A Model of the Yield Curve with Time-varying Interest Rate Targets[1]

Ganlin Chang
1-212-526-5554
gchang@lehman.com

Vasant Naik
44-20-7260-2813
vnaik@lehman.com

In this article we present a fundamentals-based model of the yield curve. The fundamental variables that drive the yield curve in our model are: the short rate, the near-term target of this rate set by the central bank and the longer-term natural rate of interest. Our model relates these variables explicitly to the term structure of interest rates. It also characterizes how the parameters governing the dynamics of these variables (in particular, the speeds of their mean reversion and their volatilities) influence the curve. The model accounts for a time-varying premium for bearing interest rate risk and can be used to separate the effects of interest rate expectations and risk premia on the shape of the yield curve. We present several examples of how such a model can help in making duration and curve positioning decisions.

1. INTRODUCTION

Most bond market participants would agree that monetary policy and the interaction of monetary policy with macro-economic variables such as inflation and growth are the fundamental determinants of the yield curve at any given time. The main policy instrument of most central banks, including the Federal Reserve Board and the European Central Bank, is the short-term interest rate. Thus, a central bank's policy typically involves setting a near-term target for short-term interest rate in light of the prevailing and expected inflation and growth conditions and the actual short rate is gradually brought near the target by open market operations. In this article we describe a fundamentals-based model of the yield curve that uses such behaviour of central banks as the key determinant of the term structure of interest rates. Our objective is to build a rigorous yet realistic framework that is useful for the analysis and the determination of the optimal duration and curve positioning of bond and swap portfolios. A class of such models has been in use at Lehman Brothers as tools in the determination of tactical and strategic views about the yield curve in global government and swap markets. We call these models Economic Factor Models of the yield curve (EFM). In this article we detail a representative example from this family.

The yield curve model that we present is a structural model of the term structure of interest rates. In other words, the yield curve and its evolution over time are derived from assumptions about economic fundamentals. The fundamental variables that are taken to be the determinants of the curve are: (i) the overnight interest rate on default-free borrowing (hereafter, the short rate), (ii) a near-term target set by the central bank for this rate, (iii) the longer-term natural rate of interest and (iv) the interest rate risk premium. We do not model expected real growth and expected inflation explicitly but instead capture the effect of these variables on the yield curve via their effect on the near-term target of the short rate and the

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longer-term natural rate. In addition, the parameters that govern the convergence of rates to their near- and longer-term expectations and the magnitude of interest rate risk also influence the shape of the yield curve. Given economically reasonable assumptions about these fundamental variables and parameters, our model provides the values of default-free bonds of various maturities at any time and how these change over time. From this, the spot and forward rate curve at any time and their temporal behaviour can be determined. Furthermore, the model can also be used to value derivatives whose payoffs may be contingent on the yield curve.

A structural model of the yield curve such as the one described here is a tool that is needed to answer what may be some of the most basic questions facing fixed-income portfolio managers. One such question, for example, is how much of the slope of the yield curve can be attributed to rate expectations and how much to risk premia. If the curve is upward sloping simply because rates are expected to rise in the future then there is no risk premium built into the curve and no reward for taking the risk. The reason why our model can be used to estimate risk premia embedded in various parts of the curve is that it explicitly and separately accounts for expected changes in rates and the compensation of bearing risk. This is in contrast to purely statistical models of the curve (say those based on Principal Components Analysis or regressions of yield on a set of explanatory variables) which provide a description of *changes* in yields but have little to say about how the *level* of yields is determined by fundamental economic variables.

Our model can also be a useful tool in scenario analysis. A portfolio manager may, for example, wish to assess how the value of his portfolio will change for a given change in the driving variables of the model (the short rate, the near-term target or the natural rate). Also present in the model are parameters that govern the magnitude of interest rate risk, the price of risk and the speeds of mean reversion of the driving variables to their equilibrium values. Thus, one could analyze the impact of a sudden shift in these parameters on portfolio values. Such scenario analysis is helpful in identifying the macro-economic risks inherent in different portfolio positions.

This article is organized as follows. In Section 2, we discuss the main features of monetary policy in major industrialized economies in recent years. This helps to identify the macro-economic variables that drive short and long horizon changes in interest rates. Section 3 shows how the most important of these features are incorporated parsimoniously in the EFM. Section 4 presents the basic properties of the yield curve that EFM can generate. Section 5 presents a number of applications of the model and Section 6 provides a concluding summary.

2. MONETARY POLICY AS A FUNDAMENTAL DETERMINANT OF THE YIELD CURVE

There is substantial empirical evidence to suggest that the behaviour of the yield curve is influenced strongly by how central banks conduct monetary policy, and expectations of how they will do so in the future. The expected future path of the central bank target rates combined with a risk premium for holding interest-rate sensitive securities are the most important considerations in pricing government bonds. Thus, a reasonable model of the yield curve ought to capture the salient features of how monetary policy is conducted. We show later in this article how our model captures these features. However, in order to motivate the detailed specifications that we use, we review in this section the conduct of monetary policy in industrialized economies in the recent years.

The long-term objective of the monetary policy of world's major central banks can be reasonably stated as long-term price stability (i.e. low, positive and a steady level inflation). It is also reasonable to assume that the goal of price stability is pursued together with a desire to ensure that the economic activity takes place at a level that is close to its full potential. Even if central bank objectives are stated expressly in terms of price stability, in reality they are likely to include a consideration for real growth simply because growth and inflation shocks are usually positively correlated. This means that the central banks would typically weigh inflation as well unemployment and economic growth in setting their monetary policy. The relative weights on these two potentially conflicting objectives could vary, depending on, for example, the stated objectives of the central bank and how independent the central bank is from the national executive body.

The key implication of the above objectives of monetary policy for term-structure modelling is that the term structure of interest rates ought to reflect inflationary and real-growth expectations. As market participants anticipate how the central banks will respond currently and in the future to the state of real growth and inflationary conditions, these anticipations will be reflected in shape of the yield curve. Any changes in these expectations will be important determinants of the changes in the yield curve over time. Moreover, to the extent that inflationary and real-growth conditions do not necessarily move together at lower frequencies, a two-factor dynamic is expected to be present in the movements of the yield curve at medium to long end.

2.1 Taylor Rules and Interest Rate Dynamics

The main policy instrument that central banks in major industrialized countries have come to use in the implementation of their objectives is the targeting of the short-term interest rate. A typical mechanism for the conduct of monetary policy seems to be first to set a target for the short-term interest rate that is thought to be consistent with the objectives of low and stable inflation and sustainable real growth. Then, open market operations are carried out to ensure that actual short-term rate is close to the target rate. In order to capture this behaviour, macroeconomists have proposed and estimated simple policy rules that relate the short rate to inflation and output growth. Clarida, Gali and Gertler (1999), for example, postulate that the short rate target, $R(t)$ is a weighted average of expected inflation and expected output gap. They suggest that this target could be modelled as:

$$R(t) = a + bE_t\pi_{t+k} + cE_t x_{t+q} \tag{1}$$

where $E_t\pi_{t+k}$ is the expected inflation (or expected change in the price level) between t and $t+k$, $E_t x_{t+q}$ denotes the expected output gap at time $t+q$ and the constants k and q denote the time horizons of the central bank in targeting inflation and output. The actual short rate $r(t)$ is then modeled as adjusting to its target gradually as follows:

$$r(t+1) = \rho r(t) + (1-\rho)R(t) + \nu\varepsilon(t+1) \tag{2}$$

where $R(t)$ denotes the target rate at time t, ρ is a parameter governing the smoothness with which the short rate converges to its target, $\varepsilon(t+1)$ is a random disturbance term and ν is the volatility of short-run changes in the short rate. The empirical evidence reported by Clarida, Gali and Gertler (1999) suggests that the above model for short rate evolution is broadly consistent with the data from the U.S., Germany, and Japan.

Monetary policy rule implied by (1) is known as a forward-looking Taylor rule. Taylor (1993) originally proposed monetary policy rules of this type which were backward looking as they related the interest rate target to past values of output gap and inflation. Forward-looking rules instead are specified in terms of expectations of the future values of these variables. Dicks (1998) shows that such forward-looking Taylor rules can be used successfully to forecast short-term interest rates in the U.S., Germany, and Japan.

Once we accept the above characterization of the objectives of monetary policy and the mechanism for its implementation, it follows that a reasonable model of the dynamics of short-term interest rates would involve two ingredients. First, it would allow a time-varying interest rate target driven by at least two factors. Second, it would postulate a short-term rate that tracks its target closely but with a lag. In the next section we describe how a yield curve model can be constructed a way that is consistent with Taylor-type rules for monetary policy. In addition to specifying dynamics of the short rate and its target, however, we show that we also need to specify a reasonable model of the price of interest rate risk.

In the specification of EFM we assume that the short rate is mean reverting, where the mean (or the interest rate target) to which it reverts also varies over time. A simple mathematical representation of this idea is constructed by assuming that the change in short rate over a small time-interval of length dt is given by:

$$dr(t) = \kappa_r \left[R(t) - r(t)\right]dt + \sigma_r dz_r(t) \tag{3}$$

where κ_r and σ are positive constants and $dz_r(t)$ is a random shock with mean 0 and variance dt. (Formally, it is the increment of a standard Brownian motion.) The parameter κ_r measures the speed of mean reversion in the short rate to its target $R(t)$ and σ measures the volatility of changes in the short rate. A simple interpretation of the mean reversion parameter κ_r is obtained by considering the time it takes a deviation of the short rate from its mean, $R(t) - r(t)$ to be halved in a world with constant target rate ($R(t) \equiv c$ where c is a constant) and with zero volatility ($\sigma_r = 0$). This is inversely related to κ_r and is given by $\frac{\ln 2}{\kappa_r}$ years.

2.2 Modelling the Interest Rate Target

While Taylor rules are concerned only with how interest rate targets are set and how the actual short rate evolves through time relative to its target, this is but a starting point for the EFM. The ultimate objective of the EFM is to determine the level and dynamics of interest rates of all maturities. For this, we need to specify a model of how the interest rate target itself changes over time.

In the modelling of the interest rate target, we assume that the evolution of the target is described by two state variables: one representing the effect of shorter-term cyclical forces on the target, and the second capturing the effect of longer-term secular forces. Accordingly, we assume that the target is also mean reverting around a longer-term natural level of the short rate. That is,

$$dR(t) = \kappa_R (L(t) - R(t))dt + \sigma_R dz_R(t) \tag{4}$$

$$dL(t) = \kappa_L (L(\infty) - L(t))dt + \sigma_L dz_L(t) \tag{5}$$

In the above, κ_R and κ_L denote the speeds of mean reversions of the rate target, $R(t)$ and of the time-varying mean $L(t)$ to which the target reverts. Thus, $R(t)$ can be interpreted as the near-term target of the central bank and $L(t)$ as the longer-term natural level of the short rate. The effect of short-term cyclical forces is likely to show up in the movement of $R(t)$ around $L(t)$ while the effect of longer-term secular forces is likely to appear in the movement of $L(t)$ around $L(\infty)$ which is a constant. As before, σ_R and σ_L denote the annualized volatilities of $R(t)$ and $L(t)$ respectively [2].

We note here for future reference that we will refer to the dynamics assumed for the driving variables in equations (3), (4) and (5) as their "natural" dynamics in the following discussion.

2.3 Modelling the Price of Interest Rate Risk

So far we have specified a model for the evolution of the short-term interest rate. If there were no risk premium for holding long-dated bonds, a model for the dynamics of short rates would be sufficient for valuation of default-free bonds for any maturity. This is because without a risk premium, expected return over a short horizon on a bond of any maturity would simply be the short rate. In reality, however, there is a risk premium embedded in default-free bond prices, and a yield curve model needs to account for this in a reasonable way.

Figure 1 provides some simple statistics that establish the existence of interest rate risk premia. Here we present the average realized excess returns (over the short rate)[3] on different maturity segments of Lehman Brothers Treasury Indices for the US and the German markets. Invariably, the average excess returns are positive and they increase with maturity. Furthermore, it is easy to argue that the risk premium embedded in default-free bonds would not only be positive but also be time varying. One expects, for example, the risk premia embedded in bonds to decrease (bonds to become more expensive) during periods of economic and financial crises- the so-called "flights-to-quality" effect. At the opposite end, these risk premia would be expected to increase during times of economic expansions and during periods when risk-bearing propensity in the market place is high. There is also considerable empirical evidence that strongly rejects the assumption of constant risk premia. For example, Fama and Bliss (1987) show that the spread between forward rates and one-year rates can predict future excess bond return, a pattern that cannot be explained if risk premia were constant.

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[2] *One also has the flexibility of modelling correlated movements in $r(t), R(t)$ and $L(t)$ in the EFM. This is done by allowing $dz_r(t), dz_R(t)$, and $dz_L(t)$ to be correlated. We set $corr[dz_r(t), dz_R(t)]$ equal to $\rho_{rR}dt$, $corr[dz_r(t), dz_L(t)]$ equal to $\rho_{rL}dt$, and $corr[dz_R(t), dz_L(t)]$ equal to $\rho_{RL}dt$. For $a = r, R, L$, $\rho_{aa} \equiv 1$ and for $a = r, R, L$ and $b = r, R, L$, $-1 \le \rho_{ab} \le 1$.*

[3] *The short rate is taken to be the fed funds target rate for the US and one month Libor for Germany.*

Figure 1. Average Monthly Excess Return (bp per month) on Lehman Brothers Treasury Indices (Jan 1994 – Sept 2002)

Maturity Bucket	US	Germany
1-3	10	12
3-5	18	19
5-7	22	25
7-10	24	29
10+	33	36

Interest rate risk premia are incorporated in the EFM by specifying the "risk-adjusted" dynamics of the driving variables, $r(t), R(t)$ and $L(t)$. The risk-adjusted dynamics are similar in form to the natural dynamics of these variables but can be governed by different parameters. Bond prices are then set equal to their discounted expected payoffs (with the short rate as the discount rate) assuming that the evolution of the driving variables is governed by the "risk-adjusted" parameters. Note that this implies that if the "risk adjusted" parameters were the same as the "natural" parameters then we would be back to the case of zero risk premium.

The assumption made in EFM about the "risk-adjusted" dynamics of $r(t), R(t)$ and $L(t)$ are the following:

$$dr(t) = \kappa_r^Q \left[R(t) - r(t)\right]dt + \sigma_r dz_r^Q(t) \tag{6}$$

$$dR(t) = \kappa_R^Q \left[L(t) - R(t)\right]dt + \sigma_R dz_R^Q(t) \tag{7}$$

$$dL(t) = \kappa_L^Q \left[L^Q(\infty) - L(t)\right]dt + \sigma_L dz_L^Q(t) \tag{8}$$

where $\kappa_r^Q, \kappa_R^Q, \kappa_L^Q$ and $L^Q(\infty)$ are given parameters and the superscript Q is used to highlight the fact that these are "risk-adjusted" parameters and can be different from the "natural" parameters that govern (3), (4) and (5).[4]

By an application of standard asset pricing theory, the natural dynamics ((3), (4) and (5)) and the "risk-adjusted" dynamics ((6), (7) and (8)) can be used to compute the risk premia embedded in the prices of any yield-curve-related security. It turns out that the risk premium for a security can be expressed as the product of the risk exposure of the security to the driving variables and the prices of risk attributable to these variables.

For illustration, consider a portfolio of default free bonds with value $V(t)$ that is only sensitive to the longer-term target, $L(t)$ (say because we have hedged out the exposure to $r(t)$ and $R(t)$). Also suppose that the only difference between the "natural" and "risk-adjusted" dynamics of $L(t)$ is in terms of the long run mean of this variable, and $L^Q(\infty) - L(\infty) > 0$. The risk premium embedded in the price of this portfolio then equals $\left(\frac{V_L}{V}\right)\left(L(\infty) - L^Q(\infty)\right)$ where the first term is the sensitivity of the portfolio value to changes in

[4] *The random shocks $z_r^Q(t), z_R^Q(t), z_L^Q(t)$ have the same properties as those used in the "natural" specification ((3), (4) and (5)).*

$L(t)$ (i.e. the portfolio's factor duration with respect to $L(t)$) and the second term captures the price of the $L(t)$ risk. Since the sensitivity for a portfolio of bonds will be negative and $L^Q(\infty)-L(\infty)>0$, the portfolio has a positive risk premium.

The above example also clarifies the idea behind "risk-adjusted" and "natural" parameters. We have noted before that bond prices are determined by their "risk-adjusted" parameters. Therefore, if $L^Q(\infty)>L(\infty)$, then bonds are priced "as if" the long-run mean of $L(t)$ (i.e. its "risk-adjusted" mean) is higher than its true long run mean. This increases bond yields or equivalently this embeds a positive risk premium in bond prices.

The more general case when the portfolio is sensitive to all the three driving variables is presented in the appendix. The main point about the general case is that if we want bond risk premia to be time-varying, then we need to allow the possibility that the "natural" and "risk-adjusted" dynamics are different not only in terms of the long-run means of $r(t)$, $R(t)$ and $L(t)$ but also in terms of the speeds of their mean reversion. For example, if we believe that the risk premium related to $L(t)$ is high when $L(t)$ is high, we must have that the bonds are priced as if the mean reversion of $L(t)$ is lower than its true mean reversion.[5]

The description of the key assumptions of the EFM and the derivation of its key pricing equation is now complete.

3. THE PRICING OF CASH INSTRUMENTS

In this section, we present an analysis of the pricing of default-free bonds using the specification described in the previous section. We show how zero coupon bonds with no default risk can be valued in the present setting. Using the zero yield curve, other objects of interest, such as the default-free par curve or the forward curve, can be constructed.

3.1 The Zero Coupon Yield Curve in the EFM

Solving the appropriate pricing equations of our model, it can be shown that the price at any date $t<T$ of a zero-coupon bond maturing at T and with a face value of 1 is given by the following equation:

$$\exp\left(-a_r(T-t)r(t)-a_R(T-t)R(t)-a_L(T-t)L(t)-b(T-t)\right) \tag{9}$$

where $a_r(.), a_R(.), a_L(.)$ and $b(.)$ are functions of the "risk-adjusted" parameters of the model and the time to maturity of the bond. A derivation of the pricing equation (9) and the expressions for $a_r(.), a_R(.), a_L(.)$ and $b(.)$ are provided in the Appendix.

Figure 2 provides an illustration of different shapes of the yield curve that the model can accommodate for a representative set of parameter values. It is seen that in addition to upward sloping and downward sloping curves, humped shaped curves are also possible in the model for different values of the driving variables, $r(t), R(t)$ and $L(t)$.

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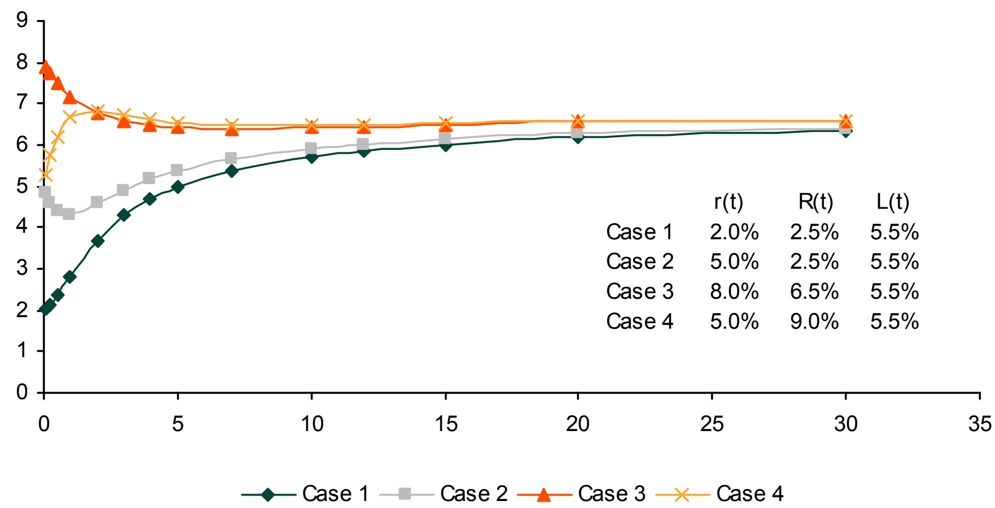
[5] *For further analysis of time-varying risk premia in bond markets, see Dai and Singleton (2002), Duffee (2002) and the references cited therein.*

Figure 2. EFM Yield Curves: Some Examples



The instantaneous forward rate for maturity at any time $t \le T$ can also be computed. From (9), this equals:

$$f(t,T) = a_r^{'}(T-t)r(t) + a_R^{'}(T-t)R(t) + a_L^{'}(T-t)L(t) + b^{'}(T-t) \qquad (10)$$

where primes denote first derivatives. It can be shown that $a_R^{'}(0) = a_L^{'}(0) = b^{'}(0) = 0$ and $a_r^{'}(0) = 1$. Therefore, $f(t,t)$ equals $r(t)$.

Equation (9) [or (10)] is one of the main results delivered by the EFM that can be a valuable tool in the analysis of bond markets. This equation provides an explicit expression relating the state variables representing fundamental macro-economic factors, the parameters governing their temporal behaviour, prices of risk and the yield curve. This is useful in carrying out a scenario analysis on the yield curve. One can, for example, quantitatively address the question of how would the yields at various terms to maturity change if the state variables $r(t), R(t)$ and $L(t)$ were to change by a given amount. One can also assess the effect on the yield curve of shifts in volatility parameters, mean reversion parameters and risk premium. The following discussion illustrates how the EFM can be used in these scenario analyses.

3.2 Sensitivity of the Yield Curve to the Underlying Factors

To use the EFM to investigate the sensitivity of yields to change in the underlying factors, we first need to set the values of the "risk-adjusted" parameters governing the mean reversion, the volatility parameters of various factors, and the correlation between the factors. In a typical implementation of the EFM, we work with mean reversion parameters that satisfy $\kappa_r^Q > \kappa_R^Q >> \kappa_L^Q > 0$. This is consistent with the empirical estimates obtained from historical bond yields or swap rates in various countries.

Figure 3. Sensitivity of the EFM Zero Curve for a One Basis Point Shift in Driving Variables

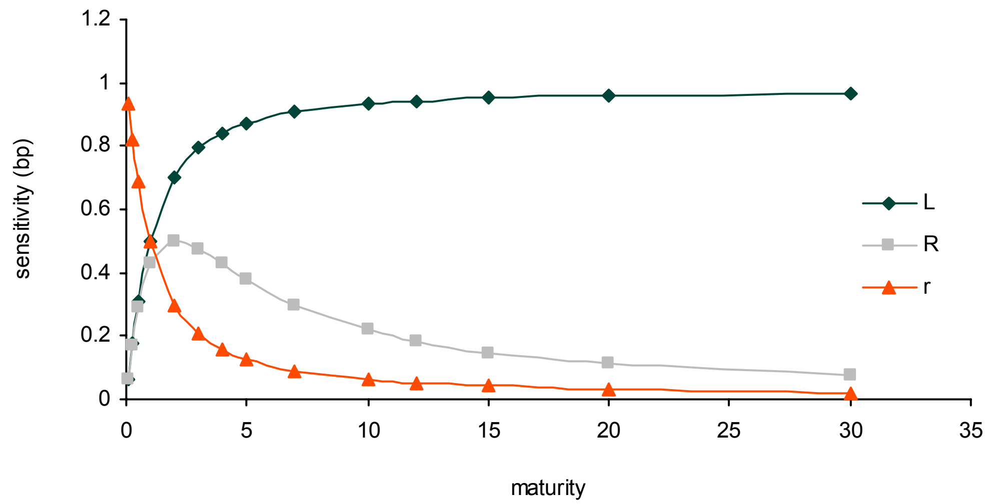


Figure 3 displays a plot of the sensitivity of the zero yields with respect to the three underlying factors for a representative set of parameter values. The three graphs in this figure represent the change (in basis points) in the zero-coupon yields of various terms to maturity for a one basis point increase in $r(t), R(t)$ and $L(t)$ respectively. It is seen from Figure 3 that $r(t)$ affects mainly the front end of the zero curve, $R(t)$ affects the intermediate end and $L(t)$ affects the long end. This derives from our choice of the relative magnitude of the mean reversion in these factors ensuring $\kappa_r^Q > \kappa_R^Q >> \kappa_L^Q > 0$. As discussed in the previous section, this choice of parameters allows the three EFM factors to be operative at different time scales. As Figure 3 shows, this is reflected in the three factors being the primary drivers of different segments of the yield curve.

As suggested in the previous section, one can think of $R(t)$ and $L(t)$ as macroeconomic factors representing respectively the near-term and longer-term drivers of expected inflation and expected real growth. If we interpret these factors in this way, then (9) [or (10)] provides a link between the yield curve and the state of the macro-economy. For example, if changes in $L(t)$ represents a secular shift in expected inflation then Figure 3 shows how the yield curve will respond to a one-basis point shift of this type.

Assessing the effect of changes in the underlying state variables on the yield curve is only one type of scenario analysis of interest. We might also want to know the effect on the yield curve of changes in the parameters of the model. For example, suppose that the volatility of changes in the short rate increases permanently. How is this reflected in the yield curve? Similarly, suppose that the current regime for output growth or expected inflation is expected to last much longer than before, resulting in a lowering of the mean-reversion parameters for these variables. What is the effect of changes in the speed of mean reversion of the underlying state variables? Again, the EFM pricing equation (9) [or (10)] can provide a consistent basis for addressing such questions.

3.3 Mean Reversion, Volatility and the Forward Curve

At the very short end (i.e. for $\tau = T - t \approx 0$), the forward curve given in (10) is closely approximated by:

$$f(t, t+\tau) \approx r(t) + \kappa^Q{}_r \left(R(t) - r(t)\right)\tau - \frac{\sigma_r^2 \tau^2}{2} \tag{11}$$

This means that at the short end, the slope of the forward curve is determined primarily by the mean reversion in the short rate and the distance of the actual short rate from its target. If the short rate is below its target, the forward curve is upward sloping for short terms to maturity and conversely if the short rate is above its target. Moreover, the higher the value of the mean reversion in the short rate, the steeper the slope. It is also seen that the forward rates at small terms to maturity are approximately linear in the time to maturity.

For long times to maturity (τ large) the following approximation to the forward curve holds, especially when κ_L is small and $\kappa_L \tau$ is also small,

$$f(t, t+\tau) \approx L(t) + \kappa_L^Q \left(L^Q(\infty) - L(t)\right)\tau - \frac{\sigma_L^2 \tau^2}{2} \tag{12}$$

Thus, at the long end of the forward curve, the parameters governing the long-term factor come into play. A higher mean reversion in $L(t)$ tends to pull up the long end of the curve if $L(t)$ is below its long-run ("risk-adjusted") mean. When $L(t)$ is higher than its long run ("risk-adjusted") mean, a higher mean-reversion pulls the curve down. More importantly, a higher volatility of $L(t)$ unambiguously pulls the curve down. When τ is large, the term that is quadratic in τ in (12) dominates. Consequently, the volatility parameter is crucial in determining the shape of the forward curve at very long maturities. The importance of σ_L in the determination of long-term forward rates, as evident from (12), reflects the role of convexity in the pricing of long-term bonds.

4. APPLICATIONS

In this section, we present some applications of EFM to the analysis of yield curves and to the determination of curve strategies. We apply the model to the US Swap curve and the Euro Government curve. For this, we estimate the parameters of the model using historical data and the method of maximum likelihood. For US Swap Curve, the data set includes monthly LIBOR rates and Swap rates with maturity varying from 3 Months to 30 Years. We assume that 6M LIBOR, 2 year and 10 year Swaps are exactly priced by the EFM. For the Euro Government curve, the data set includes fitted zeros from 6 Months to 30 Years.[6] We assume that yields for 6M, 2 year, and 10 year zeros are observed exactly. All other points on the curve are assumed to be observed with measurement error.[7] The sample period starts from June 1994 and ends in June 2002.

[6] *For the Euro market, for the period prior to January 1999, the zero curves are inferred from German treasuries. For the period after this date, these are inferred from German and French treasuries.*

[7] *The measurement errors are assumed to be independently and identically distributed over time.*

4.1 Comparing Market Yield Curves with Model Curves

We first show how well the yield curve generated by the model can fit the yield curves observed in the market. As an illustration, Figures 4 and 5 show the model-generated curve and the market curve as of April 30, 2002 for the two markets. Quite clearly, the model can fit the market curve very well on this day. It is worth noting that not only does the model fit well on one given day, it consistently fits the yield curve decently during the whole sample period, regardless of whether the curve is upward- or downward-sloping or is hump-shaped. This is evident in Figure 6 where we present the root mean squared pricing errors (RMSE) for various maturities over the sample period. It achieves the RMSE around 5-6 bps throughout the whole period from June 1994 to June 2002.

Comparing EFM Curves with Observed Curves

Figure 4.

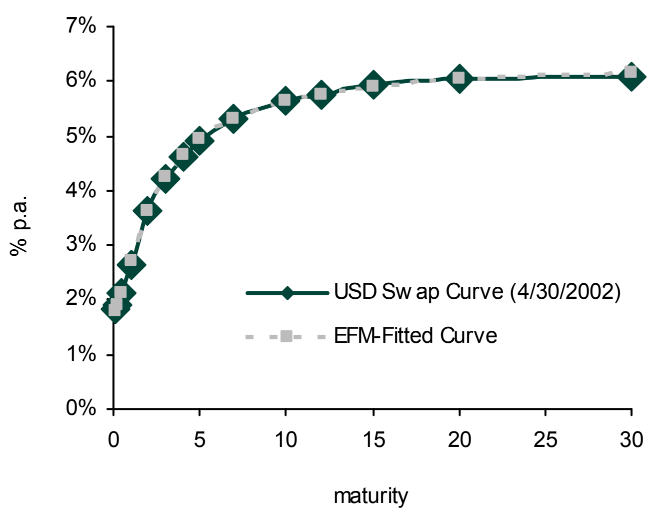


Figure 5.

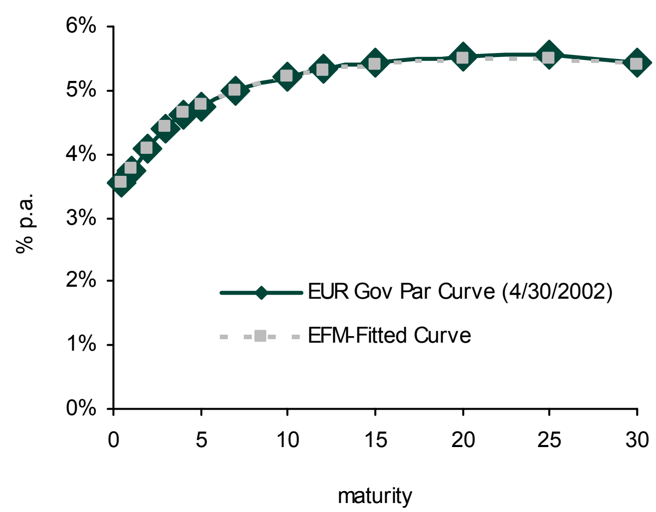


Figure 6. EFM Fitting Errors

USD Swap Curve

Maturity	3M	1Y	3Y	4Y	5Y	7Y	12Y	15Y	20Y	30Y
Root mean Squared Error (bp)	6.2	4.4	2.3	3.2	4.1	3.1	1.7	4.7	6.6	9.8

EUR Government Par Curve

Maturity	1 Yr	3 Yr	4 Yr	5 Yr	7 Yr	12 Yr	15 Yr	20 Yr	30 Yr
Root mean Squared Error (bp)	6.3	4.1	5.4	5.8	4.8	3.5	6.3	7.2	11.4

4.2 Historical Factor Realizations

As shown in the previous section, EFM yield curves are (linear) functions of the driving variables, $r(t), R(t)$ and $L(t)$. One application of the model is therefore to compute what values of these variables are priced into the yield curve at any given time. Since the factors are interpretable as the short rate, its near-term and the longer-term natural rate, one can then compare the implied values of these factors to what the investor estimates these factors to be. Any differential between the market-implied value and investor's subjective estimate would imply the existence of trades that would be profitable provided that investor forecasts are realized.

In Figures 7 and 8 we provide the time series of $r(t), R(t)$ and $L(t)$ implied by the USD swap curve and the EUR Government curve over the period June 1994 to June 2002. It is seen that the target rate tends to lead the movements in the short rate, reflecting the gradualism in central bank policy. The movements in implied $L(t)$ are also interesting. For the USD market the long rate factor has fluctuated in a stable range while in the EUR market, there is a persistent decline in $L(t)$ over the 1994-97 as realized inflation during this period came out significantly lower than was priced in the curve, and inflation expectations were revised downwards. It is also interesting that in the middle of 2002, even though concerns about deflation and global recession abound, the effect of this on the curve shows up mostly on $R(t)$ but not $L(t)$.

Realized Values of EFM Driving Variables

Figure 7. Implied Values of EFM Driving Variables from USD Swap Curve

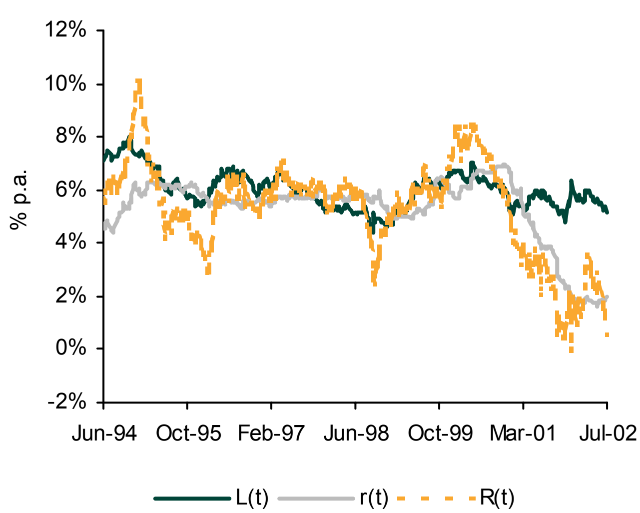
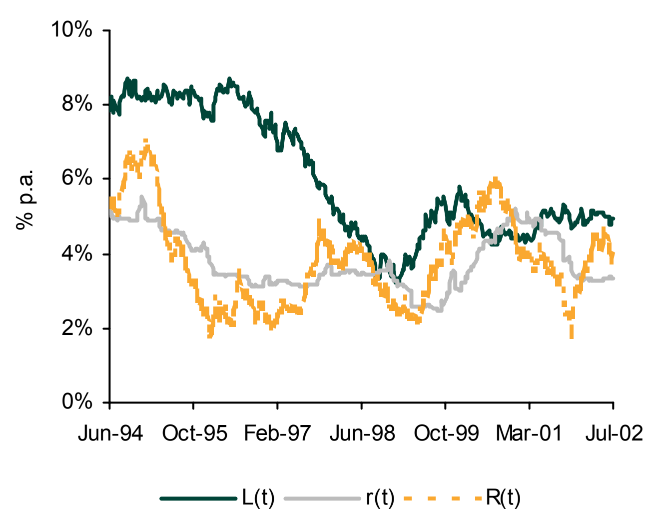


Figure 8. Implied Values of EFM Driving Variables from EUR Government Curve



4.3 Model-implied Risk Premia

One application of the EFM is in computing estimates of risk premia for various maturity sectors of the curve. Using the estimated values of the parameters of the model and the implied values of the driving variables, it is possible to compute the model implied expected excess returns for various sectors in the bond market. As an illustration, Figures 9 and 10 show the model implied ex-ante expected excess return for a 5 year zero coupon bond for the EUR and the USD markets. In the same figures, we also plot out the ex-post realized excess return. It is seen that the model expected returns are able to capture the dynamics in realized returns reasonably well.

Figure 9. Realized Excess Returns and EFM-Implied Estimates of Risk Premia for a 5 year Zero Coupon Bond: US Swaps

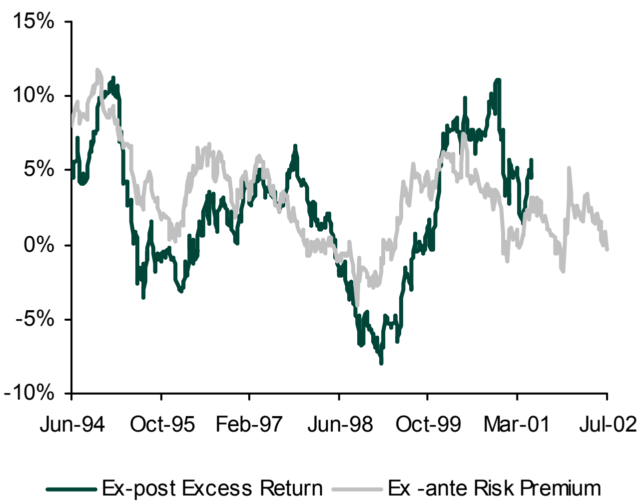
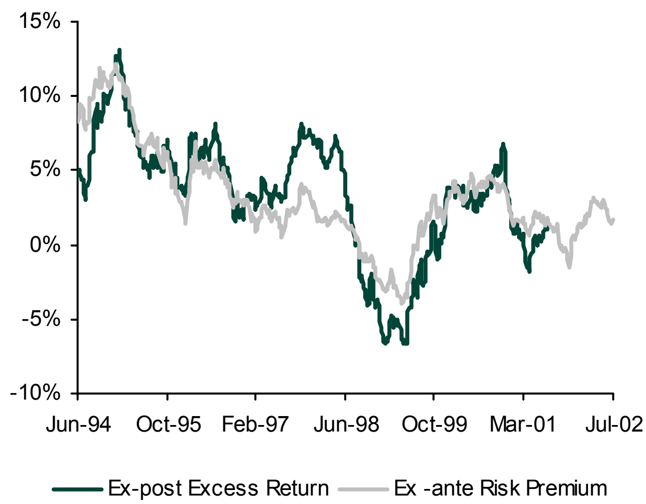


Figure 10. Realized Excess Returns and EFM-Implied Estimates of Risk Premia for a 5 year Zero Coupon Bond: EUR Government



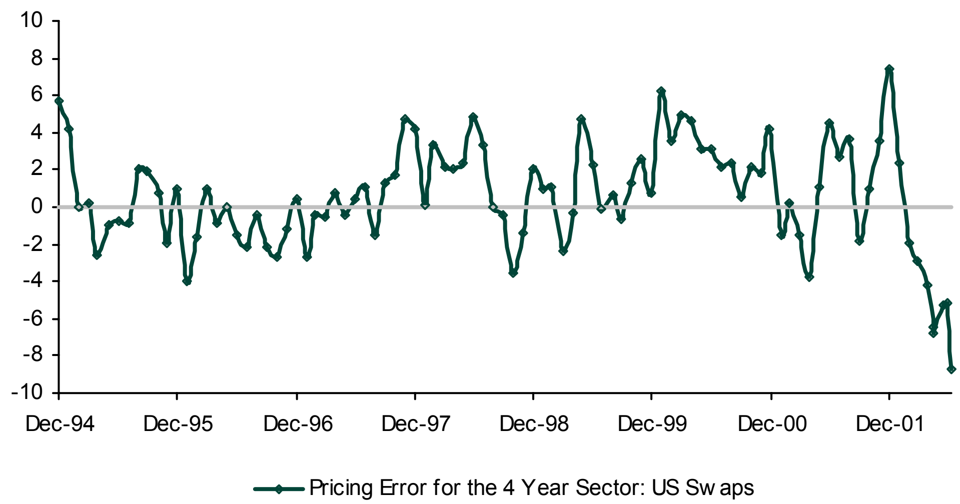
4.4 Applications of EFM in Curve Positioning Decisions

The applications mentioned above suggest that EFM should be useful as an aid in curve positioning decisions of fixed income portfolio managers. Below we illustrate this by means of two examples. The first example shows how we can use the pricing errors implied by model as a relative value indicator. The second example shows how estimates of risk premia can be used in duration decisions.

4.4.1 Relative Value Analysis: Taking Advantage of Mean Reversion in Cross-Sectional Pricing Errors

One use of the EFM is to assess if a given set of default-free bonds is fairly valued taking as given the current market prices of another set of bonds. Such analysis is frequently referred to as relative value analysis or rich-cheap analysis. To use EFM in such analysis, we can use the following procedure: (i) obtain parameter estimates for the model, and (ii) obtain the implied values of $r(t)$, $R(t)$ and $L(t)$ using (9) by fitting three rates exactly. The exact fitting of three points on the curve will leave a pricing error in most other points on the curve. Typically, these errors are small and are strongly mean-reverting. Occasionally, a large pricing error in certain sectors may imply a short-term anomaly that could represent a profitable trading opportunity, especially if supported by other analyses. For instance, Figure 11 shows the richness or cheapness of 4 year US swap rate with respect to 2 year and 10 year from Jun 1994 to July 2002 according to EFM. Historically the 4 year point is generally fair priced and any deviation from its fair price (model implied) was short-lived. However it becomes extremely rich due to a steady demand for the receiving in the belly (4-5 year) of the curve in the current steep curve environment. This suggests a butterfly trade by paying in 4s while receiving 2s and 10s.

Figure 11. Pricing Error for the 4 Year Sector: US Swaps



4.4.2 Taking Advantage of Time Variation in the Term Premium

As we have discussed before, our model can be useful to estimate risk premia in different market conditions and hence in making decisions about investment timing and curve positioning. We use the following example based on the result of Euro Government Market to illustrate how investors can take advantage of the time varying risk premium.

Suppose an investor starts investing in 1995.[8] She can choose from the following 3 strategies. First of all, she can invest the money in the short-term bond (for instance a one year bond) and roll over to the same instruments as soon as the bond matures. This strategy is similar to investing the money in a short-term government bond fund.

If the investor is more aggressive, she can choose the second strategy by taking more interest rate (duration) risk. For instance, she can buy a long bond (5 year) and roll over to the same maturity bond (5 Year) after holding it for one year. This strategy is similar to investing in a

[8] *We assume that the investor decides to distribute the investment equally along a one year period: i.e.she would invest an equal amount of capital at the beginning of each month. After 12 months,the portfolio purchased 12 months ago is unwound and a new portfolio is created.*

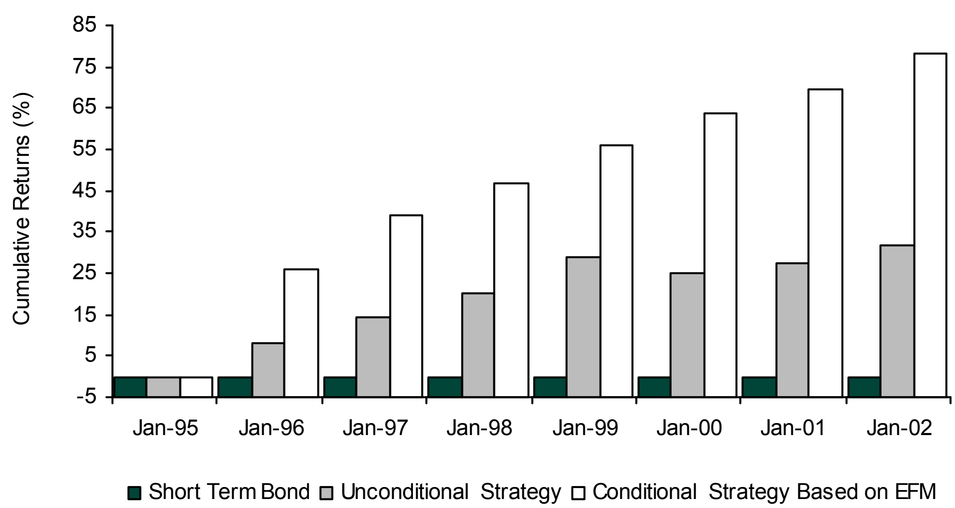
long-term government bond fund with a constant duration. Over the 7 year period (1995 – 2002), this strategy outperforms the first strategy by around 3% per year which is the realized average excess return for a 5 Year bond over this period.

However, the second duration strategy is an unconditional one since it does not make any use of time-variation in the estimates of risk premium for the 5-year bond. Hence we design the third trading strategy for the investor which is a conditional duration bet. The difference between this strategy and the unconditional one is that here the investor changes the allocation between the long bond and the short term bond based on the estimates of the risk premium instead of investing only in the long bond. In particular, she scales up (down) the weight on long bond when the estimate of its risk premium is high (low). When the risk premium is high (for example late 97 and early 98) a bigger long position is taken in the 5-year bond. On the other hand, when the estimated risk premium is low (as during the post 98 crisis), a smaller position is taken in the 5-year bond.

Figure 12 shows the performance of the three strategies. By using the unconditional strategy (#2), the investor earns a cumulative excess return of 30% (over the first strategy) over this 7 year period while the conditional strategy (#3) posts a cumulative excess return of 80%. Furthermore there was no losing year for the conditional strategy while the unconditional one lags the money market around 1999 – 2000.

The above-mentioned example should be thought of as a simple illustration of how EFM-based risk premium estimates could be used in investment decision-making. In future work, we expect to report in detail on the risk-reward characteristics of numerous strategies of the type illustrated here. Although real-life investment strategies would certainly need to be more sophisticated and would need to use signals from a number of different models, this example does suggest that models such as the EFM can have a useful role to play in devising duration strategies.

Figure 12. Evaluating an Investment Strategy Conditional on EFM-based Estimates of Risk Premia



5. CONCLUSIONS

We have presented a structural model of the yield curve that attempts to capture the behaviour of central banks and their effect on the term structure of interest rates. This is a representative example of a family of yield curve models based on time-varying interest rate targets. We have referred to this family of models as Economic Factor Models (EFM) of the yield curve. In these models, the short-term riskless rate fluctuates around a dynamic interest rate target. The target itself can potentially be affected both by near-term considerations and secular forces. Thus, there are three factors driving the yield curve: the short rate, its near term target and its longer-term natural level. We have shown how mean reversion, volatility and prices of interest rate risk are all combined in the model to yield a bond valuation equation. As rate dynamics and the price of interest rate risk are explicitly and separately modelled, we can use models such as the ones presented here to differentiate between rate expectations and risk premium embedded in the yield curve. We have presented some examples of how our model can be used as an aid in curve positioning and duration decisions. In our future work, we expect to report in detail on the empirical performance of this and related models in portfolio management and risk control.

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APPENDIX

Expected Excess Returns on Interest Rate Sensitive Claims

The (instantaneous) expected excess return (over the short rate) of any security whose value depends on the yield curve is given by:

$$\left(\frac{V_r}{V}\right)\left(\mu_r(t)-\mu_r^Q(t)\right)+\left(\frac{V_R}{V}\right)\left(\mu_R(t)-\mu_R^Q(t)\right)+\left(\frac{V_L}{V}\right)\left(\mu_L(t)-\mu_L^Q(t)\right)$$

where $\left(\frac{V_r}{V}\right)$ denotes the sensitivity of the value of the security to a small change in $r(t)$, $\mu_r(t)$ denotes the instantaneous mean of $r(t)$ under its natural dynamics (i.e. $\mu_r(t)=\kappa_r(R(t)-r(t))$ and $\mu_r^Q(t)$ denotes the instantaneous mean of $r(t)$ under its "risk-adjusted" dynamics (i.e. $\mu_r^Q(t)=\kappa_r^Q(R(t)-r(t))$). Other terms in the above equation is defined in a similar manner.

The Pricing of Zero Coupon Bond

Let us rearrange the "Risk-Adjusted" dynamics of state variables, $\left[r(t),R(t),L(t)\right]$, mentioned in Section 3 in a matrix form:

$$\begin{cases} dX_t = M^Q(X^Q{}_\infty - X_t)dt+\sigma dW_t \\ r_t = a' X_t \end{cases}$$

where $X_t=\begin{bmatrix} L_t \\ R_t \\ r_t \end{bmatrix}$, $M^Q=\begin{bmatrix} \kappa_L^Q & 0 & 0 \\ -\kappa_R^Q & \kappa_R^Q & 0 \\ 0 & -\kappa_r^Q & \kappa_r^Q \end{bmatrix}$, $X_\infty=\begin{bmatrix} L^Q{}_\infty \\ L^Q{}_\infty \\ L^Q{}_\infty \end{bmatrix}$, $\sigma=\begin{bmatrix} \sigma_{LL} & \sigma_{LR} & \sigma_{Lr} \\ 0 & \sigma_{RR} & \sigma_{Rr} \\ 0 & 0 & \sigma_{rr} \end{bmatrix}$ and $a=(0,\ 0,\ 1)'$

Since the mean-reverting speed matrix M^Q is not diagonal, we introduce the following autonomous process Z_t

$$\begin{cases} Z_t = B^Q X_t \\ dZ_t = D^Q(B^Q X_\infty - Z_t)+B^Q \sigma dW^t \end{cases}$$

where

$$B^Q=\begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{\kappa_R^Q}{\kappa_R^Q-\kappa_L^Q} & 1 & 0 \\ \dfrac{\kappa_R^Q\kappa_r^Q}{(\kappa_r^Q-\kappa_L^Q)(\kappa_r^Q-\kappa_R^Q)} & -\dfrac{\kappa_r^Q}{\kappa_r^Q-\kappa_R^Q} & 1 \end{bmatrix},\quad D^Q=\begin{bmatrix} \kappa_L^Q & 0 & 0 \\ 0 & \kappa_R^Q & 0 \\ 0 & 0 & \kappa_r^Q \end{bmatrix}$$

Under this representation, we can see that the mean reverting speed Matrix D^Q, whose elements are eigenvalues of M^Q, is diagonal and B^Q is a 3 x 3 matrix with rows equal to the eigenvectors of M^Q.

Let us define:

$$\Phi^Q(\Delta) = \begin{bmatrix} e^{-\kappa_L^Q \Delta} & 0 & 0 \\ 0 & e^{-\kappa_R^Q \Delta} & 0 \\ 0 & 0 & e^{-\kappa_r^Q \Delta} \end{bmatrix}$$

$$V^Q(\Delta) = \begin{bmatrix} \frac{\omega_{11}[1-e^{-2\kappa_L^Q \Delta}]}{2\kappa_L^Q} & \frac{\omega_{12}[1-e^{-(\kappa_L^Q+\kappa_R^Q)\Delta}]}{\kappa_L^Q+\kappa_R^Q} & \frac{\omega_{13}[1-e^{-(\kappa_L^Q+\kappa_r^Q)\Delta}]}{\kappa_L^Q+\kappa_r^Q} \\ \frac{\omega_{21}[1-e^{-(\kappa_R^Q+\kappa_L^Q)\Delta}]}{\kappa_R^Q+\kappa_L^Q} & \frac{\omega_{22}[1-e^{-2\kappa_R^Q \Delta}]}{2\kappa_R^Q} & \frac{\omega_{23}[1-e^{-(\kappa_R^Q+\kappa_r^Q)\Delta}]}{\kappa_R^Q+\kappa_r^Q} \\ \frac{\omega_{31}[1-e^{-(\kappa_r^Q+\kappa_L^Q)\Delta}]}{\kappa_r^Q+\kappa_L^Q} & \frac{\omega_{32}[1-e^{-(\kappa_r^Q+\kappa_R^Q)\Delta}]}{\kappa_r^Q+\kappa_R^Q} & \frac{\omega_{33}[1-e^{-2\kappa_r^Q \Delta}]}{2\kappa_r^Q} \end{bmatrix}$$

$$\omega_{ij} = [B^Q \sigma\sigma' B^{Q}{}']_{ij}$$

It can be shown that the price at time t of a zero coupon bond that matures at $t+\Delta$, $P(t,t+\Delta)$ is exponential-affine in state variable $X(t)$. That is, $P(t,t+\Delta) = e^{-a(\Delta)-b(\Delta)X_t}$ where

$$\begin{aligned} b(\Delta) &= (a' X_\infty - \frac{1}{2} a' M^{Q^{-1}} \sigma\sigma' M^{Q^{-1'}} a)\Delta \\ &\quad - a' M^{Q^{-1}} B^{Q^{-1}} D^{Q^{-1}} [I - \Phi^Q(\Delta)] B^Q [M^Q X_\infty - \sigma\sigma' M^{Q^{-1'}} a] \\ &\quad - \frac{1}{2} a' M^{Q^{-1}} B^{Q^{-1}} V^Q(\Delta) B^{Q^{-1'}} M^{Q^{-1'}} a \end{aligned}$$

$$a(\Delta) = -a' M^{Q^{-1}} [B^{Q^{-1}} \Phi^Q(\Delta) B^Q - I]$$

where

$$a(\Delta) = \begin{bmatrix} a_L \\ a_R \\ a_r \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{-\kappa_L^Q \Delta}-1}{\kappa_L^Q} - \frac{e^{-\kappa_L^Q \Delta} - e^{-\kappa_R^Q \Delta}}{\kappa_R^Q - \kappa_L^Q} - [\frac{\kappa_R^Q e^{-\kappa_L^Q \Delta}}{(\kappa_r^Q - \kappa_L^Q)(\kappa_R^Q - \kappa_L^Q)} - \frac{\kappa_R^Q e^{-k_R \Delta}}{(\kappa_r^Q - \kappa_R^Q)(\kappa_R^Q - \kappa_L^Q)} + \frac{\kappa_R^Q e^{-k_r \Delta}}{(\kappa_r^Q - \kappa_L^Q)(\kappa_r^Q - \kappa_R^Q)}] \\ -\frac{e^{-k_R \Delta}-1}{\kappa_R^Q} - \frac{e^{-k_R \Delta} - e^{-\kappa_r^Q \Delta}}{\kappa_r^Q - \kappa_R^Q} \\ -\frac{e^{-\kappa_r^Q \Delta}-1}{\kappa_r^Q} \end{bmatrix}$$