Fitting Yield Curves and Forward Rate Curves With Maximum Smoothness

Article *in* The Journal of Fixed Income · June 1994

DOI: 10.3905/jfi.1994.408102

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FITTING YIELD CURVES AND FORWARD RATE CURVES WITH MAXIMUM SMOOTHNESS

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The increasing richness of the theories of the term structure of interest rates, combined with the increasing complexity of derivative products whose yield is a function of interest rates and exchange rates, has highlighted the empirical challenges of fitting yield curves to current market data. The pioneering work of McCulloch [1975], using polynomial splines, and Vasicek and Fong [1982], using exponential splines, provides the foundation for traditional approaches to this complex topic.

Hull and White [1993] have shown that the single-factor term structure models like those of Vasicek [1977] and Cox, Ingersoll, and Ross [1985] can be extended by allowing selected parameters to drift over time, which generally will provide a perfect fit to observable yield curve data. Between observable data points, however, some yield curve smoothing technique is necessary for the successful application of the Hull and White approach.

This article presents a new approach to yield curve smoothing that provides a sound basis for implementation of these techniques. Previous approaches to yield curve smoothing (see Buono, Gregory-Allen, and Yaari [1992] for a recent survey) define the "best" yield curve as that function with relatively few parameters that fits a large set of data most accurately when using multiple regression techniques. We take a much different approach.

By carefully defining the criterion for the best fitting yield curve to be "maximum smoothness" for the forward rate curve, we arrive at a simple but powerful technique for yield curve fitting. This technique can be used for fitting the yield curve with one explicit function that is both consistent with all observed

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points on the yield curve and provides the smoothest possible forward rate curve consistent with the chosen functional form. In addition, a closed-form solution is derived for the smoothest of all possible functional forms that is related to but significantly different from the cubic spline approach of McCulloch.

This new maximum smoothness approach provides a simple criterion for choosing between specific functional forms for fitting the yield curve for any given set of empirical data if one chooses not to use the closed-form solution for the maximum smoothness forward rate curve. Finally, this technique can easily incorporate any number of observations from the current yield curve and can incorporate simultaneous relationships among yields when the only observable bond prices are those for coupon-bearing bonds.

After we review prior work on yield curve smoothing, we summarize the mathematical links among yields, instantaneous forward rates, and discount bond prices. The mathematical definition of smoothness is used to derive the yield curve with maximum smoothness forward rates. Finally, we compare the maximum smoothness yield curve fitting method with six other smoothing methods: the cubic spline fitting of bond prices (constrained in two ways), the cubic spline fitting of bond yields (constrained in two ways), linear yield smoothing, and exponential price smoothing.

I. REVIEW OF PRIOR WORK ON YIELD CURVE SMOOTHING

McCulloch [1975] pioneered the use of polynomial spline functions to fit the observed prices of U.S. Treasury securities. As McCulloch notes, the direct use of one polynomial function to fit the entire yield curve or to fit the zero-coupon bond prices for each maturity is satisfactory from a theoretical perspective but leads to unacceptable yield patterns on actual data. The use of cubic splines produces relatively smooth forward rates, and parameters can be estimated using normal linear regression techniques.

Vasicek and Fong [1982] state their objective in yield curve smoothing clearly: "to fit a spot rate curve (or any other equivalent description of the term structure, such as the discount function) that 1) fits the data sufficiently well, and 2) is a sufficiently smooth function." This second objective, as Vasicek and Fong note, is "less quantifiable than the first." Vasicek and Fong note the essential exponential nature of the discount function, and adapt the exponential form to fit bond price data (from which yields are derived). Like McCulloch, they select a cubic as the lowest odd order (exponential) form with continuous derivatives.

Shea [1985] summarizes the comments of many others on polynomial splines, noting that the forward rate estimates they generate are "unstable, fluctuate widely, and often drift off to very large, even negative values." Shea concludes that the same critiques apply to the exponential spline model of Vasicek and Fong as well.

While the spline approaches to fitting the term structure have considerable advantages, their many disadvantages have led many sophisticated financial market participants to use simpler approaches. For example, a survey of software vendors specializing in complex financial products reveals that twelve of thirteen vendors specifying their approach to yield curve smoothing use a straight-line interpolation to fit market yield curves for derivative product applications (see *The Mitsubishi Finance Risk Directory* [1990/1991]). Although spline techniques are gradually replacing straight-line yield curve fitting for market applications, the problems outlined by Shea remain.

Buono, Gregory-Allen, and Yaari [1992] provide a detailed survey of similar approaches to yield curve smoothing.

II. YIELD CURVE RELATIONSHIPS IN CONTINUOUS TIME

In developing a new approach to yield curve smoothing, we will rely on the standard continuous-time relationships among zero-coupon bond prices, yields, and forward rates. We set the current time to be $t = 0$, and denote the current price of a zero-coupon bond maturing at time $t \geq 0$ by $P(t)$, with $P(0) = 1$. The yield to maturity on such a bond is $y(t)$. The instantaneous forward rate for time t is denoted by $f(t)$.

We have the following relationships:

$$P(t) = e^{-ty(t)} \tag{1}$$

$$P(t) = \exp\left(-\int_0^t f(s)\,ds\right) \tag{2}$$

$$f(t) = y(t) + ty'(t) \tag{3}$$

Any one of the functions $P(t)$, $y(t)$, and $f(t)$ determines the others, and we will use the expression "term structure" interchangeably in reference to any of them.

III. MAXIMUM SMOOTHNESS

While the objective is not stated explicitly, McCulloch and Vasicek and Fong seem to have a very clear objective: to fit a smooth function with relatively few parameters to the U.S. Treasury yield curve. Shea certainly uses the degree of smoothness as a criterion of success. McCulloch and Vasicek and Fong improve on their implied criterion of smoothness by breaking the bond price function into smaller segments and fitting a lower-degree polynomial or exponential function to each segment in such a way that the first and second derivatives at the "knot points" are continuous. Bond yields and forward rates are derived from the smoothed bond price function.

By lowering the degree of the polynomial or exponential involved, McCulloch and Vasicek and Fong do away with fitted yield curves that fluctuate wildly between observable data points. In spite of these efforts, as pointed out by Shea [1985], it is still possible for the implied forward rates derived from smooth yield curves to be implausibly volatile.

A more direct way to resolve these problems is to define a different objective: to fit observable points on the yield curve with the function of time that produces the smoothest possible forward rate curve. In previous work, the intuitively obvious but mathematically vague quality of "smoothness" has been undefined.

Borrowing a common technique from numerical analysis,[1] we define the smoothest possible forward rate curve on an interval (0, T) as one that minimizes the functional

$$Z = \int_0^T [f''(s)]^2 \, ds \tag{4}$$

This expression is a common mathematical definition of smoothness used in engineering applications. It can be shown that a cubic spline fitted to the discount function produces the smoothest possible discount function according to this definition of smoothness. Likewise, a cubic spline fitted to continuous bond yields produces the smoothest possible continuous yield curve. See Schwartz [1989] for a discussion of this property of spline functions.

IV. A GENERAL SOLUTION FOR YIELD CURVES WITH MAXIMUM SMOOTHNESS

The maximum smoothness criterion is meaningless unless it is combined with observable points on the yield curve. If the observable points take the form of m given zero-coupon bond prices, $P_1, P_2, \ldots, P_m$ with maturities $t_1, t_2, \ldots, t_m$, Equation (4) will have to be minimized subject to the constraints

$$\exp\left(-\int_0^{t_i} f(s)\, ds\right) = P_i, \quad i = 1, 2, \ldots, m \tag{5}$$

We may approach the problem by expressing the forward rate curve as a function of a specified form with a finite number of parameters, and find the maximum smoothness term structure within this parametric family. Such a solution will not necessarily represent the smoothest possible forward rate curve for all potential functional forms, but it will be more smooth than that given by any other mathematical expression of the same degree and same functional form.

In this case, the solution is relatively straightforward. Let $f(t) = f(t; a_1, a_2, \ldots, a_n)$ be the forward rate curve as a known function of parameters $a_1, a_2, \ldots, a_n$. Then Equation (4), as well as the left-hand sides of Equation (5), can be calculated as functions of $a_1, a_2, \ldots, a_n$

Introducing the Lagrange multipliers λ_i, i = 1, ..., m, we can write the minimization problem as

$$\min_{a_1, a_2, \ldots, a_n} \left(Z + \sum_{i=1}^{m} \lambda_i \left[P(t_i) - P_i \right] \right)$$

The solution has the form of the equations

$$\frac{\partial}{\partial a_i}\left[\int_0^T [f''(s)]^2\, ds + \sum_{i=1}^{m} \lambda_i \times \left(\exp\left[-\int_0^{t_i} f(s)\, ds\right] - P_i\right)\right] = 0 \quad i = 1, 2, \ldots, n$$

These n equations, together with the m equations (5), provide conditions for the n + m unknown values $a_1, a_2, ..., a_n, \lambda_1, \lambda_2, ..., \lambda_m$.

Examples of this approach may include polynomial forms

$$\sum_{j=1}^{n} a_j t^{j-1}$$

or exponential forms

$$\exp\left(\sum_{j=1}^{n} a_j t^{j-1}\right)$$

as possible families for the forward rate curve or the yield curve.

We wish, however, to solve the problem in its full generality; that is, to determine the maximum smoothness term structure within all possible functional forms. The answer is given by the following theorem, provided by Oldrich Vasicek:

Theorem. The term structure f(t), $0 \le t \le T$, of forward rates that satisfies the maximum smoothness criterion

$$\min \int_0^T f''^2(s)\, ds$$

while fitting the observed prices $P_1, P_2, ..., P_m$ of zero-coupon bonds with maturities $t_1, t_2, ..., t_m$ is a fourth-order spline with the cubic term absent given by

$$f(t) = c_i t^4 + b_i t + a_i, \qquad \text{for } t_{i-1} < t \le t_i,$$

$$i = 1, 2, ..., m + 1 \tag{6}$$

where $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$. The coefficients a_i, b_i, c_i, $i = 1, 2, ..., m + 1$ satisfy the equations

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1} t_i^4 + b_{i+1} t_i + a_{i+1}, \qquad i = 1, 2, ..., m \tag{7}$$

$$4c_i t_i^3 + b_i = 4c_{i+1} t_i^3 + b_i, \qquad i = 1, 2, ..., m \tag{8}$$

$$\frac{1}{5}c_i\left(t_i^5 - t_{i-1}^5\right) + \frac{1}{2}b_i\left(t_i^2 - t_{i-1}^2\right) + a_i\left(t_i - t_{i-1}\right) = -\log\left(\frac{P_i}{P_{i-1}}\right), \quad i = 1, 2, ..., m \tag{9}$$

and $c_{m+1} = 0$. (10)

The proof of this theorem is given in the appendix. This theorem is a tribute to the intuition of McCulloch, who pioneered the use of polynomial splines in fitting the term structure of interest rates. Of course, the particular degree and form of the maximum smoothness curves differ somewhat from the third-order splines he used.

It may be noted that the solution of the smoothest term structure problem depends on the particular smoothness criterion in Equation (4). There are other meaningful measures of smoothness, such as

$$\min \int_0^T \frac{|f''(s)|}{f'^2(s) + 1}\, ds \tag{11}$$

(which, unlike (4), is invariant with respect to rotations of the curve x = f(t) in the (x, t) space) that produce different solutions.

In computational determination of the maximum smoothness term structure, it is seen that the theorem specifies 3m + 1 equations for the 3m + 3 unknown parameters a_i, b_i, c_i, $i = 1, 2, ..., m + 1$. The maximum smoothness solution is nevertheless unique.

It is easily obtained analytically as follows: The value of the objective function (4) is proportional to

$$Z = \sum_{i=1}^{m} c_i^2\left(t_i^5 - t_{i-1}^5\right)$$

which is quadratic in the parameters. Since the conditions (7), (8), (9), and (10) are all linear in the parameters, we have an unconstrained quadratic problem of the form

Minimize x'Dx

subject to

$$Ax = b$$

with the well-known solution

$$(I - A'(AA')^{-1}A)\ Dx = 0 \qquad (12)$$

Any two equations from the system (12) provide the remaining conditions on the parameters a_i, b_i, c_i, $i = 1, 2, \ldots, m + 1$.

For the asymptotic behavior of the term structure, it makes sense to include an additional requirement that

$$f'(T) = 0 \qquad (13)$$

This is equivalent to the condition

$$b_{m+1} = 0 \qquad (14)$$

since c_{m+1} is required to be zero by Equation (10). When this constraint is imposed, only one equation from the system (12) is needed. In addition, one may impose the constraint that the instantaneous forward rate at time zero be equal to an observable rate r so that $a_0 = r$. In this case, no equations from the system (12) are needed.

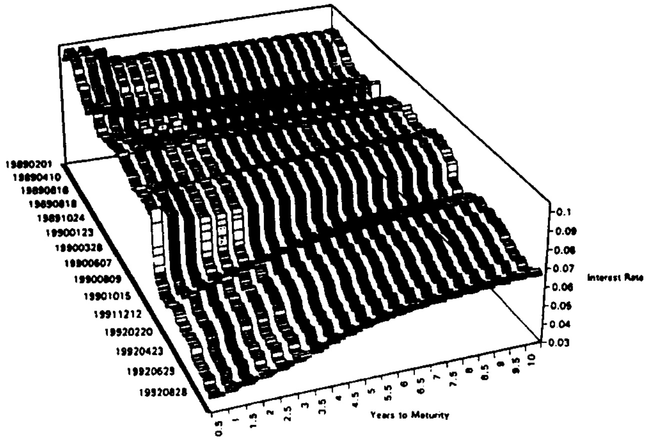
V. EMPIRICAL RESULTS

We compared this maximum smoothness approach to alternative yield curve smoothing techniques using yen interest rate swap and money market data provided by the Fuji Bank, Ltd., and U.S. dollar rates provided by the Mitsubishi Bank, Ltd. Interest rate swaps were analyzed as if the swap represented quotes for coupon-bearing bonds.

The yen data include observations for 848 business days beginning November 1, 1988. Dollar data were analyzed for 660 business days beginning February 1, 1989.

Interest rates were available for yen-yen interest rate swaps with maturities of one, two, three, four, five, seven, and ten years. Dollar-dollar interest rate swap rates included maturities of one, two, three, five, seven, and ten years. These maturities were supplemented with six-month Euro yen and Euro dollar deposit rates and one-day overnight (*gensaki* or fed funds) rates. All rates were adjusted to an actual/365-day basis and calculated as the average of bid and offered rates.

EXHIBIT 1 ■ U.S. Dollar Swap Yield Curve 1989-1992



As shown in Exhibits 1 and 2, this time period in both the yen and dollar markets includes a rich variety of yield curve shapes, including upward-sloping, downward-sloping, and flat yield curves. The data include periods of both rising and falling interest rates.

Alternative Yield Curve Smoothing Techniques

Smoothed yield curves and the related forward rates are calculated for every semiannual maturity for each observation using the maximum smoothness approach and six other yield curve smoothing techniques:

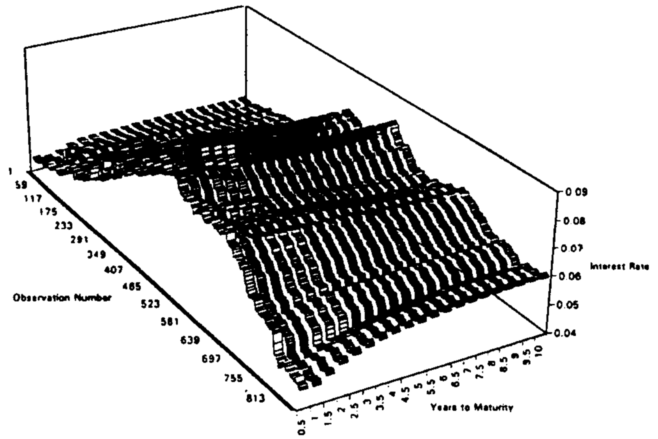
- Cubic spline smoothing of bond prices (the McCulloch approach):
  - a. With the ten-year instantaneous yield constrained to a zero first derivative, $y'(10) = 0$.
  - b. With the second derivative of the discount function at ten years constrained to zero, $p''(10) = 0$.

- Cubic spline smoothing of bond yields:
  - a. With the ten-year instantaneous yield constrained to a zero first derivative, $y'(10) = 0$.
  - b. With the second derivative of the yield function at ten years constrained to zero, $y''(10) = 0$.

- Exponential smoothing of bond prices (the Vasicek and Fong approach).

- Linear smoothing of bond yields.

EXHIBIT 2 ■ Yen Swap Yield Curve 1988-1992



In each version of cubic spline smoothing, short-term rates are constrained to converge to the one-day interest rate as the maturity approaches zero. This constraint is also imposed on the maximum smoothness technique, along with the requirement that the first derivative of the forward rate curve at the ten-year maturity point be zero.

Vasicek and Fong chose the exponential form explicitly because bond prices must be exponential in functional form; accordingly, we have not applied the exponential approach to yield smoothing. Vasicek and Fong conclude that an exponential of the following form is the best exponential spline function:

$$f(x) = a_0 + a_1 e^{-\alpha x} + a_2 e^{-2\alpha x} + a_3 e^{-3\alpha x}$$

The α parameter in the Vasicek and Fong specification stems from a transformation that Vasicek and Fong employ to construct a linear system. Its value is not determined by the need for continuous first and second derivatives of the yield curve. In what follows, we select α so that the "best" possible exponential spline yield curve is generated for this class of functions: the exponential spline yield curve with maximum smoothness.[2] Clearly, this provides a positive bias to our evaluation of the merits of the exponential functional form. We also include an analysis of the forward rates implied by direct linear interpolation of yields because of the popularity of this technique among practitioners.

The Measure of Smoothness

Section III defines smoothness in a continuous sense. In this section, we have chosen to calculate "smoothness coefficients" in discrete form by taking the sum of the squared second differences of the semiannual forward rates calculated for each technique. This approximation avoids the need to do numerical approximation of integrals in one case, and recognizes that the instantaneous forward rate curve derived from linear yield interpolation is not continuous.

Since semiannual forward rates are of greatest interest to practitioners, this empirical shortcut has some practical appeal. It does mean, however, that the maximum smoothness forward rate curves in a continuous sense will not necessarily be "most smooth" using a discrete measure of smoothness.[3]

Results of the Empirical Analysis

Exhibit 3 summarizes a comparison of the discrete smoothness statistics for semiannual forward rates and the relative rankings of the various smoothing techniques. For all 848 daily observations of the yen swap market, the exponential spline approach produces the least smooth forward rates. Accordingly, we did not implement the exponential spline approach for the U.S. dollar analysis.

The smoothness statistics reported show the very high sensitivity of results to the time period selected. In general, however, the maximum smoothness method and the cubic spline smoothing of bond prices produced forward rate curves that are consistently smoother than those produced by cubic spline smoothing of yields. This is confirmed by ranking the methods daily in terms of smoothness, with 1 indicating "most smooth." Linear yield curve smoothing produced forward rates that are more than three times "less smooth" than the forward rates produced using other yield curve smoothing methods.

The reason that Shea criticizes the smoothness of forward rates produced using prior methods is the implicit assumption that smooth forward rate curves produce interpolated yield curves that are better predictors of true market yields than curves that fluctuate drastically. Accordingly, we conducted a test of this hypothesis by omitting known data from each data set, smoothing the remaining data, and comparing the estimated yields with the true market data.[4] For both the yen data and the dollar data, we omit the known seven-year swap data and the known three-year swap data in separate tests of predictability.

The results of these two tests are shown in

EXHIBIT 3 ■ Discrete Semiannual Smoothness Statistics for Forward Rates for Yield Curve Smoothing Methods

Discrete Semiannual Smoothness Statistics for U.S. Dollar Swap Market Rates

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	0.000449	0.000454	0.000550	0.000547	0.000467	0.000737
Second Quarter	0.000121	0.000125	0.000163	0.000159	0.000126	0.000216
Third Quarter	0.000080	0.000076	0.000100	0.000090	0.000075	0.000509
Fourth Quarter	0.000065	0.000038	0.000072	0.000045	0.000038	0.000942
Total	0.000179	0.000173	0.000221	0.000210	0.000176	0.000601

Average Ranking (1 = Best) by Smoothness Statistics
U.S. Dollar Swap Market Rates

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	1.02	2.07	4.84	4.05	2.88	5.97
Second Quarter	1.21	2.65	4.96	4.05	2.12	5.79
Third Quarter	2.21	2.60	4.46	3.73	1.87	6.00
Fourth Quarter	4.21	1.66	4.64	2.68	1.80	6.00
Total	2.16	2.25	4.72	3.63	2.17	5.94

Discrete Semiannual Smoothness Statistics for Yen Swap Market Rates

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield	Exponential
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing	Smoothing
First Quarter	0.000029	0.000030	0.000071	0.000071	0.000036	0.000054	0.002280
Second Quarter	0.000045	0.000042	0.000060	0.000057	0.000045	0.000132	0.002635
Third Quarter	0.000073	0.000070	0.000087	0.000084	0.000074	0.000235	0.003004
Fourth Quarter	0.000080	0.000081	0.000087	0.000085	0.000079	0.000164	0.003147
Total	0.000057	0.000056	0.000076	0.000075	0.000059	0.000146	0.002769

Average Ranking (1 = Best) by Smoothness Statistics
Yen Swap Market Rates

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield	Exponential
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing	Smoothing
First Quarter	1.10	2.00	5.41	4.92	2.98	4.46	7.00
Second Quarter	2.19	1.30	4.97	4.08	2.74	5.68	7.00
Third Quarter	2.50	1.22	4.80	3.63	2.77	6.00	7.00
Fourth Quarter	2.25	2.94	4.50	3.44	1.66	6.00	7.00
Total	2.01	1.87	4.92	4.02	2.53	5.54	7.00

Discrete Semiannual Smoothness Statistics for Combined Swap Market Data Sample

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
Total	0.000109	0.000106	0.000139	0.000134	0.000110	0.000291

Average Ranking (1 = Best) by Smoothness Statistics
Combined Swap Market Data Sample

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
Total	2.04	1.94	4.89	3.88	2.43	5.71

Exhibit 4. For both maturities, all yield curve smoothing techniques are more effective predictors on yen swap data. In predicting seven-year yen yields (where a five-year gap in the data was filled by smoothing), the maximum smoothness approach is significantly better than all other techniques, with a mean absolute error of 1.11 basis points. For three-year yen yields, where the gap in the data was only two years, cubic spline price smoothing and the maximum smoothness technique are roughly equal, all showing a mean absolute error of about 2.95 basis points. Not too surprisingly, linear interpolation has a mean absolute error of only 2.49 basis points, attributable to the relatively short time interval between observable data points.

On U.S. dollar seven-year data, the maximum smoothness technique produces the lowest mean absolute error, 5.73 basis points. At three years, however, linear interpolation has a mean absolute error less than one-third of the next best technique, cubic spline smoothing of bond prices. All techniques are least effective in predicting U.S. dollar three-year rates because of the sharply downward-sloping yield curve, combined with a sharp "elbow" in the yield curve, in the two- to four-year maturity range during the early part of the data set. (See Exhibit 1.)

Although empirical results are highly sensitive to the time period and the type of data selected, the maximum smoothness yield curve smoothing technique seems to perform best overall when judged on the dual criteria of accuracy and smoothness.

VI. SUMMARY

The success of any yield curve smoothing technique has traditionally been judged by the smoothness of the forward rate curve it produces. Intuitively, smooth forward rates seem more likely to produce more accurate interpolated yields than volatile forward rate curves.

We have defined the integral of the squared second derivative of forward rates as a mathematical measure of smoothness. Using this objective function, one can determine the parameters of any functional form that produces the smoothest forward rate curve consistent with that functional form and observable data. We have also shown that the smoothest forward rate curves are produced by a fourth-degree polynomial with the cubic term missing.

Applying this general solution to forward rate generation, we have shown that the maximum smoothness approach outperforms other techniques when judged on the dual criteria of accuracy and smoothness. The maximum smoothness forward rate approach also provides a mathematical criterion for comparing the smoothness of the forward rates produced by any functional forms, including those representing a particular theory of the term structure of interest rates.

APPENDIX ■ Proof of the Theorem[5]

Schwartz [1989] demonstrates that cubic splines produce the maximum smoothness discount functions or yield curves if the spline is applied to discount bond prices or yields, respectively. In this appendix, we derive by a similar argument the functional form that produces the forward rate curve with maximum smoothness.

Let f(t) be the current forward rate function, so that

$$P(t) = \exp\left(-\int_0^t f(s)\,ds\right) \tag{A1}$$

is the price of a discount bond maturing at time t. The maximum smoothness term structure is a function f with a continuous derivative that satisfies the optimization problem

$$\min \int_0^T f''^2(s)\,ds \tag{A2}$$

subject to the constraints

$$\int_0^{t_i} f(s)\,ds = -\log P_i, \qquad \text{for } i = 1, 2, \ldots, m \tag{A3}$$

Here the $P_i = P(t_i)$, for i = 1, 2, ..., m are given prices of discount bonds with maturities $0 < t_1 < t_2 < \ldots < t_m < T$.

Integrating twice by parts we get

$$\int_0^t f(s)\,ds = tf(t) - \frac{1}{2}t^2f'(t) + \frac{1}{2}\int_0^t s^2f''(s)\,ds$$

Put $\quad g(t) = f''(t), \qquad 0 \le t \le T \qquad$ (A4)

$$Q_i = t_if(t_i) - \frac{1}{2}t_i^2f'(t_i), \qquad i = 1, 2, \ldots, m \tag{A5}$$

and define the step function

EXHIBIT 4 ■ Comparative Accuracy of Six Yield Curve Smoothing Methods (%)

Mean Absolute Errors when Fitting U.S. Dollar Three-Year Swap Rate

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	0.1820	0.1834	0.3551	0.3553	0.2084	0.0210
Second Quarter	0.1055	0.1065	0.1920	0.1918	0.1180	0.0259
Third Quarter	0.0934	0.0946	0.1609	0.1609	0.1036	0.0239
Fourth Quarter	0.0604	0.0567	0.0564	0.0564	0.0517	0.0482
Total	0.1078	0.1074	0.1832	0.1832	0.1165	0.0316

Mean Absolute Errors when Fitting U.S. Dollar Seven-Year Swap Rate

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	0.0405	0.0953	0.0723	0.0963	0.0506	0.0222
Second Quarter	0.0546	0.1106	0.0728	0.0926	0.0651	0.0509
Third Quarter	0.0457	0.1042	0.0581	0.0816	0.0611	0.0905
Fourth Quarter	0.0990	0.0289	0.1189	0.0871	0.0544	0.1076
Total	0.0640	0.0790	0.0851	0.0898	0.0573	0.0691

Mean Absolute Errors when Fitting Yen Three-Year Swap Rate

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	0.0126	0.0127	0.0411	0.0412	0.0164	0.0220
Second Quarter	0.0352	0.0350	0.0323	0.0323	0.0330	0.0196
Third Quarter	0.0460	0.0458	0.0393	0.0392	0.0439	0.0353
Fourth Quarter	0.0242	0.0242	0.0332	0.0331	0.0243	0.0226
Total	0.0295	0.0294	0.0365	0.0364	0.0294	0.0249

Mean Absolute Errors when Fitting Yen Seven-Year Swap Rate

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
First Quarter	0.0164	0.0277	0.0252	0.0101	0.0061	0.0082
Second Quarter	0.0492	0.0127	0.0131	0.0254	0.0138	0.0187
Third Quarter	0.0653	0.0126	0.0105	0.0290	0.0169	0.0252
Fourth Quarter	0.0364	0.0301	0.0302	0.0123	0.0074	0.0144
Total	0.0418	0.0208	0.0198	0.0192	0.0111	0.0166

Mean Absolute Errors when Fitting All Swap Rate Data

	Cubic Price Spline		Cubic Yield Spline		Maximum	Linear Yield
Portion of Data Set	$y'(10) = 0$	$p''(10) = 0$	$y'(10) = 0$	$y''(10) = 0$	Smoothness	Smoothing
Total	0.0577	0.0549	0.0745	0.0754	0.0494	0.0337

$$u(t) = 1 \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

The optimization problem can then be written as

$$\min \int_0^T g^2(s)\,ds \qquad \text{(A6)}$$

subject to

$$\frac{1}{2}\int_0^{t} s^2 u(t_i - s)\, g(s)\, ds = -\log P_i - Q_i,$$

$$i = 1, 2, \ldots, m \qquad \text{(A7)}$$

Let λ_i for $i = 1, 2, \ldots, m$ be the Lagrange multipliers corresponding to the constraints (A7). The objective then becomes

$$\min Z[g] = \int_0^T g^2(s)\,ds +$$

$$\frac{1}{2}\sum_{i=1}^{m}\lambda_i\left(\int_0^T s^2u(t_i - s)\,g(s)\,ds + \log P_i + Q_i\right) \qquad (A8)$$

According to the calculus of variations, if the function g is a solution to (A8), then

$$\frac{d}{d\varepsilon}Z[g + \varepsilon h]\,\Big|_{\varepsilon=0} = 0 \qquad (A9)$$

for any function h(t) defined on [0, T]. We get

$$\frac{d}{d\varepsilon}Z[g + \varepsilon h]\,\Big|_{\varepsilon=0} =$$

$$2\int_0^T\left[g(s) + \frac{1}{4}s^2\sum_{i=1}^{m}\lambda_i u(t_i - s)\right]h(s)\,ds$$

In order that this integral is zero for any function h, we must have

$$g(t) + \frac{1}{4}t^2\sum_{i=1}^{m}\lambda_i u(t_i - t) = 0 \qquad (A10)$$

for all $0 \le t \le T$. This means that

$$g(t) = 12c_i t^2 \qquad \text{for } t_{i-1} < t \le t_i,$$

$$i = 1, 2, \ldots, m + 1 \qquad (A11)$$

where

$$c_i = -\frac{1}{48}\sum_{j=i}^{m}\lambda_j, \qquad i = 1, 2, \ldots, m \qquad (A12)$$

$$c_{m+1} = 0 \qquad (A13)$$

and we define $t_0 = 0$, $t_{m+1} = T$.

From (A4) we get

$$f(t) = c_i t^4 + b_i t + a_i, \qquad t_{i-1} < t \le t_i,$$

$$i = 1, 2, \ldots, m + 1 \qquad (A14)$$

Continuity of f and f′ then implies that

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1}t_i^4 + b_{i+1}t_i + a_{i+1}, \qquad i = 1, 2, \ldots, m \qquad (A15)$$

$$4c_i t_i^3 + b_i = 4c_{i+1}t_i^3 + b_{i+1}, \quad i = 1, 2, \ldots, m \qquad (A16)$$

The constraints (A3) become

$$\frac{1}{5}c_i\left(t_i^5 - t_{i-1}^5\right) + \frac{1}{2}b_i\left(t_i^2 - t_{i-1}^2\right) + a_i\left(t_i - t_{i-1}\right) =$$

$$-\log\left(\frac{P_i}{P_{i-1}}\right), \quad i = 1, 2, \ldots, m \qquad (A17)$$

where we define $P_0 = 1$. This proves the theorem.

ENDNOTES

Oldrich Vasicek's contributions are central to Section IV and the appendix of this article, and only his own modesty kept him from a co-authorship. The authors are grateful to the Fuji Bank, Ltd., and the Mitsubishi Bank, Ltd., for providing the money market and interest rate swap rates used in the empirical section of the article. David Shimko's insights substantially improved that section. Yuichiro Inagaki, Tony Kobayashi, Akira Sekiyama, and Naohiko Tejima of the Kamakura Corporation provided invaluable comments and programming support.

[1]See Hildebrand [1987, p. 481]. Hildebrand says that an approximation "is 'more smooth' than any member of the relevant set of competitors in the sense that the mean-square value of its second derivative on [the interval] [a, b] is minimal."

[2]The alpha is chosen to minimize the discrete version of the smoothness statistic for six-month forward rates at semiannual intervals.

[3]We also calculated the maximum smoothness forward rates consistent with the discrete criterion of

smoothness. The forward rates produced in this manner are not significantly different from rates generated using the continuous definition of smoothness. They do rank as more smooth, however, under the discrete semiannual measure of smoothness.

[4]We are grateful to David Shimko of the University of Southern California for suggesting this test.

[5]This proof was provided by Oldrich Vasicek.

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