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The Yield Book

JULY 2004

UNITED STATES

A Look at a Variety of Duration Measures

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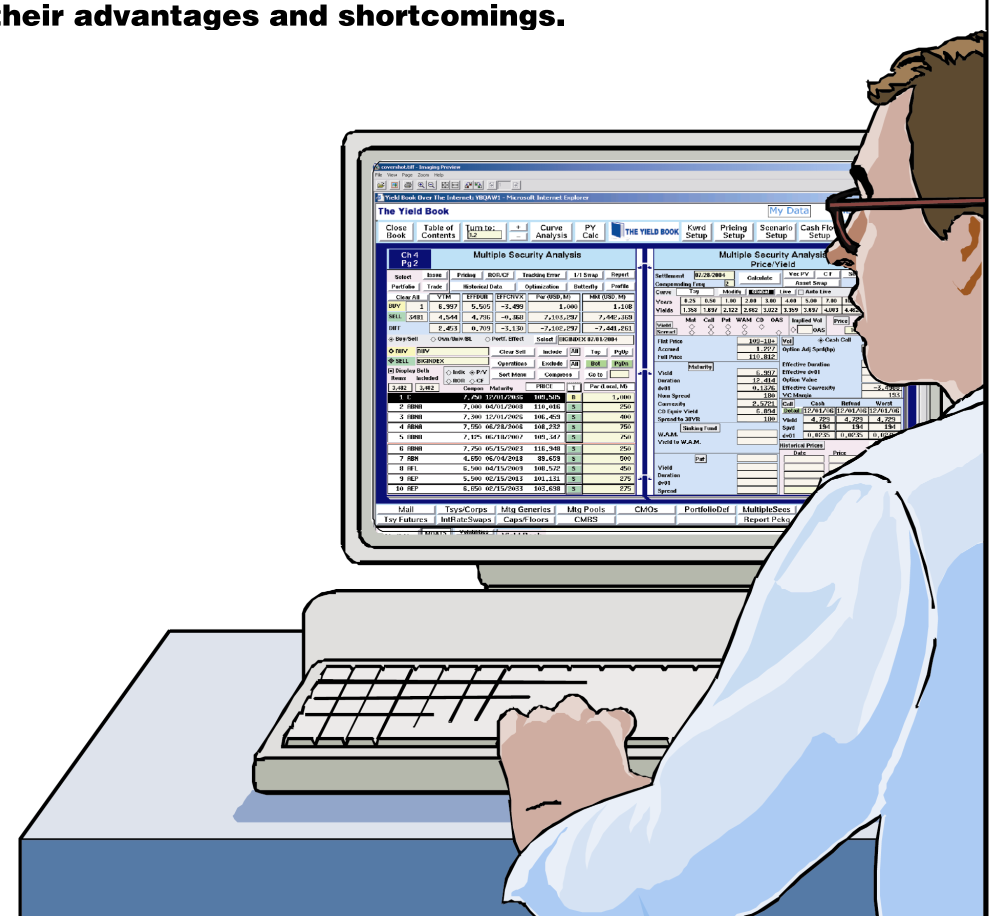
THE YIELD BOOK

- **Duration is always a fixed-income manager's first measure of the volatility of a portfolio, but there are now a variety of durations that measure different sensitivities.**

- **We examine many of these newer measures and describe their advantages and shortcomings.**



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JULY 27, 2004

A Look at a Variety of Duration Measures

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Macaulay Duration

Macaulay duration was Macaulay's original 1938 measure of the "length" of life insurance liabilities. While Macaulay hinted at the relationship between duration and the price-yield relationship, no one seemed to notice.

Modified Duration

In the 1960s, Larry Fisher presented a proof of the relationship between price change and yield change and duration.

Working Its Way Into the Mainstream

It took another 15 or 20 years for duration to become common knowledge in the industry.

Characteristics of Duration

In this section we consider the relationship of duration to yield level, time, coupon payments, etc.

Effective Duration

Modified duration was inappropriate for bonds with rate-dependent flows — mortgage-backed securities (MBSs) and callables — so this new measure was developed to substitute for modified duration for these security types. For noncallables, effective duration is usually very close to modified duration.

Portfolio Duration

Portfolio duration is simply the market-weighted duration of the constituent bonds, with a small adjustment for consistency.

Partial Durations (Key Rate Durations)

Partial durations measure the sensitivity of a bond (or portfolio) to changes in specific parts of the yield curve.

Spread Duration

Spread duration is used as a measure of the percentage price change to a change in the spread (OAS) of a bond (or portfolio). Spread duration is often very close to effective duration, but for some securities, like floaters and MBSs, it is more complicated, and the value may be quite different.

Empirical Duration

Instead of estimating how a security's price will change for a given yield change, this is a historical measure that uses actual recent price changes and changes in the level of the market (Treasuries or swaps) to measure how the security is actually responding.

"Constant Dollar" Duration

For MBSs, constant dollar duration measures the duration of securities categorized by their prices, because the duration changes as securities move from a discount price to a premium price.

Principal Component Duration

Historical changes in the yield curve can be modeled as linear combinations of "principal components," and this is a duration related to the first principal component. It is **not** calibrated to a 100bp change, as most others are.

Mortgage-Related Duration Measures

MBSs are quite complicated in valuation terms, with many factors other than "yield" level affecting them. We discuss the set of duration measures devoted to these securities in this section.

Summary

Duration has come a long way since Macaulay. The term is now used to represent sensitivities that have nothing to do with interest rate changes. Furthermore, not all durations are calibrated to a standard change in the "driver" (YTM, volatility, etc.), so it is mandatory for investment professionals to understand how the terminology differs between types of durations or even between firms who define their terms differently. We hope this paper helps that understanding.

Overview

Bob Kopprasch
The Yield Book Inc.

A number of years ago Salomon Brothers published a paper entitled *Understanding Duration and Convexity,*[1] which explained the concepts of duration and convexity by using diagrams of cash flows with fulcrums rather than relying on formulas. The paper also included narrative information on the durations of money market instruments, futures, floaters, MBSs, etc. While that paper is still correct in its descriptions of Macaulay and modified duration and it provides a useful introduction to duration and convexity, market developments related to new securities and new analytics have rendered it incomplete relative to the needs of money managers. This paper describes the newer duration measures, as well as summarizes some of the most important points on Macaulay and modified durations from the earlier paper. It also includes a little bit of history[2] to put duration in its proper historical context.

[1] *Understanding Duration and Volatility*, Robert Kopprasch, Salomon Brothers Inc, 1985.

[2] This is largely drawn from the proceedings "Pros & Cons of Immunization," a conference sponsored by Salomon Brothers on January 17, 1980.

Macaulay Duration

Introduced by Frederic Macaulay in 1938, "duration" was intended to be a better measure of the "length" of liabilities for insurance companies that had previously relied on maturity as their primary measure. In the 1938 environment (and for several decades after), there is little evidence of much interest in duration, probably because of computational difficulty at the time. But the key was that duration included both the pattern of cash flows and the level of rates in its calculation, which made it a more comprehensive measure than maturity. Macaulay's formula follows:

$$D = \frac{\sum_{t=1}^{M} \frac{t\,C_t}{(1+r)^t}}{\sum_{t=1}^{M} \frac{C_t}{(1+r)^t}}$$

Where:

M = number of periods until maturity

C_t = cash flow at time **t**

r = periodic (unannualized) discount rate

Macaulay duration is calculated as the "present-value-weighted time to receipt of cash flows," which is why we tend to measure it in "years." It is often **misquoted** as a "time-weighted present value," which it is not.[3] A look at the formula above makes this clear: each "t" (time until a cash flow) is multiplied by the present value of the associated cash flow, and then the sum of all of these terms is divided by the sum of the present values.[4] Let's also take a closer look at part of the numerator:

$$\frac{t\,C_t}{(1+r)^t} = t * \frac{C_t}{(1+r)^t}$$ Each **t** is *weighted* by the present value that the cash flow at **t** represents.

As it turns out, Macaulay duration is most useful for portfolio immunization[5] and for its relation to modified duration. Macaulay himself indicated that duration and price moves for a given rate change were approximately proportional, but immunization was the focus, not bond volatility.

In 1942, Koopmans suggested a "riskless" strategy for asset liability matching that involved matching each liability cash flow with a cash flow from an asset. But with

[3] Try typing "time-weighted present value" into Google and see how many sites have the wrong definition for duration! Naturally, if it were a present value measure, it would be quoted in dollars (or some other currency), not years.

[4] A weighted average is simply the sum of the weighted values divided by the sum of the weights. The fact that the divisor is the "price" (present value) of the bond shows that the present values are the weights.

[5] In its simplest form, immunization involves setting the duration of a portfolio equal to the duration of the liabilities (assuming that their present values are about the same). With some limitations on allowable yield curve shifts, this creates a situation in which changes in market value of the liabilities are matched by changes in the asset value. The matching must be done dynamically over time as durations change.

zero-coupon bonds (T-bills) then limited to one-year maturities, this strategy didn't offer a practical solution.[6] In 1952, Redington[7] wrote a paper in which he suggested that the criteria for a "matched" portfolio of assets and liabilities was less restrictive, and one needed to merely match the duration and present values of the assets and liabilities. However, at the time, the notion of solvency for the insurance company was defined by accounting values only, so immunization did not catch on. Finance academics — but not practitioners — gradually became aware of the duration-price relationship after 1952.

[6] Koopmans ultimately won a Nobel Prize in 1975 for his discovery and application of linear programming, which was unrelated to his work in *The Risk of Interest Fluctuations in Life Insurance Companies*, Penn Mutual Life Insurance Company Co., 1942.

[7] "Review of the Principles of Life-Office Valuations," F.M. Redington, *Journal of the Institute of Actuaries*, 78, no.3 (1952): 286–340.

Modified Duration

In the mid-1940s, both Hicks and Samuelson worked out formulaic approaches to bond price volatility.[8]

In 1966, Larry Fisher presented a proof of the relationship between duration and bond price changes. (But well into the 1970s and early 1980s, US Treasury bond traders were still using the "yield value of a 1/32" taken from the "bond basis book," commonly called the yield book.[9, 10] This was an inverse measure of volatility, because large values of YV1/32 are associated with low price volatility, and vice versa.) Researchers found that by starting with the expression for the present value of a bond, and applying some calculus and then simple algebra, it was possible to derive an expression for the percentage price volatility of a bond relative to yield changes. The expression was rather complicated looking, but could be simplified into Macaulay duration divided by (1 + periodic yield). So people talked about using Macaulay duration *modified* by the 1 + r factor. It was also called adjusted duration.[11]

Modified duration is scaled as follows: if a bond has a modified duration of 5, a 1% change in yield (100bp) would result in a 5% change in price (actually, the present value, which includes accrued interest). Thus, modified duration acts like a multiplier. Technically, the relationship can be described as:

$\Delta P/P * 100 = -\mathbf{D} \Delta Y$,

which says that the percentage change equals the negative of duration times yield change. The negative sign says that if ΔY is positive, the price change will be negative. $\Delta P/P$ is multiplied by 100 to render it into a percentage change rather than a decimal value.

In this formula, the "P" is the full price of the bond, including accrued interest, and ΔY is in absolute percent, so that 1% (i.e., 100bp) is represented as a "1."

This can be restated as:

$\Delta P = -D * \Delta Y * P/100$

Therefore, the absolute price change is equal to the negative of duration times the yield change times the full price divided by 100. For example, a bond with a duration of 5 and a full price of 90 would react to a 100bp increase in its yield by dropping about 4.5 points (5% of 90).

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[8] Hicks and Samuelson also won Nobel prizes, in 1972 and 1970, respectively.

[9] Hence the name of Homer and Leibowitz's seminal work, *Inside the Yield Book*, Sidney Homer and Martin Leibowitz, Prentice Hall, 1972, as well as the name of our portfolio analysis system.

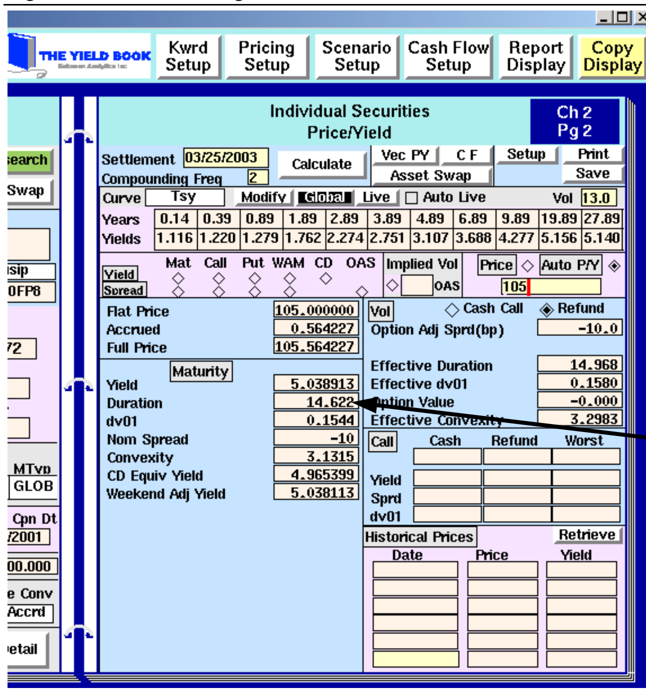
[10] Because the purpose of the yield book was to provide yields (for various coupon, maturity, and price combinations), traders worked backward to determine the volatility. For example, they would turn to the 8% coupon page, look in the ten years, three months column, and look down the page until they came to two prices (1/32 apart) that straddled the current price. The difference in yield for those two prices was the yield value of a 1/32.

[11] We show in the appendix why this measure is **not** appropriate for bonds with embedded options, rendering it **useless** in analyzing callables, putables, and MBSs.

$$\Delta P = -5*1*90/100 = -4.50.$$

This formula can be conveniently adapted to find the "dollar value of an 0.01" by changing the ΔY to 0.01. Thus, the DV01 = $-D*0.01*P/100$. In the example above, the DV01 would equal 0.045.

Figure 1. Price Yield Page



Modified duration appears under "Maturity" in The Yield Book (or under "Nominal Measures" for CMOs).

Source: Citigroup.

Working Its Way Into the Mainstream

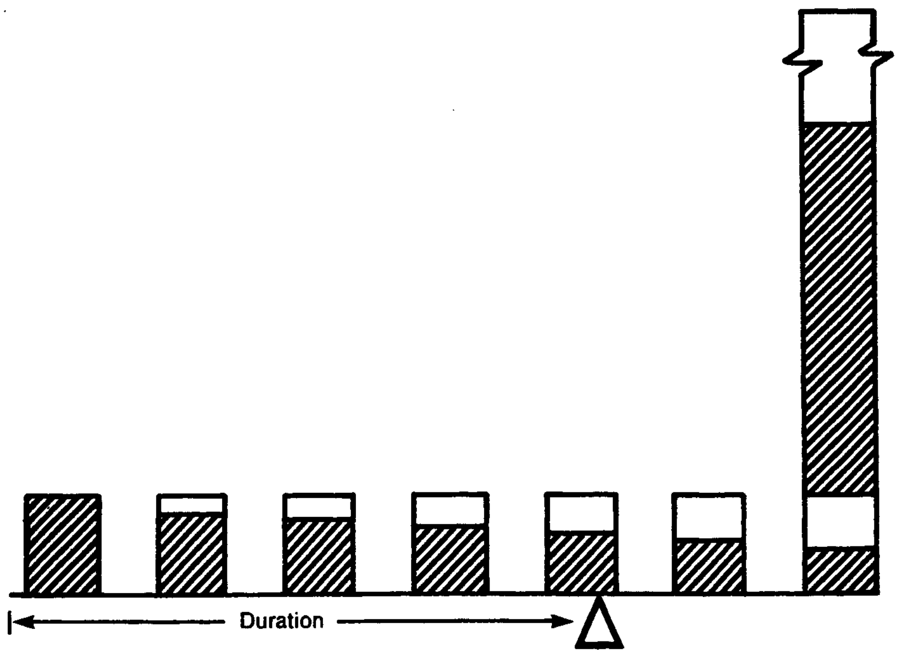
This new volatility measure, duration, was welcomed into the industry, but it was not always applied correctly. For example, investors frequently forgot that duration was to be applied to the so-called full price (including accrued interest) to determine the change. With the high coupons that prevailed in the early 1980s, this could easily cause up to an 8% error. A second common mistake was to forget that duration was a *percentage* change measure, not an *absolute* one. One famous case involved a zero-coupon trader who was long 30-year zeros (duration = 30) and hedged the position by shorting three times as many (par amount) long bonds (duration = 10). In fact, because the market value of the zeros was so low, this position was approximately 27 times overhedged! Hedging requires matching dollar volatility, or DV01s, not duration times face amount.[12] Finally, it is important to remember that duration is measured relative to changes in the yield to maturity of the bond and because

[12] The overhedge should have been apparent just by looking at the trade, without having to involve dollar values of 0.01. Let's say the 30-year zero was stripped from the then current long bond, and the same long bond was used for the hedge. If a 1:1 face value hedge was used, the trader would have already shorted an identical face amount position (completing the hedge by itself), but would also have shorted 30 years of coupons. Because the near-term coupons had a much higher value than the distant face amounts, the coupons by themselves introduced a severe overhedge, and then using a 3:1 hedge ratio simply compounded the error. There was also significant curve exposure. See the section on partial durations.

different points on the curve might move differently, a bond with a duration of 4 doesn't necessarily have price volatility twice that of a bond with a duration of 2.

The following sections review some of the characteristics and uses of duration. First, however, it would be useful to review a diagram published in the original Salomon article. It is reproduced in Figure 2 and is simply a physical analog to Macaulay's formula. It is a representation of the cash flows of a bond, with the shaded portions representing the present value of each cash flow. The distance from today (the left side of the chart) to the fulcrum (balance point) is the duration.

Figure 2. Basic Fulcrum Diagram of Duration



Source: Citigroup.

We refer to this diagram (or variants of it) repeatedly in the next section.

Characteristics of Duration

A number of general statements can be made about duration that help in understanding bond price changes (we cover these points in more detail in the earlier paper.)[13]

1 Duration tends to increase with increasing maturity, but at a decreasing rate, eventually reaching a practical maximum. Consider the effect of a longer maturity on the cash flows illustrated in Figure 2. The final principal payment is moved to the right, and additional coupons are inserted. At first, moving this large cash flow to the right necessitates moving the fulcrum to the right to keep the system in balance. But eventually, the present value of the now distant cash flow becomes very small, and moving it to the right has no further effect. This limits the extent to which duration can lengthen (see item number three).

2 The corollary to the first point is that a bond can exist for a long time without a great change to its duration, but that its duration will start to fall more rapidly *as it approaches maturity*. For example, consider a 100-year bond at 6% yield. The $100 principal is worth only $0.27 today. In the now much wider version of Figure 2, as the distant principal flow gets closer as the bond ages, it initially has little effect, so the duration stays relatively constant.[14] But eventually, as the principal flow draws closer, it pushes the fulcrum to the left because the fulcrum must be no further out than the last cash flow.

3 The maximum duration for a par bond is (1+r)/r, where r is the periodic rate of interest.[15] For corporates, Treasuries, etc., r = YTM/2. This duration is expressed in periods and would have to be divided by 2 to be expressed in years. For monthly pay instruments, r = YTM/12 (see Figure 3).

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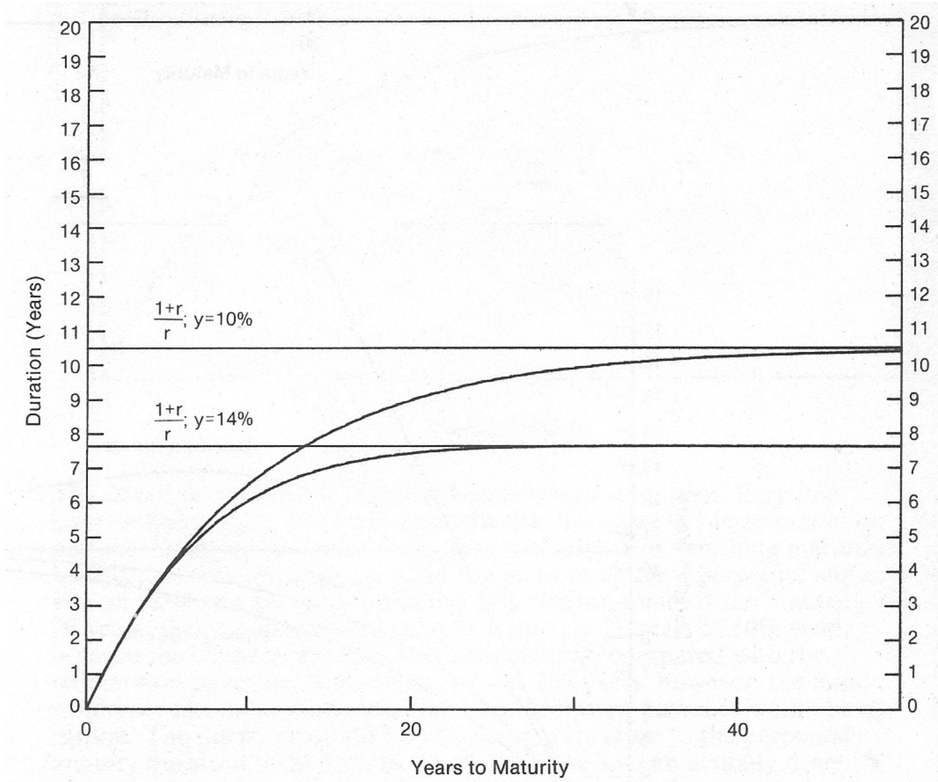
[13] See *Understanding Duration and Volatility*, 1985.

[14] Technically, this is true if we only evaluate the duration on coupon dates or at the same point in the coupon cycle each period. See point number four in this section for more detail.

[15] Note that this is for par bonds. Deep discount bonds can have longer durations, as described in *Understanding Duration and Volatility*, 1985.

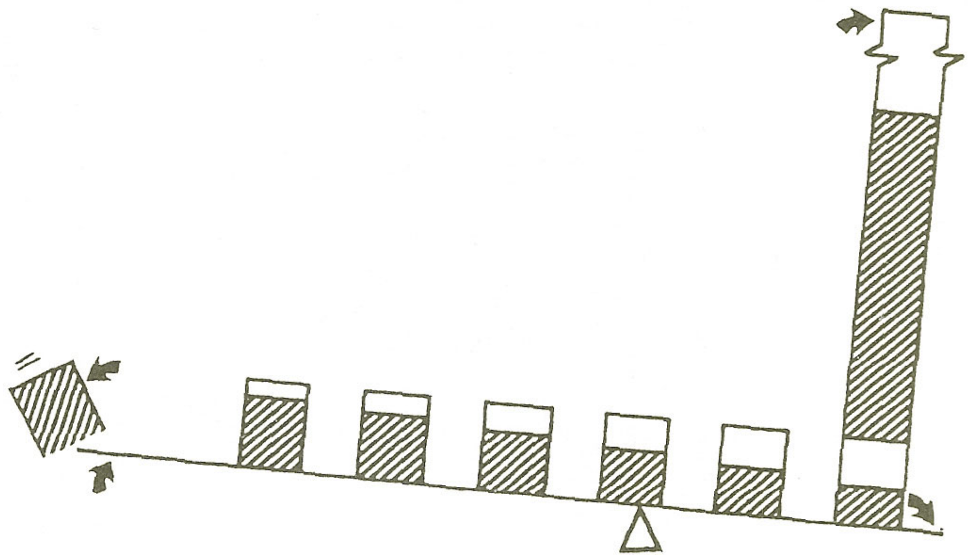
Figure 3. Effect of Yield Level



Source: Citigroup.

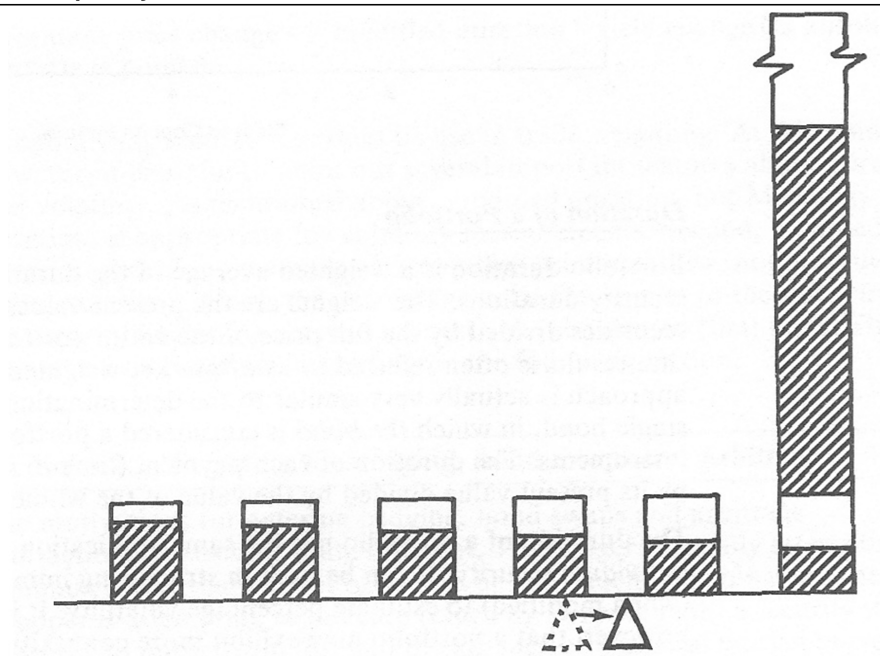
4 In a more precise version of point number two, duration declines day for day as time passes, until a coupon date is reached. Duration then jumps up after the coupon is paid. In Figure 2, (which reflects the situation just before a coupon date since the first coupon's present value is the full payment amount,) if that first coupon jumps off the diagram as it gets paid, the entire system would tilt to the right, and the fulcrum would have to move to the right to restore the balance (see Figures 4 and 5) Thus, duration increases after the coupon is paid. However, the bond's present value also declines when the accrued interest disappears, and the overall volatility remains unchanged (i.e., it does not contribute to the bond's volatility). This is because, immediately before the cash flow date, the first cash flow is at its full present value and does not change in value with changes in rates. Immediately after the cash flow, the longer flows have the same price sensitivity that they did before, and hence, the bond has the same raw price volatility. But because the accrued interest is gone, the overall value of the bond is lower, and thus, the raw price volatility represents a larger percentage volatility. Hence, duration increases.

Figure 4. The Effect of the Coupon Payment



Source: Citigroup.

Figure 5. Coupon Payment Increases Duration

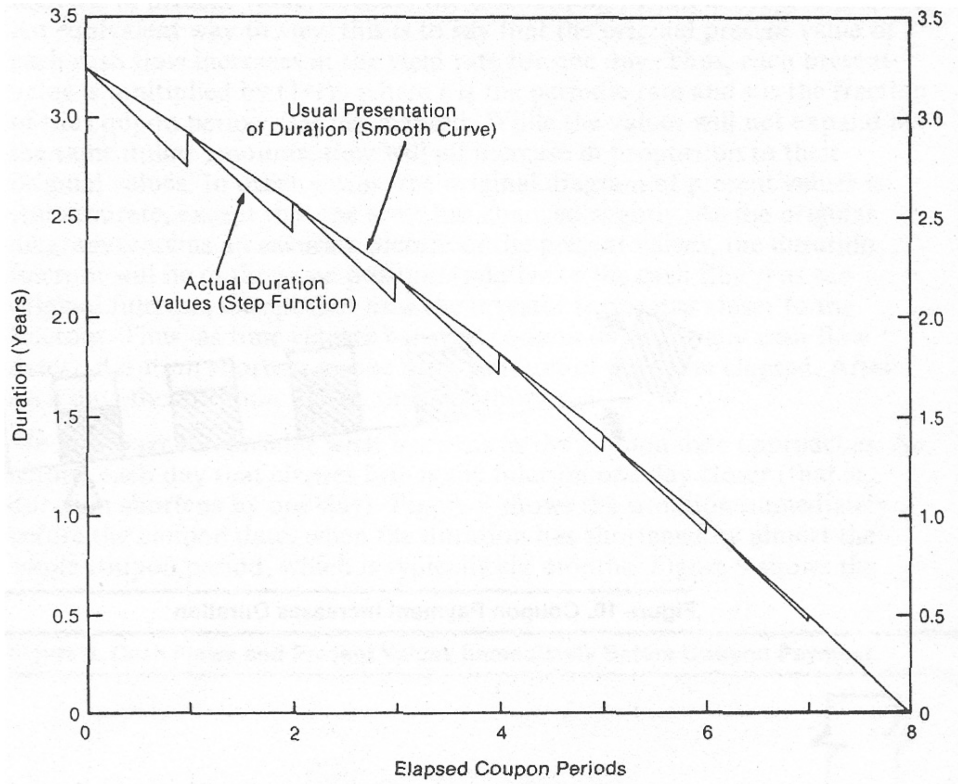


Source: Citigroup.

5 As shown in point number three, or by the Macaulay formula, duration is partly determined by the bond's yield. Other things being equal, higher rates cause the duration to be lower, and vice versa. Consider again the basic fulcrum diagram in Figure 2: in a higher rate environment, the higher rates affect the present value of distant cash flows to a greater extent, leaving the diagram more heavily loaded on the left side and the balance point shifts left. To see the effect of rates, notice how the maximum duration for a par bond is affected by rate level, as shown in Figure 3.

6 While duration decreases during the coupon period and then jumps when the coupon is paid, the volatility, as measured by the DV01, declines smoothly if the duration actually declines from coupon date to coupon date, or remains basically constant if the duration is in a “stable duration” range for a very long-maturity bond (see Figure 6).

Figure 6. Duration Through Time — Last Eight Coupon Periods of a Semiannual Bond



Source: Citigroup.

Effective Duration

As shown in the appendix, the derivation of bond price volatility relative to yield changes that results in the "modified duration" formula has embedded in it the assumption that the cash flows of the bond do not change as rates change. This means that this formula is **inappropriate** for any bond that violates this assumption and that would include any bonds with rate-dependent options: callables, putables, and especially MBSs.[16] It is also clear that ordinary yield to maturity is insufficient for these bonds, because of its use of a single set of nominal cash flows. For this large population of bonds, it was necessary to develop another analytic technique to address these shortcomings. This effort resulted in the development of option-adjusted spread (OAS) and effective duration.

For both of these measures, it is necessary to project a series of interest rate paths into the future and to evaluate the bonds along these paths. We summarize the steps of the analysis here and then follow with a detailed discussion.

1. Estimate a series of interest rate paths that are representative of possible future interest rates. It is necessary to do this in such a way that no built in arbitrage exists (that is, dominance of one security over another across all paths). This requires an interest rate model, and there are now a variety of one-, two-, and three factors models available.
2. Project the cash flows of the security along each of the paths (or at every "node" in the tree). For corporates, this requires an estimation of the incentive necessary for a corporation to call its debt, for example. For MBSs, it requires a prepayment model to estimate the changing interest and principal flows as rates change and the security ages.
3. Estimate the OAS.
4. Estimate the effective duration.
5. Estimate the effective convexity.

1. Estimating the Interest Rate Paths

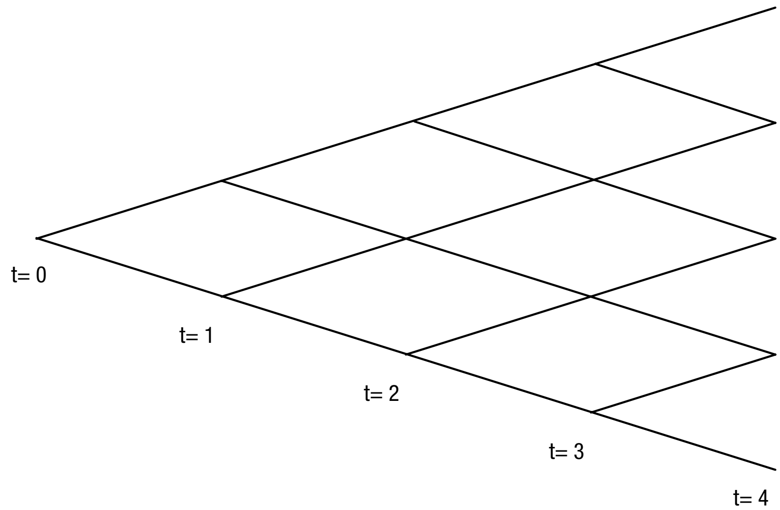
Term structure models attempt to model potential interest rate movements, taking into account yield curve shape and steepness.

Interest rate paths are generally created using one of two methods. The first is called an interest rate "tree," and it involves letting rates evolve through time by diverging up and down at (usually) regular intervals. For example, short rates begin at 5%, and at the next period they can go to 5+% and 5-%. From each of these 2 points, rates can again rise or fall. Usually, these trees have interlocking branches, so that when the 5+% outcome splits, its downward path results in the same rate as the upward path branch from the 5-% outcome. This is in Figure 7. While after two periods there are

[16] Clearly, some non-rate dependent options could also lead to misinterpretation. For example, suppose that the bond could be called after one week of consecutive days of sub-freezing temperatures in some city (some sort of weather option). We could still calculate an effective duration that took rates into effect, but the resulting value might be very misleading regarding the expected life of the security. In reality, "Chicago" bonds would be shorter than "Atlanta" bonds, which would be shorter than "Miami" bonds. So duration (the interest rate measure) might not measure all that we need to know about a bond's life.

only three possible outcomes, there are four possible paths: up-up, up-down, down-up, and down-down. After three periods, there are four outcomes, and eight possible paths, etc. If n is the number of periods, the number of endpoints is n + 1, and the number of possible paths is 2^n.

Figure 7. An Interest Rate Tree



Source: Citigroup.

In stock option modeling, these binomial trees are defined in "price space," and usually there is no bias to the tree except to gradually grow at the short-term rate of interest. In fixed income, these trees are defined in "yield space," and there are significant restrictions on the trees. Even in the simple, one factor model, which lets the short-term rate evolve over time, it is important to preserve certain characteristics. For example, the compounded value of the current short rate (let's use three months as an example) and the mean of the next period's two possible rates should be close to today's six-month rate, and so on. If these conditions are not met, there will be a set of securities that can outperform another set in all scenarios, allowing for arbitrage.

More complicated models utilize two factors (short rate and steepness of curve) or three factors (short rate, steepness, and curve curvature).[17] Even more restrictions apply to these models.

Another aspect of interest rate modeling is how volatility is incorporated. In the tree diagram, the rate at which the two paths diverge from one another is a function of volatility. If volatility is low, there won't be much chance of a large change from one period to the next. Early models used a single volatility, but more sophisticated models now use an entire volatility surface.

[17] Citigroup's new three-factor model lets rates follow a diffusion process through time, with jumps possible on FOMC meeting dates. See *A New Term Structure Model Based on the Federal Funds Target*, Y.K. Chan, Ranjit Bhattacharjee, Robert A. Russell, and Mikhail Teytel, Citigroup, May 22, 2003 (available on fi.direct.com or on www.yieldbook.com).

There are more considerations in interest rate modeling (for example, how is "mean reversion" incorporated?), but the purpose of this paper is not to discuss these in any detail.[18] The important point to remember is that not everyone faced with building such a model makes the same choices; shortcuts, subjective choices, differences in volatility and measurement intervals, etc., all combine to create differences in the ultimate rate distribution. This is only one of many reasons that different firms or vendors publish different values for OAS and duration.[19]

2. Estimating the Cash Flows

Cash flow models attempt to estimate cash flows over a wide variety of interest rate paths.

Once the interest rates have been established (that is, the paths or nodes are determined), a security's cash flows must be estimated along each of the paths. For a noncallable security, this decision is trivial, as the path won't matter. For a callable bond, there must be some rule (or trigger) incorporated to determine whether the bond will be called. And for more complex securities — like MBSs and CMOs — a path-dependent model[20] must be used. A number of large broker-dealers and several analytics vendors have proprietary prepayment models that estimate the cash flows given the interest rate path, and these models can vary significantly from each other.[21]

3. Estimating the OAS

The OAS methodology has several shortcomings that investors must be aware of.

After the cash flows have been estimated, it is time to measure the OAS. Along each path, both the short-term interest rates and the cash flows are known. Discounting the cash flows by the rates embedded in the paths will produce a set of present values, one for each path. If the **average** of these present values equals the current price (including accrued interest), then the OAS is zero (no spread added to Treasury rates). If the average is higher, as is likely, then a spread must be added to every node and the process repeated, until the average present value is forced down to the current present value. When the **average** present value equals the current present value within a desired tolerance level (i.e., "converges"), the spread that makes this happen is the OAS, *by definition*. The current methodology is to use the same spread throughout the entire tree of rates, even if there is general acknowledgement that there is a term structure of OAS (see Figure 13, which comes after the box entitled **Empirical OAS Duration**).

It's worth noting several aspects of the algorithm just described: (1) The OAS is *not* the **average spread** across the paths (it is based on average **prices**); and (2) the

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[18] For more detail on interest rate modeling, see *A Term Structure Model and the Pricing of Fixed-Income Securities*, Y.K. Chan, Salomon Brothers Inc, June 1997.

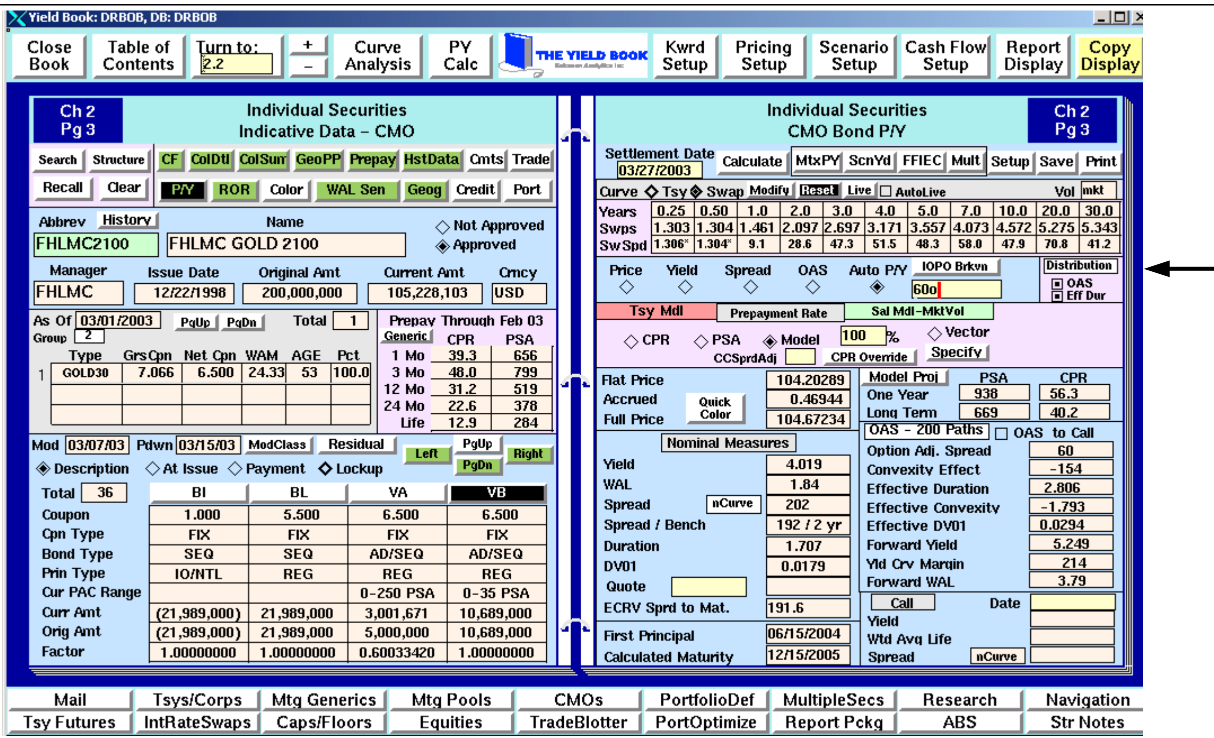
[19] See "Pitfalls in the Analysis of Option-Adjusted Spreads," D. Babbel and S. Zenios, *Financial Analysts Journal*, July-August 1992. See "Option-Adjusted Spread Analysis: Going Down the Wrong Path?," Robert W. Kopprasch, *Financial Analysts Journal*, April-May, 1993.

[20] For some securities, only the rate level matters; such securities are called state dependent. For MBSs (and related derivatives), not only does the rate at a particular node matter, but the path that rates take to get there is also important in determining the cash flow. Such securities are said to be path dependent.

[21] See, for example, *Anatomy of Prepayments: The Citigroup Prepayment Model*, March 2004. Bloomberg subscribers can type VALL<GO> to see the various predictions of many dealers. Or type VSAL<GO> for Citigroup's, VPW<GO> for UBS's (Paine Webber), VMS<GO> for Morgan Stanley's, etc.

actual spread calculated may not even be a possible outcome! For example, a derivative security might be structured so that it either pays off very handsomely, or not at all. Measured across all of the paths, the individual present values would either be very high, or very low. The spread that equated the average of these present values with the current price would likely seriously underestimate or overestimate the actual spread that would be earned, since all of the prices would be at the extremes of their distribution. This is one reason why it is useful to view the actual distribution of prices. The pattern reveals whether the outcomes fall along a smooth continuum, are somewhat irregular, or possibly even bimodal, with the most likely outcomes at the extremes. The Yield Book provides the distributions of prices and weighted-average lives for MBS and CMOs (see Figures 8–10).

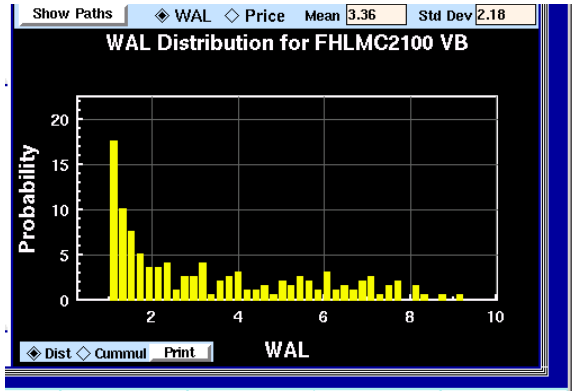
Figure 8. Menu Button for Distributions



Source: Citigroup.

A wide distribution signals greater uncertainty.

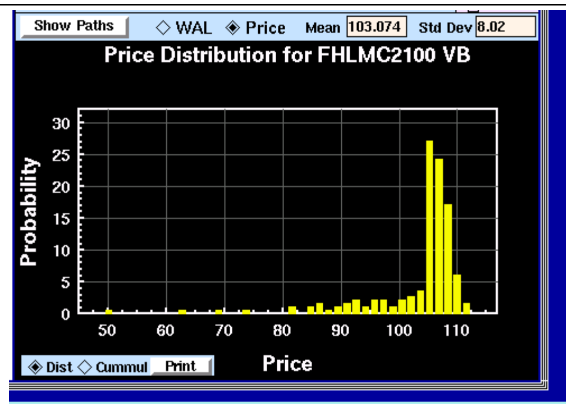
Figure 9. WAL Distribution for FHLMC2100 VB



Clicking the "Distribution" button (see arrow, above) gives a view of the either the WAL distribution or the price (present value) distribution, depending on the toggle selected.

Source: Citigroup.

Figure 10. Price Distribution for FHLMC2100 VB



Source: Citigroup.

Can OAS Be Improved?

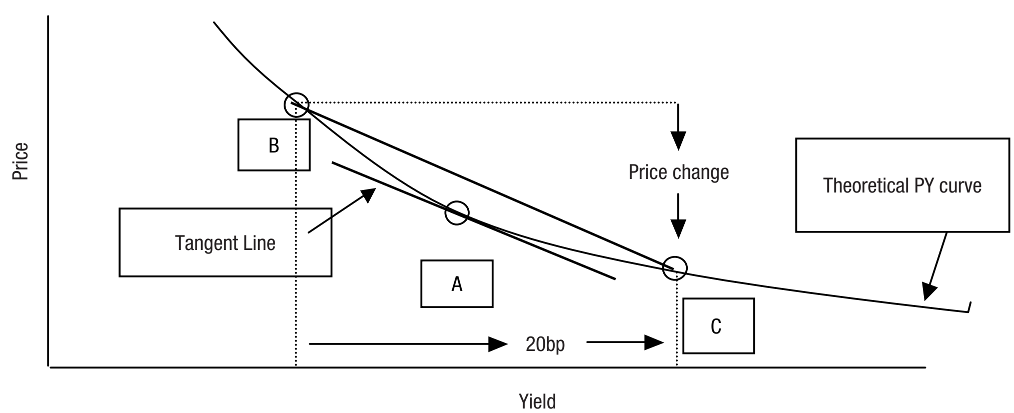
If computing power and calculation time is not an issue, the OAS calculation process can probably be improved. For example, at each node in the tree for MBSs (and based on the path followed to get to that node), a *deterministic* prepay model is normally used, that is, one specific prepayment rate is projected. In reality, the modeler could specify a distribution of possible prepayment rates based on the path. For example, a narrow distribution could be used simply to reflect possible model error. A more aggressive use might be based on the increasing refinancing sensitivity that has been exhibited over the past decade. The modeler might incorporate *some* probability of even more sensitive homeowners in the future and might even project that the sensitivity would increase over time — possibly a result of greater media effect. It would probably be necessary to increase the standard deviation of the estimate as the paths moved further into the future and the estimates become less precise. This would increase the calculation time enormously, because an entire distribution would be tied to each node in the interest rate tree, and many more "trips" through the tree would be necessary to properly evaluate all of the possibilities. As a shortcut, we calculate something called "prepayment duration" which we discuss in the mortgage-related durations section.

4. Estimate the Effective Duration

Once the OAS has been established, the next step is to estimate the effective duration. The accepted methodology is to move the starting yield curve up and down a specified amount,[22] recalculate the interest rate tree for each, re-estimate the cash flows along every path in the "up" and "down" trees, and, **holding the OAS constant**, recalculate the average price for the up and down trees. Then, the percentage price change relative to the starting price is determined, and a duration estimate is obtained by scaling the result based on a 100bp change. For example, if the curve is moved up and down by ten basis points, the percentage price change over that 20bp range would have to be multiplied by 5 to reflect a 100bp change. By doing this, the effective duration estimate for embedded-option securities can be interpreted consistently with modified duration for noncallables, that is, as a multiplier for a 100bp change.[23]

In the diagram that follows, Point A represents the original price (present value) of the security. Point B represents the new present value calculated when rates are lowered, and Point C represents the present value when the curve is moved up. In theory, if we think of the price-yield relationship as a smooth curve, we should be trying to estimate the tangent to that curve at Point A to determine volatility (and then turn it into a percentage measure to put it into duration terms). Because we do not have a formula expressing the price-yield curve, we can't easily "solve" for the tangent, so in essence we measure the "chord" between Points B and C and use it as an estimate of the tangent. So, in the vicinity of "A," the current price, the change in price for a 20bp change in rates would be the vertical distance between B and C.

Figure 11. Estimating Effective Duration

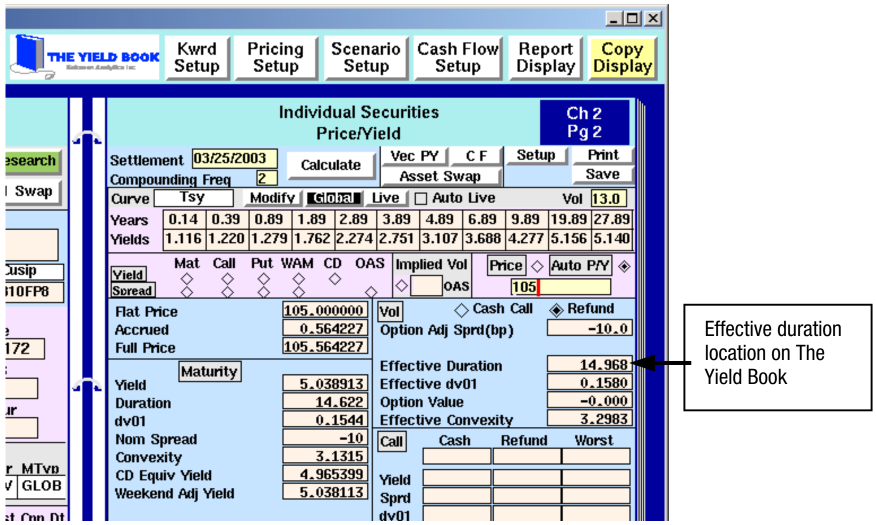


Source: Citigroup.

[22] See www.yieldbook.com for examples of apparent anomalies (that aren't).

[23] This is not quite exact, as we explain in the section "Apparent Anomalies Between Modified Duration and Effective Duration." Note that principal component duration, discussed later, is **not** calibrated to a 100bp change.

Figure 12. Effective Duration

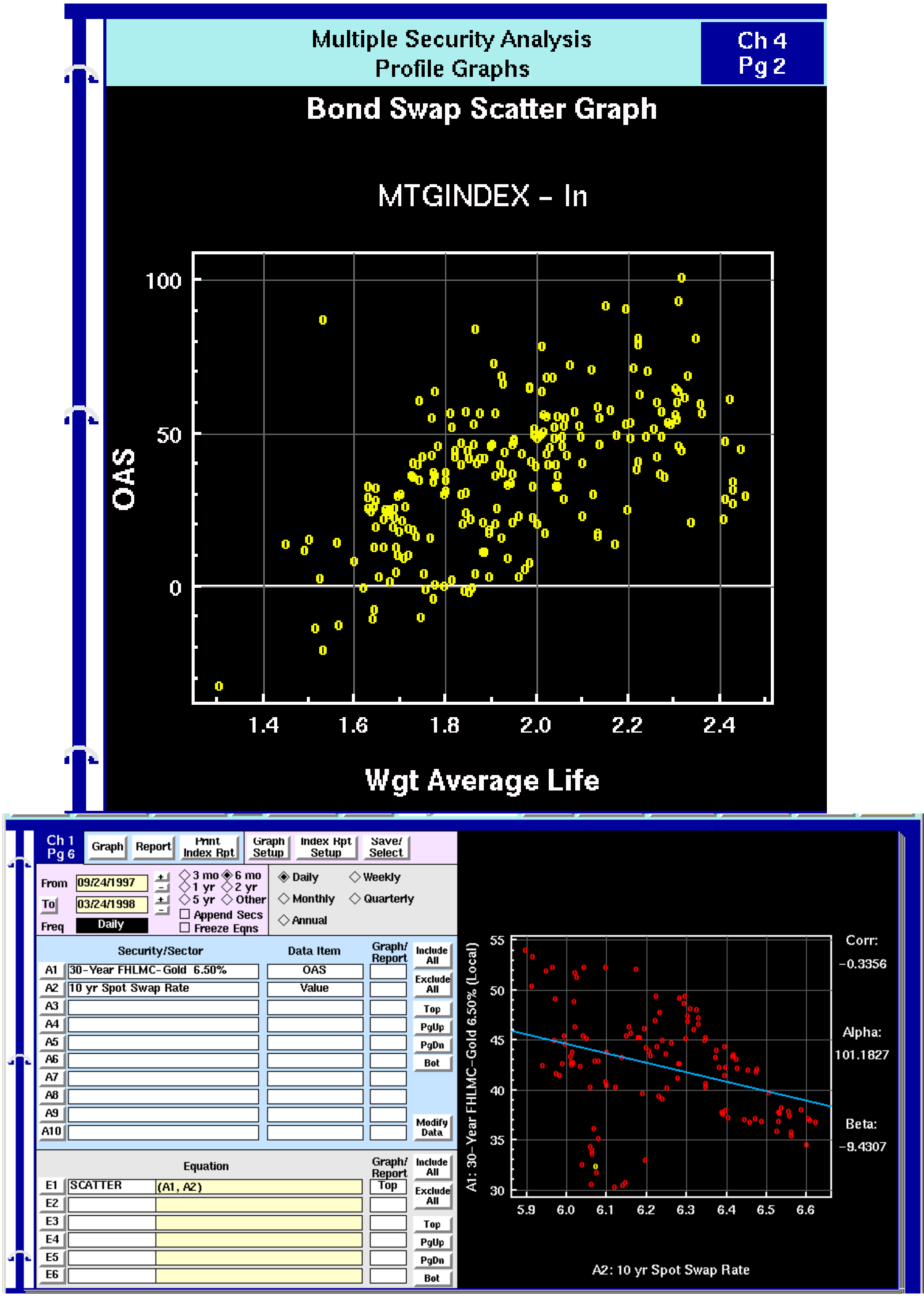


Source: Citigroup.

Empirical OAS Duration

There is another measure called **empirical OAS duration**. It is quite similar to effective duration, but with one significant difference. Instead of holding OAS constant as the curve is moved up and down, the OAS is given a directional change. It is quite typical to see MBS OAS widen when the market rallies and tighten when the market falls. (Well, this was true in the past. At the current time, it appears that the relationship is clouded by the steep OAS curve: as the market rallies, MBSs shorten, and are priced to a shorter average life. Consider Figure 13.) The provider quantifies this using regression analysis and then explicitly changes the OAS to reflect the new market level after the up or down change. Then the present values are calculated (using a new higher OAS for the "down rate" tree and a lower OAS for the "up rate" tree), and the differences are treated as described above. It is an "effective duration," because it is determined analogously, but it is *empirical* in that the OAS change is based on an empirical analysis of recent data. We mention it here because it is a type of effective duration and not an empirical duration of the type to be discussed later.

Figure 13. Multiple Security Analysis Profile Graphs

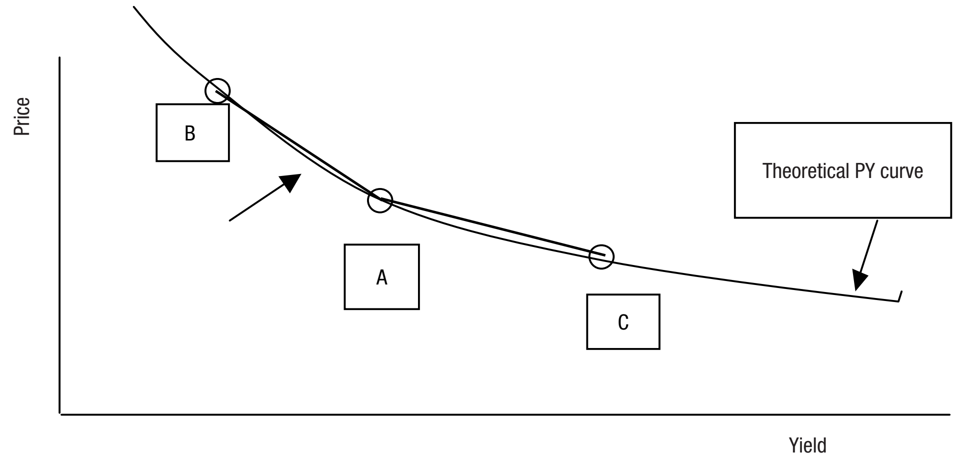


Source: Citigroup.

5. Estimate the Effective Convexity

While duration estimates a bond's price change relative to a yield change by using a linear estimate, effective convexity attempts to measure the nonlinear effects that occur to prices when yields change. In essence, effective convexity (like effective duration) is estimated using brute force. While the entire price change between points B and C is used to estimate duration, we estimate convexity by looking at the difference between the up price change (from A to B) and the down price change (from A to C). This difference (appropriately weighted) represents the convexity in the security's price yield relationship. Convexity is positive in Figure 14, but can be negative for callable bonds and MBSs.

Figure 14. Estimating Convexity



Source: Citigroup.

The amounts by which one moves the curve up and down in the duration and convexity calculations is important. If the amount is too small, it is difficult to know if the other factors that determine price are being meaningfully captured. For example, is the prepayment model for MBS sensitive enough to respond to the very small change in overall rate level? Are there "step" values in the inputs (for example, current coupon rate for GNMAs) or are they continuous? These questions lead one to use a larger change in the curve level. But if too large a change is made, we may miss the local phenomena that we are trying to measure. For example, some securities have a range of rates in which they exhibit negative convexity, but they are positively convex outside that range. If we select too large a move in the curve, we may blend the two ranges and wind up with an inexact estimate.[24] To some extent, the decision is arbitrary, but must be based on balancing the two types of problems just described. In The Yield Book, we use +/-25bp for effective duration and convexity estimation.[25, 26] The value is generally shown below the effective duration on the PY page.

[24] "Inexact estimate" sounds redundant. Throughout this paper, we try to use the word estimate rather than calculation to reflect the fact that we **are** estimating — we're using estimates of prepayments, estimates of yield curve evolution through time, estimates of volatility, etc., in our model.

[25] This is user-selectable for convenience, between +/-25bp and +/-50bp.

[26] Because of the extremely low rates for Japanese Government Bonds, we have begun to use +/-1bp in this calculation. A 25bp shift could result in negative rates.

Apparent Anomalies Between Modified Duration and Effective Duration

Effective duration was designed to capture the effects of cash flows that might vary based on changes in interest rates. Therefore, it's not unreasonable to assume that, for bonds that do not have cash flows subject to change, effective duration would produce the same values as modified Macaulay duration. Unfortunately, this is not the case, and the extent of the difference depends on several factors. (See Figure 12 in the "Effective Duration" section. Notice that the effective duration is longer than the nominal duration for a Treasury with no optionality.)

Macaulay duration measures (can be interpreted as) the proportional sensitivity of price to a 100bp change in the yield of a bond. In the Yield Book, effective duration measures the effect of a 100bp change in the level of the "par" curve, as determined by the Treasury Model curve. As an example, consider the case of zero-coupon bonds. Depending on the shape of the par (model) curve, the zero curve for various maturities will lie above or below the par curve. More important, a 100bp shift in the par curve will result in a different shift in the zero curve. Let's assume that the zero-coupon curve would move 110bp at the 20-year maturity (the response varies). Effective duration would measure the proportional price response to a 100bp shift in the par curve, which would be a 110bp shift in the yield of the bond itself. Thus, the effective duration would be approximately 110% of the Macaulay duration, even though there is no optionality in the zero.

In more general terms, when we move the par curve 100bp, we are not moving the yield of any particular bond by 100bp. For example, if the ten-year par rate is 4%, we shift the curve so that we wind up with a ten-year, 5% bond priced at par. That is not the same thing as repricing the 4% bond to yield 5%. If the curve has any slope to it, the 4% coupon bond will not yield 5% when a 5% coupon bond yields 5%. If the curve is upward sloping, the 4% bond should yield more than the 5% coupon bond, and as in the zero-coupon example, effective duration will be higher than modified duration.

See www.yieldbook.com for more "anomalies."

Portfolio Duration

Naturally, while investors are somewhat interested in the durations of the individual securities in their portfolios, they are particularly concerned with the overall portfolio duration as a summary measure, especially when compared with the duration of a benchmark. There are two possible approaches to the portfolio duration question: average the individual durations (weighted appropriately) or treat the entire portfolio as one security with a set of cash flows and calculate its duration (or estimate its effective duration).

The duration of a portfolio is almost always reported as the market-weighted average of the individual components.

If the nearly universal weighted-average method is used, one simply market weights the individual durations to get the portfolio duration. This simplicity masks one potential complication, however. Usually, when calculating Macaulay duration, the periodicity of discounting — which usually matches the cash flow frequency of the security — comes into play, and the duration is technically calculated in "periods" and then annualized. Remembering that (modified) duration can be interpreted as the percentage price response of a bond to a change of 100bp in *its* yield, a mortgage duration of 3 means something slightly different than a Treasury duration of 3 (never mind that modified duration is not appropriate at all for MBSs). In one case, we're talking about a change in a monthly yield and in another a semiannual (or bond equivalent) yield. If we assume that non-Treasury rates move in lockstep with Treasuries — after adjusting for frequency — then MBS rates wouldn't move 100bp when Treasuries did (substitute "monthly floater" or something similar if the MBS duration bothers you). So how can we interpret a portfolio duration if the components are calculated using a different metric? It's as if we're measuring one duration with a yardstick and another with a meter stick. In order to average them, they need to be on the same terms. In the Yield Book, duration is quoted and calculated for portfolio purposes according to the "common compounding frequency" found in "Pricing Setup," usually on a bond-equivalent basis.

Frankly, this small adjustment, while technically correct, is probably dwarfed by potential errors in the estimated effective durations due to nuances in the term structure model and prepayment models normally employed. Furthermore, when effective durations are calculated, they usually use a common interest rate tree, so all of the durations are calculated on the same basis from the start. But assumptions must be made, for example, about how MBS current-coupon rates will move with either Treasuries or swaps, and the "rules" employed in this relationship have a greater influence on the relative durations of different securities.

The second approach to portfolio duration treats the portfolio as one group of cash flows, with one price (total market value) and one yield. Simulations could be employed, using the cash flows already calculated for OAS and effective duration, to determine the overall portfolio fluctuation as rates change. Such a duration calculation is rarely found in current analytics. (However, the Yield Book has a new tracking error (TE) capability that does something similar, but in addition to rates changing, volatilities, currencies, etc. can change, and securities move in a correlated manner instead of in lockstep. Each TE calculation uses 10,000 scenario analyses.)[27]

Portfolio OAS

A true portfolio OAS would look at all of the cash flows of the various instruments at all points in the tree (some semiannual, some monthly, etc) and treat them as coming from one instrument. Then the OAS on this single "security" could be estimated. In reality, a number of complicating issues render this calculation more complicated than theory suggests. A close approximation is to use a spread-duration weighted OAS (weighted by market value *and* spread duration). We discuss spread duration later in the paper. This approximation can be made more accurate by incorporating spread convexity as well. We call this a "parametric estimation of true portfolio OAS." This is discussed on www.yieldbook.com.

[27] See the forthcoming paper on *Tracking Error*, The Yield Book Inc., by William S. Herman and Sang Y. Shim.

Partial Durations (Key Rate Durations)

Measuring sensitivities to specific points on the yield curve, instead of to the whole curve.

One of the legitimate criticisms leveled at duration is that it is a measure that applies only to a parallel shift in the curve.[28] Another way to express this is that it doesn't tell the investor the sensitivities to different parts of the curve in the likely event of a non-parallel shift. Investors employing a duration neutral strategy versus a benchmark know all too well that it is possible to match the duration of a benchmark but still have tremendous "curve exposure" in their portfolios. One approach to the problem is to use a vector of durations, each tied to a limited maturity range. These are known as "partial" or "key rate" durations.[29]

Normally, the sum of the partial durations closely approximates the effective duration of the security. This is obviously a necessary characteristic; otherwise, the partial durations would imply a different price change than the effective duration even for a parallel shift.

If effective duration is considered a brute force calculation, partial durations require even more brute force. The steps in calculating partial durations are:

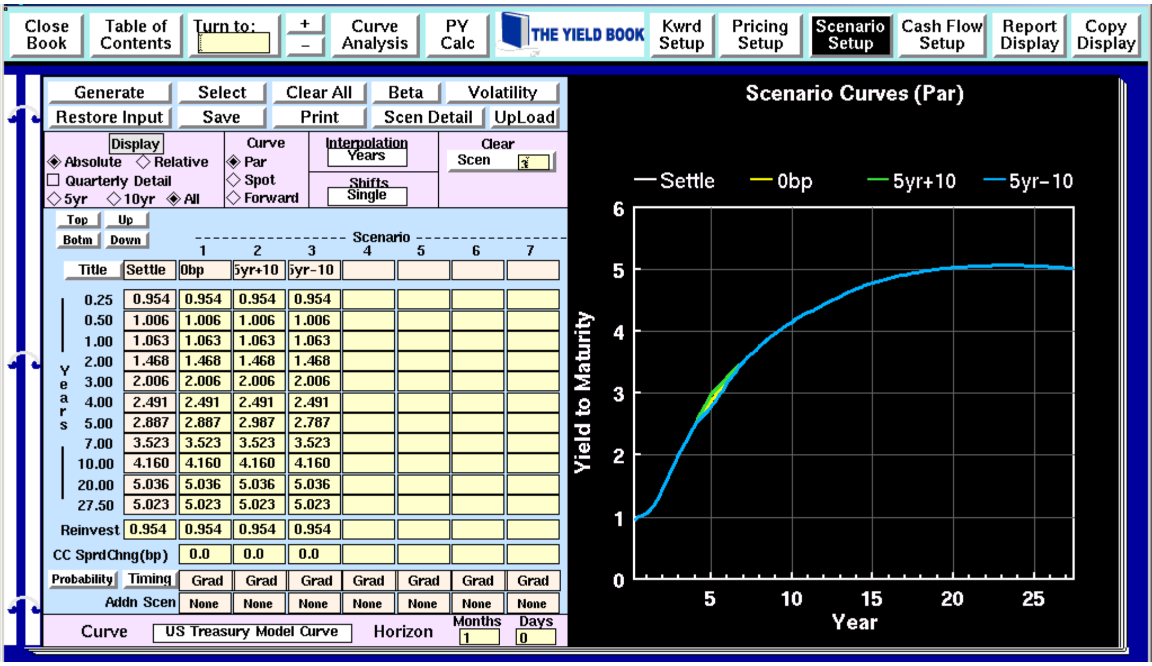
1. Measure the OAS.
2. Shift the yield curve by moving only one "key rate", such as the five-year rate. This means that rates at some nearby points (e.g., the four-year and the seven-year) are kept constant, and rates in between are interpolated (see Figures 15 and 16). Assume for a moment that we move the five-year rate **up** 10bp (see scenario 5yr+10 in Figure 15). Note that all of the rates are unchanged except for the five-year, which is 10bp higher. This can also be seen in Figure 16, which shows the **relative** yield shifts.

---

[28] For effective duration, we use a parallel shift up and down to determine the duration. Technically, Macaulay duration involves only one point on the curve — the yield of the particular bond that we are evaluating. But to compare Macaulay durations of different bonds, we are implicitly assuming that all bonds exhibit the same yield change (i.e., a parallel shift).

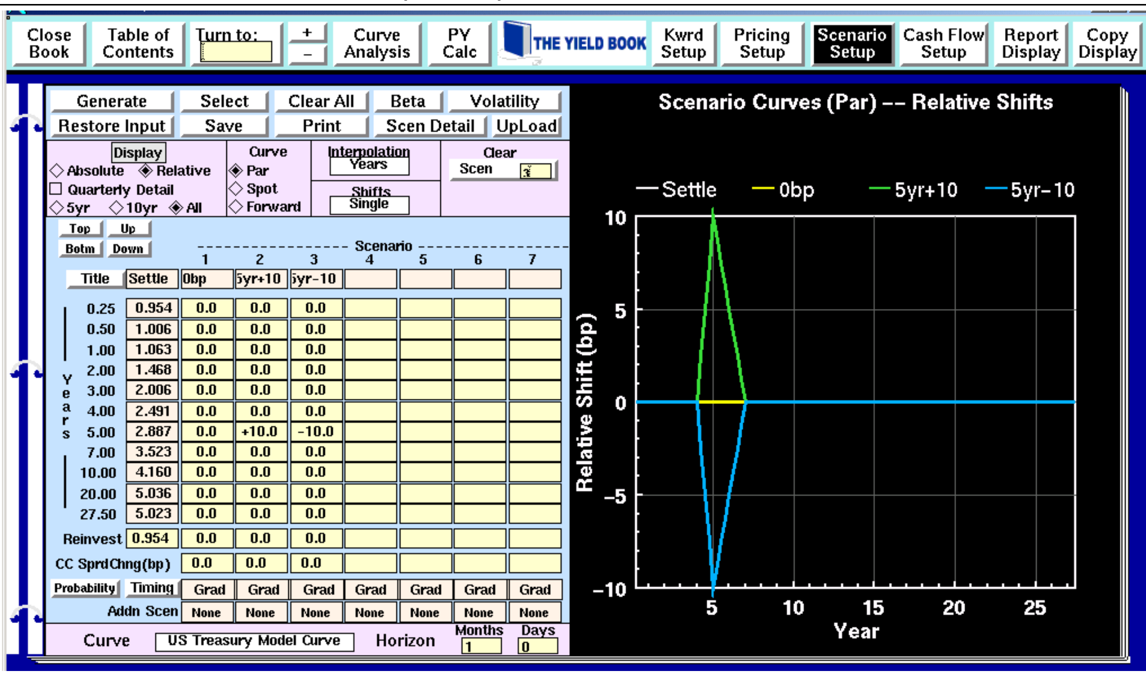
[29] Thomas Ho popularized the name key rate durations in "Key Rate Duration: A Measure of Interest Rate Risk Exposure," T. Ho, *The Journal of Fixed Income*, Vol. 2, No. 2, September 1992. The technique of measuring these partial durations was already in use on the Salomon trading floor in the early to mid-1980s.

Figure 15. Absolute Yield Shift for Partial Duration (Five-Year)



Source: Citigroup.

Figure 16. Relative Yield Shift for Partial Duration (Five-Year)



Source: Citigroup.

1 Using this new curve, recalculate the forward rates, and regenerate the interest rate tree.

2 Re-estimate the cash flows along the new interest rate tree, taking into account puts, calls, or prepayments likely to occur based on the rate path.

3 Holding OAS constant, find the present values of all of the paths, and determine the average present value.

4 Repeat the process after moving the five-year rate **down** 10bp.

5 As in the effective duration calculation earlier, evaluate the difference in the "prices" (averages of present values) of the up and down changes as a percentage of the starting price (itself an average of present values from the OAS calculation), and scale to 100bp by multiplying by 5. This is the five-year partial duration.

6 Repeat for all of the desired points on the curve.

Partials increase calculation time.

As you can see, while OAS requires evaluation across one interest rate tree, and effective duration requires two more trees, partial duration requires two additional trees and all of the associated calculations for *each* point for which it is estimated. A vector of seven partials (three-month, six-month, one-year, two-year, five-year, ten-year, 30-year) would require 14 interest rate tree evaluations in addition to the original OAS calculation. This won't take 14 times as long as the original OAS, because we don't need to iterate to find the OAS, but it still takes a (relatively) long time. On The Yield Book™, it is offered as an optional calculation that you must toggle "on."

Spread Duration

The notion of spread duration recognizes that while some aspects of bond price moves are captured by overall market level changes, there are situations in which the change in spread can affect the bond differently (and sometimes, with greater magnitude). One such situation involves floaters. In general, floaters are designed to trade near par, with coupons that reset based on some market level. It is this reset that keeps the prices generally near par. For example, a floater might reset quarterly and have a coupon formula of three-month LIBOR + 70bp. If that is the current level for new securities, then the floater should trade near par as it approaches its reset date, because at that time it will revert to a new market-level coupon. As a result, it can be thought of like a zero maturing at the next reset, and its duration will be the time to next reset,[30] even if it has 20 years to maturity. If market levels change, the bond will react like a very short bond, since it should be near par shortly.

The situation is quite different if spreads change. Let's suppose that spreads widen, so that new securities sell with a coupon of LIBOR + 100bp. What will the older LIBOR + 70bp bonds be worth? These bonds can drop significantly below par. Because the investor gives up 30bp in coupon, the price drop depends on the length of time that the 30bp is forgone. If the bond is a 20-year, it is likely to drop about 3½ points, the present value of 30bp for 20 years. The spread duration would be approximately 12 years, very similar to the Macaulay (and spread) duration of 20-year fixed-rate bonds.

In The Yield Book, spread duration is estimated by changing the OAS of the security by +/-25bp, and determining the prices at those levels. The difference in the prices, divided by the original price, is multiplied by 200 to produce the spread duration. (We multiply by 2 to normalize to a 100bp shift, and then multiply by 100 to turn the decimal into a percentage.) For example, if the starting price is 99, the -25bp price is 101, and the +25bp price is 97, the spread duration would be:

$4/99 \times 200 = 8.08$

The comments in the paragraph following Figure 14 about selecting the size of the move over which to estimate duration also apply here for spread duration.

[30] This is true for any maturing cash flow, whether from a coupon bond or a zero. This is not true if there are caps and floors on the reset.

Empirical Duration

As one can see from the description of effective duration, many assumptions regarding cash flows, rates, and volatilities are necessary to generate an effective duration. It often turns out that the market doesn't seem to trade the instruments with the predicted duration. **Empirical** duration was created to deal with these times. Rather than estimate how the security *ought* to trade, empirical duration estimates *how it has been* trading. It is normally calculated by regressing percentage price moves of the security in question versus some market benchmark, such as an on-the-run Treasury or a particular swap maturity. While its interpretation is similar to effective duration, often the creator of the empirical durations will convert these directly into empirical hedge ratios, which is probably what the money manager needs anyway. (Because empirical durations are most often used for MBSs, where investors often hedge the shortening of duration by buying Treasuries, investors need to know the "equivalent ten-year Treasury" volatility represented by their MBS position.) A hedge ratio of 0.6 ten-years means that the bond in question has a dollar volatility "equivalent" to 0.6 ten-year Treasuries (or swaps).

Even this calculation is not without its assumptions or necessary parameters. The calculation (i.e., estimation!) of empirical duration requires that the investor determine: (1) how often the data are to be recorded (usually daily, but it could be weekly or several times a day), (2) over what period the data is to be analyzed (see also the "'Constant Dollar' Duration"), and which instrument (Treasury note or swap) and which maturity will be the independent variable. Often, these durations are simply offered in a standardized way (see the Smith Barney Direct example in Figure 17) or more flexibly (see the UBS *Mortgage Strategy* website example in Figure 18).

Figure 17. Empirical Duration

citigroup

Effective/Empirical Duration Report

Mortgage Research

								Partials								1 Month					Hedge Ratios	
CPN	WAC	WAM	AGE	PX	OAS	EDUR	CNVX	2 YR	5 YR	10 YR	30 YR	VOL	SPRD	CC SPRD	PRE PAY	PX CHG	OAS CHG	EMPDUR to swaps	UPDT EMP to swaps	UPDT EMP to tsy	EFF DUR	UPDT EMPDUR
FNMA 30YR																						
4.5	5.07	354	4	94-17	6	6.5	-0.6	0.4	1.2	3.5	1.4	0.2	5.9	-0.10	-0.40	1-25	-4	6.9	6.9	7.2	0.79	0.84
5.0	5.57	356	2	97-18	0	5.7	-1.4	0.5	1.2	3.0	1.0	0.3	5.4	-0.40	-0.20	1-22	-6	6.0	5.9	6.2	0.72	0.75
5.5	5.92	356	2	100-08	-3	4.7	-2.5	0.6	1.1	2.3	0.6	0.3	4.9	-0.80	-0.10	1-12	-4	4.9	4.7	4.9	0.60	0.61
6.0	6.50	352	6	102-20	-5	3.5	-3.2	0.7	0.9	1.6	0.3	0.2	4.3	-1.20	0.00	1-03	-4	3.5	3.2	3.4	0.47	0.43
6.5	7.01	338	20	104-11	9	2.8	-2.7	0.7	0.8	1.2	0.2	0.2	3.8	-1.30	0.20	0-22	1	2.4	2.1	2.2	0.38	0.28
FHLMC 30YR																						
4.5	5.07	354	4	94-17	8	6.5	-0.6	0.4	1.2	3.5	1.3	0.2	5.9	-0.00	-0.40	1-27	-5	6.9	6.8	7.2	0.79	0.83
5.0	5.57	356	2	97-16	3	5.7	-1.3	0.5	1.2	3.0	1.0	0.3	5.4	-0.30	-0.30	1-22	-6	6.0	5.9	6.2	0.72	0.75
5.5	5.92	356	2	100-10	-2	4.7	-2.5	0.6	1.1	2.3	0.6	0.3	4.9	-0.90	-0.10	1-13	-5	4.7	4.5	4.8	0.60	0.58
6.0	6.50	352	6	102-18	-1	3.5	-3.2	0.7	0.9	1.6	0.3	0.2	4.2	-1.20	0.00	1-00	-2	3.5	3.2	3.4	0.47	0.43
6.5	7.01	338	20	104-15	7	2.7	-2.8	0.7	0.8	1.1	0.1	0.2	3.7	-1.40	0.20	0-24	0	2.2	2.0	2.1	0.37	0.27
GNMA 30YR																						
4.5	5.00	350	8	94-30	-1	6.7	-0.5	0.4	1.2	3.6	1.6	0.2	6.1	-0.00	-0.40	1-27	-5	7.1	7.0	7.3	0.82	0.86
5.0	5.50	356	2	97-31	-6	6.1	-1.3	0.5	1.2	3.2	1.3	0.3	5.9	-0.40	-0.20	1-25	-5	6.0	5.9	6.2	0.77	0.75
5.5	6.00	356	2	100-19	-7	5.0	-2.4	0.6	1.1	2.5	0.8	0.3	5.2	-0.80	-0.10	1-13	-3	4.9	4.7	5.0	0.65	0.61
6.0	6.50	352	6	102-28	-6	3.8	-3.1	0.7	0.9	1.8	0.4	0.3	4.5	-1.30	0.00	1-00	0	3.5	3.3	3.5	0.50	0.43
6.5	7.00	332	26	104-20	2	2.7	-3.1	0.7	0.8	1.1	0.2	0.2	3.8	-1.40	0.20	0-20	2	2.3	2.0	2.1	0.37	0.27
GNMA II 30YR																						
5.0	5.57	356	2	97-24	-4	6.1	-1.3	0.5	1.2	3.2	1.3	0.3	5.8	-0.40	-0.20	1-24	-4	6.1	6.0	6.3	0.77	0.75
5.5	5.97	356	2	100-13	-4	5.1	-2.3	0.6	1.1	2.6	0.9	0.3	5.2	-0.80	-0.10	1-14	-3	4.9	4.7	5.0	0.66	0.61
6.0	6.39	340	6	102-22	0	3.9	-3.0	0.7	1.0	1.8	0.4	0.3	4.6	-1.20	0.00	1-00	0	3.5	3.3	3.5	0.52	0.44
6.5	7.25	332	26	104-14	1	2.7	-2.9	0.7	0.8	1.1	0.2	0.2	3.6	-1.30	0.20	0-20	2	2.3	2.0	2.2	0.36	0.28
FNMA 15YR																						
4.0	4.65	174	4	95-30	9	4.9	-0.2	0.5	1.6	2.6	0.2	0.1	4.6	-0.00	-0.20	1-09	-3	4.7	4.7	4.8	1.10	1.05
4.5	4.98	176	2	98-13	3	4.6	-0.7	0.6	1.5	2.4	0.1	0.1	4.6	-0.30	-0.10	1-09	-3	4.1	4.1	4.2	1.05	0.95
5.0	5.40	176	2	100-23	-3	3.9	-1.5	0.6	1.4	2.0	-0.0	0.1	4.3	-0.60	-0.00	1-04	-2	3.5	3.4	3.4	0.93	0.79
5.5	5.93	160	18	102-22	6	3.1	-1.9	0.7	1.2	1.3	-0.1	0.1	3.7	-0.70	0.10	0-27	0	2.7	2.7	2.8	0.74	0.64
6.0	6.44	148	30	104-13	5	2.3	-2.2	0.7	1.1	0.8	-0.2	0.1	3.2	-0.90	0.10	0-20	1	1.9	1.7	1.8	0.57	0.42
FHLMC 15YR																						
4.0	4.65	174	4	95-29	13	4.9	-0.2	0.5	1.5	2.6	0.2	0.1	4.6	0.00	-0.20	1-09	-3	4.7	4.7	4.8	1.09	1.05
4.5	4.98	176	2	98-12	6	4.5	-0.7	0.6	1.5	2.4	0.1	0.1	4.5	-0.20	-0.10	1-10	-4	4.2	4.1	4.2	1.04	0.94
5.0	5.40	176	2	100-20	2	3.9	-1.5	0.6	1.4	2.0	-0.0	0.1	4.3	-0.60	-0.00	1-03	-2	3.5	3.4	3.5	0.93	0.80
5.5	5.93	160	18	102-20	10	3.1	-1.9	0.7	1.2	1.3	-0.1	0.1	3.6	-0.70	0.00	0-27	-1	2.8	2.7	2.8	0.73	0.65
6.0	6.44	148	30	104-11	10	2.3	-2.2	0.7	1.1	0.8	-0.2	0.1	3.2	-0.90	0.10	0-19	1	1.9	1.7	1.8	0.57	0.42
GNMA 15YR																						
4.0	4.50	178	1	96-08	-1	5.2	-0.0	0.5	1.5	3.0	0.2	0.1	4.9	0.00	-0.10	1-09	0	4.7	4.7	4.8	1.17	1.05
4.5	5.00	178	1	98-24	-4	4.8	-0.5	0.5	1.5	2.7	0.1	0.1	4.8	-0.20	-0.10	1-10	-3	4.1	4.0	4.2	1.12	0.93
5.0	5.50	178	1	101-07	-9	4.3	-1.1	0.6	1.4	2.3	0.0	0.1	4.5	-0.50	-0.00	1-04	0	3.5	3.4	3.5	1.03	0.80
5.5	6.00	177	4	103-08	-9	3.6	-1.9	0.6	1.3	1.8	-0.1	0.1	4.2	-0.70	0.10	0-27	3	2.9	2.8	2.9	0.87	0.68
6.0	6.50	168	12	105-03	-3	3.1	-1.7	0.7	1.2	1.4	-0.1	0.1	3.8	-0.80	0.10	0-17	7	1.9	1.8	1.9	0.78	0.45
IO																						
TR.340 (5.0)	5.45	345	12	27-12+	75	-16.8	-29.9	3.2	-0.2	-11.4	-8.4	0.6	4.8	-20.80	4.10	-1-11+	15	-15.6	-17.5	-18.3	-0.60	-0.63
TR.329 (5.5)	6.00	336	20	25-01	111	-31.1	-41.9	3.0	-3.8	-19.9	-10.5	-0.1	4.4	-32.60	5.00	-2-12+	59	-33.2	-36.5	-38.2	-1.02	-1.20
TR.322 (6.0)	6.55	325	29	22-20	246	-32.7	-28.3	2.4	-4.8	-20.9	-9.7	-0.6	3.9	-33.00	5.90	-2-14+	85	-43.6	-46.0	-48.5	-0.97	-1.37
TR.321 (6.5)	6.97	325	29	21-18	381	-31.7	-23.6	2.6	-4.8	-20.7	-8.9	-0.8	3.6	-31.30	6.20	-2-26+	137	-49.1	-50.9	-53.8	-0.90	-1.44

TREASURY CURVE	3MO	6MO	1YR	2YR	3YR	4YR	5YR	7YR	10YR	20YR	30YR	SWAP CURVE	3MO	6MO	1YR	2YR	3YR	4YR	5YR	7YR	10YR	20YR	30YR		Primary Mtg Rate	Sprd / 10YR Swap
YIELD	1.36	1.70	2.12	2.65	3.01	3.34	3.67	3.98	4.43	4.90	5.17	YIELD	1.68	1.95	2.40	3.11	3.57	3.90	4.16	4.54	4.92	5.47	5.54	Rate	6.11	119

Source: Citigroup.

Figure 18. UBS Empirical Hedge Ratios

UBS **Empirical Hedge Ratios (H.R.s vs 10yr, 5yr, 2yr)**

Date 7/23/2004

	Period #1	Period #2	Period #3
Days Used	10	20	40

	Treasury	Swap	Agency
Hedge	◉	○	○

Submit

Home

Choice of data period length

Choice of hedge vehicle

UBS **Empirical Hedge Ratios vs Treasury for 7/23/2004**

~ Mortgage	vs 10y 10d HR	vs 10y 20d HR	vs 10y 40d HR	vs 5y 10d HR	vs 5y 20d HR	vs 5y 40d HR	vs 2y 10d HR	vs 2y 20d HR	vs 2y 40d HR
FN30 4	0.79	0.88	0.88	1.22	1.30	1.33	3.36	3.15	3.10
FN30 4.5	0.80	0.87	0.89	1.23	1.28	1.33	3.39	3.13	3.12
FN30 5	0.72	0.78	0.81	1.10	1.14	1.21	3.04	2.80	2.84
FN30 5.5	0.58	0.65	0.70	0.89	0.96	1.05	2.48	2.36	2.49
FN30 6	0.37	0.47	0.54	0.57	0.69	0.81	1.59	1.71	1.93
FN30 6.5	0.21	0.30	0.37	0.32	0.44	0.56	0.90	1.09	1.34
FN30 7	0.14	0.19	0.21	0.20	0.28	0.33	0.56	0.70	0.79
GN30 4	0.88	1.01	0.94	1.36	1.48	1.42	3.75	3.60	3.31
GN30 4.5	0.81	0.88	0.83	1.24	1.29	1.24	3.43	3.14	2.88
GN30 5	0.73	0.78	0.81	1.12	1.14	1.22	3.09	2.80	2.86
GN30 5.5	0.60	0.66	0.71	0.92	0.97	1.06	2.55	2.38	2.50
GN30 6	0.41	0.49	0.55	0.62	0.72	0.82	1.74	1.77	1.94
GN30 6.5	0.23	0.30	0.37	0.34	0.44	0.55	0.97	1.10	1.33
GN30 7	0.14	0.13	0.20	0.21	0.20	0.31	0.55	0.48	0.74
FN15 3.5	0.77	0.83	0.78	1.18	1.22	1.17	3.24	2.98	2.71

Resulting hedge ratios displayed

Source: UBS Mortgage Strategy website (used with permission).

"Constant Dollar" Duration

For some securities (MBSs), the price level is likely to be a major factor in estimating price volatility. Constant dollar duration measures empirical duration for securities in particular price ranges.

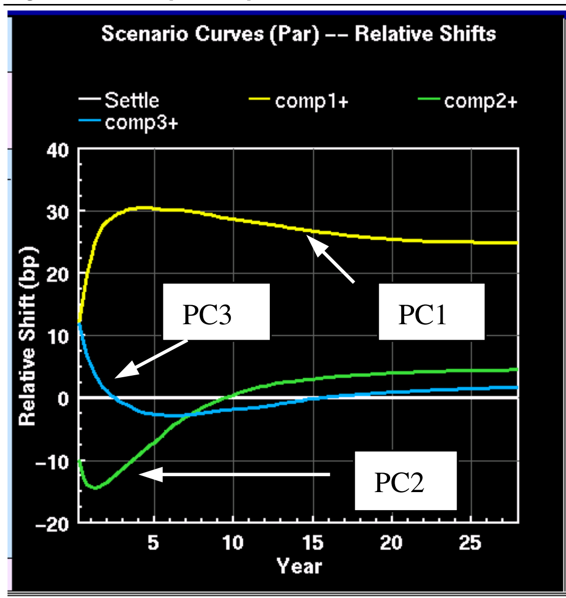
One of the potential problems with the empirical duration method is that, in an effort to include enough data points, the time period covered may include a period of directionality, or at least of different market levels. For MBSs (the usual subjects of empirical duration/hedge ratios), this can be problematic, because an MBS at 98 has a different response to interest rate moves than one at 102. If the time period covered includes such a range, the resulting statistics include a blend of differing responses, causing a loss of hedging accuracy. One method of combating this is to run the regressions of price changes of a limited dollar range versus changes in the ten-year (the five-year is often employed when looking at 15-year MBSs). For example, imagine a spreadsheet with MBS coupons going across the column headings and time going down the first column. Along each date (row), look to find an MBS (if any) priced at 97.5–98.5. If there is one, note its price change over one day compared with its benchmark, such as the ten-year Treasury. By looking over a reasonably long period of time and assembling such data, one can determine what the response of an MBS with a price near 98 is relative to the Treasury. This would then be mapped against whatever MBS is trading at (near) a dollar price of 98 today. The underlying assumption is that the price response is determined more by the price range than by a particular bond's history (which may include a much wider range of prices). At least one dealer used to calculate and publish such duration numbers.[31] Others offer a similar calculation but only for the then "current coupon" MBS. This is nothing more than the same approach, limited to a price range very close to par. This value is not provided on Yield Book.

[31] Donaldson, Lufkin, and Jenrette (now part of Credit Suisse First Boston). One factor that would also come into play is the age of the MBS, but it was calculated on TBA data only.

Principal Component Duration

Yield curve modeling is a topic that receives much interest and research focus. One of the outcomes of this focus is principal components analysis.[32] This is an attempt to measure the changes in the yield curve in "steps" by breaking most yield curve changes into several components that are independent and explain most yield curve movement. Each principal component (PC) describes a shape change and amount and has associated with it the percentage of yield curve movement that it has historically represented. For example, the first PC is similar to a parallel change, but it changes substantially less in the very short maturities and slopes slightly downward after about five years in maturity. It explains about 75% of yield curve variance (or 86% of the standard deviation). The first three "one-month"[33] PCs can be seen in Figure 19, with only the "up" scenarios shown to simplify the graph.[34]

Figure 19. Principal Component Yield Shifts



Source: Citigroup.

The first three PCs are shown in the figure to the left. These are PCs for a one-month horizon. The magnitude and shape differs somewhat for different horizons. The PCs calculated by SSB research are symmetric, although it is probably reasonable to consider that "up" moves could be larger in basis points than "down" moves.

While people think of modified duration as explaining what happens with a parallel shift, principal component duration would explain what happens with a "PC" shift. But because PC shifts are not parallel and differ in magnitude over time, they are not

---

[32] See *Principles of Principal Components*, Bülent Baygün, Janet Showers, and George Cherpelis, Salomon Smith Barney, January 31, 2000.

[33] Because these are estimated moves based on history, the projected horizon affects the magnitude and shape of the movements.

[34] While obvious for PC1, there is no unambiguous "up" scenario for PCs 2 and 3. We arbitrarily use the long end change to define the "upness" or "downness" of a change.

standardized to a 100bp shift, but rather to a one standard deviation shift.[35] For this reason, PC durations may be quite different than modified or effective durations, reflecting the fact that a one standard deviation shift might represent, on average, only 25bp–30bp. PC duration can be defined[36] as:

$$PCdur_i = \frac{P_{i,down} - P_{i,up}}{2P} * 100$$

where $P_{i,down}$ refers to the price after a one standard deviation PC shift in the down direction, and $P_{i,up}$ is defined analogously. The 2 in the denominator returns an average one standard deviation, the P renders it proportional, and the 100 turns it into a "percent." PC durations are not shown directly on the Yield Book, but can be calculated using user equations and the PC scenarios available in Scenario Setup. An example follows shortly.

Figure 20 shows the effects (in terms of ROR) of one standard deviation PC moves for the first 3 PCs for the US 5.375 of 2/15/31. Each move is considered separately. The dominance of PC1 is apparent in the size of the returns.

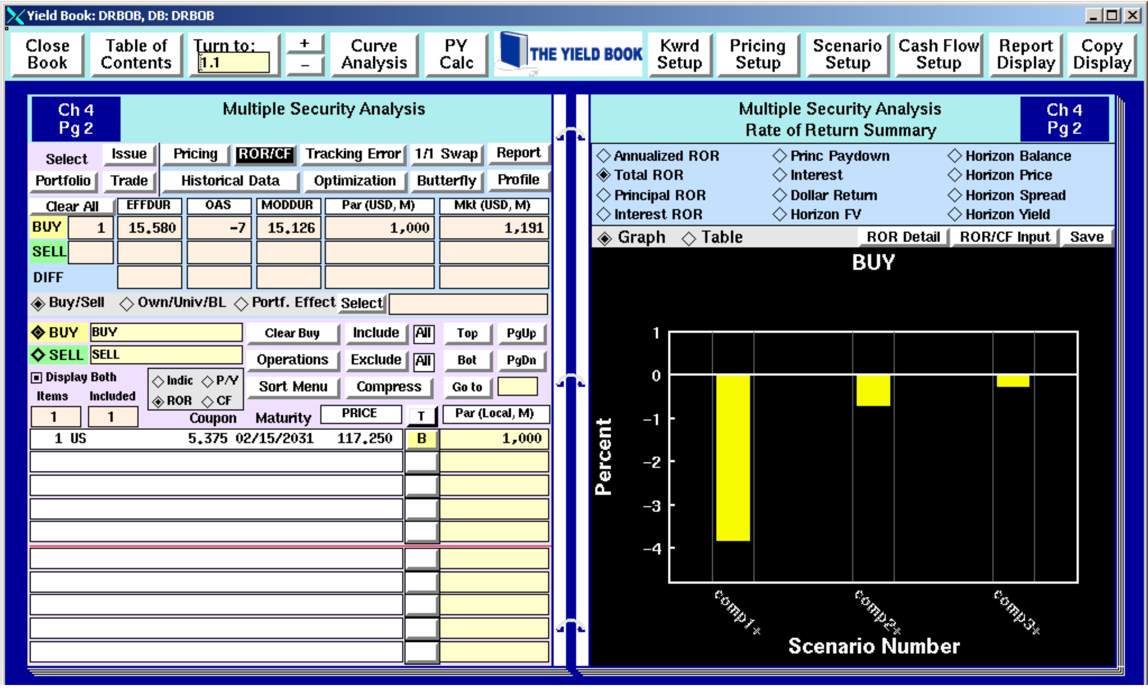
One difficulty in extending the analysis to include all 3 PCs is that there is no natural pairing of an "up" PC1 move to an "up" PC2 move, etc. Thus, one can't simply add the effects of the 3 PCs, or add the scenario yield changes, to produce a more comprehensive PC duration. Since an up PC1 move is just as likely to be accompanied by a "down" PC2 move as an up PC2 move, we can't sum the "up" moves to produce a more comprehensive one standard deviation PC "up"move. This seems to be of little consequence, however, because PC1 dominates.

It is possible to consider PC moves together for hedging purposes once a target portfolio or security has been chosen. This is described in *Principles of Principal Components*, cited in footnote 32.

[35] One difficulty with this "calibration" is that the standard deviations can change over time, so a PC duration of 5 now doesn't mean the same thing as a PC duration of 5 during a period of higher (or lower) volatility.

[36] Adapted from *Risk Management: Approaches for Fixed Income Markets*, Bennet W. Golub and Leo M. Tilman, John Wiley and Sons, 2000.

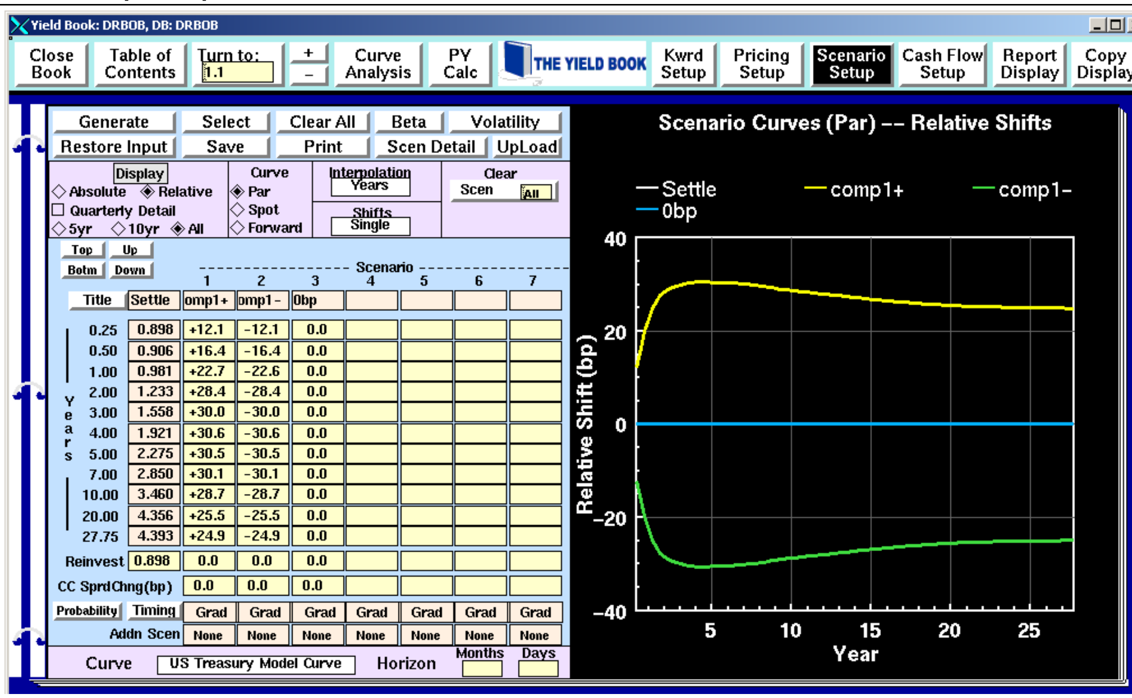
Figure 20. Rate of Return Summary



Source: Citigroup.

Consider the calculation (estimation) of PC1 duration for the US 5⅜ maturing 2/15/31. First, we can construct (selected) the appropriate scenarios in Scenario Setup. In this example, we've selected the one-month PC scenarios, and deleted PC2 and PC3 scenarios. The scenarios are shown in Figure 21. The horizon is set to zero months so that accretion of income doesn't affect the return calculations.

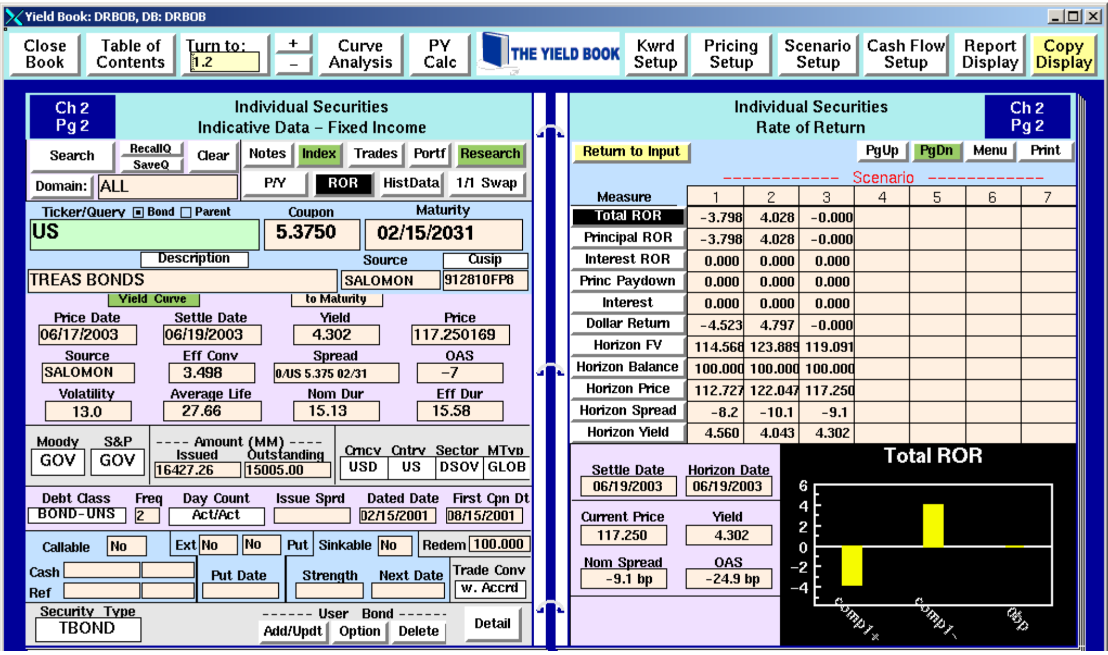
Figure 21. First Principal Component



Source: Citigroup.

The rates of return for the two scenarios "PC1 up" and "PC1 down," are shown in Figure 22.

Figure 22. Returns From a 1σ PC1 Move



Source: Citigroup.

Using the formula for PC duration shown earlier, we could calculate the PC duration by substituting prices, etc. into the equation (repeated here for convenience):

$$PCdur_i = \frac{P_{i,down} - P_{i,up}}{2P} * 100$$

This can also be restated as

$$\frac{1}{2} * \left[\frac{P_{i,down} - P}{P} + \frac{P - P_{i,up}}{P} \right] * 100$$

or, the average of the returns of the down and up scenarios. These values are shown directly in the table in Figure 22. Averaging 4.028 and 3.798 gives us 3.913, the PC1 duration.[37]

Is this consistent with values shown already? Remembering that the PC1 scenario has the long end of the curve moving about 25bp, a 3.91% change for a 25bp shift would produce a 3.91% * 4 = 15.64% change for a 100bp shift. This is quite close to the effective duration of 15.58 shown in Figure 22.

[37] While the prices could be used in the first equation, "price" actually means total present value, including accrued interest. The full price does not show in the screen shot, although it is certainly easy to obtain. Note that the sign of one of the returns changes in the equation because the direction of the price change is reversed.

Mortgage-Related Duration Measures

There are several duration measures calculated on the Yield Book that are available for MBSs only.

Volatility Duration

Volatility duration measures the percentage change in the price of an MBS that results from a change of 1% in volatility. If the security was priced using a single volatility assumption, it is repriced (holding OAS constant) with the volatility 0.5% lower and 0.5% higher. The difference in prices is expressed as a percentage of the original price. Thus, it measures the response to a 100bp shift in *volatility*. This is not the same as letting volatility vary through the interest rate tree.

Prepayment Duration

Prepayment duration measures the percentage price change that results from a 1% change in the prepay model (which is always expressed as a percentage of the model). The actual calculation shifts the model up and down by 5%, and reprices the security at constant OAS. The resulting price change is expressed as a percentage of the original price, and then divided by 10 to normalize it to a 1% change.

Current Coupon Spread Duration

Current coupon spread duration measures the percentage change in the price of an MBS that results from a change of 10bp in current coupon spread. The actual calculation shifts the spread up and down by 5bp and reprices the security at constant OAS. The resulting price change is expressed as a percentage of the original price, and *is displayed with no further normalization.* Thus, it represents the effect of a 10bp move.

Refi Duration

Refi duration is a component of prepayment duration and measures the effect of changing the refi “dial” in the Yield Book by 5%, that is, by moving the dial up to 1.05 and down to 0.95

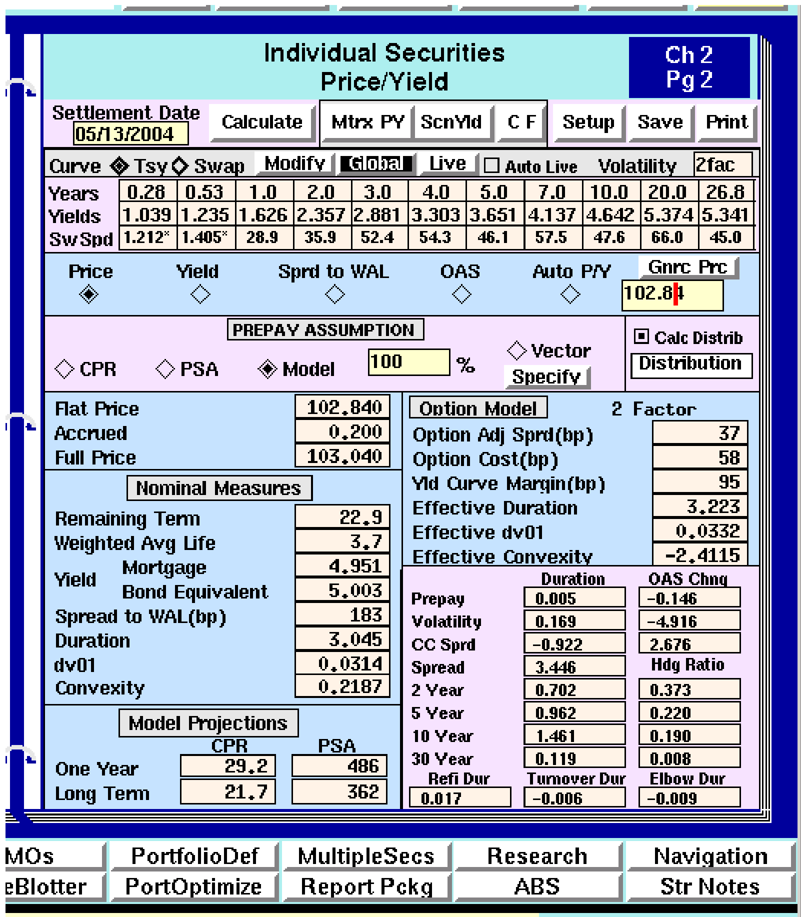
Turnover Duration

Turnover duration is a component of prepayment duration and measures the effect of changing the turnover “dial” in the Yield Book by 5%, that is, by moving the dial up to 1.05 and down to 0.95

Elbow Duration

Elbow duration is the percentage price change that results from a shift of +/-5bp to the elbow effect in the prepayment model.

Figure 23. Additional MBS Durations



Source: Citigroup.

Summary

Duration has come a long way since Macaulay. As shown in the previous section, the term is now used to represent sensitivities that have nothing to do with interest rate changes. Furthermore, not all durations are calibrated to a standard change in the "driver" (YTM, volatility, etc.), so it is mandatory for investment professionals to understand how the terminology differs between types of durations or even between firms who define their terms differently. We hope this paper helps that understanding.

Appendix

We start with the expression for the price (or present value) of the cash flows of a bond. This is shown below and is rewritten on the right for convenience of the derivation.

$$P = \sum_{t=1}^{M} \frac{C_t}{(1+r)^t} = \sum_{t=1}^{M} C_t (1+r)^{-t}$$

where:

C_t = cash flow at time t (whether principal or interest)

r = the periodic rate of interest. (For semiannual periods, it is, by convention, ½ of the annual rate.)

M = number of periods to maturity

Taking the derivative of the equation above with respect to rate, we obtain the following:

$$\frac{dP}{dr} = \sum_{t=1}^{M} C_t (-t)(1+r)^{-t-1} + \sum_{t=1}^{M} (1+r)^{-t} \frac{dC_t}{dr}$$

For fixed rate, noncallable, nonputable bonds, the cash flows are fixed and do not change with a change in rates. Therefore $\frac{dC_t}{dr}$ = 0, and the second term is eliminated. **(Naturally, for bonds with rate-dependent flows, this assumption cannot be made.)**

So we are left with the following:

$$\frac{dP}{dr} = \sum_{t=1}^{M} C_t (-t)(1+r)^{-t-1}$$

To turn this into a proportional measure, we need to divide by price:

$$\frac{dP}{dr} / P = \frac{\sum_{t=1}^{M} C_t (-t)(1+r)^{-t-1}}{\sum_{t=1}^{M} C_t (1+r)^{-t}} = \frac{1}{1+r} \frac{\sum_{t=1}^{M} C_t (-t)(1+r)^{-t}}{\sum_{t=1}^{M} C_t (1+r)^{-t}}$$

Note that the expression on the right, with the exception of the 1/(1 + r), is the same as Macaulay's formula. So the expression for the proportional interest rate sensitivity of an option-free bond is equal to Macaulay duration divided by 1 + r:

$$\frac{dP}{dr} / P = \frac{1}{1 + r} \text{ x Macaulay Duration} = \frac{\text{Macaulay Duration}}{1 + r}$$

This has come to be known as modified duration. The derivation depends on the simplifying assumption of no change in cash flows for changes in rates. For any security with cash flows that *do* depend on rates, the modified duration formula is simply inappropriate.

Disclosure Appendix

ANALYST CERTIFICATION

I, Bob Kopprasch, hereby certify that all of the views expressed in this research report accurately reflect my personal views about any and all of the subject issuer(s) or securities. I also certify that no part of my compensation was, is, or will be directly or indirectly related to the specific recommendation(s) or views in this report.

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