Does Correlation Matter in Pricing Caps and Swaptions?

A.R. Radhakrishnan*

September 1998

*Ph.D. Candidate, Department of Finance, Stern School of Business, New York University, 9-190 P, 44 West 4th Street, New York, NY 10012-1126. E-mail: aradhakr@stern.nyu.edu. The most recent version of the paper is available at http://www.stern.nyu.edu/~aradhakr.

Abstract

This paper studies the effect of forward rate correlations on caplet and swaption prices. A two-factor HJM lognormal model of forward rates that implies a realistic covariance matrix of forward rates is constructed. A one-factor lognormal model, with the same forward rate volatilities as the two-factor one, is employed for comparison purposes. The one- and two-factor models price European caplets identically. The one-factor model overprices European swaptions as expected. But the magnitude of overpricing is surprisingly small, less than three percent for at-the-money swaptions on five-year semi-annual swaps. The overpricing is less for shorter swap lengths. The surprising result is that the one-factor model underprices both American caplets and American-type swaptions. Five-year at-the-money American caplets on six-month rates are underpriced by as much as twelve percent and three-year at-the-money constant maturity Bermudan swaptions on two-year semi-annual swaps by as much as ten percent. The underpricing is relatively low for six-month and one-year options but increases with option maturity and forward rate decorrelation. Unlike constant maturity Bermudan swaptions, regular Bermudan swaptions are overpriced by the one-factor model by more than four percent in the case of three-year maturity swaptions. An intuitive explanation for the underpricing of American options under the one-factor model is offered. This explanation implies that American options on any type of interest rate security would be underpriced if perfect correlation across the forward rate term structure is assumed.

Introduction

Caps, floors, swaptions and discount bond options are some of the most widely used interest rate options in the financial market. Therefore, it is imperative that we know which term structure model to use in pricing these options - one-factor or two-factor. The advantage of the one-factor model is that it is simpler to use and computationally much faster than a two-factor model. The drawback of the one-factor model is that it assumes perfect correlation across all interest rates in the term structure potentially leading to inaccuracies in pricing of both European and American options. The objective of this paper is to examine how the assumption of perfect correlation affects the accuracy of pricing of European and American caplets and swaptions. Even though American caplets are not financially relevant, the results on American caplets can be easily extended to American discount bond options.

There are those who think that two-factor models are not required to price the above mentioned interest rate options. Hull and White (1995) say, for example, that 'the number of factors in the term structure model does not seem to be important except when pricing spread options'. Contrary to this, Rebonato (1996) seems to think that correlation is significant in pricing swaptions. He treats the swap rate as a weighted average of forward rates and hence, observes that the volatility of the swap rate should depend on the forward rate volatilities as well as their correlations. The fact that European swaptions are priced correctly in the market place using the one-factor Black's model is not inconsistent with the statement that European swaptions can only be priced correctly with a two-factor forward rate model. This is because Black's model assumes that it is the swap rate that follows a one-factor lognormal process and not the short rate or the forward rates. Even if the two-factor forward rate model is the correct one, if one uses the correct swap rate volatility derived from the two-factor model, the Black's model would give the correct European option price. But in order to price Bermudan swaptions (American swaptions that can be exercised only at reset dates), or any other American option, a term structure model is required since no closed form formula exists for American options[1]. The term structure

[1]The forward-neutral approach of Brace, Gatarek and Musiela (1997) and Jamshidian (1997) allows accurate pricing of European and American option caplets and swaptions in a one-factor context. But their approach is only an algorithmic one designed to match cap and swaption volatilities. It does not produce the true term structure of volatilities and can not price long maturity discount bond options, index amortizing swaps dependent on long rates, spread options etc accurately. It also can not give market caplet

model such as the one- or two-factor HJM forward rate model would be used to construct a forward rate tree in order to price American options. We know of no studies or published opinions on how the American option prices may be affected by correlation. The tendency seems to be to believe that whatever is true of European options would be applicable to American options as well.

We construct a two-factor HJM lognormal forward rate model that implies a realistic forward rate correlation matrix. A one-factor model is also constructed to have the same percentage forward volatilities as in the two-factor model. We find that European swaptions are overpriced by one-factor models, but not by a large amount. Surprisingly, both American caplets and American swaptions are underpriced by the one-factor model by as much as fourteen percent. The underpricing is found to increase with both the option maturity and the degree of decorrelation in forward rates. Extrapolation of swaption values estimated under one- and two-factor models in Carr and Yang (1997) support our results on American-type swaptions. We suggest that the 'twist' factor in the two-factor model makes short and long forward rates move in opposite directions leading to lower correlations over time for short maturity rates and increased early exercises by having low-value at-maturity exercise values follow high-value early exercise opportunities. This would imply lower values for all American options on all interest rate securities under the one-factor model. This explanation also accounts for the hedging errors that Canabarro (1995) finds when European caplets are hedged within one-factor models.

The next section discusses correlation effects on European and American interest rate options and gives an intuitive argument for why American options may be priced higher with forward rate decorrelation or in a two-factor model. Section three discusses one- and two-factor HJM lognormal forward rate models used in this paper to study correlation effects. Implementation and one- and two-factor comparison results are provided in the next section. The final section concludes.

---

and swaption prices simultaneously.

Discussion of correlation effects

European options

Any correlation effect on the value of European options has to be through the volatility of the underlying security. Therefore, all one has to do to see if a European option value is affected by correlation is to check if the volatility at option maturity is independent of correlation. For example, the value of a caplet is decided by the volatility of the forward rate maturing at the same time. But the forward rate volatilities are the same under one- and two-factor models for all maturities[2]. The volatility of any forward maturity is split across the two factors in the two-factor model, but they add up to the same as in the one-factor model (in fact, when squared and added they equal the squared quantity in the one-factor case). In the case of discount bond options, the discount bond volatility is a function of forward rate volatilities as can be seen in Appendix B and is more or less the same, if not exactly, under one- and two-factor formulations.

In the case of European swaptions, the relevant volatility could be either the swap value volatility or the swap rate volatility (the formulation of the expiration payoff would be different accordingly). Both volatilities are given as functions of discount bond price volatilities in Appendix C. Equation (17) in the appendix shows that the correlation among different maturity discount bond prices clearly affects the swap value volatility. But the exact effect of correlation on the swaption value needs to be determined empirically.

Any option on the spread between two interest rates of different maturities, spread option, is very much dependent on term structure correlation. Appendix D shows that the spread volatility is considerably lower in the one-factor case.

American options

We have already argued using expiration volatility that the European caplet value is independent of correlation effects. Here we will try to show that the American caplet value is somewhat dependent on correlation.

Principal component analysis of the forward rate variance matrix reveals that (Rebonato (1996)) the first factor represents change in the average level of the rates and is almost a

[2]Discussion here is in the context of one- and two-factor HJM forward rate models.

constant while the second factor represents twist or change in the slope of the term structure. This second factor has negative values for low forward rate maturities and positive values for long forward rate maturities. So the effect of the first factor is roughly uniform across all forward maturities, while the second factor effect is to decrease the short maturity rates while increasing long maturities or vice versa. Since long maturity forward rates eventually mature and become short maturity spot rates, it is likely that high-value short maturity spot rates (say, six-month spot rates) could be followed by low-value six-month spot rates and vice versa. But if low-value six-month spot rates tend to be preceded by high-value six-month spot rates, this will lead to early exercise in the case of American caplets on six-month rates.

In the one-factor case, the only factor has the same sign for all forward maturities and therefore, high short maturity rates co-exist with high long maturity forward rates and high six-month rates are less likely to be followed by low six-month rates when compared to the two-factor case. So there are fewer early exercise opportunities in the one-factor case.

But the higher chance of early exercise in the two-factor case will be relatively small since the second factor effect has to be superimposed on the larger first factor effect. Nevertheless, a higher chance of early exercise does exist and would be proportional to option maturity and decorrelation of forward rates.

The volatility of the distribution of the short maturity rate would be the same at any maturity under the one- and two-factor cases. Consider a caplet on the six-month rate under one- and two-factor models. Both models would yield the same volatility for the six-month rate and therefore, the same European option values at, say, two and five year maturities for the caplet. In the one-factor case, high-value six-month rates at two years tend to be followed by high-value six-month rates at five years, and low-value rates at two years by low value rates at five years. But in the case of the two-factor model, high-value six-month rates at two years are more likely than in the one-factor case to be followed by low-value six-month rates at five years and this provides additional early exercise opportunities in the two-factor case and therefore, a higher American call option value.

This argument can be extended to American options on any interest sensitive security that is a function of low maturity forward rates including discount bond options. Any underlying security that is a function of low maturity forward rates will be subject to the above described effect of decorrelation in short rates over time in the two-factor case. This will lead to slightly greater early exercise opportunities and hence, higher American option values for both call and put options.

Canabarro (1995) finds empirically that one-factor interest rate models price caps and discount bond options fairly accurately in a two-factor economy, but that they produce significant hedging errors over weekly intervals when hedged within the model. This finding can be explained by extending the reasoning used above to explain the underpricing of American options in one-factor models. In the two-factor model, high short rates have a higher chance of being followed by low short rates than in the one-factor model. This discrepancy would lead to hedging errors in the one-factor model since the one-factor short rates are slightly different from the 'true' short rates in each successive hedging period. In other words, it is the same phenomenon that causes hedging errors in European caps as well as underpricing in American caplet values.

Swaption values from one- and two-factor interest rate trees

We employ the Heath-Jarrow-Morton (HJM) framework to study the effect of decorrelation through a comparison of one- and two-factor models. The HJM framework has the advantage that the extension from one to two factors is very easy and intuitive. The coefficients or loadings on each factor correspond to the eigenvectors from a principal component analysis with the first factor corresponding to change in the level of the forward rates and the second corresponding to change in the slope of the rates. Given the factor loadings, the forward volatility for any maturity is easily obtained as the sum of the squares of the loadings for that maturity and the forward correlation matrix can be easily got as shown in Appendix A.

In the HJM framework, the forward rates have the following diffusion process if a two-factor model is desired.

$$df(t,T) = \alpha(t,T,.)dt + \sigma_1(t,T,f(t,T))dz_1 + \sigma_2(t,T,f(t,T))dz_2 \tag{1}$$

where 1 and 2 correspond to the two factors and $\alpha(t,T,.)$, the drift term, is given by the following no-arbitrage condition in the continuous case.

$$\alpha(t,T,.) = \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))\{\int_t^T \sigma_i(t,s,f(t,s))ds\} \tag{2}$$

In order to implement the two-factor model using a tree approach, arbitrage conditions in a discrete time framework are required. Discretization of a two-factor process requires three branches at each node. Given that you are in state s_t at time t, the three states possible at time $t+1$ are called u, m and d. Then the forward rate process can be represented as,

$$f(t{+}1,T;s_{t+1}) = \begin{cases} \text{f(t,T;s}_t) + \mu(t,T,f(t,T))h - \sigma_1(t,T,f(t,T))\sqrt{h} - \sqrt{2}\sigma_2(t,T,f(t,T))\sqrt{h} \\ \quad \text{if s}_{t+1} = s_t u \ (\textit{with probability } 1/4) \\ \text{f(t,T;s}_t) + \mu(t,T,f(t,T))h - \sigma_1(t,T,f(t,T))\sqrt{h} + \sqrt{2}\sigma_2(t,T,f(t,T))\sqrt{h} \\ \quad \text{if s}_{t+1} = s_t m \ (\textit{with probability } 1/4) \\ \text{f(t,T;s}_t) + \mu(t,T,f(t,T))h + \sigma_1(t,T,f(t,T))\sqrt{h}] \\ \quad \text{if s}_{t+1} = s_t d \ (\textit{with probability } 1/2) \end{cases} \tag{3}$$

where $\mu(t,T,f(t,T))$, and $\sigma_1(t,T,f(t,T))$ and $\sigma_2(t,T,f(t,T))$ are interpretable as the drift and volatilities of the $f(t,T;s_t)$ process and h is the time between discrete time steps as well as the forward rate maturities.

The set of discount bonds $P(t,T)$ are given by

$$P(t,T) = e^{\sum_{t_j=t}^{T-h} -f(t,t_j)h} \tag{4}$$

The martingale condition applied to discrete framework requires that

$$E_t[P(t+h,T)P(t,t+h)] = P(t,T) \tag{5}$$

Substituting equations (12) and (13) in the above condition and simplifying gives an expression for the drift term in each forward rate process as

$$e^{\mu(t,T)h} = \frac{\frac{1}{2}e^{-\sum_{t_j=t+h}^{T}\sigma_1(t,t_j)\sqrt{h}} + \frac{1}{2}e^{\sum_{t_j=t+h}^{T}\sigma_1(t,t_j)\sqrt{h}}[\frac{1}{2}e^{\sqrt{2}\sum_{t_j=t+h}^{T}\sigma_2(t,t_j)\sqrt{h}} + \frac{1}{2}e^{-\sqrt{2}\sum_{t_j=t+h}^{T}\sigma_2(t,t_j)\sqrt{h}}]}{e^{\sum_{t_j=t}^{T-h}\mu(t,t_j)h}} \tag{6}$$

Now the only items required to construct the two-factor HJM interest rate tree are the volatility functions, $\sigma_1(t,T,f(t,T))$ and $\sigma_2(t,T,f(t,T))$. If a proportional forward rate volatility is assumed, we have,

$$\sigma_i(t,T,f(t,T)) = \sigma_i(t,T)f(t,T) \tag{7}$$

where $i = 1, 2$. $\sigma_i(t, T)$ are assumed to be functions only of forward rate maturity, $T - t$, and are called the volatility loadings.

Implementation and Results

The volatility loadings in the previous section should be ideally chosen to match the desired forward rate volatilities and correlations. Suppose we have six forward rate maturities giving 21 (=6+15) volatilities and correlations to be matched. But we have only 12 ($=2\times 6$) volatility loadings at our disposal to match these 21 constraints. Therefore, no set of two-factor loadings would fit all volatilities and correlations.

If actual correlation coefficients of zero maturity forward rate with other maturities are plotted, a downward sloping exponential curve is obtained. If one were to employ principal components analysis and retain the the first two principal components to account for the first two factors, Rebonato (1996) shows that one can obtain a downward sloping correlation curve, but not an exponential one. The correlation curve implied by the two principal components is flatter than the actual exponential curve and one would need at least five factors to approximate the exponential shape.

With just two factors, one can only hope to match either the short end or the long end of the correlation curve. If the short end is matched, the long correlations will be very low or negative leading to low European swaption values on long swaps, assuming that it is swaptions that we are interested in. Matching the long end of the correlation curve will lead to too high correlation values at the short end leading to high European swaption values on short swaps.

What we have done is come up with a set of factor loadings (see Table 1) that implies reasonable forward rate volatilities and correlations (see Table 2). These values are comparable to volatility and correlation values found in the literature (Brown and Schaefer (1996), for example).

A two-factor non-recombining forward rate tree is constructed using these volatility factor loadings in equations (3) and (6). A one-factor tree is also constructed for comparison using the forward rate volatilities themselves - the square root of the sum of squares of the two loadings in the two-factor model - as the single factor loadings for each forward maturity. *This would make sure that both one- and two-factor models have the same volatility for each* *forward maturity*. Equations (3) and (6) can be used for one-factor tree also by setting $\sigma_2(.)$s equal to zero and by collapsing branches u and m into a single branch u with probability 0.5.

A flat term structure of five percent is assumed throughout this paper.

Caplets

Table 3 has European and American caplet values on the six-month rate for five different maturities ranging from six months to five years. For all maturities and all strikes, the European caplet values are very similar under both one- and two-factor models. But the American caplet values under one- and two-factor models diverge with caplet maturity. For the five year caplet, the one-factor model underprices the at-the-money American option by six percent and the out-of-the-money option by almost eight percent.

We increase the decorrelation in forward rates to see the effect on option values. This is done by multiplying the second factor loadings in Table 1 by a factor of 1.3 while keeping the first factor unchanged. The correlation matrix implied by these new set of loadings are given in Table 5. The correlations are obviously lower than in Table 1, but still not as low as the historical estimates in Brace, Gatarek and Musiela (1997). This shows that the correlation values in Table 5 are realistic as well. Panel A in Table 6 gives the five-year caplet values using the higher decorrelation values in Table 5. Once again the European values are independent of the number of factors, but the at-the-money American option is undervalued by twelve percent and the out-of-the-money American option by thirteen percent in the one-factor model.

Swaptions

Table 4 provides European swaption and constant maturity Bermudan swaption values on semi-annual swaps based on factor loadings in Table 1 for four different swaption maturities ranging from six months to three years[3]. The underlying swaps are of semi-annual frequency

[3]Constant maturity Bermuda swaptions refers to American swaptions that are exercised only at reset dates and which when exercised initiates a swap that will last the same fixed number of years as in the case of European swaptions. A regular Bermuda swaption is also exercised only at reset dates, but when exercised the swap that is initiated lasts as long as the swaption maturity plus the original swap length. So a three-year Bermuda swaption on a five-year swap that when exercised after one year will initiate a swap

and of lengths two and five years. The one-factor model overprices European swaptions with the degree of overpricing increasing with swap length and decreasing with swaption maturity. This makes sense since forward rates farther apart have lower correlations and since if the maturity between two forward rates are held constant, the correlation between them increases the farther you go into the future. The overpricing is about three percent for six-month at-the-money European swaptions on two-year swaps and about one percent for three-year at-the-money European swaptions on two-year swaps.

Constant maturity Bermudan (CMB) swaptions are underpriced by the one-factor model for all strikes of two- and three-year swaptions and in-the-money six-month and one-year swaptions. There are two correlation effects on one-factor CMB swaptions; one serves to increase the swap rate volatility and the value of the European part of the option and the second, through reduced correlation over time, serves to decrease the value of early exercise premium. At low swaption maturities, the first effect dominates and the CMB swaption is overpriced and at higher maturities, the second dominates and the CMB swaption is underpriced. So despite the fact that the two-factor model has higher early exercise premium than the one-factor one for all strikes and maturities, this premium is not enough to overcome the overpricing of European swaptions in the one-factor model for in- and out-of-the-money strikes for lower swaption maturities. The underpricing of CMB swaptions in the one-factor model clearly increases with swaption maturity reaching seven percent for at-the-money three-year swaptions on two-year swaps. The underpricing is smaller for CMB swaptions on five-year swaps since the one-factor European swaptions on five-year swaps are more overpriced than one-factor European swaptions on two-year swaps. If one looks at the early exercise premiums, the disparity between the two models is even greater. For the at-the-money three-year swaption on two-year swaps, the early exercise premium is about 17 basis points in the two-factor case and only about 9 basis points in the one-factor case.

Calculations are repeated for three-year swaptions using the increased decorrelations in Table 5. The results are reported in Panel B of Table 6. At-the-money CMB swaptions are underpriced by as much as ten percent by the one-factor model. The underpricing of CMB swaptions clearly increases with forward rate decorrelation.

---

that will last seven years.

Table 6 also gives values for regular Bermudan swaptions. Unlike CMB swaptions, the one-factor model overprices the Bermudan swaption by as much as five percent for the at-the-money strike on five-year swaps. Remember that the Bermudan swaption, if immediately exercised in this particular case, would give rise to an eight year swap. This means that if the swaption is in the money very early, immediate exercise would lead to the beneficial swap lasting eight years compared to five years in the European swaption case. So the Bermudan swaption has a very high probability of being exercised early in both one- and two-factor cases and therefore, behaves more like a longer maturity European swaption rather than one with American features. As we have already seen, the one-factor model overprices European swaptions and this overpricing increases with swap length. So the Bermudan swaption is not a typical American option and therefore, it does not exhibit underpricing in the one-factor case as other American options do.

Carr and Yang (1997) price Bermudan and CMB swaptions in the Market Model framework using Monte Carlo simulation for one- and two-factor cases. But since the one-and two-factor models do not have the same forward volatilities, their results do not allow immediate comparison between one- and two-factor models. But if one were to divide their Bermudan and CMB swaption values under one- and two-factor models by the corresponding European swaption values and compare the resulting ratios across the one- and two-factor models, results similar to ours would be obtained. This provides support for our results on American-type swaptions.

Conclusion

European and American caplet and swaption values from one- and two-factor models are empirically compared to study the effect of interest rate correlation. The two-factor model parameters are chosen such that it implies a realistic forward rate covariance matrix and the one-factor model parameters are chosen such that its forward rate volatilities are identical to the two-factor model.

Both models price the European caplets identically. One-factor overprices European swaptions since assumption of perfect forward correlation increases swap rate volatility. But the magnitude of overpricing is only three percent for at-the-money swaptions on five-year swaps, the longest swap length considered. The degree of overpricing increases with swap length and decreases with swaption maturity.

The surprising result is that the one-factor model underprices both American caplets and constant maturity Bermuda (American style security) swaptions. The underpricing is as much as twelve percent for five-year at-the-money caplets and ten percent three-year at-the-money swaptions. The underpricing increases with option maturity and the forward rate decorrelation. For swaptions, the underpricing is mostly independent of swap length. The one-factor model overprices ordinary Bermuda swaptions since ordinary Bermuda swaptions behave like European swaptions on extended-length swaps.

It is argued that the decorrelation effect on American option values comes from the 'twist' factor in the two-factor model. The opposite weights this factor has on short and long rates induces a decorrelation effect on short rates over time giving rise to more early exercise opportunities. This argument also implies higher values of varying degrees for American options on all interest rate contingent securities.

In short, forward rate correlation has an effect on European swaption values, though not by as much as some would have expected, and also on all American options on all interest rate securities.

Appendix A

Forward rate correlation matrix from proportional volatility loadings

The continuous version of the two-factor HJM model can be written for forward rates f_a and f_b of different maturities as,

$$\frac{df_a}{f_a} = \mu_a dt + \sigma_{a1} dz_1 + \sigma_{a2} dz_2 \tag{8}$$

and

$$\frac{df_b}{f_b} = \mu_b dt + \sigma_{b1} dz_1 + \sigma_{b2} dz_2 \tag{9}$$

where dz_1 and dz_2 are Wiener processes and are uncorrelated.

$$\begin{aligned} Cov(\frac{df_a}{f_a}, \frac{df_b}{f_b}) &= E[(df_a - E(df_a))(df_b - E(df_b))] \\ &= E[(\sigma_{a1} dz_1 + \sigma_{a2} dz_2)(\sigma_{b1} dz_1 + \sigma_{b2} dz_2)] \\ &= (\sigma_{a1}\sigma_{b1} + \sigma_{a2}\sigma_{b2}) \end{aligned} \tag{10}$$

$$\begin{aligned} \rho_{a,b} &= \frac{Cov(\frac{df_a}{f_a}, \frac{df_b}{f_b})}{\sqrt{Var(\frac{df_a}{f_a})}\sqrt{Var(\frac{df_b}{f_b})}} \\ &= \frac{\sigma_{a1}\sigma_{b1} + \sigma_{a2}\sigma_{b2}}{\sqrt{\sigma_{a1}^2 + \sigma_{a2}^2} + \sqrt{\sigma_{b1}^2 + \sigma_{b2}^2}} \end{aligned} \tag{11}$$

where $\rho_{a,b}$ is the correlation coefficient between f_a and f_b.

Appendix B

Discount price volatilities from forward rate volatilities

Let P_n be the price of a discount bond of maturity n. It can be written in terms the forward rates as,

$$P_n = e^{-\sum_{j=0}^{n-1} f_j h} \tag{12}$$

where f_j is the forward rate maturing at jh with h being the time between maturities. Using Ito's lemma,

$$\begin{aligned} dP_n &= \mu_{P_n} dt + \sum_{j=0}^{n-1} \frac{\partial P_n}{\partial f_j}(\sigma_{j1} f_j dz_1 + \sigma_{j2} f_j dz_2) \\ &= \mu_{P_n} dt + P_n[(\sum_{j=0}^{n-1} \sigma_{j1} f_j) h dz_1 + (\sum_{j=0}^{n-1} \sigma_{j2} f_j) h dz_2] \end{aligned} \tag{13}$$

So we have the volatility of the discount bond price as

$$\sigma_{P_n} = P_n h \sqrt{(\sum_{j=0}^{n-1} \sigma_{j1} f_j)^2 + (\sum_{j=0}^{n-1} \sigma_{j2} f_j)^2} \tag{14}$$

The correlation coefficient between P_n and P_m would be

$$\rho_{P_n,P_m} = \frac{E[(dP_n - E(dP_n))(dP_m - E(dP_m))]}{\sigma_{P_n}\sigma_{P_m}}$$

$$
\begin{aligned}
&= \frac{P_n P_m h^2 E[(\sum_{i=0}^{n-1} f_i(\sigma_{i1} dz_1 + \sigma_{i2} dz_2))(\sum_{j=0}^{m-1} f_j(\sigma_{j1} dz_1 + \sigma_{j2} dz_2))]}{P_n h\sqrt{(\sum_{i=0}^{n-1} \sigma_{i1} f_i)^2 + (\sum_{i=0}^{n-1} \sigma_{i2} f_i)^2} P_m h\sqrt{(\sum_{j=0}^{m-1} \sigma_{j1} f_j)^2 + (\sum_{j=0}^{m-1} \sigma_{j2} f_j)^2}} \\
&= \frac{\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(f_i f_j \sigma_{i1}\sigma_{j1} + f_i f_j \sigma_{i2}\sigma_{j2})}{\sqrt{(\sum_{i=0}^{n-1} \sigma_{i1} f_i)^2 + (\sum_{i=0}^{n-1} \sigma_{i2} f_i)^2}\sqrt{(\sum_{j=0}^{m-1} \sigma_{j1} f_j)^2 + (\sum_{j=0}^{m-1} \sigma_{j2} f_j)^2}} \qquad (15)
\end{aligned}
$$

Appendix C

Volatility of n-year swap value from discount bond price volatilities

Let V_{swap} be the value of a n-year swap. If c is the swap rate then,

$$V_{swap} = c\sum_{i=1}^{n-1} P_i + (1+c)P_n - 1 \qquad (16)$$

where P_i is the price of a i maturity discount bond. Then the volatility of the swap value is given by,

$$
\begin{aligned}
\sigma_{swap}^2 &= c^2\sum_{i=1}^{n-1}\sigma_{P_i}^2 + (1+c)^2\sigma_{P_n}^2 + 2c^2\sum_{i=1}^{n-1}\sum_{j=1 j\neq i}\rho_{ij}\sigma_{P_i}\sigma_{P_j} \\
&\quad +2c(1+c)\sum_{i=1}^{n-1}\rho_{in}\sigma_{P_i}\sigma_{P_n} \\
&\approx (1+c)^2\sigma_{P_n}^2 + 2c(1+c)\sum_{i=1}^{n-1}\rho_{in}\sigma_{P_i}\sigma_{P_n} \qquad (17)
\end{aligned}
$$

Volatility of swap rate

The swap rate, c, of a n-year swap is given by,

$$c = \frac{1-P_n}{\sum_{i=1}^{n} P_i} \qquad (18)$$

where P_i is the price of a discount bond maturing in year i.

Using Ito's lemma, the stochastic part of c is given by,

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{\partial c}{\partial P_i}\sigma_{P_i} dz \\
&= \frac{(1-P_n)}{(\sum_{i=1}^{n} P_i)^2}\sum_{i=1}^{n-1}\sigma_{P_i} dz + \frac{(1-P_n)+\sum_{i=1}^{n} P_i}{(\sum_{i=1}^{n} P_i)^2}\sigma_{P_n} dz \qquad (19)
\end{aligned}
$$

This gives the volatility of the swap rate as,

$$\sigma_c = \frac{1}{(\sum_{i=1}^{n} P_i)^2}\{(1-P_n)\sum_{i=1}^{n-1}\sigma_{P_i} + ((1-P_n)+\sum_{i=1}^{n} P_i)\sigma_{P_n})\} \qquad (20)$$

Appendix D

Volatility of a spread option from interest rate volatilities

Consider a spread option of two rates of maturities, n and m. The volatility of the spread option is given by,

$$\sigma^2_{R_n - R_m} = \sigma^2_{R_n} + \sigma^2_{R_m} - 2\rho_{n,m}\sigma_{R_n}\sigma_{R_m} \tag{21}$$

For the one-factor model, $\rho_{n,m}$ is 1.0 and you get,

$$\sigma^2_{R_n - R_m} = (\sigma_{R_n} - \sigma_{R_m})^2 \tag{22}$$

And the percentage difference in volatility between one- and two-factor models is given by,

$$\%\ difference = \frac{2\sigma_{R_n}\sigma_{R_m}(1 - \rho_{n,m})}{\sigma^2_{R_m} + \sigma^2_{R_n} - 2\sigma_{R_n}\sigma_{R_m}} \tag{23}$$

If $\sigma_{R_m} = a\sigma_{R_n}$ where $a > 1$. Then,

$$\%\ difference = \frac{2a(1 - \rho_{n,m})}{(a-1)^2} \tag{24}$$

Assume a one and five year spread option. Let $\sigma_{R_1} = 1.5\sigma_{R_5}$. If $\rho_{1,5} = 0.9$, the difference in volatility between one- and two-factor models is 120%. Even if $\rho_{1,5}$ only 0.98, the difference is 24%.

The difference in volatility is also affected by the relative volatilities of the two rates. In fact, as maturity difference increases the effect of a lower correlation between the rates on volatility difference is mitigated by the higher ratio of volatilities between rates. Consider a one and ten year spread option. Let the ratio of volatilities be 3. Even if the correlation coefficient is 0.80, the difference in volatility is only 30%.

Conclusion: The volatility of the spread option is highly dependent upon the correlation between the two rates. But the marginal effect of correlation is lower at higher maturities since the ratio of volatilities is higher.

References

[1] Brace, A., D. Gatarek and M. Musiela, 'The market model of interest rate dynamics', *Mathematical Finance*, 7(2), 1997, 127-155.

[2] Brown, R.H. and S.M. Schaefer, 'Why do long term forward rates (almost always) slope downwards?', London Business School working paper, November 1994.

[3] Canabarro, Eduardo, 'Where do one-factor interest rate models fail?', *The Journal of Fixed Income*, September 1995, 31-52.

[4] Carr, P. and G. Yang, 'Simulating Bermudan interest rate derivatives', *http://www.math.nyu.edu/research/carrp/index.html*, Working paper, 1997.

[5] Heath, D., R. Jarrow and A. Morton, 'Contingent claim valuation with a random evolution of interest rates', *The Review of Futures Markets*, 9(1), 1990, 54-76.

[6] Heath, D., R. Jarrow and A. Morton, 'Bond pricing and the term structure of interest rates', *Econometrica*, 60(1), 1992, 77-105.

[7] Hull, J. and A. White, 'A note on the models of Hull and White for pricing options on the term structure: Response', *The Journal of Fixed Income*, September 1995, 97-102.

[8] Jamshidian, F., 'LIBOR and swap market models and measures', *Finance and Stochastics*, 1, 1997, 293-330.

[9] Jarrow, Robert, *Modeling Fixed Income Securities and Interest Rate Options*, McGraw-Hill, 1996.

[10] Rebonato, Robert, *Interest-rate option Models : Understanding, Analyzing and Using Models for Exotic Interest-rate Options*, John Wiley, 1996.

Table 1

Values of volatility loadings for forward rates in the two-factor model

Time to Maturity (τ)	$\sigma_1(\tau)$	$\sigma_2(\tau)$	Total Volatility
0	0.207	-0.100	0.230
1	0.199	-0.020	0.200
2	0.189	0.020	0.190
3	0.173	0.050	0.180
4	0.156	0.080	0.175
5	0.138	0.100	0.170
6	0.123	0.110	0.165
7	0.113	0.117	0.163
8	0.104	0.122	0.160

Total (proportional) volatility is calculated as $\sqrt{\sigma_1^2 + \sigma_2^2}$.

Table 2

Correlation matrix implied by the volatility loadings in Table 1

Maturity	0	1	2	3	4	5	6	7	8
0	1.000	0.939	0.850	0.744	0.603	0.472	0.381	0.312	0.251
1		1.000	0.979	0.928	0.840	0.746	0.675	0.619	0.567
2			1.000	0.985	0.933	0.866	0.811	0.766	0.724
3				1.000	0.982	0.940	0.901	0.867	0.833
4					1.000	0.988	0.967	0.946	0.924
5						1.000	0.995	0.985	0.972
6							1.000	0.997	0.991
7								1.000	0.998

Table 3

Values of European and American caplets on the six-month interest rate under one- and two-factor models

One- and two-factor caplet prices are based on two- and three-branch bushy (non-recombining) HJM forward rate trees respectively. Both versions use proportional volatility functions as given in Table 1 (One-factor model uses the total volatility in the last column as the loading). The initial term structure is flat and is equal to five percent. The trees are constructed using arbitrage-free drifts given in equation (6).

Panel A: Six-month maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	4.3	4.3	4.4	4.4
5	29.3	29.4	30.2	30.2
4	99.6	99.8	102.1	103.3

Panel B: One-year Maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	11.1	11.0	11.7	11.7
5	39.4	39.5	41.6	41.9
4	101.6	101.8	106.9	109.7

Panel C: Two-year Maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	22.1	22.0	23.8	23.8
5	51.4	51.3	56.2	57.4
4	105.1	105.0	115.5	120.5

Panel D: Three-year Maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	29.8	29.6	33.4	33.7
5	58.3	58.2	66.4	68.8
4	106.1	106.4	122.3	129.5

(cont'd...)

Panel E: Five-year Maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	39.0	38.8	47.3	49.1
5	65.3	65.7	80.5	85.4
4	105.5	106.8	132.3	142.3

caplet prices are in basis points (1 bp = \$100 per \$1M face value). strike rate is in percent. since all initial forward rates are equal to five percent, a strike rate of five percent is at-the-money. caplet prices are based on fourteen steps for the two-factor tree and twenty steps for the one-factor tree.

Table 4

Values of European and constant maturity Bermudan swaptions under one- and two-factor models

One- and two-factor swaption prices on semi-annual frequency swaps are based on two- and three-branch bushy (non-recombining) HJM forward rate trees respectively. Both versions use proportional volatility functions as given in Table 1 (One-factor model uses the total volatility in the last column as the loading). The initial term structure is flat and is equal to five percent. The trees are constructed using arbitrage-free drifts given in equation (6).

Panel A: Six-month Maturity Swaptions

		European		American[1]	
swap length	strike rate	one-factor	two-factor	one-factor	two-factor
2	4	2.4	2.2	2.5	2.3
	5	46.7	46.5	48.0	48.0
	6	180.1	179.8	183.3	184.6
5	4	3.8	3.1	3.8	3.1
	5	101.0	98.2	103.1	100.3
	6	413.9	412.0	421.2	421.5

[1]since a CMB option can be exercised early only at initiation for six-month maturity, we report the value of an American swaption here.

Panel B: One-year Maturity Swaptions

		European		CMB Swaption	
swap length	strike rate	one-factor	two-factor	one-factor	two-factor
2	4	9.0	8.6	9.1	8.7
	5	65.1	64.7	66.0	66.6
	6	188.5	188.4	192.5	195.5
5	4	17.2	15.3	17.4	15.5
	5	141.0	138.2	142.3	140.5
	6	431.1	426.9	437.1	438.9

(cont'd...)

Panel C: Two-year Maturity Swaptions

		European		CMB swaption	
swap length	strike rate	one-factor	two-factor	one-factor	two-factor
2	4	22.7	22.2	23.7	23.8
	5	87.6	87.5	91.5	95.5
	6	201.6	200.6	213.3	221.2
5	4	44.5	43.4	45.8	45.0
	5	191.1	188.1	197.6	198.7
	6	458.2	450.8	482.3	488.9

Panel D: Three-year Maturity Swaptions

		European		CMB swaption	
swap length	strike rate	one-factor	two-factor	one-factor	two-factor
2	4	33.1	32.9	34.7	37.2
	5	100.9	100.9	108.4	116.7
	6	208.2	206.8	226.9	241.1
5	4	69.0	67.7	71.8	71.9
	5	221.5	219.5	235.7	239.8
	6	472.6	464.8	511.0	524.7

CMB swaption refers to constant maturity Bermuda swaption. swaption prices are in basis points (1 bp = \$100 per \$1M face value). swap frequency is semi-annual. strike rate is in percent. forward swap rate is five percent, so a strike rate of five percent is at-the-money. swaption prices are based on fourteen steps for the two-factor tree and twenty steps for the one-factor tree.

Table 5

Implied forward rate correlation matrix if second factor loadings in Table 1 are multiplied by 1.3

Maturity	0	1	2	3	4	5	6	7	8
0	1.000	0.909	0.766	0.606	0.410	0.250	0.149	0.077	0.017
1		1.000	0.965	0.883	0.753	0.632	0.548	0.487	0.433
2			1.000	0.975	0.900	0.814	0.749	0.700	0.656
3				1.000	0.974	0.922	0.877	0.840	0.806
4					1.000	0.986	0.963	0.941	0.919
5						1.000	0.995	0.985	0.972
6							1.000	0.997	0.991
7								1.000	0.998

Table 6

Caplet and swaption values based on increased correlations as per Table 5

One- and two-factor prices are based on two- and three-branch bushy (non-recombining) HJM forward rate trees respectively. Both versions use proportional volatility functions as given in Table 5 (One-factor model uses the total volatility in the last column as the loading). The initial term structure is flat and is equal to five percent. The trees are constructed using arbitrage-free drifts given in equation (6).

Panel A: Five-year Maturity Caplets

	European		American	
strike rate	one-factor	two-factor	one-factor	two-factor
6	41.3	41.4	48.7	53.9
5	67.8	68.3	81.7	92.9
4	107.5	108.8	133.4	154.3

caplet prices are in basis points (1 bp = \$100 per \$1M face value). strike rate is in percent. since all initial forward rates are equal to five percent, a strike rate of five percent is at-the-money. caplet prices are based on fourteen steps for the two-factor tree and twenty steps for the one-factor tree.

Panel B: Three-year Maturity Swaptions

		European		CM Bermudan		Bermudan	
swap length	strike rate	one-factor	two-factor	one-factor	two-factor	one-factor	two-factor
2	4	34.7	34.6	35.8	39.6	47.9	47.3
	5	103.4	103.1	109.7	121.7	176.0	170.5
	6	210.5	208.6	227.4	246.6	424.1	412.8
5	4	79.4	78.6	80.5	82.8	90.0	87.8
	5	237.2	233.8	246.7	253.4	298.0	284.9
	6	486.9	477.0	516.7	535.1	676.2	653.7

CM Bermudan refers to constant maturity Bermudan swaption and *Bermudan* refers to regular Bermudan option. swaption prices are in basis points (1 bp = \$100 per \$1M face value). swap frequency is semi-annual. strike rate is in percent. forward swap rate is five percent, so a strike rate of five percent is at-the-money. swaption prices are based on fourteen steps for the two-factor tree and twenty steps for the one-factor tree.