LEHMAN BROTHERS
Government Bond Research

Options, Volatility, and the Greeks

Part I: Introduction

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DEFINITIONS

The principal building blocks of the options markets are puts and calls. **Puts represent a right (but not an obligation) to sell an asset at a prespecified price called the strike price. A call option represents a right (but not an obligation) to buy an asset at a prespecified strike price.** When the owner of a put or call option elects to buy or sell the asset at the prespecified price, he is said to exercise the option.

Different types of options may be exercised at different times. The most common types of options are European and American options. **European puts and calls may be exercised only at the expiration date. American options may be exercised at or before the exercise date.** Listed options on Treasury and Eurodollar futures are American options.

Another type of option, called Bermudan options, is related to American options. Bermudan options may be exercised before the expiration date but only on specific dates. These options are common in the market for options on interest rate swaps (swaptions) and are often embedded in callable fixed income securities that are "discretely callable."

INTRINSIC VALUE AND TIME VALUE

The prices of options consist of two components: intrinsic value and time value.

Intrinsic value is the amount of option payoff if the option were exercised today (the term is also used for European options even though they lack the right of early exercise). For calls, this is the asset price minus the exercise price. If the asset price minus the exercise price is negative, the intrinsic value of a call option is zero. For puts, intrinsic value is the strike price minus the asset price if the strike price minus the asset price is positive or zero if it is negative.

"Time value" is the residual value of an option above its intrinsic value. For example, if a bond's price is 101, and an investor owns a 100 call on the bond that is currently worth 1.25, then the intrinsic value of the option is 1.00 and the time value of the option is 0.25.

Options with positive intrinsic value are "in the money." Options with an underlying asset price equal to the strike price are "at the money," and those that, if exercised, would produce a loss are "out of the money."

PAYOFF STRUCTURES

Because options represent the right, but not the obligation, to buy or sell an asset at the strike price, **options are limited liability assets. Therefore, if an investor buys an option on an unhedged basis, the largest possible loss is the cost of the option.** For example, for an at-the-money European call on a 2-year note, if the note's price is less than the strike price, the option offers no economic value and expires worthless. In this event, the investor loses the premium originally paid for the option but nothing else. If the 2-year note's price is greater than the strike price, the investor exercises the option. If the note's price lies between the strike price and the strike price plus the premium paid for the option, the investor still loses money, but less than the cost of the option. If the note's price is equal to the strike price plus the option cost, the investor breaks even. Above that level, the investor profits by an amount equal to the note's price minus the exercise price. This payoff diagram is shown in Figure 1.

Figure 2 shows the payoff of a European put option on a 2-year note at expiration and 60 days prior to expiration.

Figure 1. **Call Price at Expiration and 60 Days Prior**
Strike price = 98.28125; option price = 9.25/32

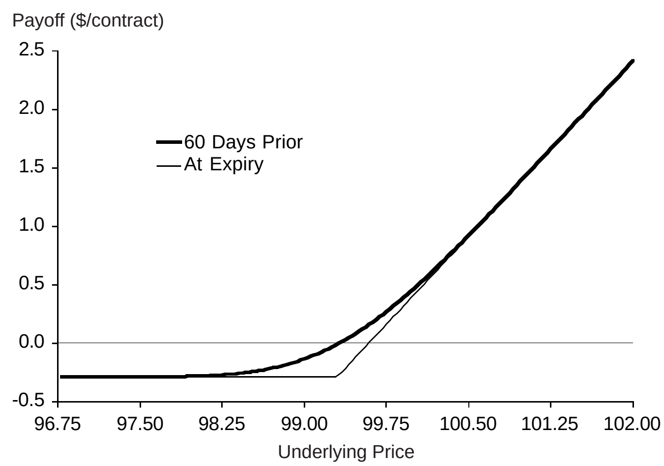
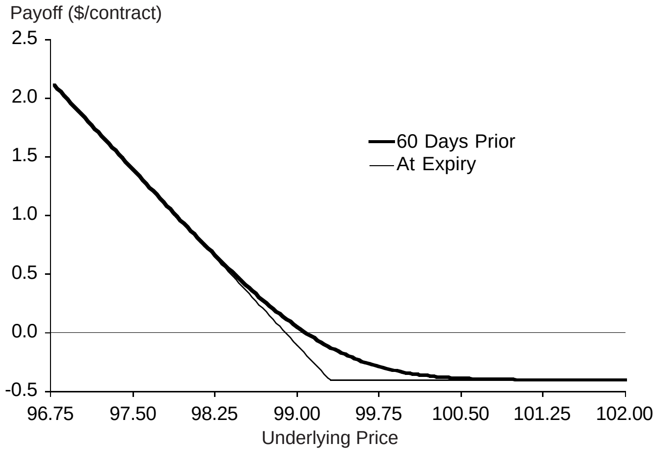


Figure 2. **Long At-the-money Spot Put Option: 60 Days before Expiration and at Expiry**
Strike price = 99.28125; option cost = 12.75/32



If, on expiration, the price of the note is greater than the strike price, the option expires worthless and the investor loses the cost of the option. If the note's price is less than the strike at expiration, the option will be exercised. If the note's price is less than the strike but not less than the strike minus the cost of the option, the option will be exercised and the investor will lose money, but less than the option cost. The investor breaks even if the underlying note's price is equal to the strike price less the option cost. If the note's price is less than the strike minus option cost, the position earns a profit.

Influences on Option Prices

Six factors influence options prices: underlying asset price, volatility of the underlying asset, financing rates, time to the option's expiration, strike price of the option, and coupons (or dividends, or more generally, cash outflows from the underlying security). This discussion is summarized in Figure 3.

If nothing else changes, the *price of a call option* increases if the price of the underlying asset increases. This is because the payoff at expiration of a call is the greater of the difference between the underlying price and the strike price, or zero. Put prices will fall because the terminal payoff of a put is the greater of the difference between the strike price and the underlying price, or zero.

Higher volatility of the underlying asset increases the values of both puts and calls because a more volatile asset is more likely to move farther away from the strike.

Time to expiration increases the value of American puts and calls but not necessarily that of European puts and calls. For American puts and calls, a longer time to expiration gives the underlying asset more time to move into the money. Because American options can be exercised early, a longer time to expiration increases opportunities of the option owner to exercise the option in the money. However, European options may not be exercised early, so an increase in time to expiration does not necessarily confer additional economic benefit on the option buyer.

Financing rates influence options prices in a more subtle way. An increase in financing rates raises the forward price of a bond. Technically speaking, the forward price determines whether an option is at the money. For a call, a higher forward price pushes the option farther into the money. For a put, a higher forward price causes the option to be farther out of the money, decreasing the price. Similar logic shows that lower (more special) financing rates decrease the value of calls and increase the value of puts.

A higher *strike price* decreases the value of calls and increases the value of puts. The most intuitive explanation hinges on the terminal payoffs for each type of option: a higher strike price reduces the intrinsic value of a call and increases the intrinsic value of a put.

Cash distributions lower the price of the underlying asset thus decreasing the value of calls and increasing the value of puts.

Figure 3. **Factors Influencing Option Prices and Direction of Effect**

Influence	**Call**	**Put**
Underlying Price	+	-
Volatility of the Underlying	+	+
Time to Expiration	+	+
Repo/Financing Rates	+	-
Strike Price	-	+
Coupon/Interim Cash Flows	-	+

Put/Call Parity

For a European call, a European put, and a forward contract all expiring on the same day, the forward price is the value of the options' strike price that equates the values of the call and the put.

We can explain put/call parity by thinking of it as three simultaneous trades. First, an investor buys a bond for forward settlement. Second, the investor sells a call expiring on the forward settlement date with the strike price equal to the forward price from the bond trade. Third, the investor buys a put expiring on the same forward settlement date with strike price equal to the same forward price. By buying a put and selling a call, the investor constructs a synthetic short position as of the expiration date, i.e., a synthetic short forward position. At the same time the investor has bought the bond forward. These two positions are exactly offsetting transactions and must result in zero profit to exclude the possibility of arbitrage.

Put/call parity in its strictest sense is true only for European options. In conjunction with other arbitrage restrictions, put/call parity can be used to create arbitrage bounds for American options. The arbitrage proof for European options is shown in the Appendix.

VOLATILITY

Definitions and Concepts

Volatility measures uncertainty. In the capital markets, **volatility refers to the random dispersion of changes in asset prices around an average, or trend, of price changes.** The volatility of a data sample generally refers to the sample's standard deviation, while the most common measure of central tendency, or trend, is the mean change.

Statistically, the standard deviation of a data sample is the square root of the sample's variance. The variance is the average of squared deviations from the mean, with the average taken over the total number of individual elements of the sample, minus one. For example, the average of the numbers -1,0, and 2 is 0.333. The differences between each number and the mean are, respectively, -1.333, -0.333, and 1.666. Squaring each number, adding the results, and dividing by 2 gives 2.333, the variance of the sample. The standard deviation is the square root of 2.333, or 1.527525.

Because the square of the deviation is always positive for price (or yield) changes, calculating the variance measures the random change, ignoring direction. Standard deviation is the square root of this number, or a directionless statistic that measures average dispersion of the asset's price about a trend.

By convention, volatility is expressed in annualized terms. Standard deviations are annualized by using the properties of variance. The key property is that for uncorrelated variables, the variance of a sum is equal to the sum of the variances.

For example, to annualize the random 6-bp daily change in a bond's 6.88% yield, we assume that the bond will behave every day as it did that day, and that one day's change has no influence on another. We square the daily change to get 36 basis points. For the 251 trading days of the year, we assume the same 36-bp change. The standard deviation is equal to the square root of 251 x 36 basis points. This is equal to 6 basis points x 15.8429 (the square root of 251), or 95.04. This method leads to a **short cut: to annualize a DAILY change, multiply it by the square root of the number of days in a (trading) year.** The result is the volatility of the bond, expressed in basis points of yield. To get a percentage, divide this result by the yield of the issue. From the preceding example, the percentage annualized volatility is 95.04/6.88, or 13.81%.

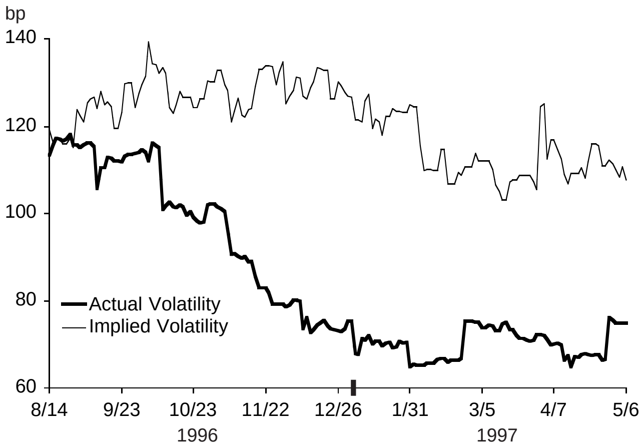
The Difference Between Historical and Implied Volatility

Historical volatility is calculated using historical price or yield data and describes the past behavior of the markets. Implied volatility is the level of volatility priced into the options markets.

To price an option, market participants depend on models using the price of the underlying security, financing rates, the expiration date of the option, strike price of the option, interim cash flows from the security (dividends or coupons), and volatility of the underlying security until option expiration. All of these, except for volatility, are observable in the marketplace. **Traders often use market option prices and other inputs to the pricing model to solve for the volatility necessary to produce the option price actually observed in the market. This volatility is the implied volatility of the option.**

Implied volatilities are the market's forecast of future volatility over the life of the option. **However, many studies have illustrated that implied volatility is a relatively poor predictor of actual volatility.** For example, Figure 4 shows that implied volatility for three-month options on the on-the-run 2-year note were consistently higher than the subsequently realized

Figure 4. **Implied versus Realized Actual Volatility: 90-day 2-year Options**



actual volatility. A standard OLS regression shows a correlation of only 45% ($R^2 = 0.206964$).

Price, Yield, and Basis Point Volatilities

Bonds with higher durations have higher price volatilities for a given change in yields. For example, price volatilities of 10.134% for the on-the-run bond and 1.413% for the on-the-run 2-year note would not automatically mean that bonds were more volatile than 2-year notes. Bonds have higher durations than 2-year notes and must be more volatile for a given yield change. Price volatilities must be corrected for duration to make meaningful comparisons among securities.

Correcting price volatility for duration produces yield volatility. The conversion is effected by dividing the price volatility of the security by its forward modified duration as of the option's expiration date. For example, on 5/6/97 the implied price volatility of the 6.25 of 2/15/2027 was 10.134% on an option expiring 8/5/97. Using a repo rate of 4.738%, the forward modified adjusted duration of the issue for 8/5/97 settlement was 12.602 and its forward yield was 6.925%. The price volatility divided by the forward modified adjusted duration was 0.804. This number, when multiplied by 100, is equal to the implied basis point volatility corresponding to the 8/5/97 option. Dividing 80.4 by the forward yield of the issue gives 11.61%, which is the option's implied yield volatility expressed as a percentage.

Are Interest Rates Normally or Lognormally Distributed?

Market practitioners have long argued about the distribution of interest rates. For two interest rate environments—one where market rates are 5.00% and the other where rates are 10.00%—where both have 10% volatility of interest rates, which environment is more volatile?

In the 5.00% interest rate environment, one standard deviation is equal to 10% of 500 basis points, or 50 bp. In the 10% interest rate environment, one standard deviation is 10% of 1000 bp, or 100 bp. **If interest rates are lognormally distributed, these two environments are equally volatile** (the change as a **proportion** of rates is constant for varying levels of market yields). **If interest rates are normally distributed, then the 10% volatility environment is more volatile** (the change in rates as an absolute number remains the same for varying yields).

TRADING VOLATILITY, HEDGING, AND THE GREEKS

Trading options is essentially the practice of segregating risks and choosing which risks to hedge away and which to retain. Market participants often use phrases such as "buying volatility (vol)" and "selling vol." Vol is often used as a synonym for "options," but these phrases also make implicit statements about which option risks the investor is willing to retain.

Volatility is not a traded asset. However, **portfolios of options and the underlying asset can be created so that the portfolios gain value from increases or decreases in volatility.** Thus, investors can be "long" or "short" volatility. Creating a portfolio of this kind entails hedging options against price moves in the underlying asset. More sophisticated hedging strategies can be constructed that refine the set of specific risks the investor retains. Options traders name the sensitivity of options prices to various influences using Greek letters.

Delta

The basic risk in an options price is underlying price risk. **The ratio for hedging underlying price risk is an option's delta.** It is the change in price of an option for a given change in price of the underlying asset.

Delta can be thought of as the probability of an option's expiring in the money (and being exercised). This rule of thumb comes from the mathematics behind the Black-Scholes-Merton family of option valuation models. To determine the appropriate delta hedge for an option, multiply the delta (expressed as a decimal) by the face amount of the underlying asset for the option.

Delta is different for calls and puts because the inherent directional risks are different for these types of options. **Because a call's price increases due to an increase in the underlying asset's price and decreases due to a decrease in the underlying asset's price,** the appropriate delta hedge for a long position in a call option is a short position in a delta-weighted amount of the underlying asset. Similarly, the appropriate delta hedge for a short position in a call option is a long position in a delta-weighted amount of the underlying asset.

Figure 5. **Call Delta as a Function of Underlying Price for June 1997 10-year Futures Contract**
Strike price = 104

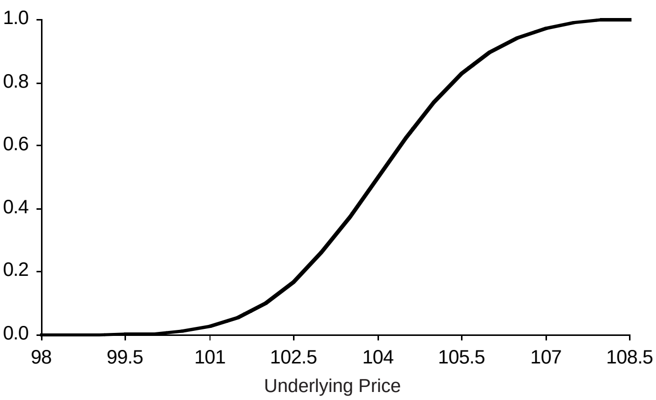
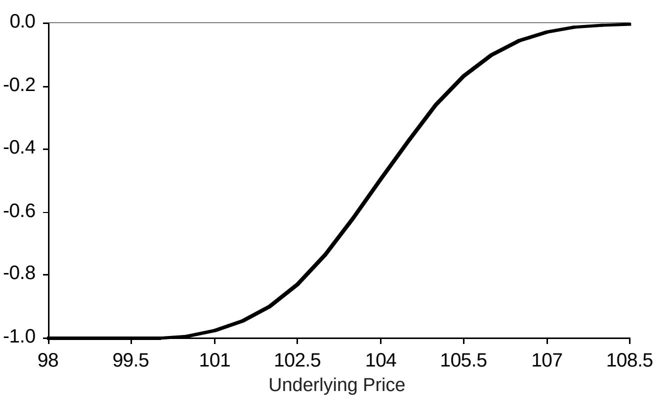


Figure 6. **Put Delta as a Function of Underlying Price for June 1997 10-year Futures Contract**
Strike price = 104



A put's price increases when the underlying price decreases and vice versa. Therefore, the appropriate hedge for a long position in a put is a long position in a delta-weighted amount of the underlying asset. The appropriate hedge for a short position in a put is a short position in the underlying asset.

Figures 5 and 6 illustrate the deltas of a call and put for different values of the underlying price. The delta of a deep out-of-the-money (OTM) call option is zero; the delta of a deep in-the-money (ITM) call is one; a call option trading at the money (ATM) will have a delta of 0.50. Similarly, the delta of a deep out-of-the-money put is zero; the delta of a deep in-the-money put is -1; the delta of an at-the-money put is -0.50.

Delta and Trade Inquiries

An options trader offering a call struck ATM spot will be short that option. To hedge directional exposure, the trader will buy a delta-weighted amount of the underlying asset. Therefore, in a sales inquiry, the trader will use the offered side of the market as the strike. A trader bidding the ATM spot call option will use the bid side of the underlying market because he must sell the appropriate delta hedge. **The trader will always use the side of the market appropriate for hedging his own delta exposure.**

To offer an at-the-money spot put, the trader will use the bid side of the underlying asset's market because he must sell the appropriate delta hedge. A trader who bids a put will use the offered side of the market because he must buy a delta-weighted amount of the underlying asset.

The Effect of other Factors on Option Deltas

The passage of time can cause the delta of an option to change. In Figure 7, deltas for in-the money, out-of-the-money, and at-the-money calls are shown as a function of time. For an ITM call option, the passage of time tends to cause the delta to converge to 1; again, delta represents the probability that the option will expire in the money. For an in-the-money option, the passage of time makes it more likely that the option will expire in the money, causing the delta to approach 1.

Likewise for out-of-the-money options: **the passage of time will decrease the probability that an out-of-the-money option will expire in the money, causing the delta to converge to zero.** This is sometimes called "delta bleed."

Figure 8 shows call deltas as a function of volatility of the underlying asset. **An increase in volatility, holding all else equal, causes the deltas of both in-the-money and out-of-the-money options to approach 0.5.** Similar to delta bleed, for in-the-money options an increase in volatility raises the probability that the option will move out of the money, causing its delta to fall. **For an out-of-the-money option, increased volatility raises the probability that the option will expire in the money, raising the delta.** This is sometimes abbreviated **ddelta/dvol.**

Figure 7. **Call Delta as a Function of Time to Expiration**

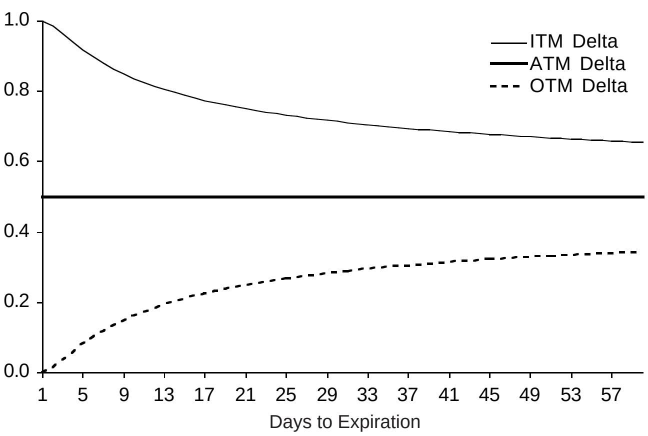
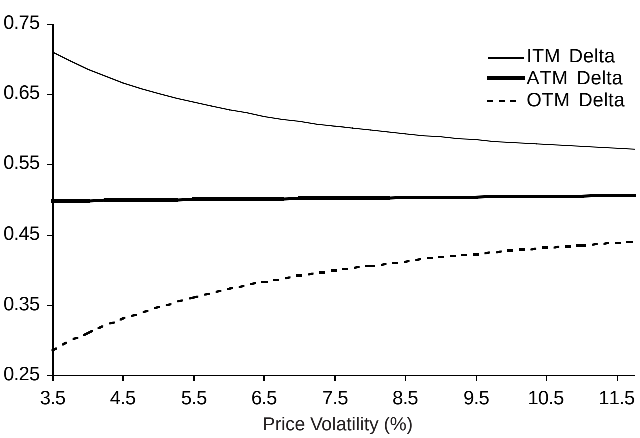


Figure 8. **Call Delta as a Function of Underlying Price Volatility**



Another way to think about the changes in delta due to a change in volatility is that increased volatility dampens the effect of the passage of time. For example, the delta of an OTM call may change from 0.40 to 0.30 over 30 days of its life, but if volatility suddenly increases sharply its delta could rise from 0.30 back to 0.40.

Gamma

The rate at which delta changes with respect to the price of the underlying is called gamma. Delta changes with movements in the underlying asset's price, causing appropriate hedge ratio to change with the underlying asset's price. Thus, delta-hedged positions can become unhedged due to price movements.

Gamma is defined as the change in delta divided by a given change in the underlying price. For an investor who is long an at-the-money spot call on $100 million of an asset currently trading at 100, the forward price of the asset is also 100 by assumption. Because the option is struck at the money, the delta is 0.50. Thus, the appropriate delta hedge is a short position in $50 million of the underlying asset.

If the gamma of the option is 0.10, meaning that for a 1-point change in the underlying price the delta increases by 10%, or 0.05, the appropriate hedge to protect against a move in the underlying asset is a short position in $55 million of the underlying asset. This makes the investor long the market by $5 million (the difference between the old and new delta-neutral hedges) of the underlying asset in a rallying market. Traders call this a **long gamma** position. **Long gamma positions get longer as the market rallies and shorter as the market falls**, analogous to the behavior of bonds with positive convexity. In fact, **options are convex functions of the underlying price, so gamma is the convexity of an option with respect to the price of the underlying asset.**

If the investor is short the option discussed above, the appropriate delta hedge is a long position in $50 million of the underlying asset. If the market goes up a point to 101, the delta changes to 0.55 and the appropriate hedge is a long position in $55 million of the underlying asset to hedge against underlying price movements. The investor is now effectively short the market by $5 million of the underlying asset in a rallying environment. This is the consequence of a **short gamma** position: **short gamma positions get shorter**

as the market goes up and longer as the market goes down.

Figure 9 illustrates option gamma as a function of the passage of time for ATM, ITM, and OTM options. For ATM options, gamma increases as time to expiration decreases. **Volatility trades using short-dated options are primarily exposed to gamma risk and are called gamma trades.**

Figure 10 shows that **gamma is highest for ATM options.** Therefore, the deltas of ATM options change the most rapidly. Thus maintaining a delta hedge can be difficult as well as expensive due to transactions costs.

The gamma of a portfolio of options is the weighted average of the gamma of each option position in the portfolio, where the weights are face values of the option positions. This property is analogous to the convexity of bonds.

Figure 9. **Option Gamma as a Function of Time to Expiration**

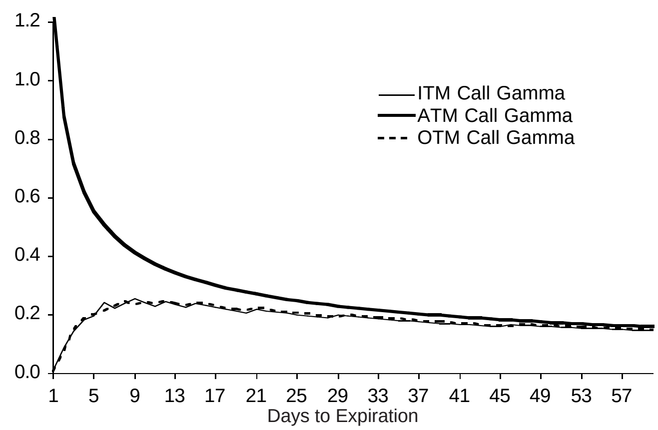
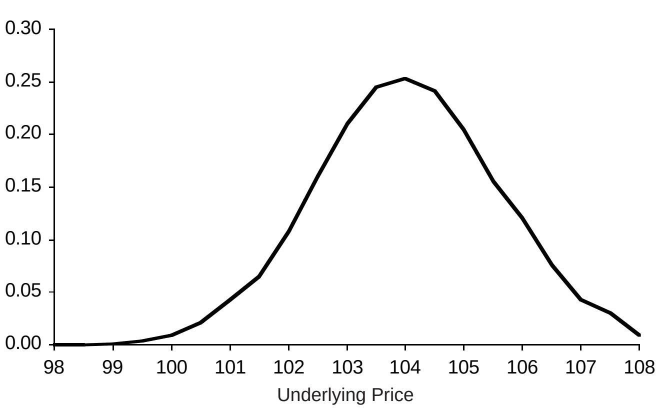


Figure 10. **Option Gamma as a Function of Underlying Price**
Strike price = 104



The gamma of a delta-neutral option position can be hedged instantaneously (i.e., for very small changes in the underlying asset) by adding another option to the position. This type of hedging can be difficult because the new option will also change the position's delta. Equations for calculating an instantaneous hedge are shown in the Appendix.

The position gamma of certain common options strategies (for example, horizontal spreads) will change sign depending on the level of the market, meaning that **the same position that is long gamma if the market rallies can be short gamma if the market falls.** In fact, for some positions, traders must be concerned about not only delta and how it changes (gamma), but also how gamma changes and how the change in gamma changes. **More complex options positions carry more complex risks with potentially larger effects.**

Vega

The change in an option's price due to a given change in volatility is called vega. It is calculated by dividing the change in an option's price by a given change in volatility. Figure 11 shows the change in three call options' prices as a function of implied volatility.

While vega is nearly linear for ATM options, at low volatilities vega displays locally positive convexity. Out-of-the-money options' prices are convex functions of implied volatility for the owners of the options. For investors who are short out-of-the-money options, the options are concave (negatively convex) functions of implied volatility. As with all negatively convex positions, vega for these options can be difficult to hedge. **If an investor owns an out-of-the-money put, he will benefit as volatility increases. In addition, the investor will make money faster when volatility increases than he will lose money when volatility decreases.** The reverse is true when the investor is short out-of-the-money options: losses will grow faster if volatility increases than will gains if volatility decreases.

As can be seen in Figures 12 and 13, vega is highest at the money for options with a long time until expiration. Options positions that would tend to benefit from increases in longer term (implied) volatility are called

long vega positions. Positions that lose money with an increase in implied volatility are called **short vega** positions. Therefore, **options positions that hinge on short-term volatility are called gamma trades, while positions that hinge on long-term volatility are called vega trades.** Although vega and gamma are present in all options, generally speaking, options with expirations under one month are predominantly subject to gamma risk. Vega exposure increases with time to expiration and gamma exposure begins to diminish in importance past 30 days.

Theta

The change in an option's price due to a given change in time to expiration is called theta. Figure 14 shows theta as a function of time. As a mirror image of gamma, **theta is time decay, generally given as the change in option price for the passage of one day.** Theta is effectively the price of gamma. That is, an investor who is long gamma (and would thus tend to benefit from an increase in short-term volatility) must expect to lose theta. An option position's daily decay can be viewed as a daily hurdle to profitability. Thus, traders sometimes say that earning enough to cover theta is "paying the rent."

Rho

The change in an option's price due to a change in (short term) interest rates is called rho. **In the fixed income markets, rho is crucial because short term (i.e., repo) rates determine the forward price of bonds.** In upwardly sloping yield curve environments, calls struck

Figure 11. **Option Price as a Function of Underlying Price Volatility**

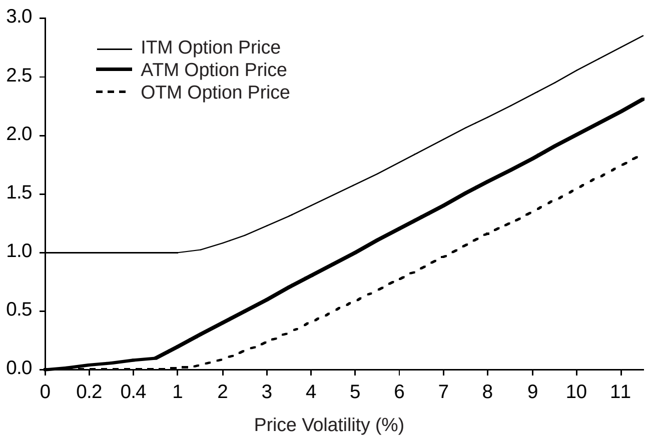


Figure 12. **Option Vega as a Function of Underlying Price,** Strike price = 104

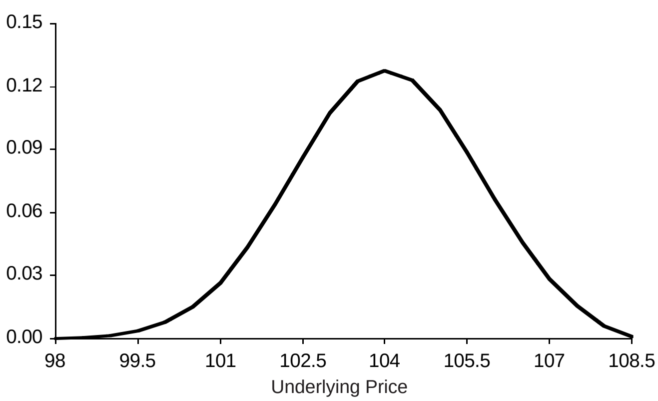


Figure 13. **Option Vega as a Function of Time to Expiration**

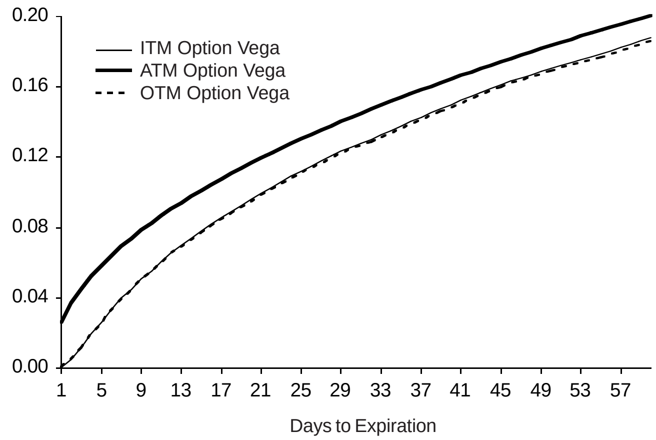
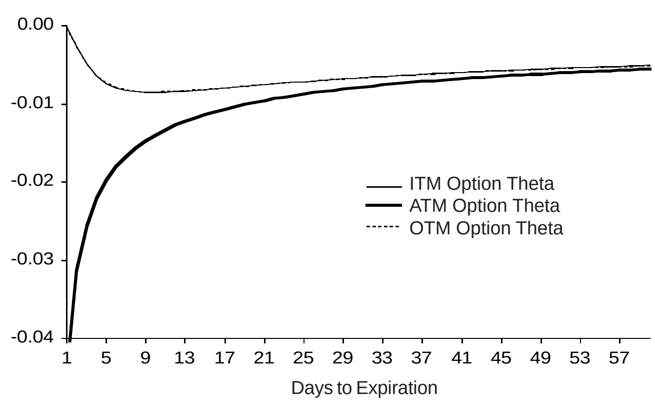


Figure 14. **Option Theta as a Function of Time to Expiration**



ATM spot will not have deltas of 0.50 because the forward price of the bond will be lower than the spot price. If the strike price of the option is the spot price of the bond at the time of the trade, then the forward price today is less than the strike price; thus the call is out of the money on a forward basis and the delta is by definition less than 0.50. Similar reasoning shows that ATM spot puts are in the money forward and thus will have deltas greater than 0.50.

PRACTICAL ISSUES FOR HEDGERS

Most options models are based on “continuous time” financial theory. These models assume that options can be continuously rehedged over infinitely small periods of time without any transaction costs, such as commissions, bid-ask spread, margin requirements, and differences in borrowing and lending rates. In the real world all these costs exist, and frequent rehedging can become quite expensive. Investors must therefore weigh the security of market neutrality gained from frequent rehedging against the costs of frequent trading.

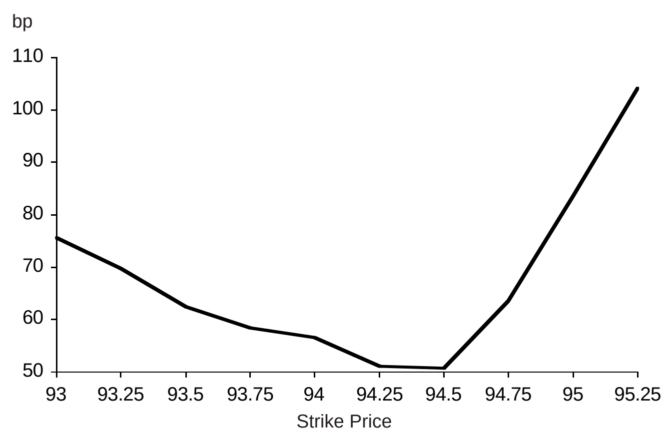
Some common strategies for hedging are simple rules such as “rehedge only once per day, but always once per day” or “rehedge if the position delta changes by a predetermined amount.” These strategies give an idea of how investors manage the risks associated with option positions. However, either the wider the time interval or the wider the price interval between rehedging, the larger the potential hedging error.

Another issue in options hedging is model misspecification. Most options models make assumptions about the ability of investors to do continuous rehedging with portfolios of the underlying asset and riskless borrowing and lending at an identical rate. However, there are other issues as well. For example, models must make assumptions about the probability distribution followed by the underlying asset of an option. For this reason, we included the brief discussion about whether rates are lognormally or normally distributed. In reality, stock and bond prices are inaccurately described by both distributions. The actual distributions of asset prices tend to display leptokurtosis: the distributions have “fat tails” and higher, thinner peaks than those of normal or lognormal distributions. One implication of this error is that Black-Scholes-Merton paradigm models tend to underprice deep out-of-the-money options and overprice at-the-money options.

Figure 15 shows what options traders call “the volatility smile.” The smile results from the fact that the Black-Scholes-Merton model’s implied volatilities of deep out-of-the-money options must be higher than those of at-the-money options in order to match actual market options prices because the model is misspecified. The smile below is taken from the December 1997 94.50 Eurodollar puts and calls.

Another source of error in options pricing has to do with the volatility of the underlying asset. Black-Scholes-Merton models assume that the volatility of the underlying asset is constant over the life of the option. Other models assume that volatility is a function of the maturity of the underlying instrument. However, empirically we observe that volatility in the marketplace changes over time. The essential problem is that in practice, options traders deal in discrete time with stochastic (randomly changing) volatility. Dealing with stochastic volatility is primarily a problem for options market makers. Investors may still use Black-Scholes-Merton implied volatilities because academic research has shown that the average volatility of an asset over the life of the option produces prices consistent with the Black-Scholes-Merton assumption of constant volatility. Implied volatility, then, should be viewed as the market’s best forecast of the average volatility of the underlying asset until option expiration.

Figure 15. **Implied Volatility Smile for December 1997 Eurodollar Contract**



APPENDIX

I. Statistical Review

A. Calculation of Standard Deviation

$$\sqrt{\frac{\sum_{t=1}^{n}(x_t - \bar{x})^2}{n-1}}$$

where x_t = observation from time t
$\bar{x}$ = sample mean
n = number of observations in sample

B. Alternative Formulation

$$\sqrt{\frac{\sum_{t=1}^{n} x_t^2}{n}}$$

where $x_t = ln\left(\frac{p_t}{p_{t-1}}\right)$
n =number of observations in sample
p_t =observation from time t

II. Proof of Put/Call Parity

A. European Bond Options

Define the following variables:

t:	present date
T:	horizon date
c:	price of one European call option
p:	price of one European put option
B_t:	price of bond at time t
X:	strike price
r:	repo rate
Coupon Income:	accrued interest until expiration date, from inception date

We consider two portfolios, A and B, with contents as follows:

Portfolio A: A long position in one European call, cash borrowings equal to $(X)e^{-r(T-t)}$+(Coupon Income)$e^{-r(T-t)}$

Portfolio B: A long position in one European put, a long position in one unit of the underlying bond.

At the expiration date, both portfolios will have the same value. To see this, consider each portfolio. If the bond price is above X, the call in Portfolio A is exercised using the borrowed money and the portfolio is worth B_T. If the bond price is below the exercise price, the call expires worthless and the portfolio is worth the strike price plus accrued interest, less the cost of borrowing.

For Portfolio B, if the bond price is below the strike price, the put is exercised and the portfolio is worth the strike price plus accrued interest. If the bond price is above the strike price, the put goes unexercised and the portfolio is worth B_T.

Both portfolios will be worth the maximum of the bond price at expiry or the strike price plus accrued interest. Because both portfolios have the same terminal value, both must have the same initial value to preclude arbitrage.

B_T must be equal to B_t minus carry. B_T is the forward price of the bond for date T as of date t. If this condition is not satisfied, there is an opportunity for arbitrage.

Put/call parity can be stated as

$c + Xe^{-r(T-t)}$ + (Coupon Income)$e^{-r(T-t)} = p + B_0$.

III. Gamma Hedging a Delta-Neutral Options Position

In the case where an investor has a portfolio consisting of one option that is delta-neutral, another option must be added to the portfolio in order to neutralize gamma exposure. We exploit the additive property of delta and gamma to create the following system of equations determining the number of units of the two options (n_1, n_2) to be combined with the underlying asset to result in a delta- and gamma-neutral portfolio:

$$1 + n_1\Delta_1 + n_2\Delta_2 = 0$$
$$0 + n_1\Delta_1 + n_2\Delta_2 = 0$$

where n_1 = number of option #1
n_2 = number of option #2
Δ_1 = delta of option #1
Δ_2 = delta of option #2
Γ_1 = gamma of option #1
Γ_2 = gamma of option #2

The solution to this system is:

$$n_1 = \frac{1}{\Delta_2 - (\Gamma_1/\Gamma_2)\Delta_1}\frac{\Gamma_2}{\Gamma_1}$$

$$n_2 = -\frac{1}{\Delta_2 - (\Gamma_1/\Gamma_2)\Delta_1}$$

LEHMAN BROTHERS