5.4.3 Term structure model and interest rate trees

Although Black's model can be used to value a wide range of instruments, it does not provide a description of the stochastic evolution of interest rates and bond prices. Consequently, for interest rate derivatives whose value depends on the exact paths of such stochastic processes, such as American-style swaptions, callable bonds, and structured notes, Black's model cannot be used. To value these instruments, RiskMetrics employs models which describe the evolution of the yield curve. These models are known as term structure models.

One group of term structure models for instruments pricing are based on no-arbitrage arguments. This group of models are consistent with today's term structure of interest rates observed in the market, meaning that market prices of bonds are recovered by the models. They take the initial term structure as input and define how it evolves. The Ho-Lee model and Hull-White model are two examples in this group which are also analytically tractable. Lognormal one-factor models, though not analytically tractable, have the advantage that they avoid the possibility of negative interest rates.

In RiskMetrics, the Black-Derman-Toy (BDT) model is used to price some complex interest rate derivatives.[11] The principal reason for using BDT model is its ease of calibration to the current interest rate and the implied term structure of volatility.

The direct application of a BDT model is to construct interest rate trees from which complex instruments can be priced. An interest rate tree is a representation of the stochastic process for the short rate in discrete time steps. The tree is calibrated to current market interest rates and volatilities. Note that in an interest rate tree, the discount rate varies from node to node. As an example, let us use a BDT model to build an interest tree, and use it to price a callable bond.

Example 5.12 BDT tree calibration

A BDT tree is fully described by a vector $r = \{r_{i0}\}$, $(i = 1, \cdots, n)$ whose elements are the lowest values of the short rate at time step i, and by a vector $\sigma = \{\sigma_i\}$, whose elements are the annualized volatilities of the short rate from time step i to time step $i + 1$. Figure 5.3 shows an example of a calibrated BDT tree.

The BDT model in its discrete version can be stated as[12]

$$r_{ij} = r_{i0} e^{2\sigma_i j \sqrt{\Delta t}}, \tag{5.49}$$

where Δt is the time step in years and j $(0, 1, \cdots, i - 1)$ is an index representing the state of the short rate at each time step.

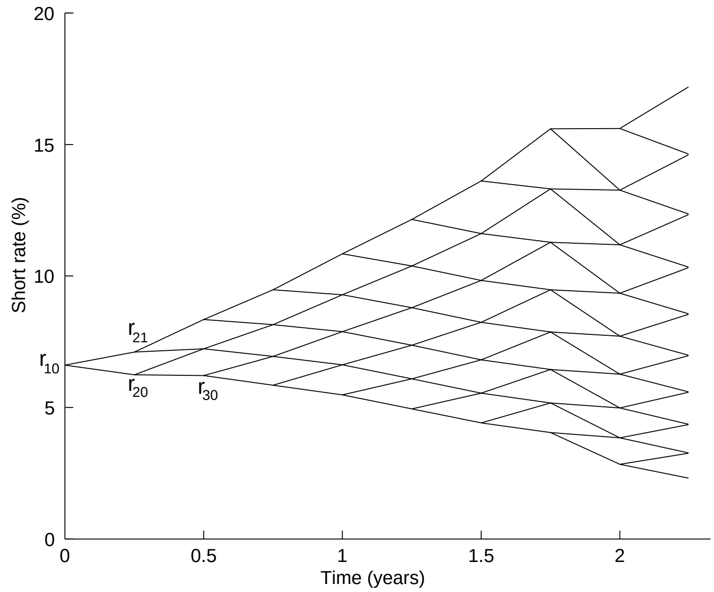
The first step for the calibration is to set each BDT volatility σ_i equal to the implied volatility of a caplet with the same maturity.[13] The short rate vector $r = \{r_{i0}\}$ is then calibrated so that the prices of the BDT

[11] See Black, Derman and Toy (1990).

[12] See Rebonato (1996).

[13] This is not theoretically correct because the forward rates in a BDT tree are only approximately lognormal. However, the induced pricing errors are very small and the time savings in the volatility calibration justify this procedure. For a more detailed discussion of calibration issues see Rebonato (1996).

Figure 5.3: An example of BDT tree



zero-coupon bonds corresponding to each of the nodes in the current yield curve exactly match the true bond prices. To do this, we start with the short maturity of the yield curve and work our way to the longest maturity.

Suppose that we want to build a tree with a step length of three months, or $\Delta t = 0.25$ years. Assume that the market prices for three-month and six-month Treasury bills are USD 98.48 and USD 96.96 respectively, and that the implied volatility of a caplet with expiry in six months is 15%. In the first step we find r_{10} by solving

$$98.48 = \frac{100}{1 + r_{10}\Delta t} = \frac{100}{1 + 0.25 r_{10}}, \tag{5.50}$$

which gives the three-month yield to be $r_{10} = 0.0617$.

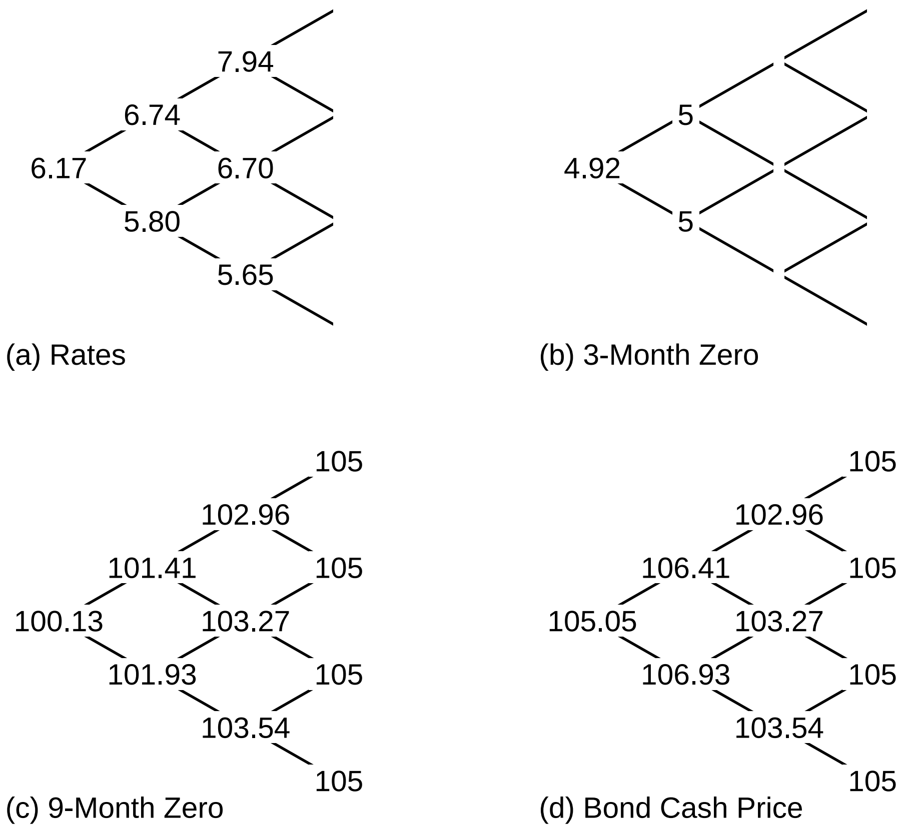
The price for six-month Treasury bills is an average discounted value over all paths out to six months. We have the following relationship for the second step

$$96.96 = \left(\frac{1}{1 + r_{10}\Delta t}\right) \times \frac{1}{2}\left(\frac{1}{1 + r_{20}\Delta t} + \frac{1}{1 + r_{21}\Delta t}\right) \times 100, \tag{5.51}$$

where the first factor on the right hand side of the equation is the discount factor for the first period, and the second factor is the average discount factor for the second period. With the relationship between r_{20} and r_{21} given by (5.49), and the forward volatility $\sigma_2 = 15\%$, r_{20} and r_{21} are found to be 0.0580 and 0.0674 respectively.

If we repeat the process until maturity time of the instrument, we will have a calibrated BDT tree as illustrated in Figure 5.4(a). We can then use it to value fixed income instruments with embedded options, including callable bonds.

Figure 5.4: A coupon bond is valued as a portfolio of zero coupon bonds using a BDT tree.



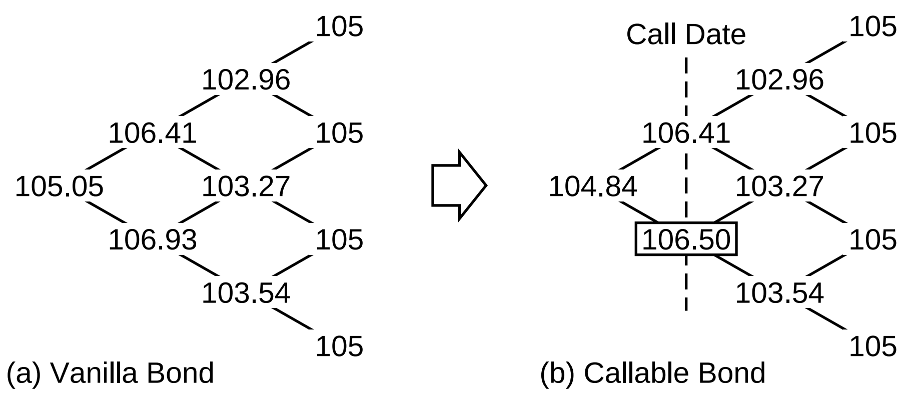
As an illustration of its application, let us use the calibrated BDT tree to price a simple fixed-coupon bond. Consider a bond with a 10% coupon, and nine months left to maturity. It can be represented by a three-month zero coupon bond with a USD 5 face value and a nine-month zero coupon bond with a USD 105 face value. As illustrated in Figure 5.4(b) and (c), we start from the maturity date and discount back cash flows according to the BDT short rates on each node. For example, in Figure 5.4(c), USD 102.96 is obtained by discounting USD 105 by 7.94%, while USD 101.41 is obtained by taking the average of USD 102.96 and USD 103.27 and discounting it by 6.74%. The price of the zero coupon bonds is obtained at the end of the discounting process when we reach time zero. Figure 5.4(d) shows the cash price of the bond is USD 105.05, which is the sum of the three-month zero coupon bond and nine-month zero zero coupon bond prices. ■

Example 5.13 Callable bond

A callable bond is a bond that the issuer can redeem before maturity on a specific date or set of dates (call dates), at specific prices (call prices).

For a callable bond, we need to check in the discounting process if a date corresponding to a node is a call date. If it is a call date and the future bond price on that node is greater than the call price, then we reset that price to the call price before continuing the discounting process.

Figure 5.5: Valuation of a callable bond with BDT tree



Consider a callable bond with the same specification as the plain vanilla bond in Example 5.12, except for a provision which allows the issuer to call the bond three months from now at a call price (clean price) of USD 101.50. We first need to convert this call price into a dirty price by adding the accrued interest of 5 USD (the full coupon because it falls on a coupon date), which means that the dirty price is USD 106.50. We then compare this call price to the prices on the three-month nodes. Since one of the prices (USD 106.93) is higher than the call price (USD 106.50), the bond on that node will be called, and as shown in Figure 5.5, this price is replaced by the call price. The cash price of the bond is then found to be USD 104.84. We can also conclude that the embedded call option is worth USD 0.21 today, which is the difference between the price of the plain vanilla and callable bonds. ■

In the parametric VaR calculation, where the derivatives of the price with respect to its underlying risk factors are required, trees can be used to compute these derivatives for complex instruments. In this case, a tree is first calibrated to current market condition and the price of the considered instrument is computed from this tree. Then a second tree is calibrated to a scenario with a small change in the risk factor. A new price of the instrument is computed from the second tree. Finally, the derivative is taken as the ratio of the price difference and the change of the risk factor.

Note that in Monte Carlo simulation, which involves the construction of interest rate trees, future scenarios of the term structure are generated using the methods described in Chapter 2. Then, in each scenario, the trees must be calibrated again to the generated term structure.

5.4.4 Analytic approximations

With the analytic methodology we covered so far, most options can be valued in closed form. For some of the exceptions whose exact analytic pricing formulas are not available, such as American options and average rate options, closed form pricing formulas can still be obtained after some analytic approximations.

Example 5.14 Average Price Option

An average price call option is a type of Asian option whose payoff is $\max(0, \bar{S} - X)$, where $\bar{S}$ is the average value of the underlying asset calculated over the life of the option, and X is the option strike price. If the underlying asset price S is assumed, as usual, to be lognormally distributed, the distribution of the arithmetic average $\bar{S}$ does not have an analytic form. However, the distribution is found to be approximately lognormal and good precision can be achieved by calculating the first two moments of $\bar{S}$ and using the standard risk-neutral valuation.[14] With the assumption that $\bar{S}$ is lognormally distributed, we can approximate the call on $\bar{S}$ as a call option on a futures contract.

Note that there is more than one method for approximating the price of an arithmetic average price option. For example, Curran's approximation adds accuracy and flexibility in handling multiple averaging frequencies.[15]
■

In Monte Carlo VaR calculations for instruments that have costly pricing functions, such as the callable bonds discussed above, quadratic approximations can be applied to vastly improve the performance. The standard simplification of complicated pricing functions is to approximate them by a second order Taylor expansion. However, Taylor series expansions are only accurate for small changes in the risk factors, and since VaR is concerned with large moves, the use of such approximations can lead to very large errors. The new method is instead based on fitting a quadratic function to the true pricing function for a wide range of underlying risk factor values. This approximation fits the range of interest rates using least square fitting techniques, and ensures a high accuracy for VaR calculations based on the approximate pricing function.

Example 5.15 Quadratic approximation of callable bonds

The valuation of complex derivative exposures can be a computationally intensive task. For risk management purposes, and particularly for VaR, we need hundreds of valuations to obtain the P&L distribution of a complex instrument. It is often the case that closed form solutions do not exist, forcing the use of more expensive numerical methods such as Monte Carlo simulation and finite difference schemes. As shown in Example 5.13,

[14] See Turnbull and Wakeman (1991).
[15] See Curran (1992).