Numerical approximation of the implied volatility under arithmetic Brownian motion

Jaehyuk Choi*, Kwangmoon Kim†, and Minsuk Kwak†

June 1, 2007

Abstract

We provide an accurate approximation method for inverting an option price to the implied volatility under arithmetic Brownian motion. The maximum error in the volatility is in the order of 10^{-10} of the given option price and much smaller for the near-the-money options. Thus our approximation can be used as a near-exact solution without further refinements of iterative methods.

Keyword : implied volatility, arithmetic Brownian motion, ABM, Bachelier, rational approximation, closed form approximation

1 Introduction

The French mathematician, Bachelier (1900) pioneered the study of financial Mathematics by investigating option pricing in his doctorate thesis several decades before Black and Scholes (1973) and Merton (1973) did. Under the Bachelier model, the forward price of an underlying security F_t follows an arithmetic Brownian motion with a volatility σ,

$$d\,F_t = \sigma\; d\,W_t. \tag{1}$$

whereas, in the Black-Scholes-Merton model, the price follows a geometric Brownian motion. The forward prices of call and put options, maturing in time T and struck at K with the current forward price F_0, are accordingly given by

$$C = E\{\max(F_T - K, 0)\} = (F_0 - K)\,N(d) + \sigma\sqrt{T}\,n(d) \tag{2}$$

$$P = E\{\max(K - F_T, 0)\} = (K - F_0)\,N(-d) + \sigma\sqrt{T}\,n(-d) \tag{3}$$

*Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

†Department of Mathematical Sciences, Korea Advanced Institute of Science and Tehonology, 335 Gwahangno, Yuseong-gu, Daejeon, Republic of Korea

Electronic copy available at: http://ssrn.com/abstract=990747

where $n(\cdot)$ and $N(\cdot)$ are the probability density function and the cumulative distribution function of the standard normal distribution respectively and d is the moneyness measured in the units of the standard deviation,

$$d = (F_0 - K) / (\sigma\sqrt{T}). \tag{4}$$

The implied volatility under arithmetic Brownian motion is the volatility σ that produces the given option price from Eqns. (2) or (3). For the rest of the article, this implied volatility will be referred as the *normal implied volatility* as opposed to the *lognormal implied volatility* from the Black-Scholes-Merton model.

Although the Bachelier model is seemingly obsolete because it allows the underlying price to become negative, its use can not be neglected. For the pricing of options whose underlying asset value is not necessarily positive, arithmetic Brownian motion is more appropriate for the stochastic process than geometric Brownian motion. Spread options actively traded in fixed income and commodity markets are such an example (Poitras, 1998; Carmona and Durrleman, 2003). In those products, the normal implied volatility is naturally used to quote the option price.

Even when the underlying price is positive, the normal implied volatility provides greater insight than the lognormal implied volatility if the price process is closer to an arithmetic Brownian motion. Interest rate derivatives are an example of such asset classes. The downward sloping volatility skew observed in the lognormal volatility space of the swaption market is much reduced in the the normal volatility space. Thus traders often use the normal volatility for the purpose of the in-house risk management.

No analytic expression is known for the normal or the lognormal implied volatility. The calculation always depends on iterative root-finding methods such as bisection or Newton-Raphson methods. With the computation powers available these days, running such methods for daily risk management takes only a fraction of a second. However, it is necessary to carefully fine-tune the solver to ensure convergence for all possible ranges of parameters. Failure to perform this task can lead to an unavailable hedge ratio, an important operational risk.

For the lognormal volatility Brenner and Subrahmanyam (1988); Chance (1996); Chambers and Nawalkha (2001); Li (2007) have made several attempts to find approximations. However, Brenner and Subrahmanyam (1988) only dealt with the at-the-money case and Chance (1996); Chambers and Nawalkha (2001); Li (2007) dealt with limited range of moneyness. Even within the proper range restricted by Li (2007), the error is still large compared to the machine precision, and thus additional refinements of iterative methods are still required.

To the best of authors' knowledge, no such research has been reported on the normal implied volatility. As it turns out, the task of inverting option prices is much simpler in the Bachelier model than in the Black-Scholes-Merton model. In Sec. 2, we show that the option pricing from the normal volatility is reduced to a one-dimensional problem in a non-dimensional form. We then transform variables so that they are better suited for approximation. In Sec. 3, we present the approximation process in detail and the error of our best approximation. In Sec. 4, we summarize with the actual coefficients.

Electronic copy available at: http://ssrn.com/abstract=990747

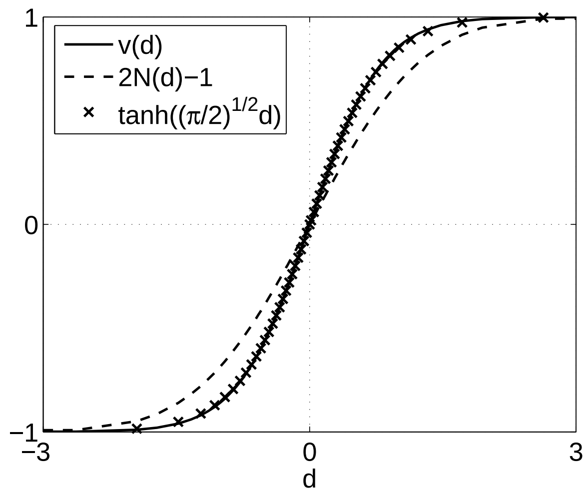

Figure 1: $v(d)$ (solid) compared with $2N(d)-1$ (dash) and $\tanh(\sqrt{\pi/2}\,d)$ (cross).

2 Non-dimensionalization and transformation

Without loss of generality we will only invert the price of the straddle, a combination of one call and one put option at the same strike, into the implied volatility. Straddle is traded as much as call or put options in the market as a bet on the volatility. If a call or put option is to be inverted, the price of the straddle with the same strike can be obtained from the put-call parity. The price of the straddle,

$$C+P = (F_0 - K)\,(2N(d) - 1) + 2\sigma\sqrt{T}\,n(d), \tag{5}$$

has the attractive property that it is symmetric on the sign of the moneyness $F_0 - K$. Thus, we only consider $F_0 \geq K$ ($d \geq 0$).

As d is non-dimensional, the natural non-dimensional form of Eq. (5) is

$$v = \frac{F_0 - K}{C+P} = \frac{d}{d(2N(d) - 1) + 2n(d)}. \tag{6}$$

We define the variable v in such a way that it is bounded in the range $[-1, 1]$, since the straddle price is always worth more than the intrinsic value $|F_0 - K|$. Thus, finding σ for given $C+P$, F_0, K and T is now equivalent to finding the inverse of $v(d)$.

The shape of $v(d)$ is similar to that of $2N(d) - 1$ (see Fig. 1); both functions asymptotically approach 1 in the same manner as d increases:

$$\log(1 - v(d)) \sim \log(1 - N(d)) \sim -d^2/2 \quad \text{as} \quad d \to \infty. \tag{7}$$

We could follow the steps used to obtain the inverse of the normal cumulative function by Acklam (2004). However, the direct inversion of $v(d)$ causes a slight

Electronic copy available at: https://ssrn.com/abstract=990747

problem for at-the-money options. When $F_0 = K$, the straddle price is given as

$$C+P = \sqrt{\frac{2T}{\pi}}\,\sigma \tag{8}$$

and the implied volatility is obvious. To ensure the continuity from the near-the-money to the at-the-money options, it is required that $\lim_{d\to 0} d(v)/v = \sqrt{2/\pi}$. As theories on the approximation are mostly on the value of the function, the condition on the derivative could be difficult to impose.

Our alternative approach is to view the not-at-the-money case as a perturbation from the at-the-money case; we modify Eq. (8) as

$$C+P = \sqrt{\frac{2T}{\pi}}\,\frac{\sigma}{h} \qquad \text{or} \qquad h = \sqrt{\frac{2}{\pi}}\,\frac{v}{d}, \tag{9}$$

and approximate h as a function of only v. The implied volatility is then calculated as

$$\sigma = \sqrt{\frac{\pi}{2T}}\,(C+P)\,h. \tag{10}$$

The introduced perturbation h is bounded between [0, 1]. Now the continuity condition, $h = 1$ at $v = 0$ ($d = 0$), becomes a part of the approximation of the function value.

To better approximate h, it is necessary to properly transform v. As shown in Fig. 2, the decay of h near $v = 1$ is extremely steep, thus making the direct approximation, $h = h(v)$, quite difficult. Under the following transformation,

$$\eta = \frac{v}{\tanh^{-1}(v)} = \frac{2v}{\log((1+v)/(1-v))}, \tag{11}$$

we find that the behavior of h as a function of η is benign (see Fig. 3). The intuition behind the transformation is that $v(d)$ and $\tanh(\sqrt{\pi/2}\,d)$ are very similar (see Fig. 1) and the inverse of $\tanh(\cdot)$ is analytically known. Therefore, η defined above is almost equal to h itself. The function $h(\eta)$ has a diverging derivative at $\eta = 0$. However, the singularity can be easily removed since we know the leading asymptotic term, $h(\eta) \sim \sqrt{\eta/(2\pi)}$, from Eqs. (7) and (9). Therefore, we approximate $h(\eta)/\sqrt{\eta}$ (see Fig. 3) rather than $h(\eta)$.

3 Numerical approximation and error

To find a good approximation of $h(\eta)/\sqrt{\eta}$, we use a rational Chebyshev approximation, (Press et al., 1992)

$$\frac{h(\eta)}{\sqrt{\eta}} \approx g(\eta) = \sum_{k=0}^{n} a_k\,\eta^k \Big/ \sum_{k=0}^{m} b_k\,\eta^k \quad (b_0 = 1). \tag{12}$$

It is known that the rational function produces a better approximation than the polynomial approximation with the same number of coefficients. The coefficients

Electronic copy available at: https://ssrn.com/abstract=990747

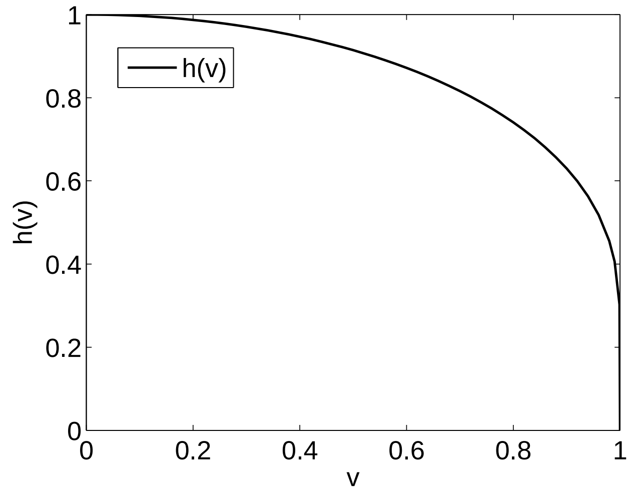

Figure 2: The perturbation h as a function of v.

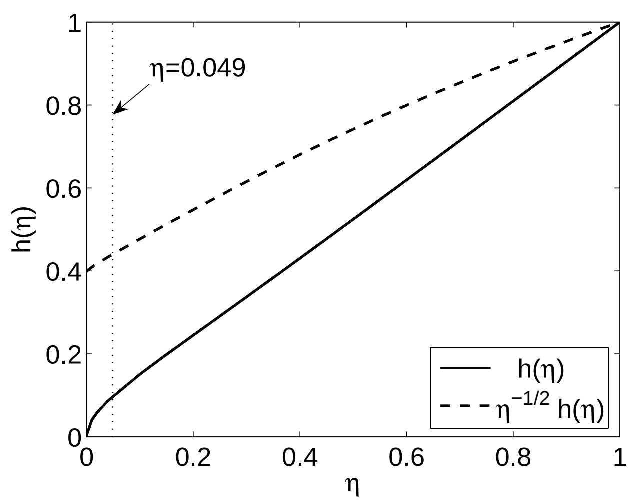

Figure 3: The perturbation h as a function of the transformation η.

Electronic copy available at: https://ssrn.com/abstract=990747

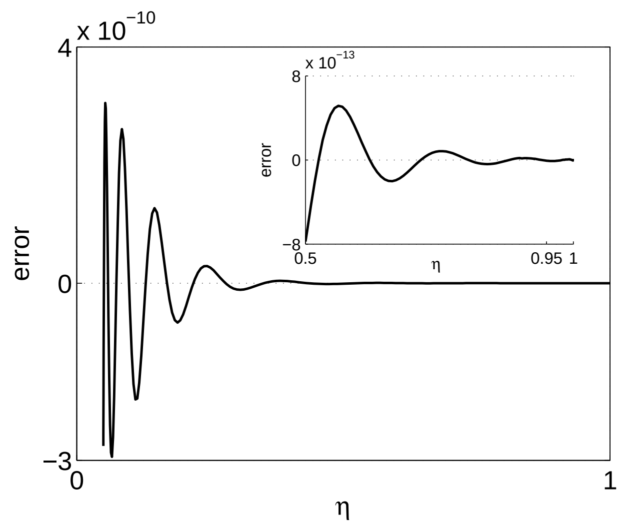

Figure 4: The error in $h(\eta)$ from the rational approximation.

are searched with an iterative algorithm so as to minimize the maximum error, $\max_\eta |\, h(\eta)/\sqrt{\eta} - g(\eta)\,|$ in the present case, in the original implementation of Press et al. (1992). However, we modify the code so that the error in $h(\eta)$, $\max_\eta |\, h(\eta) - g(\eta)\sqrt{\eta}\,|$, is minimized instead. We calculate the exact value of h for a given η up to the machine precision using bisection method.

It should be noted that it is sufficient to restrict the approximation domain of η to $[0.049,\ 1]$. When d is large, options are far-out-of-the-money. The straddle price becomes very close to the intrinsic value $|F_0 - K|$, and thus v and η approach 1 and 0, respectively. The smallest value of η occurs when the straddle price is barely distinguishable from the intrinsic value, that is, at $v = 1 - \epsilon$, where ϵ is the machine epsilon. In double precision, it corresponds to $\eta \approx 2/\log(2/\epsilon) \approx 0.054$ ($d \approx 7.7$). Our lower bound, 0.049, is the limit in approaching zero without compromising the accuracy of the approximation. Since the singularity of h at $\eta = 0$ is excluded from the domain, we could target $h(\eta)$ for approximation. However, we still use $h(\eta)/\sqrt{\eta}$ as it produces a better result. It appears that the concave shape of $h(\eta)/\sqrt{\eta}$ is better suited for a rational approximation with $m > n$. With this choice, we can also ensure that $h(\eta) \to 0$ as $\eta \to 0$ despite the error in $h(\eta)/\sqrt{\eta}$ for $\eta < 0.049$.

The set of coefficients, a_k and b_k, yielding the best accuracy is reported in the summary section below. We find the degrees, $n = 7$ and $m = 9$, provide the best fit; extra degrees did not reduce the error significantly. Fig. 4 shows the error of h. The maximum error is 3.4×10^{-10}. If one reprices a portfolio of straddles with the implied volatility from the approximation, the error in the price,

$$\Delta(C+P) = \frac{\Delta h}{h}(C+P) \leq 3.4 \times 10^{-9}\,(C+P). \tag{13}$$

Electronic copy available at: https://ssrn.com/abstract=990747

is about \$3 for a \$1 billion portfolio. For $\eta > 0.5$ ($d < 1.46$), where most practical implied volatility calculations fall, the error is much smaller as 8.2×10^{-13} (see the inset of Fig. 4). For near-the-money options, $\eta > 0.95$ ($d < 0.32$), the error is 8.8×10^{-15}. Thus, Eq. (8) is practically reduced for the at-the-money options.

4 summary

The implied volatility under arithmetic Brownian motion can be accurately obtained from the following expression:

$$\sigma = \sqrt{\frac{\pi}{2T}}\,(C+P)\,h(\eta), \tag{14}$$

where

$$\eta = \frac{(F_0 - K)/(C+P)}{\tanh^{-1}\left((F_0 - K)/(C+P)\right)}, \quad h(\eta) = \sqrt{\eta}\,\frac{\sum_{k=0}^{7} a_k \eta^k}{\sum_{k=0}^{9} b_k \eta^k}, \tag{15}$$

$$\begin{aligned}
a_0 &= \texttt{3.99496 168734 5134 e-1} & b_0 &= \texttt{1.00000 000000 0000 e+0} \\
a_1 &= \texttt{2.10096 079506 8497 e+1} & b_1 &= \texttt{4.99053 415358 9422 e+1} \\
a_2 &= \texttt{4.98034 021785 5084 e+1} & b_2 &= \texttt{3.09357 393674 3112 e+1} \\
a_3 &= \texttt{5.98876 110269 0991 e+2} & b_3 &= \texttt{1.49510 500831 0999 e+3} \\
a_4 &= \texttt{1.84848 969543 7094 e+3} & b_4 &= \texttt{1.32361 453789 9738 e+3} \\
a_5 &= \texttt{6.10632 240786 7059 e+3} & b_5 &= \texttt{1.59891 969767 9745 e+4} \\
a_6 &= \texttt{2.49341 528534 9361 e+4} & b_6 &= \texttt{2.39200 889172 0782 e+4} \\
a_7 &= \texttt{1.26645 805134 8246 e+4} & b_7 &= \texttt{3.60881 710837 5034 e+3} \\
& & b_8 &= \texttt{-2.06771 948640 0926 e+2} \\
& & b_9 &= \texttt{1.17424 059930 6013 e+1}
\end{aligned} \tag{16}$$

References

L. Bachelier. Théorie de la spéculation. *Annales Scientifiques de L'Ecole Normale Supérieure*, 17:21–88, 1900.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, 81(3):637–654, 1973.

R. C. Merton. The theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

G. Poitras. Spread options, exchange options and arithmetic brownian motion. *Journal of Futures Markets*, 18(5):487–517, 1998.

R. Carmona and V. Durrleman. Pricing and hedging spread options. *SIAM Review*, 45(4):627–685, 2003.

Electronic copy available at: https://ssrn.com/abstract=990747

M. Brenner and M. G. Subrahmanyam. A simple formula to compute the implied standard deviation. *Financial Analyst Jounal*, 5:80–83, 1988.

D. M. Chance. A generalized simple formula to compute the implied volatility. *The Financial Review*, 31:859–867, 1996.

D. R. Chambers and S. K. Nawalkha. An improved approach to computing implied volatility. *The Financial Reveiw*, 38:89–100, 2001.

M. Li. Approximate inversion of the black-scholes formula. *European Journal of Operational Research*, forthcoming, 2007.

P. J. Acklam. *An algorithm for computing the inverse normal cumulative distribution function.* `http://home.online.no/~pjacklam/notes/invnorm/`, 2004.

W. H. Press, S. A. Teukolsky, W.T. Vetterling, and Flannery B. P. *Numerical Recipes in C: The Art of Scientific Computings*, chapter 5.13, pages 204–208. Cambridge University Press, 2nd ed. edition, 1992.

Electronic copy available at: https://ssrn.com/abstract=990747