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Forecasting the interest-rate term structure:
Using the model of Fong & Vasicek, the Extended Kalman Filter and the Bollinger Bands

Abstract:

In this paper, we consider the issue of forecasting the interest-rate term structure and we present a solution. We apply the Extended Kalman Filter (EKF) to the Fong & Vasicek model to deal with the issue of computing the hidden stochastic volatility. We also introduce Bollinger bands as a variance reduction technique used to improve the Monte Carlo simulation performance[1]. Our results suggest that the forecasting technique using the unobservable component approach (EFK) to obtain values of the stochastic volatility is superior to another stochastic volatility model such as GARCH (1,1). In addition, the performance is improved when we introduce Bollinger bands.

Key-words: Term structure of interest rate, Extended Kalman Filter, Monte Carlo simulation, Root Mean Square Error, forecasting, stochastic volatility, Bollinger bands, Fong and Vasicek.

Classification JEL : C15; C63; G13

[1] This idea has been first introduced in the paper of Théorêt & Rostan 2002: improving the Monte Carlo simulation in the Fong & Vasicek framework using Bollinger bands.

1. Introduction

The "term structure" of interest rates refers to the relationship between bonds rates of different terms. When bonds interest rates are plotted against their terms, this is called the "yield curve". The theory about modeling interest-rate term structure suggests that the evolution of the yield curve shape is affected by the level of interest rates, the slope of the term structure, the curvature and the volatility of the changes, refer to Litterman and Scheinkman (1991), Chen and Scott (1993), Dai and Singleton (2000), and De Jong (2000).

Fong & Vasicek (F&V, 1992) proposed a two-factor model with a mean-reverting process and a structure that makes the short rate variance depending on the level of interest rates with a suitable restriction that the short-rate won't become negative. This model is not frequently used in practice by financial analysts because of the problem of hidden stochastic volatility which is the black box of this kind of models.

In this paper, we use the F&V model to forecast the interest-rate term structure and we apply the Extended Kalman Filter (EKF) as a tool to filter the unobserved stochastic volatility. We also introduce Bollinger bands, a well-known tool used in technical analysis, as a reduction variance technique to improve the Monte Carlo simulation performance.

The remaining of the paper is organized as follow: In section 2, we discuss the interest of forecasting the yield curve. In section 3, we present the most popular interest-rate term structure models; we also provide some details about the F&V model (1992) and about the intuition behind the EKF. In section 4, we explain the EKF scheme and the implementation of our specific model. The data and calibration are described in section 5. In section 6, we detail the approach used for the simulation. Empirical results are discussed in section 7. Finally, some interesting conclusions are offered in section 8.

2. Interest rate term structure forecasting

Forecasting the term structure is of great interest because the term structure is commonly considered as a leading indicator of the economic activity. Some findings suggest that the spread between long-term and short-term interest rates has proven to be an excellent predictor of changes in the economic activity. As a general rule, when long-term interest rates have been much above short-term rates, strong increases in output have followed within about a year; however, whenever the yield curve has been inverted for any extended period of time, a recession has followed[2]. Jim Day and Ron Lange (1997) shown that the slope of the nominal term structure from 1- to 5-year maturities is a reasonably good predictor of future changes in inflation over these horizons.

3. Interest rate term structure models

The recent literature has produced major advances in theoretical models of the term structure. Term structure models include no-arbitrage and equilibrium models. The no-arbitrage tradition focuses on perfectly fitting the term structure at a point in time to ensure that no arbitrage possibilities exist, which is essential for pricing derivatives. The equilibrium tradition focuses on modeling the dynamics of the instantaneous rate, typically using affine models that provide clear economic intuitions connecting the term structure with economic fundamentals. Equilibrium and arbitrage models have similar structures. The difference comes from the nature of the input used to calibrate the model parameters. The equilibrium models explicitly specify the market price of risk; the model parameters, assumed to be time-invariant, are estimated statistically from historical data. These models are often used by economists to understand the relationship between the shape of the term structure and its forecast for future economic conditions. Traders, however, would rather use arbitrage models because these models are calibrated to match the model price of the underlying security with its market price. Another classification of term structure models can be made according to the number of factors involved. One

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[2] An explanation of this result is that the term spread has reflected both current monetary conditions, which affect short-term interest rates, and expected real returns on investment and expectations of inflation, which are the main determinants of long-term rates. For more details see **Kevin Clinton,** The term structure of interest rates as a leading indicator of economic activity: A technical note.

should make distinction between one-factor and multifactor models. The one-factor models are popular because of their simplicity. Empirical evidence on principal component analysis has shown that almost 90 percent of the variation in the changes of the yield curve is attributable to the variation in the first factor which is considered to be the level of the interest rate[3]. Because the first factor relates to the interest rate level, any point on the yield curve may be used as a proxy for it. For most one-factor models, the factor is generally taken to be the instantaneous short rate, *r(t)*. On the other hand, the multifactor models postulate that the evolution of the interest-rate term structure is driven by the dynamics of several factors and therefore, the yields are functions of these factors. These factors can be represented by macroeconomics shocks or be related to the level, slope and curvature of the yield curve itself. Figure 1 outlines the most popular interest-rate term structure models.

Figure 1 : **Interest-rate term structure models**

Equilibrium models	No arbitrage models
Vasicek (1977)*	Ho-Lee
Cox-Ingersoll-Ross (CIR 85)*	Hull-White (90)
Brennan-Schwartz (79)**	Black-Derman-Toy (BDT 90)
Fong-Vasicek (92)**	Heath-Jarrow-Morton (HJM 92)
Longstaff-Schwartz (92)**	

*One factor models **two factors models

Interest rate forecasting is crucial for bond portfolio management and for predicting of future changes in economic activity. The arbitrage-free term structure literature has little to say about dynamics or forecasting, as it is concerned primarily with fitting the term

[3] See Champman and Pearson (2001) for a detailed discussion.

structure at a point in time. The affine equilibrium term structure literature is concerned with dynamics driven by the short rate, and so is potentially linked to forecasting.

Since the main aim of our work is to forecast the interest-rate term structure, we choose a two-factor model that belongs to equilibrium models.

<u>The Fong-Vasicek model (1992)</u>

Empirical studies have revealed that the volatility of the changes in the short rate is time varying and stochastic. To explicitly model the stochastic changes in the interest rate volatility and their effect on bond prices and option values, Fong and Vasicek (1992) proposed a two-factor extension of the Vasicek model in which the Ornstein-Uhlenbeck process is modified to include a stochastic variance that follows a square-root process:

$$\begin{aligned} dr_t &= k(\mu - r_t)dt + \sqrt{v_t}\,dW_t \\ dv_t &= \lambda(\nu - v_t)dt + \tau\sqrt{v_t}\,dWs \end{aligned} \qquad E(dW_t, dW_s) = \rho$$

where W_t and W_s are two correlated Brownian motions under the risk neutral distribution. We are considering ourselves in a risk-neutral world where investors require no compensation for risk and the expected return of securities is the risk-free interest rate.

As we can see, this model allows for a stationary mean reverting process whose variance is again a stationary stochastic process. Here μ is the unconditional average of the short rate process, k controls the degree of persistence in interest rates. In order to interpret the other parameters let us observe that the second equation is just a square root process for variance v_t. Now we can interpret parameter ν as the unconditional average variance. The parameter λ accounts for the degree of persistence in the variance. Finally, the parameter τ is the unconditional infinitesimal variance of the unobserved variance process. The hidden volatility is an obvious weakness of the model and makes it hard to use. In this case using the technique of filtering is very natural to infer the values of the unobserved volatility process.

<u>The extended Kalman filter (EKF)</u>

To use the F&V model we have to deal with the unobserved volatility process. To this end we apply the Kalman filter. This filter is a widely used methodology which recursively calculates optimal estimates of unobservable state variables, given all the information available up to some moment in time. Estimates are improved as new data arrive. The application of Kalman filtering methods in the estimation of term structure models using cross-sectional/time series data, has been investigated by Pennacchi (1991), Lund (1994, 1997), Chen and Scott (1995), Duan and Simonato (1995), Geyer and Pichler (1996), Ball and Torous (1996), Jegadeesh, and Nowman (1999), Babbs, De Jong and Santa-Clara (1999), De Jong (2000), Dewachter and Maes (2001) and Sørensen (2002).

The use of the state space formulation of term structure models and the application of the Kalman filter has the advantage to allow the underlying state variables to be handled correctly as unobservable variables compared to using a short-term rate historical volatility as a proxy.

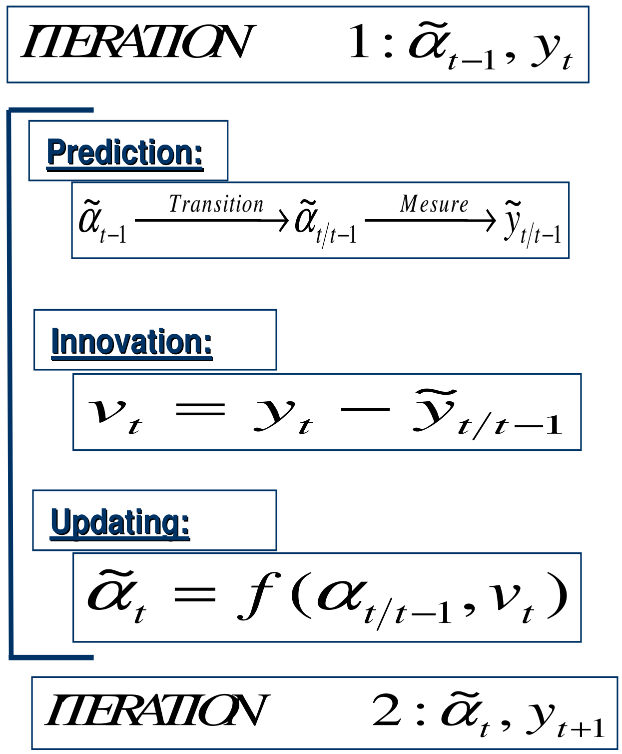
4. Methodology

In this section, we provide a brief introduction to the EKF followed by its application to the F&V model.

4.1 The Extended Kalman Filter (EKF)

The Kalman filter uses data observed in the market to infer values for unobserved state variables. The idea is to express a dynamic system in a particular form called the *state-space representation*. A state-space model is characterized by a measurement equation and a transition equation. Once this has been made, a three-step iteration process can begin. There is one iteration for each observation date *t*, and one iteration includes three steps, as is shown in figure 2.

Figure 2: The three steps of an iteration



During the first step called the prediction phase, the values of non-observable variables in (*t*-1) are used to compute their expected value in *t*, conditionally to the information available in (*t*-1). The predictions rely on the transition equation. The predicted values $\tilde{\alpha}_{t/t-1}$ are then introduced in the measurement equation to determine the measure $\tilde{y}_t$. In this equation, the errors have zero mean and are not serially nor temporarily correlated. They represent every kind of disturbances likely to lead to errors in the data. The second step or innovation phase allows for the calculation of the innovation ν_t. Lastly, non-observable variables values, which where computed in the prediction phase, are updated conditionally to the information given by ν_t. Once this calculus has been made, $\tilde{\alpha}_t$ is used to begin a new iteration.

4.2 Algorithm

The following is a formal discussion of the algorithm behind the Kalman filter.

Standard setup of the Kalman filter is applicable to the linear state-space model of the form:

$$\begin{array}{llll} \text{Measurement equation} & y_n = Z_n \alpha_n + d_n + \varepsilon_n & & Var(\varepsilon_n) = H_n, \\ \text{Transition equation} & \alpha_n = T \alpha_{n-1} + c_n + R_n \eta_n & & Var(\eta_n) = Q_n. \end{array}$$

Where (ε_n) and (η_n) are independent normal random variables with zero mean. The conditional distribution of α_n given the observations $y_1,......., y_n$ is also normal. The mean a_n and variance P_n can be calculated recursively by an application of the one-step ahead prediction equations:

$$\begin{aligned} a_{n/n-1} &= T_n a_{n-1} + c_n \\ P_{n/n-1} &= T_n P_{n-1} T'_n + R_n Q_n R'_n. \end{aligned}$$

And updating/filtering equations,

$$\begin{aligned} a_n &= a_{n/n-1} + P_{n/n-1} Z'_n F_n^{-1} (y_n - Z_n a_{n/n-1} - d_n), \\ P_n &= P_{n/n-1} - P_{n/n-1} Z'_n F_n^{-1} Z_n P_{n/n-1}, \\ F_n &= Z_n P_{n/n-1} Z'_n + H_n. \end{aligned}$$

Here $a_{n/n-1}$ and $P_{n/n-1}$ denote the conditional expectation and variance, respectively, of α_n given the observations $y_1,......., y_n$.

When the state space equation is non-linear, say

$$\begin{array}{ll} y_n = Z_n(\alpha_n) + \varepsilon_n, & Var(\varepsilon_n) = H_n, \\ \alpha_n = T_n(\alpha_{n-1}) + R_n(\alpha_{n-1})\eta_n, & Var(\eta_n) = Q_n, \end{array}$$

one can use Taylor series expansion to obtain the following approximate linearized system.

$$\begin{array}{ll} y_n = \hat{Z}_n \alpha_n + d_n + \varepsilon_n & Var(\varepsilon_n) = H_n, \\ \alpha_n = \hat{T} \alpha_{n-1} + c_n + \hat{R}_n \eta_n & Var(\eta_n) = Q_n. \end{array}$$

where $\hat{Z}_n = \frac{d}{dx} Z_n(a_{n/n-1}),\ d_n = Z_n(a_{n/n-1}) - \hat{Z}_n(a_{n/n-1}),\ \hat{T}_n = \frac{d}{dx} T_n(a_{n-1}),$
$c_n = T_n(a_{n-1}) - \hat{T}_n(a_{n-1}),\ \hat{R}_n = R_n(a_{n-1}).$

The Kalman filter for this approximate state-space model is then given by:

$$
\begin{aligned}
a_{n/n-1} &= T_n(a_{n-1}), \\
P_{n/n-1} &= \hat{T}_n P_{n-1} \hat{T}'_n + \hat{R}_n Q_n \hat{R}'_n, \\
F_n &= \hat{Z}_n P_{n/n-1} \hat{Z}'_n + H_n, \\
a_n &= a_{n/n-1} + P_{n/n-1} \hat{Z}'_n F_n^{-1}(y_n - Z_n(a_{n/n-1})), \\
P_n &= P_{n/n-1} - P_{n/n-1} \hat{Z}'_n F_n^{-1} \hat{Z}_n P_{n/n-1}.
\end{aligned}
$$

To apply this algorithm, one should proceed to a discretization and a linearization of the F&V short rate model which is discussed in next section.

4.3 Applying the Kalman filter to the F&V model

Recall that Fong and Vasicek model the short rate and its stochastic variance with the following equations:

$$
\begin{aligned}
dr_t &= k(\mu - r_t)dt + \sqrt{v_t}\, dW_t \\
dv_t &= \lambda(\nu - v_t)dt + \tau \sqrt{v_t}\, dWs
\end{aligned}
$$

As we can see, the model respects the Kalman filter state-space form. One can consider the first and the second equation as the measurement equation and the transition equation respectively. But the model is still in his continuous and non linear form. Before applying directly the algorithm of extended Kalman filter we try to put these two equations in there discreet and linear form.

4.3.1 Discretization

An application of Ito formula to the first equation yields

$$de^{kt}(r_t - \mu) = e^{kt}\sqrt{v_t}\, dW_t.$$

Integrating by parts we obtain

$$r_{t+h} = \mu + e^{-kh}(r_t - \mu) + e^{-kh}\int_t^{t+h} e^{k(s-t)}\sqrt{v_s}\,dW_s$$

Also, similarly,

$$v_{t+h} = \nu + e^{-\lambda h}(v_t - \nu) + e^{-\lambda h}\tau\int_t^{t+h} e^{\lambda(s-t)}\sqrt{v_s}\,dZ_s$$

Therefore, the discrete time specification of F&V model has the following form,

$$\begin{aligned} r_{t+h} &= \mu + e^{-kh}(r_t - \mu) + \varepsilon_t(h)_t \\ \upsilon_{t+h} &= \nu + e^{-\lambda h}(\upsilon_t - \nu) + \eta_t(h)_t \end{aligned} \qquad t = 0, h, 2h, \ldots,$$

Where h denotes the sampling interval (for example, on quarterly frequency h = 3/12), and the innovations $\varepsilon_t(h)$ and $\eta_t(h)$ are defined as

$$\varepsilon_t(h) = e^{-kh}\int_t^{t+h} e^{k(s-t)}\sqrt{\upsilon_s}\,dW_s,$$
$$\eta_t(h) = e^{-\lambda h}\int_t^{t+h} e^{\lambda(s-t)}\sqrt{\upsilon_s}\,dZ_s.$$

We approximate these innovations as

$$\varepsilon_{nh}(h) = e^{-kh}\sqrt{\upsilon_{nh}}\sqrt{h}\varepsilon_n, \quad \eta_{nh}(h) = e^{-\lambda h}\tau\sqrt{\upsilon_{nh}}\sqrt{h}\eta_n,$$

Where $\varepsilon_t(h)$ and $\eta_t(h)$ are independent standard normal random variables. Defining the transformed discrete observation to be

$$R_n = e^{kh}(r_{(n+1)h} - \mu) - (r_{nh} - \mu),\ n = 0,1,2\ldots$$

And denoting v_{nh} by Vn, we obtain the following discrete time state space system

$$\begin{aligned} R_n &= \sqrt{h}\sqrt{V_n}\varepsilon_n,\ n = 0,1,2,\ldots, \\ V_n &= e^{-\lambda h}V_{n-1} + (1 - e^{-\lambda h})\nu + e^{-\lambda h}\tau\sqrt{h}\sqrt{V_{n-1}}\eta_n,\ n = 1,2,\ldots, \end{aligned}$$

4.3.2 Linearization

To get the linear form of the F&V model, we apply the procedure of linearization, proposed by the EKF and explained above, to the discreet form of F&V model.

We consider the observation y_n to be $\ln(R_n^2/h)$.

$$y_n = \ln V_n + \ln \varepsilon_n^2.$$

Clearly $\ln \varepsilon_n^2$ is not Gaussian, but has the distribution of $\ln \chi_1^2$. To use EKF we replace this by a normal random variable with mean -1.270363 and variance 4.934802, the mean and variance, respectively, of a $\ln \chi_1^2$ random variable. We then apply the EKF methodology with

$$Z_n(x) = \ln x - 1.270363 \; ; H_n = 4.934802 \; ;$$
$$T_n(x) = e^{-\lambda h} x + (1 - e^{-\lambda h})v; R_n(x) = \tau e^{-\lambda h} \sqrt{h} \sqrt{x}; Q_n = 1.$$

Derivations:

$$\hat{Z}_n = \frac{1}{a_{n/\,n-1}}.; \qquad d_n = \ln a_{n/\,n-1} - 2.270363; \qquad \hat{T}_n = e^{-\lambda h}$$

$$c_n = (1 - e^{-\lambda h})v; \qquad \hat{R}_n = \tau e^{-\lambda h} \sqrt{h} \sqrt{a_{n-1}}$$

Finally we get the discreet and linear form of the F&V state-space model:

$$y_n = \frac{1}{a_{n/n-1}} \alpha_n + \ln a_{n/n-1} - 2.2703 + \varepsilon_n$$
$$\alpha_n = e^{-\lambda h} \alpha_{n-1} + (1 - e^{-\lambda h})\nu + \tau e^{-\lambda h} \sqrt{h} \sqrt{a_{n-1}} \eta_n$$

The last step, before applying directly the Kalman filter to infer the values of unobserved volatilities, and proceed to the forecasting of the interest rate term structure, is to estimate the values of the parameters of the model. The next section gives a data description and explains the methodology adopted for calibration.

5. Data and calibration of the Fong & Vasicek model

5.1 Data

We use Treasury bill yields provided by the Bank of Canada for the 1-, 3-, 6-month, 1-year maturity and the Canadian Government yield curve provided by Bloomberg for the 2-, 5-, 10-, 20-, 30-year maturities. We consider each yield of the Government bond term structure as the “short interest rate *r*” of the Fong & Vasicek model as well as its corresponding variance v. The span of data goes from October 23, 2002 to October 23, 2003 (250 days).

5.2 Calibration of the model

The inputs are: 1. The daily Canadian Government yield curve obtained from the Bank of Canada and from Bloomberg 2. The daily Canadian Government yield variances term structure computed from GARCH(1,1) model[4] applied to historical data (400 past daily observations of the interest rate) adjusted for the interest rate using the Campbell, Lo and Mc Kinlay methodology (1997). The yield and variance curves have been smoothed by 3rd degree polynomial functions to generate 3,000 data for each curve. The outputs of the calibration, k , μ , λ , ν and τ from the F&V model, and the correlation ρ [5] between the two factors are obtained by Full Information Maximum Likelihood Marquardt (ML) or Three-Stage Least Squares 3SLS method when the ML did not converge. The 3SLS is an appropriate technique when right-hand side variables are correlated with the error terms, and there is both heteroskedasticity, and contemporaneous correlation in the residuals.

Table 1 Example of ML parameter estimates obtained for the Dec. 10 2002

Parameter	F&V (1992)
k	0.176272
μ	0.055796
λ	3.860698
ν	0.00135
τ	0.003576
ρ	0.611125816

[4] The variance obtained from GARCH(1,1) is used only for calibration purposes. For forecasting purposes, we used the variance provided by the extended Kalman filter.

[5] We impose the correlation between the two random variables during the simulation by applying Cholesky decomposition.

6. Simulation

6.1 Evolved approach

The simulation approach adopted in this paper is based on the Monte Carlo simulation of every yield of the Canadian yield curve. We simulate a thousand trajectories for each yield. r_o and v_o, the initial values of the simulation, are respectively the yield observed at day 1 and its annualized variance obtained from EKF.

6.2 Improvement of the Monte Carlo simulation

To improve the performance of Monte Carlo simulation, we use in conjunction two variance reduction techniques: the classical antithetic variable and Bollinger bands[6], a technique borrowed to the technical analysis. Bollinger bands become narrower during less volatile periods and wider during more volatile periods. Variance reduction with Bollinger bands is obtained by forcing the simulated rate to remain inside pre-determined upper and lower bands during the simulation.

Remembering that Bollinger bands are bands usually drawn at ±2 standard deviations off the value of the 20-day moving average of the times series under study, in our paper, the standard deviation used to compute Bollinger bands is the conditional standard deviation obtained from the extended Kalman filter for each simulated yield. Moreover, instead of using the 20-day moving average of the yield that we are simulating as the central value of the bands, we use the value of the expected 3-month CDOR (Canadian Dollar Offer Rate) in 20 days[7] *minus* the 20-day historical average spread of the simulated yield over the 3-month CDOR spot. Our assumptions are that 1. The spreads between the CDOR rate and the other yields of the term structure over 20 days remain constant; 2. The implied future CDOR rates obtained from the BAX futures contract prices are a good proxy of what will be the level of the CDOR rates in 20 days.

[6] This method is detailed in Théorêt & Rostan 2002 a.

[7] The expected 3-month CDOR (Canadian Dollar Offer Rate) in 20 days is obtained from the BAX futures price traded on the Montreal Exchange (MX), using a linear interpolation of the BAX futures price. Our assumption is that the CDOR rate will vary linearly overtime. In Canada, the 3-month CDOR rate is the 3-month Bankers' acceptance rate. It is used as the floating leg rate to price plain-vanilla swap contracts. It represents the main benchmark of the Canadian Money Market.

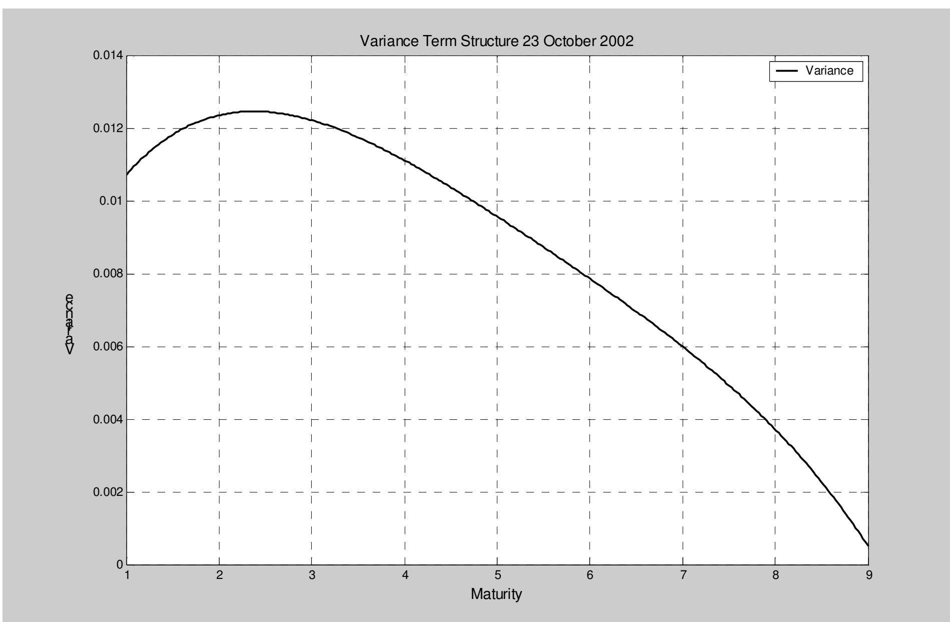
More precisely, in our paper, we test the performance of the Monte Carlo simulation with Bollinger bands drawn at ± 1 or ±2 standard deviations off the spread.

Each yield that has been computed with the Monte Carlo simulation is the forecasted yield in 20 days. Repeating the methodology for each component of the curve, we forecast the Canadian Government yield curve in 20 days.

7. Empirical results

The extended Kalman filter allows us to obtain an estimation of the stochastic volatility for each maturity and for each day in our sample. In other words, the output of the filter is a vector of 250 variance term structures (one term structure per day). Figure 3 illustrates the variance term structure for the first day of our sample (October, 23 2002).

Figure 3: Variance term structure obtained from EKF for October, 23 2002



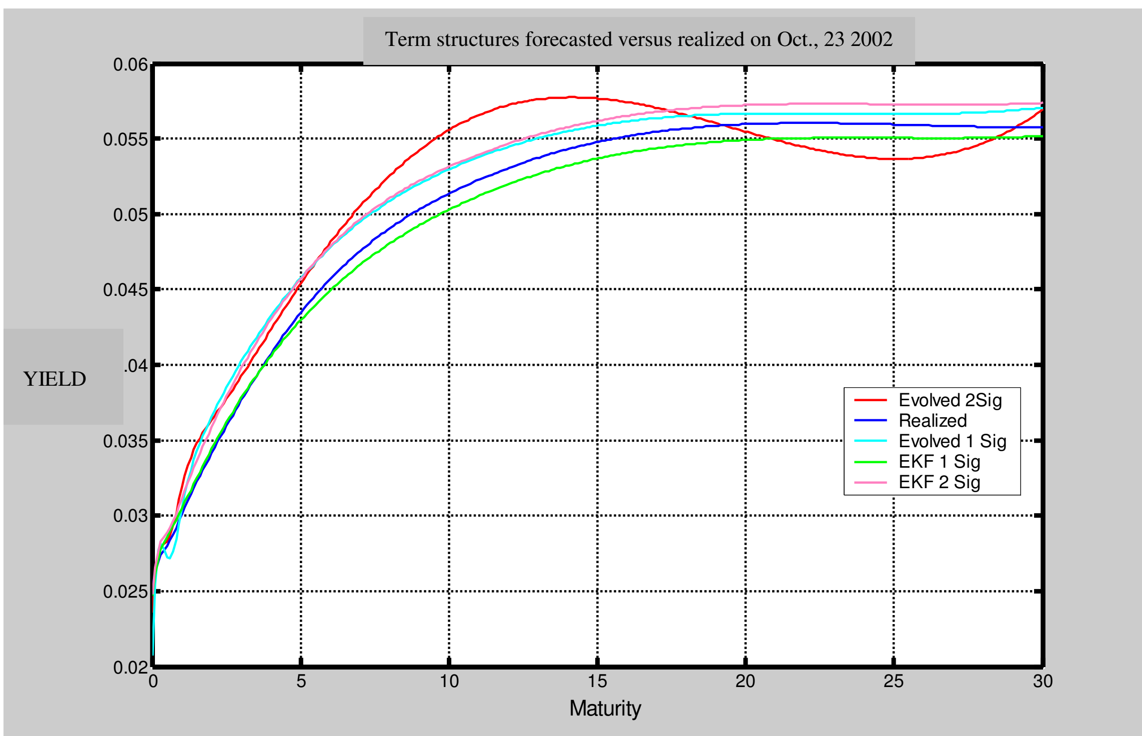
The variance term structure provided by the extended Kalman filter has the same shape of what we can empirically observe. We observe that the variance of the long term interest rate is lower than the variance of the short term interest rate which gives a downwarding curve.

We generate a thousand trajectories for each yield. The number of time steps x is computed with the following equation: $\frac{30 \quad years}{3{,}000} = \frac{\frac{20}{250}}{x}$ given x = 8 time steps for 20 days of simulation, dt = 20/250 =0.08, 30 years, the maximum maturity of the yield curve and 3,000 the number of observations used to calibrate the F&V model.

7.1 Quality of fit

The simulation procedure yields 250 interest rate term structures. Figure 4 shows the forecasted term structures obtained by various methods of simulation on October, 23 2002.

Figure 4: Interest-rate term structures forecasted versus realized on October, 23 2002



The EKF method gives the best fit to the observed interest rate term structure followed closely by the evolved method with ± 1 sigma[8]. In addition, we observe that the quality of

[8] The results of the EKF method have been compared to the results obtained from the evolved approach. In the latter, the simulation is performed in the same conditions as the EKF approach except for using GARCH(1,1) as a volatility estimation method instead of EKF.

forecast decreases as the Bollinger bands become larger. In the following section, we measure more precisely the simulation performance by computing the error of estimation.

7.2 Root Mean Square Error (RMSE)

On one hand, we obtain from the simulation 250 forecasted interest rate term structures. On the other hand, we observe 250 realized interest rate term structures, we are thus able to measure the performance of the simulation by computing the RMSE.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Forecasted\ \ yield - Realized\ \ yield)^2}$$

<u>Figure 5: RMSE for different maturities of the forecasted interest-rate term structures versus realized on the whole sample (252 days)</u>

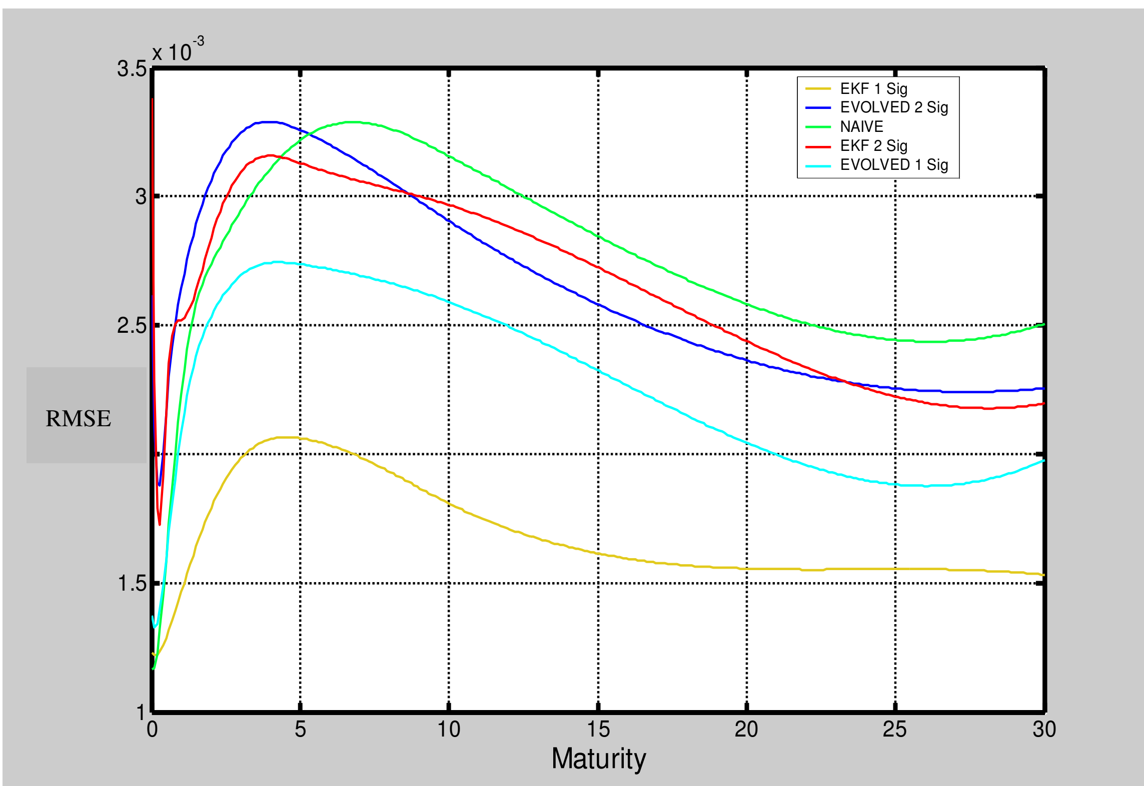


Figure 5 shows the term structures of the RMSE obtained from different methods. We observe that the EKF approach with Bollinger bands and antithetic variable with ± 1 sigma performs best. Followed by the same approach except that the volatility has been estimated by GARCH (1,1). The evolved approach with ± 2 sigma whatever the method of volatility estimation (EKF or GARCH) performs better than the naïve approach[9].

In addition, we observe that the RMSE term structures are downwarding. One of the possible reasons is that this method probably does not use all the information about the factor values contained in the cross-sectional dimension.

In order to measure the exact contribution of introducing the Bollinger bands to the F&V and EKF models, we perform the F-ratio test on the RMSE.

Table 2: **F&V+EKF** are the RMSE computed with the plain F&V model coupled to EKF. **F&V+EKF Boll 2-Sig** and **F&V+EKF Boll 1-Sig** are the RMSE obtained with F&V+EKF model improved by the reduction technique which is the Bollinger bands with two levels of sigma. We perform two different F-ratio tests.

Test 1: $H_0 : \text{RMSE}_{0\sigma} = \text{RMSE}_{1\sigma}$ versus $H_1 : \text{RMSE}_{0\sigma} \neq \text{RMSE}_{1\sigma}$

Test 2: $H_0 : \text{RMSE}_{0\sigma} = \text{RMSE}_{2\sigma}$ versus $H_1 : \text{RMSE}_{0\sigma} \neq \text{RMSE}_{2\sigma}$

With: $\alpha = .05$, $df_1 = 241$, $df_2 = 240$.

	Root Mean Square Error		
Time to maturity in years	**F&V+EKF Boll 2-Sigma**	**F&V+EKF Boll 1-Sigma**	**F&V+EKF**
0,0833	0,0021976	0,0012302	0,0017157
0,25	0,0022187	0,0014267	0,0017856
0,5	0,0023247	0,0017646	0,0023786
1	0,00244	0,0019703	0,0041471
2	0,0025822	0,0020256	0,0063822
5	0,0026933	0,0016508	0,0091518
10	0,0029729	0,001552	0,0066138
20	0,0025283	0,0015518	0,0040684
30	0,0033794	0,0015318	0,0038321
Total RMSE	**0,0233371**	**0,0147038**	**0,0400753**
F-ratio test (F_{stat})	***1,717235646***	***2,725506332***	$F_{(241,240,.05)} \approx 1.24$

[9] The naïve approach consists on computing the spreads between the 3-month CDOR over the yields composing the term structure. These spreads are assumed to be constant in the next 20 days. Only the reference 3-month CDOR will be simulated overtime to obtain the forecasted interest-rate term structure.

H_0 is rejected in both tests since $F_{stat} > F_{(241,\ 240,.05)}$. We conclude that the differences in RMSE are significant for the two levels of Sigma (One-sigma and Two-sigma) used in the Bollinger bands technique compare to F&V coupled to EKF *without* Bollinger bands. This result suggests that associating Bollinger bands to the F&V model and EKF increases the performance of the Monte Carlo simulation in term of reducing the estimation error. Moreover, the RMSE decreases as we make the interval of Bollinger bands narrower (from ± 2 sigma to ±1 sigma).

8. Conclusion

We proposed a method of forecasting the interest-rate term structure. This method is based on applying the EKF to the F&V model (1992). We found that the estimation of the unobservable component approach by EKF improved significantly the 20-day forecast of the yield curve.

Furthermore, we observed a drastic improvement of the RMSE by using the Extended Kalman Filter instead of the GARCH(1,1) method when the two methods are separately applied to the F&V model. We conclude of the superiority of the EKF method over the GARCH(1,1) method to estimate the volatility.

In addition, the test of equality of mean applied to the RMSEs provided by the F&V model coupled to EKF and provided by the addition of the Bollinger bands technique suggests that the Bollinger bands technique improves significantly the Monte Carlo simulation when it is applied to the F&V model.

However, we have seen that one of the fundamental hypotheses of the EKF is that the errors should be Gaussian which is not the case in our model. As indicated by De Jong (2000), the EKF in this situation leads to inconsistent estimation of parameters, though without high bias.

Therefore, we can suggest for future researches that other filtering techniques suitable for nonlinear models with non-Gaussian errors are necessary. Nevertheless, the use of Kitagawa method (1987) removes the inconsistency problem (see De Jong, 2000) that comes with the use of Kalman filter. This technique has been used by Danilov and Mandal (2000) to estimate stochastic volatility in two-factor short rate models.

Another way of research would be to compare our method to other methods of forecasting the interest-rate term structure such as Diebold and Li (2003).

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