Hedging Errors

We relax the Black-Scholes assumptions of known volatility and IID returns

LIHONG
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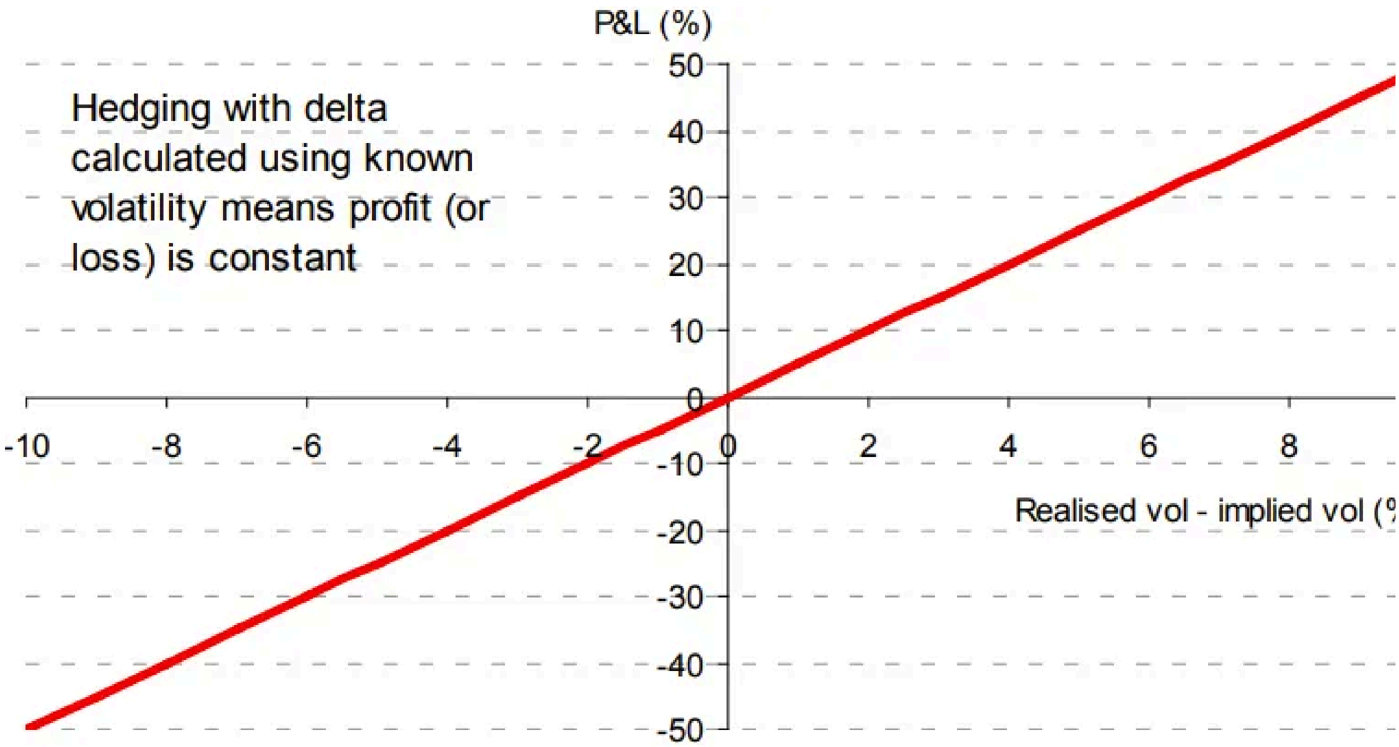
We present the main results in dynamic hedging under uncertainty and explore the implications of non-negligible correlation. We note that the optimal delta hedging schedule changes significantly based on returns correlation. In addition, estimates of historical volatility are biased under non-negligible correlation. Main references include a paper by Ahmad and Wilmott and Trading Volatility by Colin Bennett.

Overview of Hedging Errors (Graphs from Trading Volatility page 90 - 98)

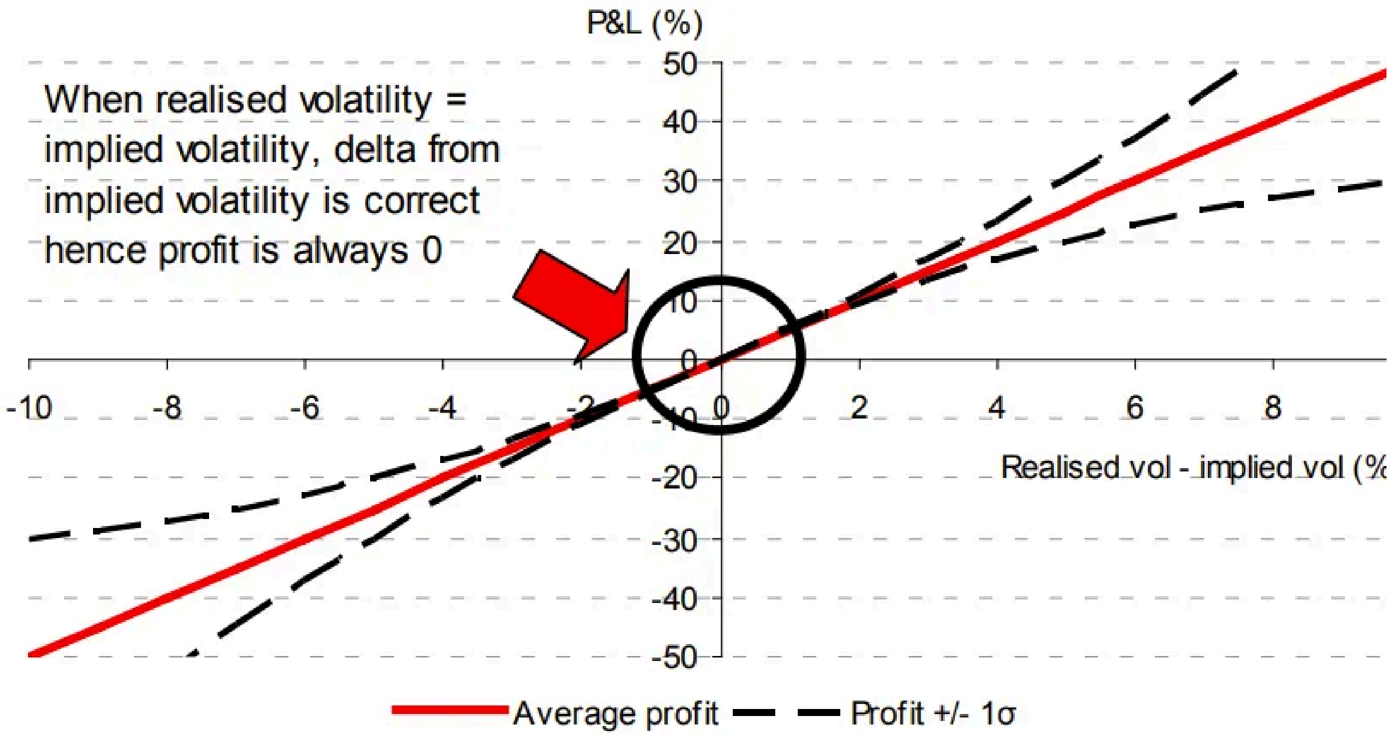
In a Black-Scholes world, the volatility of a stock is constant and known. As the position is initially delta-neutral (i.e., delta is zero), the gamma (or convexity) of the position generates a profit for both downward and upward movements. Although tl effect is consistently profitable, the position does lose time value (due to theta). If a: option is priced using the actual fixed constant volatility of the stock, these two effe balance each other out, and the position does not earn an abnormal profit or loss as the return equals the risk-free rate (theta pays for gamma).

The profit from delta hedging is equal to the difference between the actual price an the theoretical price. If an option is purchased at an implied volatility lower than tl realized volatility, the difference between the theoretical price and the actual price equal the profit of the trade. The relationship between profit and the difference in implied and realized volatility is direct. As theta and gamma are correlated, profits not path dependent when hedged with the delta implied by the true volatility.



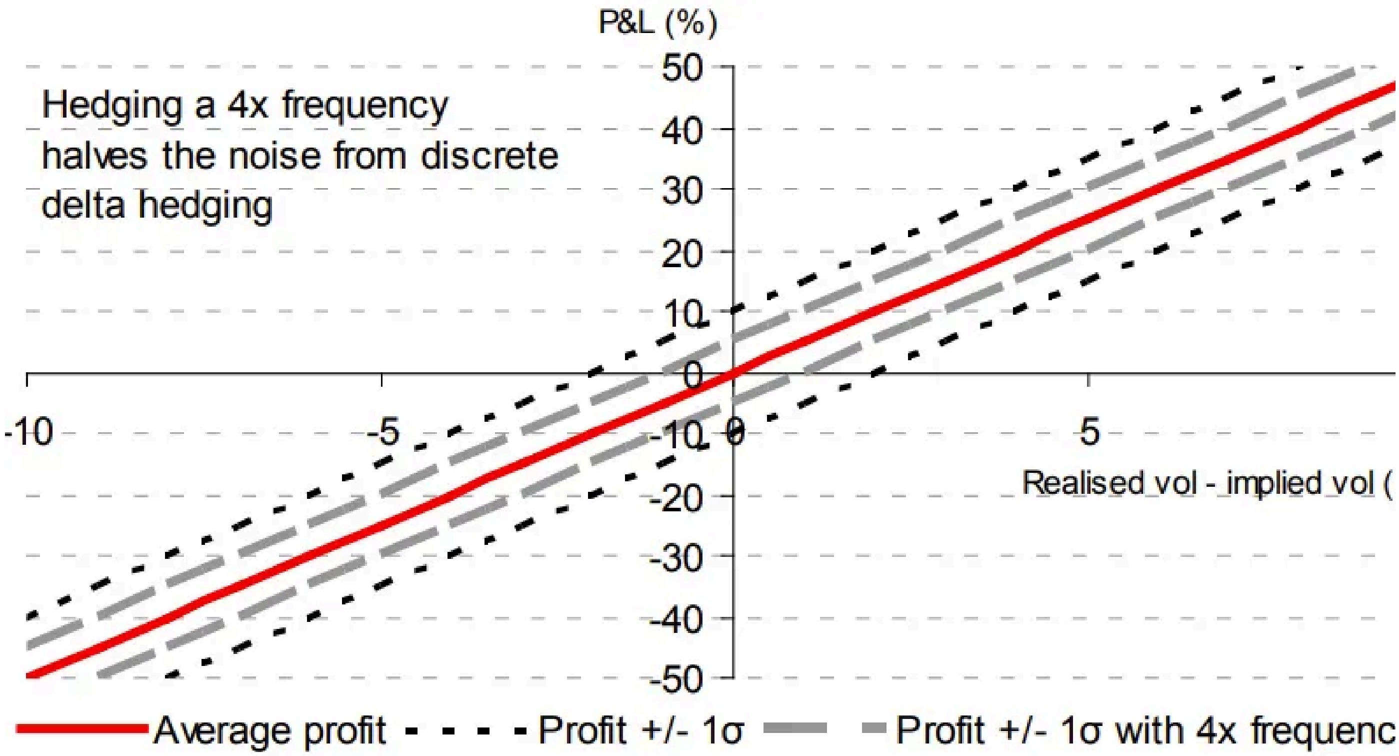
In reality, it's impossible to know the future volatility of a security in advance. As a result, one may use implied volatility to calculate deltas. However, delta hedging us this estimate introduces risk to the position, making it path dependent. Although t expected profit remains unchanged, the actual profits from delta hedging are no longer independent of the direction in which the underlying moves. This path dependency arises from the discrepancy between the correct delta (calculated using the remaining volatility to be realized over the life of the option) and the delta calculated using the implied volatility.

When there's a difference between the actual delta and estimated delta, there is risl but not enough to make a cheap option unprofitable (or an expensive option profita as long as hedging is continuous. This is because in each infinitesimally small amou of time, a cheap option will always reveal a profit from delta hedging (net of theta), although the magnitude of this profit is uncertain. The extent of potential variation profit is related to the difference between implied and realized volatility; a greater difference leads to an increased potential range of outcomes.

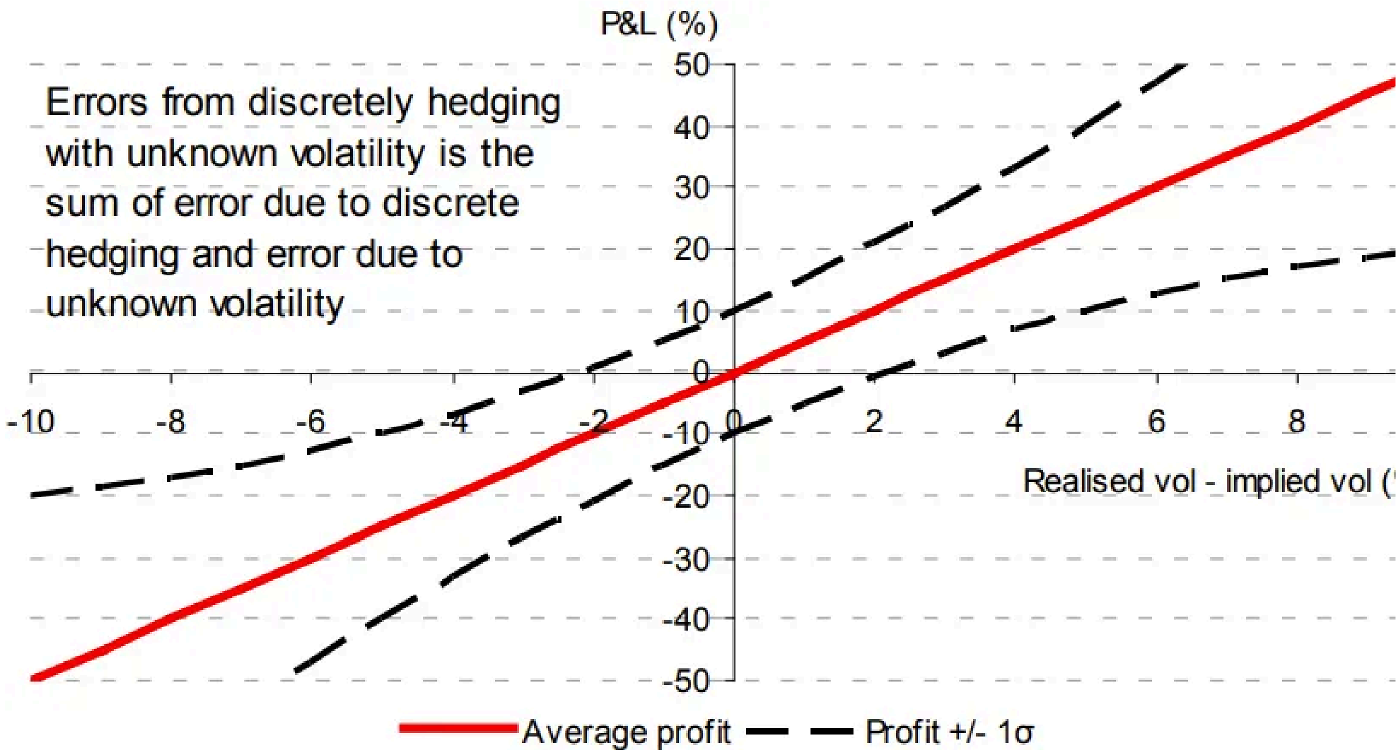


Discrete delta hedging with known volatility is a practical necessity, as continuous hedging is impossible due to trading costs, minimum trading sizes, limited trading hours, and non-trading periods like weekends. The accuracy of discrete hedging improves with increased frequency, with returns approaching those of continuous hedging in an ideal scenario of 24-hour trading and infinitely frequent hedging.

Importantly, hedging error is independent of average trade profitability. With know volatility, delta calculation errors are eliminated, and the introduced variation is essentially noise. This independence means that, unlike in continuous hedging whe it's impossible to lose money on a cheap option, discrete hedging allows for potenti losses on cheap options and gains on expensive ones.



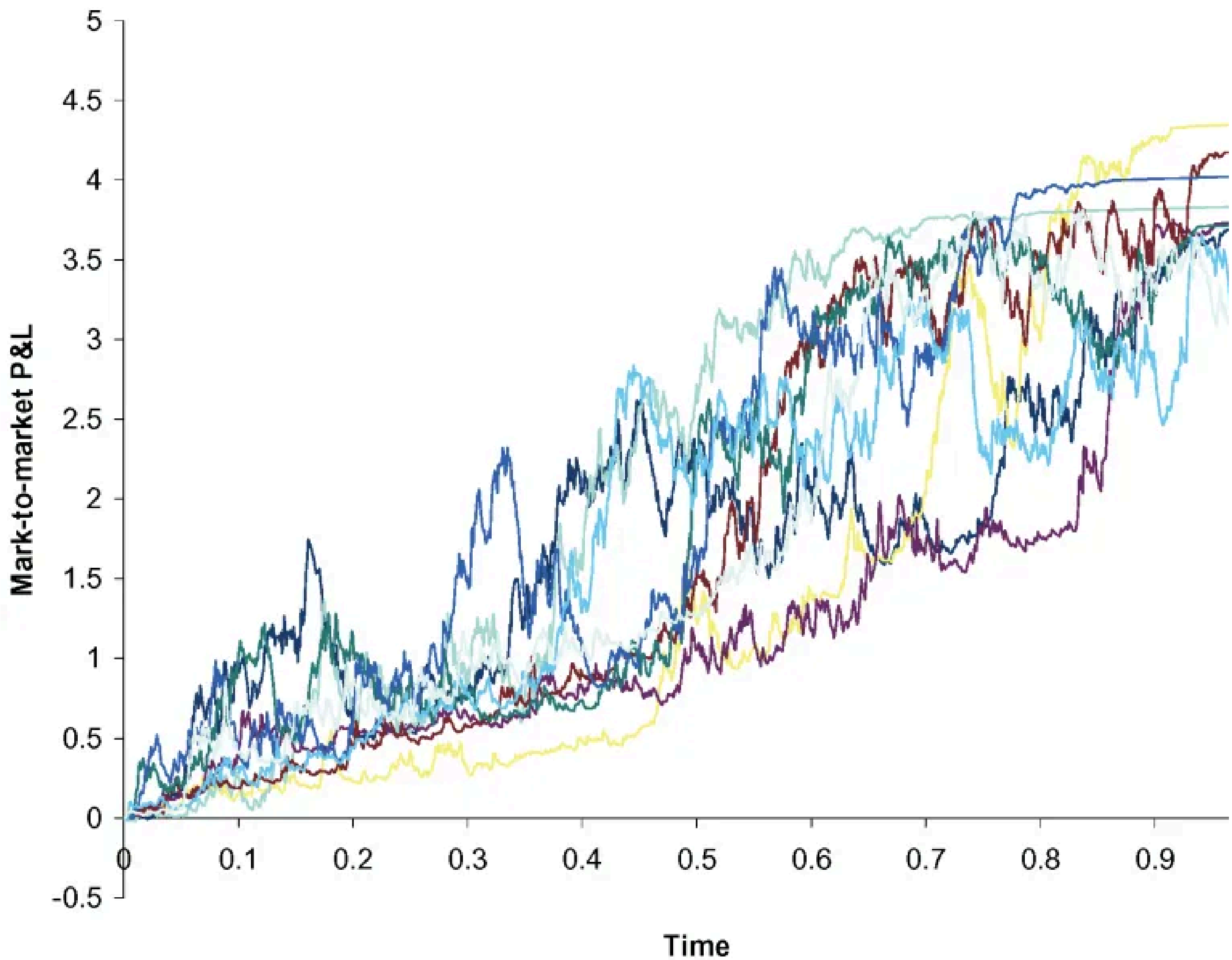
The most realistic scenario for assessing profitability in options trading combines discrete delta hedging with unknown volatility. The two primary sources of variatic in profit or loss are now combined: the inherent variation from discrete (rather thar continuous) hedging, and the potential inaccuracy of the calculated delta due to unknown future volatility.



A paper by Ahmad and Wilmott gives the formula for PnL under different hedging schemes under slightly loosened Black-Scholes assumptions. Where V(a) is the valu the option under the actual realized volatility and V(i) is the value of the option und implied volatility, the lifetime PnL for hedging with the real vol is

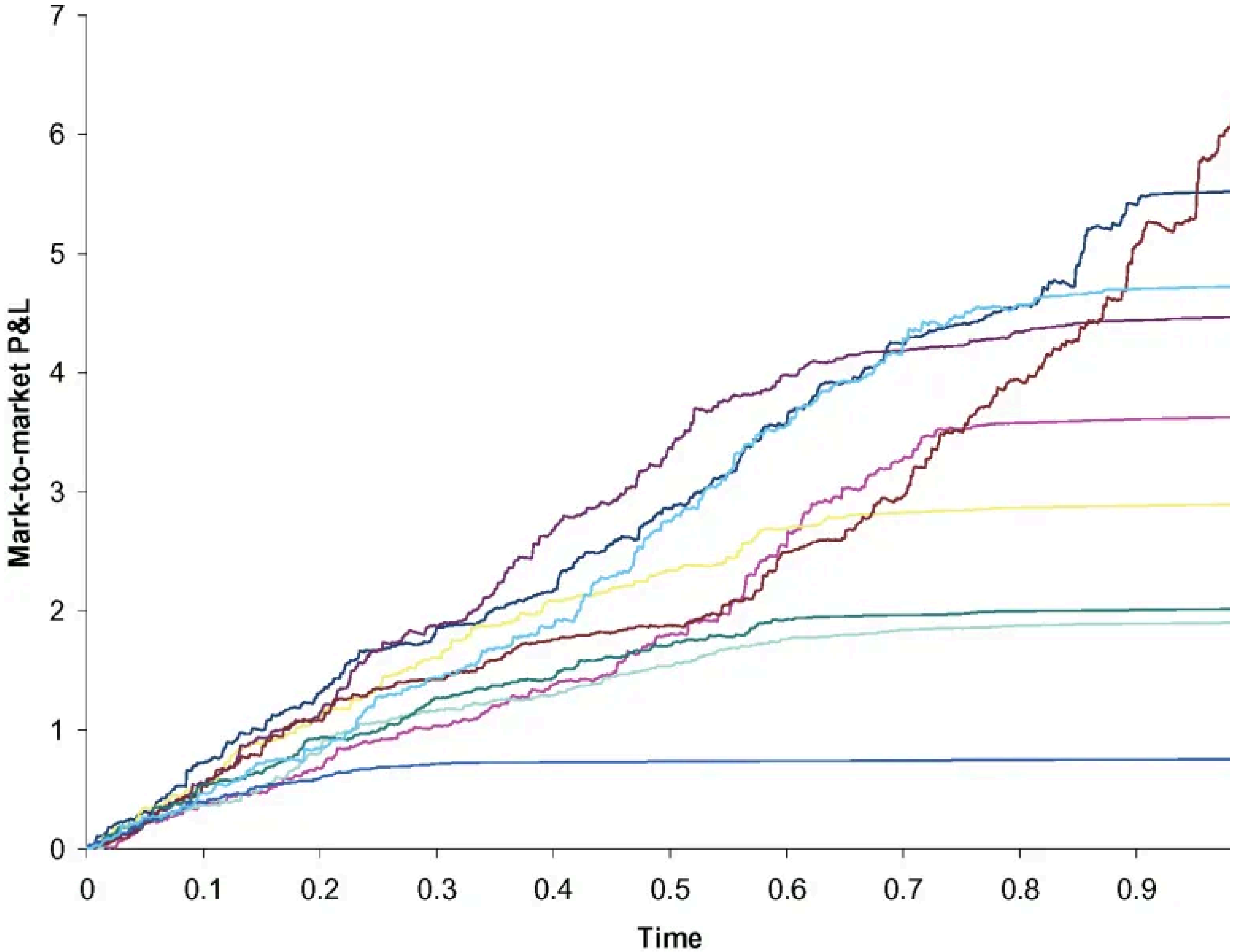
$$e^{rt_0}\int_T^{t_0} d(e^{-rt}(V_i - V_a)) = V_a - V_i$$

The final PnL is in theory not path dependent. The small discrepancies in the final PnL is due to discrete hedging in the simulation

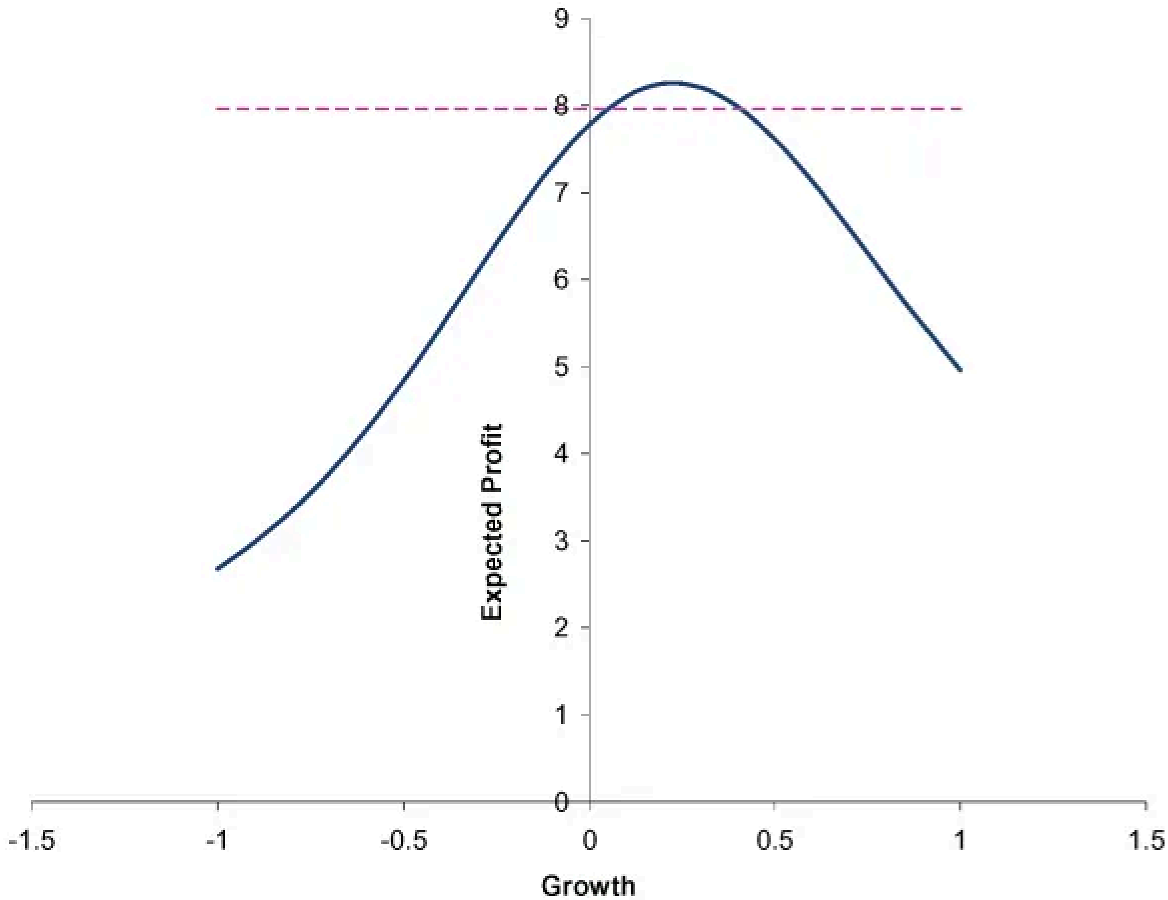


As previously noted, if we hedge with implied vol, then profits are path dependent
buying a cheap option will always net a positive profit. The PnL is given by

$$2(\sigma^2 - \tilde{\sigma}^2) \int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma_i dt$$



Is there some characteristic to the "optimal" path that gives us the most profit whe we hedge with implied vol while still assuming geometric Brownian motion? The answer is yes. We know that gamma peaks at the money as expiration comes near, a so to maximize gamma exposure **the expected profit is maximized roughly at the growth rate that ensures that the stock ends up close to at the money** at expiratio



Expected profit, hedging using implied volatility, versus growth rate μ; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. The dashed line is the profit to be made when hedging with actual volatility.

Hedging Error & Hedging Frequency

We now give some intuition to an approximation of gamma PnL variance as a funct of hedging frequency. Recall that Gamma PnL for a hedged position over n periods be given by the term

$$\frac{1}{2}\Gamma\epsilon_1^2 + \frac{1}{2}\Gamma\epsilon_2^2 + \ldots + \frac{1}{2}\Gamma\epsilon_n^2$$

The gamma PnL for an unhedged position can be given by

$$\frac{1}{2}\Gamma\left(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n\right)^2$$

Where epsilon represents small moves in the price of the underlying such that gam can be assumed to be constant during that move. If we assume that excess returns c the underlying are normally distributed with zero mean and variance sigma, we can extract those variables and instead write the above two terms as

$$\tfrac{1}{2}\Gamma S^2\sigma^2 \textstyle\sum_{i=1}^n x_i^2, \quad \text{and} \quad \tfrac{1}{2}\Gamma S^2\sigma^2 \left(\textstyle\sum_{i=1}^n x_i\right)^2$$

where the xi terms are drawn iid from the standard / unit normal distribution with zero mean and variance of 1. It follows that the square of xi for any i is drawn from Chi-squared distribution with one degree of freedom by definition. We can check tl these two terms have the same expectation since

$$\left(\textstyle\sum_{i=1}^n x_i\right)^2 = \textstyle\sum_{i=1}^n x_i + \textstyle\sum_{j=1}^n \textstyle\sum_{i=1}^{j-1} x_i x_j$$

For each i not equal to j, the expectation of their production is 0 since the terms are drawn iid. To compare their variances, it is sufficient to compare the sum of square versus the square of the sum. The sum of squares is trivial; for the square of the sun we define X to be equal to the sum form x1 to xn. Then, we note that

$$\mathrm{Var}[X^2] = \mathrm{E}[X^4] - \mathrm{E}[X^2]^2 = \mathrm{E}[X^4] - (\mathrm{Var}[X] + \mathrm{E}[X]^2)^2$$

By the zero mean and unit variance of the xi terms we can write

$$\mathrm{Var}[X^2] = \mathrm{E}[X^4] - (n\mathrm{Var}[X_1] + 0)^2 = \mathrm{E}[X^4] - n^2$$

For the fourth moment of the sum X, we have

$$\mathbb{E}[X^4] = \sum_{i,j,k,l} \mathbb{E}[X_i X_j X_k X_l].$$

We make the crucial observation is that, if even one index is different from all the others then the term vanishes by zero mean. So removing all such terms from the summation, any surviving term must satisfy that for each index i there is another in which has the same value as i. This means that either (1) all the indices are the same

(2) that the indices pair up. There are n ways for all the indices to be the same. Ther n choose 2 (equal to n(n-1)/2) ways for us to get a pair of indices, and 4 choose 2 (equ to 6) ways to arrange them in 4 spots. As such, we have

$$\mathbb{E}[X^4] = n\mathbb{E}[X_1^4] + 3n(n-1)\mathbb{E}[X_1^2]^2 = n\mathbb{E}[X_1^4] + 3n(n-1)$$

We now aim to determine the expectation of the 4th moment. We integrate the pdf the unit normal distribution over the reals twice by parts

$$E[X^4] = \frac{\int x^4 e^{-x^2/2} dx}{\int e^{-x^2/2} dx} = \frac{-x^3 e^{-x^2/2} + 3 \int x^2 e^{-x^2/2} dx}{\int e^{-x^2/2} dx} = \frac{-3xe^{-x^2/2} + 3 \int e^{-x^2/2} dx}{\int e^{-x^2/2} dx}$$

Note that the left term on the numerator of the last fraction vanishes when evaluate the value of the moment is therefore 3. For normal distributions with non-unit variance the 4th central moment we can deduce to be 3 sigma ^4. In any case, our unhedged position has PnL variance proportional to

$$3n + 3n(n-1) - n^2 = 2n^2$$

Our hedged position has PnL variance proportional to

$$n(\mathrm{E}[x_i^4] - \mathrm{E}[x_i^2]^2) = n(3-1) = 2n$$

For n > 1, we see that the hedge position has a smaller variance. The gap is proportional to n(n-1). Here we have two competing intuitions; on one hand, by inspecting the case of n = 2 we note that hedging 2x vs 1x during an interval reduce the variance of the gamma PnL by 1/n(n-1) = 1/2. Applying this argument recursivel we see that hedging 4x as often brings down the standard deviation by 1/2.

On the other hand, we may be tempted to use the fact that n(n-1) is asymptotically n squared; for this case at least, it is more appropriate to use the first intuition. The i that gamma PnL volatility is inversely proportional to the square root of hedging frequency also accords with the intuition that diffusion of price is proportional to t square root of time. Lastly, we note that hedging error accumulates linearly with respect to the number of hedging periods if frequency is held constant i.e. it accumulates linearly with respect to time but is inverse to the square root of freque (a proof can be found on the appendix of Ed Throp's 1976 paper). So

$$\sigma_{PnL} \propto \frac{1}{2} S^2 \sigma^2 (T-t) \Gamma \sqrt{\frac{1}{N}}$$

Page 95 of Trading Volatility claims, without proof, that the normalization coefficie for this to become an equation is the square root of pi; if any reader can bother to fi / figure out the proof for this I'd be happy to see it in the comments.

		Call	Put
Delta	$\frac{\partial V}{\partial S}$	$N(d_+)$	$-N(-d_+) = N(d_+) - 1$
Gamma	$\frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_+)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial V}{\partial \sigma}$	$SN'(d_+)\sqrt{T-t}$	
Theta	$\frac{\partial V}{\partial t}$	$-\frac{SN'(d_+)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_-)$	$-\frac{SN'(d_+)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d$
Rho	$\frac{\partial V}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_-)$	$-K(T-t)e^{-r(T-t)}N(-d_-)$

We can also replace Gamma with Vega for a more interpretable formula; from abov we see that Vega is spot squared times vol times expiry times Gamma. This gives

$$\sigma_{PnL} = \text{Vega}\, \sigma \sqrt{\frac{\pi}{4N}}$$

Measuring Volatility

We now see that hedging with implied volatility have severe drawbacks to hedging with the true volatility. This fact (as well as the ability to take vega positions) has le many to attempt to model the true volatility of the underlying, typically partially ba on some measurement of historical volatility.

Building a good vol model is clearly very difficult, but even the task of measuring p vol is more challenging than it first appears. For example, selecting the appropriate number of days for calculating historical volatility (to be fed into a model) is a comp craft, since recent volatility spikes due to rare events may not accurately predict fut volatility. Moreover, one may sometimes think that recent data is more representati of the future than data from the long past, and as such the number of data points o has can at times be limited.

While close-to-close volatility are the most intuitive method, it does have the drawback of being one of the least efficient ways to measure volatility (A list of alternative volatility measures can be found on page 238 of [Trading Volatility](). Here the efficiency of an estimate x is defined as the variance of the close-to-close estim over the variance of estimate x. Regarding the price codes below, the letters correspond to: open (O), high (H), low (L) and close (C).

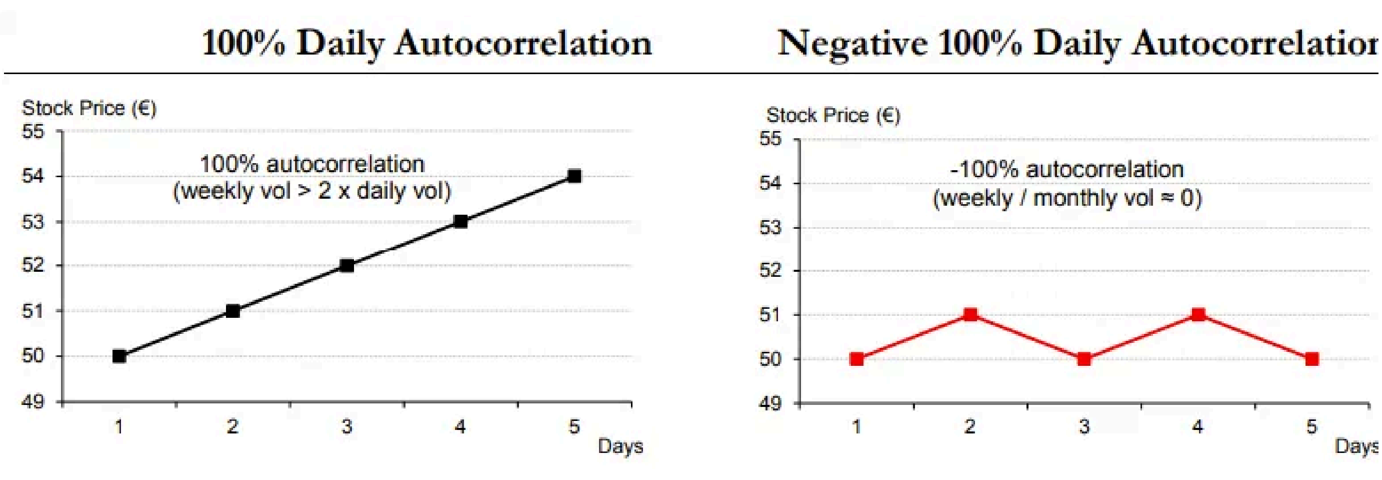
Summary of Advanced Volatility Estimates

Estimate	Prices Taken	Handle Drift?	Handle Overnight Jumps?	Efficiency (n
Close to close	C	No	No	
Parkinson	HL	No	No	
Garman-Klass	OHLC	No	No	
Rogers-Satchell	OHLC	Yes	No	
Garman-Klass Yang-Zhang ext.	OHLC	No	Yes	
Yang-Zhang	OHLC	Yes	Yes	

While variance is important in reducing the MSE of the estimate, so too is bias (whether the measure is, on average, too high or low). Generally, the close-to-close volatility estimator is biased high (it does not model overnight jumps), while many alternative estimators are biased low (they assumed continuous trading whereas discrete trading will have a smaller difference between max and min).

The Effects of Autocorrelation

The frequency of measurement—typically daily or weekly—also plays a crucial role Daily volatility is often preferred due to its larger data set, but weekly measurement can be more suitable when comparing long-term volatility across different markets they minimize the impact of varying public holidays and trading hours. The relationship between daily and weekly volatility also depends on the independence stock price returns.



In perfectly independent markets, these measures should align on average. Howeve trending markets tend to show higher weekly volatility, while mean-reverting mark may display the opposite effect (assuming markets return 1% daily with 100% correlation the weekly return is 5%, so while the daily vol is around 16% = 1 * sqrt(26 the weekly vol is around 35% = 5 * sqrt(52)) . In general, for iid random variables wit pairwise correlation rho > 0 and volatility sigma > 0 we may write

$$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \operatorname{cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i) + \sum_{i\neq j} \operatorname{cov}(X_i, X_j)$$

which indeed shows that the sum of variances under positive correlation is greater than the sum of variance under independence. In addition, given pairwise correlati rho, the average variance can be given by the term

$$\operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}(n\sigma^2 + n(n-1)\rho\sigma^2) = \rho\sigma^2 + \frac{1-\rho}{n}\sigma^2$$

The prevalence of high-frequency trading in modern markets has likely reduced the occurrence of simple momentum-based patterns, making significant autocorrelatio (either positive or negative) less common. Nevertheless, during periods of market stress, temporary negative autocorrelation may emerge as panic selling or automate trading systems exacerbate price movements. Perhaps a strategy that is long daily variance and short weekly variance will therefore usually give relatively flat returns but occasionally give a large positive return (not investment advice).

Hedging strategies are affected by autocorrelation. For instance, if an investor hedg for every 10% move in the underlying asset, they may realize larger profits in a stror trending market compared to hedging at 5% intervals. This is because allowing positions to run in trending markets can be more profitable than frequent re-hedgi However, this strategy also increases risk exposure.

In general, positive autocorrelation (i.e. momentum) suggests that price movements are more likely to continue in the same direction. In such scenarios, traders with lo gamma positions might benefit from not immediately hedging their delta exposure This is because these deltas have a positive expectancy and are likely to become mo valuable as the trend persists. Conversely, those with short gamma positions in positively autocorrelated markets might prefer more frequent hedging to protect against increasing delta exposure that could work against their position. The oppos holds true in markets exhibiting negative autocorrelation or mean-reverting tendencies. Here, traders with long gamma positions might lean towards more frequent hedging, as the deltas are prone to reversal and potential value loss. Short gamma positions in such markets might benefit from less frequent hedging, allowi the mean-reverting nature of the market to naturally reduce delta exposure over tin

Recap

We discussed the complexities of dynamic hedging under uncertainty, focusing on how profits from delta hedging relate to the difference between implied and realize volatility, and how this relationship becomes path-dependent when using estimatec volatility for hedging instead of known volatility. The discussion then delved into tl sources of variation in profit or loss, combining the effects of discrete delta hedging

We examined formulas for profit and loss under different hedging schemes and explored the concept of an "optimal" path for maximizing profit when hedging witl implied volatility. We included an analysis of hedging error and its relationship to hedging frequency, providing intuition for approximating gamma PnL variance.

Lastly, we touched on the challenges of measuring volatility accurately, discussing various estimation methods and their efficiencies. The effects of autocorrelation on volatility measurements and hedging strategies were also addressed, highlighting h market conditions can impact different hedging approaches.

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