Department of Physics
Umeå University

June 13, 2019

Master's Thesis

Implementing and testing possible hedging strategies to minimise value fluctuations in a defaulted portfolio

Gabriel Nilsson (gabriel.nilsson@hotmail.se)

June 13, 2019

Master's thesis in Engineering Physics
Supervisor: Johan Gunnars, Anton Eriksson
Examiner: Markus Ådahl

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Acknowledgements

I would like to extend a big thank you to Cinnober for giving me the opportunity to conduct my master's thesis with them. This goes out to all workers at Cinnober who has been very welcoming and helpful during my time at the office. Furthermore, I would like to thank my supervisor at Cinnober, Johan Gunnars, for his guidance and help during this thesis. I would also like to thank my internal supervisor, Anton Eriksson, for his help and feedback when writing the report, and my examiner Markus Ådahl at Umeå University for his support and discussions. Finally, I would like to express my gratitude to my family and friends who has supported me through this thesis, and throughout all the years at Umeå University.

Gabriel Nilsson (gabriel.nilsson@hotmail.se)

Abstract

A Central Counterparty (CCP) handles clearing between its members and can mutualise and reduce the counterparty and credit risk in a network. In the case of a clearing member defaulting on its obligations, the defaulted portfolio will be taken over by the CCP, which will attempt to close out the positions as quickly as possible. It is vital that the CCP minimises the losses they may suffer during the period between default and close out, the so called holding period.

This thesis investigates and tests several potential hedging strategies to minimise value fluctuations during the holding period. These include neutralising the exposure to different risk factors, as well as finding the ideal hedging position using principal component analysis. The defaulted portfolio can contain different instruments, such as options, interest rate swaps and bonds, which requires different approaches to neutralise exposure. To determine the performance of the different strategies, backtesting was performed on historical data from the years 2001 to 2013, and the results were analysed in order to determine the effectiveness and potential costs of the hedging.

The results show that significant reduction in value fluctuations can be achieved by employing these strategies, while not exceeding an affordable level of cost. Based on the findings, a function was created in Java that can recommend optimal hedging positions given a defaulted portfolio of any composition.

Sammanfattning

En Central Motpart (CCP) hanterar clearing mellan sina medlemmar och can mutualisera och reducera motpartsrisken i ett nätverk. Om en clearingmedlem inte kan uppfylla sina åtaganden, kommer medlemmens portfölj att övertas av CCP:n, vars mål då är att stänga positionerna så fort som möjligt. Det är vitalt att CCP:n minimerar förlusterna de kan drabbas av under tidsperioden mellan default och tills positionerna stängts, den så kallade holdingperioden.

Den här uppsatsen undersöker och testar flertalet möjliga hedging-strategier för att minimera värdefluktuationerna under denna period. Dessa inkluderar såväl att neutralisera exponeringen till flertalet riskfaktorer, som att med hjälp av principalkomponentanalys hitta de ideala hedging-positionerna. Den övertagna portföljen kan innehålla olika typer av instrument, som exempelvis optioner, ränteswappar och obligationer, vilket kräver olika strategier för att neutrlisera riskerna. För att testa hur strategierna presterar så testades de på historisk över åren 2001 till 2013, och resultaten analyserades för att bedöma både effektiviteten och den möjliga kostnaden av hedgningen.

Resultaten visar på att betydlig reduktion i värdefluktuationer kan uppnås utan att överskrida en rimlig kostnadsnivå. Baserat på resultaten skapades en funktion i Java som kan rekommendera optimala hedgingpositioner, givet en portfölj med någon struktur.

Glossary

At-the-money (ATM):	An instrument which is ATM is at the point of profitability.
CCP:	Central Counterparty
Coupon:	An intermediate payment during the lifespan of a bond.
Delta (Δ):	Sensitivity of a derivative to changes in the underlying price.
Derivative:	Contract that derives its value an underlying variable.
Gamma (Γ):	Sensitivity of a derivative to changes in the delta.
Greeks:	Sensitivities of a derivative.
Hedging:	Removal of risk.
Implied volatility:	The volatility of an option price implied by the market.
In-the-money (ITM):	An instrument which is ITM is currently profitable.
IRS:	Interest Rate Swap
LIBOR:	London Interbank Offered Rate.
Out-of-the-money (OTM):	An instrument which is OTM is currently not profitable.
PCA:	Principal component analysis.
Rho (ρ):	Sensitivity of a derivative to changes in the risk free rate.
Risk free rate:	Interest rate used in pricing that is considered free of risk.
Vega (ν):	Sensitivity of a derivative to changes in the implied volatility.
VIX:	Volatility index.

Contents

1 Introduction

This thesis has been conducted at Cinnober Financial Technology, at its office located in Umeå. Cinnober, now a part of Nasdaq, develops systems for trading, clearing and risk management for financial markets. Founded in 1998, the company has become a global actor with customers all over the world. Example of Cinnober customers include Japan Exchange Group (JPX) and London Metal Exchange (LME).

1.1 Objective

The main goal of this thesis is to investigate if it is viable to enter into hedging strategies to reduce the risk of value losses associated with a defaulted portfolio. In the case of a clearing member default, the clearing house will take over the member portfolio, with the goal to close out the open positions. An alternative to only focusing on closing out the positions could be to invest in hedging positions aimed at minimising value fluctuations during the period between default and close out. The defaulted portfolio can contain several different types of financial derivatives.

Given a portfolio, what reasonable hedging alternatives are there, and how do they perform? The problem to solve is to choose appropriate hedging strategies, implement them and test their effectiveness, to be able to determine what, if any, method can be employed to effectively reduce the risk of value fluctuations, while keeping potential transaction costs low.

Based on the findings, a function that can recommend viable hedging positions should also be created. This function will, given a defaulted portfolio, determine the optimal hedging positions, and provide these as a suggestion.

1.2 Limitations

There are numerous possible financial derivatives, with many variations for each kind. To limit the scope of the study, the number of possible derivatives in the portfolio was limited. Additionally, due to the unavailability of historical data for some parameters used, appropriate proxies had to be used when necessary. The data used for testing is gathered from the period July 2001 to February 2013.

2 Background

2.1 Central Counterparty

A *Central Counterparty*, commonly referred to as a CCP or a *Clearing House*, is a financial institution that serves as an intermediary between two trade participants. These organisations handle the clearing and processing of transactions between different parties and help reduce the counterparty risk, which is the risk that the opposite party in the trade will not be able to fulfill its obligations. This is done by ensuring that a trade is completed, even if one party of the trade were to fail, by serving as the buyer for the selling party, and as the seller for the buying party. This in itself can be desirable, for example if a party would prefer to be anonymous in the trade. But as the number of members in the clearing house increases, the counterparty risk can be reduced quite significantly. This is done do by simplifying the outstanding obligations between its members, by calculating the net exposure for each member [22].

This is done by calculating the net cashflows of each clearing member and reducing the total number of transactions by only requiring these new payments to be made, which reduces the exposure to credit risk from the different parties involved. For example, given five parties A, B, C, D and E, each with a certain number of commitments, the total web of transactions can be illustrated as in Figure 1.

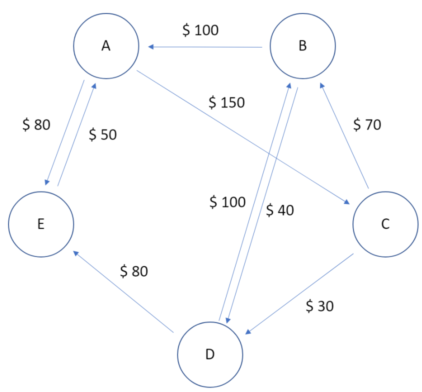


Figure 1 – Transaction commitments between different parties without central clearing.

The total amount of cash active in this network is $700. If we now introduce a central counterparty to this network, the situation can be simplified. The resulting, netted, network looks like Figure 2.

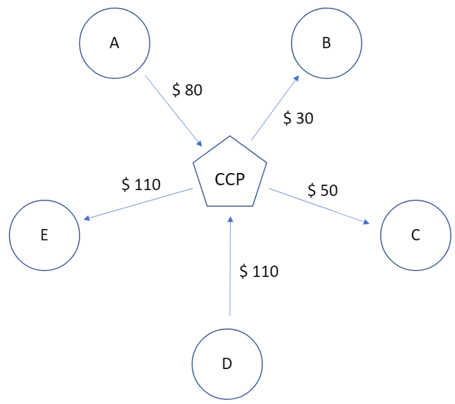


Figure 2 – Net transaction commitments between different parties with central clearing.

All transactions are now cleared through the CCP. This network has cash transactions of only $380, which is a significant reduction compared to the situation without central clearing.

In addition to reducing counterparty risk, the clearing house can also mutualise the credit risk between its members. This means that each member contributes to reduce the credit risk in the network by funds of their own. If a member were to default on a transaction, these funds could be used to maintain the stability for the clearing house and thus avoid potential chain reactions for other members.

Clearing houses has existed in its current form since the late 19th century, but throughout the 20th century, clearing houses were not seen as big institutions in the financial sector. It was not until the 2008 financial crisis, with the collapse of the Lehman Brothers firm, that clearing houses landed in the public spotlight. This happened because, as billions of dollars worth of financial instruments were at risk of being lost, many clearing houses managed to continue operations amidst the chaotic period, ensuring that their members could continue to operate without additional costs [20].

Since then, clearing houses has become a key part of the financial market. As a central partner in a vast network of traders, a clearing house has a lot of responsibility and there are many and strict regulations in place to ensure the stability of the market. For example, any clearing house wishing to establish their services in the EU, is required to apply for clearance under EMIR (European Market Infrastructure Regulation), to ensure market transparency and unnecessary risks [6].

2.2 Default Contingency

Every CCP has a contingency plan in case one or more of its members would be declared in default. The first step is that whenever a member of the CCP initialise a transaction, they have to post a collateral, also known as initial margin. This is a security offered to the CCP, to reduce the credit risk accrued from entering into the trade. If a member is declared default, their initial margin will be used in order to cover any losses resulting from the default. If this is not enough, the CCP has additional steps it can take in order to recoup the losses. The steps involved are usually referred to as the *Default Waterfall*, as it contains several layers of actions the CCP can take until the situation has been resolved. While the exact setup of the waterfall can vary between different CCPs, a general example of a waterfall is presented in Figure 3.

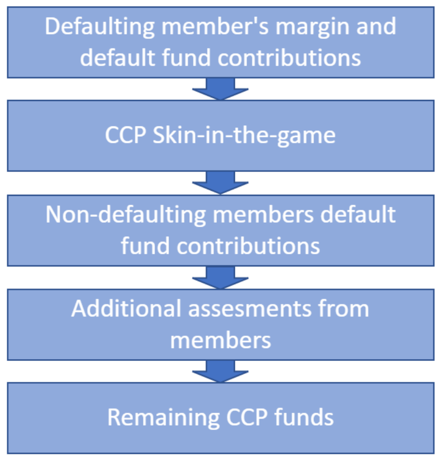


Figure 3 – Example of a typical default management waterfall.

If the initial margin is not enough to cover the losses from a member default, there is also the default fund, which is a combined fund to which all CCP members contribute a certain predetermined amount when joining the CCP, in order to mutualise the credit risk. The standard here is to at first prioritise the contributions to the default fund made by the defaulting member, and try to avoid using other CCP members' contributions. After this, the CCP usually contributes an amount of funds of their own, known as their skin-in-the-game.

After this, the remainder of the default fund is used, if the losses has still not been covered up. At this point, the other CCP members actively contribute to the effort, which is part of the mutualised risk. If the situation has still not been solved at this level, the CCP can request additional support from other members, but reaching this level is a very rare event. Finally, the last resort is to utilise the remaining resources of the CCP in order to prevent the failure and default of the CCP itself.

The very first thing that happens in a default situation however, is that the CCP takes possession of the defaulted members portfolio. While the strategy here differs between different CCPs, the immediate goal is to close out the portfolio in order to minimise losses. This is commonly done by selling the portfolio at an auction to the non-defaulting members [22].

2.3 Close Out Strategies

Every Clearing House has a predefined close out strategy they use in case of a member default. While they do differ from each other, most CCPs use some variation of the common default waterfall.

2.3.1 Nasdaq Clearing

Nasdaq Clearing has a defined default strategy to follow in the case of a member default. This strategy differs in the details depending on the market the default occurred in, with separate procedures for equity, commodities and fixed income markets, but the overall default structure is similar. The first part is to evaluate the positions and possible losses of the defaulted portfolio. Nasdaq Clearing acquires the collateral posted by the defaulting member and can run additional margin and value checks to make sure the data is up to date with the default process. The current market prices are also retrieved to make sure the next moves are enacted on up to date data [17][18][19].

After this has been done, the defaulted portfolio is inspected to determine some key factors, which include which positions bring the largest risk of value loss, the size of the positions, the price and volatility sensitivities and time to delivery. If possible, the portfolio could be divided into different parts depending on these properties to ease the next steps.

Once an adequate overview of the positions and the relevant risk measures has been obtained, the standard approach is to hedge the portfolio, in order to be able to close it without surpassing the initial margin posted by the defaulted member. There are different proposed hedging strategies depending on the type of asset. For equity derivatives, the first focus is to remove delta risk, and if deemed necessary, other risk factors such as vega. The best case scenario is to be able to completely remove all equity risk in the open portfolio. This is often possible to achieve, but when this approach fails, the solution is usually to try to minimise the sensitivities and offer the risk reduced portfolio on auction [17].

For fixed income instruments, such as swaps, the first step is to determine the sensitivity associated with a one basis point upward shift in the rate curve. With this exposure known, the following actions aim to hedge this sensitivity to zero, for example using offsetting swaps to counteract the exposure [19].

On the commodity market, the proposed hedging strategies can be divided into strategies for futures and strategies for options. For any commodity futures in the portfolio, suggested hedging strategies include moving to form a risk neutral position, buying highly liquid instruments with a high positive correlation with the instruments in the portfolio, or using options to delta hedge the portfolio. Alternatively, for options, some proposed strategies include entering long or short positions in options whose day of expiry or strike price is similar, or delta hedging through either the underlying assets or different options [18].

If these actions allow for the close out or hedging of the full portfolio, the default process stops there. But if this is not enough, the portfolio can be auctioned off, either in full to a single customer or in parts. There can be close out agreements signed between the centerparty and members, who in the case of an auction are offered the defaulted portfolio first.

2.3.2 LCH

LCH is a British clearing house active on both *Over-The-Counter* (OTC) markets as well as exchanges. While their default management process differs slightly depending on the contract type, the general method is to start by attempting to port the clients of the defaulted member to other, non-defaulted members to allow for continued service. After this, the defaulted positions are to be hedged in order to reduce the risk of further losses during the holding period.

The hedging is done in steps. First, the portfolio may be split into two or more sub-portfolios to facilitate more efficient handling, if deemed appropriate. This splitting can be done by contract type, currency or other factors determined to be important. The new portfolios will then be hedged by entering offsetting contracts. For example, LCH SwapClear, the subdivision clearing interest rate swaps, may enter into offsetting swaps in order to hedge the exposures from a defaulted members swap portfolio. The structure of the hedges are determined by the counterparty's default management group.

After the positions have been hedged, the portfolios are auctioned off to non-defaulting clearing members. During the 2008 default of Lehman Brothers, LCH successfully managed their defaulted portfolio, with a notional amount upwards of $9 trillion, without needing to delve further into their default waterfall, showing the stability of their strategy [14].

2.3.3 CME Group

CME Group, which is a combination of the *Chicago Mercantile Exchange* and the *Chicago Board of Trade*, is an American exchange that offers a wide range of products for many types of assets [5]. The default procedure in use at the CME Group follows the standard default waterfall, and includes the right to take possession of the defaulted member's portfolio. They can then act accordingly to their guidelines with this portfolio, possible actions including hedging and/or liquidating the positions within or transfer the positions to a non defaulted member. The details on how a potential hedging would be performed was not presented, but the close out strategy does include it as a possible contingency plan [4].

2.4 Historical Defaults

A CCP member default is not a common occurrence but it has happened. In September 2018, the Norwegian trader Einar Aas defaulted when he could not provide the funds for a margin call. Einar had made a bet that the German and Nordic power markets would not move away from each other, but this is exactly what happened, and at a significantly larger magnitude than normal. This resulted in additional margin calls from the CCP Nasdaq Clearing, which he was not able to cover. Einar was declared default and the CCP acted in accordance with its default waterfall to cover the losses. These turned out to be large, forcing several layers of the waterfall to be used before they could be fully covered [16]. This case is noticeable not only because of the recent occurrence but also because of the need to use additional default responses after the initial margin.

Another (in)famous event where a clearing member was declared default happened during the 2008 finance crisis, and the bankruptcy of Lehman Brothers. Lehman had markets where they operated as a clearing member, markets which became susceptible to the effects of them defaulting. Here however, the actions of the various CCPs involved are usually considered to have been very important in preventing losses, as they handled their outstanding trades all across the globe, with minimal losses accrued [3].

Depending on the size of the defaulting member, there can be adverse effects on the market they operate, or in extreme cases, the market as a whole. This is essentially what occurred in 2008, and the instability was very noticeable. The volatility, i.e the changes in price, rose to very high levels, meaning the market did experience and were expecting large price movements for financial instruments, such as options, and many of the largest stock indices suffered huge drops. The Chicago Board Options Exchange (CBOE) keeps the Volatility Index (VIX) as an indicator of the general volatility. The historical values of the index are shown together with the performance of the NASDAQ-100 stock index in Figure 4. Data was obtained from the Federal Reserve Bank of St. Louis [7].

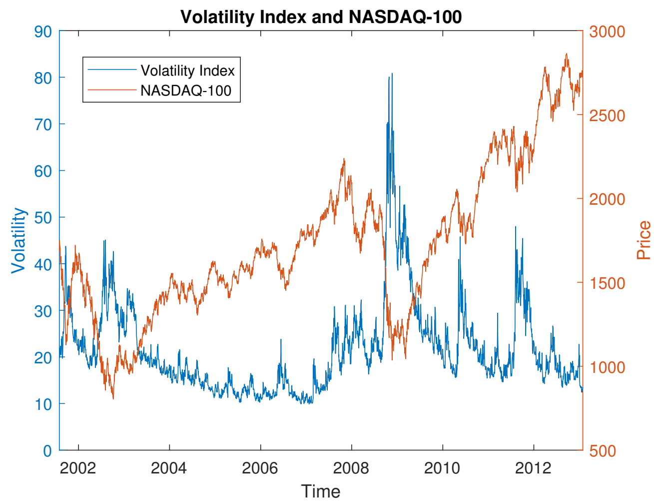


Figure 4 – Comparison between the volatility index and the NASDAQ-100 stock index.

One can clearly see how the volatility approximately quadrupled over a short time during late 2008, which is when the default of Lehman Brothers occurred. During the same period, the NASDAQ-100 fell by roughly 50%. It was not until 2010 that the index had recovered to pre-crisis levels. This period resulted in massive pressure on the CCPs that had to clear up the situation, and the state of the market did not make it easier to close out the positions in Lehman's portfolio. However, due to the pre-planned default management processes the situation was handled surprisingly well, given the circumstances, and large parts of Lehman's portfolio could be closed out within days of the default. For example, eight days post-default, LCH.Clearnet had managed to reduce the risk exposure by more than 90 %, and reported that the remaining exposure did not exceed the margin currently held by the CCP [20].

A clearing member default can have serious and lasting impact on a market. The CCP must be able to act quickly and decisively to minimise losses and prevent further instability in the clearing network. If the actions taken during the post-default period is insufficient to prevent the onset of additional losses, it may result in amplification of the effect, as other additional resources must be used from the default waterfall, and other clearing members may be affected with for example higher margin calls, resonating the effects further [22].

3 Theory

3.1 Financial Derivatives

While trading can take place directly on an asset such as a stock price or a commodity price, there exists numerous financial contracts that are more complicated in their design, and whose performance is based upon for example the price of an underlying asset or an interest rate. These kinds of contracts are known as *Derivatives*, and are very popular; in 2017, more than 25 billion future and options contracts were traded across the globe [8]. Financial derivatives can be traded both on a standardised exchange, or OTC, where the exact specifications are not predefined. For this study, the testing was not limited to one type of traded contract. As such, a few different instruments from both exchange and OTC markets were considered.

3.1.1 European Option

An option is a contract that gives its buyer, or holder, the right, but not the obligation, to buy or sell a specified amount of the underlying asset at a future time. When the contract is created, the buy or sell price, called the strike price, is defined. This price is invariant to change in the price of the underlying asset. A European call option is a type of option where the holder can only choose to exercise the option at the specified date of expiry. Consequently, the possible payoff is the difference between the strike price and the actual price of the underlying asset at the time of expiry.

Depending on if the option allows the holder to buy or sell, the type of the option is known as a *Call* or *Put* option respectively. For a European call option with a certain strike price, the holder will make a profit if the price of the underlying stock is higher than the strike price, as this allows the holder to buy the underlying asset for a lower price than the market value. The holder can then immediately sell the assets at the current market price and make a profit. If instead the underlying stock price is lower than the strike price at the time of expiry, the holder chooses not to exercise the option, which causes it to expire unused. The payoff for a European put option is reversed, such that the holder makes a profit if the underlying stock price is lower than the strike price, following similar reasoning.

In addition to buying a call option or a put option, commonly shortened simply to calls and puts, there is also the option to sell options. This is commonly referred to as having a *short position*, as opposed to the *long position* when buying an instrument. The payoff when holding a short position is the cost the holder pay when entering into the contract, assuming the underlying asset price does not change in such a way as to make the option profitable for the holder.

More formally, the payoffs for long European call and put options are defined as

$$\text{Payoff} = \max\{S_T - K, 0\}, \tag{1}$$

$$\text{Payoff} = \max\{K - S_T, 0\}, \tag{2}$$

respectively, where S_T is the price of the underlying asset at time of expiry T and K is the strike price. Since the holder will not exercise the option if the asset price is lower than the strike price, the minimum payoff for options is zero. With the same variables as Equation 1. The payoff for a short position in a call or put option is directly related to the long position payoffs, and is defined as

$$\text{Payoff} = \min\{S_T - K, 0\}, \tag{3}$$
$$\text{Payoff} = \min\{K - S_T, 0\}. \tag{4}$$

It is important to distinguish between payoff and profit; the payoff is the potential gain when the option is exercised or not, while the profit refers to the total net result, which includes the cost of buying the long position or profit from selling the short position. For example, the largest possible payoff for a short position in a European option is zero, but the largest profit from the option is not limited to zero [11].

3.1.2 Forward Contract

A forward contract gives the holder the obligation to, at a future point in time, buy or sell an asset for a predetermined price. Unlike an option, where the holder can choose not to exercise the option, the holder of a forward contract is forced to conduct the trade at the determined date. When the time of maturity arrives, the payoff of the forward contract is determined by the difference in the agreed upon price in the contract, and the actual market price. In a long position, a positive payoff is realised if the market price is larger than the contract price, as the holder can then immediately sell the asset for a profit. Conversely, a short position receives a positive payoff is the market price at expiry is lower than than the agreed upon price.

Formally, the payoff of a forward contract is similar to that of a European option, but because of the obligation to go through with the trade, it is more strict than that. The payoff of a forward contract in the long position can be written

$$\text{Payoff} = S_T - K, \tag{5}$$

where S_T is the current market price of the asset at the time of expiry, and K is the agreed upon price [11]. If one is short in a forward, the payoff is the opposite. Forward contracts are OTC derivatives, which means that they are very flexible in their setup. The contracts are not standardised, and can be customised to suit many different situations. Unlike an option, where you pay or receive the cost of the option when entering into a trade, a forward is settled at maturity, meaning no cash or goods changes hand until expiry. If forwards are used for speculation, it is also common to settle the transaction by only exchanging the net value of the contract at maturity.

3.1.3 Bonds

A *Bond* is a contract paying a predetermined amount of money at a future date. This predetermined amount is known as the bonds principal amount. A bond may also pay smaller amounts to its holder during the lifespan, which is known as coupons. These coupons are determined based on the principal amount of the bond, a known interest rate, and the number of coupons. In its simplest form, a bond has no coupons and the payment at the end is the only payment that is done throughout the life of the bond. Such a bond is known as a *Zero Coupon Bond*, as opposed to a *Coupon Bearing Bond* which pays coupons at a fixed interval during its lifespan.

The size of the coupon payments is determined by a rate, decided upon when the bond was sold. This rate can be a fixed rate, i.e the same rate is used for every payment, or it can be a floating rate, which is determined by some reference rate, for example the *London Interbank Offered Rate* or LIBOR [24]. LIBOR is the rate at which banks can obtain short term loans between each other [13].

The floating rate is usually quoted on an annual basis. As such, if the bond has more or fewer coupons per year than 1, the rate is scaled accordingly. For example, for a bond paying semiannual coupons, the annualised rate is divided in half for each coupon.

Because of the time value of money, a payment today is worth more than a payment of the same size in one year. This means the price of buying a bond is usually lower than the bonds principal amount, as this is the amount the owner will receive at a future point in time. The price of a bond is calculated by discounting the future cashflows to a present day value, and summed up to obtain a total present value of all the future cashflows.

3.1.4 Interest Rate Swaps

An *Interest Rate Swap* is an agreement between two parties to exchange interest payments on a notional principal amount. These interest rate swaps are OTC contracts and the most basic type of swap is known as a vanilla IRS. Here, two parties have interest payments to make on a certain notional amount, but they have different kinds of interest rates. One party has a fixed interest rate paid at regular intervals, for example annually, while the other party has a non-fixed interest rate that can change over time. This is referred to as a floating interest rate. A common interest rate to use as a floating rate is again LIBOR [24].

If the companies agree to enter an IRS, they are both hoping that the floating rate behaves in such a way to make it more profitable to pay their specific rate as opposed to the other. For example, the party that originally had a fixed rate is hoping that by instead paying a floating rate, they would have to pay less interest. This would be achieved by the floating rate changing to be lower than the fixed rate. Conversely, the other party is hoping that the floating rate increases to higher levels than the fixed rate. This is a zero sum contract, meaning that whatever profit one party makes on the swap, the other loses.

In a vanilla swap, the parties usually exchange payments twice per year, although generally only the net amounts are exchanged. Additionally, the upcoming payments are known in advance. The fixed rate is known throughout the entire swap, as it is fixed, and the floating rate used at an exchange date is the interest rate from six months ago. This means that at the time the swap is agreed upon, both parties immediately know the value of the first payment. After the specified time the swap is active has expired, the notional amounts are not exchanged and the total sum of the net payments determine if a party made a profit or a loss from the swap.

3.2 Valuing derivatives

To be able to conduct any form of trade, the value of the different sides must of course be determined, to ensure a fair deal for all parties involved. In order to allow analysis on the derivatives used in this study, the price and value of these contracts must be possible to determine. The availability of exact pricing models vary from instrument to instrument, and the value can in some cases be hard to determine directly. Additionally, the initial price and value of a contract will almost certainly not be the same as the value of a contract at some point during its lifespan. To be able to know the value at an intermediate time of its lifespan, the *Mark-to-Market* (MtM) value of a contract must be determinable. Models for pricing and valuing the contracts used in this study is presented below.

3.2.1 Valuing Forwards

The key element to correctly pricing a forward contract is to remove the possibility of arbitrage. Arbitrage is a guaranteed, risk-free profit, and if an instrument is priced incorrectly, these opportunities may present themselves [1]. If we assume the forward contract is written on an asset that yields no income to its holder, such as a stock that does not pay dividends, the pricing formula for this contract is

$$F(0,T) = S_0 e^{rT}, \tag{6}$$

where $F(0,T)$ is the price of the forward entered at time 0 and maturing at time T, S_0 is the current price of the underlying asset and r is the risk-free interest rate [11]. To motivate why this is true, assume $F(0,T) > S_0 e^{rT}$. A trader borrows S_0 amount of cash, having an interest rate of r on the loan. With this amount, the trader buys one share of the underlying asset and short sells a forward contract expiring in T years, for $F(0,T)$. When the time of maturity arrives, the trader supplies the share in the contract and is paid $F(0,T)$. The total payment of the loan taken is $S_0 e^{rT}$, but we assumed this was less than the cost of the forward contract. Thus, the trader makes a risk-less profit of $F(0,T) - S_0 e^{rT} > 0$, which violates the no-arbitrage requirement.

Conversely, if we assume that $F(0,T) < S_0 e^{rT}$, a trader can take reverse the previous positions, and with the same reasoning, end up making a risk-less profit of $S_0 e^{rT} - F(0,T) > 0$. This implies that the only price that will not yield arbitrage opportunities, is if $F(0,T) = S_0 e^{rT}$ [11].

At the time of its inception, the forward price is usually set such that the forward contract's value is zero, but as time progresses, and the price of the underlying changes, the value of the contract will become either positive or negative for the buyer, and the opposite for the seller. Using similar arguments based on no arbitrage as above, the formula for valuing a forward contract is

$$V(t,T) = (F(t,T) - K)e^{-r(T-t)} = (S_t e^{r(T-t)} - K)e^{-r(T-t)} = S_t - Ke^{-r(T-t)}, \tag{7}$$

where K is the agreed upon forward price, $T-t$ is the time to maturity at current time $t \in 0 \leq t \leq T$, and we have used the definition of the forward price from Equation 6 [11].

3.2.2 Black-Scholes Model

In 1973, Fischer Black and Myron Scholes published their article *The Pricing of Options and Corporate Liabilities* in the *Journal of Political Economy*, which presented the *Black-Scholes Model* for option pricing [2]. This model is based upon the Black-Scholes Equation, a partial differential equation which looks like

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{8}$$

where V is the value of an option, S is the price of the underlying asset, t is the time, σ is the volatility of the price of the underlying asset and r is the risk-free interest rate. For the full derivation of the equation, see for example the original article [2] or [1][11][24]. The concept used when deriving the model is known as *delta hedging*. Delta, which is a measure of how sensitive a derivative is, is presented in more detail in Section 3.3.

One result from solving this equation is obtaining the pricing formulas for European call and put options. A derivation of the pricing formulas is presented in Appendix A.2. The resulting expression for a European call option is

$$C = S\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{9}$$

where Φ is the cumulative density function for a standard normal variable, see appendix A.1, T is the time to expiry for the option, S is the price of the underlying asset, and the terms d_1 and d_2 are

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{10}$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{11}$$

From this formula, the valuation of a European call option can be determined exactly. The price of a European put option can be determined using an identity known as Put-Call parity, which connects the prices of a call option and a put option[1]. The price of a European put option is

$$P = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1). \tag{12}$$

When the time to maturity approaches zero, the price of an option approaches the payoff, see Equations 1 and 2. To see this, consider that when T approaches zero, d_1 and d_2 becomes very large, which makes $\Phi(d_1)$ and $\Phi(d_2)$ become close to 1. For a call option, this results in a price of

$$C \approx S \cdot 1 - 1 \cdot Ke^{-r \cdot 0} = S - K. \tag{13}$$

If K is larger than S, the option will not be exercised, meaning the price is the same as Equation 1.

3.2.3 Valuing Bonds

A bond is essentially a contract guaranteeing a future cashflow. If the bond is coupon bearing, it yields a stream of cashflows up until its maturity, in addition to the final payment of the principal amount. The simplest bond to price is the zero coupon bond. It has a single cashflow of the principal amount P at the end of its maturity, which can be discounted to a present value using the zero coupon rate r_z [24]. The value of a zero coupon bond at present time t with an arbitrary future maturity date T is

$$Z_0(t, T) = Pe^{-r_z T}. \tag{14}$$

For a coupon bearing bond with a fixed coupon rate r_c and a fixed number of annual payments d and $t_N = T$, the value is

$$Z(t, T) = \frac{r_c P}{d} \sum_{i=1}^{N} e^{r_{z,i} t_i} + Pe^{-r_z T}. \tag{15}$$

However, to be able to use these equations, the zero coupon rate r_z must be determined [24]. This rate is not directly observable on the market, but there are ways to derive it. The method used to derive the zero rates in this study uses interest rate swap par rates, and is presented in Section 3.2.5.

3.2.4 Valuing Interest Rate Swaps

In a plain vanilla interest rate swap, we have a fixed rate leg, and a floating rate leg. While the floating rate can be essentially any reference rate, here we assume it is the USD LIBOR rate. The value of an interest rate swap is determined as the net difference between the value of the two legs, and depending on your position, you are either in the money or out of the money. So in order to determine the value of a swap, both of the legs need to valued. See Figure 5 for an illustration of cashflows for the floating rate payer. The straight arrows indicate fixed (known) payments while wavy arrows indicate floating (unknown) payments.

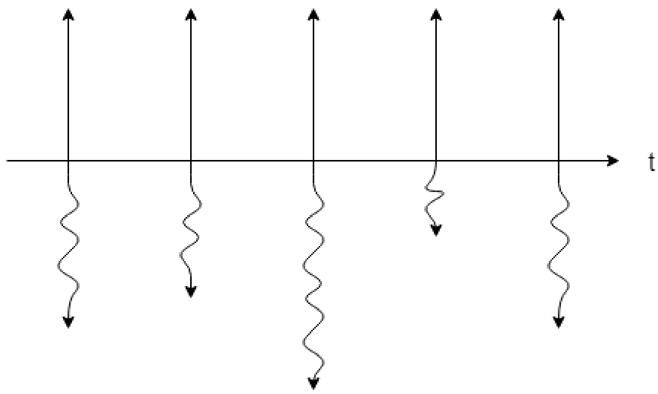


Figure 5 – Cashflows from a vanilla interest rate swap.

Valuing the fixed interest rate leg is similar to valuing a fixed rate bond. At each payment date, the fixed leg pays the fixed rate times the notional principal amount, adjusted with the relevant compounding. All payments are known immediately when the swap is initiated. To get the present value, the future payments must be discounted to today. All these payments can be seen as a sum of zero coupon bonds with increasing maturity up to the end date of the swap [24]. Assuming a notional principal value of P for simplicity, and d annual payments, the value of the fixed leg can be expressed using zero coupon bonds as

$$V_{fixed} = \frac{Pr_{fix}}{d} \sum_{i=1}^{N} e^{-r_z T}. \tag{16}$$

The floating leg of a swap can not be determined as directly, as the only known payment rate is the next. Every payment afterwards is unknown, since the rate has not been decided yet. But it turns out that this does not matter, and that we can still value the floating leg in terms of zero coupon bonds, similar to the fixed leg. Assume we are at time t_0 and have a floating leg payment at time T_i. For simplicity, assume the principal amount is 1, but these results are general and can be applied to a smaller or larger principal amount as well. Given the LIBOR rate r_i for period i, the payment occurring at T_i is $1 \cdot r_i \tau$, where τ is the time period factor $T_i - t_0$ [24].

Now by adding a positive and a negative cashflow of 1 at T_i, the value of the contract is not changed. However, a key observation here is that the LIBOR rate is the interest paid at a deposit for a certain time. Thus, a positive cashflow of $1 + r_i\tau$ at T_i, equals a cashflow of 1 today, shown in Figure 6.

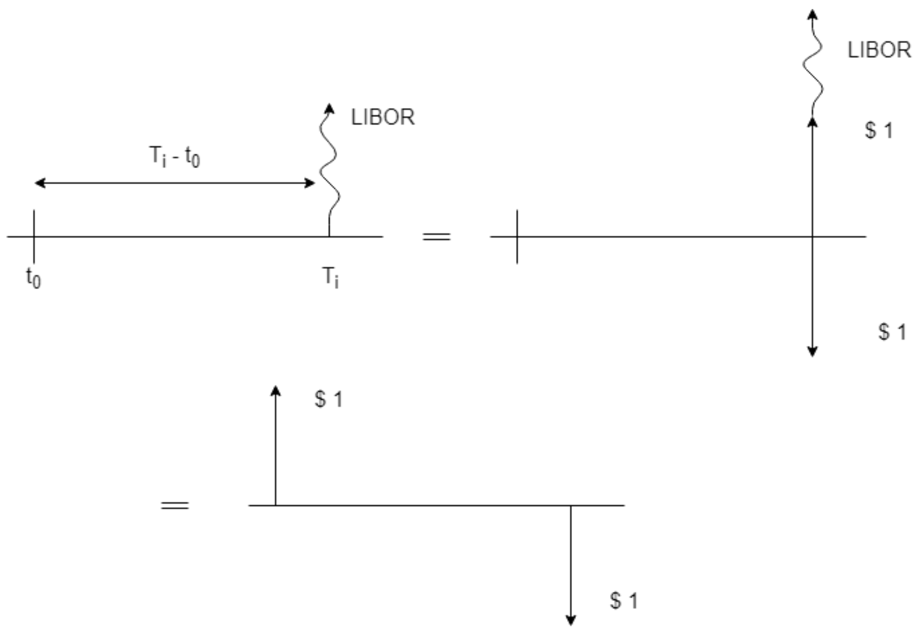


Figure 6 – Transforming a future floating leg payment.

By adding a second floating leg payment a second period τ after T_i, we can in the same way rewrite this as a positive cashflow of 1 at time T_i. This reasoning can be extended to N floating leg payments. From this, each cashflow during the life of the swap can be canceled out, except the first positive, and the last negative one. The immediate cashflow is equal to 1, as is its present value, while the final negative one can be seen as a single zero coupon bond payment maturing at T_N, see Figure 7.

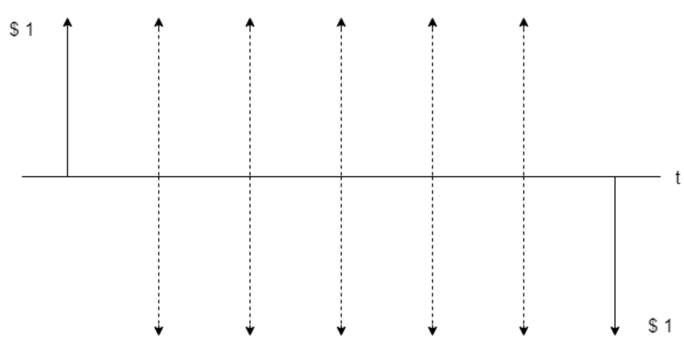


Figure 7 – Transforming all future floating cashflows.

Thus, the value of the floating leg of a swap can be written as

$$V_{floating} = 1 - e^{-r_z T}. \tag{17}$$

The total value of a swap contract with a notional principal amount of 1 for the party paying the floating leg is then

$$V_{swap} = V_{fixed} - V_{floating} = \frac{r_{fix}}{d} \sum_{i=1}^{N} e^{-r_z T} - 1 + e^{-r_z T}. \tag{18}$$

An interest rate swap on a notional principal amount P can then be expressed using zero coupon bonds from equation 14 as

$$V_{swap} = V_{fixed} - V_{floating} = \frac{r_{fix}}{d} \sum_{i=1}^{N} Z_0(t, T_i) - P + Z_0(t, T_N). \tag{19}$$

This expression holds for any maturity and any number of annual payments. When an interest rate swap is initiated, the value of the swap is usually set to zero, such that no party starts in the money or out of the money. If we set equation 19 equal to zero, and divide away the notional principal P, we can rewrite the expression to find the fixed interest rate r_{fix} that makes the swap zero valued. We get

$$r_{fix} = \frac{1 - Z_0(t, T_N)}{d^{-1} \sum_{i=1}^{N} Z_0(t, T_i)}. \tag{20}$$

This rate is known as the *swap rate*. Swap rates for many different maturities are available daily, as the swap market is very liquid [24].

3.2.5 Deriving the Zero Rates

To be able to price bonds, and by extension also interest rate swaps, the zero rate needs to be determined. These rates decide the return on an investment that has a payment only at the time of maturity. For example, investing 100 \$ over 10 years at a zero rate of 4 % yields an payment of $100e^{0.04 \cdot 10} \approx 149$ \$ at maturity. When pricing a bond, we instead know the payment at the end, and are interested in finding the fair price of the bond. To get this, the corresponding zero rates to the maturity must be used [11].

While zero rates are not directly observable, they can be deduced from available data on the market. One example of such data is swap rates, which is also the method used in this study. As historical swap rates for several different maturities are readily available online, one can use Equation 20 to derive the corresponding zero rates. This method is known as bootstrapping [24]. Starting with the lowest swap rate maturity T_1, we get

$$Z(t, T_1) = \frac{1}{1 + r_{fix}(T_1)d^{-1}}, \tag{21}$$

which is the first discount factor, given from Equation 14. A general formula to get the i:th zero rate can be expressed as

$$Z(t, T_i) = \frac{1 - r_{fix}(T_i)d^{-1} \sum_{j=1}^{i-1} Z(t, T_j)}{1 + r_{fix}(T_i)d^{-1}}. \tag{22}$$

From this expression, a zero rate curve can be derived for a desired number of maturities. For zero rates at a maturity were there are no direct swap rates, one can interpolate between existing swap rates to obtain an approximate zero rate. Additionally, in order to be able to price contracts at any time, not just at a specific maturity, the zero rates can be interpolated between the determined points, to obtain a continuous curve. An example of a series of bootstrapped zero rates can be seen in Figure 8.

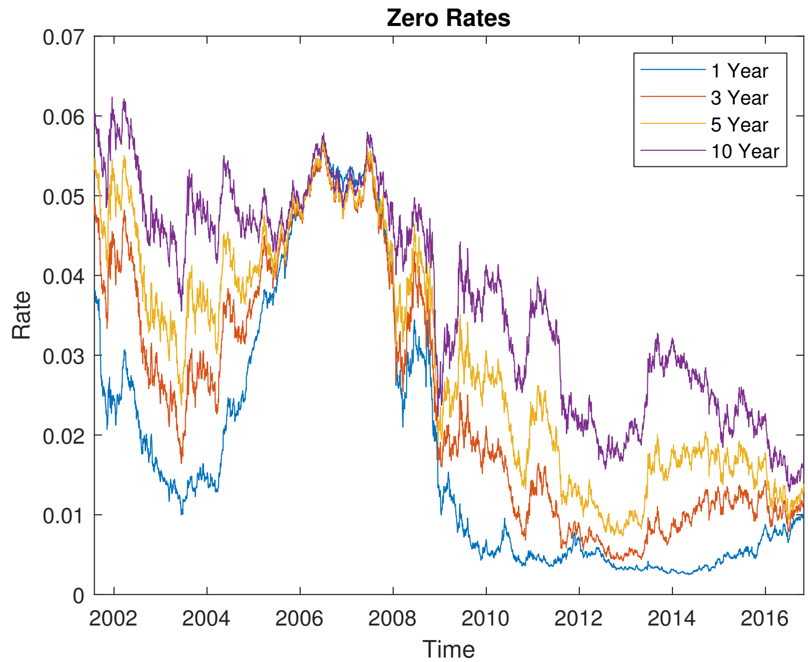


Figure 8 – Bootstrapped zero rates for four different maturities.

3.3 The Greeks

Since the market is always changing, prices rarely stay the same for long. There are many underlying parameters having an effect on prices that can fluctuate and change. To some extent, one might be able to predict movements in these parameters, but ultimately there is no certain way to know what will happen in the future, which is one of the most fundamental parts of market speculations. While no one can perfectly predict what will happen, market participants of course prefer to be as prepared as possible for future events. Analysing how sensitive a portfolio is to these potential future changes is fundamental part of portfolio management [1].

The most basic sensitivity, and arguably also the most important, is the susceptibility to changes in the price of the underlying asset. If the price of an arbitrary derivative is V and the underlying asset is S, this sensitivity is defined as

$$\frac{\partial V}{\partial S} = \Delta. \tag{23}$$

This sensitivity is commonly referred to as *delta*. Knowing the delta of an instrument, you know how its value will change if the underlying price changes.

Following the same idea as with delta, one can define the sensitivities depending on the time to maturity, the risk-free interest rate and the volatility of the underlying asset as

$$\frac{\partial V}{\partial T} = \Theta, \tag{24}$$

$$\frac{\partial V}{\partial r} = \rho, \tag{25}$$

$$\frac{\partial V}{\partial \sigma} = \nu. \tag{26}$$

These sensitivities are known as *theta*, *rho* and *vega* respectively and are, together with delta, four first order sensitivities [11]. Based on the symbols used, sensitivity measurements in finance has been collectively named *Greeks* (although note that vega is not actually a name for any Greek letter). Analysing the Greeks of ones portfolio is generally done continuously to be able to asses the risk one is exposed to at any given time.

There exists higher order sensitivities as well. The most commonly one is the second derivative of the price with respect to the price of the underlying asset, which is acquired by deriving delta a second time. From this we get

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \Gamma. \tag{27}$$

This sensitivity is known as *gamma*. Just as delta is a measurement of how the price of a derivative changes when the price of the underlying asset changes, gamma is a measurement of how quickly delta changes with respect to the price of the underlying asset [11].

There are several different approaches to determining the sensitivities of an instrument. It can be done numerically [9], but for some instruments, there are analytical expressions for the Greeks since they are derivatives of the price with respect to some underlying variable. If the price can be explicitly expressed, one can often calculate these derivatives to find the Greeks. The derivation of delta for a European call option is shown in Appendix A.3, and the other Greeks for European options can be derived similarly. The delta of a European Call and Put option is

$$\Delta_{call} = \Phi(d_1), \tag{28}$$

$$\Delta_{put} = \Phi(d_1) - 1. \tag{29}$$

The other Greeks presented in Section 3.3 are given by

$$\Theta_{call} = \frac{-S\phi(d_1)\sigma}{2\sqrt{T}} - rK\Phi(d_2)e^{-rT}, \tag{30}$$
$$\Theta_{put} = \frac{-S\phi(d_1)\sigma}{2\sqrt{T}} + rK\Phi(-d_2)e^{-rT}, \tag{31}$$

$$\rho_{call} = KT\Phi(d_2)e^{-rT}, \tag{32}$$
$$\rho_{put} = -KT\Phi(-d_2)e^{-rT}, \tag{33}$$

$$\nu = S\phi(d_1)\sqrt{T}, \tag{34}$$

$$\Gamma = \frac{\phi(d_1)}{S\sigma T}. \tag{35}$$

Vega and gamma are the same for both call and put options. Additionally, all these definitions assume the options are held in a long position. In order to get a certain Greek for a short position, one can simply change the sign in front of it [11].

While the Greeks are mainly used when analysing option instruments, it is useful to note that the delta value of a forward contract is ± 1, depending on the position in the contract. This can be seen from Equation 7, differentiating with respect to the underlying asset price. This means the value of a forward contract changes at a 1:1 rate relative to changes in the underlying price [11].

The main factor that can impact the value of bonds and interest rate swaps is the zero rates used to value the zero coupon bonds and coupon payments. As there are different zero rates for different maturities, which can be retrieved from the zero rate curve, there are potentially several zero rates to take into account for a single instrument. One common metric to use when calculating the sensitivity of an interest rate based instrument, is to alter the zero rate by a small shift. By summing up the value changes brought on by these shifts, the sensitivity of the instrument can be calculated [12].

3.4 Principal Component Analysis

Principal Component Analysis, often referred to as PCA, is a statistical method that aims to reduce the dimensionality of a data set and transform data that may be correlated, into completely uncorrelated data, while maximising the variance, i.e the explained information in each of these uncorrelated subsets.

Given a set of data $\mathbf{S}$ with n variables for m different observations, we can write the covariance matrix of $\mathbf{S}$ as

$$Cov(\mathbf{S}) = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \sigma_n\sigma_1 & \dots & \sigma_n^2 \end{pmatrix}.$$

$Cov(\mathbf{S})$ is a symmetric and positive-definite matrix, which means we can rewrite it as

$$Cov(\mathbf{S}) = \mathbf{ODO}^T, \tag{36}$$

where $\mathbf{D}$ is a diagonal matrix containing eigenvalues $\lambda_1, ..., \lambda_n$ of $Cov(\mathbf{S})$, and $\mathbf{O}$ is an orthogonal matrix with columns being the eigenvectors $\mathbf{o}_1, ..., \mathbf{o}_n$ of $Cov(\mathbf{S})$ [12]. We can assume that the the order of $\mathbf{O}$ and $\mathbf{D}$ is such that the eigenvalues occur in falling order, i.e the largest eigenvalue is found in the first diagonal position. Set $\mathbf{S}^* = \mathbf{O}^T(\mathbf{S} - E[\mathbf{S}])$, where $E[\mathbf{S}]$ is the expected value of $\mathbf{S}$. We can now look at the covariance of this transformed $\mathbf{S}$ as

$$Cov(\mathbf{S}^*) = E[\mathbf{O}^T(\mathbf{S} - E[\mathbf{S}])(\mathbf{S} - E[\mathbf{S}])^T\mathbf{O}] = \mathbf{O}^T Cov(\mathbf{S})\mathbf{O} = \mathbf{D}. \tag{37}$$

This tells us that the covariance matrix of $\mathbf{S}^*$ is diagonal with the eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ in the diagonal elements, and zeros everywhere else. This implies we can obtain completely uncorrelated components of $\mathbf{S}$ by changing the basis as above. To see this, we can write

$$\sum_{k=1}^{n} S_k \mathbf{e}_k = \mathbf{S} = \mathbf{OO}^T\mathbf{S} = E[\mathbf{S}] + \mathbf{OS}^* = E[\mathbf{S}] + \mathbf{O}\sum_{k=1}^{n} S_k^* \mathbf{e}_k = E[\mathbf{S}] + \sum_{k=1}^{n} S_k^* \mathbf{o}_k, \tag{38}$$

where $\mathbf{e}_k$ is a standard unit vector in $\mathbb{R}^n$. This illustrates that presenting $\mathbf{S}$ in terms of the standard unit basis $\mathbf{e}_1, ..., \mathbf{e}_n$ yields the standard data set, with correlation between components, but presenting it using the orthonormal basis $\mathbf{o}_1, ..., \mathbf{o}_n$, we obtain uncorrelated data, which is often mainly found in only a few of the components. The components making up this orthonormal basis are known as the *Principal Components* [12]. The variance, or the information, contained in each principal component is described by its corresponding eigenvalue λ. The total information I in the transformed data is

$$I = \sum_{k=1}^{n} \lambda_k, \tag{39}$$

and the percentage of information found in the first j principal components can be calculated as

$$I_j = \frac{\sum_{k=1}^{j} \lambda_k}{\sum_{k=1}^{n} \lambda_k}. \tag{40}$$

Usually, a very large majority of the variance in the data will be explained by the first few principal components. This property is also why principal component analysis is considered a dimension reduction technique, as it can transform a large set of data into a smaller, more manageable set without losing much information [11].

3.5 Hedging Equity Derivatives

One commonly used hedging strategy for equity derivatives is to look at the sensitivities of the portfolio and try to reduce the risk associated with these.

3.5.1 Delta Hedging

One commonly used hedging strategy for equity derivatives is to look at the sensitivities of the portfolio and try to reduce the risk associated with these. The most basic version of sensitivity hedging is delta hedging, which aims at removing the risk associated with a portfolios delta. As presented in Section 3.3, delta is the change in value stemming from a change in the price of the underlying asset. A portfolio with a nonzero delta has an exposure to price changes in the underlying asset, which should be neutralised or minimised to reduce the risk of the portfolio [1]. For a portfolio of n different contracts with m_n number of each contract, the total delta of this portfolio is

$$\Delta_{tot} = \sum_{i=1}^{n} \Delta_n m_n. \tag{41}$$

As long as all contracts are based on the same underlying asset, the delta of the portfolio is calculated as the sum of the individual deltas for each contract multiplied by the number of contracts [11]. Given suitable hedging contracts that can be acquired to counteract the risk, the problem specification can then be expressed as finding the required number w of these contracts, such that

$$\Delta_{tot} + w\Delta_{hedge} = 0. \tag{42}$$

The cost of this hedging position would be the price paid, or received, from entering into it. The value change of the portfolio is

$$dV_P = V_P + wV_h - wS, \tag{43}$$

where V_P is the value of the portfolio, V_h is the value of one hedging contract and S is the cost of one hedging contract [24]. Assuming a contract is bought or sold for its fair value, we can set $S = V_h$ and the total value of the portfolio immediately after hedging it is unchanged. This assumes no transaction costs. As time passes, the value of the original portfolio and the hedging positions will change. Assuming the hedge replicates the portfolio well, these changes should be opposite and equally sized. The value of the cash paid or received will change with time too, although generally much less. This corresponds to gains from interest obtained by placing the cash in a bank, or losses from paying interest on the loaned cash. However due to the short holding period in question, the value change of the cash position can be assumed to be zero over this time.

Equation 42 is straight forward to solve, and will directly give how many contracts that we need to enter into in order to become delta neutral. Several different contracts can be used to make a portfolio delta neutral. One alternative would be to use options, but since these derivatives have several other risk factors, see Section 3.3, this might not be optimal. An alternative to using options is to use forward contracts. A forward contract is more straight forward than an option and the delta of a forward contract is strictly ± 1. Another advantage of using forwards is that while the contract is entered immediately, all payments occur at the settlement date, which means there is no immediate cost to enter into the hedging position, excluding potential transaction costs. However, the most direct approach to delta hedge is to directly invest in shares of the underlying asset, since the delta of one share in the underlying is ± 1, and shares can be bought or sold at any time.

This hedging strategy is a simple strategy, where the risk of value changes in the portfolio due to underlying price changes is removed. Under the assumption that the only price factor that can change is the underlying price, this strategy should theoretically provide a good hedge, as long as the price changes are not too large. However this strategy ignores several other risk factors, such as sensitivity to changes in the implied volatility. Additionally, for options, delta will not remain constant over time. As soon as the underlying price changes, the portfolio will no longer be hedged as well [1].

3.5.2 Hedging multiple sensitivities

A pure delta hedge potentially leaves a lot of risk in the portfolio untouched. This can become especially important during a volatile period in the market. Finding a strategy to remove or reduce the risk associated additional risk factors would yield a more robust strategy.

The general method is, given a matrix of the desired sensitivities for the available hedging options, $\mathbf{G}$, and a vector with the current portfolio sensitivities for these sensitivities $\mathbf{g}$, to solve

$$\begin{pmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n,1} & G_{n,2} & \dots & G_{n,n} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = - \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix}$$

for $w_1, w_2, ..., w_n$ [11]. This setup works for any combination of sensitivities to neutralise. To get a well posed system of equations, we need as many hedging derivatives as there are sensitivities to neutralise. The value change in the portfolio is calculated similarly to equation 43, as

$$dV_P = V_P + \sum_{i=1}^{n} w_i (V_{h,i} - S_i). \tag{44}$$

One concern here is that for every additional equation to solve, the solution may result in increasingly large hedging positions. This can make a strategy unviable due to the sheer amount of new positions to enter into, and the corresponding transaction costs, regardless of the risk reduction. For an option portfolio, there are four first order sensitivities as well as the second order sensitivity gamma, see Section 3.3, which would require five different hedging positions with potentially large number of contracts. To reduce this, one alternative would be to instead use Equation 42 to hedge the delta exposure separately, after the other sensitivities has been neutralised. Since neutralising the other sensitivities would have to be done using options, the delta of the portfolio will have to be adjusted prior to neutralising it.

3.5.3 Different levels of hedging

The neutralisation strategies mentioned so far attempts to achieve a total removal of any and all exposure to the specified risk factors. An alternative to this would be to relax the required level of reduction, to reduce the costs of hedging, while still removing a part of the exposure. This can be integrated in the problem specification by modifying the general equation system in Section 3.5.2 to include a risk aversion parameter. This would yield

$$\begin{pmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n,1} & G_{n,2} & \dots & G_{n,n} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = -\psi \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$

where ψ is a scalar in the interval $[0, 1]$. Setting $\psi = 1$ would yield an attempt to achieve complete removal of exposure, while setting for example $\psi = 0.5$ would produce a 50% reduction in exposure to the chosen risk factors.

This can be further extended by assuming the decision is not taken manually by the owner. By making the risk aversion parameter ψ dependent on the market volatility, the process could be made more autonomous. When volatility is high, the level of neutralisation is set to full, and as volatility lowers, the hedging level is progressivley lowered.

3.5.4 Dynamic Hedging

The strategies presented so far neutralise the sensitivities of a portfolio to try to achieve a better risk reduction. However, these strategies assumes that the sensitivities are constant, and are examples of *static hedging strategies*. Once calculated, the exposures is assumed to not change as time progresses and the obtained hedge will continue to provide very good risk reduction indefinitely. This is in general an unrealistic assumption. As time passes, prices will fluctuate and sensitivities will change. This can be especially true during a default situation, during which the market can be very volatile.

An attempt to reduce the risk from this can be to try to re-balance the hedge during the holding time, to try to maintain neutrality throughout the entire period. This is an example of a *dynamic hedging strategy*. In theory, one can re-balance every hedge continuously, to maintain a perfect hedge, but in practice this is not possible. This comes both from the fact that re-balancing will have to be done at discrete times instead of continuously and that transaction costs would become very high with frequent re-balancing [24].

Because of this, a more reasonable method would be to only re-balance delta at discrete times, commonly, and in this thesis, daily. This is done by buying or shorting shares directly of the underlying stock, to adjust the portfolio delta back to zero. At the start of every day, delta is recalculated based on changes in the underlying variables. The total new delta of the portfolio is determined, and the necessary number of shares are bought or shorted.

This strategy can be applied as an extension to any of the existing previous strategies. This would result in the desired static hedge being put in place immediately upon overtaking the portfolio, with dynamic delta hedging active over the remainder of the holding period.

3.5.5 Transaction Costs

If we assume that a contract can be bought or sold exactly for its fair value, it is not very costly to acquire hedging contracts. The Black-Scholes model for pricing European options relies on the assumption that you can continuously re-balance the portfolio [24]. When there are no transaction costs present, this is feasible, but in reality, there will be a cost associated with buying or selling contracts. These costs can come from for example the price difference (spread) between bid and ask prices for the hedging contracts [11].

There is no universal way to determine transaction costs for every kind of transaction. One strategy presented by Leland [15], and further expanded upon by Hodges and Clewlow [10], is to assume transaction costs proportional to the current price of the underlying asset. Whenever a transaction is entered, a percentage of the price of the underlying asset is counted as a transaction cost, in addition to the standard cost of the contract. This kind of transaction cost would apply both when buying and selling contracts. Give a traded volume of contracts v, underlying price S and a proportional scaling factor k, the transaction costs when buying or selling an instrument can be defined as

$$T_c = k|v|S. \tag{45}$$

With the introduction of transaction costs, it adds a constraint to the hedging problem presented in Section 3.5.2. For each combination of hedging instruments, there will be an accrued transaction cost. The optimal combination is chosen as the one which minimises these.

3.6 Hedging Interest Rate Derivatives

The second part of instruments implemented in this thesis are instruments based on interest rate changes, which in this case are bonds and interest rate swaps. Three different hedging strategies were implemented for these instruments; one basic delta hedging strategy, and two strategies utilising principal component analysis, applied in two different ways.

3.6.1 Delta Hedging

The first strategy follows the same general idea as the delta hedging previously mentioned for equity derivative portfolios. As discussed in Section 3.2.5, the main dependency on an underlying variable is the zero rate used when pricing the instruments, and one direct way of hedging this risk would be to neutralise it. Given a swap or bond, one can determine the sensitivity of the instrument with respect to a change in the underlying zero curve. One basic approach is to introduce a fixed, small change in the full zero curve, and compare the value of an instrument before and after such a change. A second, more precise method would be to introduce a small change for one maturity at the time, and then sum up a total sensitivity for the instrument. This second method is how the zero rate sensitivities are determined for instruments in this study. Assume we have a vector of n zero rates $\mathbf{r} = (r_1, r_2, \dots r_n)$, for n different maturities. The sensitivity of an interest rate instrument P to a change in the zero rate can be expressed as

$$\nabla(\mathbf{r}) = \Big(\frac{\partial P}{\partial r_1}, \frac{\partial P}{\partial r_2}, \dots, \frac{\partial P}{\partial r_n}\Big)(\mathbf{r}). \tag{46}$$

The sensitivity of a portfolio with $P_1, P_2, ..., P_m$ instruments, to changes in the zero rate can be expressed as

$$\boldsymbol{\nabla}(\mathbf{r}) = \begin{pmatrix} \frac{\partial P_1}{\partial r_1}(\mathbf{r}) & \dots & \frac{\partial P_1}{\partial r_n}(\mathbf{r}) \\ \vdots & \ddots & \vdots \\ \frac{\partial P_m}{\partial r_1}(\mathbf{r}) & \dots & \frac{\partial P_m}{\partial r_n}(\mathbf{r}) \end{pmatrix}$$

The total sensitivity to changes in the zero curve for the portfolio of interest rate derivatives can be determined by summing the individual exposure in the portfolio for each zero rate maturity, i.e each column, yielding

$$\nabla_P(\mathbf{r}) = \Big(\sum_{i=1}^{m} \frac{\partial P_i}{\partial r_1}, \sum_{i=1}^{m} \frac{\partial P_i}{\partial r_2}, \dots, \sum_{i=1}^{m} \frac{\partial P_i}{\partial r_n}\Big)(\mathbf{r}). \tag{47}$$

Given hedging instruments for interest rate derivatives, the sensitivity to changes in the zero curve for each of these can be calculated in the same way that the portfolio sensitivity is determined, and the problem then becomes to construct a hedging portfolio that neutralises the exposures in the original portfolio. This means solving

$$\sum_{i=1}^{N} w_i \nabla_i(\mathbf{r})^T \delta\mathbf{r} = -\nabla_P(\mathbf{r})^T \delta\mathbf{r}, \tag{48}$$

where w_i is the required number of hedging contract i, with sensitivity ∇_i determined from Equation 46, and $\delta\mathbf{r}$ is a vector of changes in the underlying zero rate [12]. In this scenario, we consider a fixed uniform change to the entire zero curve. This way, we can eliminate the $\delta\mathbf{r}$ term and reduce the problem to only needing a single hedging instrument to fully neutralise the delta of the portfolio. This is obtained by solving

$$w_i = -\frac{\nabla_P}{\nabla_i}, \tag{49}$$

for hedging instrument i, which is straight forward. Given the availability of multiple hedging instruments, there are more than one solution to this equation. One reasonable way to determine the optimal hedging instrument would be to choose the one that results in the fewest new positions.

3.6.2 Principal Component Hedging

A different way to determine the hedging portfolio is instead to use principal component analysis to determine patterns in the swap rate, and in the behaviour of the interest rate derivatives. The first three principal components for the swap curve are usually called the level, slope and curvature of the swap curve [23]. Usually, it is enough to analyse the first few components, as these contain most of the information. The way to interpret the principal component factor loadings is that assuming a uniform zero rate shift, the factor loadings determine the size of the effect on each maturity. This gives different weights to the different maturities to more accurately describe the effects of a change compared to treating it as a uniform shift in all maturities.

There are different ways to apply this kind of analysis and this thesis applies two variants. The first method aims to determine the direct exposure to the different principal component factor loadings a portfolio has, and to determine a hedging portfolio which neutralises these [21]. The second method focuses on determining the sensitivity of each individual zero rate change as in the delta hedging, but with the added assumption that the principal components determine the way a change in the zero curve can happen [12].

The first strategy is a direct approach to neutralising exposures in the portfolio. After determining the desired number of principal components, the factor loading for each maturity is multiplied by the total number of instruments with the same time to maturity. The result would then be the exposure to that specific factor loading for that principal component. By summing up all exposures to each principal component, the total portfolio risk with respect to the principal components can be determined. The total portfolio exposure to principal component l can be expressed as ϕ_l and is determined as

$$\phi_l = \sum_{i=1}^{N} \mathbf{x}_i^T \cdot \mathbf{o}_{l,i}, \tag{50}$$

where $\mathbf{x}$ is a vector of the portfolio containing interest rate derivatives with N different possible maturities and the vector $\mathbf{o}$ contains the factor loadings of the corresponding principal component. This can be generalised to any number of principal components. To find a good hedging portfolio that neutralises this exposure, we want to enter into offsetting hedging contracts that counteracts this. For l number of principal components and n different maturities, we want to find the required number of hedging instruments w to neutralise the portfolio. Note that to get a well posed system of equations, we need $l = n$. This can be expressed as

$$\begin{pmatrix} o_{1,1} & o_{1,2} & \dots & o_{1,n} \\ o_{2,1} & o_{2,2} & \dots & o_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ o_{l,1} & o_{l,2} & \dots & o_{l,n} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_l \end{pmatrix} = - \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_l \end{pmatrix}$$

This results in a suggested hedging portfolio that neutralises exposures to the chosen number of principal components. Given more hedging instruments than equations, there will be multiple solutions. This can be used to try and keep the hedging position as small as possible [21].

The second strategy involving principal components also looks at changes in the underlying zero rate. But instead of looking at a flat increase or decrease in the rate, this strategy uses the determined principal components to better match the possible movements in the rate. This is essentially a combination of the delta hedging strategy presented in Section 3.6.1, and the principal component exposure. With q principal components, the problem is then to solve

$$\sum_{i=1}^{N} w_i \nabla_i(\mathbf{r})^T \mathbf{o}_l = -\nabla_P(\mathbf{r})^T \mathbf{o}_l, \quad \text{for } l = 1, ..., q \tag{51}$$

where ∇_i as before is the sensitivity for hedging instrument i [12]. For example, for three principal components, we want to solve

$$\begin{aligned}
w_1\nabla_1(\mathbf{r})^T\mathbf{o}_1 + w_2\nabla_2(\mathbf{r})^T\mathbf{o}_1 + w_3\nabla_3(\mathbf{r})^T\mathbf{o}_1 &= -\nabla_P(\mathbf{r})^T\mathbf{o}_1 \\
w_1\nabla_1(\mathbf{r})^T\mathbf{o}_2 + w_2\nabla_2(\mathbf{r})^T\mathbf{o}_2 + w_3\nabla_3(\mathbf{r})^T\mathbf{o}_2 &= -\nabla_P(\mathbf{r})^T\mathbf{o}_2 \\
w_1\nabla_1(\mathbf{r})^T\mathbf{o}_3 + w_2\nabla_2(\mathbf{r})^T\mathbf{o}_3 + w_3\nabla_3(\mathbf{r})^T\mathbf{o}_3 &= -\nabla_P(\mathbf{r})^T\mathbf{o}_3,
\end{aligned}$$

for w_1, w_2, w_3 in order to obtain the hedging portfolio which neutralises the original portfolio against changes in the zero curve described by the first three principal components.

4 Method

The general method used during this thesis when implementing and testing the hedging functions, can be divided into the following steps.

1. Data collection and preparation.
2. Implementing pricing and valuation models for the contract types.
3. Implementing hedging models for the contract types
4. Testing the performance of the hedging on historical data
5. Analysis and conclusions from testing.

The necessary data used during the testing was gathered from the *Federal Reserve Bank of St. Louis*, which has freely available historical financial data. Data gathered from here was specifically the NASDAQ100 stock index, the US Treasury constant maturities yields for different maturities, to use as a proxy for the risk free interest rate used in the pricing models, the volatility index, to use as a proxy for the historical implied volatility when pricing options, and the historical swap rates for maturities of 1,2,3,4,5,7 and 10 years. These were used when pricing bonds and interest rate swaps.

The first step was to prepare the necessary data. The underlying stock was assumed to be the NASDAQ100 index, scaled down by a factor ten, in order to make analysis easier. Due to occasional missing values in the data sets, linear interpolation was used to fill in missing daily values. The principal components used when hedging interest rate derivatives were calculated. These components were then linearly interpolated to allow for any payment date of up to 10 years to be used. To price interest rate derivatives, the zero rate curve also had to be created from the swap par rates, following the method presented in Section 3.2.5.

After the necessary data had been obtained, the next step was to implement the valuation formulas presented in Section 3 in MATLAB and create functions that could calculate the sensitivities of each kind of instrument. This was done according to the models presented in Section 3.3. Given an applicable instrument with any combination of valid parameters, the price and sensitivities of the instrument could be calculated.

The hedging functions were constructed to solve a system of equations given the calculated sensitivities and available hedging instruments. For equity derivative portfolios, it was assumed that there existed options with different strike prices centered on the current underlying price, and different time to maturities centered around the mean time to maturity of the portfolio instruments. For interest rate derivatives, it was assumed there existed swap contracts with time to maturities of $1, 2, 3, ..., 10$ years. These swaps were assumed to be available with a fixed rate equal to the swap par rate, making them zero valued at initiation. This property, together with the fact that there is no upfront payment was the deciding factor to use swaps as hedging instruments, instead of for example bonds. This also resulted in there being no immediate transaction costs applied to interest rate hedging, as opposed to hedging the equity derivative portfolios.

4.1 Backtesting

In order to test the effectiveness of the different hedging strategies, backtesting on historical data was applied. For each strategy, the portfolio was evaluated on 3000 business days, corresponding to historical data ranging from 2001-07-31 to 2013-01-28. For each day in the data set, the portfolio was obtained and hedged on the first day, and the performance of the hedged portfolio was monitored over a holding period of ten days. The daily profit or loss during the holding period was monitored, and on the final day, the performance of the strategy was evaluated as the difference in value today compared to the default day.

To simplify the testing procedure, separate testing runs were conducted on the different portfolios. Also, in the case of equity derivative portfolios, it was assumed that all instruments in the portfolio were based on the same underlying asset. This was deemed acceptable as performing hedging on instruments with different underlying assets would be done separately for one underlying at the time. It was assumed that different assets were uncorrelated and hedging performance was only based directly on the relevant instruments. For multi-asset portfolios, it was assumed the resulting hedging positions for each underlying asset could be added to result in a fully hedged portfolio. Note that due to all interest rate derivatives using the same zero rate curve for valuation, this is not a problem with interest rate derivatives.

4.1.1 Backtesting Equity Derivative Portfolios

The chosen hedging strategies are presented in Table 1. Due to the assumption that a default situation could potentially lead to volatile market conditions, emphasis was placed on hedging vega. Additionally, due to the short holding period, the impact of time decay was deemed to be minimal, resulting in the decision to not attempt to neutralise theta exposure.

Table 1 – The chosen strategies when hedging equity derivatives.

Sensitivities:	Delta	Gamma	Vega	Rho
Strategy 1:	**X**			
Strategy 2:	**X**		**X**	
Strategy 3:	**X**	**X**	**X**	
Strategy 4:	**X**	**X**	**X**	**X**

Two different trading strategies involving options were used for backtesting. These were a *Bull Spread* and a *Straddle*. A bull spread is profitable when the price of the underlying increases. To construct a bull spread, the original portfolio contained 1000 long call options with strike price K_1 and maturity in one year, and 1000 short call options with strike price K_2 and maturity in one year, where $K_1 < K_2$. The strategy was assumed to start out-of-the-money, which was determined to be a reasonable assumption given that the portfolio is assumed to be obtained from a defaulted party. A schematic of a bull spread is presented in Figure 9.

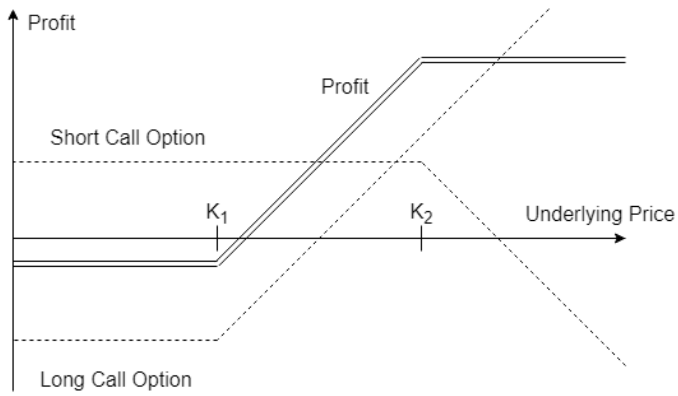


Figure 9 – Profit from a bull spread.

The straddle portfolio was constructed with 1000 long call options and 1000 long put options, with the same strike price K. This strategy is profitable if the price of the underlying asset changes significantly, either increasing or decreasing. The strategy was assumed to start out-of-the-money. See Figure 10 for a schematic of a straddle.

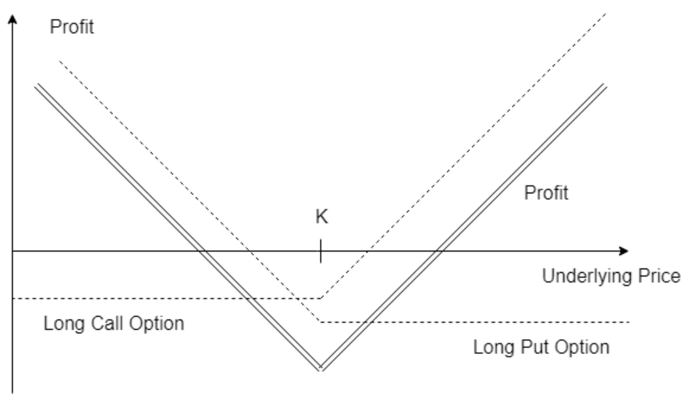


Figure 10 – Profit from a straddle.

The value of a portfolio was calculated as the Black-Scholes valuation, where a short position provided negative value. Upon acquiring hedging instruments, the immediate portfolio value was assumed to be unchanged due to perfect matching between the value of the instrument and the cost of the instrument. This does not include transaction costs, which was kept separately. Due to the short holding period of ten business days, the time value of the purchase cost was assumed to be constant.

The available hedging instruments was assumed to have strike prices of $85\%, 90\%, 95\%, 100\%, 105\%, 110\%, 115\%$ of the underlying price, and time to maturities of $80\%, 90\%, 100\%, 110\%, 120\%$ of the portfolio maturity. To reduce computational time, no more combinations were used. It was required that the acquired number of hedging positions was an integer, i.e buying 233.89 contracts was not allowed. Additionally, due to the fact that both gamma and vega are the same for call and put options, it was assumed that all available hedging contracts were call options. Hence, shorting options were widely used by the hedging strategies, while no put options were acquired.

4.1.2 Backtesting Interest Rate Portfolios

The chosen hedging strategies are presented in Table 2.

Table 2 – The chosen strategies when hedging interest rate derivatives.

Name:	Hedge:
Delta hedge	Neutralise against zero rate shift
PCA 1,2	First principal component strategy, using 2 components.
PCA 1,3	First principal component strategy, using 3 components.
PCA 2,2	Second principal component strategy, using 2 components.
PCA 2,3	Second principal component strategy, using 3 components.

Two different trading strategies were used when testing swap portfolios, and one strategy was used for a bond portfolio. The main parameter to vary for interest rate derivatives is the time to maturity. The first portfolio was assumed to have relatively short time to maturities of 1, 2, 3 and 4 years, while the second instead looked at swaps with maturities of 7, 8, 9 and 10 years. For bonds, the portfolio was assumed to contain shorter maturity bonds. The positions taken were arbitrarily chosen as the fixed rate receiver for the swap portfolios, and as buyers of the bonds.

4.1.3 Additional testing

In addition to the main strategies, several variations were tested, especially for the equity derivative portfolios. These were several of the different enhancements presented in Section 3.5. To reduce computational time due to time constraints, the testing of these strategies were performed on smaller subsets of the backtesting data. The testing were performed on the same portfolio compositions as before, for hedging strategy 3, and includes applying rehedging of delta and varying the level of neutralisation, both statically and depending on the volatility. As it was assumed that one would always want to completely neutralise delta exposure, these changes were not applied to delta.

5 Results

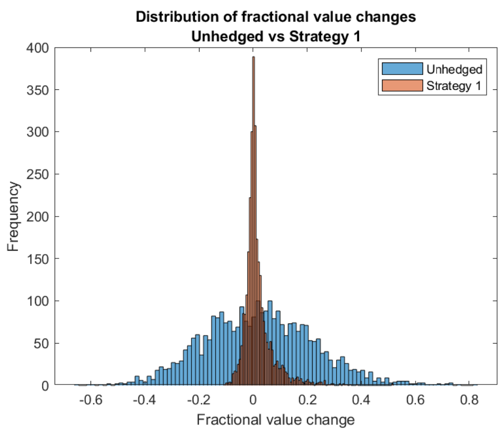
The results of the backtesting for the different portfolios are presented in their respective sections.

5.1 Equity Derivatives Portfolios

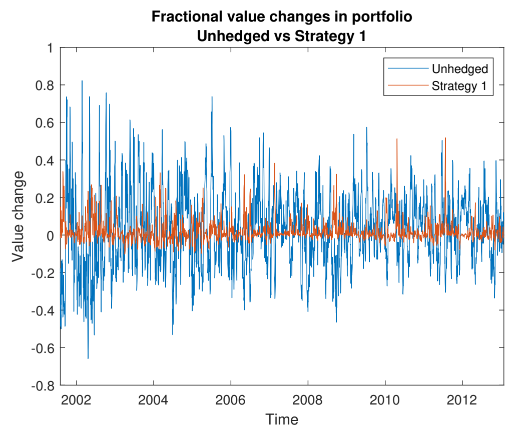
Testing was performed on two different portfolios, each employing a different trading strategy. The results for the different trading strategies are presented in their respective Sections 5.1.1 and 5.1.2. The different hedging strategies will be referred to according to Table 1.

5.1.1 Bull Spread Strategy

The first test portfolio is structured as a Bull spread with 1000 long call options and 1000 short call options. This strategy ends in the money if the underlying asset price increases. The starting position of the portfolio is out of the money.

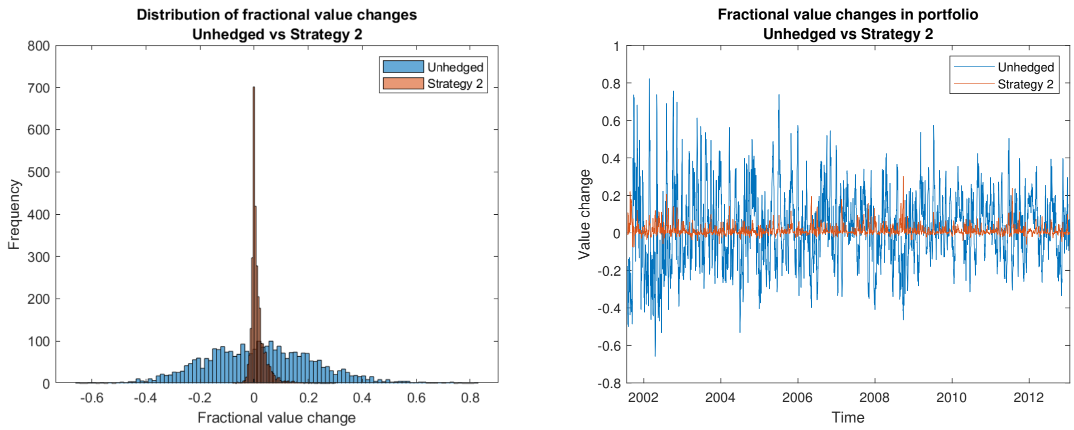


(a) Distribution of fractional value changes.



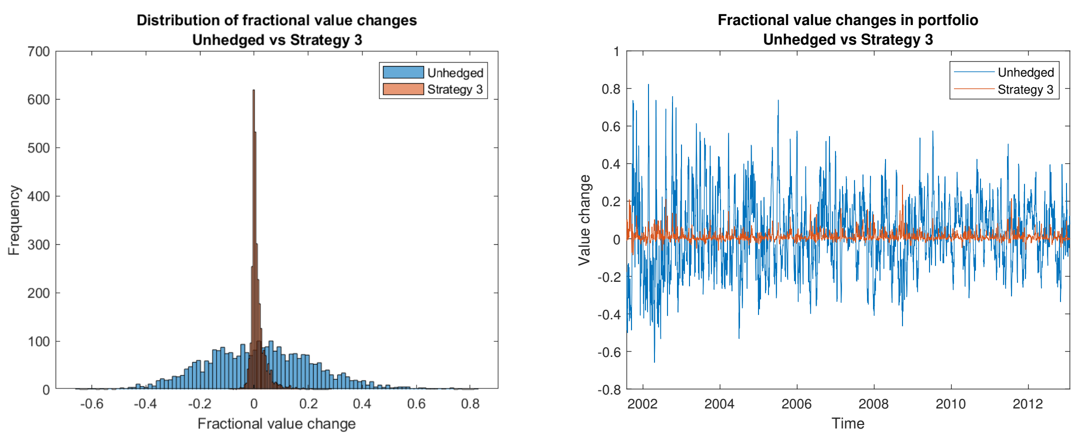
(b) Value change for each iteration.

Figure 11 – Performance of strategy 1 versus unhedged.



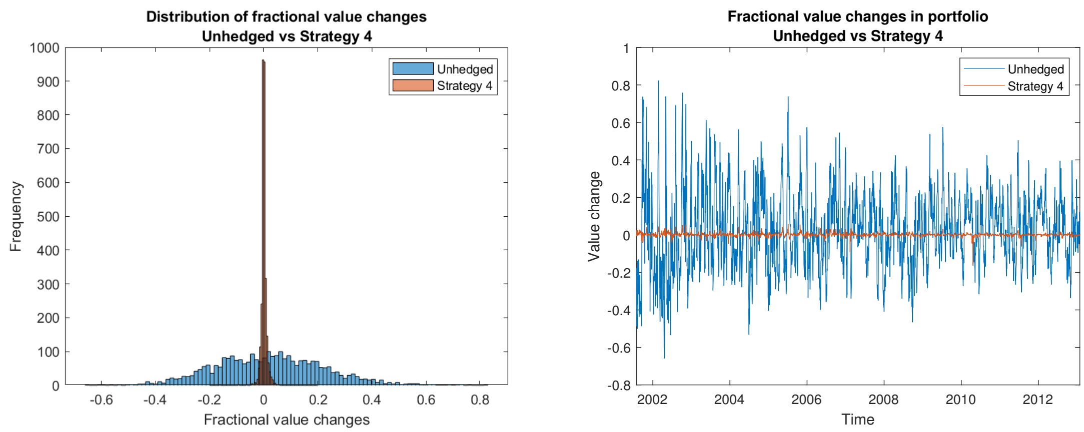
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 12 – Performance of strategy 2 versus unhedged.



(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 13 – Performance of strategy 3 versus unhedged.



(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 14 – Performance of strategy 4 versus unhedged.

The mean, median and standard deviation of the relative value changes are presented in Table 3. Additionally, since the strategies aim to minimise changes in all directions, the absolute values of the changes has also been computed and has its mean and median presented.

Table 3 – Statistical metrics of the relative value changes for each strategy.

Strategy:	**Mean**	**Median**	**Standard deviation**	**Absolute mean**	**Absolute median**
Unhedged	0.0246	0.0214	0.2049	0.1641	0.1401
Strategy 1	0.0178	0.0046	0.0549	0.0335	0.0177
Strategy 2	0.0135	0.0044	0.0298	0.0186	0.0090
Strategy 3	0.0132	0.0049	0.0283	0.0179	0.0088
Strategy 4	8.109e-4	1.981e-4	0.0098	0.0057	0.0032

An example of a hedge portfolio for equity derivatives from an arbitrarily chosen day is illustrated in table 4, where negative amounts indicate short positions. Strike prices and time to maturities is expressed as percentages of the original portfolio, as described in section 4.1.1. Note that all options acquired are call options.

Table 4 – Example of hedge portfolio for the Bull strategy on 2008-06-23.

Strategy 1	**Strike Price**	**Time to Maturity**	**Amount**
Shares:			-85
Strategy 2	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	105 %	120 %	-20
Shares:			-74
Strategy 3	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	95 %	90 %	0
Option 2:	105 %	100 %	-22
Shares:			-74
Strategy 4	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	95 %	90 %	-12
Option 2:	95 %	110 %	-246
Option 3:	110 %	110 %	213
Shares:			-3

The transaction costs accrued from each hedging strategy, relative to the original portfolio value, is illustrated in Figure 15 and the mean, median and standard deviation are presented in Table 5.

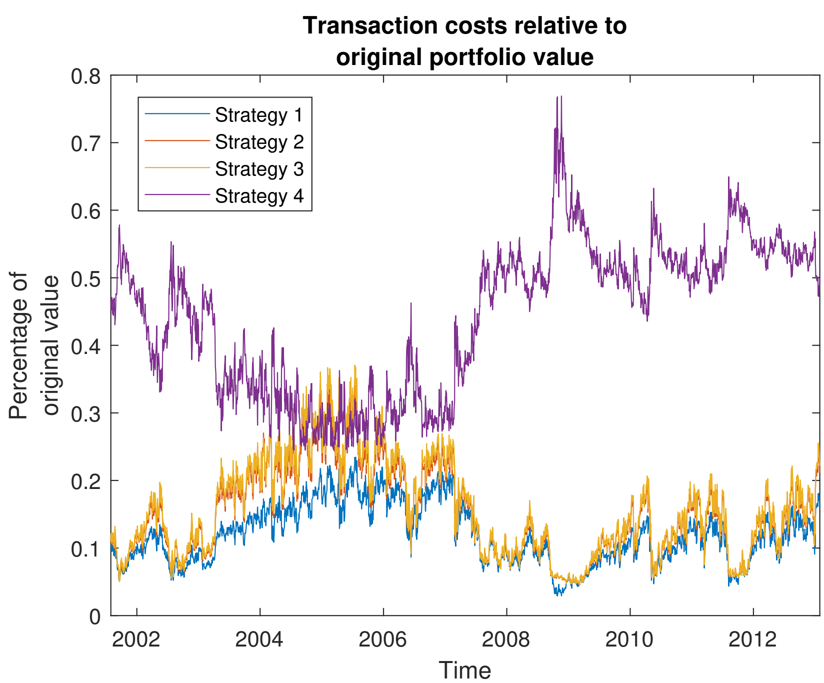


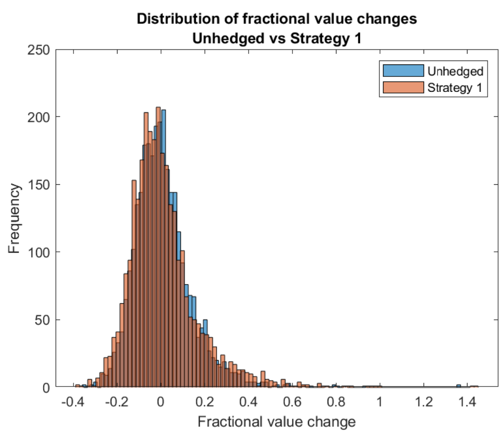
Figure 15 – Transaction costs of all hedging strategies, relative to the original portfolio value.

Table 5 – Statistical metrics of the relative transaction costs for each strategy.

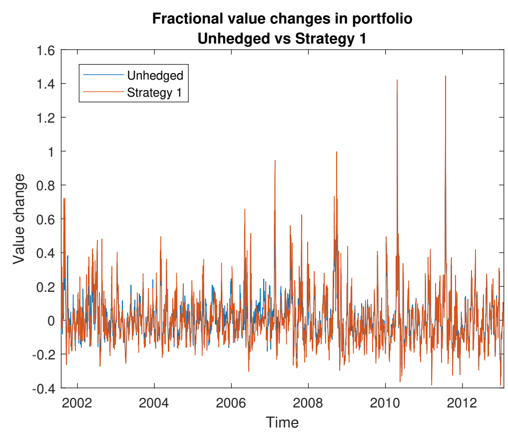
Strategy:	**Mean**	**Median**	**Standard Deviation**
Strategy 1	0.1193	0.1164	0.0432
Strategy 2	0.1511	0.1416	0.0645
Strategy 3	0.1576	0.1463	0.0699
Strategy 4	0.4417	0.4763	0.1097

5.1.2 Straddle Strategy

The second test portfolio is structured as a Straddle with 1000 long call options and 1000 long put options. This strategy ends in the money if the underlying asset price either increases or decreases. The starting position of the portfolio is out of the money, at the strike prices for the options.

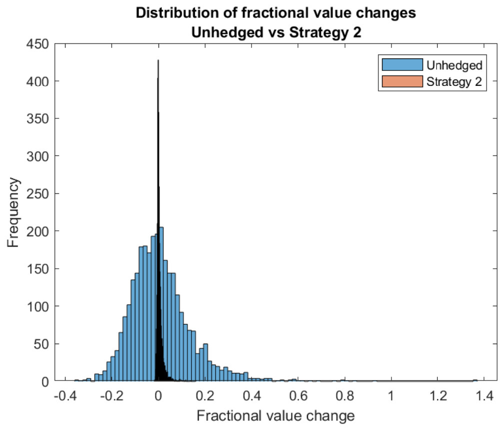


(a) Distribution of fractional value changes.

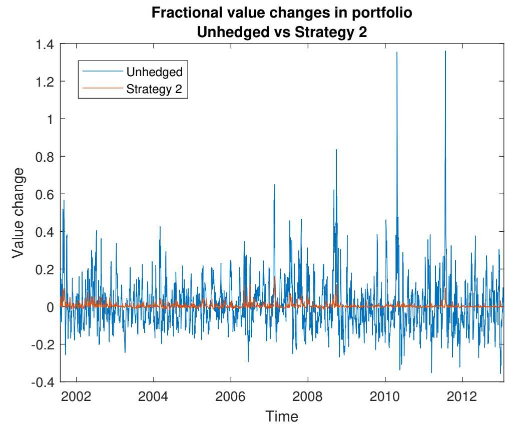


(b) Value change for each iteration.

Figure 16 – Performance of strategy 1 versus unhedged.

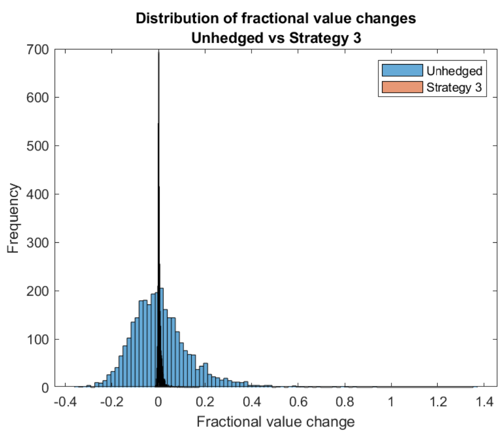


(a) Distribution of fractional value changes.

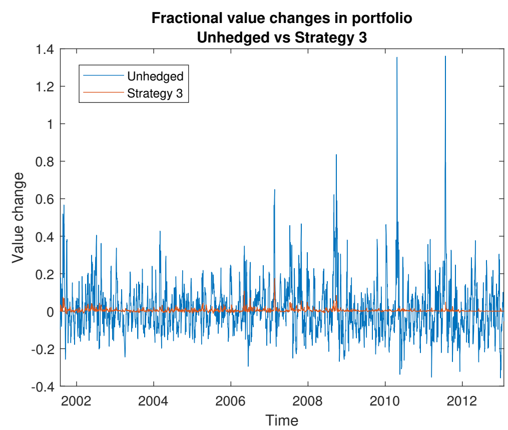


(b) Value change for each iteration.

Figure 17 – Performance of strategy 2 versus unhedged.

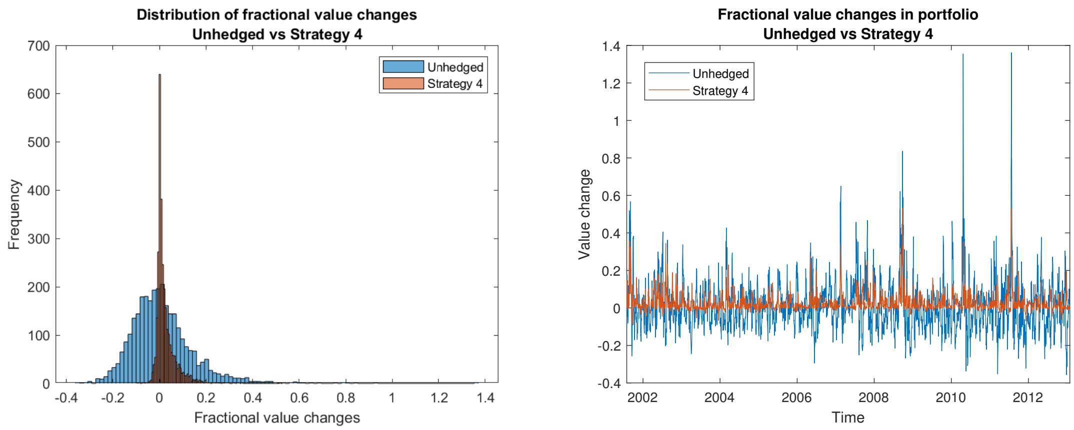


(a) Distribution of fractional value changes.



(b) Value change for each iteration.

Figure 18 – Performance of strategy 3 versus unhedged.



(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 19 – Performance of strategy 4 versus unhedged.

The mean, median and standard deviation of the relative value changes are presented in Table 6. Additionally, since the strategies aim to minimise changes in all directions, the absolute values of the changes has also been computed and has its mean and median presented.

Table 6 – Statistical metrics of the relative value changes for each strategy.

Strategy:	Mean	Median	Standard deviation	Absolute mean	Absolute median
Unhedged	0.0112	-0.0041	0.1373	0.0978	0.0748
Strategy 1	0.0109	-0.0145	0.1609	0.1123	0.0827
Strategy 2	0.0048	8.61e-4	0.0141	0.0078	0.0038
Strategy 3	0.0046	0.0018	0.0108	0.0058	0.0027
Strategy 4	0.0255	0.0096	0.0480	0.0297	0.0132

An example of a hedge portfolio for equity derivatives from an arbitrarily chosen day is illustrated in table 7, where negative amounts indicate short positions. Strike prices and time to maturities is expressed as percentages of the original portfolio, as described in section 4.1.1. Note that all options acquired are call options.

Table 7 – Example of hedge portfolio for the Straddle strategy on 2008-06-23.

Strategy 1	**Strike Price**	**Time to Maturity**	**Amount**
Shares:			-179
Strategy 2	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	105 %	120 %	-1782
Shares:			751
Strategy 3	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	105 %	90 %	-925
Option 2:	105 %	110 %	-1023
Shares:			803
Strategy 4	**Strike Price**	**Time to Maturity**	**Amount**
Option 1:	110 %	100 %	-1986
Option 2:	95 %	110 %	3187
Option 3:	110 %	110 %	-2911
Shares:			-224

The transaction costs accrued from each hedging strategy, relative to the original portfolio value, is illustrated in Figure 20 and the mean, median and standard deviation are presented in Table 8.

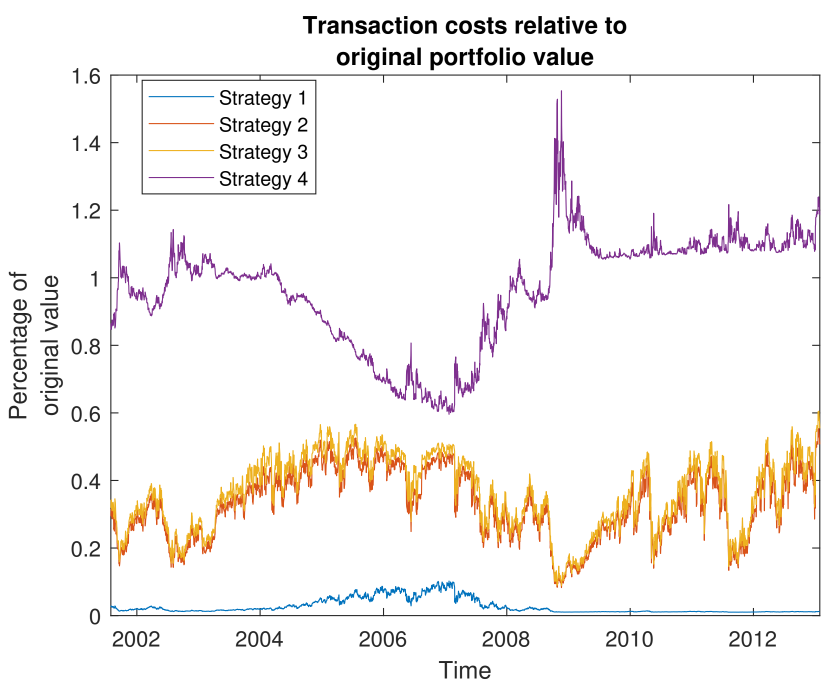


Figure 20 – Transaction costs of all hedging strategies, relative to the original portfolio value.

Table 8 – Statistical metrics of the relative transaction costs for each strategy.

Strategy:	**Mean**	**Median**	**Standard Deviation**
Strategy 1	0.0267	0.0163	0.0221
Strategy 2	0.3297	0.3299	0.1007
Strategy 3	0.3612	0.3655	0.1084
Strategy 4	0.9657	1.0074	0.1623

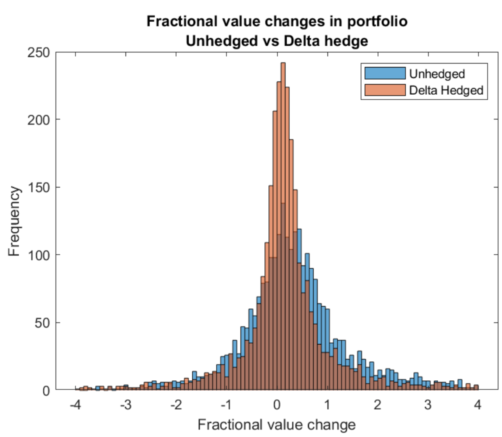
5.2 Interest Rate Swap Portfolios

Testing was performed on two different portfolios, one with short maturity swaps and one with long maturity swaps. This was done in order to see how the strategies performed both on shorter maturities and longer ones. The results for each portfolio is found in their respective Sections 5.2.1 and 5.2.2. The hedging strategies are referred to as in Table 2.

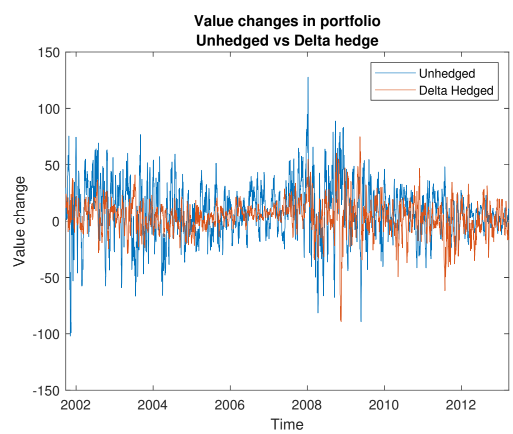
5.2.1 Short Maturity Strategy

The hedging strategies were tested on a portfolio containing 1 year, 2 year, 3 year and 4 year swaps, in a position receiving the fixed cashflows and paying the floating cashflows. The same portfolio was used in every iteration during the testing. The instruments were assumed to have been entered 20 days prior, in order to make the swap contracts have a non zero value. Due to daily changes in the zero rate curve over the 3000 testing days, the portfolio, does not necessarily have the same value on any two given days.

The fractional value change in the portfolio was calculated as the relative difference in portfolio value on the final day in the holding period compared to the starting day. This is illustrated in the left subfigures. In addition, the direct value change in the portfolio over the holding period is illustrated for each iteration and hedging strategy in the right subfigures.

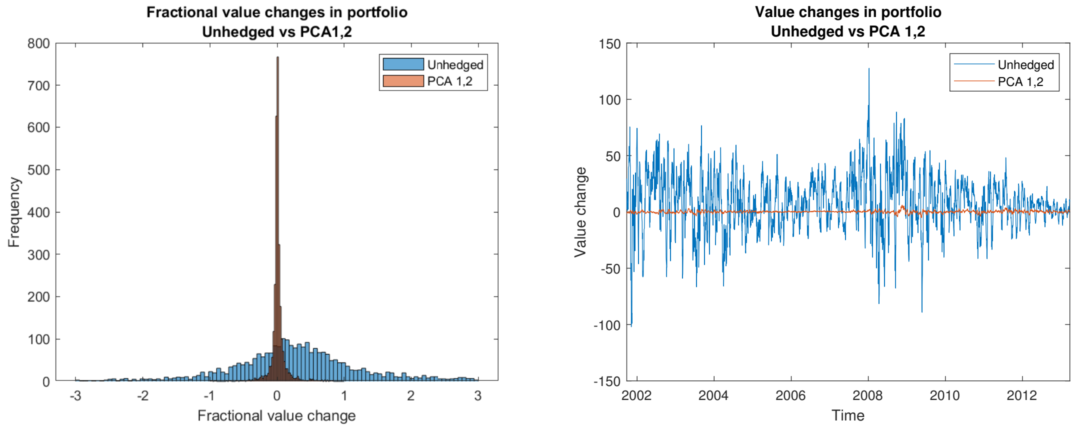


(a) Distribution of fractional value changes.



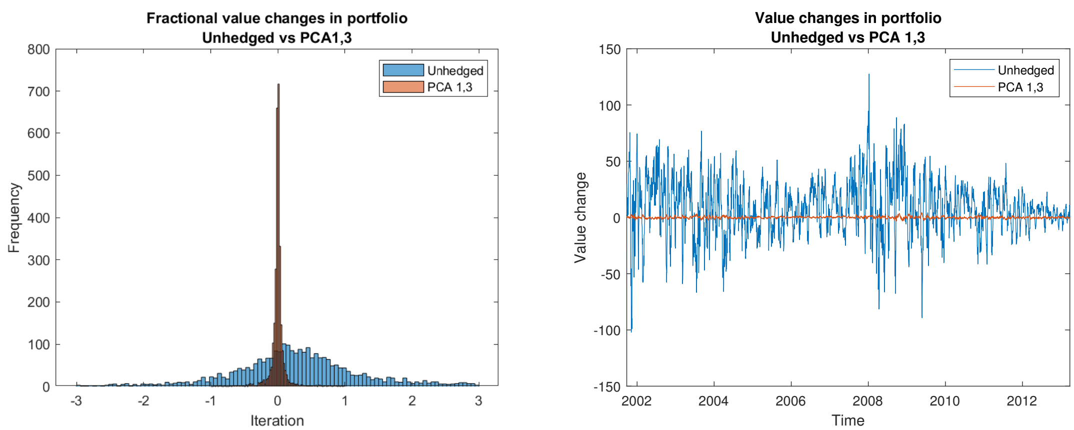
(b) Value change for each iteration.

Figure 21 – Performance of delta hedging versus unhedged.



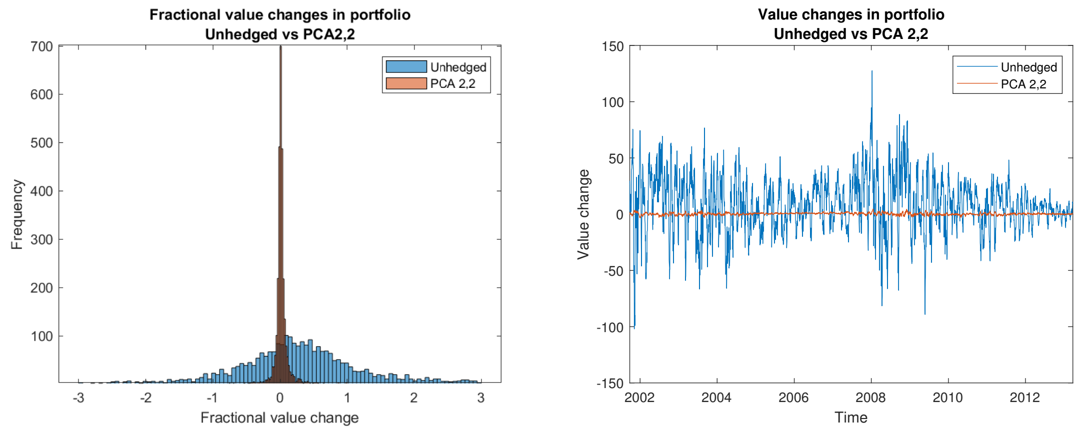
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 22 – Performance of using PCA hedging 1 with 2 components versus unhedged.



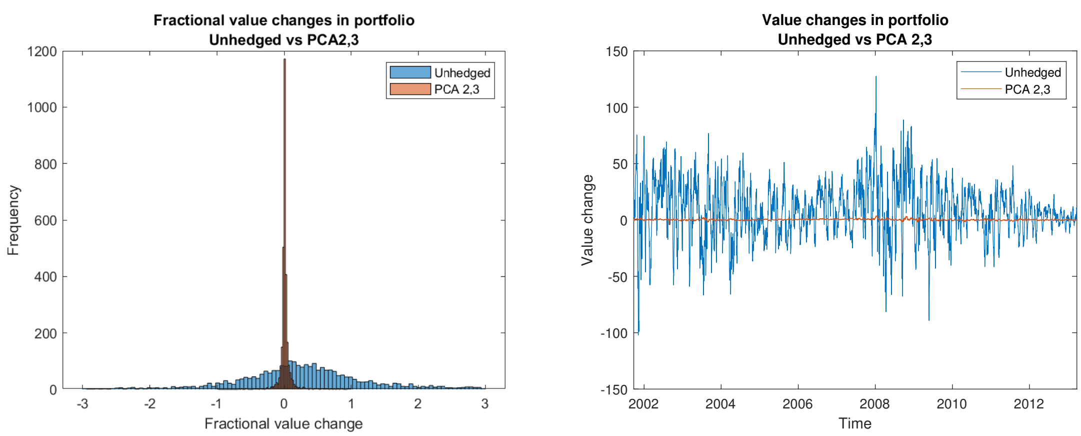
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 23 – Performance of using PCA hedging 1 with 3 components versus unhedged.



(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 24 – Performance of using PCA hedging 2 with 2 components versus unhedged.



(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 25 – Performance of using PCA hedging 2 with 3 components versus unhedged.

The mean, median and standard deviation of the relative value changes are presented in Table 9. Additionally, since the strategies aim to minimise changes in all directions, the absolute values of the changes has also been computed and has its mean and median presented.

Table 9 – Statistical metrics of the relative value changes for each strategy.

Strategy:	**Mean**	**Median**	**Standard deviation**	**Absolute mean**	**Absolute median**
Unhedged	0.3537	0.2775	1.2010	0.8711	0.5803
Delta Hedged	0.1757	0.1291	1.0593	0.6429	0.3126
PCA 1,2	0.0094	0.0057	0.2567	0.0821	0.0225
PCA 1,3	0.0030	9.116e-4	0.2300	0.0745	0.0229
PCA 2,2	0.0253	0.0117	0.2871	0.0864	0.0267
PCA 2,3	0.0251	0.0085	0.2072	0.0584	0.0168

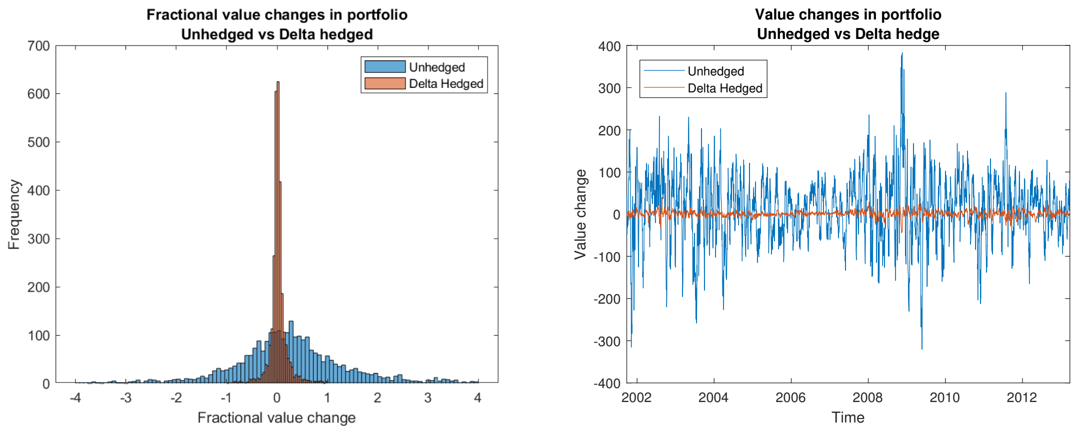
An example of a hedge portfolio for interest rate swaps is illustrated in table 10, where negative amounts indicate fixed rate paying positions.

Table 10 – Example of hedging portfolio for the short maturity strategy on 2008-06-23

Delta Hegde	**Time to Maturity**	**Amount**
Swap 1:	10	-17
PCA 1,2	**Time to Maturity**	**Amount**
Swap 1:	2	-24
Swap 2:	3	-30
PCA 1,3	**Time to Maturity**	**Amount**
Swap 1:	1	0
Swap 2:	2	-36
Swap 3:	4	-18
PCA 2,2	**Time to Maturity**	**Amount**
Swap 1:	2	-11
Swap 2:	3	-39
PCA 2,2	**Time to Maturity**	**Amount**
Swap 1:	2	-21
Swap 2:	3	-28
Swap 3:	6	-2

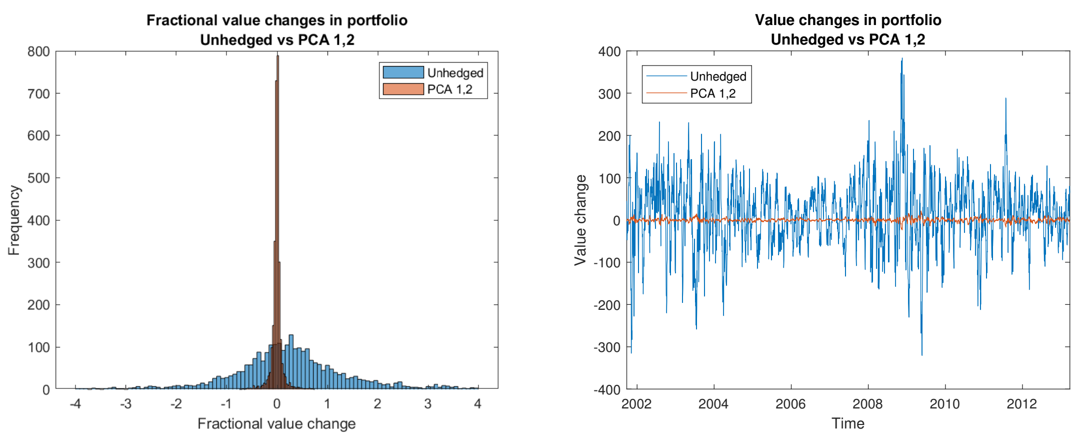
5.2.2 Long Maturity Strategy

The hedging strategies were tested on a portfolio containing 7 year, 8 year, 9 year and 10 year swaps, in a position receiving the fixed cashflows and paying the floating cashflows. The same portfolio was used in every iteration during the testing. The instruments were assumed to have been entered 20 days prior, in order to make the swap contracts have a non zero value. Due to daily changes in the zero rate curve over the 3000 testing days, the portfolio, does not necessarily have the same value on any two given days. The same metrics were measured as in Section 5.2.1.



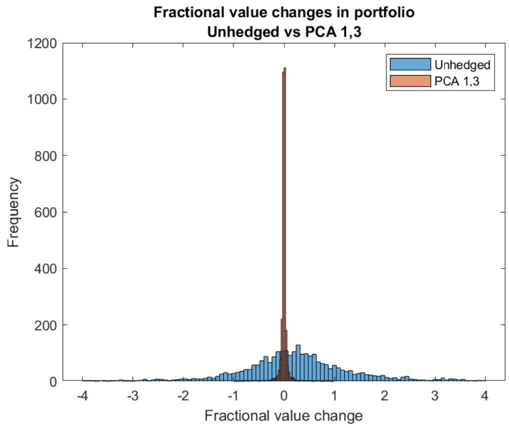
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 26 – Performance of delta hedging versus unhedged.

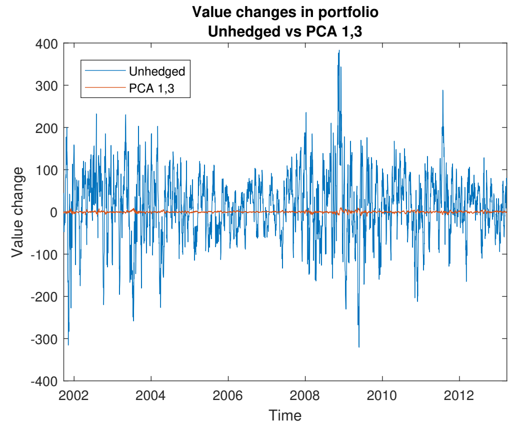


(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 27 – Performance of using PCA hedging 1 with 2 components versus unhedged.

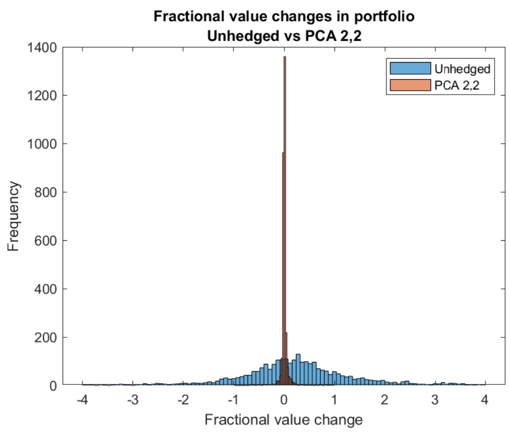


(a) Distribution of fractional value changes.

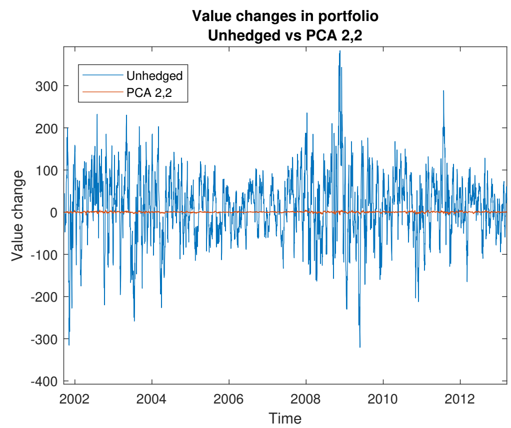


(b) Value change for each iteration.

Figure 28 – Performance of using PCA hedging 1 with 3 components versus unhedged.

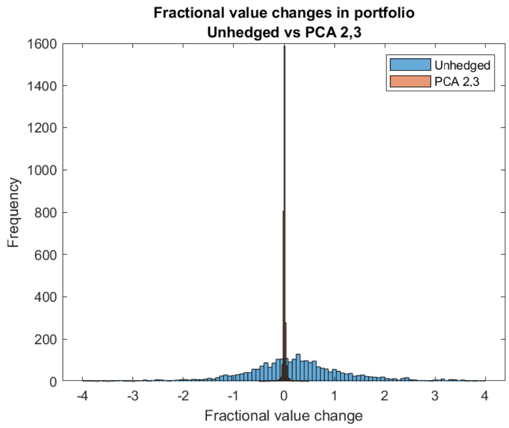


(a) Distribution of fractional value changes.

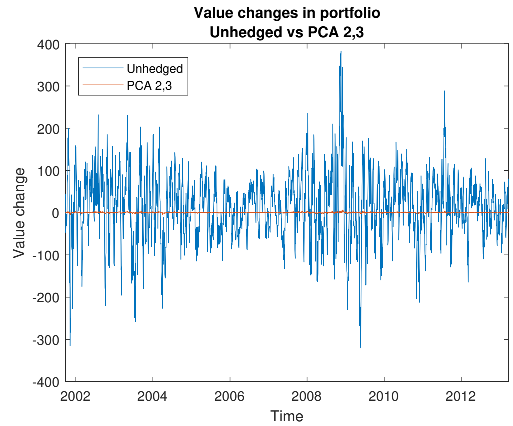


(b) Value change for each iteration.

Figure 29 – Performance of using PCA hedging 2 with 2 components versus unhedged.



(a) Distribution of fractional value changes.



(b) Value change for each iteration.

Figure 30 – Performance of using PCA hedging 2 with 3 components versus unhedged.

The mean, median and standard deviation of the relative value changes are presented in Table 9. Additionally, since the strategies aim to minimise changes in all directions, the absolute values of the changes has also been computed and has its mean and median presented.

Table 11 – Statistical metrics of the relative value changes for each strategy.

Strategy:	Mean	Median	Standard deviation	Absolute mean	Absolute median
Unhedged	0.2566	0.2127	1.2970	0.9208	0.6036
Delta Hedged	0.0232	0.0128	0.3500	0.1390	0.0518
PCA 1,2	-0.0083	-0.0016	0.2306	0.0822	0.0292
PCA 1,3	-0.0023	-5.418e-4	0.1538	0.0418	0.0133
PCA 2,2	0.0084	0.0028	0.1667	0.0390	0.0101
PCA 2,3	0.0080	0.0034	0.1306	0.0238	0.0067

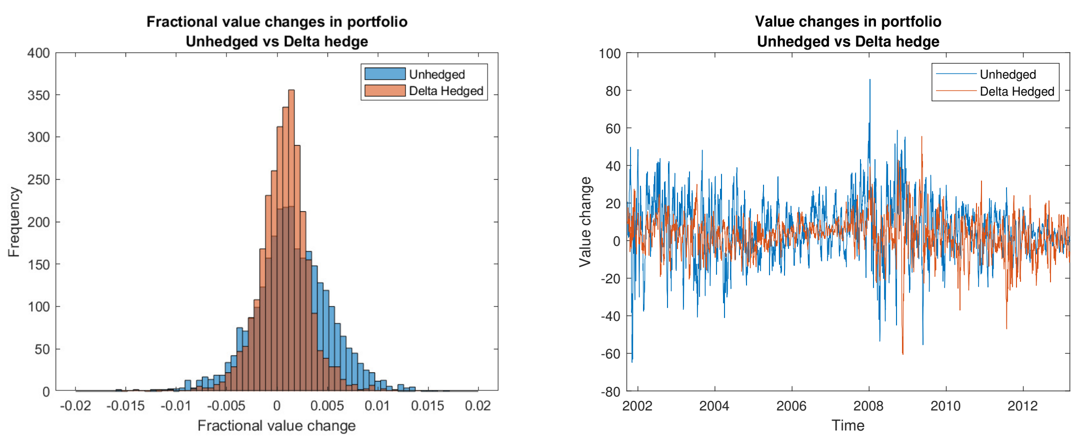
An example of a hedge portfolio for interest rate swaps is illustrated in table 12, where negative amounts indicate fixed rate paying positions.

Table 12 – Example of hedging portfolio for the long maturity strategy on 2008-06-23

Delta Hegde	**Time to Maturity**	**Amount**
Swap 1:	10	-47
PCA 1,2	**Time to Maturity**	**Amount**
Swap 1:	2	-2
Swap 2:	9	-53
PCA 1,3	**Time to Maturity**	**Amount**
Swap 1:	1	0
Swap 2:	5	-7
Swap 3:	9	-48
PCA 2,2	**Time to Maturity**	**Amount**
Swap 1:	4	-5
Swap 2:	9	-48
PCA 2,2	**Time to Maturity**	**Amount**
Swap 1:	1	1
Swap 2:	8	-29
Swap 3:	9	-24

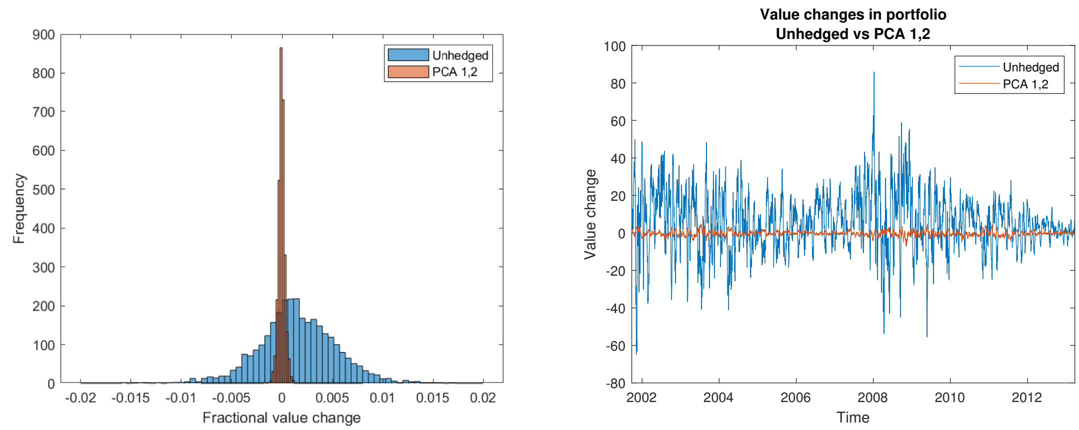
5.3 Bond Portfolio

Testing was performed on a portfolio containing shorter maturity bonds, similar to the short maturity strategy for interest rate swaps.



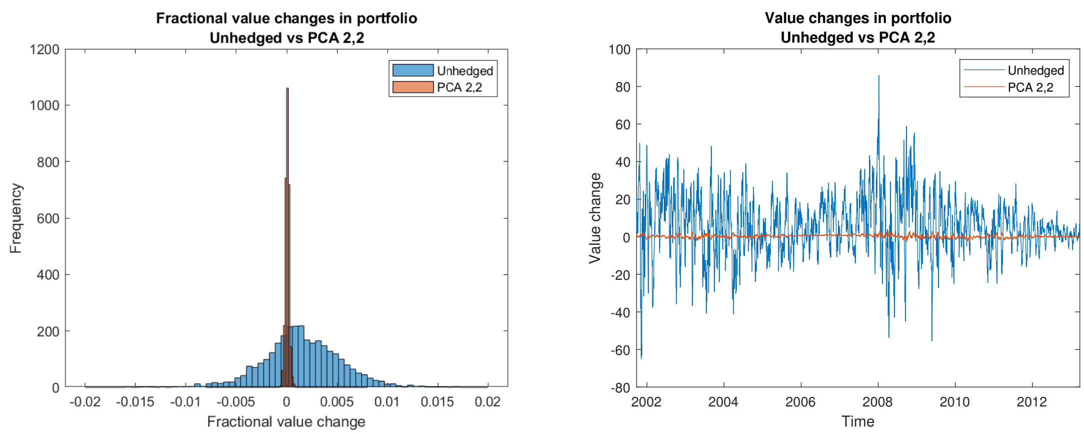
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 31 – Performance of delta hedging versus unhedged.



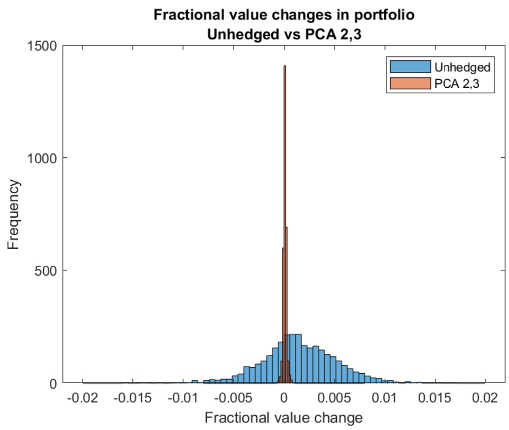
(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 32 – Performance of using PCA hedging 1 with 2 components versus unhedged.

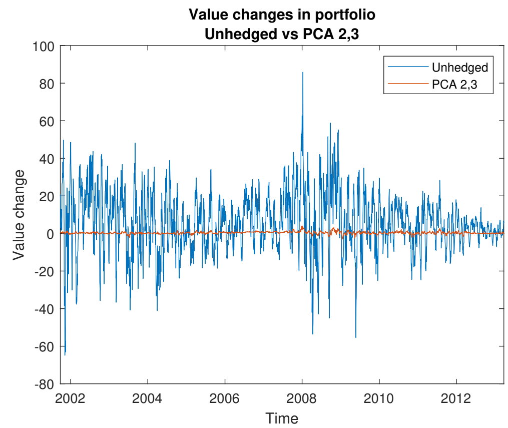


(a) Distribution of fractional value changes. **(b)** Value change for each iteration.

Figure 33 – Performance of using PCA hedging 1 with 3 components versus unhedged.



(a) Distribution of fractional value changes.



(b) Value change for each iteration.

Figure 34 – Performance of using PCA hedging 2 with 2 components versus unhedged.

The mean, median and standard deviation of the relative value changes are presented in Table 13. Additionally, since the strategies aim to minimise changes in all directions, the absolute values of the changes has also been computed and has its mean and median presented.

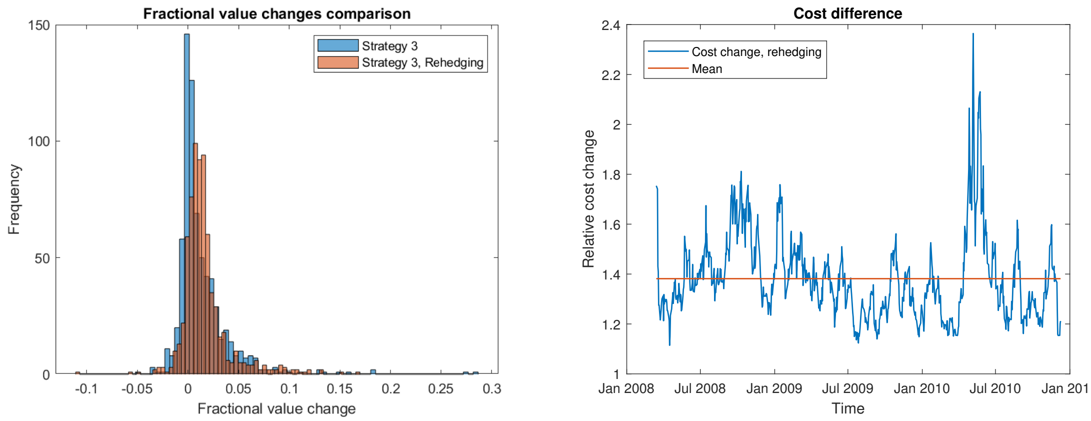
Table 13 – Statistical metrics of the relative value changes for each strategy.

Strategy	**Mean:**	**Median:**	**Standard deviation:**	**Absolute mean:**	**Absolute median:**
Unhedged:	0.0015	0.0013	0.0039	0.0032	0.0026
Delta Hedged:	6.859e-4	8.431e-4	0.0025	0.0020	0.0015
PCA 1,2	-4.735e-5	-4.814e5	3.140e-4	2.394e-4	1.816e-4
PCA 1,3	3.428e-5	3.260e-5	1.516e-4	1.165e-4	9.220e-5
PCA 2,2	6.711e-5	6.422e-5	1.801e-4	1.503e-4	1.305e-4
PCA 2,3	8.799e-5	8.057e-5	1.590e-4	1.344e-4	1.058e-4

5.4 Additional hedging

In addition to the main strategies, some additional hedging techniques were tested. Due to time constraints, these were tested on smaller subsets of the data and focus was put on the equity derivative portfolios. The backtesting interval consists of 715 days, including the most volatile periods in the full data set. To be able to compare the results, strategy 3 was chosen as the hedging strategy to test all additional hedging on.

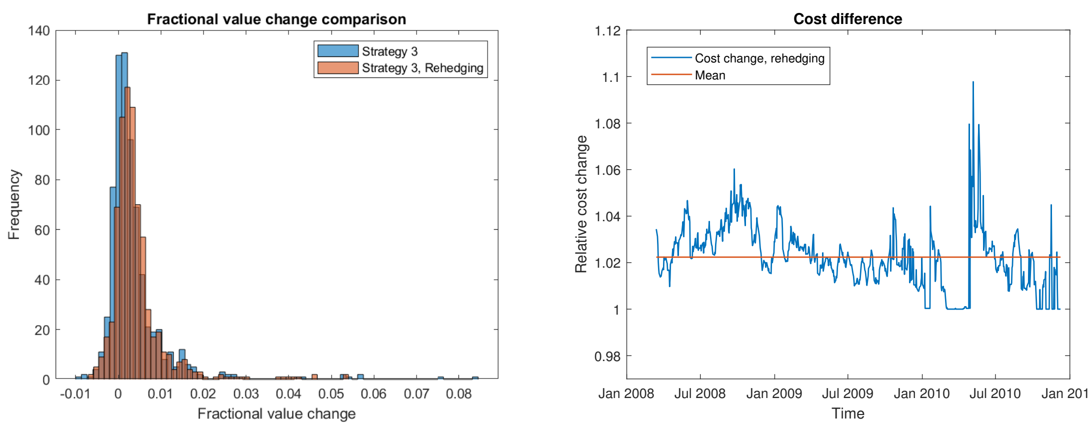
In Figures 35 and 36, the performance and cost changes of strategy 3 employing dynamic rehedging, and compared to static hedging.



(a) Performance difference.

(b) Transaction cost difference.

Figure 35 – Comparison between using dynamic delta rehedging and static hedging for the Bull spread portfolio.



(a) Performance difference.

(b) Transaction cost difference.

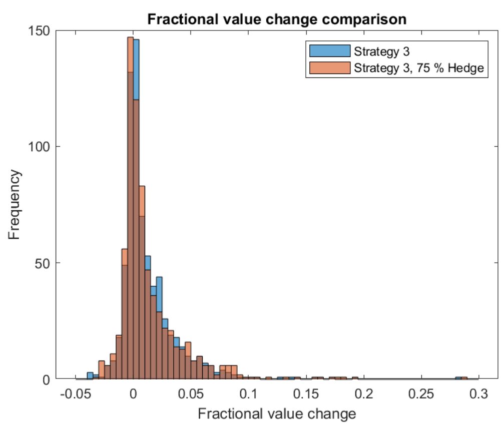
Figure 36 – Comparison between using dynamic delta rehedging and static hedging for the Straddle portfolio.

Statistical metrics for the rehedging testing is presented in Table 14. The cost is presented as a fraction of the original portfolio value, as before.

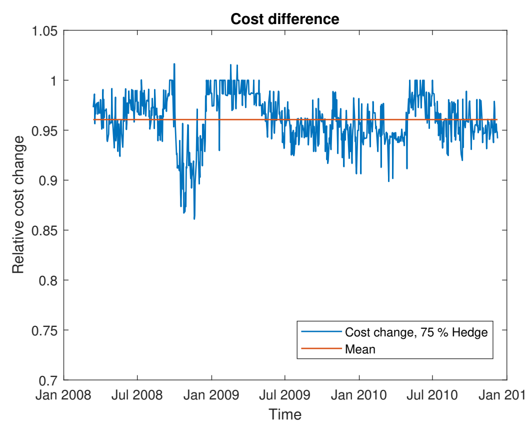
Table 14 – Statistical metrics comparison between static and dynamic hedging.

Portfolio and Hedging	Mean	Median	Standard deviation	Absolute mean	Absolute median	Mean Cost
Bull spread, static:	0.0136	0.0056	0.0284	0.0171	0.0085	0.1041
Bull spread, dynamic:	0.0168	0.0119	0.0244	0.0193	0.0129	0.1410
Straddle, static:	0.0037	0.0019	0.0079	0.0044	0.0023	0.2771
Straddle, dynamic:	0.0043	0.0029	0.0066	0.0047	0.0031	0.2826

In Figures 37, 38, 39 and 40, the performance and cost reduction of strategy 3 with lower levels of risk reduction is presented and compared to full level hedging. Note that the reduced level of hedging is not applied to delta, which is assumed to always be fully hedged.

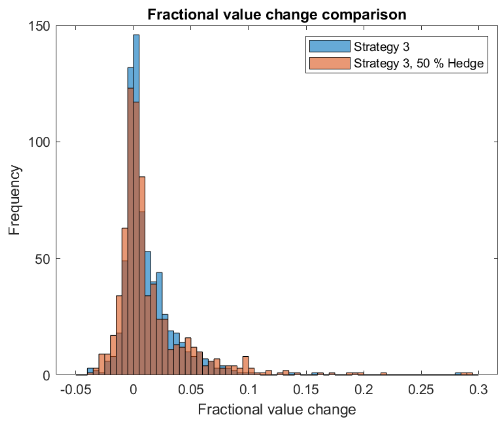


(a) Performance difference.

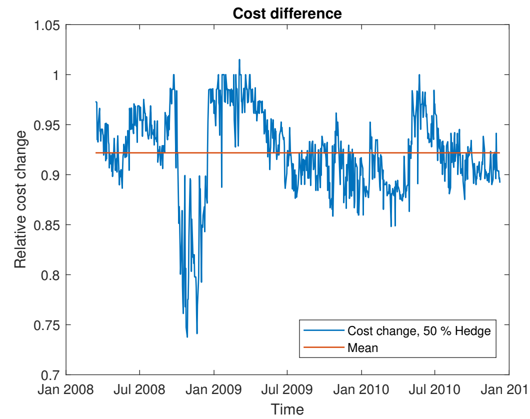


(b) Transaction cost difference.

Figure 37 – Comparison between using hedging ratios of 100 % and 75 % for the Bull spread portfolio.

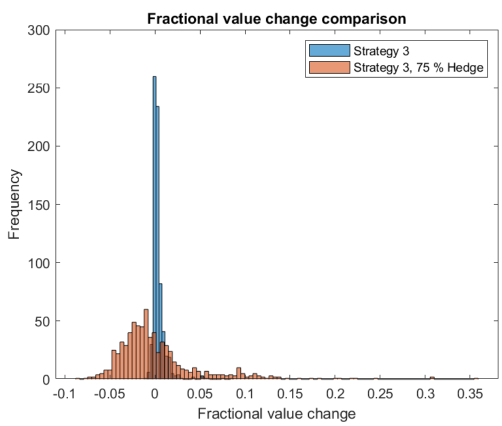


(a) Performance difference.

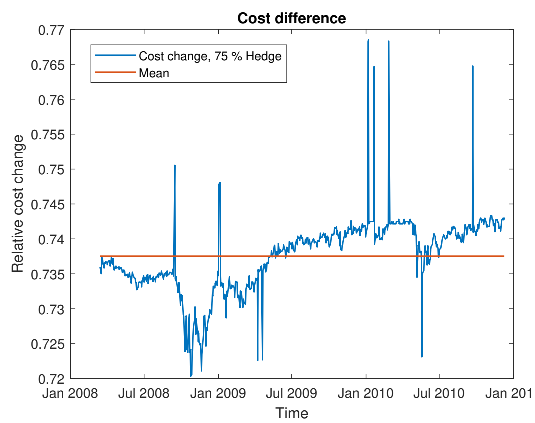


(b) Transaction cost difference.

Figure 38 – Comparison between using hedging ratios of 100 % and 50 % for the Bull spread portfolio.

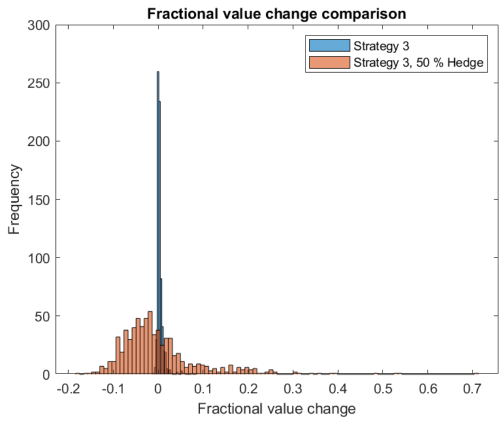


(a) Performance difference.

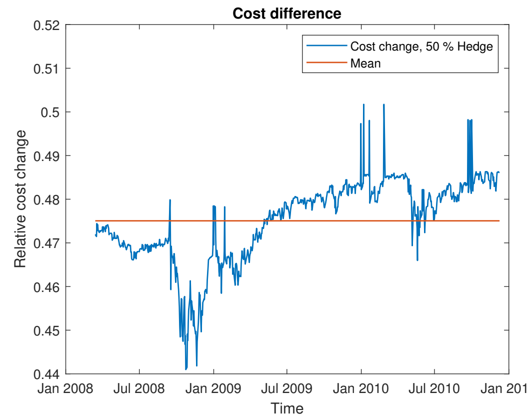


(b) Transaction cost difference.

Figure 39 – Comparison between using hedging ratios of 100 % and 75 % for the Straddle portfolio.



(a) Performance difference.



(b) Transaction cost difference.

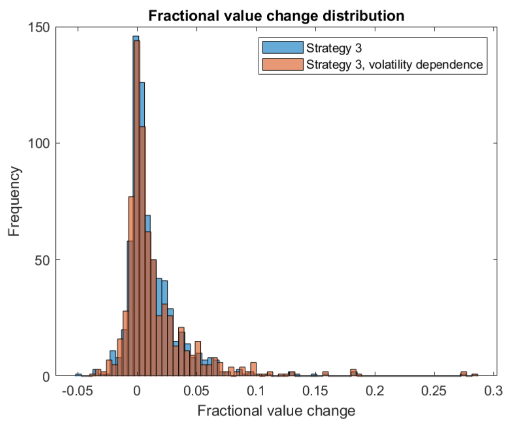
Figure 40 – Comparison between using hedging ratios of 100 % and 50 % for the Straddle portfolio.

Statistical metrics for the rehedging testing is presented in Table 15. The cost is presented as a fraction of the original portfolio value, as before.

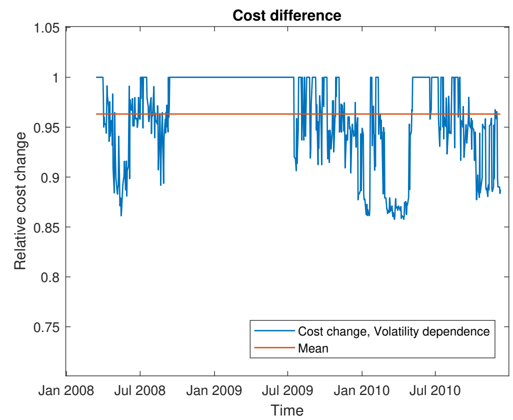
Table 15 – Statistical metrics comparison between different levels of hedging.

Portfolio and Hedging	Mean	Median	Standard deviation	Absolute mean	Absolute median	Mean Cost
Bull spread, 100 %:	0.0136	0.0056	0.0284	0.0171	0.0085	0.1041
Bull spread, 75 %:	0.0144	0.0048	0.0308	0.0182	0.0078	0.0991
Bull spread, 50 %:	0.0152	0.0044	0.0358	0.0205	0.0085	0.0942
Straddle, 100 %:	0.0037	0.0019	0.0079	0.0044	0.0023	0.2771
Straddle, 75 %:	0.0029	-0.0093	0.0493	0.0323	0.0218	0.2047
Straddle, 50 %:	0.0021	-0.0193	0.0944	0.0641	0.0452	0.1322

In Figures 41 and 42, the performance and cost changes of strategy 3 with a volatility dependent hedge level is presented and compared to full level hedging. Note that the reduced level of hedging is not applied to delta, which is assumed to always be fully hedged.

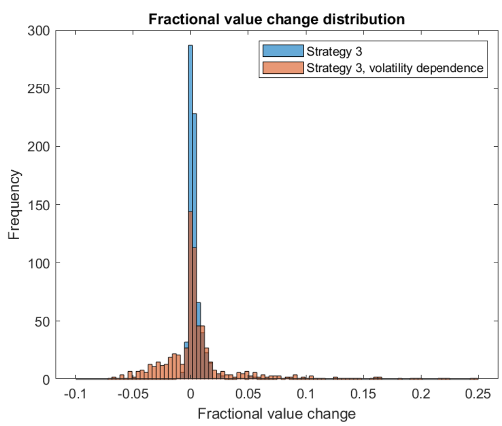


(a) Performance difference.

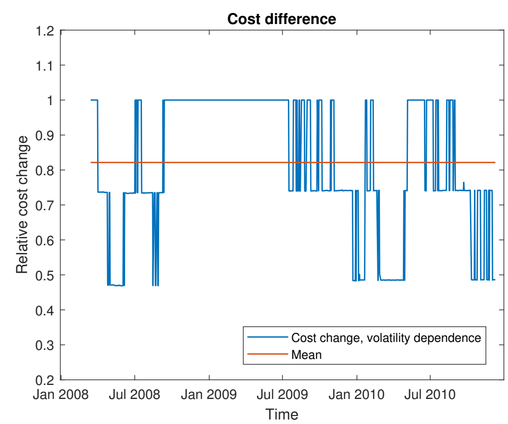


(b) Transaction cost difference.

Figure 41 – Comparison between using a volatility dependent hedging ratio for the Bull spread portfolio.



(a) Performance difference.



(b) Transaction cost difference.

Figure 42 – Comparison between using a volatility dependent hedging ratio for the Straddle portfolio.

Statistical metrics for the rehedging testing is presented in Table 16. The cost is presented as a fraction of the original portfolio value, as before.

Table 16 – Statistical metrics comparison between static and volatility dependent hedge levels.

Portfolio and Hedging	Mean	Median	Standard deviation	Absolute mean	Absolute median	Mean Cost
Bull spread, normal:	0.0136	0.0056	0.0284	0.0171	0.0085	0.1041
Bull spread, volatility dependent:	0.0157	0.0050	0.0336	0.0195	0.0083	0.0988
Straddle, normal:	0.0037	0.0019	0.0079	0.0044	0.0023	0.2771
Straddle, volatility dependent:	0.0107	0.0017	0.0535	0.0242	0.0094	0.2128

6 Discussion

6.1 Results Discussion

The general results from the testing seem to imply that most of the hedging strategies work well, with several having major and consistent improvements. It also clearly showcased the effect the portfolio structure has on the performance of a hedge, as one particular strategy could perform at one level for one portfolio composition, but a significantly different level for a second composition. One example of this is for the equity derivative portfolios, strategy 1 which focused on only hedging the delta of the portfolio performed wildly different on the two portfolios. For the bull spread portfolio, strategy 1 did perform noticeable better than the unhedged portfolio, but when applied on the straddle portfolio, there was little, if any improvement. Looking at the mean value fluctuation, it was more or less the same as the unhedged portfolio. Even worse, looking at the absolute mean and median, they are both larger for strategy 1 than for the unhedged portfolio, which definitively is not a good performance.

One interesting observation is that strategies 2 and 3 performed very similarly for both the bull spread and the straddle. This is probably due to the fact that both vega and gamma are strictly positive for a long option and strictly negative for a short option. Thus any attempt to hedge one of the sensitivities is likely to reduce exposure to the other as well. While probably not a perfect reduction, this can still mean that the step from making a portfolio gamma neutral to making it gamma and vega neutral is not generally very large, which would explain the similar and correlated behaviour of the strategies. As such, the added cost from doing this would most likely be small. Looking at the mean cost differences between the strategies, they cost very similar amounts to employ. For the bull spread, the difference is very small. For the straddle, the mean differs by a few percentage units of the original portfolio value, but is relatively close. However, by looking at the results from Tables 3 and 6, it is clear that the increased value reduction is very small, so it might also not be worth the effort for such a small gain, especially for the straddle.

Strategy 4 is the most thorough, and expensive, strategy but its performance is not flawless. For the bull spread, the strategy works very well. Even when looking at the absolute value of the changes, the mean change is around half a percent, and the standard deviation of the strategy is less than one percent. It noticeably outperforms the other strategies in every metric. However, for the straddle, it performs significantly worse. It is the strategy with the mean the furthest away from zero. While the absolute mean is still significantly better than that of the unhedged portfolio and strategy 1, it holds up poorly compared to strategies 2 and 3. This could potentially be a case of over compensation, where the acquired hedging positions neutralises the risk, but is ill prepared to deal with change during the holding period. It is also interesting to note that strategy 1 displayed a similar unevenness between the test runs, performing well on the bull spread but very poorly on the straddle. Recalling Figures 9 and 10, it may be an effect from the fact that the potential profits is much less restricted for a straddle compared to a bull spread. The bull spread is better at responding to price changes in the underlying asset; either both options are in-the-money (although with negative value for the short option), or they are both out-of-the-money. This can potentially keep the rate of growth somewhat equal which reduces drastic changes. Compare this to the straddle, where one option will always be in-the-money while the other is out-of-the-money. This can potentially result in a higher rate of change compared to the bull spread, which might also be why strategies 1 and 4 show such polarising performances.

However this pattern of worse performance for the straddle is not present for strategies 2 and 3. These strategies instead perform better, if not by much, on the straddle strategy. It is possible that the hedging of vega is the most important factor here, and that keeping the number of different hedging options and their amounts low is more important than hedging every sensitivity, at least when the potential value changes are larger. As discussed earlier, moving from a vega hedge to a gamma and vega hedge does not seem to be a large step, and the recommended hedging positions is likely to be similar.

However, these are essentially moot points when it comes to strategy 4 anyway, as the accrued transaction costs for this strategy are very high, making the strategy very unappealing. For the bull spread, the mean cost of hedging with strategy 4 was roughly half of the original portfolio value, which is a significant amount. It is even worse for the straddle, where the hedging itself regularly costs as much, if not more, than the original portfolio is worth. Combine this with the fact that it is also outperformed by other strategies and it is very hard to motivate such a strategy. The high transaction costs most likely stem from the increased complexity of the system of equations. Already with three sensitivities to hedge through options, the costs are high. Thus it feels unlikely that hedging additional sensitivities than three through this method would ever be a viable alternative.

In the case of the interest rate swaps, it is clear that the principal component hedging has a large effect on minimising the the value changes, for both portfolio compositions. It is interesting that the less advanced principal component strategy (PCA 1) has a lower mean and median compared to PCA 2. However, this is deceptive, as we can see by looking at the absolute mean and median, where the advanced strategy generally performs slightly better. Even so, the rather small difference between the two strategies is somewhat unexpected. However, the results indicating better hedging performance if three principal components is used instead of two, is expected. While it would be easy to assume using even more components could yield even better results, the rate of improvement is likely to be very small, given the variance explained by the principal components is not evenly distributed, see Appendix B. Instead it would probably be more reasonable to consider using only two components.

Unlike the principal component strategies, the delta hedging showed a big discrepancy in effectiveness between the two portfolios. For short maturities, it seems ill suited to properly reduce the risk of rate changes, with an absolute mean change of more than half of the portfolio value and a very high standard deviation. This is in stark contrast to the performance on the late maturity portfolio, where it performs significantly better. The absolute mean change is less than a quarter, and the standard deviation is reduced by a third. It still performs worse than the principal component versions, but here it at least yields a decent reduction. As it is unlikely that a change in zero rates is completely uniform, the delta strategy is outmatched since it is the only strategy with this strict assumption. The performances of the strategies also strengthens this.

In a very small amount of cases, the value of the original portfolio was very close to zero. This meant that any change in value that was larger, but not in any way out of the ordinary compared to other days during testing, was treated as an extreme fluctuation in value, hundreds or thousands of times the original value. While not incorrect, this had huge impacts on the general metrics of a strategy, and made the illustrations unintelligible. After some discussion, it was decided to ignore these values when looking at the relative changes. These had no impact when looking at the direct changes in value however, which is why the direct changes and not the relative changes is presented in the right hand figures. This to properly show the actual size of fluctuations, while still presenting the relative changes in the left hand figures.

For the bonds, one can immediately note the much smaller size of the relative value changes for the bond portfolio, and the strong similarities to in performance to the short maturity swap portfolio. These things both stem from the way the pricing of the bonds work. The swaps are treated as a long position in a fixed rate bond and a short position in a floating rate bond, or vice versa. A pure fixed bond is treated as a fixed rate bond, which is equivalent to the fixed leg of a swap. This is why the behaviour is so similar to a swap portfolio with similar composition. Secondly, the value of the bond is, similarly to a swap, all remaining payments, discounted to a present day value, but unlike a swap, there is no floating leg that counteracts the total value. While changes in the zero rate will have an impact on the value of the bond portfolio, these will be much less impactful than those to a swap portfolio.

Nevertheless, the hedging strategies showcase the same patterns for the bonds as for the swaps, which is expected due to them using the same underlying zero rate for pricing. In this case, it might be more realistic to use even the simplest hedging strategy as the relative value changes in the portfolio will be small. However, so will the gains, and the discussion might be more focused on if it is worth hedging bonds against shifts in the zero rate at all. Additionally, these tests assumed that the notional principal of the hedging swaps and the principal amount of the bonds in the portfolio has been of roughly the same order of magnitude. In reality, this is unlikely to be the case, as the notional principal of a swap is generally much larger than the bond principal. This is because of the nature of vanilla swaps, where the actual payments are on the level of a few percent of the notional principal. If the difference between principals had been more realistic, the main effect would likely have been that the number of hedging swaps acquired would be scaled accordingly, resulting in much fewer new positions. The overall value and effectiveness should not change however, at least not in theory.

In addition to the main testing performed, it was also interesting to look at different variants of the hedging employed. As this was done after the main testing was complete, there was not time to run full period testing on each strategy for this. Instead, strategy 3 was chosen as the foundation to perform these additional tests on, as this strategy had performed generally well during prior testing, has three sensitivities hedged, and did not result in too large transaction costs. The chosen testing period here was noticeably smaller to reduce computational time, but was chosen to contain some of the most volatile periods of the full set, to amplify effects.

Unsurprisingly, rehedging increase the transaction costs of the hedging, and quite significantly in the case of the Bull spread portfolio. It is important to note though that the overall relative costs of hedging was lower for the Bull spread portfolio, allowing for a change to be more noticeable as opposed to the Straddle portfolio. However the dynamic hedging seem to actually increase value changes in the portfolio slightly. This may imply that other risk factors, such as the volatility may be the more impactful factor during the holding period, as dynamic rehedging of delta would do nothing to reduce this risk. Although the rehedging does lower the standard deviation of the value changes, and did lower the impact of some of the worst days in the original hedging. It might still serve a purpose to counteract adverse movements, but the cost might not be worth the potential gain.

On the other end, one idea to reduce the transaction costs of hedging the portfolio, was to only partially hedge an exposure. This was tested for hedging levels of 75 % and 50 %. For the Bull spread, the effects of lowering the hedging levels were not that significant. The performance degraded somewhat, but the transaction costs were lowered. The effects were much more noticeable for the Straddle, where the fluctuations increased by several percentage units, see Table 15. The cost reduction was also much larger here. By looking at Figures 15 and 20, it is clear that the difference in costs between a pure delta hedge and strategy 3 is much larger for the Straddle than the Bull spread. Thus a reduction in hedging of sensitivities other than delta will be more impactful for the Straddle.

6.2 Method Discussion

While several of the chosen strategies yielded significant reduction in value changes, it is important to take into account assumptions and limitations of the underlying models. One large assumption made during the backtesting was that the implied volatility of the options could be approximated using the volatility index. This was necessary due to the extended period the backtesting was performed on, and the lack of historical volatility data, but it is a large simplification. In reality, the volatility is generally not constant for all options on a given day. It tends to vary both depending on the remaining time to maturity of the contract, and also on its moneyness, which is measured as the underlying price divided by the strike price, and is an indicator of its current profitability. In reality, this yields so called volatility surfaces, used to determine the volatility of options. Initially during the thesis, attempts were made to create a custom volatility surface, to use during the backtesting, using market data. This was successful, but after some discussion it was decided that this would completely remove the historical variations in volatility, and the volatility index was instead elected as a proxy. While this allows for more interesting volatile periods to manifest during the testing, it is nevertheless a big assumption. Ideally, one would have access to daily volatility surfaces over long time periods, but unlike for example stock data, historical volatility is not widely available.

Similar, but less impactful assumptions were made regarding other historical data. The risk free rate was approximated using the interest rate on US treasury bonds, which is usually considered risk free. An alternative here could have been to use the LIBOR, but these are limited to maturities of one year. Thus, it was decided against using these as large parts of the curve would have to be extrapolated. For the interest rate derivatives, the zero rates had to be bootstrapped from the swap rates. An alternative could have been to instead calculate the forward rates from LIBOR, but initial implementations of this method yielded unsatisfactory results.

For the backtesting, it was assumed that a number of hedging instruments with similar parameters to the portfolio instruments were available. This assumption was made in order to allow for some variability in the hedging instruments, and to allow for an additional problem dimension in choosing the optimal combination. It was further assumed that hedging contracts could be immediately acquired or shorted, which was deemed acceptable as the holding period was short, and the holder would be working to as quickly as possible hedge the defaulted portfolio.

6.3 Conclusions

Based on the findings, the best performing hedging strategies for equity derivatives are strategies 2 and 3. Strategy 1 is too sensitive with respect to the behaviour of other risk factors to be considered reliable. It can work under the right circumstances, but is probably too simple in the general case. Oppositely, strategy 4 generally performed well, but was in most cases far too expensive to be considered viable. With costs reaching or exceeding the original portfolio value, there are very few situations where this could be acceptable. Between these, strategies 2 and 3 performed well without being too expensive. The acceptable cost is of course up to the owner and may differ from case to case but generally, these strategies were not outlandishly costly. And if the holder is prepared to risk some fluctuations, the hedging could be applied with a lower level of reduction than full neutralisation.

For the interest rate derivatives, either of the principal component strategies provide good defence against zero rate changes. Due to the more involved sensitivity calculations in strategy 2, this is probably the safer bet. As there are no immediate transaction costs involved with entering into swaps, the owner can allow for proper hedging of the exposures until the positions are closed out or auctioned off. Although if the portfolio consists mainly of bonds, the owner must decide if hedging through swap contracts is worthwhile.

Overall, the testing showcases several possible hedging strategies which results in reduced risks of drastic value changes during the holding period. However it is important to remember that the testing was performed on only a few specific portfolio structures. Given larger and/or more advanced portfolios, one should not assume that the strategies will perform at the same level and additional testing might be needed to reliably expand the results to larger scenarios. Furthermore, with the assumptions made during the testing, one should be careful not to rely too heavily on the results. There are several layers to the hedging problem that are simplified away through this, but which might be present in a real scenario.

Nevertheless, these strategies can serve as a good basis to expand further testing from, and supports the general idea that short term hedging over a holding period can be worthwhile and not too expensive. By committing to a hedging strategy, a CCP can ensure the losses from a defaulted portfolio is kept from growing too large, giving some breathing room during a no doubt stressful default situation. It also reduces the risk of non-defaulting members having to contribute to the default fund which is likely to be appreciated by clearing members.

6.4 Future Studies

With the groundwork laid out by this thesis, suggestions for future studies for this scenario would be to expand the testing scenario or to focus on enhancing specific parts of the testing to obtain more accurate and reliable results. The portfolio was limited to four different derivatives in this thesis, but there are many more potential instruments to analyse. By extending the possible portfolio instruments, the decision on how to hedge will be more informed. Secondly, as mentioned before, the testing made assumptions that prevents totally realistic and reliable results. By further working with the implementation of pricing, hedging and testing, and adding additional factors to the process, the results can be further refined. Examples of this would be too look at correlation between underlying assets and perhaps different but similar instruments, to potentially reduce the dimension of the hedging problem and to implement more advanced versions of transaction costs, to more accurately replicate the real life situations.

Future studies can also be done on improving and more thoroughly testing the hedging variants proposed for the portfolios. While there were discernible patterns in the effects these had, additional testing should be performed on additional portfolio compositions and with more data before drawing any major conclusions. The findings in this study can serve as a starting point for further improvements.

6.5 Resulting Implementation

The initial implementation and testing was conducted in MATLAB, but as a result from the study, a function that can perform the valuing and risk calculations in order to recommend a suitable hedging position was written. This *Hedge Recommendation Function* was written in Java, and is designed to be used together with the existing code library.

The function is constructed in such a way that a user can specify a portfolio, containing one or several types of instruments, to be analysed and hedged. Necessary parameters such as the risk free rate, or the swap rates, must be supplied by the user. The function will then analyse the risk factors in the portfolio and, for a given level of risk aversion, the function will return suggested positions in some hedging instruments.

The decision which strategies were implemented in the function was based on the performance of the strategies during the backtesting. For equity derivative portfolios, strategies 2, 3 and 4 were implemented; however, as discussed earlier, strategy 4 is generally very expensive and does not necessarily provide better risk reduction. It is available as an option, but the main strategies recommended are strategies 2 and 3. For interest rate portfolios, principal component strategy 2 was implemented. While both principal component strategies provided similar levels of risk reduction, the more thorough strategy 2 generally performed slightly better. It can be used with both 2 and 3 principal components, specified by the user.

Since the function does not perform backtesting, it is much quicker than the MATLAB testing code. As such, the number of combinations of hedging instruments available has been increased significantly, to allow for even more possible solutions to be analysed. This can however easily be tuned if so desired, to better reflect the desired availability of hedging positions.

Similarly, the level of proportional transaction costs, if any, can easily be set to the desired level. This is only available for equity hedging derivatives, and not for swap contracts used for hedging.

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A Mathematical concepts

A.1 Normal distribution

The normal distribution is a statistical distribution with parameters μ and σ^2, representing the mean and variance. The probability density function (pdf) of a normal distribution is

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \tag{52}$$

and the cumulative density function (cdf) of a normal distribution is

$$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy, \tag{53}$$

A.2 Deriving the Black-Scholes option pricing formula

To derive the Black-Scholes equation, and the pricing formulas for options, the following assumptions are made [15].

1. Short selling of securities is allowed and instant.
2. There are no transaction costs for any party involved.
3. No arbitrage opportunities exists.
4. The price of the underlying asset follows a stochastic process, with constant drift and volatility.
5. The underlying asset pays no dividends.
6. The option can only be exercised at the time of maturity.
7. The risk-free interest rate is constant w.r.t both time and maturiy date.

The following method is presented in the book by Hull [6]. This method uses risk-neutral valuation to derive the formulas. Assume the underlying asset price S is lognormally distributed, and that the expectation can be defined as

$$E[max(S-K,0)] = \int_K^{\infty} (S-K)\phi(S)dS, \tag{54}$$

where $\phi(s)$ is the probability density function of S. Using that the mean of $ln(S)$ is

$$\text{mean}(ln(S)) = ln(E[S]) - \frac{v}{2} = M, \tag{55}$$

where v is the standard deviation of $ln(S)$. One can define a new, standard normal variable

$$Q = \frac{ln(S) - M}{v}. \tag{56}$$

Using the probability density function of the normal distribution to change the integral in equation 54 and dividing it into two parts, one obtains

$$E[max(S-K,0)] = \int_{(ln(K)-M)/v}^{\infty} e^{Qv+M}\phi(Q)dQ - K\int_{(ln(K)-M)/v}^{\infty} \phi(Q)dQ. \tag{57}$$

Expanding the probability density function $\phi(Q)$ and rewriting the terms allows us to rewrite equation 57 into

$$E[max(S-K,0)] = e^{M+v^2/2}\int_{(ln(K)-M)/v}^{\infty} \phi(Q-v)dQ - K\int_{(ln(K)-M)/v}^{\infty} \phi(Q)dQ. \tag{58}$$

The two integrals both corresponds to the two terms $\Phi(\cdot)$ in equation 9. Substituting back M yields

$$\Phi\Big(\frac{ln(E[S]/K) + v^2/2}{v}\Big) = \Phi(d_1), \tag{59}$$

And similarly for $\Phi(d_2)$. We end up with

$$E[max(S-K,0)] = E[S]\Phi(d_1) - K\Phi(d_2). \tag{60}$$

To obtain the pricing formula for a European call option, we substitute the appropriate factors into equation 60 and we get equation 9. Using the put-call parity, the corresponding formula for a put option can be found. To read more, see for example [16], [17], [19], [20].

A.3 Deriving the Delta of a European Call Option

The derivation of delta for a European call option is shown below, and the other Greeks for European options can be derived similarly.

Starting from equations 9, 10 and 11, we take the derivative with respect to the underlying S. This yields

$$\frac{\partial C}{\partial S} = \Phi(d_1) + S\frac{\partial \Phi(d1)}{\partial S} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial S}.$$

Using the chain rule and the fact that the derivative of the cumulative density function is the probability density function, we can write this as

$$\begin{aligned}\frac{\partial C}{\partial S} &= \Phi(d_1) + S\frac{\partial \Phi(d1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-rT}\frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} \\ &= \Phi(d_1) + S\phi(d_1)\frac{\partial d_1}{\partial S} - Ke^{-rT}\phi(d_2)\frac{\partial d_2}{\partial S}.\end{aligned}$$

Looking at equations 10 and 11, we can see that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}}.$$

From this we can simplify our expression for delta into

$$\begin{aligned}\frac{\partial C}{\partial S} &= \Phi(d_1) + S\phi(d_1)\frac{1}{S\sigma\sqrt{T}} - Ke^{-rT}\phi(d_2)\frac{1}{S\sigma\sqrt{T}} \\ &= \Phi(d_1) + \frac{1}{S\sigma\sqrt{T}}\Big(S\phi(d_1) - Ke^{-rT}\phi(d_2)\Big).\end{aligned} \tag{61}$$

Using the fact that $d_2 = d_1 - \sigma\sqrt{T}$, we will now show that the subtraction in the parenthesis equals zero. We start by looking at the probability density function for a standard normal variable with the above substitution.

$$
\begin{aligned}
\phi(d_2) &= \phi(d_1 - \sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}} e^{\dfrac{-(d_1 - \sigma\sqrt{T})^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{\dfrac{-d_2^2}{2}} e^{\dfrac{d_1\sigma\sqrt{T} - \sigma^2 T}{2}} \\
&= \phi(d_1) e^{d_1\sigma\sqrt{T} - \dfrac{\sigma^2 T}{2}}.
\end{aligned}
$$

Substituting in the full expression for d_1 from equation 10 into this, we obtain

$$
\begin{aligned}
\phi(d_1 - \sigma\sqrt{T}) &= \phi(d_1)\exp\left(ln(S/K) + (r + \frac{\sigma^2}{2})T - \frac{\sigma^2 T}{2} \right). \\
&= \phi(d_1)\frac{S}{K}e^{rT}
\end{aligned}
$$

Going back to equation 61, we have

$$
\frac{\partial C}{\partial S} = \Phi(d_1) + \frac{1}{S\sigma\sqrt{T}}\left(S\phi(d_1) - Ke^{-rT}\phi(d_1)\frac{S}{K}e^{rT} \right). \tag{62}
$$

Simplifying this, we finally obtain

$$
\frac{\partial C}{\partial S} = \Phi(d_1) + \frac{1}{S\sigma\sqrt{T}}\left(S\phi(d_1) - S\phi(d_1) \right) = \Phi(d_1), \tag{63}
$$

which is the analytical expression for delta of a European call option. The delta for a European put option can be found following the same procedure, starting with equation 12. To conclude, we have that

$$
\Delta_{call} = \Phi(d_1), \tag{64}
$$

$$\Delta_{put} = \Phi(d_1) - 1. \tag{65}$$

The other Greeks can be derived using the same general procedure.

B Principal Component Data

The principal components for the swap rates can be found in table 17.

Table 17 – The principal components for the swap rates.

Maturity:	**Principal Component 1**	**Principal Component 2**	**Principal Component 3**
1 year:	0.4100	0.6376	0.5402
2 year:	0.4091	0.3246	-0.1641
3 year:	0.4009	0.0732	-0.4128
4 year:	0.3869	-0.1053	-0.3718
5 year:	0.3712	-0.2341	-0.2027
7 year:	0.3426	-0.3931	0.1652
10 year:	0.3147	-0.5122	0.5516

The variance explained and the fraction of total variance explained by each principal component is presented in table 18.

Table 18 – Variance explained and total fraction of variance explained.

Principal component	**Variance**	**Fraction of variance explained**
1:	15.48535	0.95384
2:	0.70980	0.04372
3:	0.03654	0.00225
4:	0.00286	0.00018
5:	0.00021	0.00001
6:	0.00004	0
7:	0	0

The interpolated principal components for semiannual maturities of up to 10 years can be found in table 19.

Table 19 – Interpolated principal components for all used maturities.

Maturity:	Principal Component 1	Principal Component 2	Principal Component 3
0.5 year:	0.4104	0.7940	0.8923
1 year:	0.4100	0.6376	0.5402
1.5 year:	0.4095	0.4811	0.1881
2 year:	0.4091	0.3246	-0.1641
2.5 year:	0.4050	0.1989	-0.2884
3 year:	0.4009	0.0732	-0.4128
3.5 year:	0.3939	-0.0160	-0.3923
4 year:	0.3869	-0.1053	-0.3718
4.5 year:	0.3791	-0.1697	-0.2872
5 year:	0.3712	-0.2341	-0.2027
5.5 year:	0.3641	-0.2738	-0.1107
6 year:	0.3569	-0.3136	-0.0187
6.5 year:	0.3498	-0.3533	0.0732
7 year:	0.3426	-0.3931	0.1652
7.5 year:	0.3380	-0.4130	0.2296
8 year:	0.3333	-0.4328	0.2940
8.5 year:	0.3287	-0.4527	0.3584
9 year:	0.3240	-0.4725	0.4228
9.5 year:	0.3194	-0.4924	0.4872
10 year:	0.3147	-0.5122	0.5516