VOLATILITY CONVERSION CALCULATORS

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Abstract. We provide the formulas needed to translate from absolute (normal) volatilities to Black (log normal) volatilities. For good measure, we then give the formulas to convert CEV volatilities into Black and absolute volatilities

Key words. equivalent vol, deterministic vol, smile

1. Conversion between log normal and normal vol. Black's model is

$$dF = \sigma_B F dW, \qquad F(0) = f \tag{1.1a}$$

where f is today's forward swap/caplet rate and where σ_B is the implied Black (log normal) volatility. The normal model is

$$dF = \sigma_N dW, \qquad F(0) = f \tag{1.1b}$$

where σ_N is the "normal" or "absolute" or "bps per year" volatility.

1.1. Calculating normal vol from log normal vol. For a swaption with strike (fixed rate) K, the normal volatility σ_N (which gives the same price of the option) as the log normal volatility σ_B is:

$$\sigma_N = \sigma_B \frac{f-K}{\log f/K} \cdot \frac{1}{1 + \frac{1}{24}\left(1 - \frac{1}{120}\log^2 f/K\right)\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2} \tag{1.2a}$$

where

$$f = \text{ current forward swap or caplet rate,} \tag{1.2b}$$

$$K = \text{ option's strike (fixed rate),} \tag{1.2c}$$

$$\tau = \text{ time to exercise (notification) date in years,} \tag{1.2d}$$

We probably should dispense with the

$$\tfrac{1}{120}\log^2 f/K \quad \text{and} \quad \tfrac{1}{5760}\sigma_B^4\tau^2. \tag{1.3}$$

They are too small to add measureably to the accuracy.

When $f \to K$, the above formula goes to a "0 over 0." To avoid this complication, we should use the alternative formula

$$\sigma_N = \sigma_B\sqrt{fK}\,\frac{1 + \frac{1}{24}\log^2 f/K}{1 + \frac{1}{24}\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2} \tag{1.4a}$$

when

$$\left|\frac{f-K}{K}\right| < 0.001. \tag{1.4b}$$

1.2. Calculating normal vol from log normal vol. Now suppose we are given an absolute (normal) vol, and the user wants the equivalent log normal (Black) vol. For consistency, we need to invert 1.2a exactly. This should be done using a global Newton method. Let us re-write 1.2a as

$$H(\sigma_B) = \sigma_N, \tag{1.5}$$

where

$$H(\sigma_B) = \sigma_B \frac{f-K}{\log f/K} \cdot \frac{1}{1 + \frac{1}{24}\left(1 - \frac{1}{120}\log^2 f/K\right)\sigma_B^2 \tau + \frac{1}{5760}\sigma_B^4 \tau^2}.$$

The problem is to find σ_B when the normal vol σ_N is given. One should start with an initial guess of

$$\sigma_B \approx \sigma_N \frac{\log f/K}{f-K} \cdot \left\{1 + \frac{1}{24}\left(1 - \frac{1}{120}\log^2 f/K\right)\frac{\sigma_N^2 \tau}{fK}\right\} \qquad \text{if } \left|\frac{f-K}{K}\right| \geq 0.001 \tag{1.6}$$

$$\sigma_B \approx \frac{\sigma_N}{\sqrt{fK}} \frac{1 + \frac{1}{24}\sigma_B^2 \tau}{1 + \frac{1}{24}\log^2 f/K} \qquad \text{if } \left|\frac{f-K}{K}\right| < 0.001 \tag{1.7}$$

Only one, or possibly two, Newton steps will be needed. In the Newton scheme, the derivative can be approximated by

$$H'(\sigma_B) = \frac{f-K}{\log f/K}, \qquad \text{if } \left|\frac{f-K}{K}\right| \geq 0.001$$
$$H'(\sigma_B) = \frac{1}{\sqrt{fK}} \qquad \text{if } \left|\frac{f-K}{K}\right| < 0.001$$

2. Converting CEV vols to absolute or Black volatilities. Another popular skew model is the CEV model:

$$dR = \alpha R^\beta \tag{2.1a}$$

where

$$\beta = \text{ user input CEV exponent, } 0 \leq \beta \leq 1. \tag{2.1b}$$

2.1. Converting between CEV vol and normal vol. To convert the CEV vol α into a normal (absolute) vol, one can use

$$\sigma_N = \alpha \frac{(1-\beta)(f-K)}{f^{1-\beta} - K^{1-\beta}} \frac{1}{1 + \dfrac{1 - \frac{2-2\beta+\beta^2}{120}\log^2 f/K}{1 - \frac{(1-\beta)^2}{12}\log^2 f/K} \dfrac{\beta(2-\beta)\alpha^2\tau}{24(fK)^{1-\beta}}} \tag{2.2a}$$

When f is very near K, or when β is very near 1, one needs to replace the formula with one that doesn't have the singularity at $\beta = 1$ or $f = K$. To cover both possibilities, we replace the above formula with

$$\sigma_N = \alpha (fK)^{\beta/2} \cdot \frac{1 + \frac{1}{24}\log^2 f/K}{1 + \frac{(1-\beta)^2}{24}\log^2 f/K} \cdot \tag{2.2b}$$
$$\cdot \frac{1}{1 + \dfrac{1 - \frac{2-2\beta+\beta^2}{120}\log^2 f/K}{1 - \frac{(1-\beta)^2}{12}\log^2 f/K} \dfrac{\beta(2-\beta)\alpha^2\tau}{24(fK)^{1-\beta}}}$$

when

$$(1-\beta)\left|\frac{f-K}{K}\right| < 0.001 \tag{2.2c}$$

To convert the normal vol σ_N into a CEV vol, we should again use a global Newton method, to solve

$$H(\alpha) = \sigma_N. \tag{2.3a}$$

Here,

$$H(\alpha) = \alpha \frac{(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}} \frac{1}{1 + \frac{1-\frac{2-2\beta+\beta^2}{120}\log^2 f/K}{1-\frac{(1-\beta)^2}{12}\log^2 f/K}\frac{\beta(2-\beta)\alpha^2\tau}{24(fK)^{1-\beta}}} \tag{2.3b}$$

$$\text{if } (1-\beta)\left|\frac{f-K}{K}\right| \geq 0.001$$

and

$$H(\alpha) = \alpha (fK)^{\beta/2} \frac{1+\frac{1}{24}\log^2 f/K}{1+\frac{(1-\beta)^2}{24}\log^2 f/K} \frac{1}{1+\frac{\beta(2-\beta)\alpha^2\tau}{24(fK)^{1-\beta}}} \tag{2.3c}$$

$$\text{if } (1-\beta)\left|\frac{f-K}{K}\right| < 0.001$$

A superb initial guess is

$$\alpha \approx \frac{\sigma_N}{fK^{\beta/2}} \frac{1+\frac{(1-\beta)^2}{24}\log^2 f/K}{1+\frac{1}{24}\log^2 f/K}\left\{1+\frac{1}{24}\frac{\beta(2-\beta)\sigma_N^2\tau}{fK}\right\} \tag{2.4}$$

As above, the derivative for Newton's method can be taken as

$$H'(\alpha) = \frac{(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}}, \tag{2.5a}$$

$$H'(\alpha) = (fK)^{\beta/2}. \tag{2.5b}$$

2.2. Converting CEV vol to log normal vol. To convert the CEV vol α to a log normal (Black) vol, one should first translate it to the normal vol σ_N, and then use the above routine to calculate the Black vol σ_B from the normal vol σ_N.

Similarly, to convert the Black vol σ_B to the CEV vol σ, one uses the above routines to first translate it to a normal vol σ_N, and then tranlate the normal to the Black vol.

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