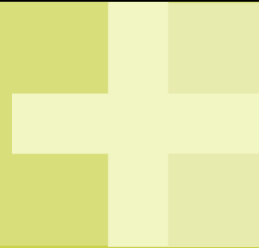


# Fitting the Yield Curve

## Cubic spline interpolation and smooth extrapolation

David Antonio, David Roseburgh  
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## Executive summary<sup>1</sup>

This document describes the current Barrie & Hibbert methodology for interpolating and extrapolating to the unconditional nominal forward rate. We use a regression spline technique for the interpolation and extrapolate using the Nelson-Siegel form. We outline the methods and present back-testing for GBP, USD, EUR, JPY and TWD for the quarters from March 2007 to March 2008 inclusive. **This methodology is applied to calibrations for all economies from the end of September 2008 onwards.**

Developing a complete yield curve from sparse market data is a fundamental aspect of the ESG calibration process. The traditional challenge when building a yield curve is that of "joining the dots" by interpolating between yields inferred from traded instruments. Of course, the task of constructing a forward rate curve (within the range of maturities for which there exist traded instruments) is greater than 'simply' interpolating – interpolation implies that we unambiguously know the forward rates at some maturities, whilst in reality we do not have this information. The inference of forward-rates (or, equivalently, zero-coupon rates) from market prices is often not trivial. We also face an additional challenge - calibration users are typically interested in time horizons beyond the longest dated market instrument, so a robust *extrapolation* methodology is also required.

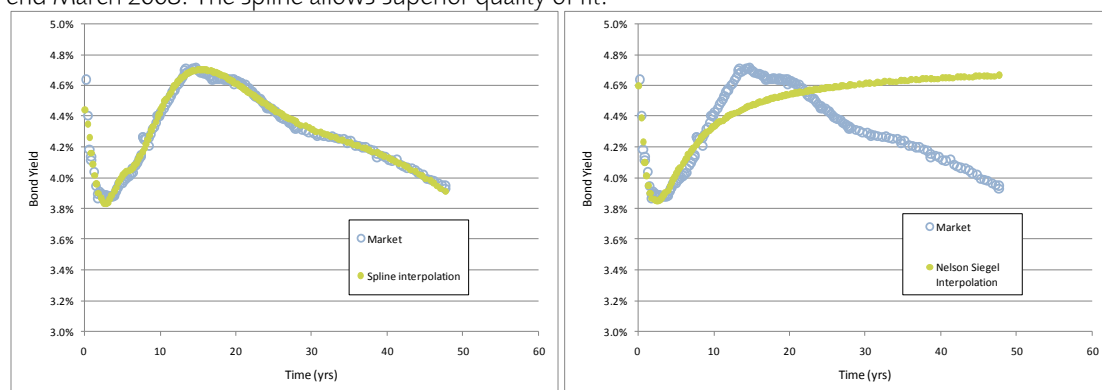
Prior to September 2008 Barrie & Hibbert have used a methodology based on the Nelson-Siegel (NS) approach, whereby the same functional form is used to both interpolate *and* extrapolate. While the NS approach has a sound economic rationale and an intuitive interpretation, it is weak in key areas. Specifically, it is insufficiently flexible to provide the required quality of fit to the market data, and the extrapolation of the forward curve is unconstrained and informed only by the market. This means small, short-maturity market changes can drive large, spurious changes in the long end of the curve.

In contrast, our new methodology introduces additional degrees of freedom, allowing an arbitrarily accurate fit as well as direct control over the volatility of long term rates- albeit at the expense of having to exogenously specify an unconditional target for extrapolation.

Superior fit quality is demonstrated in Exhibit A below. Here we see an improved fit across all maturities when using our new methodology compared with the existing NS method.

### Exhibit A

Comparison of spline interpolation (left) and the Nelson-Siegel interpolation (right) of GBP bond yields at end March 2008. The spline allows superior quality of fit.



Source: GBP\_fit\_comparison.xls

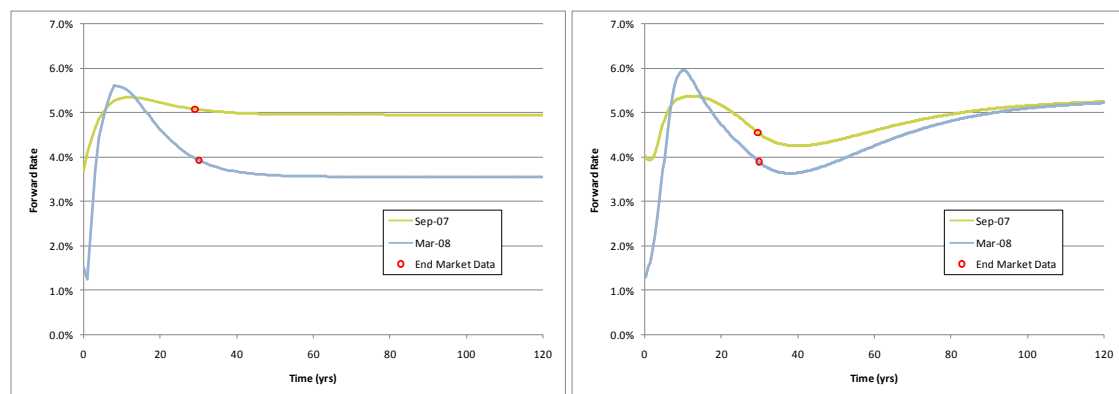
<sup>1</sup> This note was reviewed in draft form by the Technical Advisory Panel (TAP) in June 2008 and this updated document will be reviewed alongside any comments received from clients at the next meeting. The TAP is currently scheduled to meet next on November 18th 2008.

The second major flaw in the NS method, relating to stability of long term untraded forward rates, is demonstrated in Exhibit B. In this Exhibit we see forward curves at different quarters as derived by NS (left panel) and our new methodology (right panel) the red circles indicate the terminal market maturity. We see extremely large (around 150 basis points) variations in the NS case because the long-term (unconditional) forward rate is a free parameter in the fit, accordingly its level is governed by market data with sub 30 year maturity. By pinning the unconditional forward rate and extrapolating to that level using the new technique we avoid this undesirable (and spurious) volatility in long-term rates.

Sorensen (2008) explains our methodology for setting the ultra long-term unconditional nominal forward rate for the government curve as well as swap calibration, and we describe the fitting and extrapolation methodologies in this paper.

### Exhibit B

Comparison of Nelson-Siegel (left) and new methodology (right), showing variations in long term rates across end September 2007 and end March 2008 USD government bond yield curves. The red circle marks the terminal market maturity.



Source: NS\_vs\_Splining.xls

# 1 Constructing a yield curve

## 1.1 General principles

Before we construct a yield curve we define some generally desirable properties (note this list is not intended to be exhaustive):

1. **Accuracy:** Where a liquid market exists the yield curve should accurately price that market.
2. **Continuity:** The (forward rate) curve should be continuous.
3. **Smoothness:** The (forward rate) curve should be smooth. That is, its first derivative should be continuous.
4. **Volatility decay:** On average, the volatility of long (forward) rates should be lower than shorter (forward) rates.

Property 1 relates to 'interpolation' methodology, property 4 to extrapolation, and points 2 and 3 to both parts, and to the connection between them.

Additionally, the fitted curve should be stable (that is, very small changes in traded rates should not have an undue effect on the fitted forward curve) and the extrapolation method should be sufficiently flexible to allow consistency with empirically observed forward rate volatilities and correlations.

Why are these properties desirable?

1. Our task is to describe today's yield curve, as defined by a chosen set of instruments – either government ('default-free') bonds or swaps. Therefore, by definition it is clear that we must accurately match the prices of the chosen instruments.

However, the devil is in the detail – what level of accuracy is 'good enough'? In some cases the market may price similar instruments differently, perhaps because of differing tax treatment, differences in liquidity, or other 'intangible' properties – so that differences in price (or yield) are not solely attributable to differences in timing of cashflows. In addition, available data may reflect trades executed at different times, particularly in more illiquid instruments. These features might mean that prices for all (apparently similar) instruments are not compatible with a single forward curve. In such a situation, it is not clear that we ought to match every instrument's price precisely – do we believe that the market forward rate curve is discontinuous, or do we believe that price discrepancies are due to differences in the market's treatment of these bonds, and/or inaccuracies in available data? These issues are taken into account during model implementation.

2. As described above, available price data might seem to imply that the market forward curve is not smooth. Is this plausible? We might argue that forward rates are made up of three components:
  - The market's expectation of future short rates
  - Any term premia demanded by the market for investing in instruments of a particular maturity.
  - Technical 'convexity' adjustments.

It seems plausible that the market's expectation of future rates, and the premium demanded (or paid) by the market, might vary with horizon. What seems less plausible is that there are *discontinuities* in these expectations or demanded premia and we therefore assume that a forward rate curve should be a smooth function of maturity. Whilst we would acknowledge that a forward curve chosen to 'best fit' market data could result in discontinuities we believe that such features are most likely an artefact of inconsistencies in pricing data, or due to different treatment of bonds by the market, rather than due to a forward rate curve which is discontinuous.

In particular, at the terminal maturity of the market data, we do not believe that it is justifiable to have a discontinuity in the forward rate curve. This would make the very strong statement that, immediately beyond the point where the market data ends, the market's expectation and/or demanded premium changes discontinuously. Nelson and Siegel (1987) and Svensson (1995) concur with the view that this is not reasonable.

3. As with Point 2 neither expectations nor premia, should have a discontinuous slope. Accordingly the slope of the forward rate curve at the end of the available market data will be continuous.

4. The belief that long rates are set in part by expectations of future short rates, alongside the common belief that interest rates exhibit mean-reversion, adds weight to the idea that there should be less volatility in the long end of the curve than there is in the short end. An extrapolation method should produce curves in which the long end is, *on average* less volatile than the short. So, if we are able to observe only (say) 10 years of market data, it would be appropriate that the volatility of extrapolated rates should be less than that of the observable rates. The question remains of how much smaller the volatility ought to be, and this question is addressed in Liu (2008).

## 1.2 Interpolation

We fit the market bond data and swap data using a regression spline technique. We fit a spline to the instantaneous forward rate curve using a natural cubic spline and calculate the corresponding zero coupon bond prices by:

$$P(T) = \exp \left[ - \int_0^T F(t) dt \right] \quad (1)$$

Where  $P(T)$  is the price of a zero coupon bond with maturity  $T$  and  $F(t)$  are the set of instantaneous forward rates. With a set of zero coupon bonds we can evaluate swap rates and bond prices and compare with the market. We adjust the so-called "knot points" of the spline to alter the forward rate curve and achieve better agreement with the market data.

The problem of developing a forward curve in the region where traded instruments exist is a common one, and we follow McCulloch (1971, 1975), Fischer, Nychka and Zervos (1995), Waggoner (1997) and the Bank of England (Anderson and Sleath, 2001) in using regression splines to generate such a curve. This technique consists of building an instantaneous forward rate curve using a cubic spline through a set of knot points at predefined maturities. The value of the forward rate at each of these knot points is then varied until the market yield curve (in terms of a set of bond prices or swap rates) is reproduced as accurately as possible.

The selection of both the number and position of knot points is crucial. A higher number of knot points will allow a more accurate interpolation at the expense of introducing spurious oscillations (see section 1.2.3) and how the knots are spaced will affect the quality of the interpolation. Each knot point corresponds to a point along the instantaneous forward rate curve so each is specified by a maturity and the forward rate at that maturity. In order to perform the interpolation of market data we follow three steps:

1. Select the number of knots. This will depend on the number of market instruments – the fewer instruments the fewer knots required, in most cases. It will also depend on the shape of the market data – a more complex shape may require more knots. In practice the number of knots ranges from about 2 to 13.
2. Once the number of knots has been selected the maturity of each knot point (the spacing of the knots) needs to be chosen. We use two different approaches for bond and swap instruments. For bonds we space the knots such that an equal number of bonds mature between each pair of knots. This will ensure there are more knots where there are more bonds (usually at the short end of the curve) which should improve the quality of the interpolation. For swaps we use a fixed set of knot points<sup>2</sup>.
3. Having selected the number of knots and their maturities the next step is to define the forward rate at each knot point. With all this information we can uniquely define the entire instantaneous forward curve (up to the maturity of the last market instrument). We determine the forward rate at each knot point using numerical optimisation.

There are issues associated with the use for regression splines for yield curves. Because of the cubic nature of the interpolating function, and the splining constraints at each knot point, the forward curve can display a high degree of oscillation: forward rates can vary to a large degree across different maturities, giving a wavy or bumpy looking forward curve. While there is no technical reason that the forward curve could not vary like this, many practitioners find this property undesirable- particularly at the long end of the curve.

<sup>2</sup> Note that we need at least two knot points: one at maturity=0 and one at the maturity of the last market instrument. We always require knots at these maturities, no matter how many knots we select.

Minimising this variation can be approached in a couple of ways: either by choosing a low number of knot points (McCulloch (1971), (1975)), or by including a smoothness constraint when optimising the fit to market data (Fischer Nychka and Zervos (1995), Waggoner (1997), Bank of England (2001)). As far as possible we use the minimum number of knot points required to fit the market data and we also introduce smoothing parameters, as discussed in Sections 1.2.2 and 1.2.3.

### 1.2.1 Objective functions

When fitting the market swap rates we adjust the forward rate at the spline knot-points to minimise

$$X_S = \sum_{i=1}^N [S_i - \pi_i]^2 \quad (2)$$

Where  $S_i$  is the market swap rate,  $\pi_i$  is the swap rate obtained from the splined forward curve and  $N$  is the number of market data points.

When fitting the market government bond prices we minimise

$$X_G = \sum_{i=1}^M \left[ \frac{B_i - \pi_i}{D_i} \right]^2 \quad (3)$$

Where  $B_i$  is the market bond price,  $\pi_i$  is the bond price obtained from the splined forward curve,  $D_i$  is the bond duration and  $M$  is the number of market bonds. Here, we weight the objective function by the bond duration. Since the sensitivity of bond prices will vary with bond duration, we have weighted prices using their duration to give approximately the same weighting strategy as for yields. Without such weighting the fit to the market prices will tend to suffer for short-term maturities, especially when a smoothing constraint is applied.

### 1.2.2 Gradient Constraints

The point along the yield curve where the interpolation ends and the extrapolation begins is of crucial importance. As we have discussed in Section 1.1 we wish to ensure that the yield curve is smooth and continuous but we must also consider the effect of a steep gradient of the forward curve at the end of the market data. A particularly steep gradient could have a large impact on the shape of the extrapolated curve, leading to spurious volatility just beyond the terminal market maturity. It is, of course, possible that a gradient is implied by the market data so we do not seek to ensure that the gradient is zero, however, we do seek to ensure that the gradient is minimised so that it is the level of the market that is the dominant factor in informing the extrapolation. The gradient will inform the extrapolation to some extent but it will be a second order effect. It is worth reiterating that whilst a cubic spline will accurately fit the market the gradient is unconstrained at the end points and it would be foolish to assume that the gradient is economically rational. Some gradient constraints are therefore necessary in order to achieve a sensible yield curve. We use two methods to control the gradient:

#### First Derivative Smoothing

The first method is to add a term to the objective function that is reduced when the gradient (first derivative of the forward curve) is shallow. As well as minimising the squared difference of the model and market data we also minimise the square of the first derivative of the forward curve. The objective function becomes

$$X'_{S,G} = X_{S,G} + \omega_1 \int_{T_1}^{T_M} \left[ \frac{dF(t)}{dt} \right]^2 dt,$$

Where  $\omega_1$  is a constant parameter which specifies the strength of the smoothing constraint. We only employ this smoothing at the end of the forward curve and  $(T_M - T_1)$  generally corresponds to the final 20% of the available market data:  $T_M$  is the terminal market maturity and  $T_1$  is the maturity at which we begin to apply the smoothing constraint. Using a smoothness condition inevitably affects the quality of fit. ***We derive  $\omega$  empirically so as to minimise the gradient squared of the smoothed region of the forward curve whilst maintaining an appropriate fitting tolerance to the last market instrument.*** In this way we balance our need for a high quality of fit with a desire to minimise undesirable long term variation-  $\omega$  can be varied at each calibration date until our conditions are met.

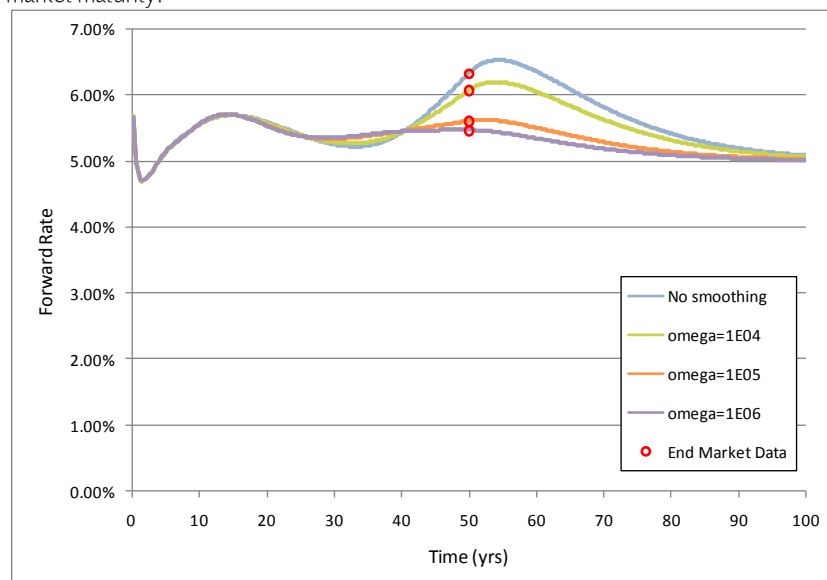
Exhibit 1.1 compares USD swap forward rate curves from March 2007 with and without the smoothing constraint. We see that without such a constraint, if the gradient of the forward curve is steep at the end of the market data this can lead to a bump in the curve. Here we make no effort to discriminate between bonds based on liquidity. Note that in Exhibit 1.1 and throughout the paper the red point in each chart labelled 'End Market Data' represents the terminal market maturity. Before this point the yield curve is specified by interpolation of the market and beyond it the yield curve is specified by extrapolation to the unconditional forward rate target.

### Gradient Constraint

The second method of controlling the gradient of the forward curve is to implement an explicit constraint on the gradient. As suggest above the smoothing strength can be altered at each calibration date until conditions are met. As the smoothing constraint appears in the objective function the value of  $\omega$  required to minimise the gradient will be dependent on the market data, which will be different at every calibration date. As such the smoothing strength cannot be kept constant and analyst judgement may be required at each calibration. The gradient constraint will ensure that very high gradients are always avoided and so reduce the degree of subjectivity. The gradient constraint method will be used in conjunction with the smoothing described above.

### Exhibit 1.1

USD: Swap forward curves at end March 2007 with varying values for the smoothing parameter  $\omega$ . The chart shows that a higher smoothing constraint can reduce spurious volatility beyond the market data. The red circle marks the terminal market maturity.



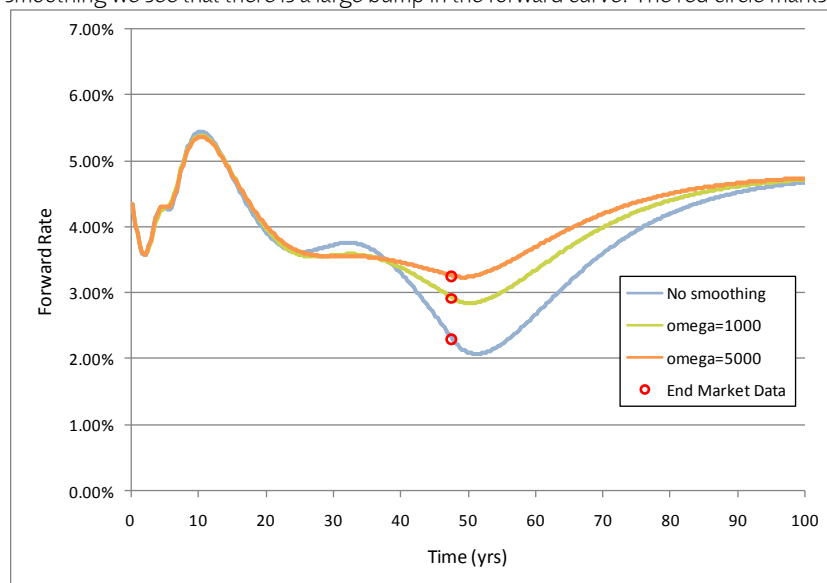
Source: EndMarch2007\_USD\_Smoothing.xls

Similarly Exhibit 1.2 compares the GBP government forward curves from March 2008 with and without the smoothing constraint. We see that the variation at the end of the forward rates is dramatically reduced, while maintaining the specified 5 basis point fitting tolerance.



## Exhibit 1.2

GBP: Government forward curves at end March 2008 with varying values for the smoothing parameter  $\omega$ . Without any smoothing we see that there is a large bump in the forward curve. The red circle marks the terminal market maturity.



Source: EndMarch2008\_GBP\_Smoothing.xls

### 1.2.3 Second Derivative Smoothing

When constructing the instantaneous forward curve over the range of maturities where we have market instruments we aim to ensure that it does not exhibit a high degree of oscillation or curvature, especially at longer maturities. The shape of the forward curve should be implied by the market data and not by our choice of interpolation method. In particular, the use of a cubic spline can introduce unwanted oscillations which we seek to minimise. Generally speaking the more knot points we use the more potential for oscillations, so one method of minimising these is to use fewer knot points, as we mentioned in Section 1.2. We have found, however, that for particularly complex market shapes we need to introduce more knot points and while this allows us to capture the essence of the market shape it can also introduce spurious oscillations. A second option is to introduce a second derivative smoothing constraint as discussed above and used by the Bank of England (Bank of England (2001)). An oscillatory forward curve will have many turning points and so the second derivative (the change in the gradient) will be large. Minimising the second derivative will therefore smooth the forward curve and reduce oscillations. Specifically we constrain the curvature (the square of the second derivative) to be minimal and the objective function, Equation (4), becomes:

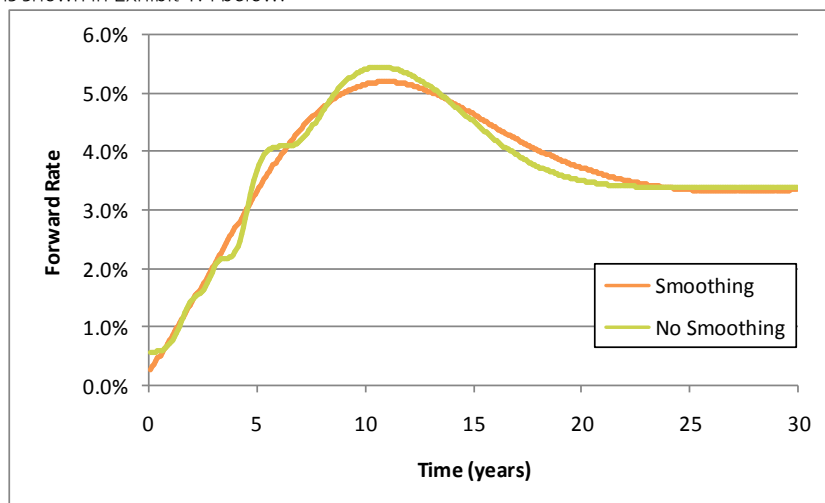
$$X'_{S,G} = X_{S,G} + \omega_1 \int_{T_1}^{T_M} \left[ \frac{dF(t)}{dt} \right]^2 dt + \omega_2 \int_{T_2}^{T_M} \left[ \frac{d^2F(t)}{dt^2} \right]^2 dt,$$

Where  $\omega_2$  is a constant parameter which specifies the strength of the smoothing constraint. We employ the second derivative smoothing over most of the forward curve: from 20% of the terminal market maturity to the terminal market maturity itself:  $(T_M - T_2) \approx 0.9T_M$ . This affords us the freedom to fit the shorter term data – where there is often rich structure – more accurately, as we do not constrain the forward curve to be smooth here. The proportion of the forward curve over which the smoothing is applied may change in some cases, depending on the nature of the market data.

As with first derivative smoothing the quality of fit to the market data will be affected with this modification to the objective function. We empirically derive the value of  $\omega_2$  required to smooth the forward curve whilst still maintaining our fitting tolerance defined in the previous section. With the added smoothness constraint we can use more knot points without introducing oscillations. We choose to set the number of knot points approximately equal to the square root of the number of bonds as suggested by McCulloch (1975). This allows sufficient degrees of freedom to fit complex market shapes but is still low enough that the optimisation problem is not too formidable. When fitting the yield curve to swap market data we space the knot points throughout the data such that there are approximately three rates between each knot point.

### Exhibit 1.3

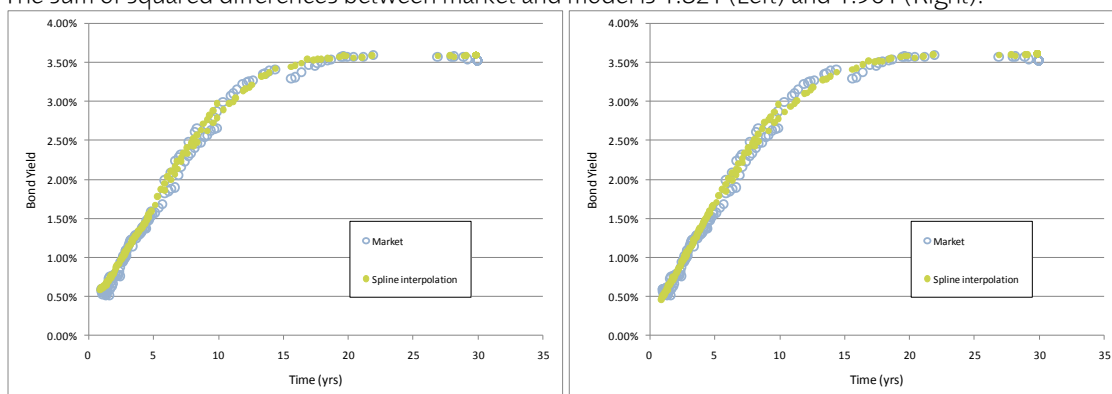
A comparison of the instantaneous forward curve with and without 2<sup>nd</sup> derivative smoothing applied. The forward curves have been derived from USD Government bond data from End March 2009. The resulting quality of fit to the bond data is shown in Exhibit 1.4 below.



Source: USD\_fwd\_smoothing.xls

### Exhibit 1.4

A comparison of the quality of fit to the USD Government bonds from End March 2009 when there is not 2<sup>nd</sup> derivative smoothing (Left) and 2<sup>nd</sup> smoothing is applied (Right). The corresponding instantaneous forward rate curves are shown in Exhibit 1.3, above. One can see that by opting to smooth the 2<sup>nd</sup> derivative we can make the forward rate curve much smoother without materially affecting the quality of fit to the market data. The sum of squared differences between market and model is 1.821 (Left) and 1.961 (Right).



Source: USD\_smoothing\_fit\_comparison\_EndMar2009.xls

## 1.3 Extrapolation

Yield levels at the long (untraded) end of the curve can have a large effect on capital calculations for ESG users with extremely long dated liabilities. Furthermore, spurious variation in these yields can feed through to give material balance sheet and valuation variation. With this in mind it becomes extremely important to develop a robust extrapolation method: one which reflects (to some degree) current market conditions and empirical views of long rate volatility, while simultaneously displaying adequate stability.

The most important principle behind our extrapolation methodology is that, in this approach, an unconditional forward rate is specified before intermediate rates between this unconditional horizon and the last observed market data are calculated. We discuss the form that this extrapolation should take below. The construction of the unconditional forward rate ('end point') is discussed in Sorensen (2008).

We considered 2 forms of extrapolation: cubic spline and Nelson Siegel (NS). Cubic spline extrapolation, as its name suggests, simply extends the cubic spline used to fit the market data to the unconditional horizon. This approach is straightforward to implement, but gives us little control over the form of the extrapolation, its variation, or its convergence on the long term estimate. For this reason we prefer to use the NS method.

The NS functional form has a number of desirable properties. At long maturities it becomes flat, and as we approach our unconditional target its second derivative is always negative (and increasing) if approaching from below and positive (and decreasing) if from above. Because forward rates are described using a very straightforward form we can analytically match the level and gradient of the curve at the terminus of the market data, and setting the unconditional level of rates is trivial. Finally, speed of convergence can be directly and intuitively controlled using the  $\lambda$  parameter, giving us an easy handle on long rate volatility and correlation.

We extrapolate the forward curve from the end of the market data to an unconditional forward rate target using the NS parametric form

$$F(t) = \beta_1 + (\beta_2 + \beta_3(t - t_{max})) \exp(-\lambda(t - t_{max})) \quad (6)$$

Here  $\beta_1$  corresponds to the unconditional forward rate,  $\lambda$  is a free parameter which we can use to control the speed of convergence and  $\beta_2$  and  $\beta_3$  can be specified by matching the value and derivative of the curve at the last market point. The terminal market maturity is denoted  $t_{max}$ .

### 1.3.1 Fixing the lambda parameter

The  $\lambda$  parameter can be interpreted as controlling the volatility of the extrapolated forward rates. A high value for  $\lambda$  will lead to faster convergence of the forward curve to the unconditional forward rate target ( $\beta_1$  in equation 5), so the forward rate volatility will decrease. A low value for  $\lambda$  will lead to slower convergence to the unconditional target so the forward rate volatility will increase. We can set the value of  $\lambda$  by estimating the forward rate volatility term structure using historical forward rate data. This analysis will be described in the knowledge base article note (Z. Liu, 2008). We will use a value of  $\lambda = 0.06$ .

## 2 Back-testing

In this section we present the yield curves (forward curves, spot curves and par yield curves) obtained from interpolating the GBP, USD, EUR, JPY and TWD market bond data and swap data using a cubic regression spline (Sections 2.1-2.5 respectively). We show results for every quarter from March 2007 to March 2008 inclusive. We extrapolate from the last market data point to an unconditional forward rate target using the Nelson-Siegel form (Equation 5).

Calculation of the unconditional forward rate targets for each economy is described in Sorensen (2008). These forward rate targets are reviewed on an annual basis, in the back-testing presented here we have used 5.7% for the swap forward curve and 5.3% for the swap forward curve. Calculation of the speed of convergence to the unconditional target is described in Liu (2008). This value is also subject to review and we use a value of  $\lambda = 0.06$  in the examples shown in this section.

For brevity, we do not show all quality of fit charts here but show a summary table in each case comparing the quality of fit achieved using the cubic spline and Nelson-Siegel methodologies. In Section 2.6, however, we do show some comparison charts to outline the typical superior fit to the market we obtain when using the cubic spline interpolation

The tables show that using the cubic spline method we get a much better fit to the market data. In all cases the standard deviation of differences between the fit and the market is markedly lower using the cubic spline method and in almost all cases the maximum difference is also lower. This is especially true of the fits to the swap data where in some cases we see the standard deviation decrease by a factor of about 35.

## 2.1 USD economy

### 2.1.1 Forward curves

#### Exhibit 2.1

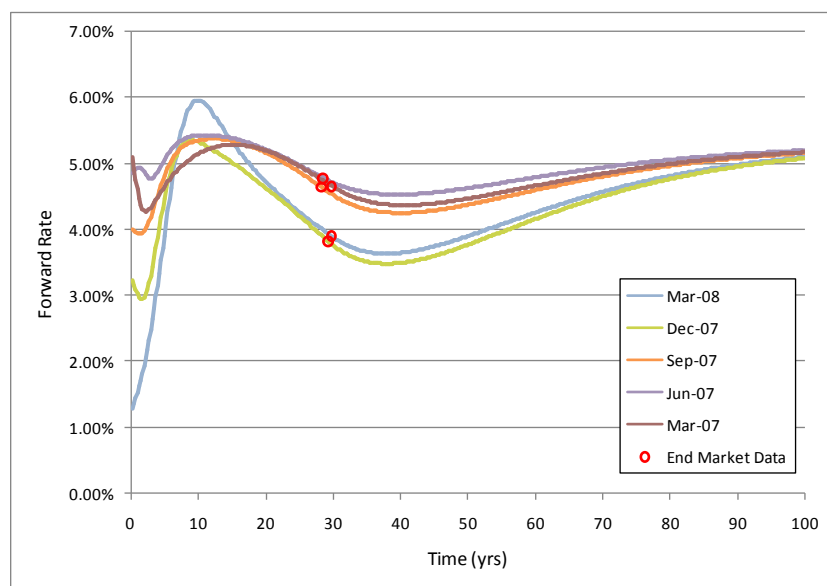
USD: A comparison of government bond yield errors from the NS and cubic spline methods. See Exhibit 2.41 for a graphical comparison of the methods at end September 2007.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	27.9	61.6	58.7	27.5	18.3
Min Error (bps)	0.1	0.2	0.0	0.0	0.0
Standard Deviation (bps)	5.3	6.8	6.3	4.1	2.4
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	44.1	148.4	64.7	83.0	25.5
Min Error (bps)	0.1	0.0	0.0	0.0	0.1
Standard Deviation (bps)	16.9	22.5	16.6	17.0	7.4

Source: Table\_all.xls. The following tables in Section 2 also come from this source.

#### Exhibit 2.2

USD: Government forward curves March 2007 to March 2008. We see that the negative gradient of the forward curve at the end of the market data leads to a dip in the forward curve, even though we have used the smoothing constraint.



Source: USD\_YC\_data.xls. The following charts in Section 2.1 also come from this source.

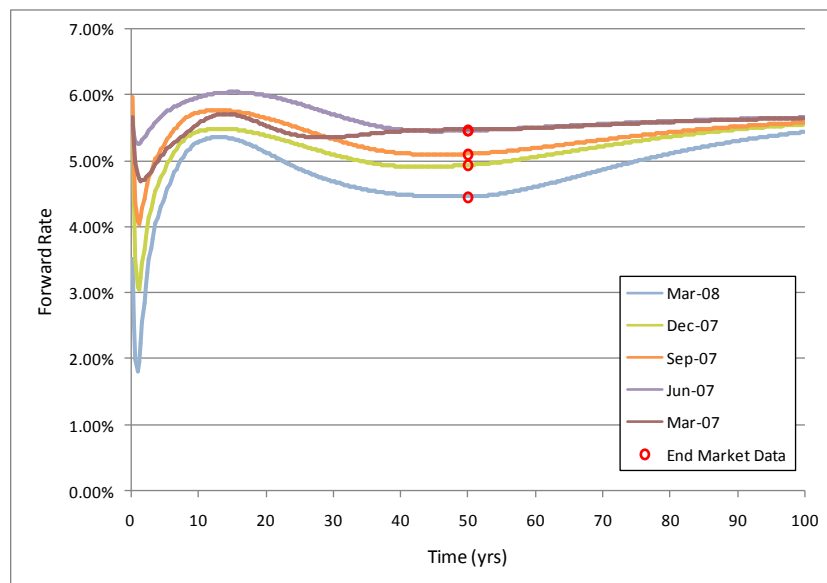
### Exhibit 2.3

USD: A comparison of the fit to swap rates using NS and a cubic spline. The standard deviation of errors using the cubic spline are much lower than the NS indicating a dramatic improvement in the quality of fit.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	1.6	1.0	0.4	0.4	1.4
<i>Min Error (bps)</i>	0.0	0.0	0.0	0.0	0.1
<i>Standard Deviation (bps)</i>	0.6	0.4	0.2	0.2	0.5
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	7.2	4.4	5.2	8.1	4.3
<i>Min Error (bps)</i>	1.2	0.1	0.3	0.1	0.1
<i>Standard Deviation (bps)</i>	4.3	2.5	2.5	2.7	2.4

### Exhibit 2.4

USD: Swap forward curves March 2007 to March 2008. In contrast to the government curves in Exhibit 2.2 the curves here are much flatter. The swap rate data tends to be flatter at the terminal maturities so we do not observe the dips noted previously.

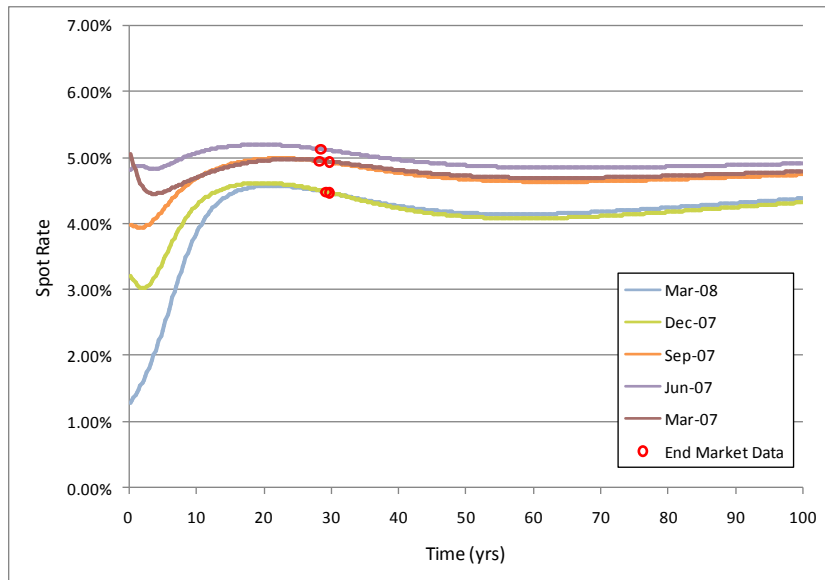


### 2.1.2 Spot curves

Here we show the spot curves corresponding to Exhibits 2.2 and 2.4. Note that large dips and fluctuations in the forward curves are not so pronounced in the spot curves.

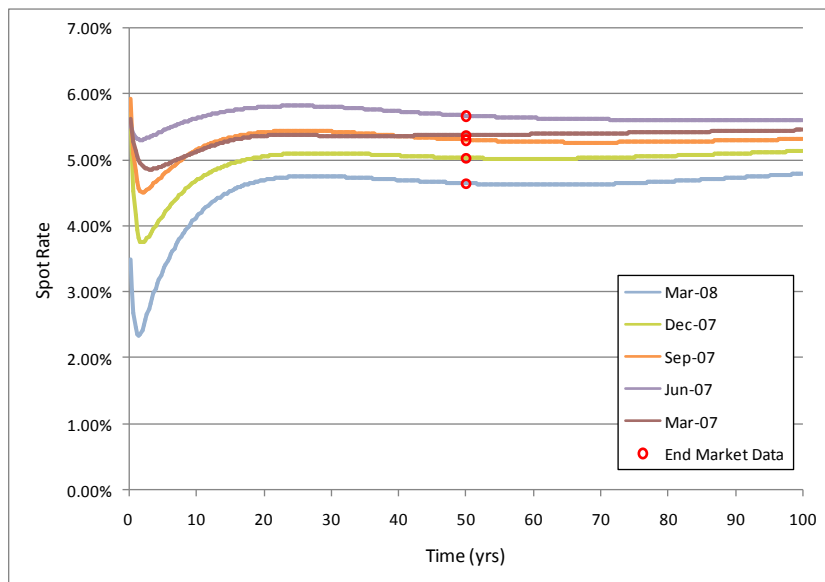
#### Exhibit 2.5

USD: Government spot curves March 2007 to March 2008.



#### Exhibit 2.6

USD: Swap spot curves March 2007 to March 2008.

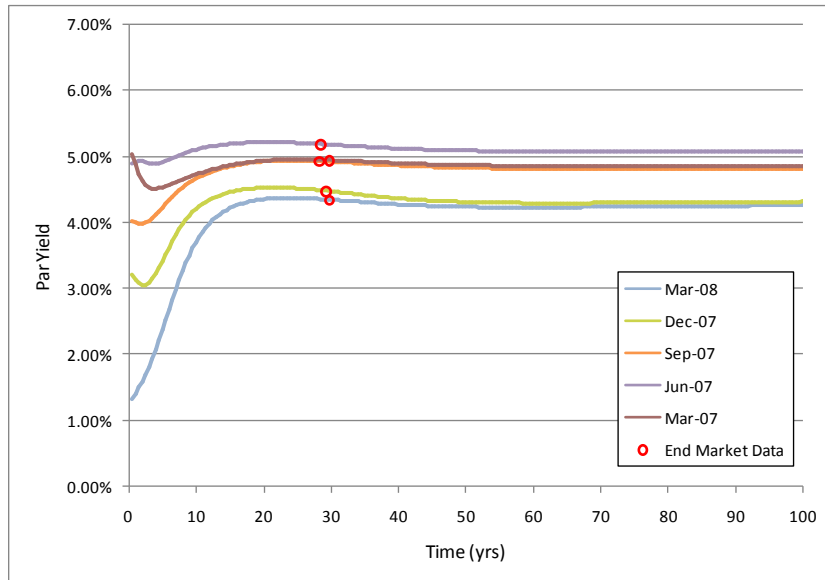


### 2.1.3 Par yield curves

Here we show the par yield curves corresponding to the forward curves in Exhibits 2.2 and 2.4.

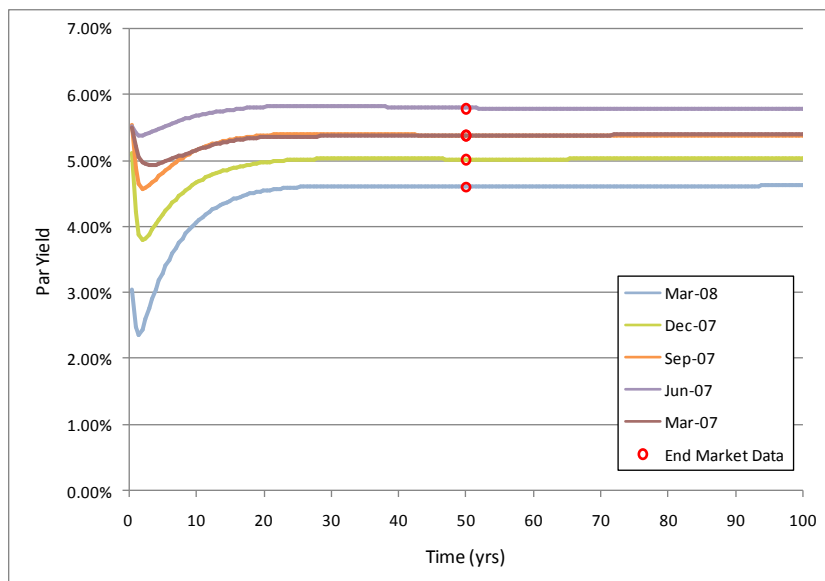
#### Exhibit 2.7

USD: Government par yield curves March 2007 to March 2008.



#### Exhibit 2.8

USD: Swap par yield curves March 2007 to March 2008.





## 2.2 TWD Economy

### 2.2.1 Forward curves

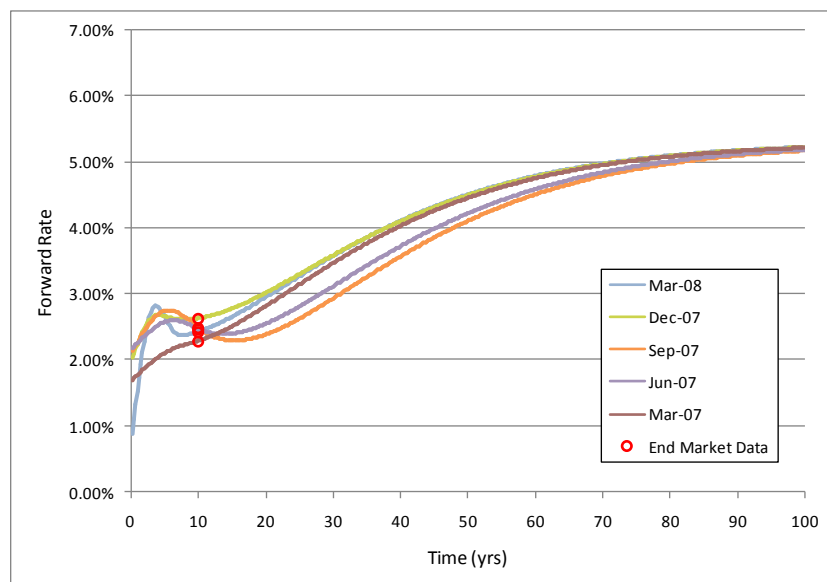
#### Exhibit 2.9

TWD: A comparison of government bond yield errors from the NS and cubic spline methods. See Exhibit 2.42 for a graphical comparison of the methods at end March 2008.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	32.2	7.2	9.8	18.4	15.3
Min Error (bps)	0.0	0.1	0.1	0.1	0.1
Standard Deviation (bps)	8.9	1.5	2.1	3.5	3.5
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	38.9	14.7	17.9	11.8	12.6
Min Error (bps)	0.3	0.1	0.0	0.2	0.2
Standard Deviation (bps)	17.7	4.5	5.2	3.3	5.7

#### Exhibit 2.10

TWD: Government forward curve March 2007 to March 2008.



Source: TWD\_YC\_data.xls. The following charts in Section 2.2 also come from this source.

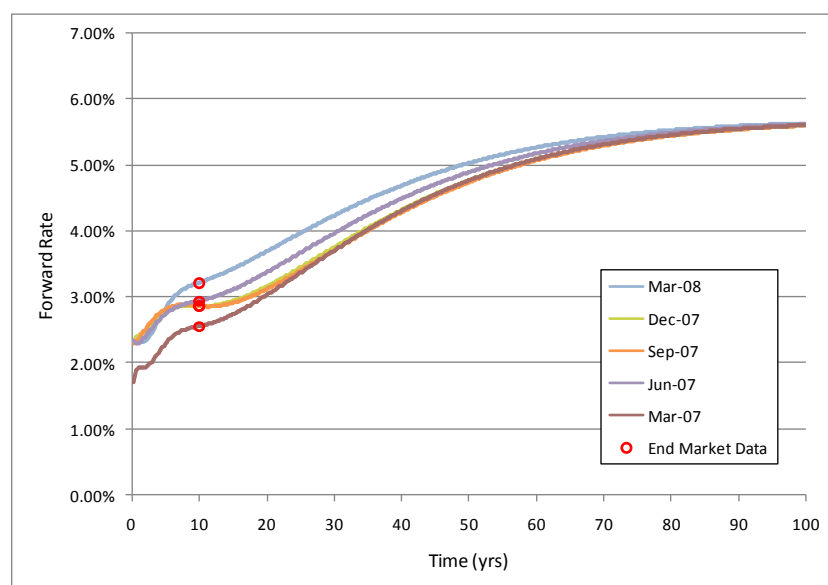
**Exhibit 2.11**

TWD: A comparison of the fit to swap rates using NS and a cubic spline.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	1.6	0.5	1.4	0.8	0.8
Min Error (bps)	0.1	0.1	0.1	0.1	0.2
Standard Deviation (bps)	0.8	0.2	0.8	0.4	0.5
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	4.9	2.3	1.8	1.5	5.2
Min Error (bps)	0.4	0.1	0.0	0.2	0.3
Standard Deviation (bps)	2.5	1.6	1.0	1.0	2.7

**Exhibit 2.12**

TWD: Swap forward curves March 2007 to March 2008.

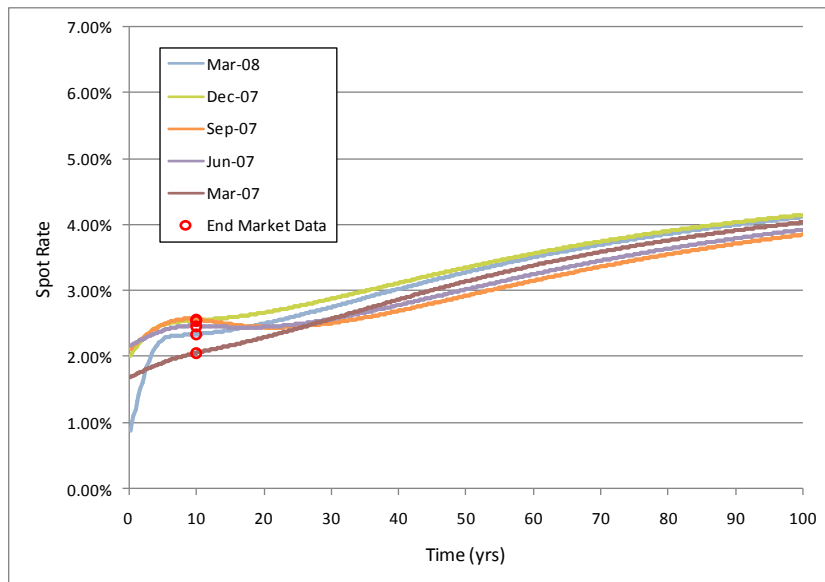


### 2.2.2 Spot curves

Here we show the spot curves corresponding to Exhibits 2.10 and 2.12. Note that large dips and fluctuations in the forward curves are not so pronounced in the spot curves.

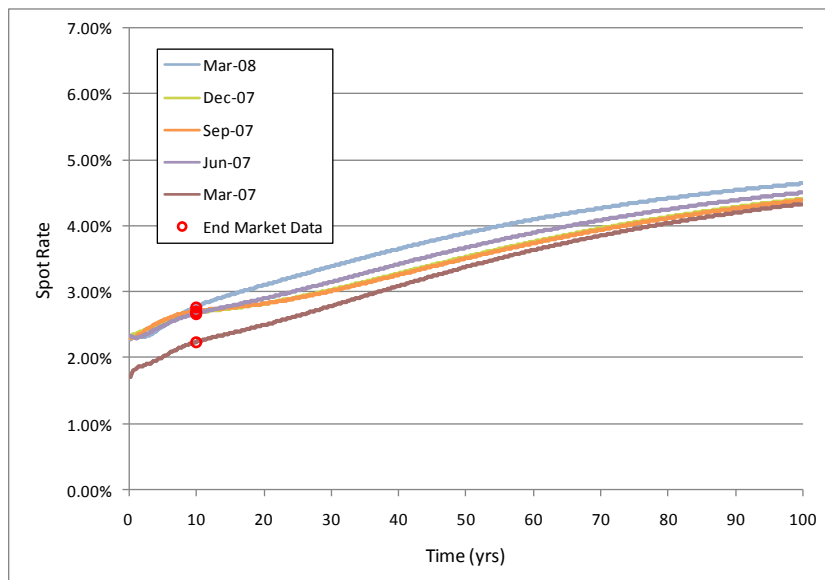
#### Exhibit 2.13

TWD: Government spot curves March 2007 to March 2008.



#### Exhibit 2.14

TWD: Swap spot curves March 2007 to March 2008.

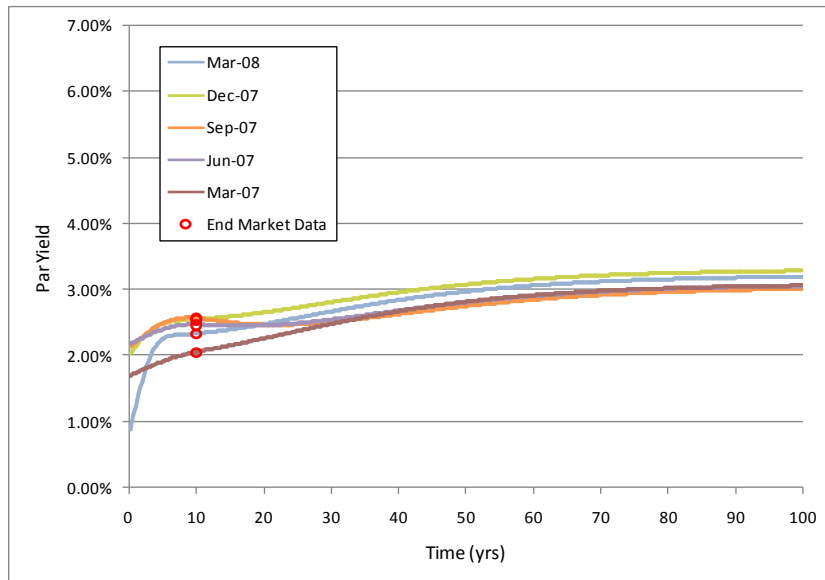


### 2.2.3 Par yield curves

Here we show the par yield curves corresponding to the forward curves in Exhibits 2.10 and 2.12.

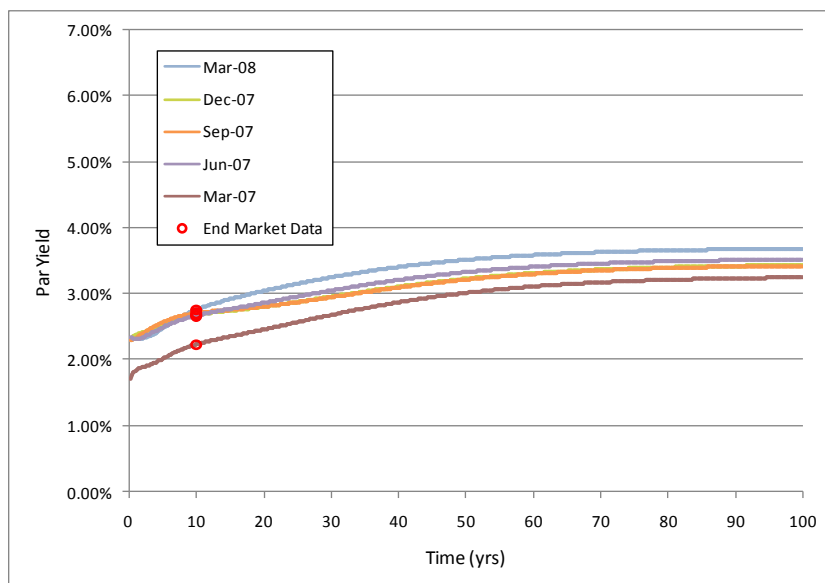
#### Exhibit 2.15

TWD: Government par yield curves March 2007 to March 2008.



#### Exhibit 2.16

TWD: Swap par yield curves March 2007 to March 2008.



## 2.3 EUR economy

### 2.3.1 Forward curves

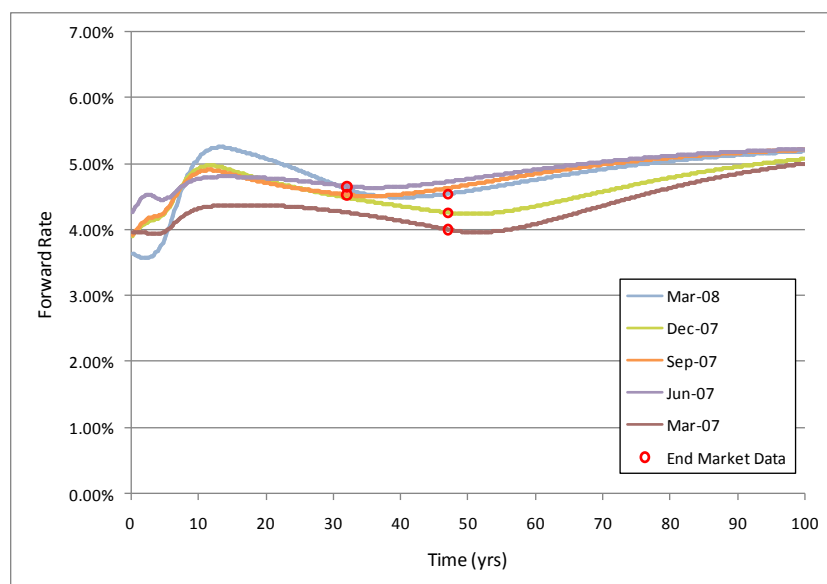
#### Exhibit 2.17

EUR: A comparison of government bond yield errors from the NS and cubic spline methods.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	20.0	7.9	9.2	6.3	5.5
Min Error (bps)	0.1	0.0	0.0	0.0	0.0
Standard Deviation (bps)	4.4	1.7	1.8	1.3	1.4
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	26.5	13.8	10.8	10.0	9.4
Min Error (bps)	0.2	0.0	0.0	0.0	0.1
Standard Deviation (bps)	9.6	5.5	4.4	4.5	4.2

#### Exhibit 2.18

EUR: Government forward curves March 2007 to March 2008.



Source: EUR\_YC\_data.xls. The following charts in Section 2.3 also come from this source.

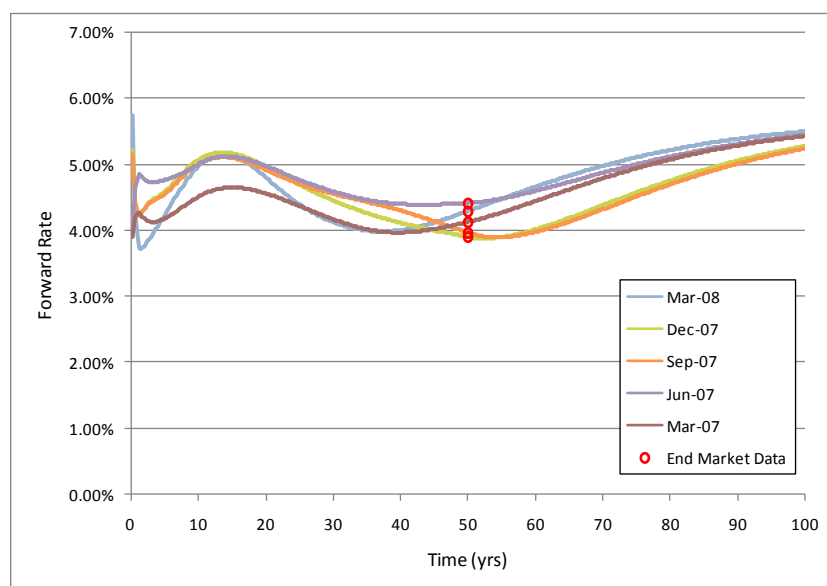
**Exhibit 2.19**

EUR: A comparison of the fit to swap rates using NS and a cubic spline. See Exhibit 2.43 for a graphical comparison of the methods at end March 2008.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	4.8	0.6	0.3	0.8	0.8
<i>Min Error (bps)</i>	0.0	0.0	0.0	0.0	0.0
<i>Standard Deviation (bps)</i>	1.7	0.2	0.1	0.3	0.3
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	6.2	7.4	7.7	6.6	6.1
<i>Min Error (bps)</i>	0.0	0.0	0.1	0.3	0.8
<i>Standard Deviation (bps)</i>	2.3	4.6	3.5	3.4	3.6

**Exhibit 2.20**

EUR: Swap forward curves March 2007 to March 2008.

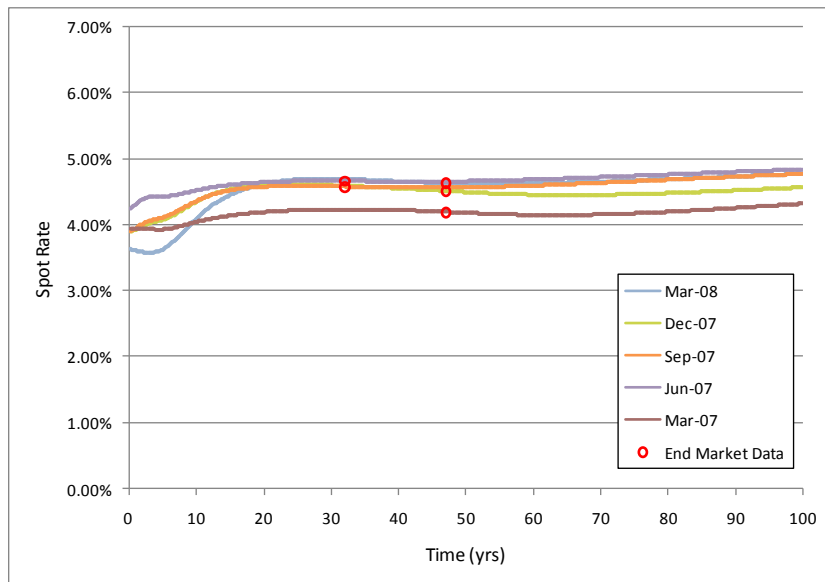


### 2.3.2 Spot curves

Here we show the spot curves corresponding to Exhibits 2.18 and 2.20. Note that large dips and fluctuations in the forward curves are not so pronounced in the spot curves.

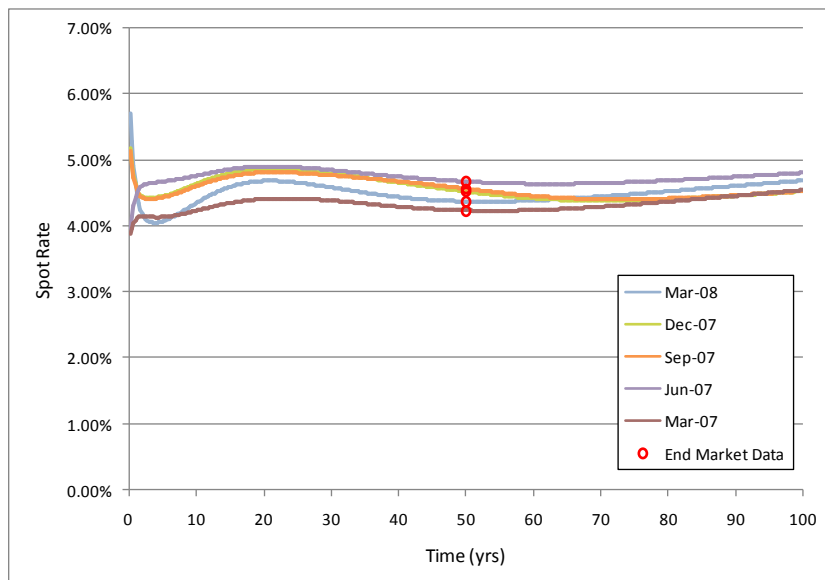
#### Exhibit 2.21

EUR: Government spot curves March 2007 to March 2008.



#### Exhibit 2.22

EUR: Swap spot curves March 2007 to March 2008.

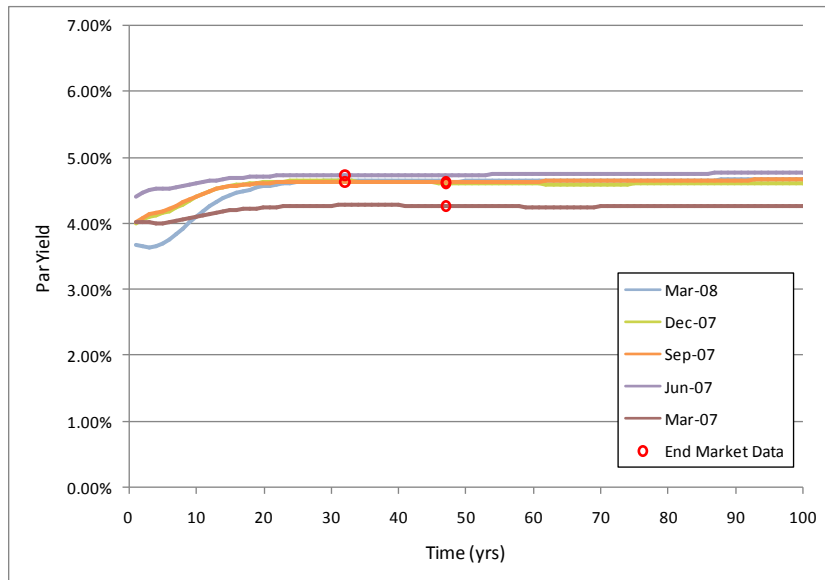


### 2.3.3 Par Yyield curves

Here we show the par yield curves corresponding to the forward curves in Exhibits 2.18 and 2.20.

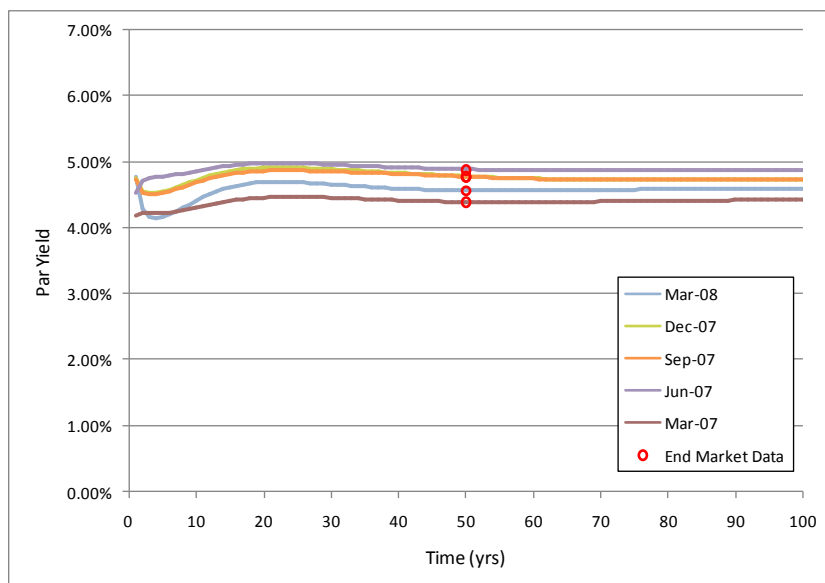
#### Exhibit 2.23

EUR: Government par yield curves March 2007 to March 2008.



#### Exhibit 2.24

EUR: Swap par yield curves March 2007 to March 2008.





## 2.4 JPY economy

### 2.4.1 Forward curves

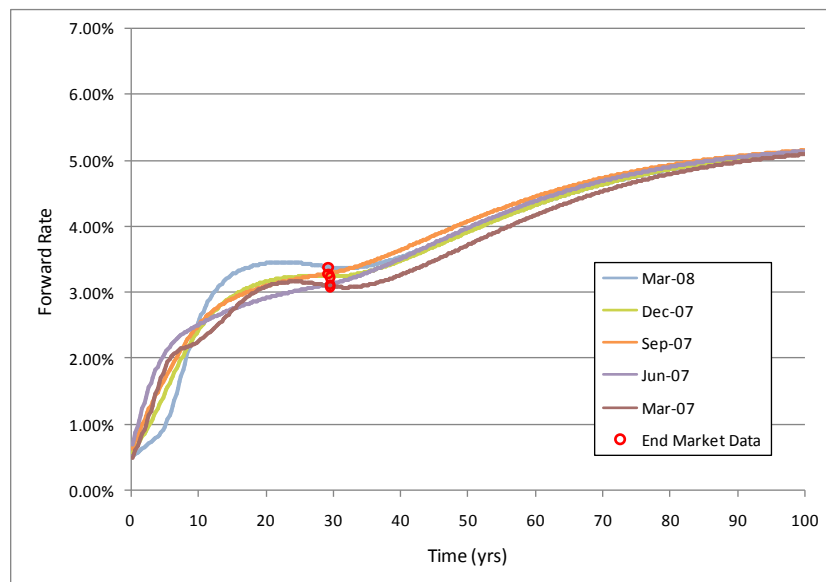
#### Exhibit 2.25

JPY: A comparison of government bond yield errors from the NS and cubic spline methods.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	10.3	10.4	9.3	6.2	13.4
Min Error (bps)	0.0	0.0	0.0	0.0	0.0
Standard Deviation (bps)	1.9	1.5	1.2	1.0	3.3
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	15.1	14.6	8.2	7.3	5.4
Min Error (bps)	0.1	0.1	0.0	0.0	0.0
Standard Deviation (bps)	5.2	3.7	2.0	2.8	1.7

#### Exhibit 2.26

JPY: Government forward curves March 2007 to March 2008.



Source: JPY\_YC\_data.xls. The following charts in Section 2.4 also come from this source.

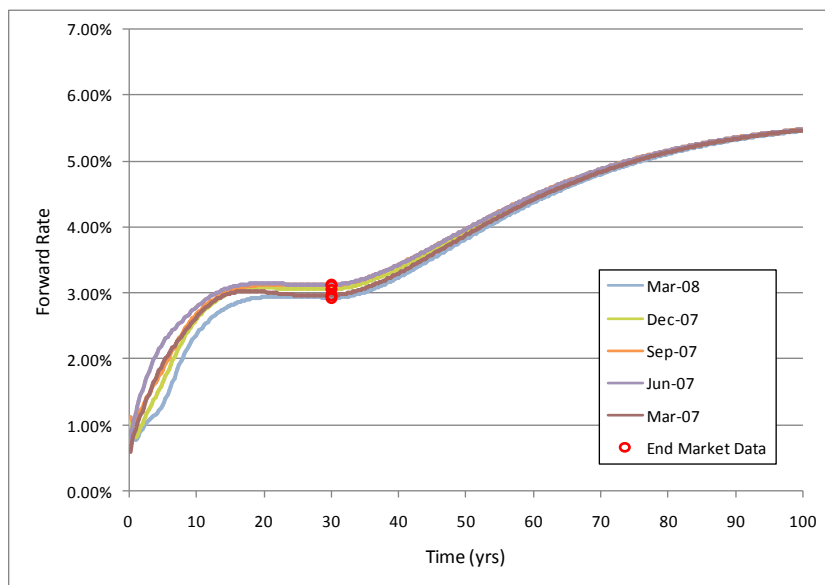
**Exhibit 2.27**

JPY: A comparison of the fit to swap rates using NS and a cubic spline. The fit to JPY swap rates using a cubic spline is excellent, with the maximum error being less than 1 bps in all cases – a significant improvement over the Nelson-Siegel method.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	0.8	0.6	0.4	0.3	0.8
<i>Min Error (bps)</i>	0.0	0.0	0.0	0.0	0.0
<i>Standard Deviation (bps)</i>	0.4	0.4	0.2	0.1	0.4
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	2.2	8.7	5.6	2.9	2.8
<i>Min Error (bps)</i>	0.3	0.5	0.1	0.1	0.1
<i>Standard Deviation (bps)</i>	1.3	4.3	3.0	1.3	1.1

**Exhibit 2.28**

JPY: Swap forward curves March 2007 to March 2008.

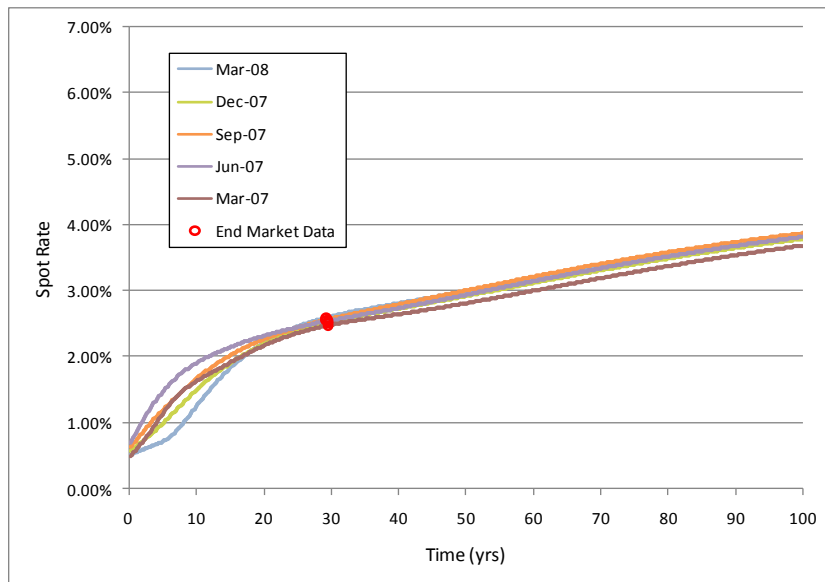


## 2.4.2 Spot curves

Here we show the spot curves corresponding to Exhibits 2.26 and 2.28. Note that large dips and fluctuations in the forward curves are not so pronounced in the spot curves.

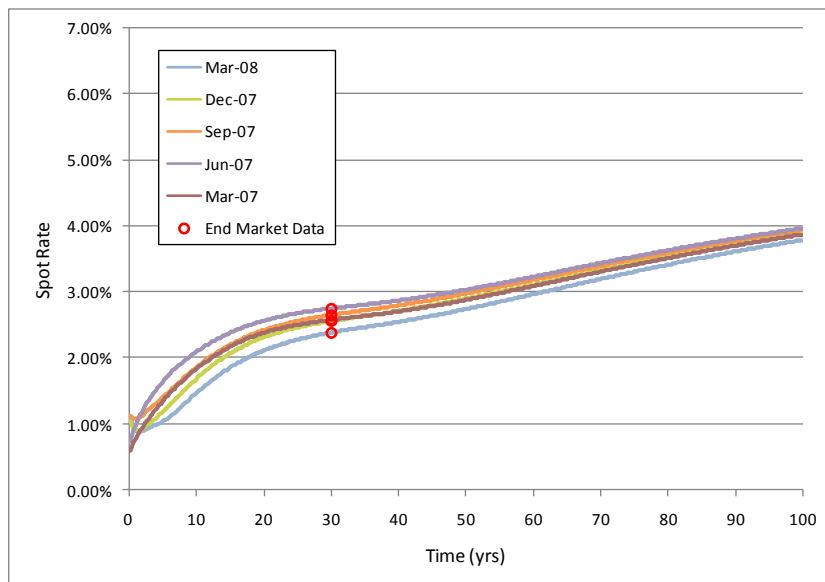
### Exhibit 2.29

JPY: Government spot curves March 2007 to March 2008.



### Exhibit 2.30

JPY: Swap spot curves March 2007 to March 2008.

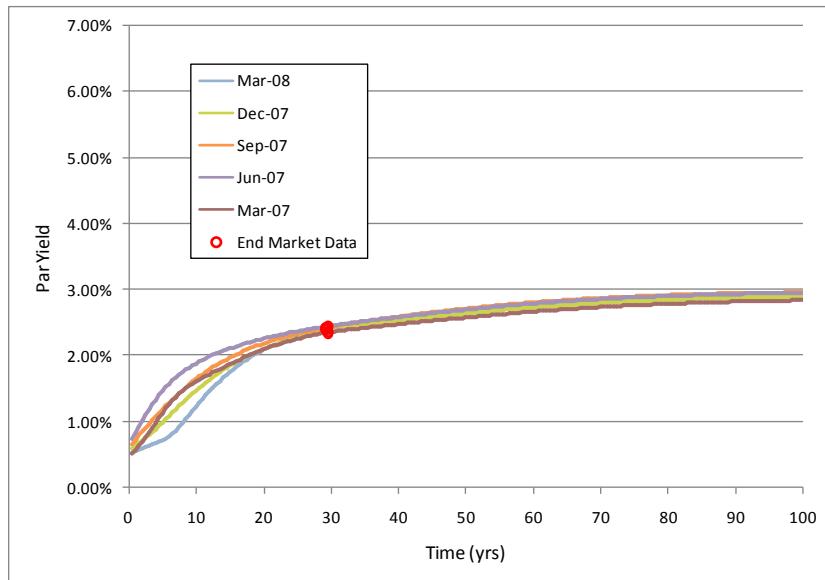


### 2.4.3 Par yield curves

Here we show the par yield curves corresponding to the forward curves in Exhibits 2.26 and 2.28.

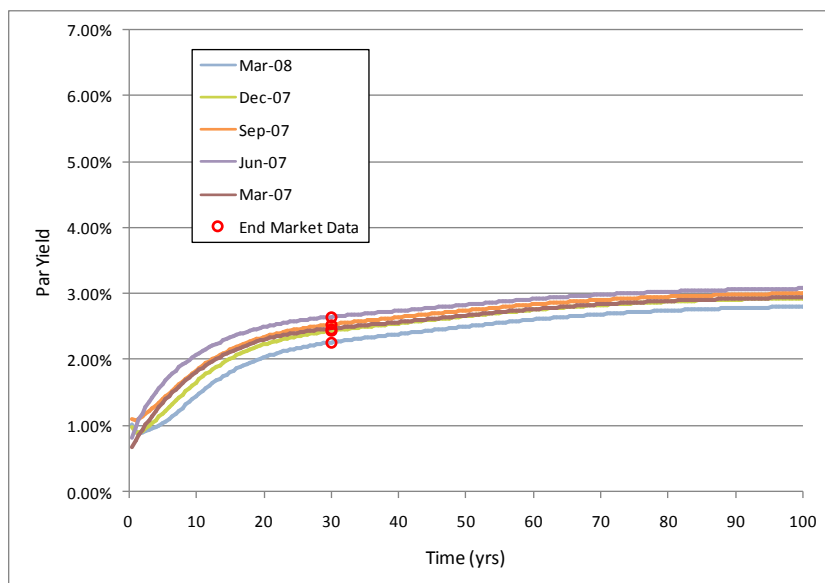
#### Exhibit 2.31

JPY: Government par yield curves March 2007 to March 2008.



#### Exhibit 2.32

JPY: Swap par yield curves March 2007 to March 2008.



## 2.5 GBP economy

### 2.5.1 Forward curves

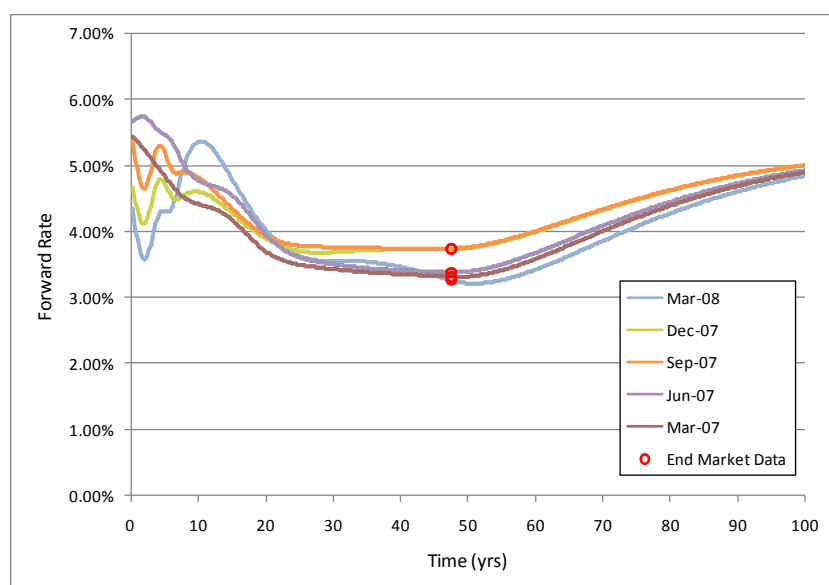
#### Exhibit 2.33

GBP: A comparison of government bond yield errors from the NS and cubic spline methods. See Exhibit A for a graphical comparison of the methods at end March 2008.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	23.9	7.0	12.4	9.7	14.3
Min Error (bps)	0.0	0.0	0.0	0.0	0.0
Standard Deviation (bps)	2.4	1.4	2.3	1.5	1.5
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
Max Error (bps)	23.8	14.0	47.8	13.5	12.7
Min Error (bps)	0.0	0.0	0.0	0.0	0.0
Standard Deviation (bps)	7.8	3.0	7.4	2.5	3.2

#### Exhibit 2.34

GBP: Government forward curves March 2007 to March 2008. Without the smoothing constraint the forward curve would have a steep negative gradient at the terminal market maturity. With the smoothing constraint the curve flattens and we avoid a large. See Exhibit 1.2.



Source: GBP\_YC\_data.xls. The following charts in Section 2.5 also come from this source.

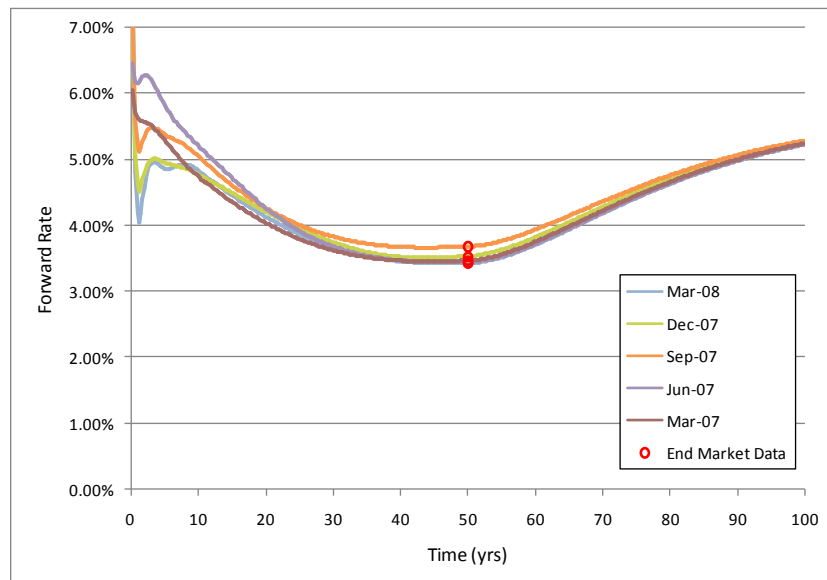
**Exhibit 2.35**

GBP: A comparison of the fit to government bonds using NS and a cubic spline. The fit is dramatically improved using the cubic spline e.g. the maximum basis point error has decreased from 46.6 to 0.5 for and the standard deviation has decreased by a factor of 36 for September 2007. See Exhibit 2.44 for a graphical comparison of the methods at end September 2007.

<b>Cubic Spline</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	4.0	1.7	0.5	1.9	0.9
<i>Min Error (bps)</i>	0.0	0.0	0.0	0.0	0.0
<i>Standard Deviation (bps)</i>	1.7	0.6	0.3	0.7	0.4
<b>Nelson-Siegel</b>	<b>31/03/2008</b>	<b>31/12/2007</b>	<b>30/09/2007</b>	<b>30/06/2007</b>	<b>31/03/2007</b>
<i>Max Error (bps)</i>	75.9	52.9	46.6	3.1	4.0
<i>Min Error (bps)</i>	0.1	0.2	0.0	0.2	0.1
<i>Standard Deviation (bps)</i>	17.6	12.3	10.8	1.7	1.4

**Exhibit 2.36**

GBP: Swap forward curves March 2007 to March 2008.

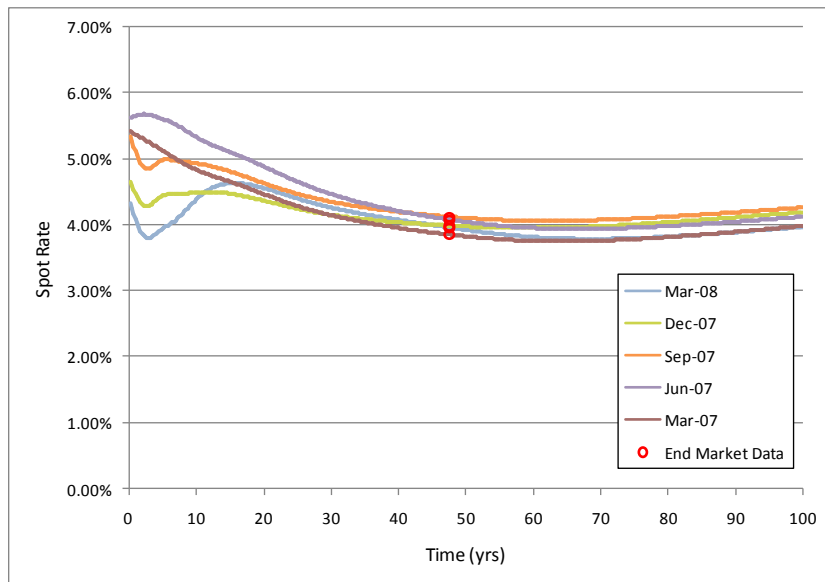


## 2.5.2 Spot curves

Here we show the spot curves corresponding to Exhibits 2.34 and 2.36. Note that large dips and fluctuations in the forward curves are not so pronounced in the spot curves.

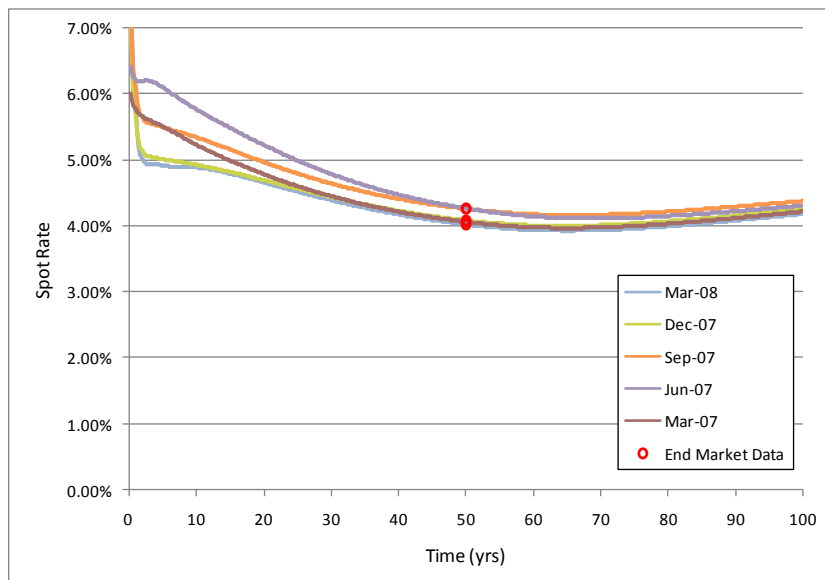
### Exhibit 2.37

GBP: Government spot curves March 2007 to March 2008.



### Exhibit 2.38

GBP: Swap spot curves March 2007 to March 2008.

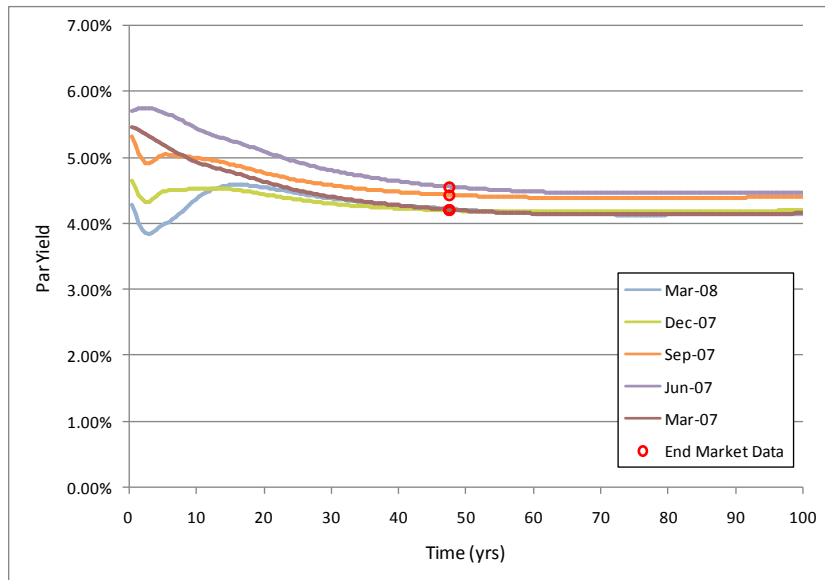


### 2.5.3 Par yield curves

Here we show the par yield curves corresponding to the forward curves in Exhibits 2.34 and 2.36.

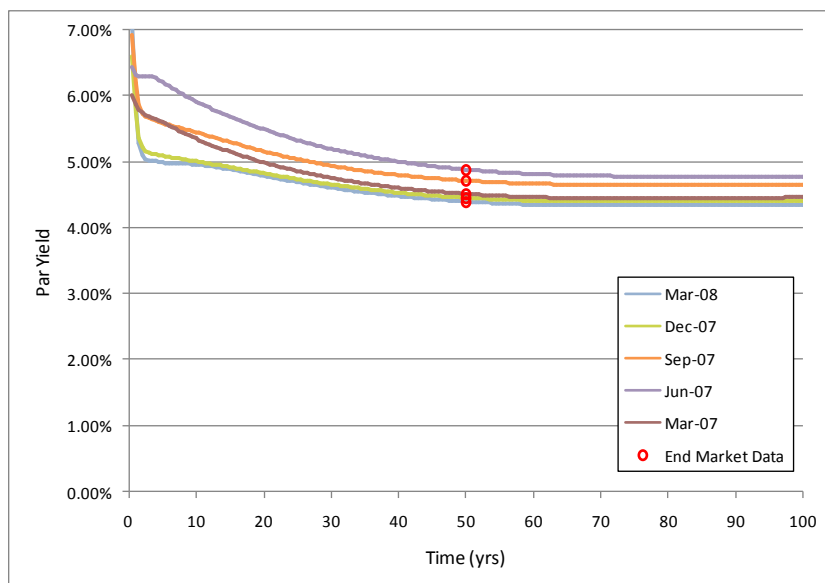
#### Exhibit 2.39

GBP: Swap par yield curves March 2007 to March 2008.



#### Exhibit 2.40

GBP: Swap par yield curves March 2007 to March 2008.



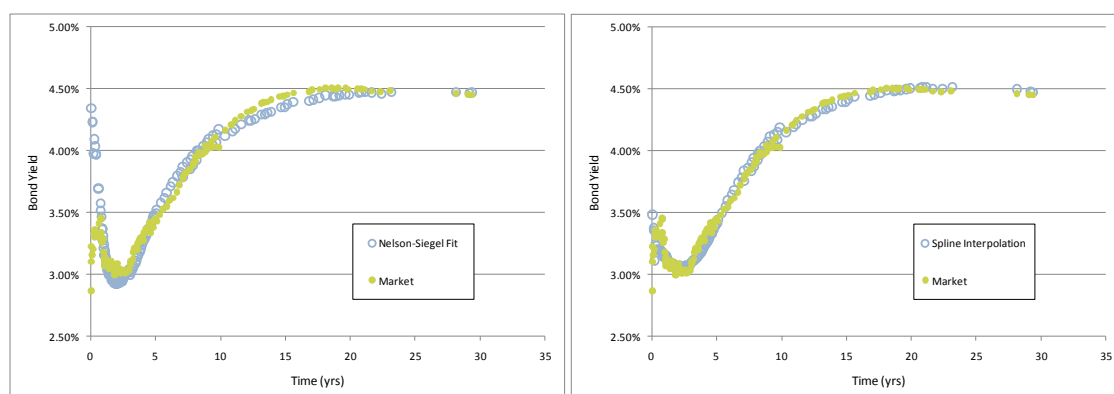


## 2.6 Quality of fit

Here we show typical swap and government bond fits and compare the new cubic spline method to the old NS method. In each case we see that the cubic spline is an improvement. In particular, we see typical swap rate calibration errors less than a basis point, while we would expect to see errors around 5 times this magnitude with NS.

### Exhibit 2.41

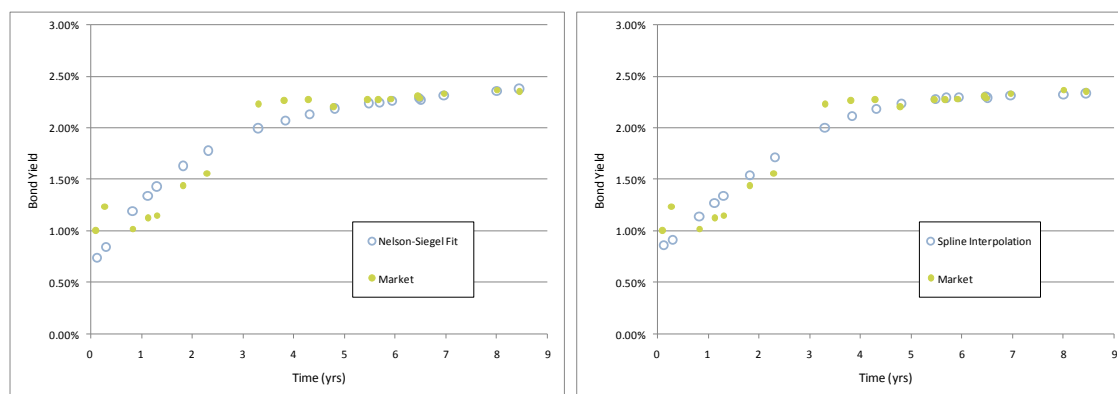
Comparison of Nelson-Siegel (left) and spline interpolation (right), showing superior quality of fit across end September 2007 USD government bond yield curve. The spline can fit the short maturity bonds with greater precision and also captures the term structure more accurately.



Source: Fit\_comparisons.xls. The following charts in Section 2.6 also come from this source.

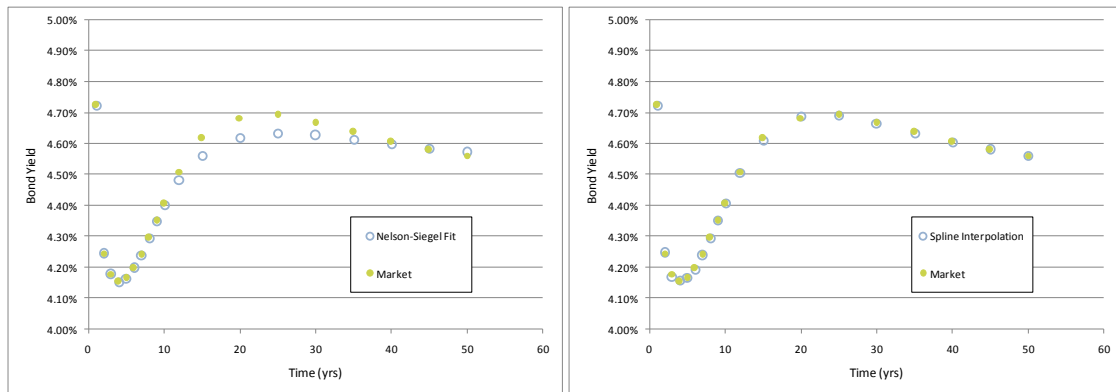
### Exhibit 2.42

Comparison of Nelson-Siegel (left) and spline interpolation (right), showing superior quality of fit across end March 2008 TWD government bond yield curve.

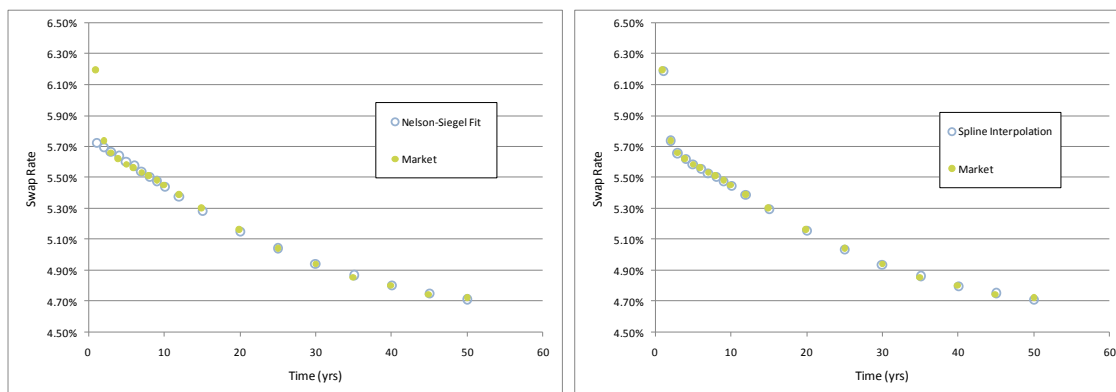


**Exhibit 2.43**

Comparison of Nelson-Siegel (left) and spline interpolation (right), showing superior quality if fit to end March 2008 EUR swap rates. The increased flexibility of the spline allows us to capture the shape of the market data more accurately.

**Exhibit 2.44**

Comparison of Nelson-Siegel (left) and spline interpolation (right), showing superior quality if fit to end September 2007 GBP swap rates. The spline can capture the initial steep shape of the market better than the Nelson-Siegel function.



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