Convexity

Greg Knudsen
International Rate Structuring & MTNs

Convexity

Convexity is a technical term used to describe the “shape” of payoffs that occur in many Financial Instruments.

In pricing certain interest rate structures, a modeller needs to take into account the convexity inherent in some of the underlying components of the structure.

We will look at …

- examples of Convexity
- how to hedge the convexity exposure
- how to estimate the “Convexity” correction

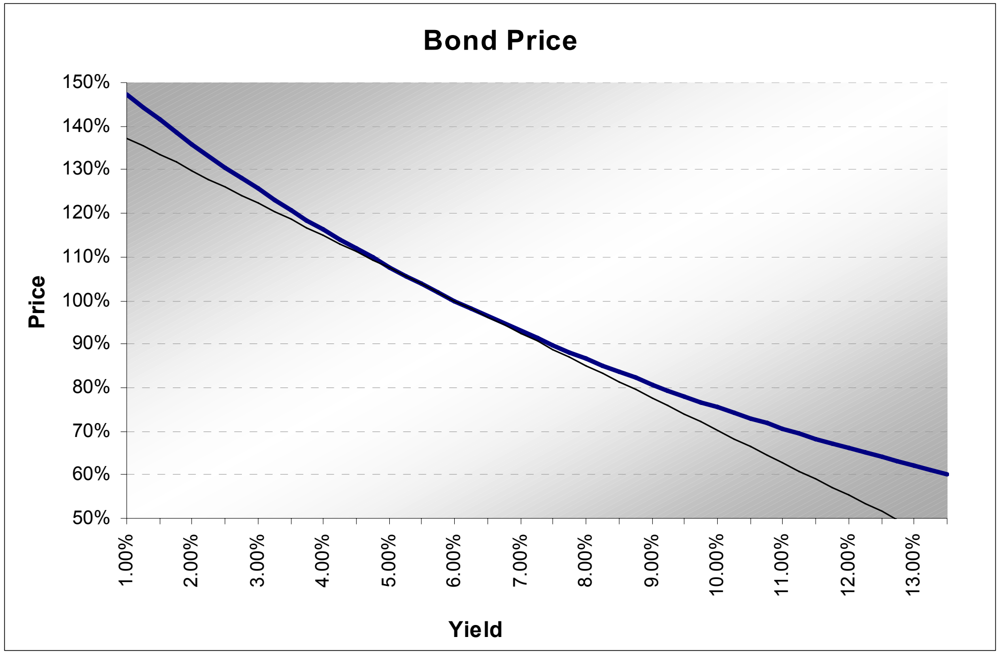
Basic Convexity of a Bond

If the Price of a standard fixed coupon Bond is plotted against the Yield, the curve displays Convexity.

This can be seen by the formula for the Price of a Bond, namely …

$$\text{Price} = \frac{c}{(1+y)^1} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \ldots + \frac{c}{(1+y)^n} + \frac{1}{(1+y)^n}$$

where c = coupon rate (%) (assumed annual)
y = yield (%) (assumed annual)
n = years to Maturity



The graph here shows the Price of a 10yr Bond, bearing a 6.00% coupon, as a function of the yield.

Note the convexity of the Price / Yield relationship …

- as yields fall the Price rises at an increasing rate

- as yields rise the Price falls at a decreasing rate

Constant Maturity Swaps

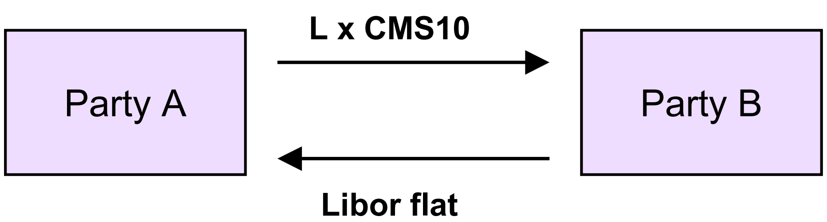
A very common structure, through which we can gain insight into the risks & pricing of convex instruments, is a Constant Maturity Swap

A Constant Maturity Swap (CMS Swap) is an interest rate swap comprised of …

- a CMS leg for which each coupon is a Leveraged CMS rate
- a standard Libor floating leg

The CMS rate can be observed either at the start (advance) or end (arrears) of each period

For example, consider the following 10yr CMS swap …



Term	**10 years**
Coupons	**Annual 30/360**
Leverage	**L**

Each CMS coupon is determined by reference to the 10yr Swap Rate (CMS10 Rate) observed at the start of each coupon period. This is an example of fixing in advance.

Alternatively, in the fixing in arrears variation, the CMS10 Rate is fixed, and paid, at the end of each coupon period.

Constant Maturity Swaps

In pricing a Constant Maturity Swap we clearly need to "predict", and then discount, the anticipated future CMS fixings.

As with many financial products, hedging considerations guide us to the correct answer.

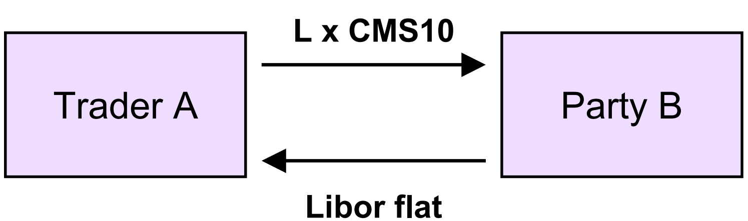
The CMS fixings in the future are realised future Swap rates. Thus a trader paying the CMS fixings could hedge the fixings by entering a series of forward starting paying swaps.

There would be one swap for each CMS fixing, and for each swap …

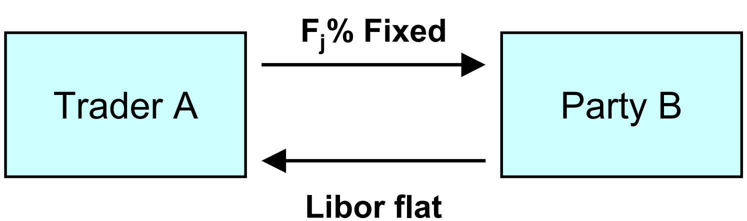
- the start date would co-incide with the fixing date of the corresponding CMS rate
- the underlying term would match the CMS term required (eg. 10 years for CMS10)
- the underlying notional used is calculated to match the PV01 of the CMS fixing

We would thus have, for each coupon period …

CMS Swap in period j on Notional N



Hedge Swap at F_j% on Notional N_j



Constant Maturity Swaps

Note the crucial difference between the CMS Swap and the Hedge Swap.

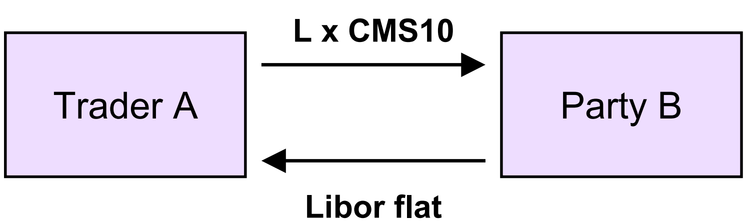
- the CMS Swap is broken up into a series of single period swaps, one for each CMS fixing.
- each corresponding Hedge Swap is a multi-period swap for which the term is the term implicit in the CMS rate being hedged.

Note also that we hedge the payment of a future CMS Rate by paying fixed at the forward swap rate F_j for the underlying period of the CMS fixing in period j.

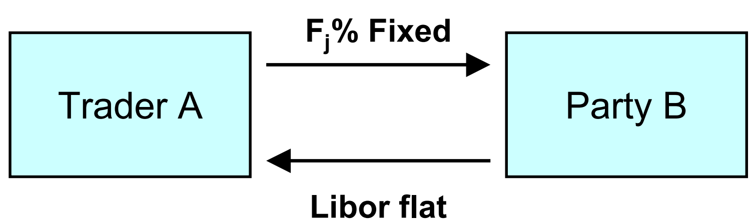
To see this, if the forward rate corresponding to the CMS fixing were to …

- rise … the CMS Swap value would decrease, the hedge swap value would increase
- fall … the CMS Swap value would increase, the hedge swap value would decrease

CMS Swap in period j on Notional N



Hedge Swap at F_j% on Notional N_j



Constant Maturity Swaps

The analysis thus far seems to suggest we should …

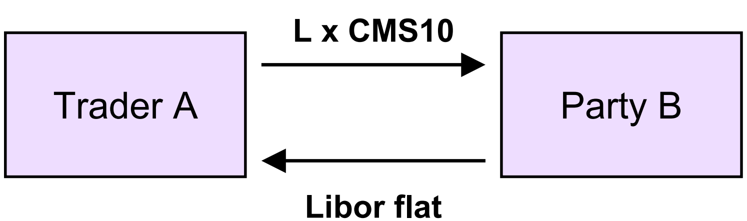
- price the CMS swap by estimating the future CMS fixings as the forward swap rates F_j
- hedge the CMS fixings by doing a series of forward starting swaps, one for each fixing
- pay fixed at rate F_j on the hedge swap for period j if we are paying the CMS fixing,
  or
  receive fixed at rate F_j on the hedge swap for period j if we are receiving the CMS fixing

The Notional N_j for the Hedge Swap j is determined by ensuring that the coupon sensitivity (PV01) of the Hedge swap is identical to the coupon sensitivity (PV01) of period j of the CMS swap.

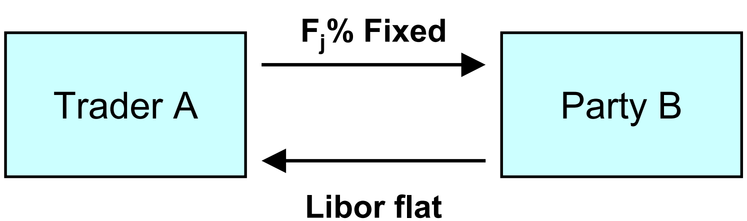
However, on closer inspection we see things are not quite so simple. The problem is convexity.

The payoff of each hedge swap is a convex function of the forward swap rate, whereas the CMS fixing is a linear function of the forward swap rate. This is best seen by looking at an example.

CMS Swap in period j on Notional N



Hedge Swap at F_j% on Notional N_j



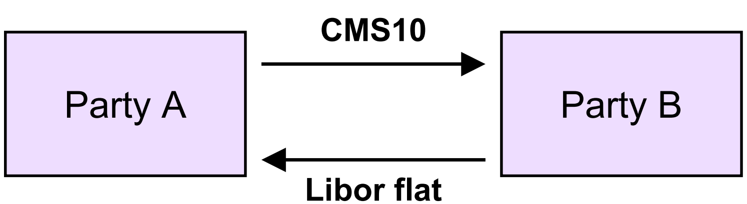
Constant Maturity Swaps

Let's look at a 10yr EUR CMS Swap, and use the 1 year period beginning in 5 years to illustrate.

Suppose the forward 10yr EUR Swap Rate, starting in 5 years, is 4.00% Annual 30/360 and that the Notional on the CMS Swap we are trying to hedge is EUR 1mm.

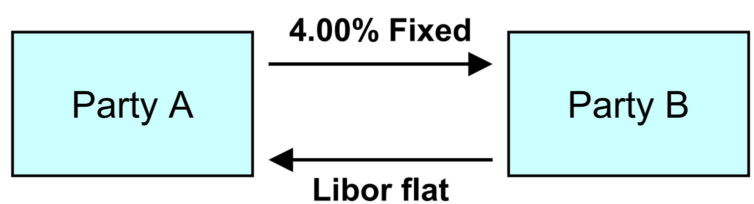
For simplification, we assume we are hedging 100% of the CMS fixing in 5 years, applicable for calculating the coupon for Year 5. We thus have …

CMS Swap in Year 5 on Notional 1mm



Start Dec 11
Maturity Dec 12
Notional EUR 1 mm
CMS10 4.00% (*)
PV01 0.01%
(*) = not Convexity adjusted

Hedge Swap at 4.00% on Notional N_6



Start Dec 11
Maturity Dec 21
Notional EUR N_6 mm
Rate 4.00%
PV01 0.081%

so that …

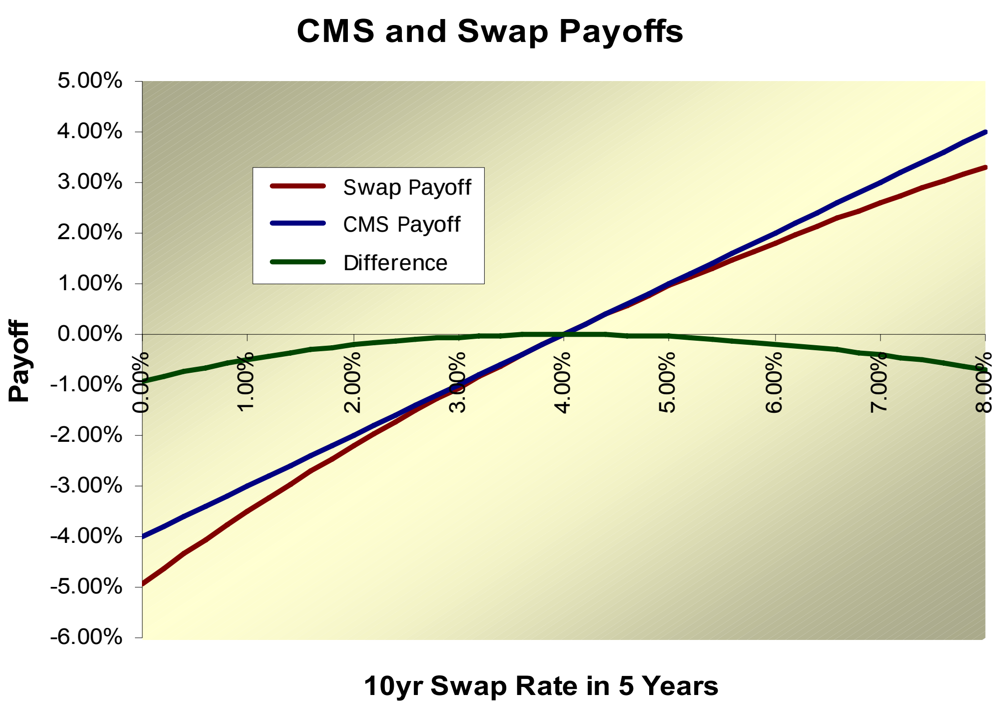
N_6 = 1mm x 0.01 / 0.081
= EUR 123,000

Convexity of the Hedge

However, as can be seen from the graph below, the swap hedge proves not to be a perfect hedge when the CMS Rate is eventually fixed, at rate R, at the end of Year 5.

The problem is that

- the CMS Payoff is a linear function of the difference (R – 4.00%)
- the Swap Hedge has a convex payoff, since the payoff is determined by discounting the annuity stream of coupon differences (R – 4.00%) over the 10yr life of the Swap at the then swap rate



CMS Payoff = N_1 x (R - 4.00%)

Swap Payoff = N_2 x (R - 4.00%) x PV_{01}

where

N_1 = Notional of CMS Swap

N_2 = Notional of Swap Hedge

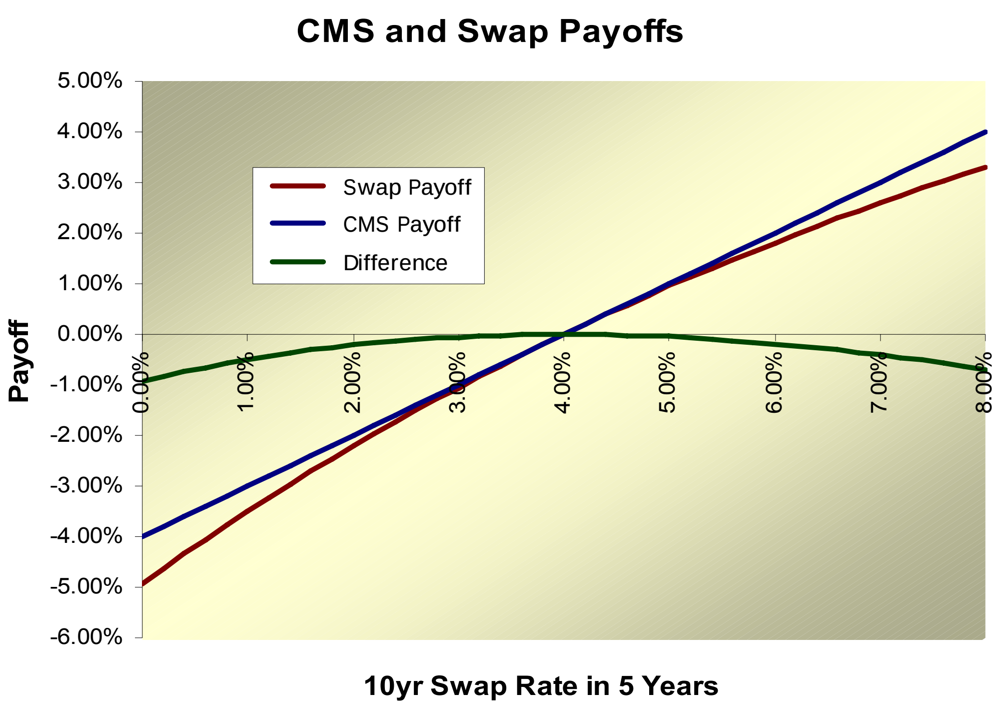
R = realised CMS Rate after 5 years

PV_{01} = PV of EUR 1 p.a. over the 10yr life of the Swap (discounted at R)

Convexity of the Hedge

The value (mark-to-market) of the Hedge Swap will always underhedge the value of the CMS payment if forward rates move in either direction

Fwd Rate Change	CMS Payoff Change linear	Swap Payoff convex
Fwd up Δ %	rises by $X = N_1 \times \Delta$	rises by Y < X
Fwd down Δ %	falls by $X = N_1 \times \Delta$	falls by Z > X



CMS Payoff $= N_1 \times (R - 4.00\%)$

Swap Payoff $= N_2 \times (R - 4.00\%) \times PV_{01}$

The CMS Payer will lose when rates move in either direction.

This is known as being short gamma

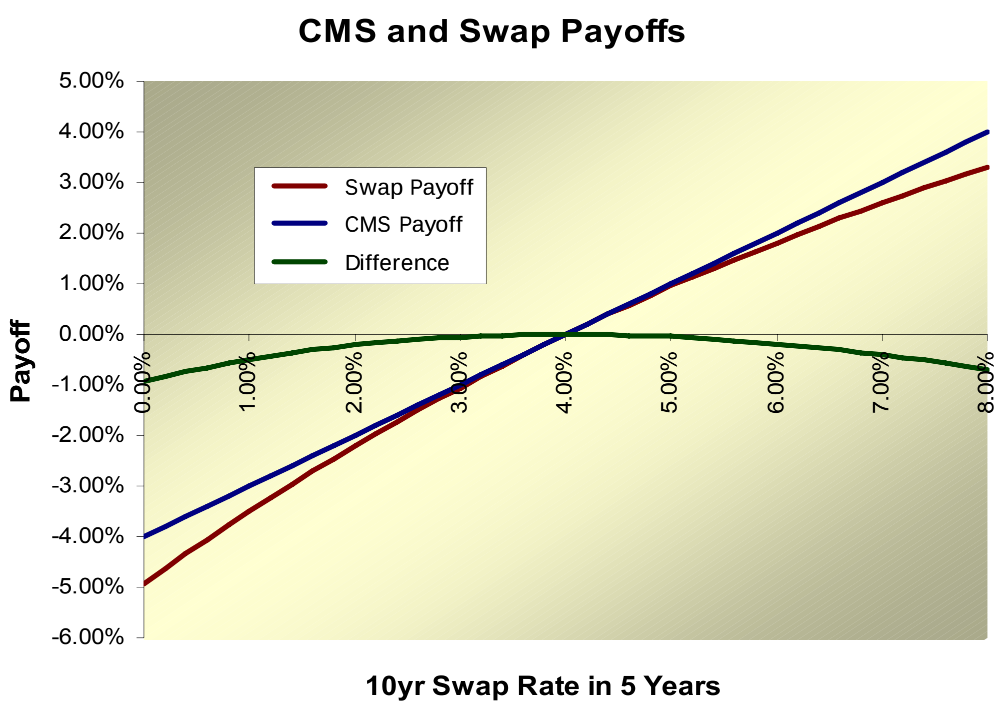
Convexity of the Hedge

Clearly then we need to modify our assumption that we can simple use the forward swap rates F_j as the estimates of the unknown future CMS fixings.

While we can lock-in these fixings at zero cost via forward starting swaps (at rates F_j) such a strategy would cause systematic losses every time rates moved.

The hedge either compensates too little against losses when forward rates rise, or loses too much when forward rates fall.

What we need to do is to adjust the forwards rate F_j up by a so-called convexity adjustment.



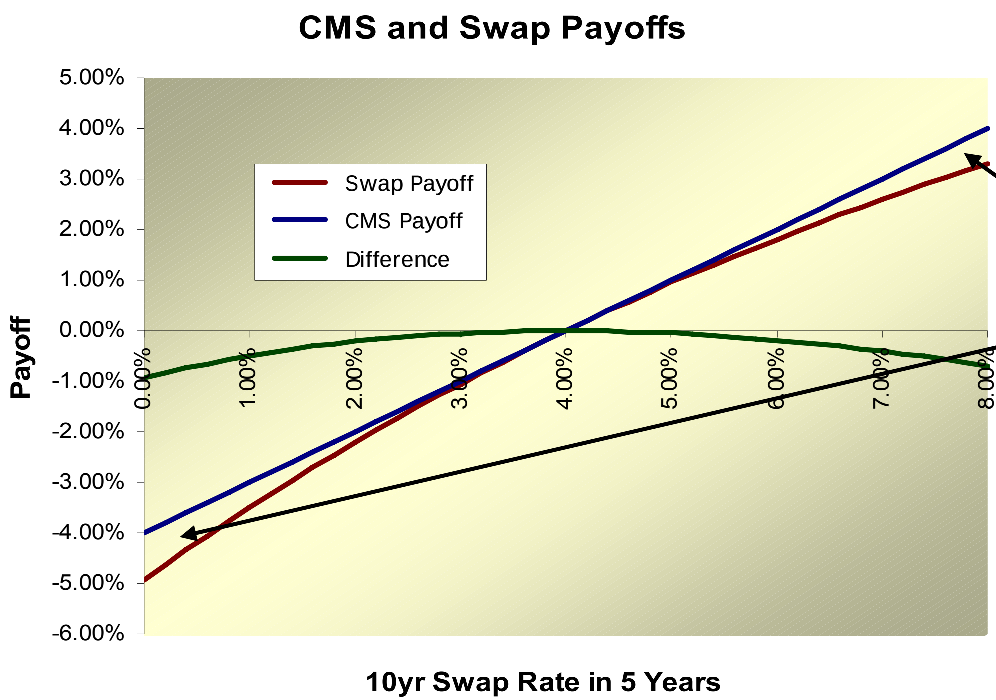
Convexity Adjustment

If the forward rates are adjusted up by a convexity correction, the fixed rate payer will value the CMS payments as more negative then if the rates were unadjusted.

This implies that the deal would need to be adjusted in some other way to compensate the CMS Payer for the additional negative value. The adjustment might be a reduced leverage, or a modified upfront payment.

In essence, the convexity adjustment should capture the anticipated losses arising from an imperfect swap hedge.

How can we calculate the correct convexity adjustment?



The Convexity Adjustment allows us to value and adjust for the mismatch

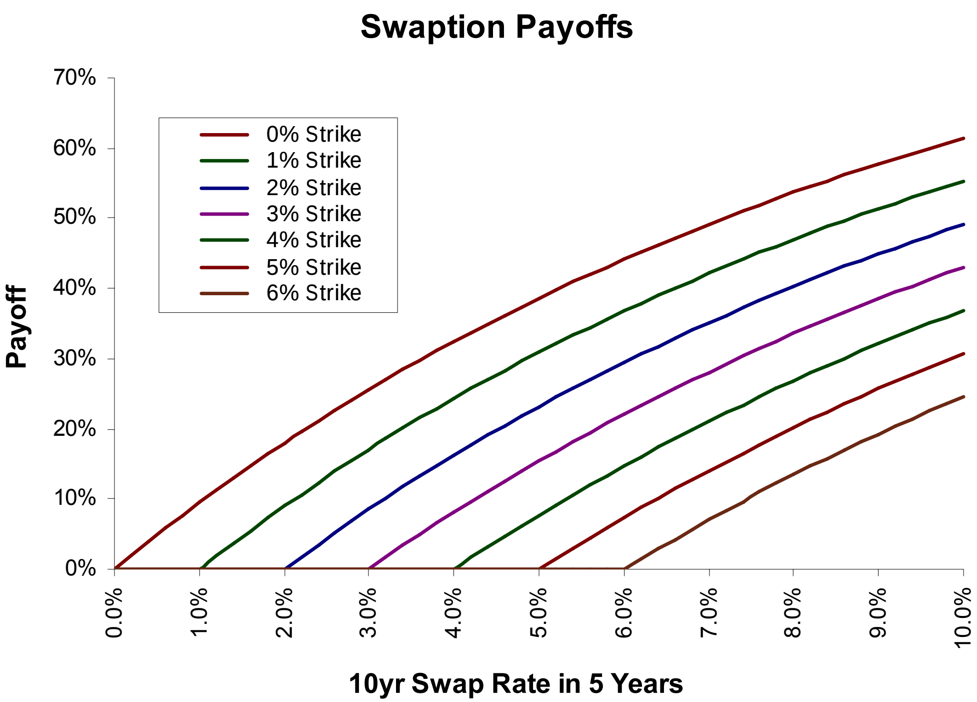
Convexity Adjustment

The Convexity Adjustment to a CMS rate can be calculated by a variety of methods.

One method which is explicit & widely used is to create a CMS payoff via a portfolio of Payer swaptions.

The value of the CMS payment is then calculated by valuing the cost of the portfolio of Payer swaptions required to replicate the payment.

We begin by looking at a typical Payer Swaption Payoffs.



The diagram on the left shows the Payoffs of 5yr into 10yr Payer Swaptions at a variety of strikes.

For example, a 1.00% 5yr into 10yr payer swaption allows the option holder to enter in 5 years a 10yr Payer Swap at 1.00% fixed vs Libor.

The option only pays off if the 10yr Swap rate in 5 years is > 1.00%

The payoffs are convex because they are discounted annuities.

CMS Replication

We can replicate the payoff of a CMS fixing by using a suitably chosen portfolio of Payer Swaptions. We will use a portfolio of swaptions restricted to 0%, 1%, .., 10% strikes.

For example, we seek to replicate the payoff in 5 years of the EUR CMS10 rate (the 10yr EUR annual swap rate observed in 5 years time)

We assume the current 5yr forward, 10yr underlying, EUR swap Rate is 4.28% Annual.

We seek to calculate the weights ω_0, ω_1,…, ω_{10} of the 5yr into 10yr EUR payer swaptions with strikes 0%, 1%, .., 10% respectively that best replicate the EUR CMS payoff in 5 years.

Party A $\xrightarrow{\text{0\% p.a.}}$ Party B

Party A $\xleftarrow[\text{Libor flat}]{}$ Party B

exercise swaption

plus

Party A $\xleftarrow{\text{1\% p.a.}}$ Party B

Party A $\xrightarrow[\text{Libor flat}]{}$ Party B

close out the swap

equals 9.47%, calculated via …

Party A $\xleftarrow{\text{1\% p.a.}}$ Party B

1% p.a. for 10 yrs discounted at 1.00%

The value of a 0% Payers in 5 years time is a function of the realised 5yr Swap rate at that time.

If the swap rate is 0% then the payoff is 0 as the swaption expires at-the-money.

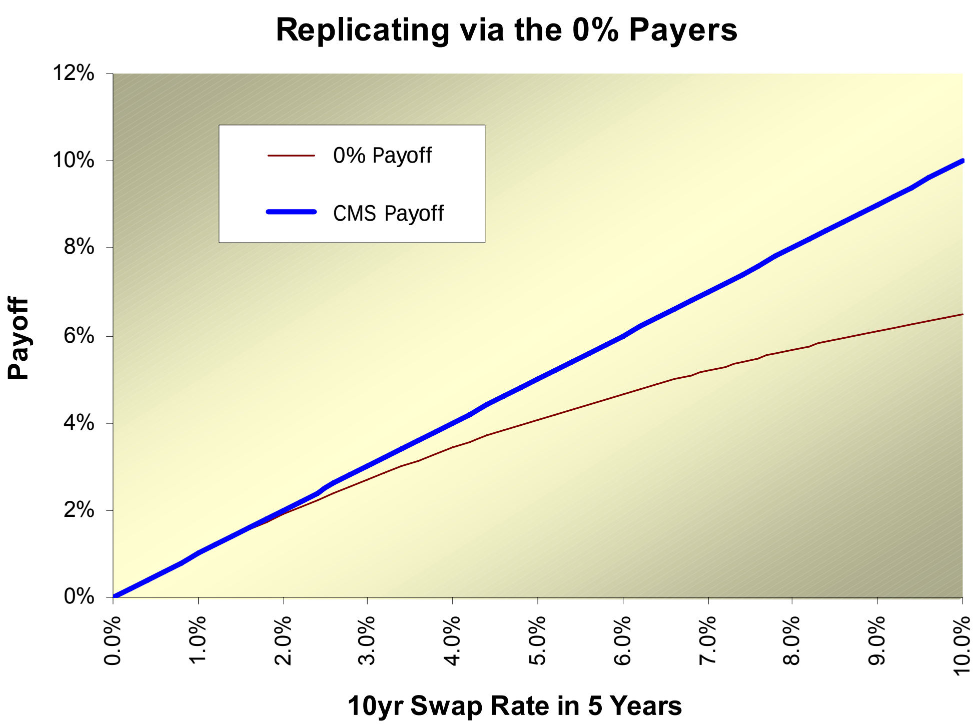
If the swap rate is 1% then the payoff is 9.47% (of the 0% Swaption Notional ω_0) since the swaption is worth the difference between 0% (the strike) and 1% (ATM) coupons over 10 years, discounted at the ATM swap rate of 1%.

CMS Replication

Crucially, the only Payer Swaption that has a positive payoff when the realised CMS rate in 5 years is 1% is the 0% Strike Swaption.

All the other Payer swaptions (with 1%, 2%, …, 10% strikes) will have 0 payoff when the realised CMS rate is 1%. This is because they expire either at the money (1% strike) or out of the money (2%, …, 10% strikes)

Thus to replicate the 1% CMS Rate we need to solve for ω_0 (and only ω_0) so that the Payoff of the 0% strike is 1% when the realised CMS rate is 1%.



Since the 0% swaption pays off 9.47% of the Swaption Notional ω_0 when the realised swap rate is 1%, we have …

ω_0 x 9.47% = 1.00%

and

ω_0 = 1.00% / 9.47% = 10.56%

The graph on the left shows the Payoff of both the 10yr CMS rate and the 0% Payer Swaption (on 10.56% of underlying Notional) as a function of the 10yr Swap rate observed in 5 years.

CMS Replication

We next try to find the Notional amount ω_1 of the 1% Payers swaption such that a portfolio comprised of

- ω_0 = 10.56% of the 0% Payer Swaption
- ω_1 of the 1% Payer Swaption

replicates the payoff of the CMS rate when the realised CMS (swap) rate in 5 years is 2.00%.
The other swaptions (2%, ..., 10% strikes) have no payoff when the realised CMS rate = 2%

Calculation of 0% Payers payoff

Party A $\xrightarrow{\text{0\% p.a.}}$ Party B

Party A $\xleftarrow{\text{Libor flat}}$ Party B

exercise swaption

plus

Party A $\xleftarrow{\text{2\% p.a.}}$ Party B

Party A $\xrightarrow{\text{Libor flat}}$ Party B

close out the swap

equals 17.96%, calculated via …

Party A $\xleftarrow{\text{2\% p.a.}}$ Party B

2% p.a. for 10 yrs discounted at 2.00%

Calculation of 1% Payers payoff

Party A $\xrightarrow{\text{1\% p.a.}}$ Party B

Party A $\xleftarrow{\text{Libor flat}}$ Party B

plus

Party A $\xleftarrow{\text{2\% p.a.}}$ Party B

Party A $\xrightarrow{\text{Libor flat}}$ Party B

equals 8.98%, calculated via …

Party A $\xleftarrow{\text{1\% p.a.}}$ Party B

1% p.a. for 10 yrs discounted at 2.00%

CMS Replication

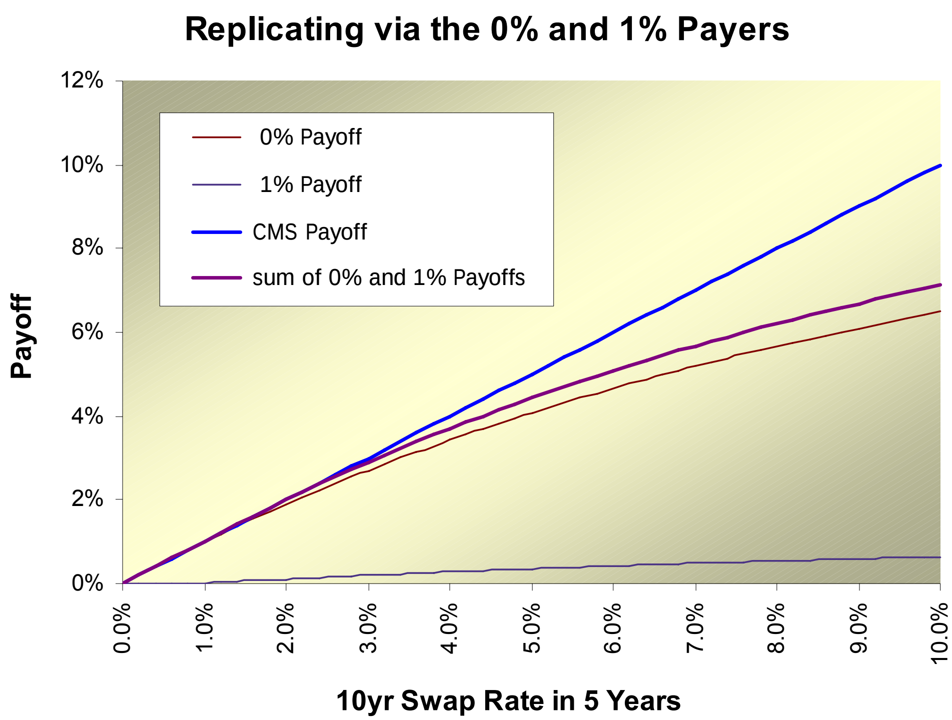
Now when the realised CMS rate is 2.00%, the payoffs of the 2 Payer swaptions are …

- 17.96% of the Notional ω_0 of the 0% Payer Swaption
- 8.98% of the Notional ω_1 of the 1% Payer Swaption

Since ω_0 = 10.56%, we thus solve the following equation for ω_1

(17.96% x ω_0) + (8.98% x ω_1) = 2.00% (the CMS payment we are replicating)

ω_1 = [2.00% - (17.96% x 10.56%)] / 8.98% = 1.15%



The graph on the left shows the Payoffs of

- the 10yr CMS rate
  (on full underlying Notional)
- the 0% Payer Swaption
  (on ω_0 = 10.56% of underlying Notional)
- the 1% Payer Swaption
  (on ω_1 = 1.15% of underlying Notional)
- the sum of the 0% and 1% payoffs
  on their respective Notionals ω_0 and ω_1

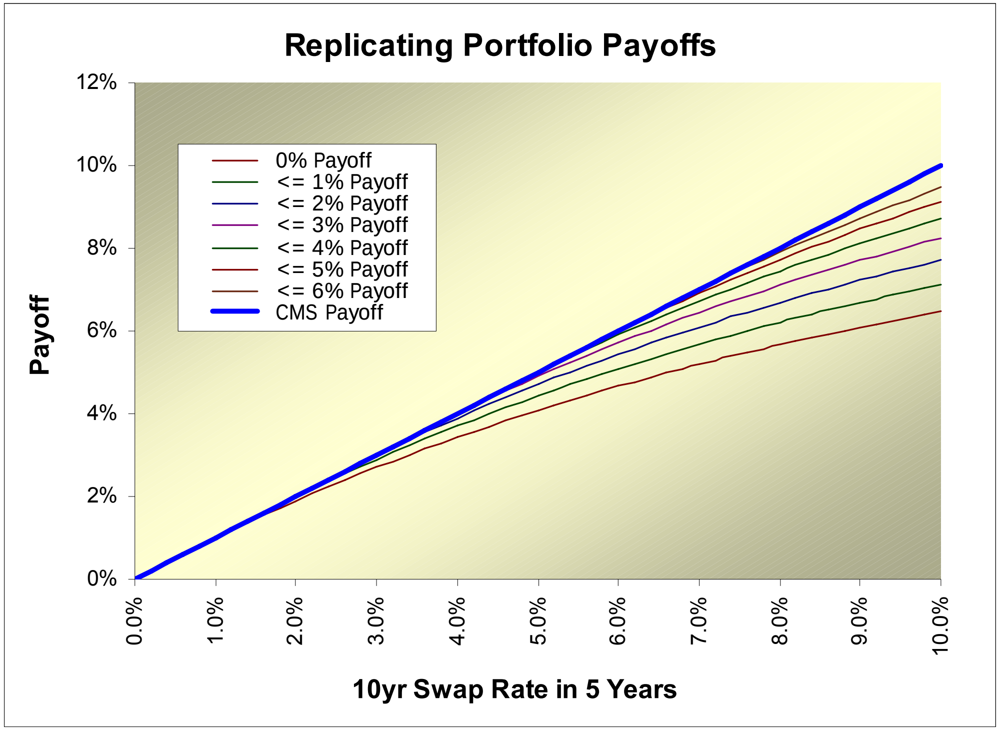
as functions of the 10yr Swap rate observed in 5 years.

CMS Replication

Continuing in this fashion for the payoff replication at 3.00%, 4.00%, …, 10.00% we calculate the sequentially the full set of weights $\omega_0, \omega_1, \ldots, \omega_9$

We find …

ω_0 = 10.56%
ω_1 = 1.15%
ω_2 = 1.20%
ω_3 = 1.24%
ω_4 = 1.29%
ω_5 = 1.33%
ω_6 = 1.37%
ω_7 = 1.42%
ω_8 = 1.46%
ω_9 = 1.49%



The graph shows the cumulative replication achieved up to and including a 6% payer Swaption. Note how the portfolio payoff gradually approaches the CMS Payoff required.

CMS Replication

So, we have now determined the weights $\omega_0, \omega_1, \ldots, \omega_9$ respectively of the Payer Swaptions (let's call them $S_0, S_1, \ldots, S_9$ where S_j has strike j%) that form our replication portfolio.

The crucial insight now is that the value today of the CMS rate we are replicating in 5 years must be the price today of the replicating swaption portfolio.

We calculate the Price of the portfolio by pricing each individual swaption …

Swaption	Strike	Vol	Value	Weight	Price
S_0	0.00%	24.55%	28.139%	10.56%	2.971%
S_1	1.00%	24.55%	21.573%	1.15%	0.248%
S_2	2.00%	21.48%	15.207%	1.20%	0.182%
S_3	3.00%	16.63%	9.199%	1.24%	0.114%
S_4	4.00%	13.11%	4.174%	1.29%	0.054%
S_5	5.00%	10.66%	1.117%	1.33%	0.015%
S_6	6.00%	9.35%	0.155%	1.37%	0.002%
S_7	7.00%	9.41%	0.025%	1.42%	0.000%
S_8	8.00%	9.88%	0.006%	1.46%	0.000%
S_9	9.00%	10.57%	0.002%	1.49%	0.000%
				Total Price = 3.587%	

Note that we performed this replication with swaptions with strikes set 1% apart, viz. 0%, 1%, 2%, ... 9%

We could do the same process with a larger portfolio set say 0.50% apart, but the answer would be almost identical.

CMS Replication

So we have determined that …

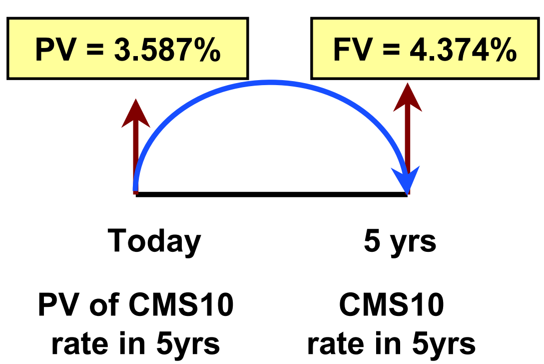
1. the current forward 10yr EUR Swap Rate in 5 years is 4.28% Annual
2. the cost today of the portfolio of swaptions replicating the CMS10 rate is 3.587%

Now, in calculating the required CMS payment, it is clear that the Present Value of the CMS payment must be the same as the Present Value of the replicating portfolio, which we see from 2 is 3.587%

Thus the CMS10 rate to be used in modelling must be the Future Value in 5 years of the 3.587% PV.

We therefore use the 5yr discount factor of 0.8200 to find the convexity adjusted forward 10yr swap rate is 3.587% / 0.8200 = 4.374%

The convexity adjustment is thus 4.374% - 4.28% = 0.094%



- **10yr swap rate in 5 years is 4.28%**
- **predicted CMS10 rate must be convexity adjusted**
- **cost today of the CMS10 replicating portfolio is 3.587%**
- **convexity adjusted CMS10 rate = 3.587% / 0.82 = 4.374%**
- **the convexity adjustment is 0.094% = 4.474% - 4.28%**