Mathematical Finance, Vol. 17, No. 1 (January 2007), 111–141

THEORY AND CALIBRATION OF SWAP MARKET MODELS

S. GALLUCCIO AND J.-M. LY

BNPParibas Fixed Income, London

Z. HUANG

JP Morgan, London

O. SCAILLET

HEC Genève and Swiss Finance Institute

This paper introduces a general framework for market models, named Market Model Approach, through the concept of admissible sets of forward swap rates spanning a given tenor structure. We relate this concept to results in graph theory by showing that a set is admissible if and only if the associated graph is a tree. This connection enables us to enumerate all admissible models for a given tenor structure. Three main classes are identified within this framework and correspond to the co-terminal, co-initial, and co-sliding model. We prove that the LIBOR market model is the only admissible model of a co-sliding type. By focusing on the co-terminal model in a lognormal setting, we develop and compare several approximating analytical formulae for caplets, while swaptions can be priced by a simple Black-type formula. A novel calibration technique is introduced to allow simultaneous calibration to caplet and swaption prices. Empirical calibration of the co-terminal model is shown to be faster, more robust, and more efficient than the same procedure applied to the LIBOR market model. We then argue that the co-terminal approach is the simplest and most convenient market model for pricing and hedging a large variety of exotic interest-rate derivatives.

KEY WORDS: swap market model, cap, swaption, calibration, graph theory

1. INTRODUCTION

In recent years, market models of interest rate dynamics have attracted much attention among academics, and have become increasingly popular among practitioners. These models look more appealing than classical short-term rate based approaches from both

We would like to thank the editor, the associate editor, and both referees for constructive criticism and comments, which have lead to a substantial improvement on previous versions. Z. Huang and O. Scaillet received support by the Swiss NSF through the NCCR Finrisk. We are grateful to J. Detemple; A. Galluccio; M. Musiela; R. Pietersz; R. Stulz; W. Ziemba; to the participants at the 2003 Swiss Doctoral Workshop (Gerzensee), 2003 ICBI Risk Management conference (Geneva), 2003 workshop on Managing Credit and Market Risk (Verona), the 2004 ICBI Global Derivatives conference (Madrid), the ESWC 2005 (London), the Solvay Finance seminar (Brussels), the USI finance seminar (Lugano), the Bachelier seminar (Paris), and the finance seminar at the University of Venice for helpful comments; and to M. Rutkowski for valuable remarks on our manuscript.

Manuscript received December 2003; final revision received September 2005.

Address correspondence to Stefano Galluccio, BNPParibas, Fixed Income Derivatives, 10 Harewood Avenue, London NW1 6AA, United Kingdom; e-mail: stefano.galluccio@bnpparibas.com.

a theoretical and a practical point of view since they are built by directly specifying arbitrage-free dynamics on a set of forward LIBOR or swap rates. Several issues on the implementation side do however exist because of the high dimensionality of the associated Markovian dynamics. Despite this, market models have recently gained an undisputed popularity thanks to the availability of new approximation and model calibration techniques.

Historically, two different (yet mathematically similar) approaches have been introduced. In the LIBOR market model (LMM) developed by Brace, Gatarek, and Musiela (1997); Goldys (1997); Miltersen, Sandmann, and Sondermann (1997); Musiela and Rutkowski (1997); arbitrage-free dynamics are assigned to a set of non-overlapping forward LIBOR rates while in the swap market model (SMM), first introduced by Jamshidian (1997), similar dynamics are assigned to a family of forward swap rates. Nonetheless, most of the available literature on the subject has focused on the LMM only. Many authors rely on the convenient assumption that consists of assigning dynamics to forward LIBOR rates driven by a d-dimensional Itô diffusion process with deterministic volatility. In this setting, Glasserman and Zhao (2000) study arbitrage-free discrete-time approximations of the LMM, while Jäckel and Rebonato (2003) and Hull and White (2000) introduce methods to approximate true LIBOR dynamics by high-dimensional lognormal processes. Schoenmakers and Coffey (1999) propose parameterized instantaneous correlation structures that simplify model implementation, while model calibration issues for LMM are extensively studied in Rebonato (2003), Brigo and Mercurio (2001), Schoenmakers and Coffey (2003), and d'Aspremont (2003). Andersen and Andreasen (2000) introduce a LMM where the instantaneous volatility of the LIBOR rates is allowed to be nonlinearly state-dependent (more precisely, of a CEV type), while stochastic volatility extensions are discussed in Andersen and Brotherton-Ratcliffe (2001), Joshi and Rebonato (2003), and Wu and Zhang (2002). In this respect, it is also worth mentioning the work of Jamshidian (1999) who studies possible extensions of the standard diffusion-based approach to general semimartingale processes, as well as Glasserman and Kou (2003), Glasserman and Merener (2003), for LMM with jumps. A vast empirical study dedicated to stochastic volatility and jump LMM is given in Jarrow, Li, and Zhao (2003), while one-factor and two-factor versions of LMM and SMM are compared in De Jong, Driessen, and Pelsser (2000). Finally, Hunter, Jäckel, and Joshi (2001) and Pietersz, Pelsser, and Van Regenmortel (2004) introduce efficient Monte Carlo methods to simulate a set of correlated LIBOR rates.

Despite the extensive literature available on the LMM, very little has been published on its swap rate counterpart until now. Models based on dynamics assigned to forward swap rates are often considered less tractable than the LMM both in theory and in the applications, although the two approaches are remarkably close in their mathematical construction (Rebonato 2003). On the practical side, many authors seem to prefer the LMM by claiming that LIBOR rates are "more fundamental" financial quantities than swap rates. This view is sometimes justified by noticing that, to a good degree of accuracy, forward swap rates may be thought of as deterministic linear combinations of forward LIBOR rates (Brigo and Mercurio 2001).

In this paper, we challenge these claims although our results are, in fact, more general. Our goal is threefold.

First, we consider a general specification of a model that concentrates on observable market rates directly. We assign arbitrage-free dynamics to a generic family of forward swap rates, and aim at finding the weakest condition under which this construction yields a unique specification in all equivalent pricing measures. In this respect, the concept of *admissibility* of a set is introduced, and its theoretical and practical implications are discussed. The Market Model Approach is introduced next. It is the most general modeling framework where arbitrage-free dynamics are assigned to an admissible set. The cornerstone of this approach is the direct modeling of forward swap rates such that absence of arbitrage and market completeness are both satisfied with respect to the chosen tenor structure. This is in contrast to a construction in the spirit of Heath–Jarrow–Morton (1992) where one starts by assigning dynamics to a set of instantaneous forward rates or a family of discount bond prices. Correspondingly, we determine the necessary and sufficient conditions for this to hold. We argue below that our approach is preferable in pricing and risk-management applications. Interestingly, we show that properties of admissible sets can be best understood with the use of graph theory. This mapping allows us to graphically characterize all admissible sets in a simple and intuitive way and, moreover, to determine the number of distinct models that are admissible for a given tenor structure. Our study shows that the class of admissible market models is very large, and contains all "standard" ones (Brace, Gatarek, and Musiela 1997; Jamshidian 1997) as special cases. We identify three major subclasses called co-initial, co-sliding, and co-terminal according to the nature of the family of forward swap rates. As an important corollary, we prove that the LMM is the only admissible model of a co-sliding type.

Second, we concentrate on the co-terminal SMM (ctSMM, in short), and discuss market situations where the use of this approach is most relevant. In particular, we demonstrate that it enjoys the same degree of mathematical tractability as the LMM. For instance, drift approximations for the pricing of vanilla options work equally well in comparison to a standard LMM framework. Obviously, this contradicts the current widespread perception about the computational burden of the SMM.

Third, we focus on the calibration of the ctSMM, and show how a joint calibration on market prices of caplets and swaptions can be achieved in a faster, more robust and more transparent way than in the LMM. We provide numerical and empirical results supporting this claim. These findings indicate that the ctSMM is a fundamental pricing and risk-management tool for a large variety of exotic interest-rate (IR) derivatives.

The rest of the paper is organized as follows. In Section 2, we introduce some notation, and define the Market Model Approach (MMA) by assigning arbitrage-free dynamics to a set of admissible (in a sense to be later defined) forward swap rates. We discuss connection with graph theory in Section 3. There, we show that a set of forward swap rates is admissible if and only if the associated graph is a tree. By borrowing results from graph theory on labeled trees we enumerate all admissible market models for a given tenor structure. In Section 4, three major subclasses of admissible models are unveiled, and a different use for each type is identified and discussed. We argue that a subclass is more appropriate to price a particular category of IR derivatives if it is directly built from the forward swap rates underlying the pay-off structure. In that way the chosen market model should be able to exploit the most relevant information for pricing and hedging purposes. From Section 5 on we concentrate on the ctSMM class, and discuss the utility of co-terminal models in the context of commonly traded exotic IR derivatives. Section 6 is dedicated to the pricing of plain-vanilla IR derivatives such as swaptions and caplets. For caplet pricing, we explore several analytical approximation methods. In Section 7, we concentrate on the calibration of the ctSMM. There, we introduce a parametric recursive approach, often called "bootstrap" algorithm by practitioners. This new methodology allows us to easily calibrate the model to market quotes of caplets and co-terminal swaptions associated to the same tenor structure. Section 8 contains all numerical results, namely a numerical comparison of the different approximation and calibration schemes developed in Sections 6 and 7. We also examine how the calibration procedure based on the parametric recursive algorithm performs empirically on weekly quotes of caplets and swaptions ranging from May 17, 2004 to May 16, 2005. Section 9 gathers some concluding remarks.

2. MARKET MODEL APPROACH

We assume that we are given a pre-specified collection of reset/settlement dates $\mathcal{T} := \{T_1, \ldots, T_M\}$, referred to as the tenor structure, with $T_j < T_k$, $1 \leq j < k \leq M$. Starting time is assumed to be $T_0 = 0$ with $T_0 < T_1$. The year fraction between any two consecutive dates is denoted by $\delta_j = T_j - T_{j-1}$, for $j = 1, \ldots, M$. Throughout the paper, we will assume that a day-count convention has been assigned, with no loss of generality.

We write $B(t, T_j), j = 1, \ldots, M$, to denote the price at time t of a discount bond that matures at time $T_j > t$. The forward swap rate $S(t, T_j, T_k)$ is defined through

$$S(t, T_j, T_k) = \frac{B(t, T_j) - B(t, T_k)}{G(t, T_j, T_k)}, \quad G(t, T_j, T_k) = \sum_{l=j+1}^{k} \delta_l B(t, T_l), \tag{2.1}$$

for $t \in [0, T_j]$.

For a generic forward swap rate, we will refer to T_j (resp. T_k) as the swap start (resp. end) date. Here we have introduced the price process of the annuity numéraire $G(t, T_j, T_k)$. From (2.1) it is easily seen that a forward swap rate is built on ratios of discount bond prices:

$$S(t, T_j, T_k) = \left(\frac{B(t, T_j)}{B(t, T_k)} - 1 \right) \Bigg/ \sum_{l=j+1}^{k} \delta_l \frac{B(t, T_l)}{B(t, T_k)}.$$

We denote the set of all forward swap rates associated to the tenor structure $\mathcal{T}$ by $\mathcal{S}_{\mathcal{T}} := \{S(t, T_j, T_k); T_j, T_k \in \mathcal{T}, j < k\}$, or simply by $\mathcal{S}$ if no confusion arises.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W a d-dimensional Wiener process. Throughout this paper, $\mathcal{F}$ will be assumed to be the natural filtration generated by W, i.e. $\{\mathcal{F}_t^W\} = \sigma(W_s, s \leq t)$, so that W is adapted to $\mathcal{F}$. A probability measure $\mathbb{P}^{T_j, T_k}$, equivalent to the historical probability measure $\mathbb{P}$, is said to be the forward swap probability measure associated with the dates T_j and T_k, or simply the forward swap measure, if for $i = 1, \ldots, M$, the relative bond price $B(t, T_i)/G(t, T_j, T_k), \forall t \in [0, T_i \wedge T_{j+1}]$, follows a local martingale process under $\mathbb{P}^{T_j, T_k}$. Here, $G(t, T_j, T_k)$ is the price of the numéraire so that the forward swap rate $S(t, T_j, T_k)$ is a $\mathbb{P}^{T_j, T_k}$-martingale. We denote the corresponding Brownian motion under $\mathbb{P}^{T_j, T_k}$ by W^{T_j, T_k}.

Since $S(t, T_j, T_k)$ is a $\mathbb{P}^{T_j, T_k}$-martingale, we postulate that, under $\mathbb{P}^{T_j, T_k}$:

$$\frac{dS(t, T_j, T_k)}{S(t, T_j, T_k)} = \lambda(t, T_j, T_k)' \, dW^{T_j, T_k}(t), \quad \forall t \in [0, T_j], \tag{2.2}$$

where the vector-valued volatility function $\lambda(t, T_j, T_k)$ is left unspecified and, in particular, it can be state-variable dependent. Here we rule out the possibility of forward swap rates with identical paths, namely with identical volatility functions and initial conditions. Since $\lambda(t, T_j, T_k)$ is deterministic, $S(t, T_j, T_k)$ will be lognormally distributed, but we point out that our results hold true even when $\lambda(t, T_j, T_k)$ is stochastic.

As we show below, however, there is no guarantee that the above dynamics are in general well-defined in their natural martingale measure for a completely generic choice of the set of forward swap rates. Therefore, there is no guarantee that these dynamics yield a unique specification in all equivalent pricing measures. For this to hold, we would need to prove the existence of a complete family of discount bond prices spanning the tenor structure. This property is however not guaranteed a priori. For a moment we will then assume that this set exists so that dynamics (2.2) are well defined.

We remark that, at least in principle, a solution could rely on building market models as particular specifications of the framework of Heath, Jarrow, and Morton (1992) (HJM) as done, for instance, in Brace, Gatarek, and Musiela (1997), within the LMM class. The problem with this "indirect" approach is that dynamics (i.e., instantaneous volatility functions) must be assigned on unobservable rates, as opposed to LIBOR or swap rates which are the natural state variables in market models. These are, in fact, the quantities one is willing to directly model and control. This also explains why we depart from a standard HJM construction in this paper.

We observe that the forward LIBOR rate $L(t, T_j), j = 1, \ldots, M-1$, defined as

$$L(t, T_j) = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta_{j+1} B(t, T_{j+1})}, \quad \forall t \in [0, T_j],$$

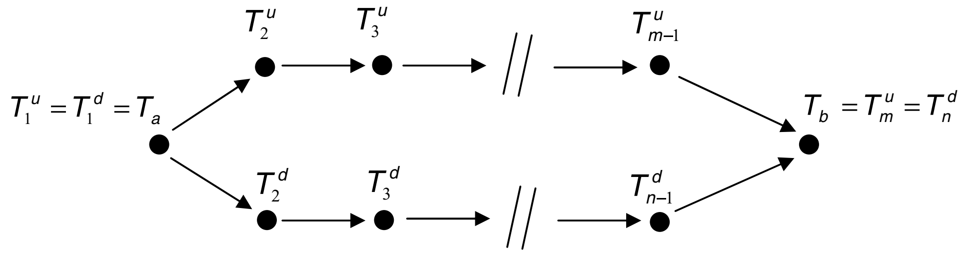
is itself a forward swap rate $S(t, T_j, T_k)$ corresponding to $k = j+1$, whose volatility function is denoted by $\lambda(t, T_j)$. Accordingly, we denote by $\mathbb{P}^{T_j}$ the corresponding forward probability measure associated to the discount bond price $B(t, T_j)$, and by W^{T_j} a Brownian motion under $\mathbb{P}^{T_j}$. Then, for every $i = 1, \ldots, M$, the relative bond price $B(t, T_i)/(\delta_{j+1} B(t, T_{j+1}))$, $\forall t \in \left[0, T_i \wedge T_{j+1}\right]$, follows a local martingale under $\mathbb{P}^{T_{j+1}}$. To simplify the exposition, we will often use the following compact notations: $B_j(t) := B(t, T_j)$, $G_{jk}(t) := G(t, T_j, T_k)$, $S_{jk}(t) := S(t, T_j, T_k)$, $\lambda_{jk}(t) := \lambda(t, T_j, T_k)$, $L_j(t) := L(t, T_j)$, $\lambda_j(t) := \lambda(t, T_j)$. Sometimes, we will also omit specifying calendar-time t dependence, if no confusion arises.

A generic arbitrage-free model is then specified by assigning arbitrage-free dynamics to a given set of forward swap rates spanning the tenor structure. Obviously, there are many possible choices regarding the selection of forward swap rates. Only a few among them are meaningful from a modeling perspective. To this aim, we introduce the following concept

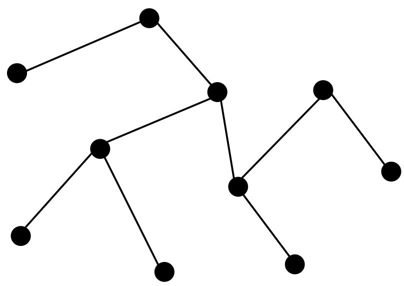
DEFINITION 2.1. Let $\mathcal{T}$ be a generic tenor structure, and $\mathcal{S}$ its associated set of forward swap rates. A collection $\{\mathcal{D}, \mathcal{S}_{\mathcal{D}}\}$ of reset/settlement dates $\mathcal{D} = \{T_a, \ldots, T_b\}$ and forward swap rates $\mathcal{S}_{\mathcal{D}} := \{S(t, T_j, T_k); T_j, T_k \in \mathcal{D}, j < k\}$ associated with $\mathcal{D}$ is called degenerate if the following conditions are satisfied

1. $\{\mathcal{D}, \mathcal{S}_{\mathcal{D}}\} \subseteq \{\mathcal{T}, \mathcal{S}\}$.
2. For any date $T_j, j = a+1, \ldots, b-1$, there exist in $\mathcal{S}_{\mathcal{D}}$ at least one forward swap rate starting at T_j and one forward swap rate ending at T_j.
3. There exist in $\mathcal{S}_{\mathcal{D}}$ at least two forward swap rates starting at T_a and two forward swap rates ending at T_b.

The thinnest configuration that is associated to a degenerate subset is depicted in Figure 2.1a. We can see that every element of $\mathcal{S}_{\mathcal{D}}$ has both start and end date contained in $\mathcal{D}$. The picture shows that the set $\mathcal{S}_{\mathcal{D}}$ yields a closed loop (or cycle) as opposed to a tree. We defer the description of the link between degenerate subsets and graph theory as well as the mathematical definitions of a cycle and a tree to the next section. The set $\mathcal{S}_{\mathcal{D}}$ is made of two disjoint subsets $\mathcal{S}_{\mathcal{D}} = \{\mathcal{S}^u_{\mathcal{D}}, \mathcal{S}^d_{\mathcal{D}}\}$ with $\mathcal{S}^u_{\mathcal{D}} := \{S^u_i\}_{\mathcal{D}}$ (upper path of the graph) and $\mathcal{S}^d_{\mathcal{D}} := \{S^d_j\}_{\mathcal{D}}$ (lower path of the graph) such that $\mathcal{H}^u = \{T^u_1, T^u_2, \ldots, T^u_m\}$ and $\mathcal{H}^d = \{T^d_1, T^d_2, \ldots, T^d_n\}$ with $T^u_1 = T^d_1 = T_a$, and $T^u_m = T^d_n = T_b$. In particular, every



(a) Degenerate set



(b) Example of a tree

FIGURE 2.1. Degenerate set and example of a tree.

date at the tip of the arrows in Figure 2.1a corresponds to a forward swap rate ending at that date. As a consequence, the set $\mathcal{S}_{\mathcal{D}}^u$ (resp. $\mathcal{S}_{\mathcal{D}}^d$) is characterized by the following graphical property: starting from date T_a, it is possible to "reach" the end date T_b by means of a unique path following arrows between dates belonging to $\mathcal{S}_{\mathcal{D}}^u$ (resp. $\mathcal{S}_{\mathcal{D}}^d$) only. From a financial point of view, this property corresponds to the following investment strategy: if an amount N_0 is invested at time T_a, there exist two distinct sequences of dates such that (*i*) at any date the notional principal is redeemed and immediately reinvested at the current interest rate, and (*ii*) the principal N_0 is redeemed at time T_b.

Let $\mathcal{A}$ be the set of all degenerate subsets $\{\mathcal{D}, \mathcal{S}_{\mathcal{D}}\}$ for a given tenor structure $\mathcal{T}$ and family of forward swap rates $\mathcal{S}$. The concept of an admissible set of forward swap rates can then be formulated.

DEFINITION 2.2. A set $\mathcal{S} := \{S(t, T_j, T_k); T_j, T_k \in \mathcal{T}, j < k\}$ of forward swap rates given a tenor structure $\mathcal{T}$ is said to be admissible if

1. The cardinality of the set is equal to the number of dates in the tenor structure minus one, i.e., $|\mathcal{S}| = M - 1$.
2. Any date $T_j, j = 1, \ldots, M$, in the tenor structure must coincide with a reset/settlement date of at least one forward swap rate in $\mathcal{S}$.
3. The set $\mathcal{A}$ is empty, i.e., $|\mathcal{A}| = 0$.

Assume that we are given forward swap rates belonging to an admissible set.[1] The following result holds:

[1] Here, we implicitly assume that the dynamics of discount bond prices are associated to equation (2.2).

PROPOSITION 2.1. *For all $t \in [0, T_1)$, if the set $\mathcal{S}$ is admissible, then a set of deflated discount bond prices $\{\frac{B(t,T_k)}{B(t,T_i)}\}_{k=1,\dots,M}$ relative to $B(t, T_i)$, $i = 1, \dots, M$, exists and is uniquely defined $\mathbb{P}$-a.s. Conversely, if for all $t \in [0, T_1)$ a set of deflated discount bond prices $\{\frac{B(t,T_k)}{B(t,T_i)}\}_{k=1,\dots,M}$ relative to $B(t, T_i)$, $i = 1, \dots, M$, exists and is uniquely specified $\mathbb{P}$-a.s. by $\mathcal{S}$, then $\mathcal{S}$ is admissible.*

Proof. See Appendix A.

Proposition 2.1 states that from an admissible set of forward swap rates it is possible to uniquely determine sets of deflated discount bond prices, relative to a given discount bond, spanning the entire tenor structure.[2] More explicitly, if the set is not admissible, three situations may occur. If $|\mathcal{S}| < M - 1$, there exist multiple choices of discount bond prices that are all compatible, at any time, with the given set of forward swap rates. In this situation, dynamics (2.2) cannot be uniquely specified. Depending on the choice of the family of discount bond prices associated to $\mathcal{S}$, multiple choices of LIBOR rate dynamics (and associated pricing measures) are possible. The model is then incomplete. If $|\mathcal{S}| > M - 1$ it is not possible to determine a family of discount bond prices that are compatible with $\mathcal{S}$, and dynamics (2.2) are not well defined. In this case, the market model is not arbitrage free. Finally if $|\mathcal{S}| = M - 1$ but the set $\mathcal{A}$ is not empty, then it is not possible to guarantee the existence of a set of deflated discount bond prices for any choice of $B(t, T_i)$, $i = 1, \dots, M$, as numéraire, as shown in Appendix A.

If the set S is admissible, Proposition 2.1 shows that it is possible, in principle, to build market models by assigning the joint dynamics of $\mathcal{S}$ in any martingale measure that is associated to the tenor structure, and to rely on the construction of Jamshidian (1997) to price LIBOR and swap derivatives by arbitrage. Examples of admissible and non-admissible sets of forward swap rates are given in Figure 2.2 for $M = 5$.

The following proposition further shows that in any admissible model all deflated discount bond prices are well behaved and, consequently, changes of numéraire are well defined.

PROPOSITION 2.2. *If the set $\mathcal{S} := \{S(t, T_j, T_k)\}$ is admissible, all deflated discount bond prices $\{\frac{B(t,T_k)}{B(t,T_i)}\}_{k=1,\dots,M}$ relative to $B(t, T_i)$, $i = 1, \dots, M$, are different from zero for all $t \in [0, T_1)$, $\mathbb{P}$-a.s.*

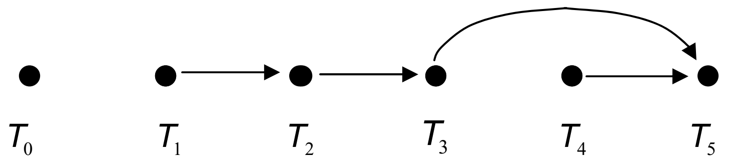
Proof. See Appendix B.

Combining Propositions 2.1 and 2.2, we deduce that the choice of an admissible set of forward swap rates is a necessary and sufficient condition for the associated market model to admit a unique specification in any equivalent probability measure associated to a discrete tenor structure. Correspondingly, we introduce the following concept.

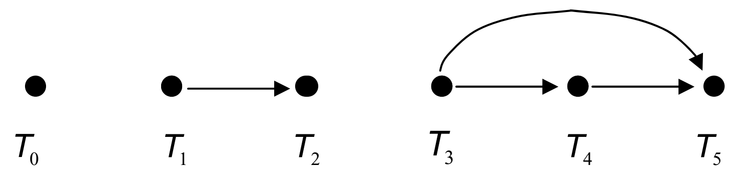
DEFINITION 2.3. A MMA is specified through assigning arbitrage free dynamics of the type (2.2) to a set of admissible forward swap rates.

We observe that Proposition 2.2 does not exclude the possibility that discount bond prices might take values larger than 1. Thus, absence of arbitrage is guaranteed in the MMA in the weak form only, after Musiela and Rutkowski (2004). The MMA is the most general framework for market models of interest rate dynamics where the starting point

[2] As a side remark, we note that the tenor structure and its associated family of forward swap rates at $t < T_1$ are different from the corresponding sets at $t > T_1$, and need to be redefined in each interval.



(a) Example of an admissible set



(b) Example of a non-admissible set

FIGURE 2.2. Examples of admissible and non-admissible sets of forward swap rates.

is the direct modeling of observable state variables. For practical purposes, however, the class of admissible sets of forward swap rates is still too large (see Proposition 3.3). In Section 4, we will select, among all admissible models, those being the most relevant from a pricing and risk-management point of view.

3. CONNECTION WITH GRAPH THEORY

The above considerations suggest an interesting way to simplify the concept of admissibility of a set of forward swap rates through the mathematical theory of graphs. Let us first recall some useful concepts related to that theory. We follow the standard text by Diestel (2000).

Let $Gr := (V, E)$ be a graph made of a given set V of vertices (or points) and a set E of edges (or lines). A generic edge e is a line connecting two adjacent vertices. If x and y are adjacent vertices, the edge connecting them will be indicated by the set $e = \{x, y\} := xy$. Pairwise, non-adjacent vertices are called independent. A vertex v is incident with an edge f if $v \in f$, e.g., x and y are incident with e. Let $Gr' \cap Gr := (V \cap V', E \cap E')$ the intersection between two graphs. If $Gr' \cap Gr = \emptyset$, then Gr and Gr' are disjoint graphs. Two disjoint graphs are graphically unconnected, i.e., there are no edges linking a vertex of Gr with one of Gr'. We say that $Gr' \subseteq Gr$ is a spanning subgraph of Gr if V' spans all vertices of Gr, that is if $V = V'$. The degree $d(v)$ of a vertex v is the number $|E(v)|$ of edges at v. A path is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$. For ease of notation, a path is usually defined by the sequence of its vertices. If P is a path with $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a cycle. From a graphical point of view, a cycle is a sequence of lines joining adjacent vertices where every vertex is simultaneously the start and the end of the path. A non-empty graph is called connected if any two of its vertices are linked by a path. A connected graph with no cycles (or acyclic graph) is called a tree (see Figure 2.1b). Spanning subgraphs that are also trees are called spanning trees. A labeled tree is a tree with its nodes labeled.

A simple link exists between sets of forward swap rates associated to a tenor structure and a graph. To this aim, we observe that, by defining $\mathcal{T} := V$ and $\mathcal{S} := E$, the set $Gr = \{\mathcal{T}, \mathcal{S}\}$ is a graph. In this formulation, a degenerate subset is equivalent to the presence of a cycle in *Gr*. More generally, it is possible to recast Definition 2.2 by means of graph theory as follows.

DEFINITION 3.1. A set $\mathcal{S}$ is said to be admissible if

1. The graph *Gr* has M vertices and $M - 1$ edges, i.e., $|\mathcal{S}| = |\mathcal{T}| - 1 = M - 1$.
2. The graph *Gr* is connected.

The following result states the equivalence between Definitions 2.2 and 3.1.

PROPOSITION 3.1. *Definition 2.2 and Definition 3.1 are equivalent.*

Proof. See Appendix C.

Definition 3.1 allows us to translate the concept of admissibility in terms of graph properties. A further step in this direction can be achieved by recalling a fundamental result in graph theory.

THEOREM 3.1. *A connected graph with M vertices is a tree if and only if it has $M - 1$ edges.*

Proof. See Diestel (2000).

As a consequence, the simplest characterisation of admissibility of a set of forward swap rates reads as follows.

PROPOSITION 3.2. *A set $\mathcal{S}$ is admissible if and only if Gr is a tree.*

Proposition 3.2 allows determining, by simple graphical inspection, whether a set of forward swap rates is admissible or not. Besides, the mapping to graph theory allows us to enumerate all admissible market models for a given tenor structure.

PROPOSITION 3.3. *For a given tenor structure $\mathcal{T} := \{T_1, \dots, T_M\}$ there are M^{M-2} admissible sets.*

Proof. See Appendix D.

These M^{M-2} admissible sets can be characterized via the Prüfer code (Prüfer 1918). This well-known algorithm is an encoding which provides a bijection between the M^{M-2} labelled trees on M nodes and strings of $M - 2$ integers chosen from an alphabet of the numbers 1 to M. Symbolic calculus packages such as Mathematica provide routines which can convert a Prüfer code to a labeled tree, and vice versa.

4. MODEL SELECTION

As Proposition 3.3 shows, the number of admissible models becomes rapidly extremely large at increasing M. The first few values are in fact 1, 1, 3, 16, 125, 1296, ... In a typical market configuration with annual reset/settlement dates and a 10-year maturity, one can introduce 10^8 different admissible market models. Thus, the choice of the set of forward swap rates that underlie model dynamics must be driven by practical considerations. Usually, dynamic IR models are used to price and hedge exotic IR derivatives for which no direct market information exists. To ensure meaningful hedging, a model should be made consistent with the term structure of interest rates as well as the volatility information provided by a set of quoted plain-vanilla derivatives. These are typically caps and swaptions. Depending on the type of exotic option to be priced, however, some of these "market" instruments may be more relevant than others. As an example, we consider a Bermudan swaption. If the option gives the holder the right to enter at times $T_1, T_2, \ldots, T_{M-1}$, into a plain-vanilla swap maturing at T_M, the only relevant European swaptions from a pricing and hedging perspective are those whose expiry dates are $T_1, T_2, \ldots, T_{M-1}$, and with underlying swap maturity T_M. In this case, the natural choice is to introduce a MMA where the relevant set coincides with the co-terminal forward swap rates. The spanning set underlying the dynamics is shown in Figure 4.1a. In this case, all forward swap rates share the same terminal date T_M but have a variable initial date. In summary, we can introduce the following model.

DEFINITION 4.1. A co-terminal swap market model (ctSMM) is built from

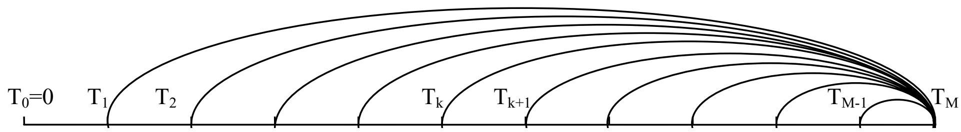
1. An admissible set $\{S_{jM}\}, j = 1, \ldots, M-1$, of forward swap rates with different start dates and equal end date T_M.
2. A collection of mutually equivalent probability measures $\mathbb{P}^{T_j,T_M}$, $j = 1, \ldots, M-1$.
3. A family W^{T_j,T_M} of processes such that: (i) for any $j = 1, \ldots, M-1$, W^{T_j,T_M} follows a d-dimensional Brownian motion under the forward swap probability measure $\mathbb{P}^{T_j,T_M}$, (ii) for any $j = 1, \ldots, M-1$, the forward swap rate satisfies the SDE, for all $t \in [0, T_j]$:

$$dS_{jM}(t) = S_{jM}(t)\lambda'_{jM}(t)\, dW^{T_j,T_M}(t), \;\; S_{jM}(0) = \frac{B_j(0) - B_M(0)}{\sum_{l=j+1}^{M} \delta_l B_l(0)}.$$

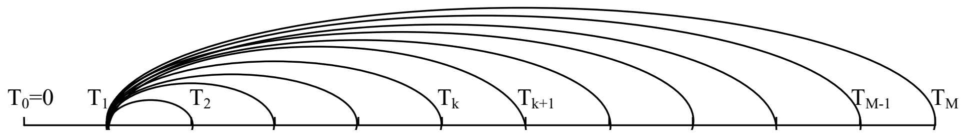
European-style derivatives give the holder the right to exercise an option at a single future date T. Therefore, the option payoff, no matter how complex, is by definition $\mathcal{F}_T$-measurable. Qualitatively speaking, a set of admissible forward swap rates sharing the same initial date T contains all the information needed to evaluate the payoff, the latter being in fact a function of a set of admissible "co-initial" forward swap rates at time T. Hence, a MMA built on a set of co-initial forward swap rates provides a powerful tool to price and hedge a large variety of European style derivatives including forward-start, amortising and zero-coupon swaptions as well as cross-currency swaptions. These issues are further discussed in Galluccio and Hunter (2004). As opposed to the ctSMM, all forward swap rates in the co-initial SMM share the same initial date T_1 but have a variable terminal date (see Figure 4.1b). We therefore arrive at the following description.

DEFINITION 4.2. A co-initial swap market model (ciSMM) is built from

1. An admissible set $\{S_{1j}\}, j = 2, \ldots, M$, of forward swap rates with equal start date T_1 and variable end dates.
2. A collection of mutually equivalent probability measures $\mathbb{P}^{T_1,T_j}$, $j = 2, \ldots, M$.



(a) Co-terminal forward swap rates



(b) Co-initial forward swap rates

(c) Co-sliding forward swap rates

FIGURE 4.1. Sets of co-terminal, co-initial, and co-sliding forward swap rates.

3. A family W^{T_1,T_j} of processes such that: (i) for any $j = 2, \ldots, M$, W^{T_1,T_j} follows a d-dimensional Brownian motion under the forward swap probability measure $\mathbb{P}^{T_1,T_j}$, (ii) for any $j = 2, \ldots, M$, the forward swap rate satisfies the SDE, for all $t \in [0, T_1]$:

$$dS_{1j}(t) = S_{1j}(t)\lambda'_{1j}(t)\, dW^{T_1,T_j}(t), \quad S_{1j}(0) = \frac{B_1(0) - B_j(0)}{\sum_{l=2}^{j} \delta_l B_l(0)}.$$

A third common class of products is that of IR derivatives, such as constant maturity swaps (or CMS), whose pay-off functions depend on a set of fixed-maturity instruments. In this case, an efficient choice consists of introducing a family of admissible forward swap rates having variable start and end date, but sharing the same period interval between tenor dates or time-to-maturity. We call this type of model "co-sliding" (see Figure 4.1c).

DEFINITION 4.3. A co-sliding swap market model (csSMM) is built from

1. An admissible set $\{S_{j,j+1}\}, j = 1, \ldots, M-1$ of forward swap rates.
2. A collection of mutually equivalent probability measures $\mathbb{P}^{T_j,T_{j+1}}, j = 1, \ldots, M-1$.
3. A family $W^{T_j,T_{j+1}}$ of processes such that: (i) for any $j = 1, \ldots, M-1$, $W^{T_j,T_{j+1}}$ follows a d-dimensional Brownian motion under the forward swap probability measure $\mathbb{P}^{T_j,T_{j+1}}$, (ii) for any $j = 1, \ldots, M-1$, the forward swap rate satisfies the SDE, for all $t \in [0, T_1]$:

$$dS_{j,j+1}(t) = S_{j,j+1}(t)\lambda'_{j,j+1}(t)\, dW^{T_j,T_{j+1}}(t), \quad S_{j,j+1}(0) = \frac{B_j(0) - B_{j+1}(0)}{\sum_{l=j+1}^{j+1} \delta_l B_l(0)}.$$

In fact Definition 4.3 introduces the LMM, since non-overlapping forward swap rates of the form $\{S_{j,j+1}\}, j = 1, \ldots, M-1$, are indeed forward LIBOR rates. Remark that other co-sliding type models can be obtained by generalizing the above equations to arbitrary families of overlapping and non-overlapping forward swap rates, like (*i*) those in the form $\{S_{j,j+n}\}, j = 1, 2, \ldots, M-n$, with $n = 2, \ldots, M-1$; (*ii*) those in the form $\{S_{j,j+m}\}, j = 1, m+1, \ldots, (km+1) \wedge (M-m)$, with $m = 2, \ldots, M-1$, and $k \in \mathbb{N}$. However, these alternative co-sliding models are not admissible, after Propositions 2.1 and 2.2. A direct consequence of the above is the following:

COROLLARY 4.1. *The LMM is the only admissible model of a co-sliding type.*

Another subclass of admissible models is that of "mixed" models obtained by assigning dynamics on admissible sets of forward swap rates where some swaps are co-sliding and some others are co-terminal. This case has been recently studied in Pietersz and Van Regenmortel (2004).

In conclusion, the MMA constitutes the most general framework to price and hedge IR derivatives where the starting point is the modeling of financial observable quantities like forward swap rates. Note, finally, that Definitions 4.1–4.3 can be easily extended to far more general classes of processes, see Jamshidian (1999).

5. THE CO-TERMINAL SMM

The co-terminal class, first introduced by Jamshidian (1997), is the one we concentrate from now on. In our opinion, its relevance has been largely overlooked in the literature until now.

We start by stressing that the underlying of many Over-the-Counter (OTC) interest rate derivatives are indeed swap (as opposed to LIBOR) rates. The simplest and most important example is the Bermudan swaption. Apart from plain-vanilla caps and swaptions, Bermudan swaptions are the most liquid IR derivatives. The owner of such a contract at a generic time t has the right to enter into a plain-vanilla swap maturing at a date T_M at any date among those in the set $\{T_1, \ldots, T_{M-1}\}$. The optimal exercise boundary can be found by using a standard optimal control approach, and the pricing formula can be formally decomposed into a hierarchical sequence of European swaptions (Jamshidian 1997). At any time T_i, $i < M-1$, the exercise decision will depend on the assessment of the volatility that will drive future swap rate dynamics, prevailing at that particular time. To price (and hedge) a Bermudan option at any time $t < T_i$ we need an arbitrage-free model for the evolution of the underlying rates. This simple example shows that a sound pricing methodology must concentrate on evolving swap rates, as they constitute the natural underlying of the Bermudan option.

The usual way, within the LMM setting, consists of trying to force the LIBOR-based dynamics to be consistent with the market information provided by the price of European swaptions. However, a perfect match of all cap and swaption prices is difficult to achieve without introducing strong assumptions in the model (Brigo and Mercurio 2001), so that global minimization algorithms are often the preferred choice. In this case, however, only a selection of options is considered for calibration (see for instance Rebonato 2003). As pointed out in a similar context by Cont and Tankov (2004), unconstrained global optimisation algorithms are in general slow, and may suffer from both accuracy and robustness problems. For these reasons, achieving a (quick and robust) simultaneous calibration to caplet and swaption prices within their bid/ask volatility spread can be problematic in the LMM setting.[3]

On the opposite, a ctSMM can be easily calibrated to a selection of caplets and "diagonal" swaptions, as shown below. These diagonal swaptions correspond to options written on co-terminal forward swap rates. The volatility risk associated to a large fraction of exotic IR derivatives is almost entirely "captured" by these two sets only. In this respect, apart from the aforementioned Bermudan swaptions, we mention Callable Cap and Reverse Floaters, Ratchet Cap Floaters and LIBOR Knock-in/out swaps. We deduce that the most appropriate framework for these complex IR derivatives is indeed the ctSMM. On the other side of the spectrum, products that are written on CMS rates, like callable CMS swaps, are better handled within a co-sliding model (i.e., the LMM).

Finally, it is worth mentioning that LIBOR rates are not directly quoted by the market, while swap rates are. Typically, forward LIBOR rates enter the valuation of forward rate agreements (FRAs). These simple instruments allow at some time t locking-in the discretely compounded interest rate that will be paid between two future dates. FRAs are OTC derivatives, and only LIBOR futures and plain-vanilla swaps are quoted on a daily basis in the market. As a consequence, the forward LIBOR rates must be derived (or "stripped") from market instruments, i.e., LIBOR futures and swaps. In Section 6, we show that a forward LIBOR rate can be formally written as a weighted sum of two consecutive forward swap rates, pretty much the same as a forward swap rate can be formally interpreted as a weighted sum of forward LIBOR rates. In this sense, the claim that LIBOR rates are "more fundamental" financial quantities than swap rates is problematic.

6. EUROPEAN OPTION PRICING

Model calibration is a reverse engineering procedure aimed at identifying the relevant model parameters from a set of liquid instruments quoted in the market. In the IR derivatives context, these instruments are plain-vanilla options written on forward swap and LIBOR rates, i.e., swaptions and caplets, respectively. To achieve a fast and stable model calibration, and avoid slow numerical procedures, we should be able to express plain-vanilla option prices in closed or quasi-closed form. In this section, we address the problem of the pricing of European options within the ctSMM. When closed-form solutions are not available, namely for caplet prices, we introduce several approximation procedures. Speed and accuracy of these different approaches are analyzed through numerical examples in Section 8.

For simplicity, we consider dynamics driven by deterministic volatility structures as in, e.g., Rutkowski (1998). The extension to more general classes of processes and the implications relating to the pricing and risk management of IR derivatives are left to future research.

To get compact notations we will make extensive use of the following auxiliary processes as in Jamshidian (1997): $\nu_{ij}(t) := \nu_{ij,M}(t) := \sum_{k=j}^{M-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l S_{lM}(t))$, $\nu_i(t) := \nu_{ii}(t)$. Note also that the following relations hold: $(i)\, G_{jM}(t)/B_M(t) = \nu_j(t)$, $(ii)\, B_j(t)/B_M(t) = 1 + \nu_j(t) S_{jM}(t)$.

[3] As many authors have emphasized, caplet and swaption markets show some degree of "inconsistency" when we try to fit a LMM to both.

6.1. Swaption Pricing

Under a deterministic volatility structure, the co-terminal forward swap rates are lognormally distributed, so that the corresponding swaption can be priced via a Black formula (Black 1976). The price of a European swaption, giving the right to enter at time T_j into a swap maturing at T_M, is given at time t by

$$\mathbf{Swn}(t, T_j, K) = G_{jM}(t)[S_{jM}(t)N(d_1) - KN(d_2)], \tag{6.1}$$

where, as usual, $d_1 = (\ln(S_{jM}(t)/K) + \frac{1}{2}\sigma_{jM}^2(T_j - t))/(\sigma_{jM}\sqrt{T_j - t})$, $d_2 = d_1 - \sigma_{jM}\sqrt{T_j - t}$, with $\sigma_{jM}^2 = \frac{1}{T_j - t}\int_t^{T_j} \lambda'_{jM}(s)\lambda_{jM}(s)\, ds$.

We use σ to indicate Black implied volatilities. Similarly to the notation used for instantaneous volatilities, σ_{jM} is the Black implied volatility of the swaption written on S_{jM}, and σ_j is the Black implied volatility of the caplet written on L_j.

6.2. Caplet Pricing

We consider the price of a caplet at time t giving the right to buy a forward LIBOR rate between T_j and T_{j+1}:

$$\mathbf{Cpl}(t, T_{j+1}, K) = \delta_{j+1} B(t, T_{j+1})\mathbb{E}_t^{T_{j+1}}[(L_j(T_j) - K)_+],$$

where the expectation is taken with respect to the forward measure $\mathbb{P}^{T_{j+1}}$ such that the forward LIBOR rate $L_j(t)$ follows a martingale process: $dL_j(t) = L_j(t)\lambda'_j(t)\, dW^{T_{j+1}}(t)$.

In the ctSMM forward LIBOR rates are not lognormally distributed and we cannot price caplets using the Black formula directly. In the LMM setting a similar situation exists: caplets can be priced in closed-form using Black formula while swaptions cannot. Quick and accurate approximation techniques to price swaptions in the LMM have been proposed by Rebonato (1998), Hull and White (2000), and Brace, Gatarek, and Musiela (1997). Their construction and accuracy are reviewed in detail in Brigo and Mercurio (2001). Here we will parallel these suggestions in the context of the ctSMM, and provide similar approximated formulae for caplet prices. In addition, we suggest a new approximation based on a spread option approach, which leads to a Margrabe-type formula (Margrabe 1978) that has no counterpart in the LMM setting.

6.2.1. Rebonato Approach. This method is similar to the one first advocated by Rebonato (1998) for the LMM. The starting point consists of observing that a forward LIBOR rate can be written as a weighted sum of two consecutive co-terminal forward swap rates:

$$L_j(u) = w_{jj}(u)S_{jM}(u) + w_{j,j+1}(u)S_{j+1,M}(u), \tag{6.2}$$

where $w_{jj}(u) = \nu_j(u)/(\nu_j(u) - \nu_{j+1}(u))$, $w_{j,j+1}(u) = -\nu_{j+1}(u)/(\nu_j(u) - \nu_{j+1}(u))$, with $w_{jj}(u) + w_{j,j+1}(u) = 1$.

The two weights can also be expressed as $w_{jj}(u) = G_{jM}(u)/(\delta_{j+1}B_{j+1}(u))$ and $w_{j,j+1}(u) = -G_{j+1,M}(u)/(\delta_{j+1}B_{j+1}(u))$. Note that they sum to one but $w_{jj}(u)$ is positive whereas $w_{j,j+1}(u)$ is negative. This is at difference with the LMM where all weights are positive in interpreting a forward swap rate as a weighted sum of forward LIBOR rates.

The two steps underlying the Rebonato approach consist of (a) "freezing" the weights at their initial value (at time t) in equation (6.2), and then differentiate both sides;

(b) "freezing" the remaining random forward LIBOR and swap rates in the volatility function. This approach provides a lognormal approximation of the LIBOR rate dynamics, and is accurate if the variability of the weights is much smaller than the variability of the forward swap rates. This hypothesis can be tested both historically and through Monte Carlo simulations (Section 8.2). The validity of the approximation can be intuitively understood by recalling that the weights are ratios of linear combinations of discount bond prices. Hence the volatility of the weights is small by construction since weight dynamics derive from ratios of highly correlated processes.

6.2.2. Hull and White Approach. A slightly more sophisticated version of the above procedure follows the path of Hull and White (HW) (2000). It consists of (a) differentiating equation (6.2) without an initial freezing of the weights, (b) freezing the remaining random forward LIBOR and swap rates in the volatility function. We finally get the lognormal dynamics:

$$\frac{dL_j(u)}{L_j(u)} \approx \sum_{l=j}^{M-1} \hat{w}_{jl}(t)\lambda_{lM}(u)'\, dW^{T_{j+1}}(u),$$

with $\hat{w}_{jl}(t) = \bar{w}_{jl}(t)S_{lM}(t)/L_j(t)$, and $\bar{w}_{jl}(t)$ is equal to

$$\begin{cases} \nu_j(t)/(\nu_j(t) - \nu_{j+1}(t)), & l = j, \\ -\delta_{j+1}\nu_{j+1}(t)(1 + \nu_{j+1}(t)S_{jM}(t))/(\nu_j(t) - \nu_{j+1}(t))^2, & l = j+1, \\ \delta_l\delta_{j+1}\nu_{j+1,l}(t)(S_{jM}(t) - S_{j+1,M}(t))\big/\big[(\nu_j(t) - \nu_{j+1}(t))^2(1 + \delta_l S_{lM}(t))\big], & j+2 \le l \le M-1, \\ 0, & \text{otherwise.} \end{cases}$$

The volatility parameter to plug into the Black caplet price can be derived from the expression

$$(T_j - t)\sigma_j^2 = \sum_{l=j}^{M-1}\sum_{k=j}^{M-1} \hat{w}_{jl}(t)\hat{w}_{jk}(t)\int_t^{T_j} \lambda_{lM}'(u)\lambda_{kM}(u)\, du.$$

Once more, the approximation is accurate provided weights $\hat{w}$ do not vary too much. Numerical experiments (Section 8.2) show that the variability of the weights is small when compared to the variability of forward swap and LIBOR rates. Besides, it is worth noticing that the first two weights are much larger than the others by a factor of about 1,000 in absolute terms. Hence, the price or the implied volatility of a caplet should only change marginally if other weights apart from the two first ones are neglected (Section 8.1). This further approximation results in a "truncated" HW approach, which will be instrumental in the recursive calibration approach of Section 7.

6.2.3. Rank-One Approach. The rank-one approach has been suggested by Brace, Gatarek, and Musiela (1997). In the LMM it starts from recognizing that a swaption can be viewed as a sum of caplets whose exercise regions depend on the forward swap rate instead of the forward LIBOR rate. The approximation then relies on freezing the drift in the forward LIBOR rate dynamics and on a rank-one approximation of the covariance matrix of the forward LIBOR rates. A similar approach can be adopted in an SMM framework since a caplet can be viewed as a sum of swaptions whose exercise regions depend on the forward LIBOR rate instead of the forward swap rate.

6.2.4. Spread-Option Approach. Recall that the forward LIBOR rate can be written as a basket of two consecutive co-terminal forward swap rates from equation (6.2). Hence, once the weight factors have been frozen at their initial values at t, the caplet can be viewed as an option on a spread between two consecutive forward swap rates. Here we use the same freezing technique as in the Rebonato approach but we do not rely on the approximation of the basket of two lognormally distributed variables by means of a single lognormal variable (which is necessary to get a Black formula). The caplet price is then akin to the formula given in Margrabe (1978).

7. MODEL CALIBRATION

The problem of the calibration of the LMM has attracted much interest recently, see for instance Brigo and Mercurio (2001), Rebonato (2003), Schoenmakers and Coffey (2003), and Wu (2002). Yet, no similar results are currently available in the ctSMM, despite the mathematical similarities existing between the two approaches.

When dealing with calibration, it is convenient to use the following scalar specification of the co-terminal forward swap rates S_{jM} for $j = 1, \dots, M-1$, under their appropriate forward swap measures:

$$\frac{dS_{jM}(t)}{S_{jM}(t)} = \Lambda_{jM}(t)\, d\bar{W}^{T_j,T_M}(t),$$

where $\Lambda_{jM}(t) := \|\lambda_{jM}(t)\|$ is a scalar time-inhomogeneous function equal to the Euclidean norm of the corresponding instantaneous volatility vector $\lambda_{jM}(t)$ and $d\bar{W}^{T_j,T_M}(t) = \lambda'_{jM}(t)\, dW^{T_j,T_M}(t)/\|\lambda_{jM}(t)\|$. Here $\bar{W}^{T_j,T_M}$ is a one-dimensional Brownian motion under the forward measure $\mathbb{P}^{T_j,T_M}$ associated with the numéraire G_{jM}. Since forward swap rate S_{jM} is undefined at times $t > T_j$, we can extend all dynamics up to T_M by requiring that $\Lambda_{jM}(t) \neq 0$ at $t \in [0, T_{j-1}]$, and $\Lambda_{jM}(t) = 0$ at $t > T_j$. The instantaneous correlation between the scalar Brownian motion is denoted by $\rho_{ij}(t)$. This specification of a ctSMM is complete once the correlation matrix $(\rho_{ij}(t))$ and the instantaneous volatility functions $\lambda_{jM}(t)$ have been both assigned. To simplify model calibration and avoid data overfitting, we will assume that the correlation matrix (ρ_{ij}) is time independent and leave alone the functions $\Lambda_{jM}(t)$ to completely specify the time dependence in the covariance matrix.

The choice of the model calibration instruments must be driven by practical (in particular hedging) considerations. If a model is calibrated to a finite set $\mathcal{A}$ of plain-vanilla options then it will show no risk sensitivity against any other set $\mathcal{B}$ disjoint from $\mathcal{A}$. Thus, the choice of $\mathcal{A}$ must be associated to that portion of the volatility matrix that we consider the most informative in capturing the volatility risk. Below we show that our ctSMM can be easily and efficiently calibrated to a set $\mathcal{A}$ made of all caplets and co-terminal swaptions associated to the tenor structure. This set is indeed optimal for a large fraction of all exotic IR derivatives, as above mentioned. This is a crucial property of the ctSMM since, in comparison, a LMM must in general be calibrated to a larger set even in situations where some calibration instruments are redundant as far as volatility risk is concerned.

The main idea to calibrate the ctSMM consists of using a parametric recursive procedure. This choice has three main advantages. First, it ensures that the model is simultaneously consistent with a set of co-terminal swaptions and caplets spanning the same tenor structure. Second, it provides smooth calibrated instantaneous volatility functions, and avoids the overfitting problem. Third, the algorithm is extremely fast and accurate, and solves the robustness problems often associated with global minimization techniques.

For illustrative purpose, we adopt a parametric form for the instantaneous volatility of the forward swap rates. The methodology is similar in spirit to the one advocated by Rebonato (2003) in the context of the LMM, although other choices are in principle possible. We thus look for shapes that replicate the initial term structure of Black implied swaption volatilities, and are (as much as possible) time stationary. The last constraint is driven by the observation that, to a large extent, the global shape of the term structure of at-the-money (ATM) volatilities tends to be preserved in time. We thus introduce the following generic form of the scalar instantaneous volatility:

$$\Lambda_{jM}(t) := \phi_j(t)\psi_j(T_j - t), \quad j = 1, \dots, M-1. \tag{7.1}$$

The stationary factor $\psi_j(T_j - t)$ is meant to reproduce the well-known "hump" of the implied volatility of the swaption with underlying S_{jM}, while the calendar-time dependent function $\phi_j(t)$ represents a perturbation mode around the stationary solution. A simple and effective parameterisation of $\psi_j(T_j - t)$ is provided by

$$\psi_j(T_j - t;\theta_j) := (a_j(T_j - t) + b_j)e^{-c_j(T_j - t)} + d_j, \quad j = 1, \dots, M-1, \tag{7.2}$$

where $\{\theta_j := (a_j, b_j, c_j, d_j), j = 1, \dots, M-1\}$ is a set of real parameters. This specification of $\psi_j(T_j - t)$ is usually suggested in the LMM literature (see Brigo and Mercurio 2001, Rebonato 1998). The factor $e^{-c_j(T_j-t)}$ is used to model the decreasing shape of the term structure at the long end while $a_j(T_j - t) + b_j$ models the upward shape at the short end. Put together, they reproduce the observed hump as a function of time-to-maturity. Finally, the parameter d_j sets a global level. Note that θ_j is indexed by j since different co-terminal swaptions have in general different shapes of implied volatility term structures.

Ideally, if $\phi_j(t)$ were unitary functions for any j, model dynamics would be perfectly stationary. Unfortunately, the constraint of perfect consistency with the initial term structure of swaption volatilities is usually too strong to allow for the model being simultaneously consistent with the cap/floor market as well. Therefore a perturbation around the stationary solution $\psi_j(T_j - t)$ is needed, and the goal will be to keep perturbation $\phi_j(t)$ as close to unity as possible in the calibration process.

The calibration is performed in two sequential steps. First, we fit $\psi_j(T_j - t)$ to the humped shape of market implied volatility of co-terminal swaptions indexed by $j = 1, \dots, M-1$, by adjusting the set $\{\theta_j\}$. Second, for a given instantaneous correlation matrix (ρ_{ij}), we calibrate $\phi_j(t)$ on both caplet and swaption market volatilities via a recursive algorithm. The correlation matrix can be statistically estimated from historical data or further modeled according to suitable parametric forms, as we discuss below. If the correlation structure is taken as an input, the only freedom left is the first component of the instantaneous volatility of forward swap rates. However, forcing (ρ_{ij}) to match its historical estimate is in general too restrictive, as we show below. Therefore, an alternative method will be introduced.

Assume that $\psi_j(T_j - t;\theta_j)$ has already been identified. We may then proceed as follows in order to make our ctSMM consistent with the caplet volatilities. Consider the following objects: the forward swap rates $S_{jM}(t)$, $S_{j+1,M}(t)$, the forward LIBOR rate $L_j(t)$ and their associated Black volatilities $\sigma_{j,M}$, $\sigma_{j+1,M}$, σ_j. In the scalar representation of the dynamics it is straightforward to show that

$$(T_j - t)\sigma_{j,M}^2 = \int_t^{T_j} \Lambda_{j,M}(s)^2\, ds, \tag{7.3}$$

$$(T_{j+1} - t)\sigma_{j+1,M}^2 = \int_t^{T_{j+1}} \Lambda_{j+1,M}(s)^2\, ds,$$

$$\begin{aligned}(T_j - t)\sigma_j^2 = {} & \hat{w}_j(t)^2 \int_t^{T_j} \Lambda_{j,M}(s)^2\, ds + \hat{w}_{j+1}(t)^2 \int_t^{T_j} \Lambda_{j+1,M}(s)^2\, ds \\ & + 2\hat{w}_j(t)\hat{w}_{j+1}(t)\rho_{j,j+1} \int_t^{T_j} \Lambda_{j,M}(s)\Lambda_{j+1,M}(s)\, ds,\end{aligned} \tag{7.4}$$

where the coefficients $\hat{w}_j(t)$, $\hat{w}_{j+1}(t)$ can be those provided by either the Rebonato or the truncated HW approximation of Section 6.2. Factor $\phi_j(t)$ is then introduced as a smooth perturbation around the stationary solution. In particular, it will be defined as follows:

$$\phi_j(t) = \phi_j f_j(t) = \begin{cases} \phi_j^a/(1+\alpha_j t), & t \in [0, T_{j-1}), \\ \phi_j^b/(1+\alpha_j t), & t \in [T_{j-1}, T_j], \end{cases} \tag{7.5}$$

where α_j, ϕ_j^a, and ϕ_j^b are positive constants. This choice for $\phi_j(t)$ is only one of the possible parametric forms, but it yields very satisfactory results since it does not introduce dramatic alterations of the initial shape associated to $\psi_j(T_j - t; \theta_j)$. Equation (7.4), in terms of $\phi_j(t)$ and the Black volatility σ_{jM}, reads

$$\begin{aligned}(T_j - t)\sigma_j^2 = {} & \hat{w}_j(t)^2\sigma_{jM}^2(T_j - t) + \hat{w}_{j+1}(t)^2\left(\phi_{j+1}^a\right)^2 \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2\, ds \\ & + 2\hat{w}_j(t)\hat{w}_{j+1}(t)\rho_{j,j+1}\phi_{j+1}^a \left[\phi_j^a \int_t^{T_{j-1}} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1}\, ds \right. \\ & \left. + \phi_j^b \int_{T_{j-1}}^{T_j} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1}\, ds\right].\end{aligned} \tag{7.6}$$

On the other hand, we have from equation (7.3):

$$(T_{j+1} - t)\sigma_{j+1,M}^2 = \left(\phi_{j+1}^a\right)^2 \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2\, ds + \left(\phi_{j+1}^b\right)^2 \int_{T_j}^{T_{j+1}} f_{j+1}^2(s)\psi_{j+1}^2\, ds. \tag{7.7}$$

Note that, if the scalar instantaneous volatility function $\Lambda_{j,M}(s)$, $s \in [t, T_j]$, of the j^{th} forward swap rate were known and if we used the above functional form of the volatility function of the $(j+1)^{\text{th}}$ forward swap rate $\Lambda_{j+1,M}(s)$, $s \in [t, T_{j+1}]$, then it would be possible to determine uniquely $\Lambda_{j+1,M}(s)$, $s \in [t, T_{j+1}]$, from the knowledge of: (*i*) the three Black volatilities $\sigma_{j,M}$, $\sigma_{j+1,M}$, σ_j and, (*ii*) the correlation $\rho_{j,j+1}$. Mathematically, this stems from the observation that the only unknown in equation (7.6) is ϕ_{j+1}^a. Assume now that ϕ_{j+1}^a has been determined from equation (7.6), then equation (7.7) can be solved for ϕ_{j+1}^b, and the volatility $\Lambda_{j+1,M}(s)$ is then uniquely identified. This procedure can thus be repeated step-by-step, i.e., by "bootstrap," until the last co-terminal swaption and caplet. In detail, the procedure goes as follows for $j = 0, \ldots, M-2$:

1. Select α_j, for $j = 1, \ldots, M-1$. We need to initialize α_j to large values in order for solutions to exist.
2. Set $\phi_1^a = 0$, and solve equation (7.7) when $j = 0$ for ϕ_1^b.

3. After (ϕ_j^a, ϕ_j^b) is known, the only remaining unknown in equation (7.6) is ϕ_{j+1}^a. ϕ_{j+1}^b can then be solved from equation (7.7). Repeat this procedure from $j = 1$ to $j = M - 2$. We note that equation (7.6) is a quadratic algebraic equation in $x := \phi_{j+1}^a$, namely $ax^2 + bx + c = 0$, with $a := \hat{w}_{j+1}^2(t) \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2 dt$, $b := 2\hat{w}_j(t)\hat{w}_{j+1}(t)\rho_{j,j+1}\phi_j^a(\int_t^{T_{j-1}} f_j(s)\psi_j f_{j+1}(s)\psi_{j+1}\, ds \;+\; \phi_j^b \int_{T_{j-1}}^{T_j} f_j(s)\psi_j f_{j+1}(s) \times \psi_{j+1}\, ds)$, and $c := [\hat{w}_j^2(t)\sigma_{jM}^2 - \sigma_j^2](T_{j+1} - t)$. The real-valued solutions are given by

$$\phi_{j+1}^a = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

$$\phi_{j+1}^b = \left(\frac{\sigma_{j+1,M}^2(T_{j+1} - t) - \left(\phi_{j+1}^a\right)^2 \int_t^{T_j} f_{j+1}^2(s)\psi_{j+1}^2\, ds}{\int_{T_j}^{T_{j+1}} f_{j+1}^2(s)\psi_{j+1}^2\, ds} \right)^{1/2}.$$

We recall that $a > 0$, $b < 0$, and $c > 0$, so there are two positive roots of equation (7.6). We favor the smaller one since it improves the stability of the recursive algorithm. It also reduces the likelihood that a real-valued ϕ_{j+1}^b does not exist.[4]
4. If we go through steps 1–3 successfully, repeat steps 1 to 3 with smaller α_j.
5. Stop when we have the set of smallest α_j for which the solution of a set of (ϕ_j^a, ϕ_j^b) exists.

Finally we remark that equation (7.4) holds for caplets written on the 12-month (12M) forward LIBOR rate. If only caplets written on the 3M or 6M forward LIBOR rate are available in the market, we can easily adapt to our case the approach introduced in Brigo and Mercurio (2001) for the LMM.

In practical applications it is also sometimes preferable to reduce the number of factors driving the dynamics of the set of co-terminal forward swap rates. All above considerations apply with little modification to the case where $W^{T_j,T_M}(t)$ is a d-dimensional Brownian motion with $d < M - 1$.

8. NUMERICAL RESULTS

The numerical tests below are conducted in the EUR market using a family of co-terminal annual swaptions over a 10-year tenor structure and a family of 1-year LIBOR caplets ($T_0 = 0 < T_1 = 1 < \cdots < T_M = 10$). The market forward swap rates and Black implied volatilities correspond to data in the EUR market observed weekly from May 17, 2004, to May 16, 2005. They have been extracted from BNP Paribas proprietary databases. Empirical tests performed in USD and GBP markets, not reported here, provided qualitatively similar results to those in EUR.

8.1. Caplet Pricing Approximations

In order to evaluate the efficiency of the caplet pricing approximations introduced in Section 6.2, we run several Monte Carlo simulations using an Euler scheme with 100,000

[4] In all our empirical tests we have not found any complex-valued ϕ_{j+1}^b. However, we cannot rule out such a possibility in general. We advocate to check the data inputs if it does happen.

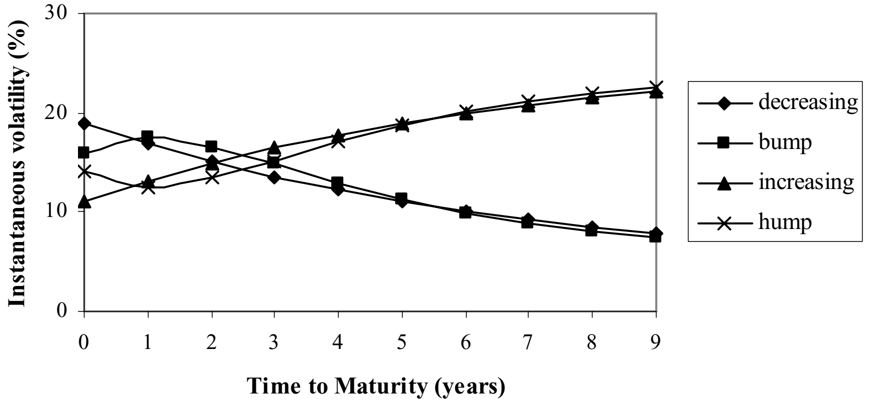


FIGURE 8.1. Instantaneous volatility term structures used in the simulations.

paths and 16 steps per period (one year) to compute the benchmark caplet prices. These benchmark results serve as a reference against which we can compare the Rebonato, the HW, the truncated HW, the spread-option, and the rank-one analytical approaches. Our Monte Carlo approach follows the methodology of Glasserman and Zhao (2000), and provides accurate results in terms of swaption prices and distribution of forward swap rates. To improve convergence of the simulation algorithm, these authors suggest to simulate quantities derived from the chosen set of forward swap rates, rather than the forward swap rates themselves, and then recover the forward swap rates from those quantities. Here we consider $Y_j = \nu_j - \nu_{j+1} - \delta_{j+1}, j = 0, \ldots, M-2$, to handle irregularly spaced tenors. Since ν_j is a martingale under the terminal (or forward) measure, Y_j is also a martingale and no drift adjustment is needed. In practice, we simulate the process $\ln Y_j$ to guarantee positiveness. It can be proved that the discretization is arbitrage-free, and the recovered forward swap rates are positive martingales under their forward swap measure (Glasserman and Zhao 2000, Theorem 3). The simulated forward swap rates are then injected in (6.2) to get the simulated forward LIBOR rates.

In our tests, we use a simple correlation parameterization of the form $\rho_{jk} = e^{-\xi|j-k|}$. More complex parameterizations have provided similar results. We set $\xi = 0.01, \phi_j = 1, \alpha_j = 0$, and $(a_j, b_j, c_j, d_j) = (a, b, c, d)$, and use four different volatility shapes that are meant to represent real market scenarios. These are plotted in Figure 8.1.

In Table 8.1 we report the Mean Absolute Relative Errors (MARE) of a 2Y maturity against 12M LIBOR caplet implied volatilities and prices. The average is computed on the nine maturities for the four different shapes. Results indicate that all approaches, except the rank-one, give rather good approximations with a maximum MARE of 2.23%. The HW method seems to be the one to be preferred in practice since it outperforms all three others in most cases. As previously anticipated, using only the first two weights in the HW approximation has a marginal impact on the accuracy of the method but has the major advantage of downsizing the expression of a caplet to a simple spread of two forward swap rates. This will enable us to exploit the full facility of a fast and accurate recursive calibration algorithm.

Whereas the rank-one approach performs reasonably well in the LMM (see Brigo and Mercurio 2001), it gives unsatisfactory results in the ctSMM, especially for short maturity caplets. Indeed, whereas the weights of the swaption approximation in the LMM are always positive and much smaller than 1 for short maturity swaptions, the weights involved in the LIBOR representation as a spread of two forward swap rates have different

TABLE 8.1
Accuracy of the Different Caplet Pricing Approximations for Different Volatility Term Structures

	HW	Truncated HW	Rebonato	Spread option	Rank one
Implied vol					
Decreasing	0.27%	0.43%	1.15%	0.74%	9.25%
Bump	0.64%	0.35%	1.59%	1.03%	17.02%
Increasing	0.79%	0.90%	1.31%	2.23%	38.73%
Hump	0.79%	1.25%	0.67%	1.40%	33.50%
Prices					
Decreasing	0.26%	0.42%	1.12%	0.73%	9.09%
Bump	0.63%	0.35%	1.56%	1.02%	16.84%
Increasing	0.79%	0.89%	1.30%	2.21%	38.64%
Hump	0.79%	1.24%	0.66%	1.39%	33.40%

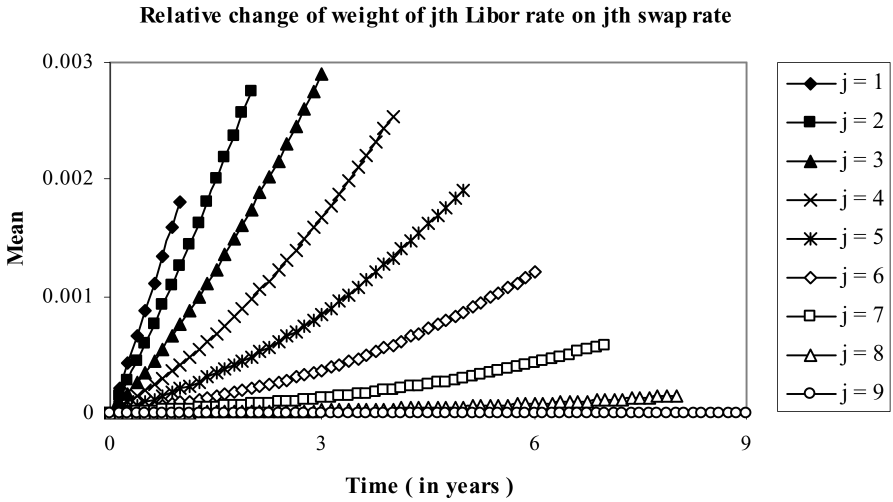
signs, and are much larger than 1 for short maturity caplets. This phenomenon leads to intrinsic numerical problems that can be best illustrated by a simple example. To fix the ideas, assume that the volatilities $\lambda_j(t) = \lambda$ are constant so that the jth caplet variance reads

$$\sigma_j^2 \approx (\hat{w}_j|\lambda| + \hat{w}_{j+1}|\lambda|)^2 - 2\hat{w}_j\hat{w}_{j+1}|\lambda||\lambda|(1 - \rho_{j,j+1}) \tag{8.1}$$
$$\approx \lambda^2 - 2\hat{w}_j\hat{w}_{j+1}\lambda^2(1 - \rho_{j,j+1}),$$

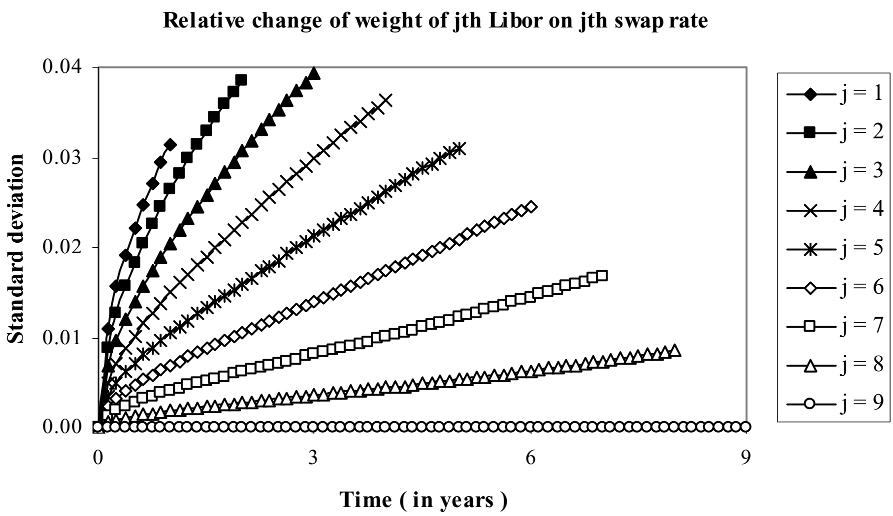
with $\hat{w}_j > 0$, $\hat{w}_{j+1} < 0$, and $\hat{w}_j + \hat{w}_{j+1} \approx 1$. The first term in equation (8.1) is recognized as the result of the rank-one approximation whereas the second term is interpreted as a perturbation around it. This second term cannot be neglected for short maturity caplets since it has the same order of magnitude of the first one unless all forward swap rates are perfectly correlated. A consequence of the above equation is that the rank-one approach systematically overprices the caplet volatility.

8.2. Weight Stability and Distributional Characteristics

In this section, we study the stability of the weights involved in the caplet pricing approximations through numerical simulations. We use an instantaneous volatility of the form $\Lambda_{jM}(t) = \phi_j f_j(t)\psi_j(T_j - t)$, where $(a_j, b_j, c_j, d_j) = (0.08, 0.10, 0.45, 0.06)$, and a correlation structure given by $\rho_{jk} = e^{-0.01|j-k|}$, for simplicity. As above mentioned, different and more consistent correlation matrix parameterizations are also possible (Schoenmakers and Coffey 2003), but the qualitative picture is not altered by this choice. The behavior of averages and standard deviations of relative changes $(w_{jj}(t) - w_{jj}(0))/w_{jj}(0)$ are plotted in Figures 8.2a and b. Results indicate that the weights present very stable averages across paths. This is similar to what is observed in the LMM. In terms of probability distribution, we have further noticed that long-term forward LIBOR rates are very close to lognormal densities whereas short-term ones do not result in such a good fit. This does not come as a surprise since all pricing approximations are much more accurate for long-term caplets than from short-term ones. We have measured a positive probability for the LIBOR rate to be negative but this probability is usually very small and negligible



(a)



(b)

FIGURE 8.2. Mean and standard deviation of relative changes of weights. Results are obtained from 400,000 Monte Carlo simulations.

for practical purposes. In addition, empirical results based on historical data instead of simulations confirm these findings.[5]

8.3. Calibration

We calibrate the ctSMM to swaption and caplet ATM volatilities by using a truncated HW method for caplet formulae. Recall that the generic form of the scalar instantaneous volatility for each forward swap rate S_{jM} is (7.1), with $\psi_j(T_j - t; \theta_j)$ and $\phi_j(t)$ parameterized as in equations (7.2) and (7.5). We anticipate that these functional choices, in conjunction with the aforementioned recursive algorithm, guarantee that the following

[5] All results not explicitly reported here are available from the authors upon request.

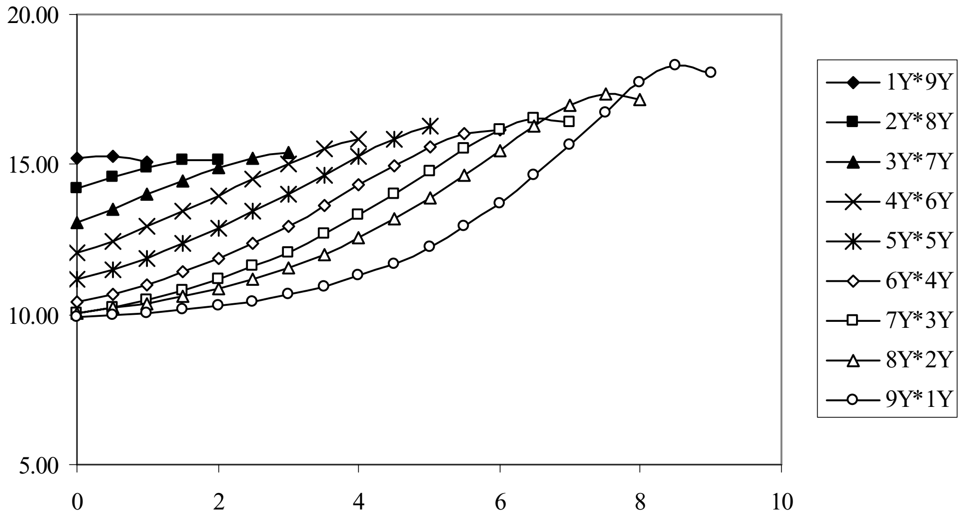


FIGURE 8.3. Instantaneous volatility curves before calibration to caplets.

objectives can be achieved: (i) The calibration algorithm is extremely fast and robust between any two consecutive dates; (ii) caplet and co-terminal swaption volatilities are matched within their market bid-ask spread; (iii) the resulting instantaneous forward volatilities $\Lambda_{j,M}(s)$, $s \in [t, T_j]$, are smooth in t for a given T_j; and (iv) the perturbation functions $\phi_j(t)$ do not dramatically alter the stationary solutions associated to $\psi_j(T_j - t; \theta_j)$. To the best of our knowledge, it is virtually impossible to match all above targets within a single calibration procedure in the LMM.

The first step of the calibration is to keep $\phi_j(t) = 1$, and to find a set of parameters $\{\theta_j\} := (a_j, b_j, c_j, d_j)$ to achieve a "best fit" of the initial term structure of swaption volatilities. This technique implies $M - 1$ independent least square minimizations, and is therefore fast and straightforward to achieve. We end up with a scalar instantaneous volatility $\Lambda_{jM}(t)$ which exhibits a shape consistent with the market hump and also matches the ATM volatilities of the set of co-terminal swaps, see Figure 8.3. Interestingly, we found that the calibrated set $\{\theta_j\}$ is rather stable with time so that one does not need to re-adjust it too often, see Table 8.2. The next logical step is to input a correlation structure (ρ_{ij}) in equations (7.6) and (7.7), and to determine $\phi_j(t) = \phi_j f_j(t)$ using the recursive algorithm to match the Black implied volatility of caplets and swaptions, as explained in Section 7.

At this stage, a few considerations are worth noting. As many authors have observed (see for instance Brigo and Mercurio 2001, Rebonato 2003) the LMM seems to be inconsistent with the market quotes of caplet and swaption volatilities in the sense that it is impossible to achieve exact calibration of both markets once an input correlation matrix has been assigned. It is often claimed that this feature is due to a "misalignment" between different swaption volatilities due to liquidity reasons. Besides, typical bid/ask spreads range between 0.25% and 0.75% in lognormal units, at least on the two most liquid markets, i.e., EUR and USD. They are therefore relatively narrow. A model that is unable to reprice vanilla options within the market bid-ask volatility spread is in principle liable to generate arbitrage in hedging exotic derivatives. Usually, the LMM is calibrated on a selection of swaption volatilities in the ATM matrix that correspond to liquid instruments and that are meant to capture, for the problem at hand, a large portion

TABLE 8.2
Results of the Calibration Methodology

Forward swap j	a_j	b_j	c_j	d_j	phi1	phi2	Alpha
	a : Means of calibrated parameters of the volatility curves						
1	0.0278	0.0412	0.4927	0.1097	0.0000	1.0023	0.0347
2	0.0256	0.0472	0.4828	0.1043	1.0417	1.0002	0.0347
3	0.0253	0.0538	0.4841	0.1002	1.0705	0.9859	0.0349
4	0.0261	0.0616	0.4848	0.0965	1.1129	0.9245	0.0349
5	0.0271	0.0701	0.4797	0.0928	1.1280	0.9373	0.0345
6	0.0400	0.0700	0.5376	0.0918	1.1131	0.9976	0.0291
7	0.0495	0.0720	0.5614	0.0923	1.0507	1.0454	0.0138
8	0.0550	0.0764	0.5705	0.0953	0.9861	1.0852	0.0096
9	0.0743	0.0825	0.7826	0.1050	0.9770	1.1274	0.0067
	b: Standard deviations of calibrated parameters of the volatility curves						
1	0.0165	0.0109	0.1593	0.0133	0.0000	0.0050	0.0094
2	0.0157	0.0107	0.1474	0.0126	0.0238	0.0142	0.0094
3	0.0153	0.0104	0.1281	0.0125	0.0196	0.0152	0.0092
4	0.0158	0.0108	0.1178	0.0126	0.0228	0.0255	0.0092
5	0.0168	0.0116	0.1149	0.0126	0.0329	0.0560	0.0096
6	0.0169	0.0155	0.0847	0.0129	0.0492	0.0760	0.0128
7	0.0187	0.0213	0.0773	0.0130	0.0498	0.0508	0.0101
8	0.0251	0.0301	0.0989	0.0139	0.0287	0.0527	0.0047
9	0.0343	0.0378	0.0481	0.0143	0.0257	0.0892	0.0019

of the actual volatility risk. In this way simultaneous calibration to caplets is possible while still keeping meaningful (i.e., smooth) instantaneous volatility functions. However, since these approaches are usually based on nonlinear global minimization algorithms (Rebonato 2003, Wu 2002), they are relatively slow. Robustness is also an issue, as above mentioned, and there is no guarantee that all selected instruments can be matched within their bid/ask spread, in general. On the other hand, full nonparametric approaches, like the one proposed in Brigo and Mercurio (2001), are aimed at matching the whole set of caplet and swaption volatilities. Nevertheless they do not allow achieving convergence unless the input swaption volatilities are artificially shocked from their mid-market values, often beyond their bid/ask spread. Even when convergence can be achieved, one has no direct control on the shape of the resulting instantaneous volatility term structures, and usually they are too rough to be used for robust and meaningful risk-management. Volatility "misalignment" problems are often advocated as the origin of the problem. However, these misalignments do not indicate that caplet/swaption arbitrage opportunities may be detected through the LMM, as already pointed out in Rebonato (2003). Instead, they show that the LMM has a rigid mathematical construction, and is not entirely consistent with the market.

In the context of the ctSMM we are faced with the same issue. In this case, however, these misalignment problems are less dramatic, and a simultaneous calibration within bid/ask spread is more easily achievable. In other words, the ctSMM is less "rigid" than the LMM. To understand the origin of this asymmetry between the two approaches, we recall that in the context of the LMM a forward swap rate can be formally decomposed as a linear combination of $M - 1$ forward LIBOR rates, while in the ctSMM a forward LIBOR can be formally written as a weighted difference between *two* consecutive forward swap rates. As a consequence, while the price of a swaption within the LMM depends on a correlation matrix among LIBOR rates of dimension $M - 1$, the price of a caplet within the ctSMM depends on a single correlation factor. Thus, calibrating the LMM to a set of caplets and swaptions is more problematic since any algorithm has to face the issue of coupling simultaneously $M(M - 1)/2$ correlations and volatilities. On the opposite, our parametric recursive algorithm for the ctSMM is essentially unidimensional since for any new maturity only one new correlation is needed and thus it only involves $M - 1$ steps.

We now describe an approach that works well in practice. The approach is based on a small perturbation of the correlation matrix around its historical estimate and a small shift of the volatilities around their mid-market values by keeping them within their bid-ask spread. The choice of slightly shocking the input matrix from the historical level can be easily justified by observing that a generic d-dimensional Itô martingale is fully specified by its covariance structure. In our case, we need to calibrate the model to the market-implied volatility. Thus, once the volatility structure is implied from caplets and swaptions quotes, the instantaneous correlation matrix needs not be consistent with its historical estimate. By slightly shocking the input correlation from historical levels, we are indirectly inferring some information from the market on the correlation itself, although a direct implied calibration of the whole correlation matrix is impossible. On this last point, we refer to Rebonato (2003).

In Table 8.2 we report the average and standard deviation of the calibrated parameters of the volatility curves over the 53 weekly observations from May 17, 2004, to May 16, 2005. The calibrated values seem to be stable as indicated by their standard deviations and, therefore, the calibration algorithm is robust. Notice, in particular, that calibration can be achieved with values of α_j always smaller than 0.035 on average, which implies that the calibrated model is almost stationary. We have also checked that differences between

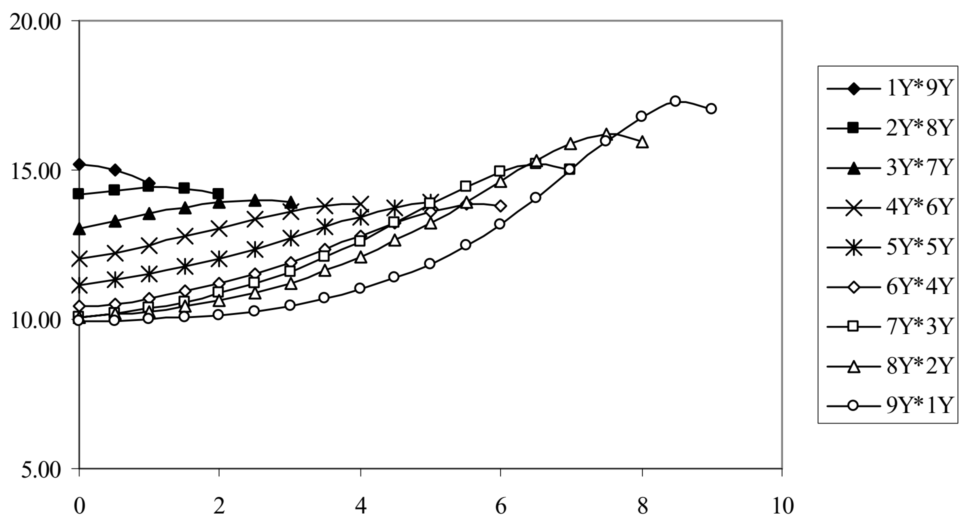


FIGURE 8.4. Instantaneous volatility curves after calibration to caplets.

market and calibrated Black volatilities are never larger than the bid/ask spread (0.25% in lognormal units).

Figure 8.3 shows the shape of the instantaneous volatility functions for the different swap rates as a function of t with fixed maturity T_j after calibration to the initial term structure of swaption Black volatilities. In Figure 8.4, we plot the same functions after performing a joint calibration on caplets and swaptions. Both figures are based on the averages reported in Table 8.2. It is worth noticing that the initial order of the curves and their shapes (consistent with a full stationary model) are still preserved to a large extent after calibration. In addition, they are smooth, which is good news on the risk-management side.

9. CONCLUDING REMARKS

In this paper we have studied a general approach suitable to price IR derivatives. This "Market Model Approach" gives birth to three major classes: the co-terminal, co-initial, and co-sliding SMM. The Market Model Approach is based on the concept of admissibility of a set of forward swap rates. We have presented and analyzed the link between these concepts and graph theory. In particular, we have shown that the LMM is the only admissible model of a co-sliding type.

By further developing the important example of the co-terminal SMM, we have shown that accurate and fast approximations are available in that setting. Besides, user-friendly calibration algorithms work efficiently in terms of speed and stability properties. They further lead to smooth and meaningful shapes for the instantaneous forward volatility of forward swap rates, while delivering an almost perfect match of both swaption and caplet implied Black volatilities. Some important theoretical extensions (and the associated calibration algorithms) related to the inclusion of stochastic volatility or the generalization to multicurrency underlying are left to future research.

APPENDIX A: PROOF OF PROPOSITION 2.1

Consider the tenor structure depicted in Section 2, namely $\mathcal{T} := \{T_1, \dots, T_M\}$. Assume that a generic family $\mathcal{S} := \{S(t)\}$ of forward swap rates is given. In general, the set $\mathcal{S}$ comprises N elements. We use Greek letters to indicate start/end dates of each forward swap rate belonging to the set, i.e.,

$$S_{\alpha_1\beta_1}(t) := \frac{B(t, T_{\alpha_1}) - B(t, T_{\beta_1})}{G_{\alpha_1\beta_1}(t)}, \dots, S_{\alpha_N\beta_N}(t) := \frac{B(t, T_{\alpha_N}) - B(t, T_{\beta_N})}{G_{\alpha_N\beta_N}(t)}.$$

We start by proving the sufficient part. Definition 2.2 implies that $N = M - 1$. Therefore, the following linear homogeneous system

$$B(t, T_{\alpha_1}) - B(t, T_{\beta_1}) = S_{\alpha_1\beta_1}(t) \sum_{k=\alpha_1+1}^{\beta_1} \delta_k B(t, T_k), \tag{A.1}$$

$$\dots$$

$$B(t, T_{\alpha_{M-1}}) - B(t, T_{\beta_{M-1}}) = S_{\alpha_{M-1}\beta_{M-1}}(t) \sum_{k=\alpha_{M-1}+1}^{\beta_{M-1}} \delta_k B(t, T_k),$$

comprises $M - 1$ equations in the n unknowns $B(t, T_{\alpha_1}), \ldots, B(t, T_{\beta_1})$; $B(t, T_{\alpha_2}), \ldots, B(t, T_{\beta_2}); \ldots; B(t, T_{\alpha_{M-1}}), \ldots, B(t, T_{\beta_{M-1}})$. To simplify the notation, we introduce the set of dates $\mathcal{U} := \cup_{i=1,\ldots,M-1}\{T_{\alpha_i}, \ldots, T_{\beta_i}\}$.

The number n of independent unknowns is fixed by Definition 2.2, and the fact that all dates must be in the tenor structure. If every date in the tenor structure coincides with at least one reset/settlement date of a swap rate in $\mathcal{S}$, then $\mathcal{U} \supseteq \mathcal{T}$. At the same time, since by construction the set $\mathcal{S}$ is defined in relation to the above tenor structure, the inclusion $\mathcal{U} \subseteq \mathcal{T}$ must also hold. Therefore, we have that $\mathcal{T} = \mathcal{U}$ identically. Since $|\mathcal{U}| = |\mathcal{T}| = M$, we deduce that the above linear system $\mathcal{L}(M-1, M)$ comprises $M - 1$ equations in M independent unknowns. Let C be the rectangular $(M - 1) \times M$ matrix associated to (A.1). In a more compact notation,

$$CB = 0, \quad B := (B(t, T_1), \ldots, B(t, T_M))'. \tag{A.2}$$

If, for instance, we consider the equation $B(t, T_{\alpha_1}) - B(t, T_{\beta_1}) = S_{\alpha_1\beta_1}(t)\sum_{k=\alpha_1+1}^{\beta_1} \times \delta_k B(t, T_k)$, then the corresponding row of C reads as

$$(\cdots 0, \quad 1, \quad -\delta_{\alpha_{1+1}} S_{\alpha_1\beta_1}(t), \cdots - \delta_{\beta_{1-1}} S_{\alpha_1\beta_1}(t), \quad -(1 + \delta_{\beta_1} S_{\alpha_1\beta_1}(t)), \quad 0\cdots),$$

the 1 entry being in column α_1.

Consider the set of deflated discount bonds relative to $B(t, T_i)$, that is $\tilde{B}_i(t, \cdot) := B(t, \cdot)/B(t, T_i)$. Since by construction $T_i \in \mathcal{T}$, by dividing both sides of all equations in (A.2) by $B(t, T_i)$, we obtain a new set of $M - 1$ linear equations in the $M - 1$ deflated discount bond prices $\tilde{B}_i(t, T_1), \ldots, \tilde{B}_i(t, T_{i-1}), 1, \tilde{B}_i(t, T_{i+1}), \ldots, \tilde{B}_i(t, T_M)$. For any $(M - 1) \times M$ homogeneous system (A.2) there exists an associated $(M - 1) \times (M - 1)$ non-homogeneous system on the corresponding deflated discount bond prices, i.e.,

$$D\tilde{B} = \Psi, \tag{A.3}$$

$$\tilde{B} := (\tilde{B}_i(t, T_1), \ldots, \tilde{B}_i(t, T_{i-1}), \tilde{B}_i(t, T_{i+1}), \ldots, \tilde{B}_i(t, T_M))',$$

where vector Ψ entries are either zero or one, with at least one non-vanishing entry.

A necessary and sufficient condition for this system to possess a unique solution is that matrix C has full rank. In fact, if $rank(C) = M - 1$, then $rank(D) = M - 1$ as well, for any choice of $B(t, T_i)$, $i = 1, \ldots, M$, as numéraire. In this case, the existence of a unique set of discount bond prices is guaranteed by elementary theorems of linear algebra. On the opposite, if two or more rows of C are linearly dependent, then $rank(C) < M - 1$. In this case, depending on the choice of the numéraire, i.e., $B(t, T_i)\, i = 1, \ldots, M$, matrix D might not have full rank, and therefore there is no guarantee that a solution exists for any choice of the numéraire. In other words, the existence and uniqueness of solutions is reduced to the study of the linear dependence among the rows of C.

We therefore analyze the structure of C when the set of forward swap rates is admissible. By construction, D is not block-diagonal, and none of the rows of C contains only zero elements.[6] Furthermore, the first column of C, namely the column made of the coefficients associated to the shortest discount bond $B(t, T_1)$, contains either a single 1 entry (corresponding to a single swap starting at T_1) or multiple 1 entries (corresponding to

[6] If the matrix were block-diagonal, then the graph associated to the tenor structure would be made of two separate subgraphs. Proposition 3.1 shows that this is incompatible with the notion of admissibility since the graph should be connected.

many swaps starting at T_1). In the first case the remaining $M-2$ elements of the first column entries are all zeros. Hence, it is impossible to find a linear combination of the rows reducing to a zero vector. If there are multiple 1 entries then, once again, the remaining entries of the column must fill in with zeros. Unlike before, it is possible to find linear combinations of the rows which will annihilate the first entry of the resulting vector. However, once such a combination is held fixed, it cannot annihilate the other entries and yield a zero vector for any realisation of the diffusion processes $S_{\alpha_1\beta_1}(t), \ldots, S_{\alpha M-1\beta M-1}(t)$. Note, however, that this argument does not exclude the possibility that a zero vector is indeed obtained by a linear combination of the rows for one particular realization of the set of forward swap rates, but this event occurs with probability 0 with respect to $\mathbb{P}$ only. In summary, if the set of forward swap rates is admissible, we have $\mathbb{P}[\det D \neq 0] = 1$, and the system (A.3) admits a unique solution in terms of deflated discount bond prices, $\mathbb{P}$-a.s. This holds for any t in the specified time interval. This ends the proof of the sufficient part.

The proof of the necessary part is as follows. If a system of $M-1$ deflated discount bond prices admits a unique solution as a function of a set of forward swap rates, then necessarily it must be a linear non-homogeneous system $\mathcal{L}(M-1, M-1)$. This in turn implies that $\mathcal{T} = \mathcal{U}$ and that $|\mathcal{S}| = M-1$. Condition 1 in Definition 2.2 is then satisfied. Then, inclusions $\mathcal{U} \supseteq \mathcal{T}$ and $\mathcal{U} \subseteq \mathcal{T}$ must simultaneously hold. The former constraint is equivalent to Condition 2 in Definition 2.2.

To prove that Condition 3 is also satisfied, we proceed by contradiction. Assume that set $\mathcal{S}$ is not admissible. Given that Conditions 1 and 2 are already satisfied, this is equivalent to assume that Condition 3 is not, i.e., there exists at least one degenerate subset $\mathcal{C}$ in $\mathcal{T}$. Now recall the definitions of $\mathcal{H}^u$ and $\mathcal{H}^d$ given in the main text. For every pair of consecutive dates T^u_{i-1}, T^u_i in $\mathcal{H}^u$ the following equation holds by absence of arbitrage opportunities $S^u_{i-1,i}(t)G^u_{i-1,i}(t) = B(t, T^u_{i-1}) - B(t, T^u_i)$ with obvious notations. A similar result holds for all dates in the set $\mathcal{H}^d$. The following two identities:

$$\sum_{i=2}^{m} \left[B\left(t, T^u_{i-1}\right) - B\left(t, T^u_i\right) \right] = B\left(t, T^u_1\right) - B\left(t, T^u_m\right),$$

$$\sum_{i=2}^{n} \left[B\left(t, T^d_{i-1}\right) - B\left(t, T^d_i\right) \right] = B\left(t, T^d_1\right) - B\left(t, T^d_n\right),$$

imply that $\sum_{i=2}^{m}[B(t, T^u_{i-1}) - B(t, T^u_i)] = \sum_{i=2}^{n}[B(t, T^d_{i-1}) - B(t, T^d_i)]$, since $B(t, T^u_1) = B(t, T^d_1) = B(t, T_a)$ and $B(t, T^u_m) = B(t, T^d_n) = B(t, T_b)$. By absence of arbitrage opportunities, we deduce that

$$\sum_{i=2}^{m} S^u_{i-1,i}(t) G^u_{i-1,i}(t) = \sum_{i=2}^{n} S^d_{i-1,i}(t) G^d_{i-1,i}(t).$$

This identity implies that one equation in the system (A.2), among those associated to $\mathcal{H}^u$ and $\mathcal{H}^d$, is redundant. Therefore, the $M-1$ equations in the system do not form a linearly independent set, and $rank(C) < M-1$. In that case there is no guarantee that a solution exists for a generic choice of $B(t, T_i)$, $i = 1, \ldots, M$, as numéraire. This contradicts the hypothesis.

APPENDIX B: PROOF OF PROPOSITION 2.2

If the set S is admissible then, after Proposition 2.1, there exists a unique set of deflated discount bond prices $\tilde{B}_i(t, T_1), \ldots, \tilde{B}_i(t, T_{i-1}), 1, \tilde{B}_i(t, T_{i+1}), \ldots, \tilde{B}_i(t, T_M)$ relative to $B(t, T_i)$. This set is the unique solution of a non-homogeneous linear system

$$D\tilde{B} = \Psi, \tag{B.1}$$

$$\tilde{B} := (\tilde{B}_i(t, T_1), \ldots, \tilde{B}_i(t, T_{i-1}), \tilde{B}_i(t, T_{i+1}), \ldots, \tilde{B}_i(t, T_M))',$$

and vector Ψ entries are either zero or one, with at least one non-vanishing entry. The solution can be easily found by means of the Cramer's rule: the jth solution $\tilde{B}_i(j)$ to (B.1) can be symbolically expressed as

$$\tilde{B}_i(j) = \frac{\det\left[D_i^{(j)}\right]}{\det[D]}; \quad j = 1, \ldots, M-1,$$

where $D^{(j)}$ is the square matrix obtained from D by replacing its j-th. column with vector Ψ. A sufficient condition for $\tilde{B}_i(j)$ to be non-zero is that $\det[D_i^{(j)}] \neq 0$. A necessary and sufficient condition for a determinant to be zero is that its rows are linearly independent. By assumption, matrix C (associated to the homogeneous system A.2) has rank $r = M - 1$. This implies that its rows are linearly independent. Also, we notice that $-1 \times \Psi$ coincides with one of the columns of C. Thus, the columns of $D_i^{(j)}$ coincide with the $M - 1$ columns of C, apart from an irrelevant sign in one of the columns. From a general property of linear spaces, we deduce that the rows of matrix $D_i^{(j)}$ must necessarily be linearly independent, too. For the same reason as above, it might happen that for a given realization of the set of forward swap rates, two (or more) rows of $D^{(j)}$ are linearly dependent. This event, however, occurs with probability 0 with respect to $\mathbb{P}$.

APPENDIX C: PROOF OF PROPOSITION 3.1

We need to prove that Conditions 1, 2, and 3 in Definition 2.2 are necessary and sufficient for Gr to be connected and have M vertices linked by $M - 1$ edges.

We start by proving the necessary part. We observe that Condition 2 in Definition 2.2 means that the graph Gr has no isolated vertices. In particular all vertices of Gr have a minimum degree of 1. Since a connected graph has a minimum degree of 1 by definition, Condition 2 of Definition 3.1 implies Condition 2 of Definition 2.2. The following result can be found in Diestel (2000): If Gr is a connected graph, then Gr has a spanning tree subgraph Gr'. Thus Gr' has M vertices and $M - 1$ edges. By Condition 1 of Definition 3.1 this implies that Gr is itself a tree. Since a tree is acyclic, this implies that Condition 3 of Definition 2.2 is satisfied too. Conditions 1 of the two definitions are trivially the same. This ends the first part of the proof.

To prove the sufficient part we assume, by contradiction, that Gr is not connected. In this case, let $Gr = \{Gr_1, Gr_2, \ldots, Gr_l\}$ be its decomposition in disjoint subgraphs $Gr_1, Gr_2, \ldots, Gr_l$. From Condition 2 of Definition 2.2, none of them has an isolated vertex. From Condition 1 of Definition 2.2, we deduce then that one among $Gr_1, Gr_2, \ldots, Gr_l$ is a graph with a number of edges equal to the number of vertices minus 1. All other subgraphs, on the other side, must be restricted to have an equal number of edges and vertices. This is the only partition that is compatible with the total number of edges, $M - 1$, and vertices, M. We assume, with no loss in generality, that graph $Gr_i = \{V_i, E_i\}$ has $n_i - 1$

edges and n_i vertices. Since it is connected then, from the same argument as before, Gr_i is a tree. Thus, it is acyclic. We next consider one among the remaining $l - 1$ graphs $Gr_j = \{V_j, E_j\}$ with n_j edges and vertices. Since it is connected, let Gr'_j be its spanning tree subgraph having $n_j - 1$ edges. The following result can be found in Diestel (2000): a tree H is maximally acyclic. This means that a tree H contains no cycle but $H + xy$ does, for any two non-adjacent vertices x and $y \in H$. Since Gr_j is obtained by the tree Gr'_j by adding one edge between non-adjacent vertices,[7] it must contain a cycle which is in contrast with Condition 3 in Definition 2.2. The contradiction comes from having assumed that Gr is not connected. This ends the second part of the proof.

APPENDIX D: PROOF OF PROPOSITION 3.3

From Proposition 3.2 we know that a set is admissible if and only if its graph is a tree. Since a tenor structure is made of ordered dates, we must attach date labels $\{T_1, \ldots, T_M\}$ to all nodes of the tree. In this way, isomorphic trees are considered distinct for the purpose of enumeration. Such a graph is known in graph theory as a "labeled" tree. The number of labeled trees on M nodes is known to be M^{M-2} after Cayley (1889), which yields the stated result.

REFERENCES

ANDERSEN, L., and J. ANDREASEN (2000): Volatility Skews and Extensions of the LIBOR Market Model, *Appl. Math. Finance* 7, 1–32.

ANDERSEN, L., and R. BROTHERTON-RATCLIFFE (2001): Extended LIBOR Market Models with Stochastic Volatility, Working Paper, Gen Re Financial Products.

D'ASPREMONT, A. (2003): Interest Rate Model Calibration Using Semidefinite Programming, *Appl. Math. Finance* 10, 183–213.

BLACK, F. (1976): The Pricing of Commodity Contracts, *J. Financial Econ.* 3, 167–179.

BRACE, A., D. GATAREK, and M. MUSIELA (1997): The Market Model of Interest Rate Dynamics, *Math. Finance* 7, 127–155.

BRIGO, D., and M. MERCURIO (2001): *Interest Rates Models: Theory and Practice*. Heidelberg: Springer-Verlag.

CAYLEY, A. (1889): A Theorem on Trees, *Quat. J. Math.* 23, 376–378.

CONT, R., and P. TANKOV (2004): Non-parametric Calibration of Jump-Diffusion Option Pricing Models, *J. Comput. Finance* 7, 1–43.

DE JONG, F., J. DRIESSEN, and A. PELSSER (2000): Libor and Swap Market Models for the Pricing of Interest Rate Derivatives: An Empirical Analysis, *Eur. Financial Rev.* 5, 201–237.

DIESTEL, R. (2000): *Graph Theory*. Heidelberg: Springer-Verlag.

GALLUCCIO, S., and C. HUNTER (2004): The Co-Initial Swap Market Model, *Econ. Notes* 33, 209–232.

GLASSERMAN, P., and S. KOU (2003): The Term Structure of Simple Forward Rates with Jump Risk, *Math. Finance* 3, 383–410.

GLASSERMAN, P., and N. MERENER (2003): Cap and Swaption Approximations in LIBOR Market Models with Jumps, *J. Comput. Finance* 7, 1–11.

GLASSERMAN, P., and X. ZHAO (2000): Arbitrage-Free Discretization of Lognormal Forward Libor and Swap Rate Models, *Finance Stochast.* 4, 35–68.

[7] This is a direct consequence of the hypothesis on the set $\mathcal{S}$ comprising distinct swap rates. In fact, if all swap rates are distinct, two vertices cannot be both incident on two different edges.

GOLDYS, B. (1997): A Note on Pricing Interest Rate Derivatives When Forward LIBOR Rates Are Lognormal, *Finance Stochast.* 1, 345–352.

HEATH, D., R. JARROW, and A. MORTON (1992): Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica* 60, 77–105.

HULL, J., and A. WHITE (2000): Forward Rate Volatilities, Swap Rate Volatilities and the Implementation of the LIBOR Market Model, *J. Fixed Income* 10, 46–62.

HUNTER, C., P. JÄCKEL, and M. JOSHI (2001): Getting the Drift, *RISK*, 14(7).

JÄCKEL, P., and R. REBONATO (2003): The Link between Caplet and Swaption Volatilities in a Brace-Gatarek-Musiela/Jamshidian Framework: Approximate Solutions and Empirical Evidence, *J. Comput. Finance* 6, 35–45.

JAMSHIDIAN, F. (1997): LIBOR and Swap Market Models and Measures, *Finance Stochast.* 1, 293–330.

JAMSHIDIAN, F. (1999): LIBOR Market Model with Semimartingales, Working Paper, NetAnalytic Ltd., London.

JARROW, R., H. LI, and F. ZHAO (2003): Interest Rate Caps ‘Smile’ Too! But Can the LIBOR Market Models Capture It? Working Paper, Cornell University.

JOSHI, M., and R. REBONATO (2003): A Stochastic Volatility Displaced-Diffusion Extension of the LIBOR Market Model, *Quant. Finance* 3, 458–469.

MARGRABE, W. (1978): The Value of an Option to Exchange One Asset for Another, *J. Finance* 33, 177–186.

MILTERSEN, K., K. SANDMANN, and D. SONDERMANN (1997): Closed-form Solutions for Term Structure Derivatives with Lognormal Interest Rates, *J. Finance* 70, 409–430.

MUSIELA, M., and M. RUTKOWSKI (1997): Continuous-Time Term Structure Models: Forward Measure Approach, *Finance Stochast.* 1, 261–291.

MUSIELA, M., and M. RUTKOWSKI (2004): *Martingale Methods in Financial Modeling*, 2nd. edn. Berlin: Springer-Verlag.

PIETERSZ, R., A. PELSSER, and M. VAN REGENMORTEL (2004): Fast Drift-Approximated Pricing in the BGM Model, *J. Comput. Finance* 8, 93–124.

PIETERSZ, R., and M. VAN REGENMORTEL (2004): Generic Market Models. Erasmus University preprint.

PRÜFER, H. (1918): Neuer Beweis eines Satzes über Permutationen, *Arch. Math. Phys.* 27, 742–744.

REBONATO, R. (1998): *Interest Rate Option Models*, 2nd edn. Chichester, UK: Wiley.

REBONATO, R. (2003): *Modern Pricing of Interest-Rate Derivatives*. Princeton, NJ: Princeton University Press.

RUTKOWSKI, M. (1998): Dynamics of Spot, Forward and Futures LIBOR Rates, *Int. J. Theor. Appl. Finance* 1, 425–445.

SCHOENMAKERS, J., and B. COFFEY (1999): LIBOR Rate Models, Related Derivatives and Model Calibration, Working Paper 480, Weirstraß Institut Berlin.

SCHOENMAKERS, J., and B. COFFEY (2003): Systematic Generation of Correlation Structure for the Libor Market Model, *Int. J. Theor. Appl. Finance* 6, 1–13.

WU, L. (2002): Fast At-the-Money Calibration of the LIBOR Market Model through Lagrange Multipliers, *J. Comput. Finance* 6, 33–45.

WU, L., and F. ZHANG (2002): LIBOR Market Model: From Deterministic to Stochastic Volatility, Working Paper, Claremont Graduate University and University of Science and Technology Hong-Kong.