# Three Alternative Methods for Estimating Hedge Ratios

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**Abstract**

This chapter first discusses four different theoretical models, which include minimum variance, mean-variance, expected utility, and value-at-risk method. Then we use S&P 500 data to show how three alternative estimation methods can be used to estimate hedge ratio. These three methods include OLS method, GARCH method, and cointegration and error correction method. We found that OLS method is not sufficient for estimating hedge ratio.

**Keywords**

Hedge ratio · Cointegration · Minimum variance · Mean-variance hedge ratio · ARCH GARCH model

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## Introduction

A **future contract** is a standardized legal agreement between a buyer and a seller, who promise now to exchange a specified amount of money for goods or services at a future time. Of course, there is nothing really unusual about a contract made in advance of delivery. For instance, whenever something is ordered rather than purchased on the spot, a futures (or forward) contract is involved. Although the price is determined at the time of the order, the actual exchange of cash for the merchandise takes place later. For some items, the lag is a few days, while for others (such as a car) it may be months. Moreover, a futures contract imparts a legal obligation to both parties of the contract to fulfill the specifications. To guarantee fulfillment of this obligation, a "good-faith" deposit, also called **margin,** may be required from the buyer (and the seller, if he or she does not already own the product).

Future contracts are frequently used to hedge spot contracts to review the risk. Therefore, hedge ratio estimation is important for risk management. One of the best uses of derivative securities such as futures contracts is used to hedge the risk. For the last half century, both academicians and practitioners have shown great interest in the issue of hedging with futures, therefore theoretical analysis and empirical investigation for future hedge ratio are important for both academicians and practitioners. This is quite evident from the large number of articles written in this area.

The main theoretical issue in hedging involves the determination of the optimal hedge ratio. However, the optimal hedge ratio depends on the particular objective function to be optimized. In this chapter, we will carefully discuss different objective functions used to derive optimal hedge ratio. For example, one of the most widely-used hedging strategies is based on the minimization of the variance of the hedged portfolio (e.g., see Johnson 1960; Ederington 1979; and Myers and Thompson 1989). This so-called minimum-variance (MV) hedge ratio is simple to understand and estimate. However, the MV hedge ratio completely ignores the expected return of the hedged portfolio. Therefore, this strategy is inconsistent with the mean-variance tradeoff framework unless the individuals are infinitely risk-averse or the futures price follows a pure martingale process (i.e., expected futures price change is zero).

Some more realistic strategies that incorporate both the expected return and risk (variance) of the hedged portfolio have been developed (e.g., see Howard and D'Antonio 1984; Cecchetti et al. 1988; and Hsin et al. 1994). These strategies are consistent with the mean-variance framework. In fact, it can be shown that if the future price follows a pure martingale process, then the optimal mean-variance hedge ratio will be reduced to the MV hedge ratio.

Another aspect of the mean-variance-based strategies is that even though they are an improvement over the MV strategy, for them to be consistent with the expected utility maximization principle, either the utility function needs to be quadratic or the returns should be jointly normal. Unless these assumptions are valid, then the hedge ratio may not be optimal with respect to the expected utility maximization principle. Some research has solved this problem by deriving the optimal hedge ratio based on the maximization of the expected utility (e.g., see Cecchetti et al. 1988; and Lence 1995, 1996). It should be noted that this approach requires the use of specific utility function and specific return distribution.

Attempts have been made to eliminate these specific assumptions regarding the utility function and return distributions. Some of them involve the minimization of the mean extended-Gini (MEG) coefficient, which is consistent with the concept of stochastic dominance (e.g., see Cheung et al. 1990; Kolb and Okunev 1992, 1993; Lien and Luo 1993a; Shalit 1995; and Lien and Shaffer 1999). Shalit (1995) shows that if the prices are normally distributed, then the MEG-based hedge ratio will be the same as the MV hedge ratio.

Recently, hedge ratios based on the generalized semivariance (GSV) or lower partial moments have been proposed (e.g., see De Jong et al. 1997; Lien and Tse 1998, 2000; and Chen et al. 2001). These hedge ratios are also consistent with the concept of stochastic dominance. Furthermore, these GSV-based hedge ratios have another attractive feature, whereby they measure portfolio risk by the GSV, which is consistent with the risk perceived by managers, because of its emphasis on the returns below the target return (see Crum et al. 1981; and Lien and Tse 2000). Lien and Tse (1998) show that if the futures and spot returns are jointly normally distributed, and if the futures price follows a pure martingale process, then the minimum-GSV hedge ratio will be equal to the MV hedge ratio. Finally, Hung et al. (2006) have proposed a related hedge ratio that minimizes the value-at-risk associated with the hedged portfolio when choosing hedge ratio. This hedge ratio will also be equal to MV hedge ratio if the futures price follows a pure martingale process. However, the future price in general does not follow the pure martingale process; therefore, generalized hedge ratio should be considered to improve the effectiveness of hedging decision.

It should be noted that most of the studies mentioned above (except Lence (1995, 1996)) ignore transaction costs as well as investments in other securities. Lence (1995, 1996) derives the optimal hedge ratio where transaction costs and investments in other securities are incorporated into the model. Using a CARA utility function, Lence finds that under certain circumstances, the optimal hedge ratio is zero; that is, the optimal hedging strategy is not to hedge at all.

In addition to the use of different objective functions in the derivation of the optimal hedge ratio, previous studies also differ in terms of the dynamic nature of the hedge ratio. For example, some studies assume that the hedge ratio is constant over time. Consequently, these static hedge ratios are estimated using unconditional probability distributions (e.g., see Ederington 1979; Howard and D'Antonio 1984; Benet 1992; Kolb and Okunev 1992, 1993; and Ghosh 1993). On the other hand, several studies allow the hedge ratio to change over time. In some cases, these dynamic hedge ratios are estimated using conditional distributions associated with models such as ARCH and GARCH (e.g., see Cecchetti et al. 1988; Baillie and Myers 1991; Kroner and Sultan 1993; and Sephton 1993). The GARCH based method has recently been extended by Lee and Yoder (2007) where regime-switching model is used. Alternatively, the hedge ratios can be made dynamic by considering a multi-period model where the hedge ratios are allowed to vary for different periods. This is the method used by Lien and Luo (1993b).

When it comes to estimating the hedge ratios, many different techniques are currently being employed, ranging from simple to complex ones. For example, some of them use such a simple method as the ordinary least squares (OLS) technique (e.g., see Ederington 1979; Malliaris and Urrutia 1991; and Benet 1992). However, others use more complex methods, such as the conditional heteroscedastic (ARCH or GARCH) method (e.g., see Cecchetti et al. 1988; Baillie and Myers 1991; and Sephton 1993), the random coefficient method (e.g., see Grammatikos and Saunders 1983), the cointegration method (e.g., see Ghosh 1993; Lien and Luo 1993b; and Chou et al. 1996), or the cointegration-heteroscedastic method (e.g., see Kroner and Sultan 1993). Recently, Lien and Shrestha (2007), Maghyereh et al. (2019), and Sultan et al. (2019) have suggested the use of wavelet analysis to match the data frequency with the hedging horizon. Finally, Lien and Shrestha (2010) also suggest the use of multivariate skew-normal distribution in estimating the minimum variance hedge ratio.

By now it is clear that there are several different ways of deriving and estimating hedge ratios. In this chapter, we will review these different techniques and approaches, and examine their relations. In addition, we also use S&P 500 index future data to show how three different econometrics methods can be used to estimate hedge ratios.

There are four sections in this chapter. In section "Alternative Theories Used for Deriving the Static Optimal Hedge Ratios," four alternative theories for deriving the static optimal hedge ratios are carefully reviewed. Section "Applications of OLS, GARCH, and CECM Models to Estimate Optimal Hedge Ratio" will use three alternative estimation methods to perform an empirical study to estimate minimal hedge ratio. Finally, in section "Conclusion" we provide a summary. Appendix A presents the monthly data of S&P 500 index and its future from January 2005 to August 2020, and Appendix B explains the applications of R language in estimating three optimal hedge ratios.

## Alternative Theories Used for Deriving the Static Optimal Hedge Ratios

By combining investments in the spot market and futures market to form a portfolio, we can eliminate (or reduce) fluctuations in its value by hedging. Consider a portfolio consisting of $C_s$ units of a long spot position and $C_f$ units of a short futures position.[1] Following Lee et al. (2013) and Chen et al. (2003), let $S_t$ and $F_t$ denote the spot and futures prices at time $t$, respectively. Since the futures contracts are used to reduce the fluctuations in spot positions, the resulting portfolio is known as the hedged portfolio. If the hedge ratio is discussed in terms of price changes (profits),

[1] Without loss of generality, we assume that the size of the future contract is 1.

then the profit on the hedged portfolio, $\Delta V_H$, and the hedge ratio, $H$, are, respectively, given by:

$$\Delta V_H = C_s \Delta S_t - HC_f \Delta F_t \text{ and } H = \frac{C_f}{C_s}, \tag{1}$$

where $\Delta S_t = S_{t+1} - S_t$ and $\Delta F_t = F_{t+1} - F_t$.

The main objective of hedging is to determine and estimate the optimal hedge ratio (either $h$ or $H$). It should be noted that optimal hedge ratio will depend on a particular objective function to be used. Hedge ratio can be classified into either static hedge ratio or dynamic hedge ratio.

It is important to note that in the above setup, the cash position is assumed to be fixed, and we only look for the optimum futures position. Most of the hedging literature assumes that the cash position is fixed, a setup that is suitable for financial futures. However, when we are dealing with commodity futures, the initial cash position becomes an important decision variable that is tied to the production decision, which has been discussed by Lence (1995, 1996).

Static hedge ratio means the hedge ratio remains constant over time. In general, there are nine static hedge ratios as follows: (a) Minimum-Variance Hedge Ratio, (b) Optimum Mean-Variance Hedge Ratio, (c) Sharpe Hedge Ratio, (d) Maximum Expected Utility Hedge Ratio, (e) Minimum Mean Extended-Gini Coefficient Hedge Ratio, (f) Optimum Mean-MEG Hedge Ratio, (g) Minimum Generalized Semi-variance Hedge Ratio, (h) Optimum Mean-Generalized Semivariance Hedge Ratio, and (i) Minimum Value-at-Risk Hedge Ratio. In this section, we will discuss static hedge ratios (a), (b), (c), and (i) in detail. The information for the other five static hedge ratios can be found in Chen et al. (2003).

## Minimum-Variance Hedge Ratio

The most widely used static hedge ratio is the minimum-variance (MV) hedge ratio. Johnson (1960) derives this hedge ratio by minimizing the portfolio risk, where the risk is given by the variance of changes in the value of the hedged portfolio as follows:

$$Var(\Delta V_H) = C_s^2 Var(\Delta S) + C_f^2 Var(\Delta F) - 2C_s C_f H Cov(\Delta S, \Delta F).$$

The MV hedge ratio, in this case, is given by:

$$H_J^* = \frac{C_f}{C_s} = \frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)}. \tag{2a}$$

Alternatively, if we use $Var(R_h)$ to represent the risk of hedged portfolio return ($R_h$), then the MV hedge ratio is obtained by minimizing $Var(R_h)$ which is given by:

$$Var(R_h) = Var(R_s) + h^2 Var\left(R_f\right) - 2hCov\left(R_s, R_f\right).$$

where $h$ is the so-called hedge ratio, $R_s$ and $R_f$ are the so-called one-period returns on the spot and futures positions, respectively. In this case, the MV hedge ratio is given by:

$$h_J^* = \frac{Cov\left(R_s, R_f\right)}{Var\left(R_f\right)} = \rho \frac{\sigma_s}{\sigma_f}, \tag{2b}$$

where $\rho$ is the correlation coefficient between $R_s$ and $R_f$, and $\sigma_s$ and $\sigma_f$ are standard deviations of $R_s$ and $R_f$, respectively.

The attractive features of the MV hedge ratio are that it is easy to understand and simple to compute. However, in general, the MV hedge ratio is not consistent with the mean-variance framework since it ignores the expected return on the hedged portfolio. For the MV hedge ratio to be consistent with the mean-variance framework, either the investors need to be infinitely risk-averse or the expected return on the futures contract needs to be zero.

## Optimum Mean-Variance Hedge Ratio

Various studies have incorporated both risk and return in the derivation of the hedge ratio. For example, Hsin et al. (1994) derive the optimal hedge ratio that maximizes the following utility function:

$$\underset{C_f}{Max} \;\; V(E(R_h), \sigma; A) = E(R_h) - 0.5A\sigma_h^2, \tag{3}$$

where $A$ represents the risk aversion parameter. It is clear that this utility function incorporates both risk and return. Therefore, the hedge ratio based on this utility function would be consistent with the mean-variance framework. The optimal number of futures contract and the optimal hedge ratio are, respectively, given by:

$$h_2 = -\frac{C_f^* F}{C_s S} = -\left[\frac{E\left(R_f\right)}{A\sigma_f^2} - \rho \frac{\sigma_s}{\sigma_f}\right]. \tag{4}$$

One problem associated with this type of hedge ratio is that in order to derive the optimum hedge ratio, we need to know the individual's risk aversion parameter. Furthermore, different individuals will choose different optimal hedge ratios, depending on the values of their risk aversion parameter.

Since the MV hedge ratio is easy to understand and simple to compute, it will be interesting and useful to know under what condition the above hedge ratio would be the same as the MV hedge ratio. It can be seen from Eqs. (2b) and (4) that if $A \to \infty$ or $E(R_f) = 0$, then $h_2$ would be equal to the MV hedge ratio $h_J^*$. The first condition is simply a restatement of the infinitely risk-averse individuals. However, the second condition does not impose any condition on the risk-averseness, and this is important. It implies that even if the individuals are not infinitely risk averse, then the MV hedge ratio would be the same as the optimal mean-variance hedge ratio if the expected return on the futures contract is zero (i.e., futures prices follow a simple martingale process). Therefore, if futures prices follow a simple martingale process, then we do not need to know the risk aversion parameter of the investor to find the optimal hedge ratio.

## Sharpe Hedge Ratio

Another way of incorporating the portfolio return into the hedging strategy is to use the risk-return tradeoff (Sharpe measure) criteria. Howard and D'Antonio (1984) consider the optimal level of futures contracts by maximizing the ratio of the portfolio's excess return to its volatility:

$$\underset{C_f}{Max} \ \theta = \frac{E(R_h) - R_F}{\sigma_h}, \tag{5}$$

where $C_f$ is the level of futures contracts. $\sigma_h^2 = Var(R_h)$ and $R_F$ represents the risk-free interest rate.

The classic one-to-one hedge is a naïve strategy based upon a broadly defined objective of risk minimization. The strategy is naïve in the sense that a hedging coefficient of one is used regardless of past or expected correlations of spot- and futures-price changes. Working's (1953) strategy brings out the speculative aspects of hedging by analyzing changes in the basis and, accordingly, exercising discrete judgment about when to hedge and when not to hedge. The underlying objective of Working's (1953) decision rule for hedgers is one of profit maximization. Finally, Johnson (1960), in applying the mean-variance criteria of modern portfolio theory, emphasizes the risk-minimization objective but defines risk in terms of the variance of the hedged position. Although Johnson's method improves on the naïve strategy of a one-to-one hedge, however, it essentially disregards the return component associated with a particular level of risk. Rutledge (1972) uses both mean and variance information to derive hedge ratio.

Howard and D'Antonio (1984) use an objective function defined in Eq. 5 to derive the hedge ratio as defined in Eq. 6.

In deriving this hedge ratio, they use a mean-variance framework, and their strategy begins by assuming that the "agent" is out to maximize the expected return for a given level of portfolio risk. With a choice of putting money into three assets – a spot position, a futures contract, and a risk-free asset – the agent's optimal portfolio will depend on the relative risk-return characteristics of each asset. For a hedger, the optimal portfolio may contain a short futures position, a long futures position, or no futures position at all. In general, the precise futures position to be entered into will be determined by (1) the risk-free rate, (2) the expected returns and the standard deviations for the spot and futures positions, and (3) the correlation between the return on the spot position and the return on the futures.

$$\text{Hedge ratio } H = \frac{(\lambda - \rho)}{\gamma\pi(1 - \lambda\rho)} \tag{6}$$

where:

$\pi = \sigma_f/\sigma_s$ = relative variability of futures and spot returns;
$\alpha = \overline{r}_f/(\overline{r}_s - i)$ = relative excess return on futures to that of spot;
$\gamma = P_f/P_s$ = current price ratio of futures to spot;
$\lambda = \alpha/\pi = \left(\overline{r}_f/\sigma_f\right)/[(\overline{r}_s - i)/\sigma_s]$ = risk-to-excess-return relative of futures versus the spot position;
$P_s$, $P_f$ = the current price per unit for the spot and futures, respectively;
$\rho$ = simple correlation coefficient between the spot and futures returns;
$\sigma_s$ = standard deviation of spot returns;
$\sigma_f$ = standard deviation of futures returns;
$\overline{r}_s$ = mean return on the spot over some recent past interval;
$\overline{r}_f$ = mean return on the futures over some recent past interval; and
$i$ = risk-free rate.

Based upon Eqs. 6 and 1, the optimal number of futures positions, $C_f^*$, is given by:

$$C_f^* = -C_s \frac{\left(\frac{S}{F}\right)\left(\frac{\sigma_s}{\sigma_f}\right)\left[\frac{\sigma_s}{\sigma_f}\left(\frac{E\left(R_f\right)}{E(R_s)-R_F}\right) - \rho\right]}{\left[1 - \frac{\sigma_s}{\sigma_f}\left(\frac{E\left(R_f\right)\rho}{E(R_s)-R_F}\right)\right]}. \tag{7}$$

From the optimal futures position, we can obtain the following optimal hedge ratio:

$$h_3 = -\frac{\left(\frac{\sigma_s}{\sigma_f}\right)\left[\frac{\sigma_s}{\sigma_f}\left(\frac{E\left(R_f\right)}{E(R_s)-R_F}\right) - \rho\right]}{\left[1 - \frac{\sigma_s}{\sigma_f}\left(\frac{E\left(R_f\right)\rho}{E(R_s)-R_F}\right)\right]}. \tag{8}$$

Again, if $E(R_f) = 0$, then $h_3$ reduces to:

$$h_3 = \left(\frac{\sigma_s}{\sigma_f}\right)\rho, \tag{9}$$

which is the same as the MV hedge ratio $h_J^*$.

By analyzing the properties of λ Howard and D'Antonio (1984) discern some important insights for the coordinated use of futures in a hedge portfolio. Numerically, λ expresses the relative attractiveness of investing in futures versus the spot position. When $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$, the futures contract offers less, the same, and more excess return per unit of risk than the spot position, respectively. Since this analysis is being undertaken from a hedger's point of view, it is assumed $\lambda < 1$. An assumption that $\lambda > 1$ would inappropriately imply that theoretically it is possible to hedge the futures position with the spot asset. In addition, they also consider the effect of the relationship of λ to ρ on the optimal hedge ratio, *H*. Other detailed analysis can be found in Table 4 of Howard and D'Antonio's (1984) paper.

## Minimum Value-at-Risk Hedge Ratio

Hung et al. (2006) suggest a new hedge ratio that minimizes the Value-at-Risk of the hedged portfolio. Specifically, the hedge ratio $h$ is derived by minimizing the following Value-at-Risk of the hedged portfolio over a given time period $\tau$:

$$VaR(R_h) = Z_\alpha \sigma_h \sqrt{\tau} - E[R_h]\tau \tag{10}$$

The resulting optimal hedge ratio, which Hung et al. (2006) refer to as zero-VaR hedge ratio, is given by

$$h^{VaR} = \rho \frac{\sigma_s}{\sigma_f} - E\left[R_f\right] \frac{\sigma_s}{\sigma_f} \sqrt{\frac{1-\rho^2}{Z_\alpha^2 \sigma_f^2 - E\left[R_f\right]^2}} \tag{11}$$

It is clear that, if the futures price follows martingale process, the zero-VaR hedge ratio would be the same as the MV hedge ratio.

## Applications of OLS, GARCH, and CECM Models to Estimate Optimal Hedge Ratio

In this section, we will discuss different approaches to estimating the optimum hedge ratios and apply these methods in practice to estimate the optimal stock index futures hedge ratio for S&P 500 index. We first introduce how to apply the ordinary least square (OLS) technique to estimating the optimum hedge ratios and discuss its restrictions. Secondly, considering the heteroscedastic nature of the error term, generalized autoregressive conditional heteroskedasticity (GARCH) technique is introduced to estimating the hedge ratio. In addition, we describe cointegration and error correction method to take into account the possibility of long-run equilibrium between spot and futures prices. Finally, we provide an empirical study using OLS, GARCH, and CECM models to estimate the optimal stock index futures hedge ratio for S&P 500 index through R language.

## OLS Method

The conventional approach to estimating the MV hedge ratio involves the regression of the changes in spot prices on the changes in futures price using the OLS technique (e.g., see Junkus and Lee 1985). Specifically, the regression equation can be written as:

$$\Delta S_t = a_0 + a_1 \Delta F_t + e_t, \tag{12}$$

where the estimate of the MV hedge ratio, $H_j$, is given by $a_1$. The OLS technique is quite robust and simple to use. However, for the OLS technique to be valid and efficient, assumptions associated with the OLS regression must be satisfied. One case where the assumptions are not completely satisfied is that the error term in the regression is heteroscedastic. This situation will be discussed later.

Another problem with the OLS method, as pointed out by Myers and Thompson (1989), is the fact that it uses unconditional sample moments instead of conditional sample moments, which use currently available information. They suggest the use of the conditional covariance and conditional variance in Eq. (2a). In this case, the conditional version of the optimal hedge ratio (Eq. (2a)) will take the following form:

$$H_J^* = \frac{C_f}{C_s} = \frac{Cov(\Delta S, \Delta F) \mid \Omega_{t-1}}{Var(\Delta F) \mid \Omega_{t-1}}. \tag{13}$$

Suppose that the current information ($\Omega_{t-1}$) includes a vector of variables ($X_{t-1}$) and the spot and futures price changes are generated by the following equilibrium model:

$$\Delta S_t = X_{t-1}\alpha + u_t,$$

$$\Delta F_t = X_{t-1}\beta + v_t.$$

In this case, the maximum likelihood estimator of the MV hedge ratio is given by (see Myers and Thompson (1989)):

$$\hat{h} \mid X_{t-1} = \frac{\hat{\sigma}_{uv}}{\hat{\sigma}_v^2}, \tag{14}$$

where $\hat{\sigma}_{uv}$ is the sample covariance between the residuals $u_t$ and $v_t$, and $\hat{\sigma}_v^2$ is the sample variance of the residual $v_t$. In general, the OLS estimator obtained from Eq. (12) would be different from the one given by Eq. (14). For the two estimators to be the same, the spot and futures prices must be generated by the following model:

$$\Delta S_t = \alpha_0 + u_t, \qquad \Delta F_t = \beta_0 + v_t.$$

In other words, if the spot and futures prices follow a random walk, then with or without drift, the two estimators will be the same. Otherwise, the hedge ratio estimated from the OLS regression (12) will not be optimal.

## GARCH Method

Ever since the development of GARCH models, the OLS method of estimating the hedge ratio has been generalized to consider the heteroscedastic nature of the error term in Eq. (12). In this case, rather than using the unconditional sample variance and covariance, the conditional variance and covariance from the GARCH model are used in the estimation of the hedge ratio. As mentioned above, such a technique allows an update of the hedge ratio over the hedging period.

Consider the following bivariate GARCH model (see Cecchetti et al. 1988; and Baillie and Myers 1991):

$$\begin{bmatrix} \Delta S_t \\ \Delta F_t \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \text{ and } e_t \mid \Omega_{t-1} \sim N(0, H_t)$$

where $\Omega_{t-1}$ is all the available information up to time $t - 1$. $N$ denotes a bivariate normal distribution, and $H_t$ is a time-varying $2 \times 2$ positive-definite conditional variance-covariance matrix as follows.

$$H_t = \begin{bmatrix} H_{11,t} & H_{12,t} \\ H_{12,t} & H_{22,t} \end{bmatrix}$$

The conditional MV hedge ratio at time $t$ is given by $h_{t-1} = H_{12,t}/H_{22,t}$. This model allows the hedge ratio to change over time, resulting in a series of hedge ratios instead of a single hedge ratio for the entire hedging horizon.

In practice, it is necessary to model the conditional variance-covariance matrix and ensure the positive definiteness of $H_t$. Different models for $H_t$ have been proposed in the literature, for example, Bollerslev (1988), Bollerslev et al. (1988) and Engle and Kroner (1995). Bollerslev (1988) built up a VECH-GARCH (1,1) model assuming that $H_t$ is a linear combination of its own squared residual $e'_{t-1}$ and past conditional variance-covariance variance $H_{t-1}$ as follows.

$$vec(H_t) = C + A\ vec\left(e_{t-1}e'_{t-1}\right) + \mathrm{B}\ vec(H_{t-1}).$$

where *vec* is the operator that stacks a matrix as a column vector. A and B are both a $3 \times 3$ parameter matrices.

$$vec(H_t) = \begin{bmatrix} H_{11,t} \\ H_{12,t} \\ H_{22,t} \end{bmatrix}$$
$$= \begin{bmatrix} c_{01,t} \\ c_{02,t} \\ c_{03,t} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{3,t-1}^2 \end{bmatrix}$$
$$+ \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} H_{11,t-1} \\ H_{12,t-1} \\ H_{13,t-1} \end{bmatrix}$$

To simplify calculation, Bollerslev et al. (1988) suggest a diagonal VECH-GARCH (1,1) model that parameter matrices A and B are restricted to be diagonal.

$$vec(H_t) = \begin{bmatrix} H_{11,t} \\ H_{12,t} \\ H_{22,t} \end{bmatrix}$$
$$= \begin{bmatrix} c_{11}^* \\ c_{12}^* \\ c_{22}^* \end{bmatrix} + \begin{bmatrix} a_{11}^* & 0 & 0 \\ 0 & a_{12}^* & 0 \\ 0 & 0 & a_{22}^* \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}^2 \end{bmatrix}$$
$$+ \begin{bmatrix} g_{11}^* & 0 & 0 \\ 0 & g_{12}^* & 0 \\ 0 & 0 & g_{22}^* \end{bmatrix} \begin{bmatrix} H_{11,t-1} \\ H_{12,t-1} \\ H_{22,t-1} \end{bmatrix}$$

## Cointegration and Error Correction Method (CECM)

The techniques described so far do not take into consideration the possibility that spot price and futures price series could be non-stationary. If these series have unit roots, then this will raise a different issue. If the two series are cointegrated as defined by Engle and Granger (1987), then the regression Eq. (12) will be mis-specified and an error-correction term must be included in the equation. Since the arbitrage condition ties the spot and futures prices, they cannot drift far apart in the long run. Therefore, if both series follow a random walk, then we expect the two series to be cointegrated in which case we need to estimate the error correction model. This calls for the use of the cointegration analysis. Therefore, the cointegration and error correction method should be used to estimate the hedge ratio.

The cointegration analysis involves two steps. First, each series must be tested for a unit root (e.g., see Dickey and Fuller 1981; and Phillips and Perron 1988). Second, if both series are found to have a single unit root, then the cointegration test must be performed (e.g., see Engle and Granger 1987; Johansen and Juselius 1990; and Osterwald-Lenum 1992).

If the spot price and futures price series are found to be cointegrated, then the hedge ratio can be estimated in two steps (see Ghosh 1993; and Chou et al. 1996). The first step involves the estimation of the following cointegrating regression:

$$S_t = a + bF_t + u_t. \tag{15}$$

The second step involves the estimation of the following error correction model:

$$\Delta S_t = \rho u_{t-1} + \beta \Delta F_t + \sum_{i=1}^{m} \delta_i \Delta F_{t-i} + \sum_{j=1}^{n} \theta_i \Delta S_{t-j} + e_j, \tag{16}$$

where $u_t$ is the residual series from the cointegrating regression. The estimate of the hedge ratio is given by the estimate of $\beta$. Some researchers (e.g., see Lien and Luo 1993b) assume that the long-run cointegrating relationship is $(S_t - F_t)$, and estimate the following error correction model:

$$\Delta S_t = \rho(S_{t-1} - F_{t-1}) + \beta \Delta F_t + \sum_{i=1}^{m} \delta_i \Delta F_{t-i} + \sum_{j=1}^{n} \theta_i \Delta S_{t-j} + e_j. \tag{17}$$

Alternatively, Chou et al. (1996) suggest the estimation of the error correction model as follows:

$$\Delta S_t = \alpha \widehat{u}_{t-1} + \beta \Delta F_t + \sum_{i=1}^{m} \delta_i \Delta F_{t-i} + \sum_{j=1}^{n} \theta_i \Delta S_{t-j} + e_j, \tag{18}$$

where $\widehat{u}_{t-1} = S_{t-1} - (a + bF_{t-1})$; that is, the series $\widehat{u}_t$ is the estimated residual series from Eq. (15). The hedge ratio is given by $\beta$ in Eq. (17).

Kroner and Sultan (1993) combine the error-correction model with the GARCH model considered by Cecchetti et al. (1988) and Baillie and Myers (1991) in order to estimate the optimum hedge ratio. Specifically, they use the following model:

$$\begin{bmatrix} \Delta \log_e(S_t) \\ \Delta \log_e(F_t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \alpha_s(\log_e(S_{t-1}) - \log_e(F_{t-1})) \\ \alpha_f(\log_e(S_{t-1}) - \log_e(F_{t-1})) \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}, \tag{19}$$

where the error processes follow a GARCH process. As before, the hedge ratio at time $(t-1)$ is given by $h_{t-1} = H_{12,t}/H_{22,t}$.

## Empirical Studies

Here, we apply OLS, GARCH, and CECM models in practice to estimate optimal hedge ratios through R language. Monthly data for S&P 500 index and its futures were collected from Datastream database, the sample consisted of 188 observations from January 31, 2005, to August 31, 2020. Summary statistics of this data are presented in Table 1.

First, in Eq. (12), we use the OLS method by regressing the changes in spot prices on the changes in futures prices to estimate the optimal hedge ratio. The estimate of hedge ratio obtained from the OLS technique are reported in Table 2. As shown in Table 2, we can see that the hedge ratio of S&P 500 index is significantly different from zero, at a 1% significance level. Moreover, the estimated hedge ratio, denoted by the coefficient of $\Delta F_t$, is generally less than unity.

Secondly, we apply a conventional regression model with heteroscedastic error terms to estimate the hedge ratio. Here, an AR(2)-GARCH(1,1) model for the changes in spot prices regressed on the changes in futures prices is specified as follow,

$$\Delta S_t = a_0 + a_1 \Delta F_t + e_t, e_t = \varepsilon_t - \varphi_1 e_{t-1} - \varphi_2 e_{t-2}$$

$$\varepsilon_t = \sqrt{h_t}\,\epsilon_t, h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$

where $\epsilon_t \sim N(0, 1)$. The estimated result of AR(2)-GARCH(1,1) model is shown in Table 3. The coefficient estimates of AR(2)-GARCH(1,1) model, as shown in

**Table 1** Summary statistics

| | Min | Max | Mean | Std Dev | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| SPOT | 735.09 | 3500.31 | 1781.57 | 656.57 | 0.66 | 2.30 |
| FUTURES | 734.20 | 3498.90 | 1780.09 | 656.05 | 0.66 | 2.31 |
| $\Delta S_t$ | −369.63 | 327.84 | 12.17 | 79.23 | −0.76 | 7.92 |
| $\Delta F_t$ | −381.40 | 332.70 | 12.16 | 80.34 | −0.77 | 8.08 |
| Observations | 188 | | | | | |

**Table 2** Hedge ratio coefficient using the conventional regression model

| Variable | Estimate | Std. Error | t-ratio | p-value |
|---|---|---|---|---|
| Intercept | 0.1984 | 0.2729 | 0.73 | 0.4680 |
| $\Delta F_t$ | 0.9851 | 0.0034 | 292.53 | <0.0001 |

**Table 3** Hedge ratio coefficient using the conventional regression model with heteroscedastic errors

| Variable | Estimate | Std. Error | t-ratio | p-value |
|---|---|---|---|---|
| Intercept | 0.0490 | 0.0144 | 3.41 | 0.0007 |
| $\Delta F_t$ | 0.9994 | 0.0008 | 1179.59 | <0.0001 |
| $e_{t-1}$ | −0.9873 | 0.0109 | −90.29 | <0.0001 |
| $e_{t-2}$ | −0.9959 | 0.0145 | −68.83 | <0.0001 |
| $\omega$ | 0.0167 | 0.0098 | 1.71 | 0.0866 |
| $\varepsilon_{t-1}^2$ | 0.3135 | 0.0543 | 5.78 | <0.0001 |
| $h_{t-1}$ | 0.6855 | 0.0530 | 12.94 | <0.0001 |

Table 3, are all significantly different from zero, at a 1% significance level. This finding suggests that the importance of capturing the heteroscedastic error structures in conventional regression model. In addition, the hedge ratio of conventional regression with AR(2)-GARCH(1,1) model is higher than the OLS hedge ratio for S&P 500 futures contract.

Next, we will apply the CECM model to estimate optimal hedge ratio. Here, standard-augmented Dickey-Fuller (ADF) unit roots and Phillips and Ouliaris (1990) residual cointegration tests are performed, and the optimal hedge ratios estimated by error correction model (ECM) will be presented. Here, we apply the augmented Dickey-Fuller (ADF) regression to test for the presence of unit roots. The ADF test statistics, as shown in Panel A of Table 4, indicates that the null hypothesis of a unit root cannot be rejected for the levels of the variables. Using differenced data, the computed ADF test statistics shown in Panel B of Table 4 suggested that the null hypothesis is rejected, at the 1% significance level. As differencing one produces stationarity, we may conclude that each series is integrated of order one, I(1), process which is necessary for testing the existence of cointegration. We then apply Phillips and Ouliaris (1990) residual cointegration test to examine the presence of cointegration. The result of Phillips-Ouliaris cointegration test shown is reported in Panel C of Table 4. The null hypothesis of the Phillips-Ouliaris cointegration test is that there is no cointegration present. The result of Phillips-Ouliaris cointegration test indicates the null hypothesis of no cointegration is rejected, at the 1% significance level. This suggests that the spot S&P 500 index is cointegrated with the S&P 500 index futures.

Finally, we apply the ECM model in term of Eq. (16) to estimate optimal hedge ratio. Table 5 shows that the coefficient on the error-correction term, $\widehat{u}_{t-1}$, is significantly different from zero, at a 1% significance level. This suggests that the importance of estimating error correction model, and in particular the long-run equilibrium error term cannot be ignored in conventional regression model. In

**Table 4** Unit roots and residual cointegration tests results

| Variable | ADF statistics | lag parameter | p-value |
|---|---|---|---|
| *Panel A. Level data* | | | |
| Spot | −1.3353 | 1 | 0.8542 |
| Futures | −1.3458 | 1 | 0.8498 |
| *Panel B. First-order differenced data* | | | |
| Spot | −10.104 | 1 | <0.01 |
| Futures | −10.150 | 1 | <0.01 |
| *Panel C. Phillips-Ouliaris Cointegration Test* | | | |
| Phillips-Ouliaris demeaned | −60.783 | 1 | <0.01 |

**Table 5** Error correction estimates of hedge ratio coefficient

| Variable | Estimate | Std. Error | t-ratio | p-value |
|---|---|---|---|---|
| $\Delta F_t$ | 0.9892 | 0.0031 | 316.60 | <0.001 |
| $\widehat{u}_{t-1}$ | −0.3423 | 0.0571 | −5.99 | <0.001 |

addition, the ECM hedge ratio is higher than the conventional OLS hedge ratio for S&P 500 futures contract. This finding is consistent with the results in Lien (1996, 2004) who argued that the MV hedge ratio will be smaller if the cointegration relationship is not considered.

## Conclusion

In this chapter, we have reviewed four approaches for deriving the optimal hedge ratio. These approaches can be divided into the minimum-variance approach, the mean-variance based approach, and the expected utility maximizing approach. All these approaches will lead to the same hedge ratio as the conventional minimum-variance (MV) hedge ratio, if the futures price follows a pure martingale process and if the futures and spot prices are jointly normal. However, if these conditions do not hold, then the hedge ratios based on the various approaches will be different, therefore the user should carefully examine what kind of assumption they will use.

Since the statistical properties of the MV hedge ratio are well known, it is easy to estimate and perform the statistical hypothesis test. For example, we can test whether the optimal MV hedge ratio is the same as the naïve hedge ratio. Since the MV hedge ratio ignores the expected return, it will not be consistent with the mean-variance analysis unless the futures price follows a pure martingale process. In general, if the martingale and normality conditions do not hold, then the MV hedge ratio will not be consistent with the expected utility maximization principle in terms of mean and variance.

In order to make the hedge ratio consistent with the expected utility maximization principle, we can derive the optimal hedge ratio by maximizing the expected utility in terms of mean and variance. However, to implement such an approach, we need to assume a specific utility function such as quadratic utility function, and we need to make an assumption regarding the return distribution. Therefore, different utility functions will lead to different optimal hedge ratios.

It should be noted that the derivation of the optimal hedge ratio discussed in this chapter does not incorporate transaction costs. Furthermore, these derivations do not allow investments in securities other than the spot and corresponding futures contracts. As shown by Lence (1995), once we relax these conventional assumptions, the resulting optimal hedge ratio can be quite different from the ones obtained under the conventional assumptions. Lence's (1995) results are based on a specific utility function and some other assumption regarding the return distributions.

In this chapter, we have also reviewed various ways of estimating the optimum hedge ratio. As far as the estimation of the conventional MV hedge ratio is concerned, there are a large number of methods that have been proposed in the literature. These methods range from a simple regression method to complex cointegrated heteroscedastic methods with regime-switching, and some of the estimation methods include a kernel density function method as well as an empirical distribution method.

In this chapter, we have also discussed about the relationship between the optimal MV hedge ratio and the hedging horizon. We feel that this relationship has not been fully explored and can be further developed in the future. For example, we would like to know if the optimal hedge ratio approaches the naïve hedge ratio when the hedging horizon becomes longer.

In this chapter, we used S&P 500 index futures from January 2005 to August 2020 in terms of three estimation methods to estimate hedge ratio. These three methods are OLS regression method, GARCH method, and CECM method. We also compared hedge ratio estimates from these three methods in some detail. The R programs used to estimate this empirical result can be found in Appendix B.

For those who are interested in research in this area, we would like to finally point out that one requires a good understanding of financial economic theories and econometric methodologies. In addition, a good background in data analysis and computer programing would also be helpful.

## Appendix A. Monthly Data of S&P500 Index and Its Futures (January 2005 – August 2020)

| Date | SPOT | FUTURES | C_spot | C_futures |
|---|---|---|---|---|
| 1/31/2005 | 1181.27 | 1181.7 | −30.65 | −32 |
| 2/28/2005 | 1203.6 | 1204.1 | 22.33 | 22.4 |
| 3/31/2005 | 1180.59 | 1183.9 | −23.01 | −20.2 |
| 4/29/2005 | 1156.85 | 1158.5 | −23.74 | −25.4 |
| 5/31/2005 | 1191.5 | 1192.3 | 34.65 | 33.8 |
| 6/30/2005 | 1191.33 | 1195.5 | −0.17 | 3.2 |
| 7/29/2005 | 1234.18 | 1236.8 | 42.85 | 41.3 |
| 8/31/2005 | 1220.33 | 1221.4 | −13.85 | −15.4 |
| 9/30/2005 | 1228.81 | 1234.3 | 8.48 | 12.9 |
| 10/31/2005 | 1207.01 | 1209.8 | −21.8 | −24.5 |
| 11/30/2005 | 1249.48 | 1251.1 | 42.47 | 41.3 |
| 12/30/2005 | 1248.29 | 1254.8 | −1.19 | 3.7 |
| 1/31/2006 | 1280.08 | 1283.6 | 31.79 | 28.8 |
| 2/28/2006 | 1280.66 | 1282.4 | 0.58 | −1.2 |
| 3/31/2006 | 1294.83 | 1303.3 | 14.17 | 20.9 |
| 4/28/2006 | 1310.61 | 1315.9 | 15.78 | 12.6 |
| 5/31/2006 | 1270.09 | 1272.1 | −40.52 | −43.8 |
| 6/30/2006 | 1270.2 | 1279.4 | 0.11 | 7.3 |
| 7/31/2006 | 1276.66 | 1281.8 | 6.46 | 2.4 |
| 8/31/2006 | 1303.82 | 1305.6 | 27.16 | 23.8 |
| 9/29/2006 | 1335.85 | 1345.4 | 32.03 | 39.8 |
| 10/31/2006 | 1377.94 | 1383.2 | 42.09 | 37.8 |
| 11/30/2006 | 1400.63 | 1402.9 | 22.69 | 19.7 |
| 12/29/2006 | 1418.3 | 1428.4 | 17.67 | 25.5 |
| 1/31/2007 | 1438.24 | 1443 | 19.94 | 14.6 |

(continued)

| Date | SPOT | FUTURES | C_spot | C_futures |
|---|---|---|---|---|
| 2/28/2007 | 1406.82 | 1408.9 | −31.42 | −34.1 |
| 3/30/2007 | 1420.86 | 1431.2 | 14.04 | 22.3 |
| 4/30/2007 | 1482.37 | 1488.4 | 61.51 | 57.2 |
| 5/31/2007 | 1530.62 | 1532.9 | 48.25 | 44.5 |
| 6/29/2007 | 1503.35 | 1515.4 | −27.27 | −17.5 |
| 7/31/2007 | 1455.27 | 1461.9 | −48.08 | −53.5 |
| 8/31/2007 | 1473.99 | 1476.7 | 18.72 | 14.8 |
| 9/28/2007 | 1526.75 | 1538.1 | 52.76 | 61.4 |
| 10/31/2007 | 1549.38 | 1554.9 | 22.63 | 16.8 |
| 11/30/2007 | 1481.14 | 1483.7 | −68.24 | −71.2 |
| 12/31/2007 | 1468.35 | 1477.2 | −12.79 | −6.5 |
| 1/31/2008 | 1378.55 | 1379.6 | −89.8 | −97.6 |
| 2/29/2008 | 1330.63 | 1331.3 | −47.92 | −48.3 |
| 3/31/2008 | 1322.7 | 1324 | −7.93 | −7.3 |
| 4/30/2008 | 1385.59 | 1386 | 62.89 | 62 |
| 5/30/2008 | 1400.38 | 1400.6 | 14.79 | 14.6 |
| 6/30/2008 | 1280 | 1281.1 | −120.38 | −119.5 |
| 7/31/2008 | 1267.38 | 1267.1 | −12.62 | −14 |
| 8/29/2008 | 1282.83 | 1282.6 | 15.45 | 15.5 |
| 9/30/2008 | 1166.36 | 1169 | −116.47 | −113.6 |
| 10/31/2008 | 968.75 | 967.3 | −197.61 | −201.7 |
| 11/28/2008 | 896.24 | 895.3 | −72.51 | −72 |
| 12/31/2008 | 903.25 | 900.1 | 7.01 | 4.8 |
| 1/30/2009 | 825.88 | 822.5 | −77.37 | −77.6 |
| 2/27/2009 | 735.09 | 734.2 | −90.79 | −88.3 |
| 3/31/2009 | 797.87 | 794.8 | 62.78 | 60.6 |
| 4/30/2009 | 872.81 | 870 | 74.94 | 75.2 |
| 5/29/2009 | 919.14 | 918.1 | 46.33 | 48.1 |
| 6/30/2009 | 919.32 | 915.5 | 0.18 | −2.6 |
| 7/31/2009 | 987.48 | 984.4 | 68.16 | 68.9 |
| 8/31/2009 | 1020.62 | 1019.7 | 33.14 | 35.3 |
| 9/30/2009 | 1057.08 | 1052.9 | 36.46 | 33.2 |
| 10/30/2009 | 1036.19 | 1033 | −20.89 | −19.9 |
| 11/30/2009 | 1095.63 | 1094.8 | 59.44 | 61.8 |
| 12/31/2009 | 1115.1 | 1110.7 | 19.47 | 15.9 |
| 1/29/2010 | 1073.87 | 1070.4 | −41.23 | −40.3 |
| 2/26/2010 | 1104.49 | 1103.4 | 30.62 | 33 |
| 3/31/2010 | 1169.43 | 1165.2 | 64.94 | 61.8 |
| 4/30/2010 | 1186.69 | 1183.4 | 17.26 | 18.2 |
| 5/31/2010 | 1089.41 | 1088.5 | −97.28 | −94.9 |
| 6/30/2010 | 1030.71 | 1026.6 | −58.7 | −61.9 |
| 7/30/2010 | 1101.6 | 1098.3 | 70.89 | 71.7 |
| 8/31/2010 | 1049.33 | 1048.3 | −52.27 | −50 |
| 9/30/2010 | 1141.2 | 1136.7 | 91.87 | 88.4 |

(continued)

| Date | SPOT | FUTURES | C_spot | C_futures |
|---|---|---|---|---|
| 10/29/2010 | 1183.26 | 1179.7 | 42.06 | 43 |
| 11/30/2010 | 1180.55 | 1179.6 | −2.71 | −0.1 |
| 12/31/2010 | 1257.64 | 1253 | 77.09 | 73.4 |
| 1/31/2011 | 1286.12 | 1282.4 | 28.48 | 29.4 |
| 2/28/2011 | 1327.22 | 1326.1 | 41.1 | 43.7 |
| 3/31/2011 | 1325.83 | 1321 | −1.39 | −5.1 |
| 4/29/2011 | 1363.61 | 1359.7 | 37.78 | 38.7 |
| 5/31/2011 | 1345.2 | 1343.9 | −18.41 | −15.8 |
| 6/30/2011 | 1320.64 | 1315.5 | −24.56 | −28.4 |
| 7/29/2011 | 1292.28 | 1288.4 | −28.36 | −27.1 |
| 8/31/2011 | 1218.89 | 1217.7 | −73.39 | −70.7 |
| 9/30/2011 | 1131.42 | 1126 | −87.47 | −91.7 |
| 10/31/2011 | 1253.3 | 1249.3 | 121.88 | 123.3 |
| 11/30/2011 | 1246.96 | 1246 | −6.34 | −3.3 |
| 12/30/2011 | 1257.6 | 1252.6 | 10.64 | 6.6 |
| 1/31/2012 | 1312.41 | 1308.2 | 54.81 | 55.6 |
| 2/29/2012 | 1365.68 | 1364.4 | 53.27 | 56.2 |
| 3/30/2012 | 1408.47 | 1403.2 | 42.79 | 38.8 |
| 4/30/2012 | 1397.91 | 1393.6 | −10.56 | −9.6 |
| 5/31/2012 | 1310.33 | 1309.2 | −87.58 | −84.4 |
| 6/29/2012 | 1362.16 | 1356.4 | 51.83 | 47.2 |
| 7/31/2012 | 1379.32 | 1374.6 | 17.16 | 18.2 |
| 8/31/2012 | 1406.58 | 1405.1 | 27.26 | 30.5 |
| 9/28/2012 | 1440.67 | 1434.2 | 34.09 | 29.1 |
| 10/31/2012 | 1412.16 | 1406.8 | −28.51 | −27.4 |
| 11/30/2012 | 1416.18 | 1414.4 | 4.02 | 7.6 |
| 12/31/2012 | 1426.19 | 1420.1 | 10.01 | 5.7 |
| 1/31/2013 | 1498.11 | 1493.3 | 71.92 | 73.2 |
| 2/28/2013 | 1514.68 | 1513.3 | 16.57 | 20 |
| 3/29/2013 | 1569.19 | 1562.7 | 54.51 | 49.4 |
| 4/30/2013 | 1597.57 | 1592.2 | 28.38 | 29.5 |
| 5/31/2013 | 1630.74 | 1629 | 33.17 | 36.8 |
| 6/28/2013 | 1606.28 | 1599.3 | −24.46 | −29.7 |
| 7/31/2013 | 1685.73 | 1680.5 | 79.45 | 81.2 |
| 8/30/2013 | 1632.97 | 1631.3 | −52.76 | −49.2 |
| 9/30/2013 | 1681.55 | 1674.3 | 48.58 | 43 |
| 10/31/2013 | 1756.54 | 1751 | 74.99 | 76.7 |
| 11/29/2013 | 1805.81 | 1804.1 | 49.27 | 53.1 |
| 12/31/2013 | 1848.36 | 1841.1 | 42.55 | 37 |
| 1/31/2014 | 1782.59 | 1776.6 | −65.77 | −64.5 |
| 2/28/2014 | 1859.45 | 1857.6 | 76.86 | 81 |
| 3/31/2014 | 1872.34 | 1864.6 | 12.89 | 7 |
| 4/30/2014 | 1883.95 | 1877.9 | 11.61 | 13.3 |
| 5/30/2014 | 1923.57 | 1921.5 | 39.62 | 43.6 |

(continued)

| Date | SPOT | FUTURES | C_spot | C_futures |
|---|---|---|---|---|
| 6/30/2014 | 1960.23 | 1952.4 | 36.66 | 30.9 |
| 7/31/2014 | 1930.67 | 1924.8 | −29.56 | −27.6 |
| 8/29/2014 | 2003.37 | 2001.4 | 72.7 | 76.6 |
| 9/30/2014 | 1972.29 | 1965.5 | −31.08 | −35.9 |
| 10/31/2014 | 2018.05 | 2011.4 | 45.76 | 45.9 |
| 11/28/2014 | 2067.56 | 2066.3 | 49.51 | 54.9 |
| 12/31/2014 | 2058.9 | 2052.4 | −8.66 | −13.9 |
| 1/30/2015 | 1994.99 | 1988.4 | −63.91 | −64 |
| 2/27/2015 | 2104.5 | 2102.8 | 109.51 | 114.4 |
| 3/31/2015 | 2067.89 | 2060.8 | −36.61 | −42 |
| 4/30/2015 | 2085.51 | 2078.9 | 17.62 | 18.1 |
| 5/29/2015 | 2107.39 | 2106 | 21.88 | 27.1 |
| 6/30/2015 | 2063.11 | 2054.4 | −44.28 | −51.6 |
| 7/31/2015 | 2103.84 | 2098.4 | 40.73 | 44 |
| 8/31/2015 | 1972.18 | 1969.2 | −131.66 | −129.2 |
| 9/30/2015 | 1920.03 | 1908.7 | −52.15 | −60.5 |
| 10/30/2015 | 2079.36 | 2073.7 | 159.33 | 165 |
| 11/30/2015 | 2080.41 | 2079.8 | 1.05 | 6.1 |
| 12/31/2015 | 2043.94 | 2035.4 | −36.47 | −44.4 |
| 1/29/2016 | 1940.24 | 1930.1 | −103.7 | −105.3 |
| 2/29/2016 | 1932.23 | 1929.5 | −8.01 | −0.6 |
| 3/31/2016 | 2059.74 | 2051.5 | 127.51 | 122 |
| 4/29/2016 | 2065.3 | 2059.1 | 5.56 | 7.6 |
| 5/31/2016 | 2096.96 | 2094.9 | 31.66 | 35.8 |
| 6/30/2016 | 2098.86 | 2090.2 | 1.9 | −4.7 |
| 7/29/2016 | 2173.6 | 2168.2 | 74.74 | 78 |
| 8/31/2016 | 2170.95 | 2169.5 | −2.65 | 1.3 |
| 9/30/2016 | 2168.27 | 2160.4 | −2.68 | −9.1 |
| 10/31/2016 | 2126.15 | 2120.1 | −42.12 | −40.3 |
| 11/30/2016 | 2198.81 | 2198.8 | 72.66 | 78.7 |
| 12/30/2016 | 2238.83 | 2236.2 | 40.02 | 37.4 |
| 1/31/2017 | 2278.87 | 2274.5 | 40.04 | 38.3 |
| 2/28/2017 | 2363.64 | 2362.8 | 84.77 | 88.3 |
| 3/31/2017 | 2362.72 | 2359.2 | −0.92 | −3.6 |
| 4/28/2017 | 2384.2 | 2380.5 | 21.48 | 21.3 |
| 5/31/2017 | 2411.8 | 2411.1 | 27.6 | 30.6 |
| 6/30/2017 | 2423.41 | 2420.9 | 11.61 | 9.8 |
| 7/31/2017 | 2470.3 | 2468 | 46.89 | 47.1 |
| 8/31/2017 | 2471.65 | 2470.1 | 1.35 | 2.1 |
| 9/29/2017 | 2519.36 | 2516.1 | 47.71 | 46 |
| 10/31/2017 | 2575.26 | 2572.7 | 55.9 | 56.6 |
| 11/30/2017 | 2647.58 | 2647.9 | 72.32 | 75.2 |
| 12/29/2017 | 2673.61 | 2676 | 26.03 | 28.1 |
| 1/31/2018 | 2823.81 | 2825.8 | 150.2 | 149.8 |

(continued)

| Date | SPOT | FUTURES | C_spot | C_futures |
|---|---|---|---|---|
| 2/28/2018 | 2713.83 | 2714.4 | −109.98 | −111.4 |
| 3/30/2018 | 2640.87 | 2643 | −72.96 | −71.4 |
| 4/30/2018 | 2648.05 | 2647 | 7.18 | 4 |
| 5/31/2018 | 2705.27 | 2705.5 | 57.22 | 58.5 |
| 6/29/2018 | 2718.37 | 2721.6 | 13.1 | 16.1 |
| 7/31/2018 | 2816.29 | 2817.1 | 97.92 | 95.5 |
| 8/31/2018 | 2901.52 | 2902.1 | 85.23 | 85 |
| 9/28/2018 | 2913.98 | 2919 | 12.46 | 16.9 |
| 10/31/2018 | 2711.74 | 2711.1 | −202.24 | −207.9 |
| 11/30/2018 | 2760.17 | 2758.3 | 48.43 | 47.2 |
| 12/31/2018 | 2506.85 | 2505.2 | −253.32 | −253.1 |
| 1/31/2019 | 2704.1 | 2704.5 | 197.25 | 199.3 |
| 2/28/2019 | 2784.49 | 2784.7 | 80.39 | 80.2 |
| 3/29/2019 | 2834.4 | 2837.8 | 49.91 | 53.1 |
| 4/30/2019 | 2945.83 | 2948.5 | 111.43 | 110.7 |
| 5/31/2019 | 2752.06 | 2752.6 | −193.77 | −195.9 |
| 6/28/2019 | 2941.76 | 2944.2 | 189.7 | 191.6 |
| 7/31/2019 | 2980.38 | 2982.3 | 38.62 | 38.1 |
| 8/30/2019 | 2926.46 | 2924.8 | −53.92 | −57.5 |
| 9/30/2019 | 2976.74 | 2978.5 | 50.28 | 53.7 |
| 10/31/2019 | 3037.56 | 3035.8 | 60.82 | 57.3 |
| 11/29/2019 | 3140.98 | 3143.7 | 103.42 | 107.9 |
| 12/31/2019 | 3230.78 | 3231.1 | 89.8 | 87.4 |
| 1/31/2020 | 3225.52 | 3224 | −5.26 | −7.1 |
| 2/28/2020 | 2954.22 | 2951.1 | −271.3 | −272.9 |
| 3/31/2020 | 2584.59 | 2569.7 | −369.63 | −381.4 |
| 4/30/2020 | 2912.43 | 2902.4 | 327.84 | 332.7 |
| 5/29/2020 | 3044.31 | 3042 | 131.88 | 139.6 |
| 6/30/2020 | 3100.29 | 3090.2 | 55.98 | 48.2 |
| 7/31/2020 | 3271.12 | 3263.5 | 170.83 | 173.3 |
| 8/31/2020 | 3500.31 | 3498.9 | 229.19 | 235.4 |

## Appendix B. Applications of R Language in Estimating Three Different Optimal Hedge Ratios

In this appendix, we show the estimation procedure on how to apply OLS, GARCH, and CECM models to estimate optimal hedge ratios through R language. R language is a high-level computer language that is designed for statistics and graphics. Compared to alternatives, SAS, Matlab or Stata, R is completely free. Another benefit is that it is open source. Users could head to http://cran.r-project.org/ to download and install R language. Based upon monthly S&P 500 index and its futures as presented in Appendix A, the estimation procedures of applying R language to estimate hedge ratio are provided as follows.

First, we use OLS method in term of Eq. (12) to estimate minimum variance hedge ratio. By using linear model (`lm`) function in R language, we obtain the following program code.

```
SP500= read.csv(file="SP500.csv")
OLS.fit <- lm(C_spot~C_futures, data=SP500)
summary(OLS.fit)
```

Next, we apply a conventional regression model with an AR(2)-GARCH(1,1) error terms to estimate minimum variance hedge ratio. By using `rugarch` package in R language, we obtain the following program.

```
library(rugarch)
fit.spec <- ugarchspec(
variance.model = list(model = "sGARCH",
 garchOrder = c(1, 1)),
mean.model = list(armaOrder = c(2, 0),include.mean = TRUE,
 external.regressors= cbind(SP500$C_futures)),
 distribution.model = "norm")

GARCH.fit <- ugarchfit(data = cbind(SP500$C_spot),
 spec = fit.spec)
GARCH.fit
```

Third, we apply the ECM model to estimate minimum variance hedge ratio. We begin by applying an augmented Dickey-Fuller (ADF) test for the presence of unit roots. The Phillips and Ouliaris (1990) residual cointegration test is applied to examine the presence of cointegration. Finally, the minimum variance hedge ratio is estimated by the error correction model. By using `tseries` package in R language, we obtain the following program.

```
library(tseries)
```

# Augmented Dickey-Fuller Test
# Level data

```
adf.test(SP500$SPOT, k = 1)
adf.test(SP500$FUTURES, k = 1)
```

# First-order differenced data

```
adf.test(diff(SP500$SPOT), k = 1)
adf.test(diff(SP500$FUTURES), k = 1)
```

# Phillips and Ouliaris (1990) residual cointegration test

```
po.test(cbind(SP500$FUTURES,SP500$SPOT))
```

```
# Engle-Granger two-step procedure
## 1.Estimate cointegrating relationship
```

```
reg <- lm(SPOT~FUTURES, data=SP500)
```

```
## 2. Compute error term
```

```
Resid <- reg$resid
```

```
# Estimate optimal hedge ratio using the error correction model
```

```
n=length(resid)
ECM.fit <-lm(diff(SPOT) ~ -1 + diff(FUTURES) + Resid[1:n-1],
data=SP500)
summary(ECM.fit)
```

Finally, we apply a multivariate GARCH(1,1)-BEKK model to estimate dynamic minimum variance hedge ratio.

```
library(mgarchBEKK)
estimated=BEKK(as.matrix(cbind(SP500$C_futures,SP500$C_spot)),
 order = c(1, 1))
estimated$est.params
estimated$asy.se.coef
```

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