The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Interest Rate Volatility
IV. The SABR-LMM model

Andrew Lesniewski
Baruch College and Posnania Inc

First Baruch Volatility Workshop
New York

June 16 - 18, 2015

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Outline

1 The LMM methodology

2 Dynamics of the SABR-LMM model

3 Covariance structure of SABR-LMM

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

The LMM methodology

- The main shortcoming of short rate models is that they do not allow for close calibration to the entire volatility cube.
- This is not a huge concern on a trading desk, where locally calibrated term structure models allow for accurate pricing and executing trades.
- It is, however, a concern for managers of large portfolios of fixed income securities (such as mortgage backed securities) which have exposures to various segments of the curve and various areas of the vol cube.
- It is also relevant in enterprise level risk management in large financial institutions where consistent risk aggregation across businesses and asset classes is important.
- A methodology that satisfies these requirements is the LIBOR market model (LMM) methodology, and in particular its stochastic volatility extensions.
- That comes at a price: LMM is less tractable than some of the popular short rate models. It also tends to require more resources than those models.
- We will discuss a natural extension of stochastic volatility LMM, namely the SABR-LMM model.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Classic LMM

- Let us begin with a very brief review of the classic LMM model, see eg. [1].
- We consider a sequence of approximately equally spaced dates $0 = T_0 < T_1 < \ldots < T_N$ which will be termed the *standard tenors*. A standard LIBOR forward rate L_j, $j = 0, 1, \ldots, N - 1$ is associated with a FRA which starts on T_j and matures on T_{j+1}.
- Usually, it is assumed that $N = 120$ and the L_j's are 3 month LIBOR forward rates. Note that these dates refer to the actual start and end dates of the contracts rather than the LIBOR "fixing dates", i.e. the dates on which the LIBOR rates settle. To simplify the notation, we shall disregard the difference between the contract's start date and the corresponding forward rate's fixing date.
- Each LIBOR forward L_j is modeled as a continuous time stochastic process $L_j(t)$. Clearly, this process has the property that it gets killed at $t = T_j$. The dynamics of the forward process is driven by an $N - 1$-dimensional, correlated Wiener process $W_1(t), \ldots, W_{N-1}(t)$.
- We let ρ_{jk} denote the correlation coefficient between $W_j(t)$ and $W_k(t)$:

$$dW_j(t) \, dW_k(t) = \rho_{jk} dt. \tag{1}$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition

- To motivate the form of the LIBOR forwards dynamics, consider first a single LIBOR forward L_k. Under the measure Q_k, L_k is a martingale so that

$$
\begin{aligned}
dL_k(t) &= C(t)\, dW_k(t), \\
L_k(0) &= L_{k0},
\end{aligned}
\tag{2}
$$

where L_{k0} is the current value of the forward (as given by the curve model).

- The coefficient

$$
C_k(t) = C_k(t, L_k(t))
\tag{3}
$$

is the instantaneous volatility and it determines the internal volatility smile dynamics.

- For $j \neq k$, L_j *is not* a martingale under Q_k, and so we must have

$$
\begin{aligned}
dL_j(t) &= \Delta_j^{(k)}(t)\, dt + C_j(t)\, dW_j(t), \\
L_j(0) &= L_{j0}.
\end{aligned}
\tag{4}
$$

- We shall determine the drifts $\Delta_j^{(k)}(t)$ by requiring absence of arbitrage.
- The *no arbitrage* requirement of asset pricing imposes a relationship between the drift term and the diffusion term, whose form depends on the choice of k.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Change of numeraire formula

- This is accomplished by means of the change of numeraire technique: it allows us to modify the probability law (the measure) of the process so that, under the new measure, the process is driftless, i.e. it is a martingale.
- Before we proceed, let us take a minute to recall the *change of numeraire formula*. It is a consequence of Girsanov's theorem, and its derivation can be found e.g. in [1].
- Consider a financial asset whose dynamics is given in terms of the state variable X. Under the measure P, associated with the numeraire $\mathcal{N}$, its dynamics reads:

$$dX(t) = \Delta^{P}(t)\, dt + C(t)\, dW^{P}(t). \tag{5}$$

- We can relate this dynamics to the dynamics of the same asset under an equivalent martingale measure Q, associated with the numeraire $\mathcal{M}$:

$$dX(t) = C(t)\, dW^{Q}(t) \tag{6}$$

(remember that the diffusion coefficients in these equations are unaffected by the change of measure!).

- Namely, the drift coefficient $\Delta^{P}(t)$ can be expressed in terms of $\mathcal{N}$ and $\mathcal{M}$ as follows:

$$\Delta^{P}(t)\, dt = dX(t)\, d\Big(\log \frac{\mathcal{N}(t)}{\mathcal{M}(t)}\Big). \tag{7}$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition

- We shall now apply this formula to LMM.
- Let us first assume that $j < k$. The numeraires for the measures Q_j and Q_k are the prices $P(t, T_{j+1})$ and $P(t, T_{k+1})$ of the zero coupon bonds expiring at T_{j+1} and T_{k+1}, respectively.
- Explicitly,

$$P(t, T_{j+1}) = P(t, T_{\gamma(t)}) \prod_{\gamma(t) \leq i \leq j} \frac{1}{1 + \delta_i F_i(t)} , \qquad (8)$$

where F_i denotes the OIS forward[1] spanning the accrual period $[T_i, T_{i+1})$, and where $\gamma : [0, T_N] \to \mathbb{Z}$ is defined by

$$\gamma(t) = m + 1, \quad \text{if } t \in [T_m, T_{m+1}) .$$

- Notice that $P(t, T_{\gamma(t)})$ is the "stub" discount factor over the incomplete accrual period $[t, T_{\gamma(t)}]$.

---

[1] Recall that all discounting is done on OIS.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition

● Since the drift of $L_j(t)$ under Q_j is zero, formula (7) yields:

$$
\begin{aligned}
\Delta_j(t)\, dt &= dL_j(t)\, d\log \frac{P(t, T_{j+1})}{P(t, T_{k+1})} \\
&= -dL_j(t)\, d\log \prod_{j+1 \leq i \leq k} (1 + \delta_i F_i(t)) \\
&= -\sum_{j+1 \leq i \leq k} dL_j(t)\, \frac{\delta_i dF_i(t)}{1 + \delta_i F_i(t)} \\
&= -C_j(t) \sum_{j+1 \leq i \leq k} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt,
\end{aligned}
\tag{9}
$$

where, in the third line, we have used the fact that the spread between L_j and F_j is deterministic, and thus its contribution to $dL_j dF_i$ is zero.

● Similarly, for $j > k$, we find that

$$
\Delta_j(t) = C_j(t) \sum_{k+1 \leq i \leq j} \frac{\rho_{ji} \delta_i C_i(t)}{1 + \delta_i F_i(t)}.
\tag{10}
$$

A. Lesniewski   **Interest Rate Volatility**

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition

- We can thus summarize the above discussion as follows. The dynamics of the LMM model is given by the following system of SDEs: for $t < \min(T_k, T_j)$,

$$
dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1 \leq i \leq k} \dfrac{\rho_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \dfrac{\rho_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j > k. \end{cases} \quad (11)
$$

- In addition to the forward measures discussed above, it is convenient to use the spot measure. It is expressed in terms of the rolling banking account numeraire:

$$
B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{1 \leq i \leq \gamma(t)} P(T_{i-1}, T_i)}. \quad (12)
$$

- Under the spot measure, the LMM dynamics reads:

$$
dL_j(t) = C_j(t) \Big(\sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dW_j(t) \Big). \quad (13)
$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Classic LMM

- Before proceeding to the stochastic volatility extension of LMM, let us make a few observations.
- LMM allows exact calibration to the current LIBOR / OIS multi-curve. Unlike the short rate models discussed earlier, this calibration is done *separately* from the calibration of the dynamic aspects of the model: the current curve is simply the initial condition for the dynamics of the model.
- LMM is intrinsically multi-factor, meaning that it is capable of capturing accurately various aspects of the curve dynamics: parallel shifts, steepenings / flattenings, butterflies, etc.
- Its specification allows for a rich variety of volatility regimes.
- LMM is not Markovian and all valuations require Monte Carlo simulations.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

LMM and smile dynamics

- The classic LMM model has a severe drawback: while it is possible to calibrate it to match at the money option prices, it generally misprices out of the money options.
- The reason for this is its specification. While the market uses stochastic volatility models in order to price out of the money vanilla options, LMM is incompatible with such models.
- In order to remedy the problem, we describe a model that combines the key features of the LMM and SABR models.
- To this end, we assume that the instantaneous volatilities $C_j(t)$ of the forward rates L_j are of the form

$$C_j(t) = C_j(t, L_j(t), \sigma_j(t)), \tag{14}$$

with stochastic volatility parameters $\sigma_j(t)$.

- The SABR specification corresponds to the choice

$$C_j(t) = \sigma_j(t) L_j(t)^{\beta_j}, \tag{15}$$

with $\beta_j \leq 1$.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

LMM and smile dynamics

- Furthermore, we assume that, under the T_{k+1}-forward measure Q_k, the full dynamics of the forward is given by the stochastic system:

$$
\begin{aligned}
dL_k(t) &= C_k(t)\, dW_k(t), \\
d\sigma_k(t) &= D_k(t)\, dZ_k(t).
\end{aligned}
\tag{16}
$$

where the diffusion coefficient $D_k(t)$ of the process $\sigma_k(t)$ is a process of the form

$$
D_j(t) = D_j(t, L_j(t), \sigma_j(t)). \tag{17}
$$

- The SABR specification corresponds to the choice

$$
D_k(t) = \alpha_k(t)\, \sigma_k(t). \tag{18}
$$

- We assume that the instantaneous vol of vol $\alpha_k(t)$ is a *deterministic function* of t rather than a constant. This extra flexibility is added in order to make sure that the model can be calibrated to market data.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

LMM and smile dynamics

- From the general principles discussed above, we expect that the Q_k-dynamics of j-th LIBOR forward is given by

$$
\begin{aligned}
dL_j(t) &= \Delta_j^{(k)}(t)\, dt + C_j(t)\, dW_j(t), \\
d\sigma_j(t) &= \Gamma_j^{(k)}(t)\, dt + D_j(t)\, dZ_j(t),
\end{aligned}
\tag{19}
$$

- These equations are supplemented by the initial conditions:

$$
\begin{aligned}
L_j(0) &= L_{j0}, \\
\sigma_j(0) &= \sigma_{j0},
\end{aligned}
\tag{20}
$$

where L_{j0}'s and σ_{j0}'s are the currently observed values.

- We impose the following correlation structure among the Brownian motions:

$$
\begin{aligned}
dW_i(t)\, dZ_j(t) &= r_{ij}\, dt, \\
dZ_i(t)\, dZ_j(t) &= \eta_{ij} dt.
\end{aligned}
\tag{21}
$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

LMM and smile dynamics

- We note that a pair of SDEs for a fixed j describes the dynamics of a forward maturing at T_{j+1}. The parameter σ_{j0} is simply the SABR σ_0-parameter, while the correlation r_{jj} is the ρ-parameter. The SABR α-parameter α_j is given by

$$\alpha_j = \sqrt{\frac{1}{T_j} \int_0^{T_j} \alpha_j(t)^2 \, dt} \, . \tag{22}$$

- The model can thus be calibrated exactly to the cap / floor market.
- The matrix $\{r_{ij}\}$ is not symmetric. Its off-diagonal elements do not affect cap / floor valuation.
- The block correlation matrix

$$\Pi = \left[\begin{array}{cc} \rho & r \\ r^{\mathrm{T}} & \eta \end{array} \right] \tag{23}$$

is assumed to be strictly positive definite.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition continued

- We determine the drifts $\Delta_j^{(k)}(t)$ and $\Gamma_j^{(k)}(t)$ by requiring absence of arbitrage.
- The calculation in the case of $\Delta_j^{(k)}(t)$ is essentially identical to the derivation of the drift terms for the classic LMM.
- As a result, under Q_k,

$$dL_j(t) = C_j(t) \times \begin{cases} -\sum_{j+1 \leq i \leq k} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j < k, \\ dW_j(t), & \text{if } j = k, \\ \sum_{k+1 \leq i \leq j} \dfrac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t), & \text{if } j > k. \end{cases} \quad (24)$$

- Similarly, under the spot measure, the SABR-LMM dynamics reads:

$$dL_j(t) = C_j(t) \left(\sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t) \right). \quad (25)$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition continued

- Let us now compute the drift term $\Gamma_j^{(k)}(t)$ in the SDE for $\sigma_j(t)$.
- We proceed as in the case of calculating $\Delta_j^{(k)}(t)$. Assuming first that $j < k$ and using formula (7):

$$
\begin{aligned}
\Gamma_j(t)\, dt &= d\sigma_j(t)\, d\log \frac{P(t, T_{j+1})}{P(t, T_{k+1})} \\
&= -d\sigma_j(t)\, d\log \prod_{j+1 \leq i \leq k} \left(1 + \delta_i F_i(t)\right) \\
&= -\sum_{j+1 \leq i \leq k} d\sigma_j(t)\, \frac{\delta_i dF_i(t)}{1 + \delta_i F_i(t)} \\
&= -D_j(t) \sum_{j+1 \leq i \leq k} \frac{r_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt.
\end{aligned}
$$

- Similarly, for $j > k$, we find that

$$
\Gamma_j(t) = D_j(t) \sum_{k+1 \leq i \leq j} \frac{r_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, .
$$

A. Lesniewski   **Interest Rate Volatility**

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition continued

● This leads to the following stochastic dynamics. Under the measure Q_k,

$$
d\sigma_j(t) = D_j(t) \times
\begin{cases}
-\sum_{j+1 \le i \le k} \dfrac{r_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j < k, \\
dZ_j(t)\,, & \text{if } j = k, \\
\sum_{k+1 \le i \le j} \dfrac{r_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dZ_j(t)\,, & \text{if } j > k\,.
\end{cases}
\tag{26}
$$

● Similarly, under the spot measure,

$$
d\sigma_j(t) = D_j(t) \Big(\sum_{\gamma(t) \le i \le j} \frac{r_{ji}\delta_i D_i(t)}{1 + \delta_i F_i(t)}\, dt + dZ_j(t) \ \Big).
\tag{27}
$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition continued

- We now substitute the volatility specification (14) and (17). Under the measure Q_k, the dynamics of the full model reads:

$$
dL_j(t) = \sigma_j(t) L_j(t)^{\beta_j}
$$

$$
\times \begin{cases}
-\sum_{j+1 \leq i \leq k} \dfrac{\rho_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)} \, dt + dW_j(t), & \text{if } j < k, \\
dW_j(t), & \text{if } j = k, \\
\sum_{k+1 \leq i \leq j} \dfrac{\rho_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)} \, dt + dW_j(t), & \text{if } j > k,
\end{cases}
$$

$$
d\sigma_j(t) = \alpha_j(t) \sigma_j(t)
$$

$$
\times \begin{cases}
-\sum_{j+1 \leq i \leq k} \dfrac{r_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)} \, dt + dZ_j(t), & \text{if } j < k, \\
dZ_j(t), & \text{if } j = k, \\
\sum_{k+1 \leq i \leq j} \dfrac{r_{ji}\delta_i\sigma_i(t) L_i(t)^{\beta_i}}{1+\delta_i F_i(t)} \, dt + dZ_j(t), & \text{if } j > k.
\end{cases}
\tag{28}
$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

No arbitrage condition continued

● Similarly, under the spot measure Q_0, the dynamics is given by the stochastic system:

$$
\begin{aligned}
dL_j(t) &= \sigma_j(t) L_j(t)^{\beta_j} \Big(\sum_{\gamma(t) \leq i \leq j} \frac{\rho_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dW_j(t) \Big), \\
d\sigma_j(t) &= \alpha_j(t) \sigma_j(t) \Big(\sum_{\gamma(t) \leq i \leq j} \frac{r_{ji} \delta_i \sigma_i(t) L_i(t)^{\beta_i}}{1 + \delta_i F_i(t)} dt + dZ_j(t) \Big).
\end{aligned}
\tag{29}
$$

● The stochastic dynamics given by the equations above defines the SABR-LMM model. It was proposed in [3].

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Other specifications

- Other SABR-style extensions of LMM have also been developed. The model discussed in [4] assumes one common factor driving stochastic volatility for all LIBOR forwards.

- Specifically, the Q_k-dynamics is assumed to be of the form:

$$\begin{aligned} dL_j\left(t\right) &= \Delta_j^k\left(t\right)dt + \sigma_j v\left(t\right)L_j\left(t\right)^\beta dW_j\left(t\right), \\ dv\left(t\right) &= \alpha v\left(t\right)dZ\left(t\right), \end{aligned} \tag{30}$$

with

$$dW_j\left(t\right)dZ\left(t\right) = r_j dt. \tag{31}$$

- The coefficients σ_j and α are deterministic constants.

- The no-arbitrage condition determines the form of the drift terms in (30); they are of the form similar to (9) - (10).

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Calibratable parameters

- In order to fully calibrate the SABR-LMM, the following parameters have to be determined:
    - (i) Initial values of the LIBOR forwards L_{j0}.
    - (ii) LIBOR/OIS basis curve.
    - (iii) Initial values of the volatility parameters σ_{j0}.
    - (iv) Instantaneous vol of vol curves $\alpha_j(t)$.
    - (v) CEV exponents β_j.
    - (vi) Correlation matrix Π.
- As already pointed out, calibration of items (i) and (ii) is straightforward, as the relevant numbers can be directly read off the LIBOR / OIS multi-curve.
- Calibration of the remaining parameters on the list requires cross-sectional fitting to the prices of swaptions and caps / floors.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Choosing the β's

- SABR-LMM specifies the values of the CEV exponents β_j for each LIBOR forward L_j but it does not explicitly refer to the CEV exponents β_{mn} for the benchmark forward swap rates S_{mn}.
- The latter are internally implied by the model and should be tied to the values implied by the vanilla swaption markets.
- There is no simple relation between β_j's and β_{mn}'s. An approximation which works well in practice is given by the following formulas:

$$\begin{aligned}
\beta_{mn} &= \sum_{m \leq j \leq n-1} a_{mn,j}\beta_j + b_{mn} \\
&\triangleq a_{mn}^{\mathrm{T}}\beta + b_{mn},
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
a_{mn,k} &= \frac{2 \log L_{k0}}{(n-m)^2} \sum_{m \leq j \leq n-1} \frac{1}{\log L_{j0} + \log L_{k0}} \,, \\
b_{mn} &= \frac{1}{(n-m)^2} \sum_{m \leq j,k \leq n-1} \frac{\log \rho_{jk}}{\log L_{j0} + \log L_{k0}} \,.
\end{aligned} \tag{33}$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

Choosing the β's

● Note that

$$\sum_{m \leq j \leq n-1} a_{mn,j} = 1. \tag{34}$$

● Consequently, the CEV exponent of a swaption is a weighted average of the CEV powers of the spanning forwards plus a convexity correction. Under a perfectly flat forward curve we would have

$$a_{mn,j} = 1/(n-m), \tag{35}$$

for all j.

● The convexity correction b is rather small. On a typical market snapshot it is of the order of magnitude 10^{-3}, and thus, for all practical purposes, it can be assumed zero.

● For reasons of practicality, we should parameterize the the curve β_j, by a small numver of parameters.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

SABR-LMM volatility matrix

- Volatility calibration of SABR-LMM requires setting the values of:
    - the volatility parameters σ_{j0},
    - the deterministic instantaneous vol of vol functions $\alpha_j(t)$.
- Parameters σ_{j0} are simply the σ-parameters in the SABR model for the caplet on the LIBOR forward L_j.
- For the vol of vol functions, we choose

$$\alpha_j(t) = h(T_j - t), \tag{36}$$

where $h(t)$ is the hump function discussed in the Presentation I. It is linked to the α-parameter α_j for the caplet on L_j via

$$\alpha_j = \sqrt{\frac{1}{T_j} \int_0^{T_j} \alpha_j(t)^2 \, dt} . \tag{37}$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

SABR-LMM volatility matrix

- In order to help the intuition, we represent the instantaneous stochastic volatility as a lower triangular matrix:

	$t \in [T_0, T_1)$	$t \in [T_1, T_2)$	$\ldots$	$t \in [T_{N-1}, T_N)$
$\sigma_0(t)$	0	0	$\ldots$	0
$\sigma_1(t)$	$\sigma_{1,0}$	0	$\ldots$	0
$\sigma_2(t)$	$\sigma_{2,0}$	$\sigma_{2,1}$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma_{N-1}(t)$	$\sigma_{N-1,0}$	$\sigma_{N-1,1}$	$\ldots$	0

Table: 1. SABR-LMM volatility matrix

- The nonzero entries in Table 1 are given by

$$\sigma_{i,j}(t) = \sigma_i(t), \text{ for } T_j \leq t < T_{j+1}. \tag{38}$$

- We refer to the lower triangular matrix in Table 1, SABR-LMM's internal representation of volatility, as the *SABR-LMM volatility matrix*. We will denote it with the symbol $\mathcal{V}$.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

SABR-LMM volatility matrix

- This representation allows us to map the domain of the SABR-LMM volatility matrix which affects a given instrument.
- For example, the domain affecting a 6M by 5Y cap is marked with ● in the table below:

	0	1	2	. . .	17	18	19	. . .	$N-1$
0	0	0	0	. . .	0	0	0	. . .	0
1	○	0	0	. . .	0	0	0	. . .	0
2	●	●	0	. . .	0	0	0	. . .	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	●	●	●	. . .	0	0	0	. . .	0
18	●	●	●	. . .	●	0	0	. . .	0
19	●	●	●	. . .	●	●	0	. . .	0
20	○	○	○	. . .	○	○	○	. . .	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$N-1$	○	○	○	. . .	○	○	○	. . .	0

Table: 2. The domain of vol sensitivity of a cap

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

SABR-LMM volatility matrix

- On the othe hand, a related instrument, 6M into 4.5Y swaption has the following vol sensitivity domain on the SABR-LMM vol matrix:

	0	1	2	...	17	18	19	...	$N-1$
0	0	0	0	...	0	0	0	...	0
1	○	0	0	...	0	0	0	...	0
2	●	●	0	...	0	0	0	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
17	●	●	○	...	0	0	0	...	0
18	●	●	○	...	○	0	0	...	0
19	●	●	○	...	○	○	0	...	0
20	○	○	○	...	○	○	○	...	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$N-1$	○	○	○	...	○	○	○	...	0

Table: 3. The domain of vol sensitivity of a swaption

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

The correlation matrix

- Simultaneous calibration to the cap / floor and swaption markets in a stable and consistent manner requires calibrating the high dimensional correlation matrix Π. The dimensionality of Π is $O(N^2)$, far exceeding the number of instruments available for calibration.
- A small subset of elements of Π can be directly calibrated to the cap / floor market. Namely, as already pointed out, the diagonal elements r_{jj} of the matrix r are the ρ-parameters of the SABR model for the caplet on L_j.
- The off-diagonal elements of r (which have no impact on cap / floor pricing) should be approximately constant along each row, monotone declining as the distance from the diagonal increases. A natural parametric choice is

$$r_{ij} = r_{ii} e^{-\nu |T_i - T_j|}, \tag{39}$$

where $\nu > 0$ is a calibratable parameter.

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

The correlation matrix

- There is no reliable way of calibrating the correlation matrices ρ and η to market implieds. In particular, using "no arbitrage" relations between caps and swaptions may lead to wrong results.

- Instead, we estimate ρ and η from the historical data. A relatively long observation window (at least 6 months) should be used.

- To assure the stability of the calibrated parameters, it is common to model the correlation parameters in a parametric form. A convenient and intuitive parameterized form of ρ_{ij} is given by the formula:

$$\rho_{ij} = \overline{\varrho}_{\min(i,j)} + \left(1 - \overline{\varrho}_{\min(i,j)}\right) \exp(-\lambda_{\min(i,j)}|T_i - T_j|), \tag{40}$$

where

$$\overline{\varrho}_k = \varrho \tanh(\mu T_k), \tag{41}$$

and

$$\lambda_k = \lambda T_k^{-\kappa}. \tag{42}$$

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

The correlation matrix

- The meaning of the parameters is as follows:
    - (i) ϱ is the asymptotic level of correlations,
    - (ii) μ is a measure of speed at which ϱ is approached,
    - (iii) λ is a the decay rate of correlations,
    - (iv) κ is an asymmetry parameter.
- Intuitively, positive κ means that two consecutive forwards with short maturities are less correlated than two such forwards with long maturities.
- The parameters in this formula are fixed by fitting to the historical data.
- A word of caution: this parameterization produces a matrix that is only approximately positive definite[2].
- The matrix η is parameterized in a similar fashion.

---

[2] typically it has a few tiny negative eigenvalues

The LMM methodology
Dynamics of the SABR-LMM model
Covariance structure of SABR-LMM

References

📄 Brigo, D., and Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer Verlag (2006).

📄 Crispoldi, C., Wigger, G., and Larkin, P.: *SABR and SABR LIBOR Market Models in Practice*, Palgrave MacMillan (2015).

📄 Hagan, P., and Lesniewski, A.: LIBOR market model with SABR style stochastic volatility, working paper (2006).

📄 Mercurio, F., and Morini, M.: Joining the SABR and Libor models together, *Risk*, 2009.

📄 Rebonato, R., McKay, K., and White, R.: *The SABR / LIBOR Market Model: Pricing, Calibration and Hedging for Complex Interest-Rate Derivatives*, Wiley (2009).