EVALUATING AND HEDGING EXOTIC SWAP INSTRUMENTS VIA LGM

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Abstract. Here we use the one factor LGM model to price the standard IR exotic deals: callable swaps (including amortizing swaps), callable inverse floaters, callable super-floaters, callable range notes, autocaps and revolvers, and range notes. We lay out the complete pricing of these deals: how to represent the deals, how to select the calibration instruments, how to determine the appropriate calibration strategies and algorithms, the different deal evaluation algorithms for each deal type, and the usage of adjusters to obtain the best possible prices and hedges.

We then extend this analysis to the world of bonds. By incorporating a second, credit spread factor into the LGM model, we extend our valuation/hedging analysis to bonds with embedded options

Key words. LGM, interest rate models, calibration, exotic options

(Part head:)Introduction

In this paper we specify how to price and hedge common exotic interest rate deals: callable swaps (including amortizing swaps), callable inverse floaters, callable capped- floaters and super-floaters, callable range notes, autocaps, revolvers, and captions.

Here we introduce our notation and the mathematics of swaps and vanilla options.

In Part II, we work out "best practices" for using the one factor LGM mode for pricing and hedging: We introduce the LGM model, we determine how to select effective calibration instruments, we derive efficient calibration strategies and algorithms, we then discuss evaluation methodologies and determine efficient algorithms for evaluating the prices of exotic instruments under the calibrated model. In each of the subsequent sections, we treat a different exotic: callable swaps (with Bermudan and American), callable amortizing swaps, callable inverse floaters, callable capped-floaters and super-floaters, callable range notes and accrual, autocaps, and revolvers. Throughout these sections we use best practices in choosing the calibration instruments, calibrating the model, evaluating the deal, and using internal adjustors to obtain risks to the proper instrumentsand clean up the prices.

In Part III, we extend the model to include a second factor for credit. The extended LGM model is used to price bond, focussing on bonds with embedded options.

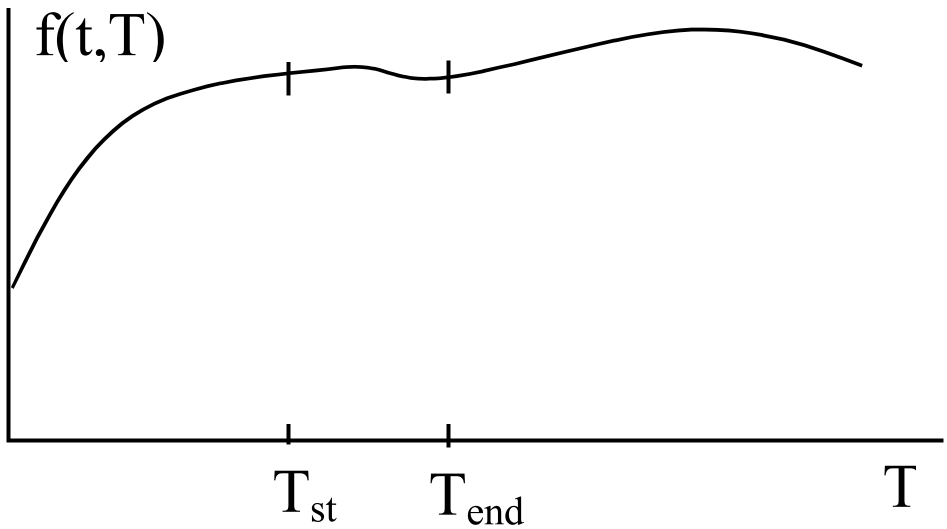
1. Discount factors, zeros, and FRAs. Suppose at date t, one agrees to loan out \$1 at date T, and get repaid the next day:

$$\text{(1.1a)} \qquad 1 \qquad \text{paid at } T,$$

$$\text{(1.1b)} \qquad 1 + f(t,T)\Delta T \qquad \text{received at } T + \Delta T.$$

By definition, the fair interest rate to charge is

$$\text{(1.1c)} \qquad f(t,T) = \text{instantaneous forward rate for date } T \text{ as seen at date } t.$$



Instantaneous rate for date T as seen at date t

Now suppose at date t one agrees to loan out \$1 on T_{st}, with the money repaid on T_{end}. Economically this is equivalent to loaning out \$1 on T_{st}, getting repaid \$1 plus interest the next day, re-loaning out the \$1 plus interest, getting repaid \$1 plus interest plus interest on the interest, ... Clearly, if one agrees at date t to loan out

$$1 \qquad \text{paid at } T_{st}, \tag{1.2a}$$

the agreement should specify getting repaid

$$e^{\int_{T_{st}}^{T_{end}} f(t,T')dT'} \qquad \text{received at } T_{end} \tag{1.2b}$$

for the deal to be fair. Alternatively, we can re-phrase this as

$$e^{-\int_{T_{st}}^{T_{end}} f(t,T')dT'} \qquad \text{paid at } T_{st}, \tag{1.3a}$$

$$1 \qquad \text{received at } T_{end}. \tag{1.3b}$$

This type of single payment deal is equivalent to a FRA (forward rate agreement).

Suppose we imagine that we are at date t, and we ask how much would I need to pay immediately to receive \$1 at date T. Clearly the fair amount is

$$e^{-\int_{t}^{T_{end}} f(t,T')dT'} \qquad \text{paid at } t, \tag{1.4a}$$

$$1 \qquad \text{received at } T_{end}. \tag{1.4b}$$

By definition,

$$\hat{Z}(t;T) = e^{-\int_t^T f(t,T')dT'} = \text{value at } t \text{ of \$1 paid at } T, \tag{1.5}$$

is the value of a zero coupon bond for maturity T on date t.

Today is always $t = 0$ in our notation. *Discount factors* are today's values of the zero coupon bonds:

$$D(T) = \hat{Z}(0;T) = e^{-\int_0^T f_0(T')dT'}, \tag{1.6a}$$

where

$$f_0(T) = f(0,T) = \text{today's instantaneous forward rate curve.} \tag{1.6b}$$

The discount factors and today's forward curve are not random. We can always get their values by stripping the current swap curve. On the other hand, $f(t,T)$ and $\hat{Z}(t;T)$ will fluctuate until *now* catches up to date t. This is why we use different notation for discount factors and zero coupon bonds in general.

2. Swaps.

2.1. Fixed leg. Consider a swap with start date t_0, fixed leg pay dates $t_1, t_2, \ldots, t_n$, and fixed rate R^{fix}. The fixed leg makes the payments

$$\alpha_i R^{fix} \qquad \text{paid at } t_i \qquad \text{for } i = 1, 2, \ldots, n-1, \tag{2.1a}$$

$$1 + \alpha_n R^{fix} \qquad \text{paid at } t_n, \tag{2.1b}$$

where

$$\alpha_i = \mathrm{cvg}(t_{i-1}, t_i, \beta) \tag{2.1c}$$

is the coverage (day count fraction) for interval i computed according to the appropriate day count basis β. On any given day t, these payments have the value

$$\hat{V}_{fix}(t) = R^{fix} \sum_{i=1}^{n} \alpha_i \hat{Z}(t;t_i) + \hat{Z}(t;t_n). \tag{2.2}$$

The mechanics of the swap market is covered in Appendix A. There we explain how to correctly construct date sequences and compute coverages.for the major currencies.

2.2. Floating leg. Let us now consider the swap's floating leg. Floating legs usually have a different frequency than the fixed legs, so let this leg's start and pay dates be

$$t_0 = \tau_0, \tau_1, \ldots, \tau_m = t_n \tag{2.3}$$

The floating leg pays

$$\tilde{\alpha}_j r_j \qquad \text{paid at } \tau_j \qquad \text{for } j = 1, 2, \ldots, m-1, \tag{2.4a}$$

$$1 + \tilde{\alpha}_m r_m \qquad \text{paid at } \tau_m = t_n, \tag{2.4b}$$

where

$$\tilde{\alpha}_j = \mathrm{cvg}(\tau_{j-1}, \tau_j, \tilde{\beta}) \tag{2.4c}$$

is the coverage for interval j computed according to the appropriate day count basis $\tilde{\beta}$. Here r_j is generally the Libor or Euribor floating rate for interval j. This rate is set on the *fixing date*; for most floating legs, the fixing date is two London business days before the interval starts on τ_{j-1}.

The floating rate represents a deal in which one invests 1 unit of the ccy at τ_{j-1} and receives $1 + \tilde{\alpha}_j r_j$ units back at τ_j. *In principle,* the value of the two payments must be the same at the fixing date τ_j^{fix}, so

$$\hat{Z}(\tau_j^{fix}; \tau_{j-1}) = (1 + \tilde{\alpha}_j r_j)\, \hat{Z}(t; \tau_j). \tag{2.5a}$$

In theory, then, the value of the interest payment $\tilde{\alpha}_j r_j$ would be

$$V_j^{theor}(\tau_j^{fix}) = \hat{Z}(\tau_j^{fix}; \tau_{j-1}) - \hat{Z}(\tau_j^{fix}; \tau_j) \tag{2.5b}$$

on the fixing date τ_j^{fix}. If the value of the floating rate payment is the difference of two freely tradeable securities (the zero coupon bonds) at the fixing time, then the value must equal this difference for all earlier times as well. So *in principle*, the value of the j^{th} floating interest rate payment is

$$V_j^{theor}(t) = \hat{Z}(t;\tau_{j-1}) - \hat{Z}(t;\tau_j) \qquad \text{for } t < \tau_j^{fix} \tag{2.6}$$

for any date t, at least until the rate is fixed. At any date t, the forward *fair* or *true* rate $r_j^{true}(t)$ is defined so that the value of the interest payment exactly equals the theoretical value:

$$\tilde{\alpha}_j r_j^{true}(t)\hat{Z}(t;\tau_j) = \text{theoretical value of interest rate payment} = \hat{Z}(t;\tau_{j-1}) - \hat{Z}(t;\tau_j). \tag{2.7a}$$

So

$$r_j^{true}(t) = \frac{\hat{Z}(t;\tau_{j-1}) - \hat{Z}(t;\tau_j)}{\tilde{\alpha}_j \hat{Z}(t;\tau_j)}. \tag{2.7b}$$

In practice, floating rates are not set at the fair rate, they are set at the fair rate plus a small offset s_j, the *forward basis spread*, due to credit considerations and supply and demand. The value of the (forward) basis spread depends on which index is used for the floating rate (3m USD Libor, 1m FedFunds, 6m Euribor, etc.), and on the starting date. The value of the floating rate payment paid at τ_j is

$$\hat{V}_j(t) = \hat{Z}(t;\tau_{j-1}) - \hat{Z}(t;\tau_j) + \tilde{\alpha}_j s_j \hat{Z}(t;\tau_j). \tag{2.8a}$$

By definition, the forward rate for the floating rate is defined by,

$$\tilde{\alpha}_j r_j^{fwd}(t)\hat{Z}(t;\tau_j) = \text{value of interest rate payment}, \tag{2.8b}$$

so

$$r_j^{fwd} = r_j^{true}(t) + s_j = \frac{\hat{Z}(t;\tau_{j-1}) - \hat{Z}(t;\tau_j)}{\tilde{\alpha}_j \hat{Z}(t;\tau_j)} + s_j. \tag{2.8c}$$

Basis spread curves are obtained by stripping basis swaps. One can show that forward basis spreads are Martingales in the appropriate forward measures. Since they are very small, usually just $1-2$ bps, and since they seldom vary, one always assumes they are constant. That is, one assumes that the gamma of the forward spread is inconsequential.

Summing these payments together, the value of the floating leg is

$$\hat{V}_{flt}(t) = \hat{Z}(t;t_0) + \sum_{j=1}^{m} \tilde{\alpha}_j s_j \hat{Z}(t;\tau_j). \tag{2.9}$$

This is true regardless of the model used. The value of the receiver swap (receive the fixed leg, pay the floating leg) is

$$\hat{V}_{rec}(t) = R^{fix}\sum_{i=1}^{n} \alpha_i \hat{Z}(t;t_i) + \hat{Z}(t;t_n) - \hat{Z}(t;t_0) - \sum_{j=1}^{m} \tilde{\alpha}_j s_j \hat{Z}(t;\tau_j). \tag{2.10a}$$

The value of the payer swap (pay the fixed leg, receive the floating leg) is

$$\hat{V}_{pay}(t) = -\hat{V}_{rec}(t) = \hat{Z}(t;t_0) + \sum_{j=1}^{m} \tilde{\alpha}_j s_j \hat{Z}(t;\tau_j) - R^{fix}\sum_{i=1}^{n} \alpha_i \hat{Z}(t;t_i) - \hat{Z}(t;t_n). \tag{2.10b}$$

2.3. Handling the basis spread. Basis spreads are a nuisance. They are large enough that one cannot neglect them entirely (except for USD 3m Libor), and small enough that they are nearly irrelevent. One way of handling them is to treat them as another, very small, fixed leg. There is really nothing wrong with that approach, although one usually has twice as many fixed leg pay dates.

We use a second, common approach in which each interval's fixed rate is adjusted to account for the value of the basis spreads. I.e, if the basis spread is 0.625bps in an interval, then we subtract 0.625bps from the fixed leg's rate instead of adding it to the floating leg. More precisely, today's value of the swap is,

$$\hat{V}_{rec}(0) = \sum_{i=1}^{n} \alpha_i R^{fix} D(t_i) + D(t_n) - D(t_0) - \sum_{j=1}^{m} \tilde{\alpha}_j s_j D(\tau_j). \tag{2.11}$$

which is the same as

$$\hat{V}_{rec}(0) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) D(t_i) + D(t_n) - D(t_0)./ \tag{2.12a}$$

Here S_i is the basis spread expressed with the same frequency and day count basis as the fixed leg. If the floating leg frequency is the same or higher than the fixed leg frequency, then

$$S_i = \frac{\sum_{j \in I_i} \tilde{\alpha}_j s_j D(\tau_j)}{\alpha_i D(t_i)}, \tag{2.12b}$$

where "$j \in I_i$" represents the floating leg intervals which are part of the i^{th} fixed leg interval. That is, the floating leg intervals whose theoretical dates τ_j^{th} are contained in the i^{th} fixed leg theoretical interval $t_{i-1}^{th} < \tau_j^{th} \le t_i^{th}$. If the fixed leg frequency is shorter than the floating leg frequency (this is rare), then the same S_i is used for all fixed leg intervals forming part of each floating leg interval. So,

$$S_i = \frac{\tilde{\alpha}_j s_j D(\tau_j)}{\sum_{i \in I_j} \alpha_i D(t_i)}, \tag{2.12c}$$

where, "$i \in I_j$" represents the fixed leg intervals i with $\tau_{j-1}^{th} < t_i^{th} \le \tau_j^{th}$.

Either way, we approximate the swap values as

$$\hat{V}_{rec}(t) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) \hat{Z}(t; t_i) + \hat{Z}(t; t_n) - \hat{Z}(t; t_0). \tag{2.13a}$$

$$\hat{V}_{pay}(t) = -\hat{V}_{rec}(t) = \hat{Z}(t; t_0) - \hat{Z}(t; t_n) - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) \hat{Z}(t; t_i). \tag{2.13b}$$

for all dates t, where the strike R^{fix} and effective spread S_i are known constants. We are neglecting any evolution of the basis spreads and any minor differences due to the differences between the legs' day count bases and frequencies. We will use this approach throughout. Computationally, it would be just as easy to modify the code to add another fixed leg, but this would make the formulas messier and debugging more difficult.

2.4. Swap rate and level. At any time t, the swap rate $R^{sw}(t)$ is defined to be the break even rate, the value of R^{fix} which would make the swap worth zero. Clearly,

$$R^{sw}(t) = \frac{\hat{Z}(t, x; t_0) - \hat{Z}(t, x; t_n) + \sum_{i=1}^{n} \alpha_i S_i \hat{Z}(t; t_i)}{L(t)}, \tag{2.14a}$$

where the *level* (also known as the *PV01*, the *DV01*, or the *annuity*) is

$$L(t) = \sum_{i=1}^{n} \alpha_i \hat{Z}(t;t_i). \tag{2.14b}$$

We can re-write the swap values in terms of the swap rate and level as

$$\hat{V}_{rec}(t) = \left[R^{fix} - R^{sw}(t)\right] L(t), \qquad \hat{V}_{pay}(t) = \left[R^{sw}(t) - R^{fix}\right] L(t). \tag{2.14c}$$

In particular, today's swap rate and level are

$$R^0 = \frac{D(t_0) - D(t_n) + \sum_{i=1}^{n} \alpha_i S_i D(t_i)}{L^0}, \tag{2.15a}$$

$$L^0 = \sum_{i=1}^{n} \alpha_i D(t_i), \tag{2.15b}$$

and the swap values are

$$\hat{V}_{rec}(t) = \left[R^{fix} - R^0\right] L^0, \tag{2.15c}$$

$$\hat{V}_{pay}(t) = \left[R^0 - R^{fix}\right] L^0. \tag{2.15d}$$

3. Swaptions. A swaption is a European option on a swap. Consider a receiver swaption with *notification date* t^{ex}. If one exercises on this date, one obtains the receiver swap. Clearly

$$\hat{V}_{rec}^{opt}(t^{ex}) = \left[R^{fix} - R^{sw}(t^{ex})\right]^{+} L(t^{ex}) \tag{3.1}$$

is the value of the receiver swaption on the exercise date.

Swaption prices are almost always quoted in terms of Black's model.To introduce this model, suppose we choose the level $L(t)$ as our numeraire. (It is just the sum of a bunch zero coupon bonds, and hence is a tradable instrument). There exists a probability measure in which the value of all tradeable instruments (including the swaption) divided by the numeraire is a Martingale. So

$$\hat{V}_{rec}^{opt}(t) = L(t)\,\mathrm{E}\left\{ \left. \frac{\hat{V}_{rec}^{opt}(T)}{L(T)} \right| \mathfrak{F}_t \right\} \qquad \text{for any } T > t \tag{3.2a}$$

If we evaluate the expected value at $T = t_{ex}$, we see that

$$\hat{V}_{rec}^{opt}(t) = L(t)\,\mathrm{E}\left\{ \left. \left[R^{fix} - R^{sw}(t^{ex})\right]^{+} \right| \mathfrak{F}_t \right\} \tag{3.2b}$$

Moreover, the swap rate

$$R^{sw}(t) = \frac{\hat{Z}(t,x;t_0) - \hat{Z}(t,x;t_n) + \sum_{i=1}^{n} \alpha_i S_i \hat{Z}(t;t_i)}{L(t)} \tag{3.3a}$$

is clearly a tradeable market instrument (a bunch of zero coupon bonds) divided by the numeraire. So the swap rate is also a Martingale. By the Martingale representation model, then, we conclude that

$$dR^{sw} = A(t,*)dW, \tag{3.3b}$$

where dW is Brownian motion, and $A(t,*)$ is some measureable coefficient.

At this point we know that

$$\hat{V}_{rec}^{opt}(t) = L(t)\,\mathrm{E}\left\{ \left. \left[R^{fix} - R^{sw}(t^{ex})\right]^{+} \right| \mathfrak{F}_t \right\} \tag{3.4a}$$

where

$$dR^{sw} = A(t,*)dW, \tag{3.4b}$$

for some $A(t,*)$. Fundamental theory can take us no further. We now have to *model* $A(t,*)$. Black proposed that $A(t,*) = \sigma R^{sw}$, so that the swap rate is log normal:

$$dR^{sw} = \sigma R^{sw} dW. \tag{3.5}$$

Finding the expected value $\mathrm{E}\left\{ \left[R^{fix} - R^{sw}(t^{ex})\right]^+ \middle| R^{sw}(t) \right\}$ under this model yields *Black's formula*,

$$\hat{V}^{opt}_{rec}(t) = \left\{ R^{fix}\mathfrak{N}(d_1) - R^{sw}(t)\mathfrak{N}(d_2) \right\} L(t), \tag{3.6a}$$

with

$$d_{1,2} = \frac{\log R^{fix}/R^{sw}(t) \pm \frac{1}{2}\sigma^2 \left(t^{ex} - t\right)}{\sigma\sqrt{t^{ex} - t}} \tag{3.6b}$$

Today's market price of the swaption is

$$\hat{V}^{mkt}_{rec}(0) = \left\{ R^{fix}\mathfrak{N}(d^0_1) - R^0\mathfrak{N}(d^0_2) \right\} L^0, \tag{3.7a}$$

where

$$d^0_{1,2} = \frac{\log R^{fix}/R^0 \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}} \tag{3.7b}$$

and

$$R^0 = \frac{D_0 - D_n + \sum_{i=1}^n \alpha_i S_i D_i}{L^0}, \qquad L^0 = \sum_{i=1}^n \alpha_i D_i \tag{3.7c}$$

Here $D_i = D(t_i)$ are today's discount factors. A payer swaption is a European option to pay the fixed leg and receiver the floating leg. The value of the payer swaption is obtained by reversing R^{fix} and R^0 in the above formulas:

$$\hat{V}^{mkt}_{pay}(0) = \left\{ R^0\mathfrak{N}(-d^0_2) - R^{fix}\mathfrak{N}(-d^0_1) \right\} L^0 \tag{3.8a}$$

$$= \hat{V}^{mkt}_{rec}(0) - \left\{ R^{fix} - R^0 \right\} L^0. \tag{3.8b}$$

If one analyzes Black's formula, one discovers that the receiver and payer swaption values are both increasing functions of the volatility σ. Instead of quoting swaption prices in terms of dollar values, one can just as well quote the price in terms of the value of σ that needs be inserted into Black's formula to obtain the market price. This value of the volatility is known as the implied volatility.

3.1. Caplets and floorlets. Consider a floorlet for the interval τ_0 to τ_1. The floating rate r for the interval is set on the fixing date t_{ex} two (London) business days before the interval starts at τ_0, and the floorlet pays the difference between the strike (fixed rate) and the floating rate at the end of it's period, provided this difference is positive:

$$\tilde{\alpha}(R_{fix} - r)^+ \qquad \text{paid at } \tau_1. \tag{3.9a}$$

Here $\tilde{\alpha}$ is the coverage (day count fraction) of the interval τ_0 to τ_1. As above, the value of the floating rate payment is

$$\hat{Z}(t_{ex};\tau_0) - \hat{Z}(t_{ex};\tau_1) + \tilde{\alpha}s_1\hat{Z}(t_{ex};\tau_1), \tag{3.9b}$$

on the fixing date, where s_1 is the (forward) basis spread for the interval. The floorlet's payoff is

$$\hat{V}_{floorlet}(t_{ex}) = \left[\tilde{\alpha}\left(R_{fix} - s_1\right)\hat{Z}(t_{ex};\tau_1) + \hat{Z}(t_{ex};\tau_1) - \hat{Z}(t_{ex};\tau_0)\right]^{+}. \tag{3.9c}$$

This is the same payoff as a 1 period receiver swaption. Similarly, caplet payoffs are identical to the payoffs of 1 period payer swaptions.

The analysis of caplets and floorlets parallels the analysis for swaptions exactly. We define the *forward* or *FRA* rate as

$$R^{FRA}(t) = \frac{\hat{Z}(t,x;\tau_0) - \hat{Z}(t,x;\tau_1) + \tilde{\alpha}s_1\hat{Z}(t;\tau_1)}{\tilde{\alpha}\hat{Z}(t;\tau_1)}, \tag{3.10a}$$

and choose the zero coupon bond $\hat{Z}(t;\tau_1)$ as our numeraire. The value of the floorlet is

$$\hat{V}_{floorlet}(t_{ex}) = \tilde{\alpha}\hat{Z}(t;\tau_1)\,\mathrm{E}\left\{\left[R^{fix} - R^{FRA}(t^{ex})\right]^{+}\Big|\,\mathfrak{F}_t\right\}, \tag{3.10b}$$

where the forward FRA rate is a Martingale in this measure. Modeling this rate as log normal

$$dR^{FRA} = \sigma R^{FRA}dW. \tag{3.10c}$$

again yields *Black's formula*,

$$\hat{V}_{floorlet}(t) = \left\{R^{fix}\mathfrak{N}(d_1) - R^{FRA}(t)\mathfrak{N}(d_2)\right\}\tilde{\alpha}\hat{Z}(t;\tau_1), \tag{3.11a}$$

with

$$d_{1,2} = \frac{\log R^{fix}/R^{FRA}(t) \pm \frac{1}{2}\sigma^2\left(t^{ex} - t\right)}{\sigma\sqrt{t^{ex} - t}} \tag{3.11b}$$

Today's market price of the floorlet is

$$\hat{V}^{mkt}_{floorlet}(0) = \left\{R^{fix}\mathfrak{N}(d^0_1) - R^0\mathfrak{N}(d^0_2)\right\}\tilde{\alpha}D(\tau_1), \tag{3.12a}$$

where

$$d^0_{1,2} = \frac{\log R^{fix}/R^0 \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}}, \tag{3.12b}$$

and where today's forward FRA rate is

$$R^0 = \frac{D_0 - D_1 + \tilde{\alpha}s_1 D_1}{\tilde{\alpha}D_1}. \tag{3.12c}$$

The value of the caplet is obtained by reversing R^{fix} and R^0 in the above formulas:

$$\hat{V}^{mkt}_{caplet}(0) = \left\{R^0\mathfrak{N}(-d^0_2) - R^{fix}\mathfrak{N}(-d^0_1)\right\}\tilde{\alpha}D(\tau_1) \tag{3.13a}$$

$$= \hat{V}^{mkt}_{rec}(0) - \left\{R^{fix} - R^0\right\}\tilde{\alpha}D(\tau_1). \tag{3.13b}$$

Note that the caplet and floorlet values are special cases of the payer and receiver swaptions with $n = 1$. As before, the implied vol σ is the value of the volatility which makes the above formulas match the actual market values of the floorlet and caplet.

3.2. Digital caplets/floorlets. Digital caplets/floorlets are identical to a regular caplets/floorlets, except that the payoff is 1 paid at the end date τ_1 if the floating rate ends up in the money:

$$V_{caplet}^{dig}(0) = \begin{cases} Z(t_{fix};\tau_1) & \text{if } r \geq R^{fix} \\ 0 & \text{if } r < R^{fix} \end{cases}, \tag{3.14a}$$

$$V_{floorlet}^{dig}(0) = \begin{cases} 0 & \text{if } r \geq R^{fix} \\ Z(t_{fix};\tau_1) & \text{if } r < R^{fix} \end{cases}. \tag{3.14b}$$

Following the above line of reasoning shows that the value of these digitals is

$$\hat{V}_{caplet}^{dig}(t) = \mathfrak{N}(-d)\hat{Z}(t;\tau_1), \qquad \hat{V}_{floorlet}^{dig}(t) = \mathfrak{N}(+d)\hat{Z}(t;\tau_1) \tag{3.15a}$$

with

$$d = \frac{\log R^{fix}/R^{FRA}(t) \pm \frac{1}{2}\sigma^2\left(t^{ex}-t\right)}{\sigma\sqrt{t^{ex}-t}} \tag{3.15b}$$

Today's value of the digitals is

$$\hat{V}_{caplet}^{dig}(0) = \mathfrak{N}(-d)D(\tau_1), \qquad \hat{V}_{floorlet}^{dig}(0) = \mathfrak{N}(+d)D(\tau_1) \tag{3.16a}$$

with

$$d = \frac{\log R^{fix}/R^{FRA}(t) + \frac{1}{2}\sigma^2 t^{ex}}{\sigma\sqrt{t^{ex}}} \tag{3.16b}$$

Again, the implied digital caplet vol is the value of σ which makes the above theoretical digital prices match their market prices.

(Part head:)Pricing exotics via LGM

4. The LGM model. A modern interest rate model consists of three parts: a numeraire, a set of random evolution equations in the risk neutral world, and the Martingale pricing formula. The one factor LGM model has a single state variable, $\hat{X}$. It starts at 0 today and satisfies

$$d\hat{X} = \alpha(t)d\hat{W}, \qquad \hat{X}(0) = 0. \tag{4.1}$$

This is the evolution under the risk neutral measure induced by the numeraire, which will be named shortly. Clearly $\hat{X}(t)$ is Gaussian with the transition density

$$p(t,x;T,X)dX = \text{prob}\left\{ X < \hat{X}(T) \leq X + dX \middle| \hat{X}(t) = x \right\} \tag{4.2a}$$

given by

$$p(t,x;T,X) = \frac{1}{\sqrt{2\pi\Delta\zeta}} e^{-\frac{1}{2}(X-x)^2/\Delta\zeta}$$

Here

$$\zeta(\tau) = \int_0^\tau \alpha^2(t')dt', \qquad \Delta\zeta = \zeta(T) - \zeta(t) = \int_t^T \alpha^2(t')dt'. \tag{4.2b}$$

We choose the numeraire to be

$$\hat{N}(t,x) = \frac{1}{D(t)} e^{+H(t)x + \frac{1}{2}H^2(t)\zeta(t)}.$$

Note that the value of the numeraire is 1 today: $\hat{N}(0,0)=1$.

The last part of the model is the Martingale valuation formula. Suppose at time t the economy is in state $\hat{X}(t)=x$. If $\hat{V}(t,x)$ is the value of any freely tradeable security, then $\hat{V}(t,x)/\hat{N}(t,x)$ is a Martingale:

$$\begin{aligned}\hat{V}(t,x) &= \hat{N}(t,x)\,\mathrm{E}\left\{ \frac{\hat{V}(T,X)}{\hat{N}(T,X)} \middle| \hat{X}(t)=x \right\} \\ &= \frac{\hat{N}(t,x)}{\sqrt{2\pi\Delta\zeta}} \int \frac{\hat{V}(T,X)}{\hat{N}(T,X)} e^{-\frac{1}{2}(X-x)^2/\Delta\zeta} dX \qquad \text{for any } T>t.\end{aligned} \tag{4.3}$$

If the security throws off cash payments, then we would need to modify this formula appropriately.

The LGM model can be written most simply in terms of the *reduced prices*

$$V(t,x) \equiv \frac{\hat{V}(t,x)}{\hat{N}(t,x)}. \tag{4.4a}$$

Since the value of the numeraire is 1 today, values and reduced values are equal today: $\hat{V}(0,0)=V(0,0)$. As we shall see, we *only* have to calculate the reduced prices $V(t,x)$ and *never* have to calculate the full prices $\hat{V}(t,x)$. This simplifies our formulas substantially. In terms of the reduced prices $V(t,x)$, the LGM model is

$$V(t,x) = \frac{1}{\sqrt{2\pi\Delta\zeta}} \int V(T,X) e^{-\frac{1}{2}(X-x)^2/\Delta\zeta} dX \qquad \text{for any } T>t. \tag{4.4b}$$

with

$$\Delta\zeta = \zeta(T)-\zeta(t) \tag{4.4c}$$

as always.

4.1. Zero coupon bonds and the forward curves.. Let us go a bit further before summarizing. The (reduced) value of a zero coupon bond is

$$Z(t,x;T) = \frac{\hat{Z}(t,x;T)}{\hat{N}(t,x)} = \frac{1}{\sqrt{2\pi\Delta\zeta}} \int \frac{1}{\hat{N}(T,X)} e^{-\frac{1}{2}(X-x)^2/\Delta\zeta} dX \tag{4.5}$$

Substituting for the numeraire and carrying out the integration yields the (reduced) zero coupon price

$$Z(t,x;T) = D(T) e^{-H(T)x-\frac{1}{2}H^2(T)\zeta(t)}. \tag{4.6}$$

At $t=0$, the state variable is $x=0$, by definition. Since $Z(0,0;T)=D(T)$, the LGM model automatically matches today's discount curve $D(T)$.

At t,x the instantaneous forward rate for maturity T, namely $f(t,x;T)$, is defined via

$$\hat{Z}(t,x;T) = Z(t,x;T)\hat{N}(t,x;T) = e^{-\int_t^T f(t,x;T')dT'}. \tag{4.7a}$$

Similarly, the discount factor can be written in terms of today's instantaneous forward rate $f_0(T)$ as

$$D(T) = e^{-\int_0^T f_0(T')dT'}. \tag{4.7b}$$

So eqs. 4.7a, 4.7b) show that for the LGM model,

$$f(t,x;T) = f_0(T) + H'(T)x + \left[H'(T)\right]^2 \zeta(t) \tag{4.7c}$$

The last term $\left[H'(T)\right]^2 \zeta(t)$ is a small convexity correction; although it is needed for pricing, it does not affect the qualitative behavior of the model. The other terms show that at any date t, the forward curve is made up of today's forward curve $f_0(T)$ plus an amount x of the curve $H'(T)$. The amount x of the shift is a Gaussian random variable with mean zero and variance $\zeta(t)$.

The curve $H'(T)$ is a *model parameter*; as we shall see, it replaces the mean reversion coefficient $\kappa(t)$ in the Hull-White model. The other model parameter is the variance $\zeta(t)$. It takes the place of the volatility $\sigma(t)$. As always, model parameters have to be set *a priori* during the calibration procedure by combining both theoretical reasoning (guessing) and calibration of vanilla instruments.

4.2. Aside: Connection to the Hull White model. Under the Hull White model, deals are valued according to the expectation

$$\hat{V}(t,r) = \mathrm{E}\left\{ e^{-\int_t^T \hat{r}(t')dt'} \hat{V}(T, \hat{r}(T)) \Big| \hat{r}(t) = r \right\} \qquad \text{for any } T > t \tag{4.8a}$$

under the risk neutral measure. In this measure, the short rate $\hat{r}(t)$ is presumed to evolve according to

$$d\hat{r} = [\theta(t) - \kappa(t)\hat{r}]\, dt + \sigma(t)dW. \tag{4.8b}$$

The *mean reversion* $\kappa(t)$ and *local volatility* $\sigma(t)$ are the model parameters of this model. Given the mean reversion and local volatility, $\theta(t)$ is then chosen to match today's discount curve $D(T)$.

In appendix A we derive the LGM model from the Hull White model. Thus, the LGM model is exactly the Hull-White model written in a more convenient form. There we also examine the connection between the Hull-White and LGM parameters. This shows that Hull-White parameters can be written in terms of the LGM parameters by

$$\kappa(t) = -\frac{H''(t)}{H'(t)}. \tag{4.9a}$$

$$\sigma(t) = H'(t)\sqrt{\zeta'(t)}. \tag{4.9b}$$

As we shall see, the value of any vanilla option depends only on the value of the variance at the exercise date, $\zeta(t_{ex})$, and on mean reversion function $H(t_j)$ at the deal's pay dates t_j. Calibration determines the functions $\zeta(t)$ and $H(t)$ fairly directly. Obtaining the mean reversion parameter $\kappa(t)$ requires differentiating $H(t)$ twice, which is an inherently noisy procedure. Similarly, obtaining $\sigma(t)$ also requires differentiating $\zeta(t)$. This is why calibrating directly on the Hull-White model (instead of the LGM formulation of the model) is often an inherently unstable procedure.

4.3. Model invariances. The LGM parameters can be written in terms of the Hull-White parameters as

$$H(t) = A \int_0^t e^{-\int_0^{t'} \kappa(t")dt"} dt' + B, \tag{4.10a}$$

$$\zeta(t) = \frac{1}{A^2} \int_0^t \sigma_1^2(t') e^{+2\int_0^{t'} \kappa(\tau)d\tau} dt', \tag{4.10b}$$

where A and B are arbitrary positive constants. Since different A and B yield the same Hull-White model, and thus yield the identical prices, the LGM model has two invariances.

First, all market prices remain unchanged if we change the model parameters by:

$$H(T) \longrightarrow CH(T), \qquad \zeta(t) \longrightarrow \zeta(T)/C^2. \tag{4.11a}$$

for any positive constant C. To prove this, note that if we make the above transformation and then transform the internal variables x and X by

$$x \longrightarrow x/C, \qquad X \longrightarrow X/C, \tag{4.11b}$$

we obtain the same transition probabilities and zero coupon bond prices that we started with.

Second, all market prices remain unchanged if

$$H(T) \longrightarrow H(T) + K \qquad \zeta(T) \longrightarrow \zeta(T) \tag{4.12a}$$

for any constant K. To prove this, note that if we make the above transformation, and then transform the internal variables x and X by

$$x \longrightarrow x + K\zeta, \qquad X \longrightarrow X + K\zeta, \tag{4.12b}$$

we obtain the same transition probabilities and zero coupon bond prices $\hat{Z}(t, x; T) = N(t, x)Z(t, x; T)$ as before.

It is critical to pin down these invariances (by arbitrarily choosing some value of $H(t)$ and of $\zeta(t)$) before calibration. Otherwise convergence would be infinitely slow, with numerical roundoff determining which of the equivalent sets of model parameters is chosen.

4.4. Scaling. On average, interest rates in G7 countries change by ± 80 bps or so over the course of a year. Equivalently, the standard deviation of $H'(T)x$ should be about 1% or less each year. We choose to use the time scale of years (so $\Delta T = 1$ means an elapsed time of 1 year) and we scale $H(T)$ and $H'(T)$ to be $O(1)$. Then x is of order $O(1\% \times \sqrt{t})$ at date t, and $\zeta(t)$ is of the order of $O(10^{-4}t)$. This makes the last term in (4.7c) $O(10^{-4}t)$ also.

More precisely, suppose we have chosen $H(0) = 0$, and have scaled $H(T)$ so it increases by 1 or so every year. Then:

$$H'(T) \sim O(1), \qquad H(T) \sim O(T), \qquad \zeta(t) \sim O\left(0.64 \times 10^{-4}\right) t \tag{4.13a}$$

$$x, X \sim \left(0.8 \times 10^{-2}\right) \sqrt{t} \tag{4.13b}$$

$$H'(T)H(T)\zeta(t) \sim \left(0.64 \times 10^{-4}\right) tT \tag{4.13c}$$

4.5. Summary of the LGM model. The complete LGM model can be summarized as

$$V(t, x) = \frac{1}{\sqrt{2\pi\Delta\zeta}} \int V(T, X) e^{-\frac{1}{2}(X-x)^2/\Delta\zeta} dX_1 dX_2 \qquad \text{for any } T > t, \tag{4.14a}$$

with $\Delta\zeta \equiv \zeta(T) - \zeta(t)$, and with the (reduced) zero coupon bond formula being

$$Z(t, x; T) = D(T) e^{-H(T)x - \frac{1}{2}H^2(T)\zeta(t)}, \tag{4.14b}$$

and with

$$x = 0 \qquad \text{at } t = 0 \tag{4.14c}$$

Consequently, $\zeta(0) = 0$.

These equations are the only facts about the model we need to price any security. This model automatically reproduces the discount curve $D(T)$. The functions $H(T)$ and $\zeta(t)$ are model parameters, which are set during the calibration step, where the model prices are matched to the market prices of selected vanilla instruments, usually caplets and swaptions. Once the model is calibrated, $H(T)$ and $\zeta(t)$ are known functions, and the price of exotic deals can be determined from the Martingale formula 4.14a, using the zero coupon formula 4.14b to calculates the payoffs. Later we will present the calibration and pricing steps in exquisite detail.

5. Calibration.

5.1. Calibration and hedging. Model calibration is the most critical step in pricing. It determines not only the price obtained for an exotic deal, but also the hedges of the exotic. To see this, suppose we have some model $\mathfrak{M}$. It invariably contains unknown mathematical parameters which are set by calibration. To calibrate, one

- selects a set of vanilla instruments whose volatilities (prices) are known from market quotes. Let these volatilities be $\sigma_1, \sigma_2, \ldots, \sigma_n$.

The calibration procedure picks the model parameters by

- matching the model's yield curve to today's discount factors $D(T)$; and

- matching the model's price of the selected vanilla instruments to their maket volatilities, either exactly of in a least squares sense.

The calibrated model $\mathfrak{M}'$ is a function of today's discount factors and these n volatilities. The calibrated model is now used to price the exotic deal. The *only* step in this procedure which uses market information is the calibration step. This means that the price of the deal is a function of today's yield curve $D(T)$ and the n volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$. The price of the deal depends on *no other market information.*

Consider what happens at the nightly mark-to-market. The model is calibrated and deal is priced as above. Next the vega risks are calculated by bumping the vols in the volatility matrix (cube) one-by-one. After each bump, the deal is priced using the identical software, and the difference between the new price and the base price is the bucket vega risk for the the bumped volatility. *Unless* the bumped vol is one of the n vols used in calibration, it has no effect on the calibration of the model, so it does not affect the price of the instrument. An exotic deal *only has vega risks to the n vanilla instruments used in calibration.* After the vega risks are calculated for all the deals on the books, enough of each vanilla instrument is bought/sold to neutralize the corresponding bucket vega risk. This means that in the normal course of events, *an exotic deal will be hedged by a linear combination of the vanilla instruments used during calibration.* If the span of the vanilla instruments provide a good representation of the exotic, then the hedges should exhibit rock solid stability, with the day-to-day amounts of the hedges changing only as much as necessary to account for the actual changes in the market place. If the vanilla instruments do not provide a good representation of the exotic, then the hedges may exhibit instabilities, with day to day amounts of the hedges changing substantially even for relatively minor market changes. This latter is highly undesirable as the increased hedging costs gradually eliminate any initial profit from the exotic. (The nice term for this is leaking away your P&L). Indeed, in practice, even small improvements in the algorithm for matching the hedges to the exotics pay off disproportionately in the adroitness of the hedging.

One can move most, and possibly all, the vega risk from the calibration instruments to a different (and presumably better) set of hedging instruments by using *risk migration.* This is also known as applying an external adjuster. As a side benefit, this method also improves the pricing, often dramatically. This technique will also be discussed as it pertains to the different exotics.

5.2. Exact forumulas for swaption/caplet pricing.

5.2.1. Swaps. Consider a swap with start date t_0, fixed leg pay dates $t_1, t_2, \ldots, t_n$, and fixed rate R^{fix}. The fixed leg makes the payments

$$\alpha_i R^{fix} \qquad \text{paid at } t_i \qquad \text{for } i = 1, 2, \ldots, n-1, \tag{5.1a}$$

$$1 + \alpha_n R^{fix} \qquad \text{paid at } t_n, \tag{5.1b}$$

where $\alpha_i = \mathrm{cvg}(t_{i-1}, t_i, \beta)$ is the coverage for period i according to the fixed leg's day count basis β. On any given day t, the fixed leg's value is

$$\hat{V}_{fix}(t,x) = R^{fix} \sum_{i=1}^{n} \alpha_i \hat{Z}(t,x;t_i) + \hat{Z}(t,x;t_n) \tag{5.2a}$$

As discussed earlier, the value of the floating leg is

$$\hat{V}_{flt}(t,x) = \hat{Z}(t,x;t_0) + \sum_{i=1}^{n} \alpha_i S_i \hat{Z}(t,x;t_i), \tag{5.2b}$$

where S_i is the floating rate's basis spread, adjusted to the fixed legs day count basis and frequency. The value of the receiver swap is

$$\hat{V}_{rec}(t,x) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) \hat{Z}(t,x;t_i) + \hat{Z}(t,x;t_n) - \hat{Z}(t,x;t_0). \tag{5.2c}$$

where the strike R_{fix} and effective spread S_i are known constants. Under the LGM model, the reduced value of the swap is clearly

$$V_{rec}(t,x) = \frac{\hat{V}_{rec}(t,x)}{\hat{N}(t,x)} = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) Z(t,x;t_i) + Z(t,x;t_n) - Z(t,x;t_0) \tag{5.3a}$$

where

$$Z(t,x;t) = D_i e^{-H_i x - \frac{1}{2}H_i^2 \zeta(t)} \tag{5.3b}$$

is the reduced value of the zero coupon bonds. Here

$$D_i = D(t_i), \qquad H_i = H(t_i) \tag{5.3c}$$

are the discount factors and values of $H(t)$ at the swap's pay dates t_i.

Payer swap values are the negative of the receiver swap values.

5.2.2. Swaptions. A swaption is a European option on a swap. Consider a receiver swaption with *notification date* t_e. If one exercises on this date, one receives the fixed leg and pays the floating leg, so clearly

$$V_{rec}^{opt}(t_e,x_e) = \left[\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) Z(t_e,x_e;t_i) + Z(t_e,x_e;t_n) - Z(t_e,x_e;t_0)\right]^+ \tag{5.4}$$

is the (reduced) value of the receiver swaption on the exercise date. Under the LGM model, today's value of the swaption is given by

$$V_{rec}(0,0) = \frac{1}{\sqrt{2\pi\zeta(T)}} \int e^{-\frac{1}{2}X^2/\zeta(T)} V_{rec}^{opt}(T,X)dX \tag{5.5}$$

for any $T > 0$. Evaluating the integral at $T = t_e$ yields

$$\begin{aligned}
&(5.6) \qquad \hat{V}_{rec}^{opt}(0,0) \\
&= \frac{1}{\sqrt{2\pi\zeta_e}} \int e^{-\frac{1}{2}X^2/\zeta_e} \left[\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) Z(t_e,X;t_i) + Z(t_e,X;t_n) - Z(t_e,X;t_0)\right]^+ dX \\
&= \frac{1}{\sqrt{2\pi\zeta_e}} \int e^{-X^2/2\zeta_e} \left[\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i e^{-H_i X - \frac{1}{2}H_i^2\zeta_e} + D_n e^{-H_n X - \frac{1}{2}H_n^2\zeta_e} - D_0 e^{-H_0 X - \frac{1}{2}H_0^2\zeta_e}\right]^+ dX \\
&= \frac{1}{\sqrt{2\pi\zeta_e}} \int e^{-y^2/2\zeta_e} \left[\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i e^{-(H_i-H_0)y - \frac{1}{2}(H_i-H_0)^2\zeta_e} + D_n e^{-(H_n-H_0)y - \frac{1}{2}(H_n-H_0)^2\zeta_e} - D_0\right]^+ dy
\end{aligned}$$

Here $\zeta_e = \zeta(t_e)$ and $H_i = H(t_i)$as always.

We now assume wolog *that* $H(T)$ *is an increasing function*, so that $H'(T) > 0$. We can always ensure this is so by using the invariance $H(T) \longrightarrow -H(T)$, $\zeta(t) \longrightarrow \zeta(t)$ to change the sign of $H'(T)$. The quantity inside the square brackets $[\ ]^+$ is clearly a decreasing function of y since $H_i - H_0 > 0$ for all i. So define the break-even point y^* as the unique solution of

$$\sum_{i=1}^{n} \alpha_i \left(R_{fix} - S_i\right) D_i e^{-(H_i-H_0)y^* - \frac{1}{2}(H_i-H_0)^2\zeta_e} + D_n e^{-(H_n-H_0)y^* - \frac{1}{2}(H_n-H_0)^2\zeta_e} = D_0. \tag{5.7a}$$

Working out the interval shows that the receiver swaption's price is

$$\hat{V}^{opt}_{rec}(0,0) = \sum_{i=1}^{n} \alpha_i \left(R_{fix} - S_i\right) D_i \mathfrak{N}\left(\frac{y^* + [H_i - H_0]\,\zeta_e}{\sqrt{\zeta_e}}\right) + D_n \mathfrak{N}\left(\frac{y^* + [H_n - H_0]\,\zeta_e}{\sqrt{\zeta_e}}\right) - D_0 \mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_e}}\right) \tag{5.7b}$$

under the 1 factor LGM model. A payer swaption is a European option to pay the fixed leg and receiver the floating leg. Repeating the above derivation for a payer swaption shows that its value today is

$$\hat{V}^{opt}_{pay}(0,0) = D_0 \mathfrak{N}\left(\frac{-y^*}{\sqrt{\zeta_e}}\right) - \sum_{i=1}^{n} \alpha_i \left(R_{fix} - S_i\right) D_i \mathfrak{N}\left(\frac{-y^* - [H_i - H_0]\,\zeta_e}{\sqrt{\zeta_e}}\right) - D_n \mathfrak{N}\left(\frac{-y^* - [H_n - H_0]\,\zeta_e}{\sqrt{\zeta_e}}\right) \tag{5.7c}$$

Using $\mathfrak{N}(-x) = 1 - \mathfrak{N}(x)$, this can be written as

$$\hat{V}^{opt}_{pay}(0,0) = \hat{V}^{opt}_{rec}(0,0) + D_0 - \sum_{i=1}^{n} \alpha_i \left(R_{fix} - S_i\right) D_i - D_n, \tag{5.7d}$$

which is call/put parity.

Caplets and floorlets. Consider a floorlet for the interval τ_0 to τ_1. The floating rate r for the interval is set on the fixing date t_e two (London) business days before the interval starts at τ_0, and floorlet pays the difference between the strike (fixed rate) and the floating rate at the end of it's period, provided this difference is positive. As discussed earlier, the floorlet's payoff is can be written as

$$\hat{V}_{floorlet}(t_e) = \left[\left(1 + \tilde{\alpha}\left[R^{fix} - s_1\right]\right)\hat{Z}(t_e;\tau_1) - \hat{Z}(t_e;\tau_0)\right]^+, \tag{5.8}$$

independent of the model being used. This is the same payoff as a 1 period receiver swaption, so the value of a floorlet is the special case of a receiver swaption, with $n = 1$. Under the LGM model, we can solve for y^* explicitly when $n = 1$, obtaining

$$\hat{V}_{floorlet}(0,0) = (1 + \tilde{\alpha}\left[R^{fix} - s_1\right])D_1 \mathfrak{N}(d_1^*) - D_0 \mathfrak{N}(d_2^*) \tag{5.9a}$$

with

$$d^*_{1,2} = \frac{\log \dfrac{1 + \tilde{\alpha}\left[R^{fix} - s_1\right]}{1 + \tilde{\alpha}\left[R^0 - s_1\right]} \pm \frac{1}{2}\left(H_1 - H_0\right)^2 \zeta_e}{\left(H_1 - H_0\right)\sqrt{\zeta_e}}. \tag{5.9b}$$

and R^0 is the forward FRA rate, defined by

$$R^0 = \frac{D(\tau_0) - D(\tau_1)}{\tilde{\alpha} D(\tau_1)} + s_1. \tag{5.9c}$$

Note that the $\tilde{\alpha}$ and s_1 are the orginal coverage and basis spread for the floating leg; there is no adjustment to a "fixed leg frequency and day count basis."

Similarly, caplets are single period payer swaptions. Under the LGM model, there value is

$$\hat{V}_{caplet}(0,0) = D_0 \mathfrak{N}(d_1) - (1 + \tilde{\alpha}\left[R^{fix} - s_1\right])D_1 \mathfrak{N}(d_2) \tag{5.10a}$$

with

$$d_{1,2} = \frac{\log \dfrac{1+\tilde{\alpha}\left[R^0 - s_1\right]}{1+\tilde{\alpha}\left[R^{fix} - s_1\right]} \pm \frac{1}{2}\left(H_1 - H_0\right)^2 \zeta_e}{\left(H_1 - H_0\right)\sqrt{\zeta_e}}, \tag{5.10b}$$

and again

$$R^0 = \frac{D(\tau_0) - D(\tau_1)}{\tilde{\alpha} D(\tau_1)} + s_1. \tag{5.10c}$$

5.2.3. Summary of 1 factor LGM exact pricing formulas. *Under the LGM model, the prices of vanilla swaptions, caplets, and floorlets depend on* $\zeta(t)$ only *through* $\zeta(t_{ex})$*, its value at the notification date. The swaption prices depend on* $H(T)$ only *through the differences* $H(t_j) - H(t_0)$ *for the pay dates* t_j *of the fixed leg. This will be the key to creating lightning fast, stable calibration schemes.*

Under the one factor LGM model, the exact pricing formulas for swaptions are

$$\begin{aligned} \hat{V}^{opt}_{rec}(0,0) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i \mathfrak{N}\left(\frac{y^* + \left[H_i - H_0\right]\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \\ + D_n \mathfrak{N}\left(\frac{y^* + \left[H_n - H_0\right]\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) - D_0 \mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_{ex}}}\right) \end{aligned} \tag{5.11a}$$

$$\begin{aligned} \hat{V}^{opt}_{pay}(0,0) = D_0 \mathfrak{N}\left(-\frac{y^*}{\sqrt{\zeta_{ex}}}\right) \quad & - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i \mathfrak{N}\left(-\frac{y^* + \left[H_i - H_0\right]\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \\ & - D_n \mathfrak{N}\left(-\frac{y^* + \left[H_n - H_0\right]\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \end{aligned} \tag{5.11b}$$

$$= \hat{V}^{opt}_{rec}(0,0) + D_0 - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i - D_n. \tag{5.11c}$$

Here y^* is obtained by solving

$$\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i e^{-(H_i - H_0)y^* - \frac{1}{2}(H_i - H_0)^2 \zeta_{ex}} + D_n e^{-(H_n - H_0)y^* - \frac{1}{2}(H_n - H_0)^2 \zeta_{ex}} = D_0. \tag{5.11d}$$

Newton's method requires the derivatives of the prices with respect to the model parameters. We observe that

$$\frac{\partial}{\partial H_i}\hat{V}^{opt}_{rec}(0,0) = \frac{\partial}{\partial H_i}\hat{V}^{opt}_{pay}(0,0) = \sqrt{\zeta_{ex}}\alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{5.12a}$$

$$\frac{\partial}{\partial H_n}\hat{V}^{opt}_{rec}(0,0) = \frac{\partial}{\partial H_n}\hat{V}^{opt}_{pay}(0,0) = \sqrt{\zeta_{ex}}\left[1 + \alpha_n\left(R^{fix} - S_n\right)\right] D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{5.12b}$$

$$\begin{aligned}\frac{\partial}{\partial\sqrt{\zeta_{ex}}}\hat{V}^{opt}_{rec}(0,0) = \frac{\partial}{\partial\sqrt{\zeta_{ex}}}\hat{V}^{opt}_{pay}(0,0) &= \sum_{i=1}^{n}[H_i - H_0]\,\alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)\\ &\quad + [H_n - H_0]\, D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)\end{aligned} \tag{5.12c}$$

$$\begin{aligned}\frac{\partial}{\partial H_0}\hat{V}^{opt}_{rec}(0,0) = \frac{\partial}{\partial H_0}\hat{V}^{opt}_{pay}(0,0) &= -\sqrt{\zeta_{ex}}\sum_{i=1}^{n}\alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)\\ &\quad -\sqrt{\zeta_{ex}} D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)\end{aligned} \tag{5.12d}$$

The caplet/floorlet prices are given by

$$\hat{V}_{caplet}(0,0) = D_0\mathfrak{N}(d_1) - (1 + \tilde{\alpha}\,[R_{fix} - s_1])D_1\mathfrak{N}(d_2) \tag{5.13a}$$

with

$$d_{1,2} = \frac{\log\dfrac{1 + \tilde{\alpha}\left[R^0 - s_1\right]}{1 + \tilde{\alpha}\left[R_{fix} - s_1\right]} \pm \frac{1}{2}\left(H_1 - H_0\right)^2\zeta_{ex}}{\left(H_1 - H_0\right)\sqrt{\zeta_{ex}}} \tag{5.13b}$$

and

$$\begin{aligned}\hat{V}_{floorlet}(0,0) &= (1 + \tilde{\alpha}\,[R_{fix} - s_1])D_1\mathfrak{N}(d_1^*) - D_0\mathfrak{N}(d_2^*)\\ &= \hat{V}^{opt}_{caplet}(0,0) + (1 + \tilde{\alpha}\,[R_{fix} - s_1])D_1 - D_0\end{aligned} \tag{5.13c}$$

with $d^*_{1,2} = -d_{2,1}$:

$$d^*_{1,2} = \frac{\log\dfrac{1 + \tilde{\alpha}\left[R_{fix} - s_1\right]}{1 + \tilde{\alpha}\left[R^0 - s_1\right]} \pm \frac{1}{2}\left(H_1 - H_0\right)^2\zeta_{ex}}{\left(H_1 - H_0\right)\sqrt{\zeta_{ex}}}. \tag{5.13d}$$

Here, R^0 is the forward FRA rate

$$R^0 = \frac{D(\tau_0) - D(\tau_1)}{\tilde{\alpha}D(\tau_1)} + s_1. \tag{5.13e}$$

The caplet/floorlet prices are clearly Black's formulas for call/put prices for an asset with forward value D_0, strike $(1 + \alpha_1 R^{adj}_{fix})D_1$, and implied volatility satisfying

$$\sigma_{imp}\sqrt{t_{ex}} = \left(H_1 - H_0\right)\sqrt{\zeta_{ex}} \tag{5.14}$$

5.2.4. Approximate vanilla pricing formulas for the 1 factor LGM model. It is useful to develop approximate formulas for one factor LGM model, *even though we have exact* closed form formulas. Recall that market prices for swaptions are usually quoted in terms of Black's formula

$$\hat{V}^{mkt}_{rec}(0) = \left\{R^{fix}\mathfrak{N}(d^0_1) - R^0\mathfrak{N}(d^0_2)\right\} L^0, \tag{5.15a}$$

$$\begin{aligned}\hat{V}^{mkt}_{pay}(0) &= \left\{R^0\mathfrak{N}(-d^0_2) - R^{fix}\mathfrak{N}(-d^0_1)\right\} L^0 \\ &= V^{mkt}_{rec}(0) - L_0\left\{R^{fix} - R^0\right\}\end{aligned} \tag{5.15b}$$

where

$$d^0_{1,2} = \frac{\log R^{fix}/R^0 \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}} \tag{5.15c}$$

and

$$R^0 = \frac{D_0 - D_n + \sum_{i=1}^n \alpha_i S_i D_i}{L^0}, \qquad L^0 = \sum_{i=1}^n \alpha_i D_i \tag{5.15d}$$

Here $D_i = D(t_i)$ are today's discount factors at the pay dates. By using *equivalent vol techniques* (or direct asymptotics), one discovers that under the LGM model, the implied (Black) volatility of the swaption is approximately:

$$\sigma_B\sqrt{t_{ex}} \approx \frac{\sqrt{\zeta_{ex}}}{\sqrt{R^{fix}R^0}} \frac{\sum_{i=1}^n \alpha_i(R^0 - S_i)D_i\,(H_i - H_0) + D_n\,(H_n - H_0)}{\sum_{i=1}^n \alpha_i D_i}. \tag{5.15e}$$

This provides a good way to use market quotes of the implied volatility to obtain initial guesses for calibration.

One can re-write these quotes more simply in terms of the implied *normal* volatility. Under the Gaussian (normal) swap rate model, the value of the swaption is

$$V^{mkt}_{rec}(0) = L^0\left\{\left(R^{fix} - R^0\right)\mathfrak{N}\left(\frac{R^{fix} - R^0}{\sigma_N\sqrt{\tau_{ex}}}\right) - \sigma_N\sqrt{\tau_{ex}}G\left(\frac{R^{fix} - R^0}{\sigma_N\sqrt{\tau_{ex}}}\right)\right\} \tag{5.16a}$$

$$\begin{aligned}V^{mkt}_{rec}(0) &= L^0\left\{\left(R^0 - R^{fix}\right)\mathfrak{N}\left(\frac{R^0 - R^{fix}}{\sigma_N\sqrt{\tau_{ex}}}\right) - \sigma_N\sqrt{\tau_{ex}}G\left(\frac{R^0 - R^{fix}}{\sigma_N\sqrt{\tau_{ex}}}\right)\right\} \\ &= V^{mkt}_{rec}(0) - L^0\left\{R^{fix} - R^0\right\}.\end{aligned} \tag{5.16b}$$

The equivalent vol work shows that the implied *normal* or *absolute* vol σ_N, is approximately

$$\sigma_N\sqrt{t_{ex}} \approx \sqrt{\zeta_{ex}}\frac{\sum_{i=1}^n \alpha_i(R^0 - S_i)D_i\,(H_i - H_0) + D_n\,(H_n - H_0)}{\sum_{i=1}^n \alpha_i D_i}. \tag{5.16c}$$

5.2.5. Forward volatility. Forward volatility is a key concern of calibrated models. Suppose that we calibrate a model and then ask what the swaption volatilities will look like at a date t in the future. If the volatilities are increasing with t, we may be buying future volatility at too dear a price, and if volatilities are decreasing with t, we may be selling future volatility too cheaply.

If we repeat the above equivalent vol analysis at a date t in the future, then we discover that the (normal) swaption volatility at that date is

$$\sigma_N \approx \sqrt{\frac{\zeta(\tau_{ex}) - \zeta(t)}{\tau_{ex} - t}} \frac{\sum_{i=1}^n \alpha_i(R^0 - S_i)D_i\,[H(t_i) - H(t_0)] + D_n\,[H(t_n) - H(t_0)]}{\sum_{i=1}^n \alpha_i D_i}, \tag{5.17a}$$

where

$$R^0 = \frac{D_0 - D_n + \sum_{i=1}^n \alpha_i S_i D_i}{L^0}. \tag{5.17b}$$

If $H(t)$ is decreasing exponentially, then $\zeta(\tau_{ex})$ should be increasing exponentially to compensate.

6. Calibration strategy. The most critical aspect of pricing is choosing the right set of vanilla instruments for calibrating the model. Even small improvements in matching the vanilla instruments to the exotic deal often lead to significant improvements in the price and the stability of the hedge. For each type of exotic, the best calibration strategy often cannot be determined from purely theoretical considerations. Instead, one needs to determine which method leads to the best (the most "market fit") prices and risks.

Here we briefly discuss calibration strategies, illustrating the different strategies with a simple Bermuda swap (Bermudan swaptions and callable swaps are considered *much* more carefully in a later section).

Consider a Bermudan receiver with start date t_0, end date t_n, and strike R_{fix}. Let the fixed leg dates be $t_0, t_1, \ldots, t_n$, and let the exercise dates be $\tau_1, \tau_2, \ldots, \tau_n$. If the Bermudan is exercised at τ_j, then the holder receives the fixed leg payments

$$\alpha_i R^{fix} \qquad \text{paid at } t_i, \qquad i = j, j+1, \ldots n-1 \tag{6.1a}$$

$$1 + \alpha_n R^{fix} \qquad \text{paid at } t_n, \tag{6.1b}$$

where $\alpha_i = \text{cvg}(t_{i-1}, t_i, \beta)$ is the coverage for interval i. In return, the holder makes the floating leg payments, which are worth the same as

$$1 \qquad \text{paid at } t_{j-1}, \tag{6.1c}$$

$$\alpha_i S_i \qquad \text{paid at } t_i, \qquad i = j, j+1, \ldots n. \tag{6.1d}$$

Here we have adjusted the basis spread to the fixed leg's frequency and day count basis as discussed above. Therefore, if the Bermudan is exercised at τ_j, one receives/makes the payments

$$\begin{array}{rll} -1 & \text{at } t_{j-1}, & \\ \alpha_i \left(R^{fix} - S_i\right) & \text{at } t_i, & \text{for } i = j, j+1, \ldots n-1, \\ 1 + \alpha_n \left(R^{fix} - S_n\right) & \text{at } t_n. & \end{array} \tag{6.2a}$$

Clearly at any point t, x the j^{th} payoff is worth:

$$\hat{V}_j^{pay}(t,x) = \sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) \hat{Z}(t,x;t_i) + \hat{Z}(t,x;t_n) - \hat{Z}(t,x;t_{j-1}). \tag{6.2b}$$

6.1. Characterizing the exotic. The first step in calibration is to characterise the exotic, extracting its essential features. If the Bermudan is exercised on exercise date τ_j, one receives a swap worth

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) \hat{Z}(\tau_j,x;t_i) + \hat{Z}(\tau_j,x;t_n) - \hat{Z}(\tau_j,x;t_{j-1}) \qquad \text{at } \tau_j \tag{6.3}$$

Suppose we evaluate this swap using today's yield curve with a parallel shift of size γ,

$$\hat{Z}(\tau_j,x;t_i) \longrightarrow D(t_i) e^{-\gamma t_i} = D_i e^{-\gamma t_i}. \tag{6.4}$$

The shift γ_j at which the j^{th} swap is at-the-money is found by solving:

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i e^{-\gamma_j (t_i - t_{j-1})} + D_n e^{-\gamma_j (t_n - t_{j-1})} = D_{j-1} \tag{6.5}$$

The Bermudan is characterized by
a) the set of exercise dates $\tau_1, \tau_2, \ldots, \tau_n$;
b) the set of parallel shifts γ_j for $j = 1, 2..., n$; and
c) the length $t_n - t_0$ of the longest swap.

The second step is to select a calibration strategy and choose the calibration instruments. As we shall see, since we have two functions of time to calibrate, we can calibrate two seperate series of vanilla instruments. *Under the LGM model, the prices of vanilla swaptions, caplets, and floorlets depend on* $\zeta(t)$ only *through* $\zeta(t_{ex})$*, its value at the notification date. The swaption prices depend on* $H(T)$ only *through the differences* $H(t_j) - H(t_0)$ *for the pay dates* t_j *of the fixed leg. This will be the key to creating lightning fast, stable calibration schemes.* The trick is to calibrate on vanilla instruments whose pay dates line up exactly. We now go through the various calibration strategies for this Bermudan.

6.2. Calibration to the diagonal with constant mean reversion.. Consider the swaptions with exercise (notification) dates τ_j, start dates t_{j-1}, end dates t_n, and strike R_{fix} for $j = 1, 2, \ldots, n$. These are called the "diagonal set" of swaptions for the Bermudan. More briefly, the "j into $n-j$" swaptions struck at R_{fix}. Surely we should calibrate our LGM model to these swaptions, for if the model failed to price these swaptions accurately, we could have *no* confidence in the Bermudan price.

In the normal course of business, our hedges will be a linear combination of our calibration instruments, in this case the diagonal swaptions. As market conditions change, the particular combination will also change. Intuitively, the Bermudan should be well represented by a (dynamic) linear combination of these swaptions, so one expects the amount of the hedges to be stable.

For this calibration technique, we take the mean reversion coefficient κ to be constant. Empirically taking $\kappa = 0$ gives decent prices for exotic deals, although these prices can usually be improved by taking the mean reversion κ to be positive (generally $\kappa < 6\%$) or slightly negative (generally $\kappa > -1\%$).

Recall that $H"(T)/H(T) = -\kappa$, so that $H(T) = Ae^{-\kappa T} + B$ for some constants A and B. At this point we use the model invariants $H(T) \longrightarrow CH(T)$ and $H(T) \longrightarrow H(T) + K$ to set

$$H(T) = \frac{1 - e^{-\kappa T}}{\kappa}, \tag{6.6}$$

without loss of generality, where T is measured in years. With $H(T)$ known, we need to determine $\zeta(t)$ by calibrating to the diagonal swaptions.

Under the LGM model, the value of the j^{th} swaption is

$$\begin{aligned} V_j^{mod} = \sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i \mathfrak{N}\left(\frac{y_j^* + [H_i - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \\ + D_n \mathfrak{N}\left(\frac{y_j^* + [H_n - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) - D_j \mathfrak{N}\left(\frac{y_j^*}{\sqrt{\zeta_j}}\right). \end{aligned} \tag{6.7a}$$

Here y_j^* is given implicitly by

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i e^{-(H_i - H_{j-1}) y_j^* - \frac{1}{2}(H_i - H_{j-1})^2 \zeta_j} + D_n e^{-(H_n - H_{j-1}) y_j^* - \frac{1}{2}(H_n - H_{j-1})^2 \zeta_j} = D_{j-1}, \tag{6.7b}$$

and $\zeta_j = \zeta(\tau_j)$ is the value of ζ on the exercise date of the j^{th} swaption. For future reference, we note that

$$\begin{aligned} \frac{\partial}{\partial \sqrt{\zeta_j}} V_j^{mod} = \sum_{i=j}^{n} [H_i - H_{j-1}]\, \alpha_i (R^{fix} - S_i) D_i G\left(\frac{y_j^* + [H_i - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \\ + [H_n - H_{j-1}]\, D_n G\left(\frac{y_j^* + [H_n - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \end{aligned} \tag{6.7c}$$

The market value of the j^{th} swaption is

$$V_j^{mkt} = L_j^0 \left\{ R_{fix} \mathfrak{N}\left(d_1^j\right) - R_j^0 \mathfrak{N}\left(d_2^j\right) \right\}, \tag{6.8a}$$

where the level and current swap rate for the j^{th} swap are

$$L_j^0 = \sum_{i=j}^{n} \alpha_i D_i, \qquad R_j^0 = \frac{D_{j-1} - D_n + \sum_{i=j}^{n} \alpha_i S_i D_i}{L_j^0}, \tag{6.8b}$$

and

$$d_{1,2}^j = \frac{\log R_{fix}/R_j^0 \pm \frac{1}{2}\sigma_j^2 \tau_j}{\sigma_j \tau_j^{1/2}}. \tag{6.8c}$$

Here σ_j is the implied volatility of the j^{th} swaption obtained from, e.g., the volatility cube.

For each swaption j, the value $\sqrt{\zeta_j}$ needs to be chosen to match the LGM value V_j^{mod} of the swaption to its market value V_j^{mkt}. This value is unique since V_j^{mod} is an increasing function of $\sqrt{\zeta_j}$. Since the derivative $\partial V_j^{mod}/\partial\sqrt{\zeta_j}$ is known explicitly, $\sqrt{\zeta_j}$ can be found easily by using a global Newton's method. (Note: It is usually more efficient to solve for $\sqrt{\zeta_j}$ instead of ζ_j).

Since $\zeta(0) = 0$, once the swaptions are calibrated, we have $\zeta(t)$ at $0, \tau_1, \tau_2, \ldots, \tau_n$. Piecewise linear interpolation should be used to get values of $\zeta(t)$ for dates t between these points. As we shall see, if $\tau_1, \tau_2, \ldots, \tau_n$ are the Bermudans exercise dates, evaluating the Bermudan does *not* require knowing $\zeta(t)$ at intermediate dates.

6.2.1. Aside: Initial guess. An accurate initial guess for $\sqrt{\zeta_j}$ can be found from the equivelent vol formula 5.15e. This yields

$$\sqrt{\zeta_j} \approx \sigma_j\sqrt{\tau_j}\sqrt{R^{fix}R_j^0}\frac{\sum_{i=j}^{n} \alpha_i D_i}{\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i \left(H_i - H_{j-1}\right) + D_n \left(H_n - H_{j-1}\right)} \tag{6.9}$$

6.2.2. Aside: Global Newton's method for one parameter fits. Suppose one is trying to solve

$$f(z) = \text{target} \tag{6.10}$$

for z. Normally one starts from an intial guess z_0, and expands $f(z_{n+1}) = f(z_n + \delta z) \approx f(z_n) + f'(z_n)\delta z$ to obtain a Newton's method:

$$\delta z = z_{n+1} - z_n = \frac{\text{target} - f(z_n)}{f'(z_n)}. \tag{6.11}$$

Provided this algorithm converges, it converges very rapidly. Unfortunately, this algorithm sometimes diverges.

The global Newton method differs in only one respect: after calculating the Newton step δz, one checks to see if taking this step decreases the error. If it does, one accepts the step. If it does not, then one cuts the step in half, and then again checks to see if the error decreases. Eventually the error will decrease, and the step is accepted. The next Newton step is then calculated.

6.2.3. Aside: Infeasible market prices. Since

$$\zeta(t) = \int_0^t \alpha^2(t')dt', \tag{6.12}$$

clearly $\zeta(t)$ must be an increasing function of t :

$$0 = \zeta(0) \le \zeta_1 \le \zeta_2 \le \cdots \le \zeta_n. \tag{6.13}$$

Since each ζ_j is calibrated seperately, it may happen that $\zeta_j < \zeta_{j-1}$. (In practice this happens very, very rarely, but it *does* happen). One should test to see that the condition $\zeta_j \ge \zeta_{j-1}$is true after each ζ_j is found, and when this conditiona is violated, one should replace ζ_j by ζ_{j-1}, its minimum feasible value:

$$\zeta_j \longrightarrow \zeta_{j-1} \qquad \text{if } \zeta_j < \zeta_{j-1}. \tag{6.14}$$

This means that the j^{th} swaption will be priced at the closest possible price to the market price attainable within the calibrated LGM model, but it will not match the price exactly.

6.2.4. Aside: Where do the $\kappa's$ come from?. Suppose we set κ, calibrate the model to the diagonal, and then price the Bermudan. The resulting Bermudan price is a slightly increasing function of κ. Selecting the right κ ensures that we match the market price for the Bermudan. Desks often use a matrix to keep track of the κ needed to price a "y NC x" Bermudan correctly. That is, they fill in the κ's for the liquid Bermudans, and use "continuity" obtain the other entries in the matrix. Empirically, the κ change very, very slowly. market makers keep track of the mean reversion κ

6.3. Calibration to the diagonal with $H(T)$ specified. Suppose that $H(T)$ is specified *a priori.* (A possible source of such curves $H(T)$ is indicated below). Typically $H(T)$ is given at discrete points $H(T_1), H(T_2), \ldots, H(T_N)$. In that case, piecewise linear interpolation is used between nodes. This is equivalent to assuming that all shifts of the forward rate curve are piecewise constant curves. See 4.7c.

With $H(T)$ set, we can use the preceding procedure and formulas to calibrate on the diagonal swaptions. This determines the value of $\zeta(t)$ at $\tau_1, \tau_2, \ldots, \tau_n$. As above, one adds the point $\zeta(0) = 0$, one ensures that the $\zeta_j = \zeta(\tau_j)$ are increasing, and uses piecewise linear interpolation to obtain $\zeta(t)$ at other values of t.

6.3.1. Origin of the $H(T)$. Suppose one had the set of Bermudan swaptions 30 NC 20, 30 NC 15, 30 NC 10, 30 NC 5 and 30 NC 1. Wouldn't it be nice if the same curve $H(T)$ were used for each of these Bermudans? The 30 NC 10 Bermudan includes the 30 NC 15 and the 30 NC 20 Bermudans. It would be satisfying if our valuation procedure for the 30 NC 15 and 30 NC 20 assigned the same price to these Bermudans regardless of whether they were individual deals or part of a larger Bermudan.

One could arrange this by first using a constant κ, let's call it κ_4, to calibrate and price the 30 NC 20 Bermudan. Without loss of generality, we could select

$$\begin{aligned} H'(T) &= e^{\kappa_4(T_{30}-T)} \\ H(T) &= -\tfrac{e^{\kappa_4(T_{30}-T)}-1}{\kappa_4} \end{aligned} \qquad \text{for } T_{20} \leq T \leq T_{30}. \tag{6.15a}$$

We would calibrate on the diagonal to find $\zeta(t)$ at expiry dates $\tau_m, \tau_{m+1}, \ldots$ beyond 20 years, and then price the 30 NC 20 Bermudan. Selecting the right value of κ_4 would match the Bermudan price to its market value. Neither the swaption prices nor the Bermudan prices depend on $H(T)$ or $\zeta(t)$ for dates before the 20 year point.

To price the 30 NC 15, one could use the $H(T)$ obtained from κ_4 for years 20 to 30, and choose a different kappa, say κ_3, for years 15 to 20:

$$\begin{aligned} H'(T) &= e^{\kappa_3(T_{20}-T)}e^{\kappa_4(T_{30}-T_{20})} \\ H(T) &= -\frac{e^{\kappa_3(T_{20}-T)}-1}{\kappa_3}e^{\kappa_4(T_{30}-T_{20})} - \frac{e^{\kappa_4(T_{30}-T_{20})}-1}{\kappa_4} \end{aligned} \qquad \text{for } T_{15} \leq T \leq T_{20}. \tag{6.15b}$$

Calibrating would produce the same $\zeta(t)$ values for years 20 to 30 as before. In addition, for each κ_3 it would determine $\zeta(t)$ for years 15 to 20. By selecting the right κ_3, one could match the 30 NC 15 Bermudan's market price.

Continuing in this way, one produces the values of $\zeta(t)$ and $H(T)$ for years 10 to 15, for years 5 to 10, and finally for years 1 to 5. This $\zeta(t)$ and $H(T)$ would then yield a model which matches all the diagonal swaptions and happens to correctly price all the liquid, 30y co-terminal Bermudans. These $\kappa(t)$'s turn out to be extremely stable, only varying very rarely, and then by small amounts. Typically a desk would remember the $\kappa(t)$'s as a function of the co-terminal points, relying on the same $\kappa(t)$'s for years.

In general, if T_n is the co-terminal point and $T_0, T_1, \ldots, T_{n-1}$ are the "no call" points, then $H(T)$ is:

$$H(T) = -\frac{e^{\kappa_j(T_j-T)}-1}{\kappa_j}\prod_{i=j+1}^{n} e^{\kappa_i(T_i-T_{i-1})} - \sum_{k=j+1}^{n}\frac{e^{\kappa_k(T_k-T_{k-1})}-1}{\kappa_k}\prod_{i=k+1}^{n} e^{\kappa_i(T_i-T_{i-1})} \qquad \text{for } T_{j-1} \leq T \leq T_j \tag{6.16}$$

After $H(T)$ and $\zeta(t)$ have been found, one can use the invariants to re-scale them if desired.

6.4. Calibration to the diagonal with linear $\zeta(t)$. This is an idea pioneered by Solomon brothers. Let us use a constant local volatility α. Then

$$\zeta(t) = \int_0^t \alpha^2 dt' = \alpha^2 t \tag{6.17}$$

is linear. By using the invariance $\zeta(t) \longrightarrow \zeta(t)/C^2, H(T) \longrightarrow CH(T)$ we can choose α to be any arbitrary constant without affecting any prices. So we choose

$$\zeta(t) = \alpha_0^2 t, \tag{6.18}$$

where t is measured in years, and the dimensionless constant α_0 is typically 10^{-2}. For this calibration, we use the other invariant to set $H_n = H(t_n) = 0$. We now determine the values of H_i for other values of i by calibrating on the diagonal swaptions, starting with the last swaption.

Recall that the price of the j^{th} diagonal swaption is

$$\begin{aligned} V_j^{mod} = R_{fix} \sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i \mathfrak{N}\left(\frac{y_j^* + [H_i - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \\ + D_n \mathfrak{N}\left(\frac{y_j^* + [H_n - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) - D_j \mathfrak{N}\left(\frac{y_j^*}{\sqrt{\zeta_j}}\right) \end{aligned} \tag{6.19a}$$

and it's derivative with respect to H_{j-1} is

$$\begin{aligned} \frac{\partial}{\partial H_{j-1}} V_j^{mod} = -\sqrt{\zeta_j} \sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i G\left(\frac{y_j^* + [H_i - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \\ -\sqrt{\zeta_j} D_n G\left(\frac{y^* + [H_n - H_{j-1}]\,\zeta_j}{\sqrt{\zeta_j}}\right) \end{aligned} \tag{6.19b}$$

Here y_j^* is given implicitly by

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i e^{-(H_i - H_{j-1}) y_j^* - \frac{1}{2}(H_i - H_{j-1})^2 \zeta_j} + D_n e^{-(H_n - H_{j-1}) y_j^* - \frac{1}{2}(H_n - H_{j-1})^2 \zeta_j} = D_{j-1}. \tag{6.19c}$$

Consider the last swaption, $j = n$. It depends on $\zeta_n = \zeta(\tau_n)$, on H_n, and H_{n-1}. Of these, ζ_n is known, H_n has been set to zero, so only H_{n-1} is unknown. Since V_n^{mod} is a decreasing function of H_{n-1}, there is a unique value of H_{n-1} which matches the model price to the market price. This can be found easily using a global Newton's method. We can then move onto the $j = n - 1$ swaption. This swaption depends on H_{n-2}, which is unknown, and ζ_{j-1}, H_{n-1}, and H_n, which are known. Working backwards like this, we can calibrate all of the swaptions, and for each calibration there will only be a single unknown parameter, H_{j-1}.

This calibration procedure will yield $H_0, H_1, ..., H_n$ on the dates $t_0, t_1, ..., t_n$. One uses linear interpolation/extrapolation to get $H(t)$ at other values of t.

6.4.1. Infeasible values. In deriving the swaption formulas, we assumed that $H(T)$ was an increasing function of T. (This assumption was stronger than we needed: inspection of the above argument shows that one only needs to assume that there is a unique break-even point y^*, with in-the-moneyness on the left.) Since we are calibrating the H_j's seperately, it may happen that H_{j-1} may exceed H_j. (In practice, this has never happened to my knowledge. Still one must be prepared.) After each H_{j-1} is found, one should check to see that

$$H_{j-1} \leq H_j. \tag{6.20}$$

If this condition is violated, one should reset $H_{j-1} = H_j$. This means the j^{th} swaption would not match its market price exactly. Instead it would be the closest feasible price.

6.5. Calibration to diagonals with prescribed $\zeta(t)$. Suppose $\zeta(t)$ is a known function which is increasing and has $\zeta(0) = 0$. We could carry out the preceding calibration procedure to determine $H(T)$ from the diagonal swaptions; the procedure doe *not* depend on $\zeta(t)$ being linear.

6.6. Calibration to caplets with constant mean reversion. One can also calibrate the LGM model to a series of caplets/floorlets instead of swaptions. This is inadvisable for pricing the Bermudan, since it means that we would invariably be hedging swaption risks with caplets. However, we present the caplet (floorlet) calibration methodology here, since this methodology is sensible for other deal types, such as autocaps and revolvers.

Recall that the Bermudan has exercise date $\tau_1, \tau_2, ..., \tau_n$. To hedge each of these vega risks, we choose caplets/floorlets whose fixing dates exactly correspond to these exercise dates. So consider the n floorlets with

$$\tau_j = \text{fixing date}, \qquad t_j^0 = \text{start date}, \qquad t_j^1 = \text{end date}, \qquad R_j^{fix} = \text{strike} \tag{6.21}$$

for $j = 1, 2, \ldots, n$. Here standard market practice for the currency determines the floorlet's start dates and end dates in terms of the fixing date τ_j.

For each of these reference caplets, we need to select the strike R_j^{fix} appropriately. Surely we shouldn't just blindly pick the floorlet strikes R_j^{fix} equal to the Bermudan's strike R_j^{fix}. Unless the yield curve happens to be flat, the short dated floorlets are likely to be way in the money, and the long dated ones are likely to be way out of the money. Rather, to match the Bermudan as closely as possible, we need to choose floorlets which are at-the-money precisely when the swaps underlying the Bermudan are at the money.

Consider the Bermudan's payoff from exercising at τ_j. Recall that this payoff would be at-the-money under a parallel shift γ_j of the yield curve. To match the Bermudan as closely as possible, we need to pick the strike R_j^{fix} so that the floorlet is at-the-money for the same parallel shift γ_j. The floorlet's payoff is at

$$\left[\left\{1 + \tilde{\alpha}_j \left(R_j^{fix} - s_j\right)\right\} \hat{Z}(\tau_j; t_j^1) - \hat{Z}(\tau_j; t_j^0)\right]^+ \tag{6.22a}$$

where $\tilde{\alpha}_j = \text{cvg}(t_j^0, t_j^1)$ is the day count fraction, and

$$s_j = \text{floating rate's basis spread fo start date } t_j^0. \tag{6.22b}$$

See equation 3.9a *et seq.* Under the shifted yield curve,

$$\hat{Z}(\tau_j, x; t) \longrightarrow D(t) e^{-\gamma_j t}, \tag{6.23}$$

the value of the floorlet's payoff is

$$\left\{1 + \tilde{\alpha}_j \left(R_j^{fix} - s_j\right)\right\} D(t_j^1) e^{-\gamma_j (t_j^1 - t_j^0)} - D(t_j^0). \tag{6.24a}$$

For this to be at-the-money, we need to choose

$$R_j^{fix} = \frac{D(t_j^0) e^{+\gamma_j (t_j^1 - t_j^0)} - D(t_j^1)}{\tilde{\alpha}_j D(t_j^1)} + s_j \tag{6.24b}$$

for the caplet's strike.

6.6.1. Calibration to caplets. For this calibration method, we choose a constant mean reversion κ. This means that $H(T) = Ae^{-\kappa T} + B$ for some constants A and B, and using the invariants we can set

$$H(T) = \frac{1 - e^{-\kappa T}}{\kappa}, \tag{6.25}$$

without loss of generality. Here T is measured in years. We now obtain $\zeta(t)$ by calibrating to the floorlets

$$\tau_j = \text{fixing date}, \qquad t_j^0 = \text{start date}, \qquad t_j^1 = \text{end date}, \qquad R_j^{fix} = \text{strike} \tag{6.26a}$$

with strikes

$$R_j^{fix} = \frac{D(t_j^0)e^{+\gamma_j(t_j^1 - t_j^0)} - D(t_j^1)}{\tilde{\alpha}_j D(t_j^1)} + s_j \tag{6.26b}$$

for $j = 1, 2, ..., n$.

The market value of the j^{th} floorlet is

$$V_j^{mkt} = \tilde{\alpha}_j D(t_j^1) \left\{ R_j^{fix} \mathfrak{N}\left(d_1^j\right) - R_j^0 \mathfrak{N}\left(d_2^j\right) \right\}, \tag{6.27a}$$

where the current break-even rate for the j^{th} floorlet is

$$R_j^0 = \frac{D(t_j^0) - D(t_j^1)}{\tilde{\alpha}_j D(t_j^1)} + s_j, \tag{6.27b}$$

and

$$d_{1,2}^j = \frac{\log R_j^{fix}/R_j^0 \pm \frac{1}{2}\sigma_j^2 \tau_j}{\sigma_j \tau_j^{1/2}}. \tag{6.27c}$$

Here σ_j is the implied volatility of the j^{th} floolet obtained from, e.g., the volatility cube.

Under the LGM model, the value of the j^{th} floorlet is

$$V_j^{mod}(0,0) = (1 + \tilde{\alpha}_j \left[R_j^{fix} - s_j\right]) D(t_j^1) \mathfrak{N}\left(\tilde{d}_1^j\right) - D(t_j^0) \mathfrak{N}\left(\tilde{d}_2^j\right) \tag{6.28a}$$

with

$$\tilde{d}_{1,2}^j = \frac{\log \dfrac{1 + \tilde{\alpha}_j \left[R_j^{fix} - s_j\right]}{1 + \tilde{\alpha}_j \left[R_j^0 - s_j\right]} \pm \frac{1}{2}\left(H_j^1 - H_j^0\right)^2 \zeta_j}{\left(H_j^1 - H_j^0\right)\sqrt{\zeta_j}}. \tag{6.28b}$$

and its derivative is

$$\frac{\partial}{\partial \sqrt{\zeta_j}} V_j^{mod} = \left(H_j^1 - H_j^0\right)(1 + \tilde{\alpha}_j R_j^{fix}) D(t_j^1) G\left(\tilde{d}_1^j\right). \tag{6.28c}$$

Here

$$\zeta_j = \zeta(\tau_j), \qquad H_j^0 = H(t_j^0), \qquad H_j^1 = H(t_j^1). \tag{6.28d}$$

For each caplet j, we determine ζ_j by using a global Newton procedure to match the floorlet price under the LGM model price to its market price. Since the model price is an increasing function of ζ_j, this solution is unique. Alternatively, we note that V_j^{mod} is just Black's formula:

$$V_j^{mod} = D(t_j^1)\left\{F\mathfrak{N}\left(d_1^*\right) - K\mathfrak{N}\left(d_2^*\right)\right\} \tag{6.29a}$$

with

$$F = 1 + \tilde{\alpha}_j \left(R_{fix}^j - s_j\right) \tag{6.29b}$$

$$K = D(t_j^0)/D(t_j^1) = 1 + \tilde{\alpha}_j \left(R_j^0 - s_j\right) \tag{6.29c}$$

$$\tilde{d}_{1,2} = \frac{\log F/K \pm \frac{1}{2}\sigma^2 \tau_j}{\sigma\sqrt{\tau_j}}, \tag{6.29d}$$

and

$$\sqrt{\zeta_j} = \frac{\sigma\sqrt{\tau_j}}{H_j^1 - H_j^0}. \tag{6.29e}$$

So one can use an existing implied vol routine to obtain σ, and then ζ_j.

After fitting all the caplets, one needs to ensure that

$$0 = \zeta(0) \le \zeta_1 \le \cdots \le \zeta_n. \tag{6.30}$$

It is conceivable that $\zeta_j < \zeta_{j-1}$ for some j; if this ever occurs, then one raises ζ_j up to equal ζ_{j-1}. As always, we use linear interpolation to fill in the values at other times t.

6.7. Calibration to caplets with $H(T)$ specified. There are occasionally times when one wishes to calibrate the model with the curve $H(T)$ is specified *a priori*. Typically $H(T)$ is given at discrete points $H(T_1), H(T_2), \ldots, H(T_N)$, and piecewise linear interpolation is used between nodes. (This is equivalent to assuming that all shifts of the forward rate curve are piecewise constant curves. See 4.7c.

With $H(T)$ set, we choose the same floorlets as above and calibrate to determine the value of $\zeta(t)$ at $\tau_1, \tau_2, \ldots, \tau_n$. One then ensures that the $\zeta(t)'s$ are increasing, and uses piecewise linear interpolation to obtain $\zeta(t)$ at other values of t.

6.8. Calibration to caplets with linear $\zeta(t)$. For this calibration strategy, we assume a constant local volatility α so that $\zeta(t)$ is linear in t. By using the invariance $\zeta(t) \longrightarrow \zeta(t)/C^2$,we can choose $\zeta(t)$ to be

$$\zeta(t) = \alpha_0^2 t, \tag{6.31}$$

for any constant α_0, without loss of generality. Here t is measured in years, and we choose the dimensionless constant α_0 to be 10^{-2}.

We now choose the same floorlets as above, and determine the $H(T)$ by calibrating to the caplets. The value of the j^{th} floorlet is

Under the LGM model, the value of the j^{th} floorlet is

$$V_j^{mod}(0,0) = (1 + \tilde{\alpha}_j \left[R_j^{fix} - s_j\right]) D(t_j^1) \mathfrak{N}\left(\tilde{d}_1^j\right) - D(t_j^0)\mathfrak{N}\left(\tilde{d}_2^j\right) \tag{6.32a}$$

with

$$\tilde{d}_{1,2}^j = \frac{\log \dfrac{1 + \tilde{\alpha}_j \left[R_j^{fix} - s_j\right]}{1 + \tilde{\alpha}_j \left[R_j^0 - s_j\right]} \pm \frac{1}{2}\left(H_j^1 - H_j^0\right)^2 \zeta_j}{\left(H_j^1 - H_j^0\right)\sqrt{\zeta_j}}. \tag{6.32b}$$

and

$$\zeta_j = \zeta(\tau_j), \qquad H_j^0 = H(t_j^0), \qquad H_j^1 = H(t_j^1). \tag{6.32c}$$

With $\zeta_j$known, the only unknown is $\Delta H_j = H(t_j^1) - H(t_j^0)$ for each floorlet. Since the price is an increasing function of ΔH_j, we can use a global Newton routine (or an implied vol routine) to find the unique value of ΔH_j which matches the model price to the market price. We can then take $H(T)$ to be piecewise linear, with $H(0) = 0$, with breaks in the slope at $t_2^0, t_3^0, ..., t_n^0$, and with the slopes chosen to match the $\Delta H_j's$ implied by the market prices. This yields

$$H(T) = (\Delta H_1)T \qquad \text{for } T \le t_2^0 \tag{6.33a}$$

$$H(T) = (\Delta H_k)(T - t_k^0) + \sum_{j=2}^{k-1} (\Delta H_j)\left(t_{j+1}^0 - t_j^0\right) + (\Delta H_1)t_2^0 \qquad \text{for } t_k^0 \le T \le t_{k+1}^0 \tag{6.33b}$$

for $k = 1, 2, ..., n-1$ and

$$H(T) = (\Delta H_n)(T - t_n^0) + \sum_{j=2}^{n-1} (\Delta H_j)\left(t_{j+1}^0 - t_j^0\right) + (\Delta H_1) t_2^0 \qquad \text{for } t_n^0 \leq T \tag{6.33c}$$

After calibration, one should ensure that $H(T)$ is increasing, modifying the values of $H(T)$ as needed.

6.9. Calibration to caplets with prescribed $\zeta(t)$. Suppose $\zeta(t)$ is a known function which is increasing and has $\zeta(0) = 0$. We could carry out the preceding calibration procedure to determine $H(T)$ from the caplets; the procedure doe *not* depend on $\zeta(t)$ being linear.

6.10. Calibration to diagonal swaptions and a row of swaptions. The one factor LGM model has two model parameters, $\zeta(t)$ and $H(T)$, so it can be calibrated to two distinct sequences of vanilla instruments. Here we simultaneously calibrate the LGM model on the "diagonal" swaptions

$$\text{exercise date } \tau_j, \qquad \text{start date } t_{j-1}, \qquad \text{end date } t_n \qquad \text{for } j = 1, 2, ...n, \tag{6.34a}$$

and on the "row" of swaptions with common exercise date τ_1, common start date t_0, and varying end dates:

$$\text{exercise date } \tau_1, \qquad \text{start date } t_0, \qquad \text{end date } t_k \qquad \text{for } k = 1, 2, ...n. \tag{6.34b}$$

We choose the strike of the diagonal swaptions to be the fixed rate R^{fix} of the Bermudan, for clearly the diagonal swaptions would be at-the-money whenever the respective Bermudan payoff is at-the-money. To select the strike R_k^{fix} of the k^{th} row swaption, note that this swaption has the payoff

$$V_k^{row}(t, x) = \sum_{i=1}^{k} \alpha_i (R_k^{fix} - S_i) Z(t, x; t_i) + Z(t, x; t_k) - Z(t, x; t_0), \tag{6.35a}$$

at any t, x. We choose R_k^{fix} so that this swaption is at-the-money under same the parallel shift γ_1,

$$\hat{Z}(t, x; t_i) \longrightarrow D(t) e^{-\gamma_1 t_i}, \tag{6.35b}$$

as the Bermudan exercise with the exercise date τ_1. This requires

$$R_k^{fix} = \frac{D_0 - D_k e^{-\gamma_1 (t_k - t_0)} + \sum_{i=1}^{k} \alpha_i S_i D_i e^{-\gamma_1 (t_i - t_0)}}{\sum_{i=1}^{k} \alpha_i D_i e^{-\gamma_1 (t_i - t_0)}}. \tag{6.36}$$

We first use the multiplicative invariant to set

$$\zeta_1 = \zeta(\tau_1) = \alpha_0^2 \tau_1 \tag{6.37a}$$

without loss of generality, where we arbitrarily choose $\alpha_0 = 0.01$. We use the additive invariant to set

$$H_0 = H(t_0) = 0. \tag{6.37b}$$

Under the LGM model, today's value of the k^{th} row swaption is

$$V_k^{row}(0, 0) = \sum_{i=1}^{k} \alpha_i \left(R_k^{fix} - S_i\right) D_i \mathfrak{N}\left(\frac{y^* + [H_i - H_0]\,\zeta_1}{\sqrt{\zeta_1}}\right) + D_k \mathfrak{N}\left(\frac{y^* + [H_k - H_0]\,\zeta_1}{\sqrt{\zeta_1}}\right) - D_0 \mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_1}}\right), \tag{6.38a}$$

and its derivative with respect to H_k is

$$\frac{\partial}{\partial H_k} V_k^{row}(0,0) = \sqrt{\zeta_1}\left[1+\alpha_k\left(R_k^{fix}-S_k\right)\right] D_k G\left(\frac{y^*+[H_k-H_0]\,\zeta_1}{\sqrt{\zeta_1}}\right). \tag{6.38b}$$

Here y^* is the solution of

$$\sum_{i=1}^{k}\alpha_i(R_k^{fix}-S_i)D_i e^{-(H_i-H_0)y^*-\frac{1}{2}(H_i-H_0)^2\zeta_1}+D_k e^{-(H_k-H_0)y^*-\frac{1}{2}(H_k-H_0)^2\zeta_1}=D_0. \tag{6.38c}$$

The $k=1$ swaption depends on H_0, H_1, and ζ_1, of which only H_1 is unknown. Since the swaption's price is an increasing function of H_1, we can use a global Newton scheme to obtain the unique value of H_1 at which the swaption's price will match its market value. The $k=2$ swaption then introduces one new parameter, H_2, which we select by using the global Newton scheme to match this swaption to its market value. In very rare situations, it may occur that $H_2 < H_1$, in which case we set $H_2 = H_1$. (In these cases we cannot match the swaption price exactly, so we set H_2 to its closest feasible value). We continue taking the swaptions in turn, with each swaption j introducing one additional value of H_j to be set by calibrating the swaption to its market value. We then ensure that $H_j \geq H_{j-1}$, adjusting the value of H_j if needed, and then continue to the next swaption. In this way we find

$$H_0 = H(t_0), \qquad H_1 = H(t_1), \ldots, H_n = H(t_n). \tag{6.39}$$

We use linear interpolation/extrapolation to get $H(T)$ as other values of T.

Having found $H(T)$, we can now use the "calibration to the diagonal with $H(T)$ specified" method (and code!) to get $\zeta(t)$.

6.10.1. Aside: Using τ_2. For some deals, τ_1 is too short for the τ_1 into k swaptions to be used for calibration. Instead of blindly taking τ_1 as the exercise date, many firms take the Bermudan's first exercise date which is at least, say, 6 months from today.

6.11. Calibration to diagonal swaptions and caplets. A Bermudan swaption can be viewed as the most expensive of its component European swaptions, plus an option to "switch" to a different swaption should market conditions change. The component swaptions are just the diagonal swaptions, so calibrating to the diagonals accounts for this part of the pricing. On any exercise date, "switch" option is the option to exercise immediately, or to delay the exercise decision until the next exercise date. Since these delays are short, typically six months, one may believe that the switch option can best be represented by short underlyings. Accordingly, one could argue that one should calibrate to either a column of caplets or a column of 1 year underlyings, as well as the diagonal swaptions. Here we calibrate on the caplets and swaptions simultaneously; in the next section we calibrate to the diagonal swaptions and the swaptions with one year underlyings.

6.12. Calibration to diagonal swaptions and a column of swaptions. One may could argue that caplet and swaption markets have distinct indentities, and that mixing the two markets introduces small, but needless, noise. Instead one could calibrate on the diagonal swaptions and a column of swaption's with 1 year tenors. (In most currencies, these are the swaptions with the shortest underlying avaliable).

6.13. Other calibration strategies. There are many other simple calibration strategies; although they are not overly appropriate for pricing a Bermudan, they may well be appropriate for other deal types.

Calibrate on swaptions with constant κ or specified $H(T)$. Suppose we have chosen a constant mean reversion parameter κ, or have otherwise specified $H(T)$. Then the calibration procedure just needs to find $\zeta(t)$. Suppose we have selected an arbitrary set of n swaptions to be our calibration instruments. In LGM valuation of each swaption the only unknown parameter is $\zeta(t)$ at the swaption's exercise date. Using a global Newton's method to calibrate each swaption to its market value thus determines $\zeta(t)$ and the exercise dates $\tau_1, \tau_2, \ldots, \tau_n$ of the n swaptions. After obtaining the $\zeta_j = \zeta(\tau_j)$, we need to ensure that $\zeta(\tau_j)$ are non-decreasing, alterring the offending values if necessary. We then include the value $\zeta_0 = \zeta(0) = 0$, and use piecewise linear interpolation to obtain $\zeta(t)$ at other dates.

Note that this method fails if two swaptions share the same exercise date τ; calibration would either yield the same ζ, in which case one of the swaptions is redundant, or differing ζ, in which case our data is contradictory. If the exercise dates of any two swaptions are too close, say within 1-2 months, the results may be problematic. For this reason one usually ensures that the swaption exercise dates are, say, at least $2\frac{1}{2}$ months apart, excluding instruments from the calibration set to achieve this spacing, if necessary.

Calibrate on swaptions with specified $\zeta(t)$. Suppose we have chosen a linear $\zeta(t)$, or otherwise specified parameter $\zeta(t)$. The calibration procedure just needs to find $H(T)$. Suppose we have selected an arbitrary set of n swaptions to be our calibration instruments. We can then arrange the swaptions in increasing order of their final pay dates. Let these final pay dates be $T_1, T_2, \ldots, T_n$. Suppose we use our invariance to set $H_0 = H(0) = 0$, and we use piecewise linear interpolation:

$$H(T) = \Delta_1 T \qquad \text{for } T < T_1, \tag{6.40a}$$

$$H(T) = \sum_{i=1}^{k-1} \Delta_i (T_i - T_{i-1}) + \Delta_k (T - T_{k-1}) \qquad \text{for } T_{k-1} < T < T_k, \tag{6.40b}$$

$$H(T) = \sum_{i=1}^{n-1} \Delta_i (T_i - T_{i-1}) + \Delta_n (T - T_{n-1}) \qquad \text{for } T_n < T, \tag{6.40c}$$

where $T_0 = 0$.

For the first swaption, the slope Δ_1 determines the value of $H(T)$ at all the swaption's pay dates. Since $\zeta(t)$ is known, the LGM value of the swaption depends only on a single unknown quantity, Δ. It is easily seen that the value is an increasing function of Δ_1, so one can use a global Newton scheme to find the unique Δ_1 which matches the swaption's price to its market value.

The value of $H(T)$ at the second swaption's pay dates is determined by both Δ_1 and Δ_2, of which only Δ_2 is unknown at this stage. Again a global Newton scheme can be used to find the Δ_2 needed to calibrate the swaption to its market value. (In rare cases it may occur that $\Delta_2 < 0$; in this case we need to set $\Delta_2 = 0$, its minimum feasible value).

We then continue in this way, calibrating the swaptions and obtaining the Δ_j's in succession. This method will fail only if two deals have the same final pay date, and will work poorly if the final pay dates are too near together. For this reason one usually ensures that the final pay dates are, say, at least $2\frac{1}{2}$ months apart, excluding instruments from the calibration set to achieve this spacing, if necessary.

7. Evaluation of exotics. dldldl
(Part head:)Valuation of exotic swaps and caps

8. Bermudan swaptions and callable swaps (bullet).

8.1. Deal definition and encapsulation.

8.1.1. Mid-period exercises.

8.1.2. Encapsulation of deal data.

8.2. Characterization.

8.3. Evaluation.

8.4. Comparision of results.

8.4.1. Constant mean reversion and market implied $\kappa's$.

8.4.2. Given $H(t)'s$ matching the Bermudan market.

8.5. Extension to american swaptions.

9. Bermudan swaptions and callable swaps (amortizing).

9.1. Deal definition and encapsulation.

9.2. Characterization.

9.3. Evaluation.

9.4. Comparision of results.

10. Callable inverse floaters.

11. Callable capped floaters and superfloaters.

12. Callable accrual swaps.

13. Autocaps.

14. Revolvers.

15. Captions (European options on caps). (Part head:)Valuation of bonds

16. Extending the LGM model to include credit.

17. Valuation of noncallable bullet bonds.

18. Valuation of callable bonds. Appendix A. Mechanics and standard practices of swap markets.

A.1. Date arithmetic and business day conventions.

A.2. Year fractions, day count bases, and "adjusted" interest rate payments.

A.3. Fixing dates, spot dates, and floating legs.

A.4. Notification dates and the swaption market.

A.5. Cap markets. Appendix B. Connection between LGM and Hull White.

The one factor LGM model is completely equivalent t the Hull-White model; it is just written in a vastly more convenient form. Here we show this by deriving the LGM model from the Hull-White model. This then determine the relationship between the Hull White model parameters and the LGM model parameters.

B.1. Hull-White model. The Hull-White model is written using the money market numeraire. With this numeraire, the value $\hat{V}(t,y)$ of any tradeable security at t,y is

$$\hat{V}(t,y) = \mathrm{E}\left\{ e^{-\int_t^T r(t',\hat{Y}(t'))dt'}\hat{V}(T,Y) \middle| \hat{Y}(t) = y \right\}, \tag{B.1a}$$

where the short rate is given by

$$r(t,\hat{Y}) = \theta(t) + \hat{Y}, \tag{B.1b}$$

and where the state variables $\hat{Y}$ start at 0 and evolve according to

$$d\hat{Y} = -\kappa(t)\hat{Y}dt + \sigma(t)d\hat{W}, \qquad \hat{Y}(0) = 0, \tag{B.1c}$$

(Often authors use the short rate $\hat{R} = r(t,\hat{Y}) = \theta(t) + \hat{Y}$ as the fundamental process. For convenience we have seperated the short rate $\hat{R}(t)$ into its mean $\theta(t)$ and random component $\hat{Y}$). The model parameters are the mean reversion $\kappa(t)$ and the local volatility $\sigma(t)$. The other model parameter, $\theta(t)$, is reserved to make the model match today's discount curve.

To derive the LGM model from Hull-White, define the new variable

$$\hat{U}(t) = \hat{Y}(t)e^{+\int_0^t \kappa(\tau)d\tau}. \tag{B.2}$$

Then the short rate is given by

(B.3a) $$r(t) = \theta(t) + h(t)\hat{U},$$

where $\hat{U}$ evolves according to

(B.3b) $$d\hat{U} = \alpha(t)d\hat{W}, \qquad \hat{U}(0) = 0$$

Here $h(t)$ is defined by

(B.4a) $$h(t) = e^{-\int_0^t \kappa(t')dt'},$$

and $\alpha(t)$ is related to the Hull-White parameters by

(B.4b) $$\alpha(t) = \sigma(t)e^{+\int_0^t \kappa(\tau)d\tau}.$$

The value $\hat{Z}(t,u;T)$ of a zero coupon bond can be calculated by solving the Feynman-Kac equation

(B.5a) $$\hat{Z}_t + \tfrac{1}{2}\alpha^2\hat{Z}_{uu} = r(t,u)\hat{Z} = [\theta(t) + h(t)u]\,\hat{Z}$$

for $T < t$, with the boundary condition

(B.5b) $$\hat{Z}(T,u;T) = 1 \qquad \text{at } t = T.$$

Solving we obtain

(B.6a) $$\hat{Z}(t,u;T) = e^{-[H(T)-H(t)]u - A(t,T)}$$

where

(B.6b) $$H(\tau) = \int_0^\tau h(t')dt' + K,$$

and

(B.6c) $$A(t,T) = \int_t^T \theta(t')dt' - \tfrac{1}{2}\int_t^T [H(T) - H(t')]^2\,\zeta'(t')dt'.$$

Here the function $\zeta(t)$ is given by

(B.6d) $$\zeta(\tau) = \int_0^\tau \alpha^2(t')dt'$$

and K is an arbitrary constant.

At $t,u = 0,0$, the zero coupon bond must match today's discount factors $D(T)$. Recalling that the discount factors can be written as

(B.7) $$D(T) = e^{-\int_0^T f_0(t')dt'},$$

we see that this requires

(B.8) $$\theta(T) = f_0(T) + H'(T)\cdot\int_0^T \zeta'(t')\,[H(T) - H(t')]\,dt'.$$

Substituting for θ allows us to write the zero coupon bond formula as

(B.9a) $$\hat{Z}(t,x;T) = \frac{D(T)}{D(t)}e^{-[H(T)-H(t)]x - \frac{1}{2}\left[H^2(T)-H^2(t)\right]\zeta(t)},$$

where

$$x(t) = u - \int_0^t \zeta'(t')H(t')dt'. \tag{B.9b}$$

We have nearly derived the LGM model from the Hull-White model. Consider the variables

$$\begin{aligned} \hat{X}(t) &= \hat{U} - \int_0^t \zeta'(t')H(t')dt' \\ &= \hat{Y}(t)e^{+\int_0^t \kappa(\tau)d\tau} - \int_0^t \zeta'(t')H(t')dt' \end{aligned} \tag{B.10}$$

The new variable evolves according to

$$d\hat{X}_1 = -\left\{\alpha^2(t)H(t)\right\}dt + \alpha(t)d\hat{W}, \qquad \hat{X}(0) = 0, \tag{B.11a}$$

The Hull-White model can now be written as

$$\hat{V}(t,\mathbf{x}) = \mathrm{E}\left\{ e^{-\int_t^T r(t',\hat{X}(t'))dt'} \hat{V}(T,X) \middle| \hat{X}(t) = x \right\}, \tag{B.12a}$$

where

$$\begin{aligned} r(t',\hat{X}(t')) &= \theta(t) + H'(t)\hat{X}(t) + H'(t)\int_0^t H(t')\zeta'(t')dt' \\ &= f_0(t) + H'(t)\hat{X}(t) + H'(t)H(t)\zeta(t). \end{aligned} \tag{B.12b}$$

Accordingly, the value of any freely tradeable security (between cash flows) satisfies the Feynman-Kac equation

$$\hat{V}_t - H(t)\zeta'(t)\hat{V}_x + \tfrac{1}{2}\zeta'(t)\hat{V}_{xx} = \left\{f_0(t) + H'(t)x + H'(t)H(t)\zeta'(t)\right\}\hat{V} \tag{B.13}$$

One can verify by direct substitution that the zero coupon bonds $\hat{Z}(t,x;T)$ satisfy this PDE. Another solution is

$$\hat{N}(t,x) = \frac{1}{D(t)} e^{H(t)x + \frac{1}{2}H^2(t)\cdot\zeta(t)}, \tag{B.14}$$

as can again be verified by direct substitution. Let us define the reduced price

$$V(t,x) = \frac{\hat{V}(t,\mathbf{x})}{\hat{N}(t,x)}. \tag{B.15a}$$

Substituting $\hat{V}(t,x) = \hat{N}(t,x)V(t,x)$ into equation **??**, and using the fact that $\hat{N}(t,x)$ also satisfies **??**, one finds that $V(t,x)$ satisfies

$$V_t + \tfrac{1}{2}\zeta'(t)V_{xx} = 0 \tag{B.15b}$$

But this is the backwards Kolmogorov equation for

$$V(t,x) = \mathrm{E}\left\{ V(T,X) \middle| \hat{X}(t) = x \right\}, \tag{B.16a}$$

provided we take $\hat{X}(t)$ as evolving according to the process

$$d\hat{X}(t) = \alpha(t)d\hat{W}. \tag{B.16b}$$

Here

$$\zeta(\tau) = \int_0^\tau \alpha^2(t')dt', \tag{B.16c}$$

as before. Together, B.14, B.15a and B.16a - B.16c are LGM model.

B.2. Connection between LGM and HW parameters. The LGM model parameters are the (accumulated) variance $\zeta(t)$, and the *response* or *mean reversion* function $H(t)$. The Hull-White model has the local vol function $\sigma(t)$, the mean reversion rate $\kappa(t)$, and $\theta(t)$. The mean reversion *function* $H(t)$ can be obtained from the Hull-White mean reversion *parameter* by integrating:

$$H(t) = \int_0^t e^{-\int_0^{t'} \kappa(t")dt"} dt' + K. \tag{B.17a}$$

In principle, the mean reversion parameter can be obtained by differentiating the observed mean reversion function:

$$\kappa(t) = -\frac{H''(t)}{H'(t)}. \tag{B.17b}$$

As we shall see, the value of a vanilla instrument depends only on the value of the mean reversion function $H_j(t)$ at the pay dates t of the instrument, so calibration determines the function $H(t)$ fairly directly. Obtaining the mean reversion parameter $\kappa(t)$ from this function requires differentiation, which is an inherently noisy procedure. This is why calibrating directly on the Hull-White formulation of the model (instead of LGM formulation) is often inherently unstable.

The variance $\zeta(t)$ can also be determined from the HW local volatility $\sigma(t)$:

$$\zeta(t) = \int_0^t \sigma_1^2(t') e^{+2\int_0^{t'} \kappa(\tau)d\tau} dt', \tag{B.18a}$$

Alternatively, the HW parameters can be determined by differentiating the $\zeta_M(t)$ matrix:

$$\sigma(t) = H'(t)\sqrt{\zeta'(t)}. \tag{B.18b}$$

We shall find that prices of European options depend on the value of $\zeta(t)$ at the exercise date of the option, and not on the value at other times. This allows us to cleanly determine $\zeta(t)$ by calibrating to vanilla deals with different exercise dates. Determining the local volatility $\sigma(t)$ from $\zeta(t)$ requires differentiating. This is another reason why calibrating the Hull-White formulation of the model is often unstable, while calibrating the LGM formulation is stable.