FAST SWAPTION PRICING UNDER A MARKET MODEL WITH STOCHASTIC VOLATILITY

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Abstract. In this paper we study a LIBOR market model with a volatility multiplier, which follows a square-root process. This model captures downward volatility skews through using negative correlations between forward rates and the multiplier. Approximate pricing formula is developed for swaptions, and the formula is implemented via fast Fourier transform. Numerical results on pricing accuracy are presented, which support the approximations made in deriving the formula.

Key words: LIBOR model, stochastic volatility, square-root process, swaptions, Fast Fourier transform (FFT)

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1. Introduction

This paper introduces a LIBOR market model with stochastic volatility. Over the past a few years, standard market model (Brace, Gatarek and Musiela, 1997; Jamshidian, 1997; and Miltersen, Sandmann and Sondermann, 1997) has established itself as the benchmark model for interest-rate derivatives. Several virtues of the market model are responsible for its popularity. First, state variables of the model, LIBOR, are directly observable. Second, the model justifies the use of Black's formula for caplets and even swaptions (so-called the benchmark derivative instruments). Third, as a multi-factor model, the market model can conveniently incorporate exogenous forward-rate correlations. The closed-form pricing of the benchmark derivatives enables efficient calibration of the model, making it possible to implement the model in real time. Nonetheless, the standard market model suffers from insufficient capacity: it can not generate implied volatility smiles or skews[1] for those benchmark derivatives. The phenomenon of smiles/skews is a very pronounced reality of LIBOR markets. For more consistent pricing and more effective hedging, market participants have been interested in extensions of the standard model that address, in particular, the issue of volatility smiles and skews.

Extensions to the standard market model have been made largely through adding at least one of the following features or ingredients: level-dependent volatilities, stochastic volatilities, and jumps. Andersen and Andreasen (2000) adopt constant-elasticity-variance (CEV) processes, which generates volatility skews by taking non-1 elasticity parameters. On top of the CEV model, Andersen and Brotherton-Ractliffe (2001) superimpose a stochastic volatility dynamics, which effectively produces additional curvature to the otherwise monotonic volatility skews. Zhou (2003) develops a theory parallel to those of Andersen, Andreasen and Brotherton-Ractliffe, using some unconventional specifications of volatility processes. With these models, swaptions can be priced in closed-form or, in special cases, Fourier transformation method (Andersen and Andreasen, 2002). In the other line of research, Glasserman and Kou (2003) develop a comprehensive term structure theory with the jump-diffusion dynamics. Based on their theory, Glasserman and Merener (2003) derive approximate closed-form formulae for caplets and swaptions. Parallel theories were extended to a LIBOR model based on general Lévy processes by Eberlein and Ožkan (2004). Jarrow, Li and Zhao (2003) combine the two types of models, and examine empirically the model's

[1]In literature both "skew" and "smirk" are used to name a slanted smile.

ability to fit actual volatility smiles/skews. Other interesting extensions include a model based on displaced-diffusion (DD) processes (Joshi and Rebonato, 2001), and a model based on mixed lognormal densities for LIBOR (Brigo and Mercurio, 2003).

In this paper we introduce a correlation-based model for LIBOR. This development is motivated by the belief that "leveraging effect" [2] (Black, 1976) may be behind the volatility smiles and skews in the LIBOR markets. Specifically, we adopt a set-up similar to that of Andersen and Brotherton-Ratcliffe (2001), yet, contrary to it, we exclude state dependent diffusions but include correlations between forward rates and a stochastic multiplier (hereafter *rate - multiplier correlations*). The model so developed can be regarded as the LIBOR version of Heston's model (1993), the most popular equity option model with stochastic-volatility[3]. Several other reasons are also behind this extension. First, it is observed that time series of interest rates display statistical properties similar to those of equities (as well as other financial time series) (Chen and Scott, 2001). A major property is leptokurtic feature[4], which can be captured by the Heston's type models. Second, among stochastic volatility models (see e.g. Lewis, 2000), the Heston's model carries nice analytical tractability that renders closed-form option formula. Third and most importantly, Heston's model establishes a direct correspondence between a downward skew to a negative correlation between (the change of) the state variable and its stochastic volatility[5]. Ironically, such correlation might have been deemed counter-productive for the Heston's type model in the context of LIBOR, as under forward measures it causes dependence of volatility process on forward rates, a circular dependence that undermines analytical tractability. A key question for developing a Heston's type model for LIBOR is whether it is possible to relegate such dependence without sacrificing accuracy.

The answer to the above question is positive. Inspiring all technical build-up of this paper is the crucial insight that, under its corresponding forward swap measure, a swap-rate process retains its Heston's type, with however state-dependent coefficients for its volatility process, yet the time variability of those coefficients is small. This insight has led to the use of "frozen coefficients" to get rid of the circular dependance, and the consequence is "closed-form" swaption pricing. The theory of this paper evolves as follows. Starting from a parsimonious specification of risk neutralized forward-rate processes, we derive forward-rate

[2] It usually means the negative correlation between a state variable and its volatility.

[3] A lognormal process whose volatility follows a square-root process (Cox *et al*; 1985).

[4] Higher peak and fatter tails than that of a normal distribution.

[5] Such a correspondence in fact is well known to exist in stochastic volatility models, e.g. Zhou (2003).

and swap-rate processes under their corresponding forward measures and forward swap measures, respectively. Through selectively freezing coefficients, according to time variability, we end up with approximate processes of the Heston's type. For a Heston's processes there exists moment generating function in closed form, and in term of which we express the Laplace transform of a swaption. The swaption price then follows from an inverse Laplace transform, and the latter is evaluated via fast Fourier transform (FFT). Through pricing comparisons we confirm the remarkable accuracy of the transformation method, and, unambiguously, we demonstrate the "leveraging effect" on volatility smiles and skews.

To a large extent, the theory developed in this paper is a product of re-engineering guided by a valuable insight. It has made use of a number of models or techniques as building blocks, especially the Heston's model (1993) for equity option, the volatility modeling technique of Chen and Scott (2001) and Andersen and Brotherton-Ractliffe (2001), the swap-rate process approximation under the standard market model of Andersen and Andreasen (2000), and the FFT evaluation technique of Carr and Madan (1998). The new model reinstates the role of the rate - factor correlations in the formation of volatility smiles/skews, which is appealing in finance. In addition, the new model lends itself for further extensions, for example, to incorporate jump risk in state variables.

The remaining part of the paper is organized as follows. In section 2 we set up the LIBOR market model with stochastic volatility, and develop an approximate caplet formula following Heston's approach. Section 3 is for swaption pricing, where we introduce necessary treatments/approximations to retain analytical tractability, present analytical moment-generating function for piecewise constant model parameters, and describe a transformation method for numerical option valuation. In section 4 we make pricing comparisons between our transformation method and Monte Carlo simulation method, and demonstrate the correspondence between the rate - multiplier correlations and the skews. Finally in section 5 we conclude. Most technical details are put in the appendix.

2. The Market model with stochastic volatility

The derivation of our market model with stochastic volatility starts from the price process of zero-coupon Treasury bonds. Let $P(t,T)$ be the zero-coupon Treasury bond maturing at $T(\geq t)$ with par value \$1. Without loss of generality, we assume the risk-neutralized process for $P(t,T)$ to be

$$dP(t,T) = P(t,T)\left[r_t dt + \sigma(t,T)\cdot d\mathbf{Z}_t\right], \tag{1}$$

where r_t is the stochastic risk-free rate, $\sigma(t,T)$ is the volatility vector of $P(t,T)$, and $\mathbf{Z}_t$ is a finite dimensional vector of independent Brownian motions under the risk-neutral measure, which we denote by $\mathbb{Q}$, and "$\cdot$" is the usual vector product. Let $f_j(t) = f(t;T_j,T_{j+1})$ be the arbitrage-free forward lending rate seen at time t for the period (T_j,T_{j+1}), which is an observable and tradable quantity in the interest-rate markets (through e.g. a *forward-rate agreement*). The forward term rate relates to zero-coupon bond prices by

$$f_j(t) = \frac{1}{T_{j+1}-T_j}\left(\frac{P(t,T_j)}{P(t,T_{j+1})}-1\right).$$

As a function of two zero-coupon bonds, the dynamics of the forward term rate is determined by that of zero-coupon bonds. Using Ito's lemma we can derive

$$df_j(t) = f_j(t)\gamma_j(t)\cdot[d\mathbf{Z}_t - \sigma(t,T_{j+1})dt], \qquad 1\leq j\leq N. \tag{2}$$

Here $\gamma_j(t)$, intuitively regarded as the volatility vector of $f_j(t)$, is a function of zero-coupon bond volatilities:

$$\gamma_j(t) = \frac{1+\Delta T_j f_j(t)}{\Delta T_j f_j(t)}[\sigma(t,T_j)-\sigma(t,T_{j+1})], \tag{3}$$

where $\Delta T_j = T_{j+1}-T_j$. This dependence relationship can be viewed conversely. In fact, if one instead begins with prescribing the volatilities of the forward rates, then the volatilities of the zero-coupon bonds follow from

$$\sigma(t,T_{j+1}) = -\sum_{k=\eta(t)}^{j}\frac{\Delta T_k f_k(t)}{1+\Delta T_k f_k(t)}\gamma_k(t) + \sigma(t,T_{\eta(t)}), \tag{4}$$

where $\eta(t)$ is the smallest integer such that $T_{\eta(t)}\geq t$. For $\gamma_j(t)$ satisfying usual regularity conditions, it is proved (Brace *et al*, 1997) that $f_j(t)$ does not blow up, and one can put $\sigma(t,T_{\eta(t)}) = 0$ for $T_{\eta(t)-1}\leq t\leq T_{\eta(t)}$ without causing a trouble. Equations (2,3) constitutes the so-called *market model* of interest rates. Roughly speaking, the stochastic evolution of the N forward rates is governed by their covariance defined by

$$Cov^i_{jk} = \int_{T_{i-1}}^{T_i}\gamma_j(t)\cdot\gamma_k(t)dt, \qquad i\leq j,k\leq N, \quad 1\leq i\leq N. \tag{5}$$

Note that $\gamma_j(t)=0$ for $t\geq T_j$ since f_j is fixed from the time T_j becomes "dead". The market model has the capacity to build in desirable correlation structure between the forward rates (See Wu (2003) for instance).

To accommodate volatility smiles/skews, we, following Chen and Scott (2001) and Andersen and Brotherton-Ratcliffe (2001), adopt a stochastic multiplier to the risk neutralized processes of the forward rates:

$$\begin{aligned} df_j(t) &= f_j(t)\sqrt{V(t)}\gamma_j(t) \cdot \left[d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt\right], \\ dV(t) &= \kappa(\theta - V(t))dt + \epsilon\sqrt{V(t)}dW_t. \end{aligned} \tag{6}$$

Here, κ, θ and ϵ are state-independent variables[6], and W_t is an additional 1-D Brownian motion under the risk-neutral measure. As a distinct feature of our modeling, we allow correlations between the stochastic multiplier and forward rates:

$$E^Q\left[\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\mathbf{Z}_t\right) \cdot dW_t\right] = \rho_j(t)dt, \quad \text{with} \quad |\rho_j(t)| \leq 1. \tag{7}$$

Here, $\left(\frac{\gamma_j(t)}{\|\gamma_j(t)\|} \cdot d\mathbf{Z}_t\right)$ is equivalent to a single Brownian motion that drives $f_j(t)$. Note that for the model above, (3) remains the arbitrage-free condition.

Introducing the rate - multiplier correlations is motivated by the belief that implied volatility smiles/skews in the interest-rate options can also be attributed to the "leveraging effect", analogous to the situation in the equity options[7]. Technically, adopting a uniform volatility multiplier for all rates rather than one multiplier for each rate renders great advantages for analytical swaption pricing, and in addition, has very positive implications for model calibration.

We now address caplet pricing under the extended LIBOR model (6). A caplet is a call option on a forward rate. Assume that the notional value of a caplet is one dollar, then the payoff of the caplet on $f_j(T_j)$ is

$$\Delta T_j(f_j(T_j) - K)^+ \triangleq \Delta T_j \max\{f_j(T_j) - K, 0\},$$

To price the caplet we choose, in particular, $P(t, T_{j+1})$ to be the numeraire, and let $\mathbb{Q}^{j+1}$ denote the corresponding forward measure (i.e. the martingale measure corresponding to $P(t, T_{j+1})$). The next proposition establishes the relationship between Brownian motions under the risk-neutral and under the forward measures.

[6]The distributional properties of $V(t)$ are well understood (e.g., Avellaneda and Laurence, 2000). When $2\kappa\theta > \epsilon^2$, in particular, $V(t)$ has a stationary distribution and stays strictly positive.

[7]The empirical results of Chen and Scott (2001) suggest zero rate - multiplier correlation only for the nearest-term forward rate. In early versions of this paper, we had included plots for implied caplet volatilities of USD for the date of July 5, 2002. While the implied volatility curve of longer maturities appear like downward skews, the implied volatility curve of the six-month caplets is a smile, which is consistent to the claim of independence. The plots are omitted for brevity.

Proposition 2.1. *Let $\mathbf{Z}_t$ and W_t be Brownian motions under $\mathbb{Q}$, then $\mathbf{Z}_t^{j+1}$ and W_t^{j+1}, defined by*

$$\begin{aligned} d\mathbf{Z}_t^{j+1} &= d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt, \\ dW_t^{j+1} &= dW_t + \xi_j(t)\sqrt{V(t)}dt, \end{aligned} \tag{8}$$

are Brownian motions under $\mathbb{Q}^{j+1}$, where

$$\xi_j(t) = \sum_{k=1}^{j} \frac{\Delta T_k f_k(t)\rho_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)} \quad \square$$

In terms of $\mathbf{Z}_t^{j+1}$ and W_t^{j+1}, the extended market model (6) becomes

$$df_j(t) = f_j(t)\sqrt{V(t)}\gamma_j(t)\cdot d\mathbf{Z}_t^{j+1}, \tag{9}$$

$$dV(t) = [\kappa\theta - (\kappa + \epsilon\xi_j(t))V(t)]dt + \epsilon\sqrt{V(t)}dW_t^{j+1}. \tag{10}$$

In formalism, the multiplier process remains a square-root process under $\mathbb{Q}^{j+1}$. Yet $\xi_j(t)$ depends on forward rates, and such dependence prohibits analytical option valuation. The time variability of $\xi_j(t)$, however, is small. In fact,

$$\begin{aligned} \xi_j(t) = \sum_{k=1}^{j} \frac{\Delta T_k f_k(0)\rho_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(0)} &+ \frac{\rho_k(t)\|\gamma_k(t)\|\Delta T_k}{(1+\Delta T_k f_k(0))^2}(f_k(t)-f_k(0)) \\ &+ O(\rho_k(t)\|\gamma_k(t)\|\Delta T_k^2(f_k(t)-f_k(0))^2). \end{aligned} \tag{11}$$

In light of the martingale property $E^{Q_{j+1}}[f_j(t)|\mathcal{F}_0] = f_j(0)$, we see that

$$\begin{aligned} E^{Q^{j+1}}[\xi_j(t)|\mathcal{F}_0] &= \sum_{k=1}^{j} \frac{\Delta T_k f_k(0)\rho_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(0)} + O(\rho_k(t)\|\gamma_k(t)\| \; Var(\Delta T_k f_k(t))) \\ Var(\xi_j(t)|\mathcal{F}_0) &\approx (\rho_k(t)\|\gamma_k(t)\|)^2 \; Var(\Delta T_k f_k(t)). \end{aligned}$$

According to the model, $Var(\Delta T_k f_k(t)) \sim \Delta T_k f_k^2(t)\|\gamma_k(t)\|^2 V(t)t$. Since $\Delta T_k f_k(t)$ is mostly under 5%, the expansion in (11) is dominated by the first term. Hence, to remove the dependence of $V(t)$ on $f_j(t)$'s, we choose to ignore higher order terms in (11) and consider the approximation

$$\xi_j(t) \approx \sum_{k=1}^{j} \frac{\Delta T_k f_k(0)\rho_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(0)}. \tag{12}$$

This is close to the technique of "freezing coefficients". For notational simplicity we denote

$$\tilde{\xi}_j(t) = 1 + \frac{\epsilon}{\kappa}\xi_j(t)$$

and thus retain a neat equation for the process of $V(t)$:

$$dV(t) = \kappa[\theta - \tilde{\xi}_j(t)V(t)]dt + \epsilon\sqrt{V(t)}dW_t^{j+1}. \tag{13}$$

For the processes joint by (9) and (13), caplet pricing can be achieved along the approach pioneered by Heston (1993). According to arbitrage pricing theory (APT) (Harrison and Pliska, 1981), the price the caplet on $f_j(T_j)$ can be expressed as

$$\begin{aligned} C_{let}(0) &= P(0,T_{j+1})\Delta T_j E^{Q^{j+1}}\left[(f_j(T_j)-K)^+|\mathcal{F}_0\right] \\ &= P(0,T_{j+1})\Delta T_j f_j(0)\left(E^{Q^{j+1}}\left[e^{X(T_j)}\mathbf{1}_{X(T_j)>k}|\mathcal{F}_0\right]-e^k E_0^{Q^{j+1}}\left[\mathbf{1}_{X(T_j)>k}|\mathcal{F}_0\right]\right), \end{aligned}$$

where $X(t)=\ln f_j(t)/f_j(0)$ and $k=\ln K/f_j(0)$. The two expectations above can be valuated using the moment generating function of $X(T_j)$, defined by

$$\phi(X(t),V(t),t;z)\triangleq E\left[e^{zX(T_j)}|\mathcal{F}_t\right], \quad z\in C.$$

In terms of $\phi_T(z)\triangleq\phi(0,V(0),0;z)$, it is shown that (e.g. Kendall (1994) or more recently Duffie, Pan and Singleton (2000))

$$\begin{aligned} E^{Q^{j+1}}\left[\mathbf{1}_{X(T_j)>k}|\mathcal{F}_0\right] &= \frac{\phi_T(0)}{2}+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\{e^{-iuk}\phi_T(iu)\}}{u}du, \\ E^{Q^{j+1}}\left[e^{X(T_j)}\mathbf{1}_{X(T_j)>k}|\mathcal{F}_0\right] &= \frac{\phi_T(1)}{2}+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}\{e^{-iuk}\phi_T(1+iu)\}}{u}du. \end{aligned} \tag{14}$$

The integrals above can be evaluated numerically. For later reference we call this approach Heston method.

When the Brownian motions $\mathbf{Z}_t^{j+1}$ and W_t^{j+1} are independent, one can work out the moment generating function directly. In general, one can solve for $\phi(x,V,t;z)$ from the Komogrov backward equation corresponding to the joint processes:

$$\begin{aligned} &\frac{\partial\phi}{\partial t}+\kappa(\theta-\tilde{\xi}_jV)\frac{\partial\phi}{\partial V}-\frac{1}{2}\|\gamma_j(t)\|^2V\frac{\partial\phi}{\partial x} \\ &\qquad +\frac{1}{2}\epsilon^2V\frac{\partial^2\phi}{\partial V^2}+\epsilon\rho_jV\|\gamma_j(t)\|\frac{\partial^2\phi}{\partial V\partial x}+\frac{1}{2}\|\gamma_j(t)\|^2V\frac{\partial^2\phi}{\partial x^2}=0, \end{aligned} \tag{15}$$

subject to terminal condition

$$\phi(x,V,T_j;z)=e^{zx}. \tag{16}$$

It is known that the solution is of the form

$$\phi(x,V,t;z)=e^{A(t,z)+B(t,z)V+zx}, \tag{17}$$

and A and B are available analytically for constant coefficients (Heston, 1993). The analytical solutions can be extended to the case of piece-wise coefficients through recursion[8]. The proof of the next proposition is provided in appendix for completeness.

[8]This is also suggested in Andersen and Andreasen (2002).

Proposition 2.2. *For piece-wise constant coefficients, A and B are given by recursive expressions*

$$\begin{cases} A(t,z) = A(T_k,z) + \dfrac{\kappa\theta}{\epsilon^2}\left\{(a_k+d_k)(T_k-t) - 2\ln\left[\dfrac{1-g_k e^{d_k(T_k-t)}}{1-g_k}\right]\right\}, \\ B(t,z) = B(T_k,z) + \dfrac{(a_k+d_k-\epsilon^2 B(T_k,z))(1-e^{d_k(T_k-t)})}{\epsilon^2(1-g_k e^{d_k(T_k-t)})}, \\ \qquad \text{for } \; T_{k-1} \le t < T_k, \quad k = j, j-1, \ldots, 1, \end{cases} \tag{18}$$

where

$$a_k = \kappa\xi - \rho_j(T_k)\epsilon\|\gamma_j(T_k)\|z, \quad d_k = \sqrt{a^2 - \|\gamma_j(T_k)\|^2\epsilon^2(z^2-z)}, \quad g_k = \frac{a_k+d_k-\epsilon^2 B(T_k,z)}{a_k-d_k-\epsilon^2 B(T_k,z)}.$$

3. Swaption Pricing

The equilibrium swap rate for a period (T_m, T_n) is defined by

$$R_{m,n}(t) = \frac{P(t,T_m) - P(t,T_n)}{B^S(t)},$$

where

$$B^S(t) = \sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1})$$

is an annuity. The payoff of a swaption on $R_{m,n}(T_m)$ can be expressed as

$$B^S(T_m) \cdot \max(R_{m,n}(T_m) - K, 0),$$

where K is the strike rate.

The swap rate can be regarded as the price of a tradable portfolio, consisting of one long T_m-maturity zero-coupon bond and one short T_n-maturity zero-coupon bond, measured by the annuity $B^S(t)$. According to APT, the swap rate is a martingale under the measure corresponding to the numeraire $B^S(t)$. This measure is called the *forward swap measure* (Jamshidian, 1997) and is denoted by $\mathbb{Q}^S$ in this paper. Similar to pricing under a forward measure, we need to characterize the Brownian motions under the forward swap measure.

Proposition 3.1. *Let $\mathbf{Z}_t$ and W_t be Brownian motions under $\mathbb{Q}$, then $\mathbf{Z}_t^S$ and W_t^S, defined by*

$$\begin{aligned} d\mathbf{Z}_t^S &= d\mathbf{Z}_t - \sqrt{V(t)}\sigma^S(t)dt, \\ dW_t^S &= dW_t + \sqrt{V(t)}\xi^S(t)dt \end{aligned} \tag{19}$$

are Brownian motions under $\mathbb{Q}^S$*, where*

$$\sigma^S(t) = \sum_{j=m}^{n-1} \alpha_j \sigma(t, T_{j+1}), \qquad \xi^S(t) = \sum_{j=m}^{n-1} \alpha_j \xi_j, \tag{20}$$

with weights

$$\alpha_j = \alpha_j(t) = \frac{\Delta T_j P(t, T_{j+1})}{B^S(t)} \quad \square$$

Using Ito's lemma one can show that, under the forward swap measure, the swap rate process becomes

$$\begin{aligned} dR_{m,n}(t) &= \sqrt{V(t)} \sum_{j=m}^{n-1} \frac{\partial R_{m,n}(t)}{\partial f_j(t)} f_j(t) \gamma_j(t) \cdot d\mathbf{Z}^S(t), \\ dV(t) &= \kappa[\theta - \tilde{\xi}^S(t) V(t)]dt + \epsilon \sqrt{V(t)} dW^S(t), \end{aligned} \tag{21}$$

Here

$$\tilde{\xi}^S(t) = 1 + \frac{\epsilon}{\kappa} \xi^S(t).$$

For the partial derivatives of the swap rate with respect to forward rates, we have

Proposition 3.2. *Let*

$$R_{m,n}(t) = \sum_{k=m}^{n-1} \alpha_k f_k, \quad \alpha_k = \frac{\Delta T_k P(t, T_{k+1})}{B^S(t)},$$

there is

$$\frac{\partial R_{m,n}(t)}{\partial f_j(t)} = \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j(t)} \left[\sum_{l=m}^{j-1} \alpha_l (f_l - R_{m,n}(t)) \right], \quad m \le j \le n-1 \quad \square$$

Parallel to swaption pricing under the standard market model (e.g. Sidennius, 2000; Andersen and Andreasen, 2000), we approximate the swap rate process by a lognormal distribution with a stochastic volatility:

$$\begin{aligned} dR_{m,n}(t) &= R_{m,n}(t) \sqrt{V(t)} \sum_{j=m}^{n-1} w_j(0) \gamma_j(t) \cdot d\mathbf{Z}^S(t), \qquad 0 \le t < T_m, \\ dV(t) &= \kappa[\theta - \tilde{\xi}_0^S(t) V(t)]dt + \epsilon \sqrt{V(t)} dW^S(t), \end{aligned} \tag{22}$$

where

$$\begin{aligned} w_j(t) &= \frac{\partial R_{m,n}(t)}{\partial f_j} \frac{f_j(t)}{R_{m,n}(t)}, \\ \xi_0^S(t) &= \sum_{j=m}^{n-1} \alpha_j(0) \xi_j(t). \end{aligned}$$

In above approximations, we have removed the dependence of $\xi_0^S(t)$ on forward rates through taking full advantage of the negligible time variability of $w_j(t)$ and $\alpha_j(t)$ (in comparing with that of forward rates). The approximate swap-rate process has moment generating function in closed form, analogous to those of forward rates. In fact, when $n = m + 1$, $R_{m,m+1}(t) = f_m(t)$ and $B^S(t) = \Delta T_m P(t, T_{m+1})$, i.e., the swap rate reduces to a forward rate, and the swaption reduces to a caplet[9]. Theoretically, we can treat a caplet as a special case of swaptions.

Instead of taking the Heston's approach, we adopt a transformation method developed by Carr and Madan (1998). Under the forward swap measure, there is the following expression for swaptions

$$PS(0) = B^S(0)R_{m,n}(0)E^S\left[\left(e^{X(T_m)} - e^k\right)^+ |\mathcal{F}_0\right], \tag{23}$$

where $E^S[\cdot]$ stands for expectation under the forward swap measure, $X(T_m) = \ln R_{m,n}(T_m)/R_{m,n}(0)$ and $k = \ln K/R_{m,n}(0)$. Let $G(k) = E^S\left[\left(e^{X(T_m)} - e^k\right)^+ |\mathcal{F}_0\right]$. Carr and Madan (1998) link up the Laplace transform of $G(k)$ with the moment-generating function of $X(T_m)$:

$$\psi(u) = \frac{\phi_{T_m}(1 + a + iu)}{(a + iu)(1 + a + iu)} \quad \text{for } a > 0,$$

where $\phi_{T_m}(v) = \phi(0, V(0), 0, v)$, characterized in (17) and detailed in Proposition 2.2 (with m, ρ^S and $\gamma_{m,n}$ in places of j, ρ_j and γ_j). The swaption price follows from an inverse Laplace transform

$$G_T(k) = \frac{exp(-ak)}{\pi}\int_0^\infty e^{-iuk}\psi(u)du, \tag{24}$$

which is evaluated numerically using FFT. Details are refereed to Carr and Madan (1998). For easy reference we call this method the FFT method.

A rigorous error analysis of the lognormal approximation poses an open challenge. For the standard market model, a recent paper by Brigo et. al. (2004) studies the quality of the approximation using entropy distance. The analysis seems applicable to the approximation (22) for the case of zero rate - multiplier correlation. In the paper, we will examine the validity of the lognormal approximation from the perspective of swaption pricing accuracy.

We finish the section with comments on calibration. First of all, it is favorable to decouple estimating the multiplier process from estimating the forward-rate processes. Estimating the multiplier process can be achieved using the time series data of implied Black's volatilities of at-the-money caplets, as is suggested in Chen and Scott (2001). Once the

[9] For convenience we have taken the same ΔT for both caps and swaptions. Note that in reality caps and swaptions can have different interval between cash flows. In such case, we may take the smallest interval for ΔT.

process for $V(t)$ is specified, we can proceed to determine the pair of $(\|\gamma_j\|, \rho_j)$ through calibrating $f_j(t)$ to caplet smile or skew of maturity T_j. This leads to a bi-variate optimization problem, which should be quite manageable. In addition, according to Figure 14 and 15 in section 4, we may take the implied Black's volatility of the at-the-money (ATM) option as an initial guess for $\|\gamma_j\|$, and start an iterative process to solve for ρ_j and $\|\gamma_j\|$, alternatively. Once ρ_j's are obtained, we can proceed to calibrate the model to ATM swaptions by taking time-dependent $\|\gamma_j\|$'s. If one want to calibrate to swaption smiles/skews, then he/she may have to let ρ_j's be time dependent as well. This is likely to result in a mid-scale optimization problem. Depending on applications, the difficulty sometimes may be alleviated. For example, when we calibrate the model only to swaptions, we may consider the approximation

$$\rho^S \approx \sum_{j=m}^{n-1} w_j(0)\rho_j,$$

which effectively replaces the weights $\{w_j(0)\|\gamma_j(t)\|/\|\gamma_{m,n}(t)\|\}$ by $\{w_j(0)\}$, and thus remove $\|\gamma_j\|$'s explicitly from the calibration procedure. Given, in addition, forward-rate correlations exogenously, we can solve for $\|\gamma_j\|$'s from the already obtained $\|\gamma_{m,n}\|$'s. The components of γ_j are determined by matching model correlations to the input correlations of forward rates, using the technique developed by Wu (2003). Specifically, taking the rate - multiplier correlations into account, we can derive the following equation for $(\gamma_j/\|\gamma_j\|)$'s,

$$(1-\rho^2(t_i))\left(\frac{\gamma_j}{\|\gamma_j\|}\right)\cdot\left(\frac{\gamma_k}{\|\gamma_k\|}\right)+\rho^2(t_i)=C_{jk}(t_i).$$

where C_{jk} is the correlation between the time series data of f_j and f_k. The existence of $(\gamma_j/\|\gamma_j\|)$'s requires that the matrix with components

$$\frac{C_{jk}(t_i)-\rho^2(t_i)}{1-\rho^2(t_i)}, \quad i \le j\wedge k,$$

be non-negative definite. In our opinion, this constraint is by no means excessive. Instead, it reflects part of the reality of the market.

4. Numerical Results

In this section we present results on swaption pricing using the FFT method. Under scrutiny are two issues: pricing accuracy and capability to generate volatility smiles and skews. For given forward-rate and multiplier processes, we compute swaption prices (including caplets as special cases) by both FFT method and Monte Carlo (MC) simulation method, and then examine their differences. In addition, we will check on the accuracy of the FFT method under both weak and strong effects of stochastic volatility. As will be seen, the differences in implied volatilities are mostly under 1%, suggesting the soundness of the approximations taken by the FFT method.

Let us briefly describe the Monte Carlo simulation method for the extended market model. The MC method is implemented under the risk neutral measure. To build in the correlation between the forward rates and the stochastic factor, we recast the equation for the forward rates into

$$\frac{df_j(t)}{f_j(t)} = -V(t)\gamma_j(t)\cdot\sigma(t,T_{j+1})dt + \sqrt{V(t)}\left(\sqrt{1-\rho_j^2(t)}\gamma_j(t)\cdot d\hat{\mathbf{Z}}_t + \rho_j(t)\|\gamma_j(t)\|_2 dW_t\right), \tag{25}$$

where $(\hat{\mathbf{Z}}_t, W_t)$ is a vector of independent Brownian motions. Treated as a lognormal process, $f_j(t)$ is advanced by the so-called *log-Euler* scheme:

$$f_j(t+\Delta t) = f_j(t)e^{-V(t)(\gamma_j(t)\cdot\sigma(t,T_{j+1})+\frac{1}{2}\|\gamma_j(t)\|^2)\Delta t+\sqrt{V(t)}\left(\sqrt{1-\rho_j^2(t)}\gamma_j(t)\cdot\Delta\hat{\mathbf{Z}}_t+\rho_j(t)\|\gamma_j(t)\|_2\Delta W_t\right)}.$$

The evolution of volatility, meanwhile, takes a step-wise moment-matched log-normal scheme (Andersen and Brotherton-Ratecliffe, 2001). The use of the lognormal time-stepping scheme avoids the possible breakdown of an Euler type scheme (e.g. Kloeden and Platen, 1992) for the square-root process, but it is less straightforward and not too much is known regarding the order of accuracy of the scheme. In order to achieve higher accuracy, we have taken a small time-step size ($\Delta t = 1/12$) and a large number of paths (100,000) for the simulation method.

Example 1: The term structure of interest-rates, forward-rate processes and multiplier process are described below.

- Spot forward-rate curve : $\Delta T_j = 0.5$, $f_j(0) = 0.04 + 0.00075j$, for all j.
- Volatility term structure of a two-factor model

$$\gamma_j(T_k) = (0.08 + 0.1e^{-0.05(j-k)}, 0.1 - 0.25e^{-0.1(j-k)}), \quad k \le j.$$

- Multiplier process: $V(0) = \theta = \kappa = 1$ and $\epsilon = 1.5$.
- Rate - multiplier correlations: $\rho_j = -0.5$ or 0 for all j.

This volatility term structure corresponds to a short rate volatility of about 25%. The initial interest-rate term structure and multiplier dynamics are taken from Andersen and Brotherton-Ratecliffe (2001), which may have a practical background. In this example, $\kappa = 1$ corresponds to a half life of mean reversion equal to $\ln(2)/\kappa = 0.69$, which represents a strong mean revision that results in a weak effect of stochastic volatility for long horizons.

Figure 1 - 6 display the implied Black's volatilities of swaptions, where "x" is for the FFT method and "o" is for the MC method. It can be seen that for at-the-money and out-of-the-money swaptions, the two sets of implied volatilities are indistinguishable. As the strike goes deep into the money, however, such indistinguishness is gradually lost, and it appears that the problem is in the MC method. To get a complete picture of pricing accuracy, we also take a look at dollar prices. Table 3a and 3b detail swaption prices in basis points (bps) for the case $\rho = -0.5$. Also included in the tables are implied Black's volatilities, the difference between the implied volatilities[10], and the radius (or half of the width) of 95% confidence interval (CI) for the Monte Carlo prices. These quantities are presented for pairs of maturity and strike under the following format:

Table 1. The ordering of cell entries

	Strike	
Maturity	FFT Price	(Implied Vol.)
	MC Price	(Implied Vol.)
	Radius of 95% CI	(Difference of implied volatilities.)

From the tables one can see that, for deeply in-the-money swaptions, the percentage differences in dollar prices are in the magnitude of 0.1%, even smaller than those of the at-the-money swaptions (with exceptions amongst in-10-to-10 swaptions). Based on the very small price differences, we believe that the poor display of the implied volatilities by the MC method is attributed to the high sensitivity of implied volatilities to roundoff errors for small strikes. The bigger errors in the few swaptions with long maturity plus long tenor may just reflect the approximation nature of both FFT and MC method. We want to draw attentions to the fact that, as maturity gets longer, the smiles and skews become flattened. This is caused by the weakening stochasticity of volatilities. Note that on Monte Carlo simulations, we have tried a hybrid scheme out of an Euler method and the lognormal moment-matching scheme, and we have also applied antithetic variates technique (e.g. Boyle *et al*, 1997), yet only to find no significant improvement.

For the FFT method we have taken dampening parameter $\alpha = 2$, truncation range $A = 50$, and number of divisions $N = 100$. This selection was made after several trials. Figure 7 and 8 display the real and imaginary parts of $\psi_T(u)$ for the in-1-to-1 swaption with

[10] We use "—" to indicate irrelevance when an implied volatility by the MC method is zero.

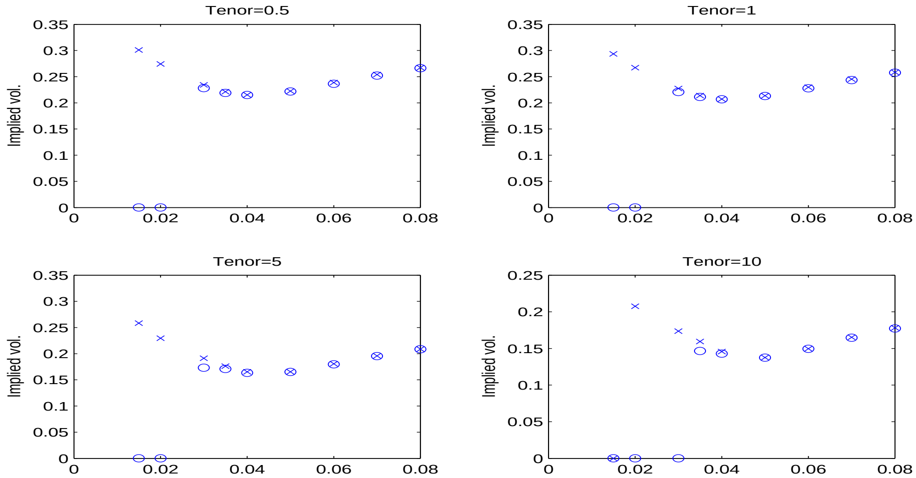

FIGURE 1. Implied volatilities of 1-year swaptions.

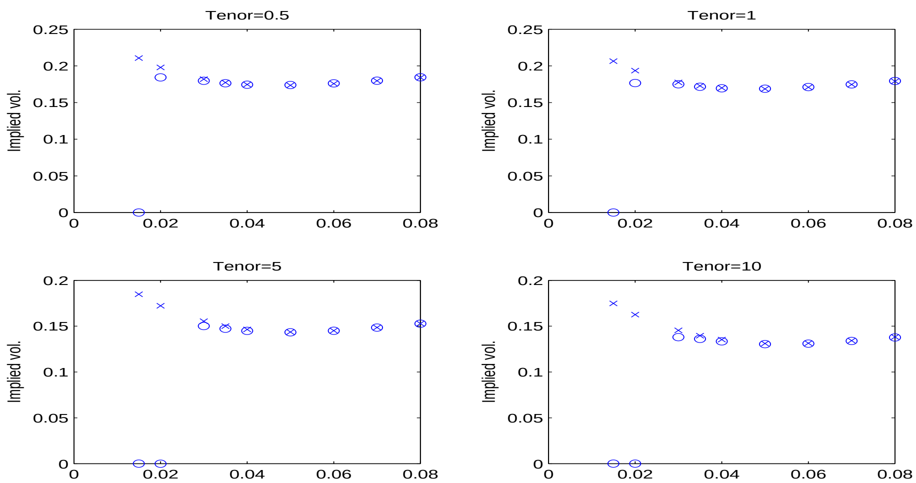

FIGURE 2. Implied volatilities of 5-year swaptions.

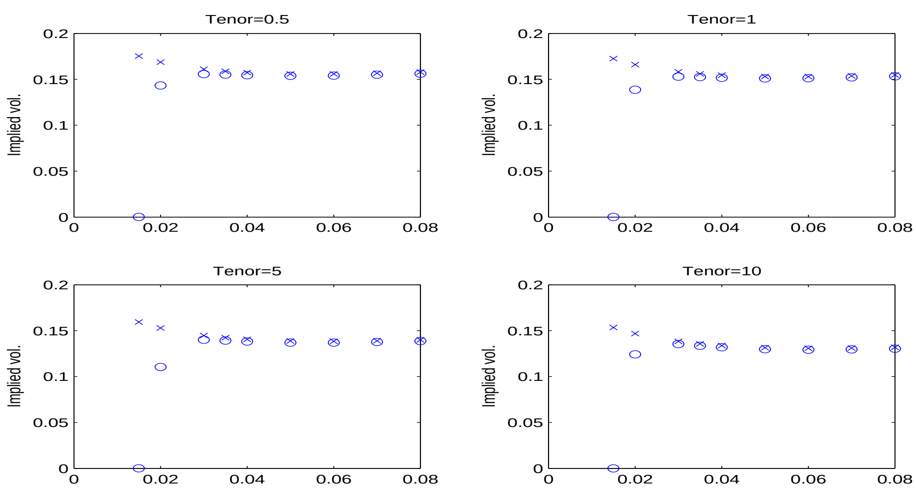

FIGURE 3. Implied volatilities of 10-year swaptions.

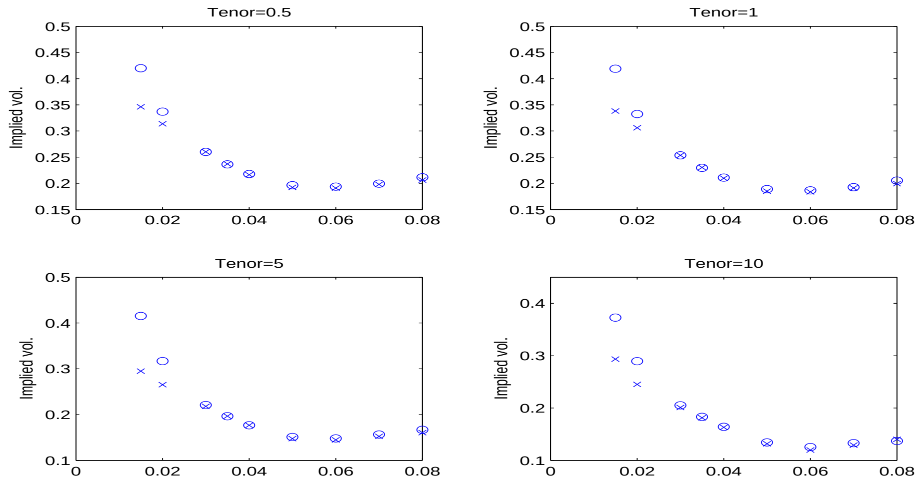

FIGURE 4. Implied volatilities of 1-year swaptions.

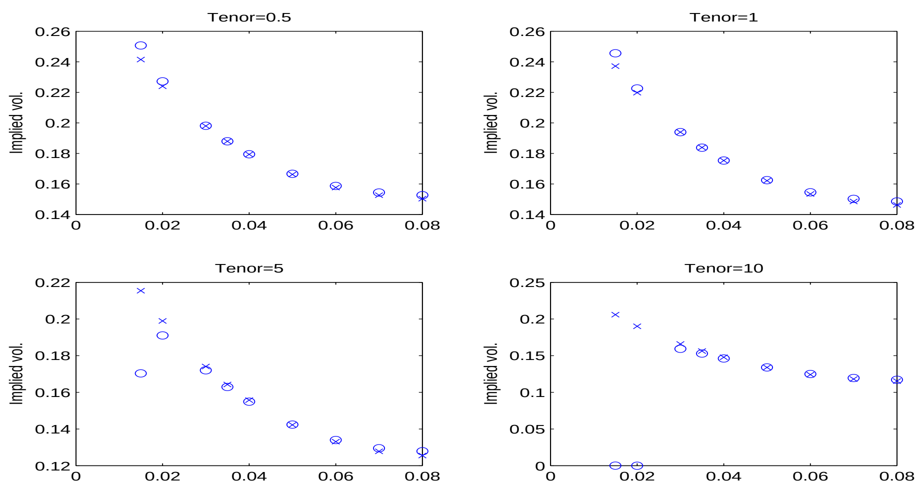

FIGURE 5. Implied volatilities of 5-year swaptions.

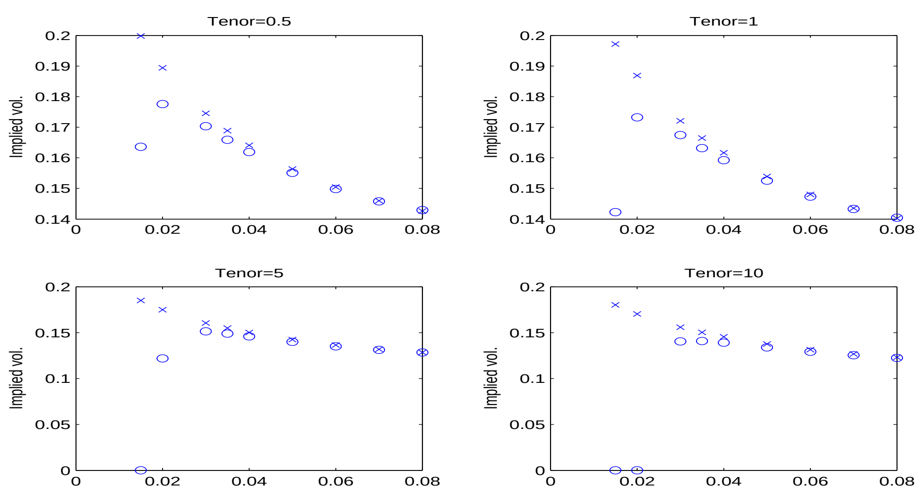

FIGURE 6. Implied volatilities of 10-year swaptions.

notional value equal to one dollar. Beyond $A = 50$, both real and imaginary parts are well under the magnitude of 10^{-4}. Laplace transforms of other swaptions look similar. We have also computed the prices using Heston's method, and found that, under the same resolution of discretization, the Heston's prices are almost indistinguishable from their FFT counterparts. We thus omit the Heston's prices in presentation.

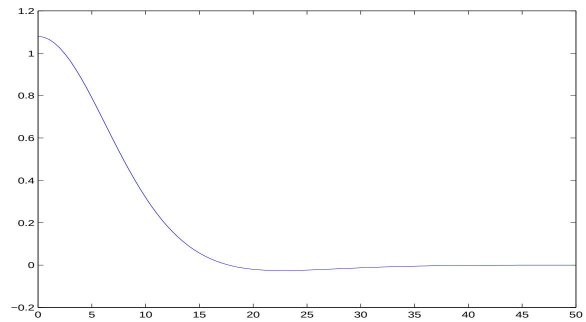

FIGURE 7. Real part of $\phi_T(u)$ for the in-1-to-1 swaption.

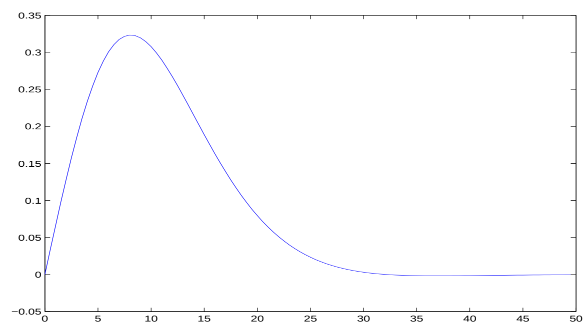

FIGURE 8. Imaginary part of $\phi_T(u)$ for the in-1-to-1 swaption.

In Table 2, we report the CPU times for FFT, Heston's and the Monte Carlo methods. The computations are done under MATLAB-5.3 in a PC with 1.1 GHz Intel Celeron CPU. Note that each execution of FFT method produces $N = 100$ prices for about 71% of the CPU time taken by the Heston's method (which produces only one price). The ratio of 71% is consistent with the fact the Heston's method evaluates two integrals, while the FFT method evaluates only one.

	FFT	Heston	Monte Carlo
CPU(seconds)	0.93	1.3	39486

TABLE 2. CPU times of the three methods with $\rho = -0.5$.

Example 2: We redo the calculation with the same input data of Example 1 except $\kappa = 0.15$. This *kappa* corresponds to a half life of mean reversion of $\ln(2)/\kappa = 4.6$ years, and it represents a stronger effect of stochastic volatility for longer time horizon[11]. For brevity, we only report the implied Black's volatilities for the case of negative rate - multiplier correlations, $\rho_j = -0.5$. Figure 9 - 11 display patterns similar to those of Figure 4 - 6. Apparently, the accuracy of the FFT method is about the same as that of Example 1. Comparing Figure 11 with Figure 6, we see the former has obviously steeper skews for in-5-to-10 and in-10-to-10 swaptions. Note that in this example we have applied antithetic variates technique to the MC method. The same accuracy suggests the robustness of the FFT method with regard to the strength of stochastic volatilities.

From modeling point of view, it makes sense to check the impact of stochastic local volatility on the level of an implied Black's volatility curve. For the same swap-rate variance $Var(X(T_m))$, Figure 12 and 13 show the implied volatility curves of swaption prices across strikes for market models with and without stochastic volatility. In producing the two figures we have taken $\rho_j = -0.5$ for the latter model. The horizontal lines in the figures correspond to *root mean variances*, defined by

$$RMV = \sqrt{\frac{1}{T_m} Var(X(T_m))} = \sqrt{\frac{1}{T_m}[\phi''_{T_m}(0) - (\phi'_{T_m}(0))^2]}.$$

Not that for the standard market model, the RMV is identical to implied Black's volatility. One can see that the Black's implied volatility curve stays at the same level of RMV, while tilts near at-the-money strike. This characteristic feature suggests an initialization strategy for calibration.

Finally, we take a closer look at the role of rate - multiplier correlations on the formation of volatility smiles or skews, through examining the variation of volatility smiles/skews in response to changes in the correlations. Figure 14 is for caplets, where the downward sloping skew corresponds to a negative correlation of $\rho = -0.5$, the upward sloping skew corresponds to a positive correlation of $\rho = 0.5$, and the nearly symmetric smile corresponds to zero correlation, $\rho = 0$. Without surprise, similar correspondence exists in swaptions, as is depicted in Figure 15. These figures show that through the extended model we can attribute volatility smiles/skews directly to the "leveraging effect". To some practitioners, this is a very plausible feature.

[11]We thank the anonymous referee for suggesting this test.

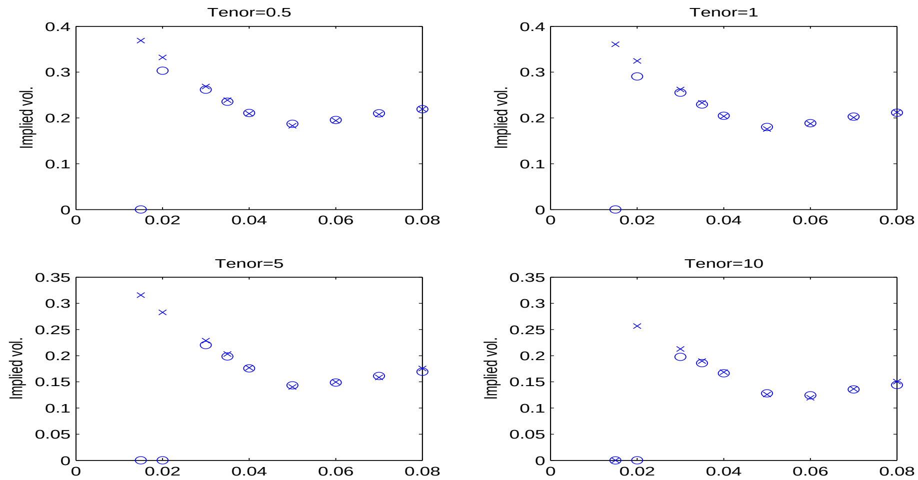

FIGURE 9. Implied volatilities of one-year swaptions.

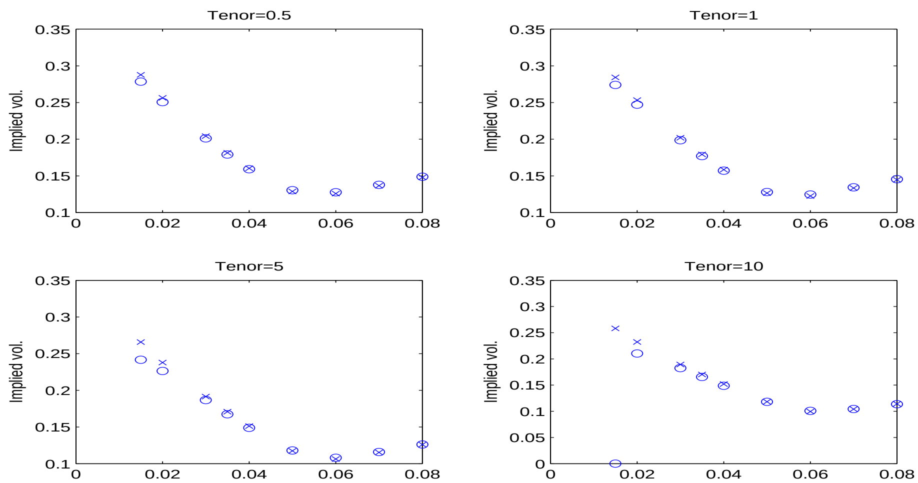

FIGURE 10. Implied volatilities of five-year swaptions.

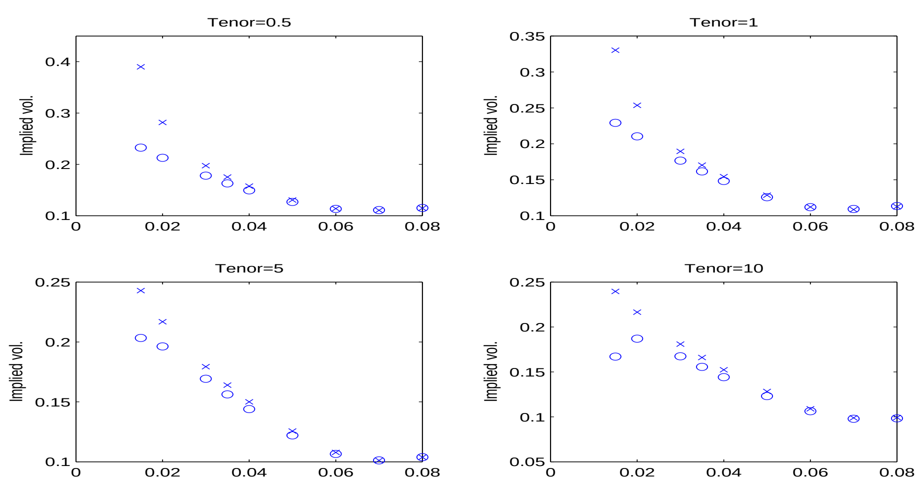

FIGURE 11. Implied volatilities of ten-year swaptions.

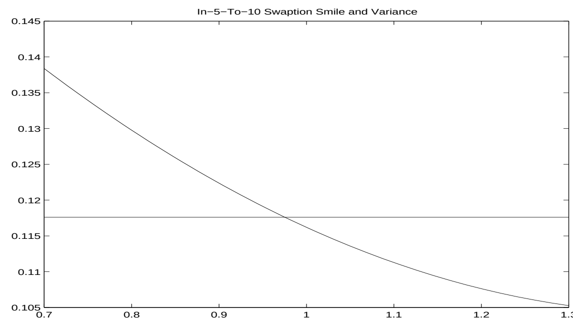

FIGURE 12. Volatility skew vs swap-rate volatility.

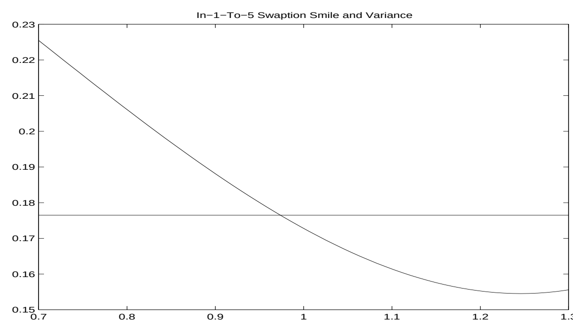

FIGURE 13. Volatility skew vs swap-rate volatility.

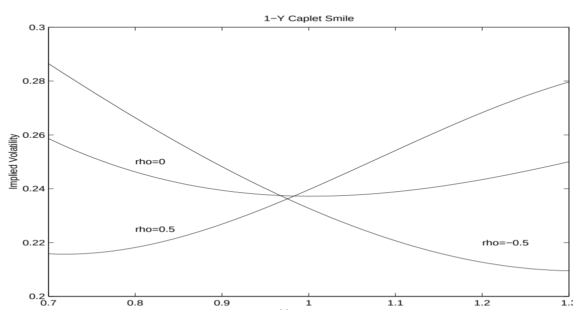

FIGURE 14. Volatility smile and skews for one-year caplets.

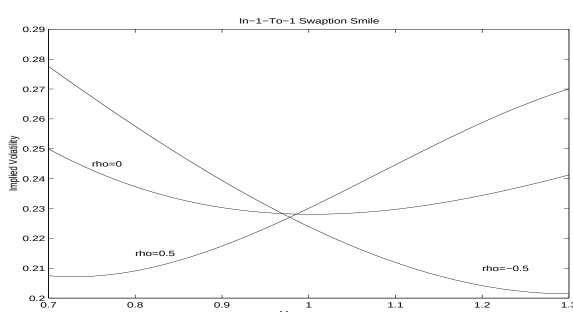

FIGURE 15. Volatility smile and skews for one-year swaptions.

5. Conclusion

This paper introduces a correlation-based model for LIBOR derivatives. By adopting a multiplicative volatility multiplier that follow a square-root process, we develop a LIBOR version of the Heston's model. With such a model, we can generate either volatility smiles or skews, using appropriate correlation between the stochastic multiplier and forward rates. Approximate swaption pricing is achieved through an inverse Laplace transform, and a high accuracy of the transformation method is confirmed through pricing comparisons. The outcomes of the comparisons are strongly supportive of the entire treatment. The preliminary success of the model introduces other interesting problems, including a rigorous accuracy analysis for the approximations made in pricing, and the calibration of the model.

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Appendix A. Details of some Derivations

Proof of Proposition 2.1:

The Radon-Nikodym derivative of $\mathbb{Q}^{j+1}$ with respect to $\mathbb{Q}$ is

$$\begin{aligned}\frac{d\mathbb{Q}^{j+1}}{d\mathbb{Q}} &= \frac{P(t,T_{j+1})/P(0,T_{j+1})}{B(t)} \\ &= e^{\int_0^t -\frac{1}{2}V(\tau)\sigma_{j+1}^2(\tau)d\tau+\sqrt{V(\tau)}\sigma_{j+1}\cdot d\mathbf{Z}_t} \\ &\triangleq m_{j+1}(t), \quad t \le T_{j+1}.\end{aligned}$$

Clearly we have

$$dm_{j+1}(t) = m_{j+1}(t)\sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_t.$$

Let $< \cdot, \cdot >$ denote covariance. By Girsanov theorem (e.g. Hunt and Kennedy, 2000), we obtain the Brownian motions under $\mathbb{Q}^{j+1}$:

$$\begin{aligned}d\mathbf{Z}_t^{j+1} =& d\mathbf{Z}_t - < d\mathbf{Z}_t, dm_{j+1}(t)/m_{j+1}(t) > \\ =& d\mathbf{Z}_t - \sqrt{V(t)}\sigma_{j+1}(t)dt, \\ dW_t^{j+1} =& dW_t - < dW_t, dm_{j+1}(t)/m_{j+1}(t) > \\ =& dW_t - < dW_t, \sqrt{V(t)}\sigma_{j+1}(t)\cdot d\mathbf{Z}_t > \\ =& dW_t + \sqrt{V(t)}\sum_{k=1}^{j}\frac{\Delta T_k f_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)} < dW_t, \frac{\gamma_k(t)}{\|\gamma_k(t)\|}\cdot d\mathbf{Z}_t > \\ =& dW_t + \sqrt{V(t)}\sum_{k=1}^{j}\frac{\Delta T_k f_k(t)\|\gamma_k(t)\|}{1+\Delta T_k f_k(t)}\rho_k(t)dt \qquad \square\end{aligned}$$

Proof of Proposition 2.2:

For clarity we let $\tau = T - t$ and $\lambda = \|\gamma_j\|$. Substituting the formal solution (17) to (15), we obtain the following equations for the undetermined coefficient:

$$\begin{aligned}\frac{dA}{d\tau} &= a\theta B, \\ \frac{dB}{d\tau} &= b_2B^2 + b_1B + b_0,\end{aligned} \tag{A.1}$$

where

$$a = \kappa\theta, \quad b_0 = \frac{1}{2}\lambda^2(z^2 - z), \quad b_1 = (\rho\epsilon\lambda z - \kappa\xi), \quad b_2 = \frac{1}{2}\epsilon^2.$$

Now consider (A.1) with constant coefficients and general initial conditions

$$A(0) = A_0, \qquad B(0) = B_0.$$

Since B is independent of A, it is solved first. In the special case when

$$b_2 B_0^2 + b_1 B_0 + b_0 = 0,$$

we have a easy solution

$$\begin{aligned} B(\tau) &= B_0, \\ A(\tau) &= A_0 + a_0 B_0 \tau. \end{aligned} \tag{A.2}$$

Otherwise, let Y_1 be the solution to

$$b_2 Y^2 + b_1 Y + b_0 = 0,$$

Assume $b_2 \neq 0$, then

$$Y_1 = \frac{-b_1 \pm d}{2b_2}, \quad \text{with} \quad d = \sqrt{b_1^2 - 4b_0 b_2}. \tag{A.3}$$

Without loss of generality we will take the "+" sign for Y_1. We then consider the difference between Y_1 and B:

$$Y_2 = B - Y_1.$$

Clearly Y_2 satisfies

$$\begin{aligned} \frac{dY_2}{d\tau} &= \frac{d(Y_1 + Y_2)}{d\tau} \\ &= b_2(Y_1 + Y_2)^2 + b_1(Y_1 + Y_2) + b_0 \\ &= b_2 Y_2^2 + (2b_2 Y_1 + b_1) Y_2 \\ &= b_2 Y_2^2 + d Y_2, \end{aligned} \tag{A.4}$$

with initial condition

$$Y_2(0) = B_0 - Y_1.$$

Note in the last equality of (A.4) we have used the equation (A.3). Equation (A.4) belongs to the class of Bernoulli equations which can be solved explicitly. One can verify that the solution is

$$Y_2 = \frac{d}{b_2} \frac{g e^{d\tau}}{(1 - g e^{d\tau})}, \quad \text{with} \quad g = \frac{-b_1 + d - 2B_0 b_2}{-b_1 - d - 2B_0 b_2}. \tag{A.5}$$

It follows that

$$\begin{aligned} B(\tau) &= Y_1 + Y_2 \\ &= \frac{-b_1 + d}{2b_2} + \frac{d}{b_2} \frac{g e^{d\tau}}{(1 - g e^{d\tau})} \\ &= B_0 + \frac{(-b_1 + d - 2b_2 B_0)(1 - e^{d\tau})}{2b_2(1 - g e^{d\tau})}. \end{aligned}$$

Having obtained B, we integrate the first equation of (A.1) to get A:

$$
\begin{aligned}
A(\tau) &= A_0 + a_0 \int_0^\tau B(s) ds \\
&= A_0 + a_0 B_0 \tau + \frac{a_0(-b_1 + d - 2b_2 B_0)}{2b_2} \int_0^\tau \frac{1 - e^{d\tau}}{1 - g e^{d\tau}} d\tau \\
&= A_0 + a_0 B_0 \tau + \frac{a_0(-b_1 + d - 2b_2 B_0)}{2b_2} \left[\tau - \int_0^\tau \frac{(1-g) e^{d\tau}}{1 - g e^{d\tau}} d\tau \right] \\
&= A_0 + \frac{a_0(-b_1 + d)\tau}{2b_2} - \frac{a_0(-b_1 + d - 2b_2 B_0)}{2b_2 d} \int_1^{e^{d\tau}} \frac{(1-g)}{1 - gu} du \\
&= A_0 + \frac{a_0(-b_1 + d)\tau}{2b_2} - \frac{a_0(-b_1 + d - 2b_2 B_0)}{2b_2 d} \frac{(g-1)}{g} \ln\left(\frac{1 - g e^{d\tau}}{1 - g} \right) \\
&= A_0 + \frac{a_0}{2b_2} \left[(-b_1 + d)\tau - 2 \ln\left(\frac{1 - g e^{d\tau}}{1 - g} \right) \right].
\end{aligned}
$$

Letting

$$
\begin{aligned}
A_0 &= A(\tau_j, z), \\
B_0 &= B(\tau_j, z),
\end{aligned}
$$

and replacing τ by $\tau - \tau_j$, we arrive at (18). The solution $\phi(z)$ so obtained belongs to $\mathbb{C}^1$ and hence is a weak solution to (15) $\square$

Proof of Proposition 3.1:

Denote the forward swap measure by $\mathbb{Q}^S$. The Radon-Nikodym derivative for $\mathbb{Q}^S$ is

$$
\begin{aligned}
\frac{d\mathbb{Q}^S}{d\mathbb{Q}} &= \frac{B^S(t)/B^S(0)}{B(t)} \\
&= \frac{1}{B^S(0)} \sum_{j=m}^{n-1} \Delta T_j P(0, T_{j+1}) e^{\int_0^t -\frac{1}{2} V(\tau) \sigma_{j+1}^2(\tau) d\tau + \sqrt{V(\tau)} \sigma_{j+1} \cdot d\mathbf{Z}_t} \\
&\triangleq m_S(t), \quad t \le T_m.
\end{aligned}
$$

There is

$$
\begin{aligned}
dm_S(t) &= \frac{1}{B^S(0)} \sum_{j=m}^{n-1} \Delta T_j P(0, T_{j+1}) e^{\int_0^t -\frac{1}{2} V(\tau) \sigma_{j+1}^2(\tau) d\tau + \sqrt{V(\tau)} \sigma_{j+1} \cdot d\mathbf{Z}_t} \sqrt{V(t)} \sigma_{j+1}(t) \cdot d\mathbf{Z}_t \\
&= \frac{1}{B^S(0) B(t)} \sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1}) \sqrt{V(t)} \sigma_{j+1}(t) \cdot d\mathbf{Z}_t \\
&= m_S(t) \sum_{j=m}^{n-1} \alpha_j \sqrt{V(t)} \sigma_{j+1}(t) \cdot d\mathbf{Z}_t.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
d\mathbf{Z}_t^S =& d\mathbf{Z}_t - < d\mathbf{Z}_t, dm_S(t)/m_S(t) > \\
=& d\mathbf{Z}_t - \sqrt{V(t)} \sum \alpha_j \sigma_{j+1}(t) dt, \\
=& d\mathbf{Z}_t - \sqrt{V(t)} \sigma^S(t) dt, \\
dW_t^S =& dW_t - < dW_t, dm_S(t)/m_S(t) > \\
=& dW_t - < dW_t, \sqrt{V(t)} \sum \alpha_j \sigma_{j+1}(t) \cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \sum_{k=1}^{j} \frac{\Delta T_k f_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(t)} < dW_t, \frac{\gamma_k(t)}{\|\gamma_k(t)\|} \cdot d\mathbf{Z}_t > \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \sum_{k=1}^{j} \frac{\Delta T_k f_k(t) \|\gamma_k(t)\|}{1 + \Delta T_k f_k(t)} \rho_k(t) dt \\
=& dW_t + \sqrt{V(t)} \sum_{j=m}^{n-1} \alpha_j \xi_j(t) dt \\
=& dW_t + \sqrt{V(t)} \xi^S(t) dt \qquad \square
\end{aligned}
$$

Proof of Proposition 3.2:

Differentiating the swap rate with respect to a forward rate we literally have

$$
\frac{\partial R_{m,n}(t)}{\partial f_j} = \alpha_j + \sum_{k=m}^{n-1} \frac{\partial \alpha_k}{\partial f_j} f_k. \tag{A.6}
$$

From the price-yield relation we obtain

$$
P(t, T_{k+1}) = \frac{P(t, T_k)}{1 + \Delta T_k f_k} = \ldots = \frac{P(t, T_m)}{\Pi_{l=m}^{k} (1 + \Delta T_l f_l)}. \tag{A.7}
$$

Apparently

$$
\begin{aligned}
\frac{\partial P(t, T_{k+1})}{\partial f_j} =& \begin{cases} \frac{-\Delta T_j}{1+\Delta T_j f_j} \cdot \frac{P(t,T_m)}{\Pi_{l=m}^{k}(1+\Delta T_l f_l)}, & k \geq j, \\ 0, & k < j \end{cases} \\
=& \frac{-\Delta T_j}{1 + \Delta T_j f_j} \cdot P(t, T_{k+1}) \cdot H(k - j),
\end{aligned} \tag{A.8}
$$

where $H(\cdot)$ is the Heaviside function defined such that $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ otherwise. Using the above derivatives as building blocks, we proceed with

$$\begin{aligned}
\frac{\partial \alpha_k}{\partial f_j} &= \Delta T_k \cdot \left(\frac{\partial P(t, T_{k+1})}{\partial f_j} B^S(t) - P(t, T_{k+1}) \frac{\partial B^S(t)}{\partial f_j} \right) \Big/ (B^S(t))^2, \\
&= \Delta T_k \cdot \left(\frac{-\Delta T_j}{1 + \Delta T_j f_j} \cdot P(t, T_{k+1}) \cdot H(k-j) B^S(t) \right. \\
&\qquad \left. - P(t, T_{k+1}) \sum_{l=m}^{n-1} \Delta T_l \frac{-\Delta T_j}{1 + \Delta T_j f_j} \cdot P(t, T_{l+1}) \cdot H(l-j) \right) \Big/ (B^S(t))^2 \\
&= \frac{-\Delta T_j}{1 + \Delta T_j f_j} \alpha_k \left(H(k-j) - \sum_{l=j}^{n-1} \alpha_l \right) \\
&= \frac{\Delta T_j}{1 + \Delta T_j f_j} \alpha_k \left(1 - H(k-j) - \sum_{l=m}^{j-1} \alpha_l \right)
\end{aligned} \tag{A.9}$$

Substituting the above expression to equation (A.6) we end up with

$$\begin{aligned}
\frac{\partial R_{m,n}(t)}{\partial f_j} &= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \sum_{k=m}^{n-1} \alpha_k \left(1 - H(k-j) - \sum_{l=m}^{j-1} \alpha_l \right) f_k \\
&= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \left\{ \sum_{k=m}^{n-1} \alpha_k f_k [1 - H(k-j)] - (\sum_{l=m}^{j-1} \alpha_l)(\sum_{k=m}^{n-1} \alpha_k f_k) \right\} \\
&= \alpha_j + \frac{\Delta T_j}{1 + \Delta T_j f_j} \sum_{k=m}^{j-1} \alpha_k \left(f_k - R_{m,n}(t) \right) \qquad \square
\end{aligned} \tag{A.10}$$

Table 3a. Swaption prices (bps) by FFT and MC methods, $\rho_j = -0.5$.

	Strikes																	
	0.015		0.020		0.030		0.035		0.040		0.050		0.060		0.070		0.080	
Exp.	Tenor=0.5 year																	
1	124.74	(0.346)	101.33	(0.314)	56.31	(0.260)	36.46	(0.237)	20.30	(0.217)	3.63	(0.192)	0.46	(0.191)	0.06	(0.198)	0.01	(0.206)
	124.84	(0.420)	101.43	(0.337)	56.31	(0.260)	36.41	(0.236)	20.40	(0.218)	3.85	(0.196)	0.50	(0.194)	0.07	(0.199)	0.02	(0.211)
	0.26	(-0.074)	0.26	(-0.023)	0.24	(0.000)	0.21	(0.001)	0.17	(-0.001)	0.08	(-0.004)	0.03	(-0.003)	0.01	(-0.001)	0.01	(-0.005)
5	128.40	(0.241)	109.39	(0.224)	73.99	(0.198)	58.37	(0.188)	44.63	(0.179)	23.67	(0.166)	11.18	(0.158)	4.91	(0.153)	2.11	(0.150)
	128.50	(0.251)	109.47	(0.227)	74.00	(0.198)	58.37	(0.188)	44.64	(0.180)	23.74	(0.167)	11.35	(0.159)	5.10	(0.154)	2.26	(0.153)
	0.40	(-0.009)	0.40	(-0.003)	0.38	(-0.000)	0.36	(-0.000)	0.33	(-0.000)	0.26	(-0.000)	0.18	(-0.001)	0.13	(-0.002)	0.09	(-0.002)
10	122.56	(0.200)	108.01	(0.189)	80.96	(0.175)	68.84	(0.169)	57.86	(0.164)	39.51	(0.156)	25.93	(0.150)	16.48	(0.146)	10.24	(0.143)
	122.26	(0.164)	107.69	(0.178)	80.62	(0.170)	68.50	(0.166)	57.54	(0.162)	39.27	(0.155)	25.79	(0.150)	16.44	(0.146)	10.29	(0.143)
	0.42	(0.036)	0.42	(0.012)	0.41	(0.004)	0.40	(0.003)	0.38	(0.002)	0.33	(0.001)	0.28	(0.001)	0.23	(0.000)	0.19	(-0.000)
Exp.	Tenor=1 year																	
1	250.34	(0.338)	203.95	(0.306)	114.25	(0.253)	74.24	(0.230)	41.33	(0.210)	7.08	(0.185)	0.81	(0.184)	0.11	(0.191)	0.02	(0.199)
	250.54	(0.419)	204.15	(0.332)	114.25	(0.254)	74.15	(0.230)	41.49	(0.211)	7.52	(0.189)	0.89	(0.187)	0.11	(0.193)	0.02	(0.205)
	0.49	(-0.081)	0.49	(-0.026)	0.46	(-0.000)	0.41	(0.001)	0.33	(-0.001)	0.15	(-0.004)	0.05	(-0.003)	0.02	(-0.002)	0.01	(-0.006)
5	256.57	(0.237)	218.84	(0.220)	148.23	(0.194)	116.90	(0.184)	89.25	(0.175)	46.95	(0.162)	21.80	(0.154)	9.33	(0.149)	3.90	(0.146)
	256.72	(0.246)	218.96	(0.223)	148.23	(0.194)	116.88	(0.184)	89.24	(0.175)	47.05	(0.163)	22.12	(0.155)	9.69	(0.150)	4.18	(0.149)
	0.77	(-0.008)	0.77	(-0.003)	0.73	(-0.000)	0.69	(0.000)	0.63	(0.000)	0.49	(-0.000)	0.35	(-0.001)	0.24	(-0.002)	0.16	(-0.002)
10	243.95	(0.197)	215.12	(0.187)	161.37	(0.172)	137.21	(0.166)	115.26	(0.162)	78.52	(0.154)	51.30	(0.148)	32.40	(0.144)	19.96	(0.140)
	243.31	(0.142)	214.45	(0.173)	160.65	(0.167)	136.49	(0.163)	114.58	(0.159)	78.00	(0.153)	51.00	(0.147)	32.29	(0.143)	20.05	(0.140)
	0.81	(0.055)	0.81	(0.014)	0.79	(0.005)	0.77	(0.003)	0.73	(0.002)	0.65	(0.001)	0.55	(0.001)	0.45	(0.000)	0.36	(-0.000)

Table 3b. Swaption prices (bps) by FFT and MC methods, $\rho_j = -0.5$.

	Strikes																	
	0.015		0.020		0.030		0.035		0.040		0.050		0.060		0.070		0.080	
Exp.	Tenor=5 year																	
1	1270.92	(0.295)	1057.33	(0.265)	634.28	(0.218)	432.98	(0.197)	253.24	(0.177)	38.80	(0.148)	2.52	(0.145)	0.17	(0.152)	0.01	(0.160)
	1271.51	(0.415)	1057.92	(0.317)	634.62	(0.221)	432.90	(0.196)	253.34	(0.177)	40.86	(0.151)	2.89	(0.148)	0.23	(0.156)	0.03	(0.167)
	1.78	(-0.120)	1.81	(-0.052)	1.80	(-0.003)	1.70	(0.000)	1.45	(-0.000)	0.66	(-0.003)	0.18	(-0.003)	0.05	(-0.004)	0.02	(-0.006)
5	1263.09	(0.215)	1088.81	(0.199)	753.89	(0.174)	599.73	(0.164)	459.81	(0.156)	237.79	(0.142)	102.99	(0.133)	38.86	(0.128)	13.79	(0.126)
	1262.32	(0.170)	1088.01	(0.191)	752.89	(0.172)	598.68	(0.163)	458.88	(0.155)	238.01	(0.142)	104.36	(0.134)	40.51	(0.129)	15.06	(0.128)
	2.95	(0.045)	2.99	(0.008)	2.95	(0.002)	2.83	(0.001)	2.64	(0.001)	2.08	(-0.000)	1.45	(-0.001)	0.92	(-0.002)	0.57	(-0.002)
10	1166.33	(0.185)	1035.00	(0.175)	785.51	(0.161)	670.88	(0.155)	565.15	(0.150)	384.83	(0.142)	248.67	(0.137)	153.53	(0.132)	91.45	(0.128)
	1161.75	(0.000)	1030.23	(0.122)	780.44	(0.152)	665.75	(0.149)	560.22	(0.146)	380.75	(0.140)	246.01	(0.135)	152.36	(0.131)	91.47	(0.129)
	3.09	(—)	3.16	(0.053)	3.18	(0.009)	3.12	(0.006)	3.02	(0.004)	2.69	(0.002)	2.28	(0.001)	1.84	(0.001)	1.45	(-0.000)
Exp.	Tenor=10 year																	
1	2520.39	(0.289)	2138.85	(0.245)	1377.56	(0.201)	1003.41	(0.181)	648.36	(0.163)	130.06	(0.132)	6.01	(0.120)	0.29	(0.129)	0.02	(0.140)
	2520.58	(0.373)	2139.06	(0.290)	1377.88	(0.205)	1003.89	(0.183)	649.08	(0.164)	133.89	(0.134)	7.79	(0.125)	0.39	(0.133)	0.02	(0.137)
	2.69	(-0.084)	2.77	(-0.045)	2.90	(-0.004)	2.86	(-0.002)	2.65	(-0.001)	1.44	(-0.003)	0.36	(-0.006)	0.08	(-0.004)	0.01	(0.003)
5	2427.65	(0.206)	2118.31	(0.190)	1514.76	(0.166)	1229.49	(0.156)	963.78	(0.148)	521.08	(0.134)	231.57	(0.124)	86.03	(0.118)	28.81	(0.115)
	2423.17	(0.000)	2113.86	(0.000)	1510.78	(0.159)	1225.84	(0.153)	960.89	(0.146)	520.90	(0.134)	233.99	(0.125)	89.00	(0.120)	30.97	(0.117)
	4.60	(—)	4.75	(—)	4.87	(0.007)	4.78	(0.003)	4.56	(0.002)	3.74	(0.000)	2.66	(-0.001)	1.68	(-0.001)	1.00	(-0.002)
10	2171.88	(0.180)	1942.00	(0.170)	1499.81	(0.156)	1293.14	(0.150)	1099.88	(0.146)	762.93	(0.138)	500.62	(0.132)	312.17	(0.127)	186.54	(0.123)
	2160.78	(0.000)	1930.60	(0.000)	1487.89	(0.140)	1281.16	(0.141)	1088.17	(0.139)	752.57	(0.134)	492.75	(0.129)	307.48	(0.125)	184.62	(0.123)
	4.76	(—)	4.94	(—)	5.12	(0.016)	5.10	(0.009)	5.00	(0.006)	4.56	(0.004)	3.92	(0.002)	3.20	(0.001)	2.51	(0.001)