_{\tau\tau\tau} + \tfrac{8}{3}\tau^3 G_{\tau\tau} + \tfrac{1}{2}\tau^2 G_\tau\big] + \nu_2\big[\tfrac{2}{3}\tau^3 G_{\tau\tau} + \tfrac{1}{2}\tau^2 G_\tau\big].$$

To understand the solution, let us use (A.15) to write Q^2 in a convenient way,

(A.20) $$Q^2 = \tfrac{1}{2}\nu_1^2\tau^2 x^2 G_{\tau\tau} + \tfrac{1}{12}\nu_1^2(x^2 - \tau)\tau G_\tau + \tfrac{1}{6}\nu_2(2x^2 + \tau)\tau G_\tau.$$

Then through $O(\epsilon^2)$, our solution is

(A.21) $$\begin{aligned}\tilde{Q} &= Q^0 + \epsilon Q^1 + \epsilon^2 Q^2 + \dots \\ &= G + \epsilon\nu_1\tau x G_\tau + \tfrac{1}{2}\epsilon^2\nu_1^2\tau^2 x^2 G_{\tau\tau} + \epsilon^2\big(\frac{4\nu_2 + \nu_1^2}{12}x^2 + \frac{2\nu_2 - \nu_1^2}{12}\tau\big)\tau G_\tau,\end{aligned}$$

which we can re-write as

(A.22a) $$\tilde{Q}(\tau, x) = G(\tilde{\tau}, x),$$

with

(A.22b) $$\tilde{\tau} = \tau\big[1 + \epsilon\nu_1 x + \epsilon^2\big(\frac{4\nu_2 + \nu_1^2}{12}x^2 + \frac{2\nu_2 - \nu_1^2}{12}\tau\big) + \dots\big].$$

To obtain the option value, recall that

(A.23a) $$V(t, f) = D(t, t_s)\epsilon\tilde{Q}(\tau, x) = D(t, t_s)\epsilon G(\tilde{\tau}, x)$$

and note that

(A.23b) $$\epsilon G(\tilde{\tau}, x) = G(\epsilon^2\tilde{\tau}, \epsilon x) = G(A^2(K)\tilde{\tau}, f - K).$$

Thus the value of the option is

(A.24a) $$V(t, f) = D(t, t_s)G(\tau^*, f - K),$$

where

(A.24b) $$\tau^* = A^2(K)\tau\big[1 + \nu_1(f - K) + \frac{4\nu_2 + \nu_1^2}{12}(f - K)^2 + \frac{2\nu_2 - \nu_1^2}{12}A^2(K)\tau + \dots\big].$$

A.2 Implied Volatility

Although (A.24) gives a closed form expression for the price of the option, it is not very convenient. So we compute the equivalent Black volatility implied by this price. To simplify the calculation, we take the square root of (A.24b) and obtain

(A.25) $$\sqrt{\tau^*} = A(K)\sqrt{\tau}\Big[1 + \tfrac{1}{2}\nu_1(f - K) + \frac{2\nu_2 - \nu_1^2}{12}(f - K)^2 + \frac{2\nu_2 - \nu_1^2}{24}A^2(K)\tau + \dots\Big].$$

Recall that $\nu_1 = A'(K)/A(K)$, $\nu_2 = A''(K)/A(K)$, so the first two terms of (A.25) are $\sqrt{\tau}[A(K) + \frac{1}{2}A'(K)(f - K) + \dots]$. This suggests expanding A around the average

(A.26) $$f_{av} = \tfrac{1}{2}(f + K)$$

instead of K. Accordingly, we define

$$\gamma_1 = A'(f_{av})/A(f_{av}), \quad \gamma_2 = A''(f_{av})/A(f_{av}), \quad \ldots, \tag{A.27}$$

and re-write (A.25) in terms of γ_1 and γ_2 instead of ν_1 and ν_2. This then shows that the option price is

$$V(t, f) = D(t, t_s) G(\tau^*, f - K), \tag{A.28a}$$

with

$$\sqrt{\tau^*} = A(f_{av})\sqrt{\tau}\Big[1 + \frac{\gamma_2 - 2\gamma_1^2}{24}(f - K)^2 + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{av})\tau + \ldots\Big]. \tag{A.28b}$$

Now suppose we had started with Black's model

$$dF(t) = \sigma_B F(t) dW \tag{A.29}$$

instead of $dF = \alpha(t)A(F)dW$. Repeating the preceding analysis for the special case $\alpha(t) = \sigma_B$, $A(F) = F$, shows that the option price is

$$V(t, f) = D(t, t_s) G(\tau_B, f - K), \tag{A.30a}$$

with

$$\sqrt{\tau_B} = \sigma_B f_{av}\sqrt{t_{ex} - t}\Big[1 - \frac{(f - K)^2}{12 f_{av}^2} - \frac{\sigma_B^2(t_{ex} - t)}{24} + \ldots\Big]. \tag{A.30b}$$

Since $G(\tau_B, f - K)$ is an increasing function of τ_B, the Black price (A.30) matches the correct price (A.28) if and only if

$$\sqrt{\tau_B} = \sqrt{\tau^*}. \tag{A.31}$$

Equating yields the implied volatility

(A.32a)

$$\sigma_B = a\frac{A(f_{av})}{f_{av}}\Big\{1 + (\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{av}^2})\frac{(f - K)^2}{24} + (2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2})\frac{a^2 A^2(f_{av})(t_{ex} - t)}{24} + \ldots\Big\},$$

where

$$f_{av} = \tfrac{1}{2}(f + K), \quad \gamma_1 = \frac{A'(f_{av})}{A(f_{av})}, \quad \gamma_2 = \frac{A''(f_{av})}{A(f_{av})}, \tag{A.32b}$$

and

$$a = \frac{1}{t_{ex} - t}\int_t^{t_{ex}} \alpha^2(t')dt'. \tag{A.32c}$$

Setting $t = 0$ at $f = F_0$ yields (1.5).

Equation (A.32) gives the implied volatility for the European call option for the model

$$dF(t) = \alpha(t) A(F) dW(t). \tag{A.33}$$

A similar analysis shows that the implied volatility for a European put option is given by the same formula.

B Higher order results

The above analysis can be carried out to arbitrarily high order. Carrying it out through $O(\epsilon^4)$ yields the more accurate (but more complicated) equivalent vol formula

(B.1)

$$\begin{aligned}\sigma_B = \frac{aA(f_{av})}{f_{av}}\Big\{1 &+ \frac{\Delta}{24}\big(2\gamma_2 - \gamma_1^2 + \tfrac{1}{f_{av}^2}\big) + \frac{\theta^2}{24}\big(\gamma_2 - 2\gamma_1^2 + \tfrac{2}{f_{av}^2}\big)\\ &+ \frac{\Delta^2}{480}\big(2\gamma_4 + 4\gamma_1\gamma_3 + 3\gamma_2^2 - 3\gamma_1^2\gamma_2 + \tfrac{3}{4}\gamma_1^4 - \tfrac{3}{4f_{av}^4} + \frac{10\gamma_2 - 5\gamma_1^2 + 5/f_{av}^2}{2f_{av}^2}\big)\\ &+ \frac{\Delta\theta^2}{2880}\big(6\gamma_4 - 18\gamma_1\gamma_3 + 14\gamma_2^2 - 29\gamma_1^2\gamma_2 + 11\gamma_1^4 - \tfrac{11}{f_{av}^4} + \frac{35\gamma_2 - 40\gamma_1^2 + 40/f_{av}^2}{f_{av}^2}\big)\\ &+ \frac{\theta^4}{1440}\big(\tfrac{3}{4}\gamma_4 - 6\gamma_1\gamma_3 - 2\gamma_2^2 + 17\gamma_1^2\gamma_2 - 8\gamma_1^4 + \tfrac{8}{f_{av}^4} + \frac{5\gamma_2 - 10\gamma_1^2 + 10/f_{av}^2}{f_{av}^2}\big) + \dots\Big\}.\end{aligned}$$

Here

(B.2a)
$$f_{av} = \tfrac{1}{2}(f+K), \quad \theta = f - K,$$

(B.2b)
$$a = \Big(\frac{1}{t_{ex}-t}\int_t^{t_{ex}} \alpha^2(t')dt'\Big)^{1/2}, \quad \Delta = a^2A^2(f_{av})(t_{ex}-t),$$

and

(B.2c)
$$\gamma_1 = \frac{A'(f_{av})}{A(f_{av})}, \quad \gamma_2 = \frac{A''(f_{av})}{A(f_{av})}, \quad \gamma_3 = \frac{A'''(f_{av})}{A(f_{av})}, \quad \gamma_4 = \frac{A''''(f_{av})}{A(f_{av})}.$$

For the special case of a power-law model, $dF = \alpha(t)F^\beta dW$, the equivalent Black vol reduces to

(B.3)

$$\begin{aligned}\sigma_B = \frac{a}{f_{av}^{1-\beta}}\Big\{1 &+ \frac{(1-\beta)^2\Delta}{24f_{av}^2} + (1-\beta)(2+\beta)\frac{\theta^2}{24f_{av}^2} + \frac{(1-\beta)^2\Delta^2}{1920f_{av}^4}(7 - 54\beta + 27\beta^2)\\ &+ \frac{(1-\beta)^2\Delta\theta^2}{2880f_{av}^4}(29 - 13\beta - 16\beta^2) + \frac{(1-\beta)\theta^4}{5760f_{av}^4}(72 + 34\beta - 9\beta^2 - 7\beta^3) + \dots\Big\}.\end{aligned}$$

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Figure 1: Error in the equivalent vol for a normal model ($\beta = 0$). Shown is the difference between the equivalent vol given by (1.5) and the exact implied volatility as a function of the strike K for 1 year, 2 year, 5 year, and 10 year options. The case shown has $F_0 = 100$ and an at-the-money Black vol of 20%. The graph is scaled so that a 1 bp error would change 20% to 20.01%.

Figure 2: The error in the option price caused by using the equivalent vol formula (1.5). Shown is the error as a function of the strike K for 1 year, 2 year, 5 year, and 10 year options. Here 1 bp corresponds to an error of $10^{-4}F_0$ for the same case as Figure 1.

Figure 3: The error in the option price caused by using the equivalent vol formula (1.5). Shown is the ratio of the error to the time value of the option as a function of the strike K for 1 year, 2 year, 5 year, and 10 year options. Same case as Figure 1. Note that when the error is an appreciable percent of the time value, the time value of the option is near zero.

Figure 4: Error in the equivalent vol for a normal model ($\beta = 0$). Shown is the difference between the more accurate equivalent vol given by (B.3) and the exact implied volatility as a function of the strike K for 1 year, 2 year, 5 year, and 10 year options. Same case as Figure 1. Note that the graph is scaled so that a 2% error would change 20% to 20.0002%.