Statistical modeling of credit default swap portfolios

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Abstract

We undertake a systematic study of the univariate and multivariate properties of CDS spreads using the CDS spread time series of CDX Investment Grade index constituents from 2005 to 2009. We find that CDS spread returns appear to be stationary and exhibit positive autocorrelations, conditional heteroscedasticity, two-sided heavy tails, serial dependence in extreme values, and large co-movements not necessarily linked to credit events. The first principal component of CDS spread returns corresponds to heavy-tailed parallel shifts in CDS spreads across obligors.

We then propose a heavy-tailed multivariate time series model for CDS spreads, which can be used as a framework for measuring and managing the risk of CDS portfolios. This model is shown to capture adequately the observed statistical properties of CDS spreads and has better performance than affine jump-diffusion or random walk models for predicting loss quantiles of various CDS portfolios. In particular, loss quantiles estimated using the heavy-tailed multivariate model adjust more rapidly to the market shocks in late 2008, with a consistent performance across normal and extreme market environments.

Keywords: credit default swaps, credit risk, stylized properties, risk management, autocorrelation, heavy tails, conditional heteroscedasticity, principal component analysis, credit events, loss distribution, Value-at-Risk, expected shortfall, CDS.

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Contents

1 Introduction

The first generation of credit risk models has primarily focused on the modeling of *default risk*, either by modeling the capital structure of firms [49] or through reduced-form models for 'hazard rates' and default probabilities [9, 15, 19, 17, 30, 36, 42, 43]. However, during the recent financial crisis, investors in credit derivatives experienced substantial losses even in absence of any defaults in the portfolios underlying these contracts. Volatility in market values of credit-sensitive instruments in absence of defaults is mainly due to

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the change of credit quality of the underlying obligors, which is reflected in their credit spreads. This *spread risk* turns out to be the major risk faced by investors in credit derivatives.

Consider for instance the CDX.NA.IG index, an equally weighted portfolio of 125 5-year credit default swaps (CDS) (see section 2). Assume that the protection premium of each CDS is equal to 100bps. If the recovery rate for each CDS is equal to 40%, which is a standard assumption for pricing credit derivatives, default of a single obligor will generate a loss of $0.6/125 = 0.48\%$ of the total portfolio notional value. In an investment grade index, such a default is a rare event: 8 defaults have been observed in the CDX.NA.IG since 2003[1]. Table 1 shows that this loss is equivalent to the loss generated by a change in the CDS spreads which corresponds to the 99th percentile of daily changes, an event which occurs more than twice a year! This example shows that, at least for investment grade credit portfolios, spread risk is at least as important as default risk, if not more. Not surprisingly, credit risk models which did not accurately capture spread risk performed poorly during the recent crisis [16].

	Percentile of daily P&L								
Period	1%	5%	10%	30%	50%	70%	90%	95%	99%
2005-09	-0.431%	-0.206%	-0.103%	-0.015%	0.000%	0.011%	0.084%	0.192%	0.459%
2005-07	-0.070%	-0.031%	-0.020%	-0.004%	0.003%	0.008%	0.020%	0.026%	0.049%
2007-09	-0.700%	-0.297%	-0.207%	-0.071%	-0.011%	0.038%	0.202%	0.313%	0.542%

Table 1: Percentiles of the daily profit-and-loss (percentage of the total portfolio notional value) of an equally weighted credit portfolio consisting of the 5-year CDS written on the constituents in CDX.NA.IG.12.

The importance of spread risk calls for a better understanding of variations in credit spreads and models which accurately reflect the characteristics of these variations. The empirical literature shows that credit spreads have particular statistical features which need to be incorporated in a model for spread risk [2, 3, 7, 12, 54]. Collin-Dufresne et al [12], Blanco, Brennan and Marsh [7] and Alexander and Kaeck [1] explore the determinants of credit spreads and their relation with other economic variables. Rahman [54] and Almer, Heidorn and Schmaltz [2] study empirical properties of credit spreads for financial obligors; Rahman [54] proposed a multivariate DCC-GARCH model for credit spreads.

As argued by Blanco, Brennan and Marsh [7], the CDS market has become the main forum for credit risk price discovery. This work contributes to the previous empirical literature by undertaking a systematic study of the relationship between CDS spreads and credit yield credit spreads. In particular, CDS prices appear to be better integrated with firm-specific variables in the short run, and both CDS and bond markets equally reflect those factors in the long run.

The risk management of credit sensitive instruments calls for models which are capable of adequately reproducing the statistical properties of CDS spreads. Any default risk model implies some dynamics for credit spreads, but most existing default risk models have focused on analytical tractability rather than statistical properties of (CDS) spreads, and spread dynamics implied by these models do not necessarily correspond to observed dynamics of spreads. This results in poor performance of these models for hedging and risk management [16]. Another context which requires joint statistical modeling of CDS spreads is the risk management of CDS clearinghouses. Regulatory reform in the light of the 2008 crisis has moved CDS trading from over-the-counter bilateral trading to central clearing. Central counterparties require a deposit (initial margin) from clearing participants, based on the risk of their CDS portfolios [14]. The consistent computation of such margin requirements requires a multivariate model for (co-)movements in CDS spreads.

As shown by Collin-Dufresne et al [12], credit spread changes are principally driven by supply/demand fluctuations that are independent from factors traditionally considered in credit risk modeling and standard proxies for liquidity. This observations suggests that direct stochastic modeling of CDS spread returns is more effective than trying to explain spread movements in terms of other economic variables. This is the approach we adopt here: we propose, in the second part of this paper, a heavy-tailed multivariate time series model for the dynamics of CDS spreads which reflects the observed empirical properties of CDS spreads yet is easy to estimate and use. We compare our model with previously proposed –random walk and affine jump-diffusion– models and show that the model adequately predicts the distribution of losses for a variety of CDS portfolios with long and short positions, making it a useful tool for the risk management of CDS

[1]This includes both on-the-run and off-the-run series.

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portfolios.

In this paper, we undertake a systematic study of univariate and multivariate properties of CDS spread returns in (Section 3). Based on these observations, we propose a multivariate time series model which is suitable for measuring and managing the risk of CDS portfolios. Below is a summary of the main contributions of our study.

- The study of statistical properties of CDS spreads for CDX index constituents in the period 2005-09 reveals that:
  - CDS spread returns can be modeled as stationary processes with positive autocorrelations, positive serial correlations in extreme values, conditional heteroscedasticity and two-sided heavy tails.
  - Large co-movements are observed in the CDS spread series, indicating the presence of heavy-tailed common factors; these large co-movements are not necessarily linked to credit events.
  - Correlations across obligors of CDS spread returns increase substantially in 2007-09.
  - Principal component analysis suggests that the main contribution to the variance of CDS spread returns comes from idiosyncratic jumps.
  - Credit events do not necessarily lead to large upward moves in the CDS spreads.
- Section 4 shows that commonly used affine jump-diffusion models [24, 29] are not able to match the observed serial dependence properties and the two-sided heavy-tailed distributions of CDS spread returns and tend to overestimate the probability of having (large) co-movements in the CDS spreads.
- In section 5, we propose a heavy-tailed multivariate time series model for CDS spread returns and show that this model is able to reproduce the observed statistical properties of CDS spread returns as well as their dependence structures adequately. We also propose a quasi maximum likelihood estimation method for the model.
- In section 6, we show that the heavy-tailed multivariate model compares favorable to the affine jump-diffusion model [24] and a random walk model [57]: it provides more accurate prediction for the loss quantiles of a wide variety of CDS portfolios, in 2005-07 and also during the market turmoil of late 2008.

2 Data

We consider daily observations of the 5-year CDS par spreads (or CDS spreads for simplicity) from 4 April 2005 to 17 July 2009, where the reference obligors belong to Markit CDX North America Investment Grade Series 12 portfolio (CDX.NA.IG.12). CDS with other maturities such as 1, 3, 7, and 10 years are sometimes available but less liquidly traded. Table 2 shows the distribution of the obligors in terms of industrial sectors, credit ratings and number of available observations. There are a total of 125 obligors in the CDX.NA.IG.12 portfolio which can be divided into eight industrial sectors as defined by Markit. The largest sector is the consumer cyclical sector which contains 29 obligors and the smallest is the materials sector which contains 6 obligors. As of 5 December 2009, all obligors are investment grade except for the CIT Group Inc whose S&P long-term local currency issuer rating is equal to D. Indeed, CIT Group Inc has been removed from CDX.NA.IG.12 on 3 November 2009 due to default.

Our sample covers the period before and during the subprime crisis. For each obligor whose data are available for the full sample period, it contains a total of 1109 daily observations. The data set spans the period 2005-2009 and spans a reasonably long period to provide a meaningful basis for the statistical analysis of CDS spreads

The CDS spread (log-)return over a time interval Δt (equal to 1 or 5 days) is defined as

$$r_t = \ln(s_t / s_{t-\Delta t})$$

where s_t is the CDS spread observed at time t. Figure 1 shows the CDS spreads and the daily spread returns of Conoco Philips. The behavior of the CDS spreads can be clearly divided into two regimes: before and after the onset of the subprime crisis in 2007. In particular, the CDS spreads are substantially larger and more volatile after 2007. Therefore, our analysis will focus on two sample periods:

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Sector	No. of obligors
Financial	23
Materials	6
Consumer Stable	13
Utilities	8
Energy	8
Industrial	21
Consumer Cyclical	29
Communications and Technology	17

S&P rating as of 5-Dec-2009	No. of obligors
AA- to AA+	5
A- to A+	42
BBB- to BBB+	73
BB- to BB+	4
D	1

Available no. of sample days	No. of obligors
> 1000	115
800 - 1000	3
500 - 800	6
300 - 500	1

Table 2: Distribution of obligors in terms of sectors, Standard and Poor's credit ratings and number of available daily observations.

1. Pre-subprime period (2005-07): 4 April 2005 to 30 June 2007
2. Subprime crisis (2007-09): 1 July 2007 to 17 July 2009

We consider obligors whose available CDS history exceeds 50 days, which leaves us with 121 obligors for analysis for 2005-2007. In the 2007-09 period, all obligors have more than 50 sample days.

Table 3 and 4 show the summary statistics of the CDS spreads and the spread returns respectively. We choose to present one obligor in each sector to demonstrate our observations. The summary statistics confirm our earlier observations that the CDS spreads and spread returns are significantly more volatile in 2007-09. For instance, the daily standard deviation of spread returns of MetLife in 2007-09 is twice its value in 2005-07. One interesting observation is that, the sample skewness of spread returns in 2007-09 is generally smaller and closer to 0. Moreover, although the spread returns appear to be leptokurtic in both periods, the sample kurtosis is generally smaller in 2007-09.

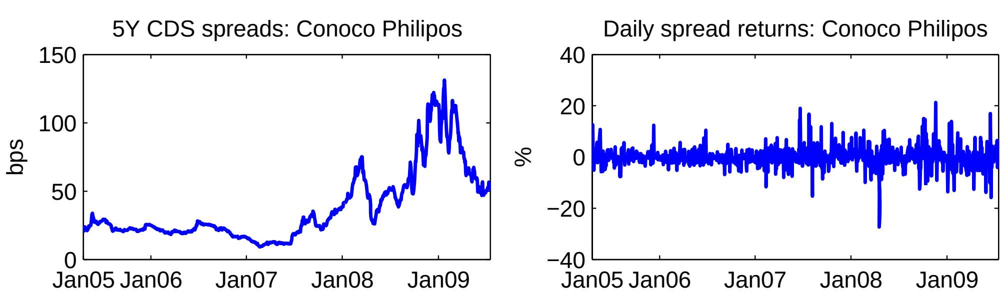

Figure 1: 5-year CDS spreads (left) and daily spread returns (right) of Conoco Philips.

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Obligor	Period	Conoco Phillips	Eastman Chemical	First Energy	HP	JC Penny	MetLife	Motorola	Pfizer
Rating		A	BBB	BBB	A	BB	A-	BB+	AA
Sector		Energy	Materials	Utilities	Com Tech	Con Cyclical	Financial	Industrial	Con Stable
Mean (bps)	2005-09	38.02	65.79	63.97	34.71	156.04	151.12	137.84	25.52
	2005-07	20.16	49.46	37.34	21.39	69.02	20.86	30.41	7.13
	2007-09	57.17	83.32	92.54	49.01	249.40	290.88	253.10	45.26
Stdev (bps)	2005-09	26.79	35.40	40.77	22.42	140.06	229.35	159.10	28.97
	2005-07	5.19	7.59	12.41	10.11	33.05	7.11	9.03	2.76
	2007-09	27.37	44.09	41.28	23.19	150.53	266.99	163.47	31.29
Skewness	2005-09	1.46	2.05	1.23	1.31	1.69	1.75	1.40	1.71
	2005-07	-0.34	0.28	-0.42	0.84	1.64	-0.03	0.51	0.53
	2007-09	0.77	1.04	0.54	0.86	1.00	0.72	0.53	0.90
Kurtosis	2005-09	4.38	6.65	3.96	4.80	5.38	4.86	4.13	5.26
	2005-07	2.39	2.20	2.05	3.66	5.65	1.65	2.62	1.58
	2007-09	2.69	3.06	2.75	3.68	3.24	2.21	2.76	3.07
Max (bps)	2005-09	131.27	227.02	206.60	134.80	701.04	989.99	682.17	127.27
	2005-07	33.77	70.75	65.80	57.18	222.00	34.58	53.69	12.33
	2007-09	131.27	227.02	206.60	134.80	701.04	989.99	682.17	127.27
Min (bps)	2005-09	9.23	32.56	12.51	7.75	35.95	10.68	14.14	3.80
	2005-07	9.23	35.74	12.51	7.75	35.95	10.68	14.14	3.80
	2007-09	17.95	32.56	23.16	13.55	53.30	14.04	33.87	6.52

Table 3: Summary statistics of 5-year CDS spreads (daily observations).

Obligor	Period	Conoco Phillips	Eastman Chemical	First Energy	HP	JC Penny	MetLife	Motorola	Pfizer
Mean (%)	2005-09	0.08	0.04	0.08	0.04	0.00	0.27	0.16	0.13
	2005-07	-0.02	-0.01	-0.14	-0.14	-0.24	-0.10	-0.03	-0.11
	2007-09	0.19	0.10	0.32	0.23	0.26	0.67	0.36	0.39
Stdev (%)	2005-09	3.70	4.14	3.75	3.83	4.43	4.86	4.27	3.88
	2005-07	2.72	2.87	2.39	2.84	3.52	2.03	3.62	3.63
	2007-09	4.52	5.17	4.79	4.65	5.23	6.65	4.86	4.11
Skewness	2005-09	0.16	1.28	1.24	0.13	0.91	0.66	1.76	1.35
	2005-07	1.22	1.96	1.78	0.44	3.04	0.40	3.04	0.34
	2007-09	-0.15	0.98	0.88	-0.04	0.08	0.36	1.07	2.06
Kurtosis	2005-09	10.08	17.81	13.46	9.04	17.55	16.36	21.58	12.41
	2005-07	11.05	16.16	17.71	9.01	51.69	7.14	24.78	8.75
	2007-09	7.87	13.54	9.18	7.26	7.60	9.50	18.45	14.15
Max (%)	2005-09	21.32	43.89	28.29	21.01	44.08	42.15	41.55	31.23
	2005-07	19.00	25.58	20.73	15.14	44.08	10.86	32.17	22.04
	2007-09	21.32	43.89	28.29	21.01	28.36	42.15	41.55	31.23
Min (%)	2005-09	-27.32	-20.54	-21.27	-22.94	-26.72	-32.28	-33.51	-18.75
	2005-07	-11.65	-8.74	-9.72	-16.27	-20.49	-8.43	-14.62	-18.75
	2007-09	-27.32	-20.54	-21.27	-22.94	-26.72	-32.28	-33.51	-13.05

Table 4: Summary statistics of daily spread returns.

3 Stylized properties of CDS spreads

3.1 Stationarity and unit root tests

Property 1 (Stationarity of CDS spread returns)**.** *CDS spread returns appear to be stationary, whereas CDS spreads themselves are not.*

We consider three tests for stationarity: (1) Augmented Dickey-Fuller (ADF) test [23, 35], (2) Phillips-Perron (PP) test [53] and (3) Kwiatkowski, Phillips, Schmidt and Shin (KPSS) test [41]. For ADF test and PP test, the null hypothesis assumes the time series has a unit root. KPSS test is an inverse of the PP test in which the null hypothesis assumes that the time series does not have a unit root. From Table 5, we observe that the spread returns of almost all obligors reject the null hypothesis of ADF and PP tests and cannot reject the null hypothesis of KPSS test. On the other hand, CDS spread series of only a small number of obligors can reject the null hypothesis of ADF and PP and cannot reject the null hypothesis of KPSS. This suggests that our statistical analysis should focus on CDS spread returns which appear to be stationary.

3.2 Linear serial dependence

Property 2 (Autocorrelation)**.** *CDS spread returns exhibit positive autocorrelations at 1 to 3 days, especially during the crisis period 2007-09. The autocorrelations diminish when the observation interval increases.*

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CDS spreads

Tests and decisions	1-day		5-day	
	2005-07	2007-09	2005-07	2007-09
Reject H_0^{adf}	45	52	35	24
Reject H_0^{pp}	48	43	45	46
Cannot reject H_0^{kpss}	5	1	32	16
Total number of obligors	121	125	118	125

CDS spread log returns

Tests and decisions	1-day		5-day	
	2005-07	2007-09	2005-07	2007-09
ADF: Reject H_0	121	125	115	125
Philips-Perron: Reject H_0	121	125	118	125
KPSS: Cannot reject H_0	121	118	117	125
Total number of obligors	121	125	118	125

Table 5: Number of obligors that reject the null hypothesis of Augmented Dickey-Fuller (ADF) test, reject the null hypothesis of Phillips-Peron (PP) test and cannot reject the null hypothesis of Kwiatkowski, Phillips, Schmidt and Shin (KPSS) test at 95% confidence level for (1) CDS spreads and (2) spread returns time series. CDS spreads and spread returns are observed at 1- and 5-day time interval.

Linear autocorrelations of asset returns are often insignificant except for very small intraday time scales [13]. However, Figure 2 shows that daily spread returns exhibit positive autocorrelation for small time lags. Illiquidity may be one cause for the presence of positive autocorrelations.

Table 6 shows that the positive autocorrelations appear to diminish when the observation interval increases. In particular, while we can reject the null hypothesis of Ljung-Box test for all daily spread return series, we can no longer reject the test in many cases for 5-day and 10-day spread returns at 95% level. Our result is further supported by Figure 3 which shows the number of obligors that have significant autocorrelation coefficients at different time lags. Partial autocorrelations also exhibit similar features but they will not be shown this paper.

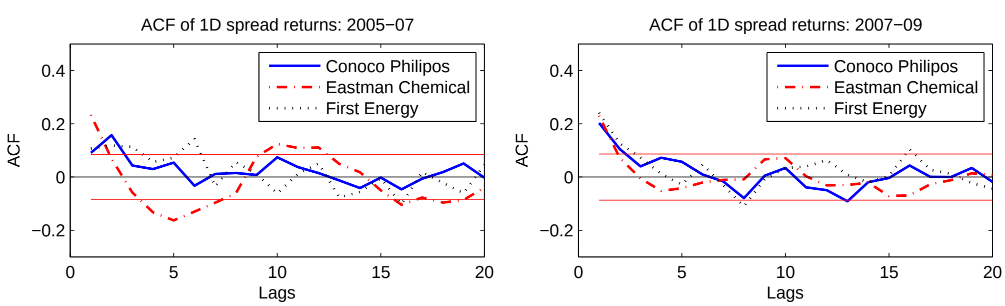

Figure 2: Sample autocorrelation function of daily spread returns in 2005-07 and 2007-09. Obligors: Conoco Philips, Eastman Chemical and First Energy . The 95% confidence interval bounds are computed under the hypothesis that the time series is an i.i.d. process.

3.3 Heavy tails

As early as the 1960s, Mandelbrot [13, 47] pointed out the insufficiency of the normal distribution for modeling the marginal distribution of asset returns and their heavy-tailed character. We observe similar features in CDS spreads:

Property 3 (Heavy-tailed distribution). *CDS spread returns appear to have two-sided heavy tails. In 2005-07, daily spread returns have heavier right tails which have tail indices in the range of* $2 \sim 4$*, and lighter left*

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	Obs interval (day)	ACF (lags) 1	2	3	4	5	10	20	Ljung-Box test Q(5)		Q(10)		Q(20)	
2005-07														
Conoco Philips	1	0.09	0.16	0.04	0.03	0.05	0.07	0.00	22.30	(0.00)	26.40	(0.00)	31.49	(0.05)
	5	-0.08	0.19	-0.12	0.06	-0.10	0.02	-0.08	8.42	(0.13)	10.56	(0.39)	26.16	(0.16)
	10	0.13	-0.11	0.01	-0.07	-0.04	-0.21	-0.06	2.22	(0.82)	19.46	(0.03)	27.10	(0.13)
Eastman Chemical	1	0.23	0.07	-0.06	-0.13	-0.16	0.12	-0.04	61.46	(0.00)	91.31	(0.00)	129.26	(0.00)
	5	-0.36	0.28	-0.20	-0.01	-0.06	0.02	-0.07	29.53	(0.00)	31.65	(0.00)	47.80	(0.00)
	10	0.00	-0.16	-0.11	-0.06	0.05	0.21	-0.07	2.76	(0.74)	12.18	(0.27)	21.41	(0.37)
First Energy	1	0.11	0.12	0.11	0.06	0.07	-0.06	0.05	27.09	(0.00)	43.90	(0.00)	60.84	(0.00)
	5	0.41	0.04	-0.15	-0.16	-0.15	0.17	-0.09	28.50	(0.00)	44.06	(0.00)	66.17	(0.00)
	10	0.14	-0.23	-0.30	-0.18	0.21	-0.13	0.09	15.25	(0.01)	25.81	(0.00)	32.18	(0.04)
2007-09														
Conoco Philips	1	0.20	0.11	0.04	0.07	0.06	0.03	-0.02	33.61	(0.00)	37.86	(0.00)	46.65	(0.00)
	5	0.19	-0.11	-0.07	0.04	-0.03	-0.08	-0.09	6.00	(0.31)	8.89	(0.54)	24.45	(0.22)
	10	-0.11	0.02	0.00	-0.08	-0.15	-0.19	0.21	2.40	(0.79)	12.22	(0.27)	23.82	(0.25)
Eastman Chemical	1	0.23	0.07	-0.01	-0.05	-0.04	0.07	0.01	33.71	(0.00)	39.24	(0.00)	46.57	(0.00)
	5	0.09	0.00	-0.06	-0.05	0.04	-0.10	0.09	1.58	(0.90)	5.58	(0.85)	15.22	(0.76)
	10	-0.08	-0.05	0.14	0.03	-0.13	0.02	0.07	2.63	(0.76)	6.65	(0.76)	12.88	(0.88)
First Energy	1	0.24	0.13	0.07	0.01	-0.03	0.03	-0.04	43.39	(0.00)	52.37	(0.00)	63.66	(0.00)
	5	0.03	-0.09	0.20	-0.11	-0.16	-0.05	0.01	9.64	(0.09)	15.34	(0.12)	21.83	(0.35)
	10	0.00	-0.01	-0.09	-0.15	-0.16	-0.10	0.15	3.30	(0.65)	7.25	(0.70)	21.30	(0.38)

Table 6: Autocorrelation coefficients of 1, 5 and 10-day spread returns. Ljung-Box test statistics and p-values (in brackets) for 5, 10 and 20 lags.

tails which have tail indices in the range of $3 \sim 6$. *In 2007-09, CDS spread daily return tails are symmetric and have tail indices in the range of* $3 \sim 6$.

The heavy-tailed character of CDS spread returns can be clearly seen from the quantile plots in Figure 4 and the Hill estimators [37] for the tail indices in Figure 5. Since both left and right tails appear to be heavier than those implied by the normal distribution, models which only allow upward jumps, such as affine-jump diffusion models [24] and non-Gaussian Ornstein-Uhlenbeck models with positive jumps [10] may not be sufficient to explain the two-sided heavy-tailed distribution.

We observe that almost all obligors have tail indices are larger than 2, which suggests that the spread returns have finite variances. Hypothesis testing on tail indices is complicated by the fact that spread returns are autocorrelated: we will perform a more detailed analysis below for conditional spread returns.

3.4 Nonlinear serial dependence

Property 4 (Volatility clustering in CDS spreads)**.** *Daily spread returns exhibit volatility clustering and conditional heteroscedasticity. In particular, absolute values of CDS spread returns exhibit significant positive autocorrelation.*

This effect, illustrated in Figure 6 and Figure 7, is a quantitative signature of volatility clustering: large price variations are more likely to be followed by large price variations.

We further investigate this property by performing the White test [58] on the CDS spread returns. For the daily spread returns, most of the obligors, 83 out of 121 in 2005-07 and 75 out of 125 in 2007-09, reject the null hypothesis that the residual variance is constant at 95% confidence level. On the other hand, for 5-day spread returns, only a relatively small number of obligors, 50 out of 118 in 2005-07 and 20 out of 125 in 2007-09, reject the null hypothesis.

This property is traditionally modeled using ARMA-GARCH models [8, 27]. Thus, we estimate an ARMA-GARCH model with i.i.d. Student-t innovations for 1-day and 5-day spread returns by using maximum likelihood. Orders of the models are chosen based on Akaike information criterion (AIC). For the daily spread returns, most obligors, 105 out of 121 in 2005-07 and 119 out of 125 in 2007-09, have at last one statistically significant GARCH coefficients at 95% level. On the other hand, for the 5-day spread returns, there are 74 out of 118 obligors in 2005-07 and 36 out of 125 obligors in 2007-09 have statistically significant GARCH coefficients at 95% level, which is significantly less than the case for the daily spread returns.

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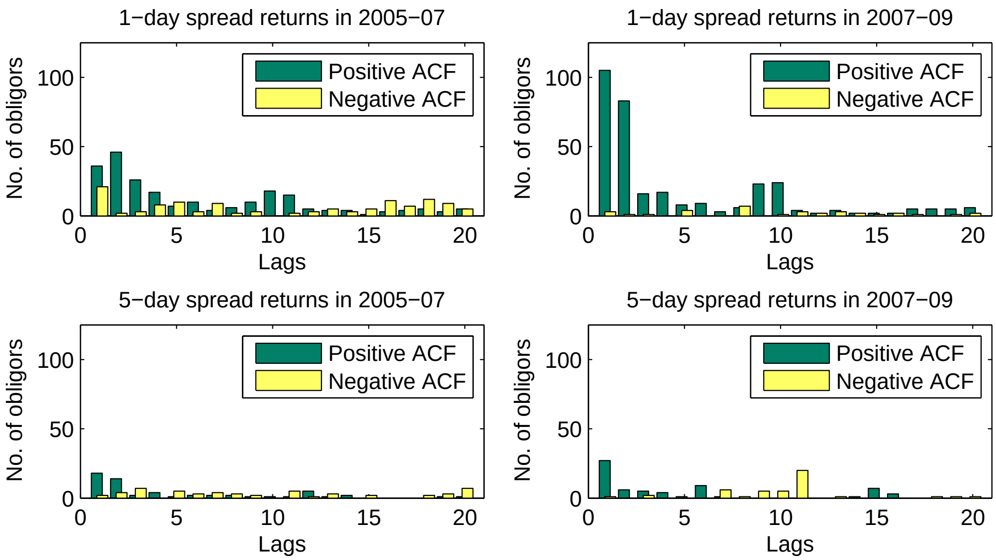

Figure 3: Number of obligors whose spread returns have statistically significant positive and negative autocorrelations at each lag at 95% confidence level. The critical values for statistical testing is computed under the hypothesis that the time series is an i.i.d. process.

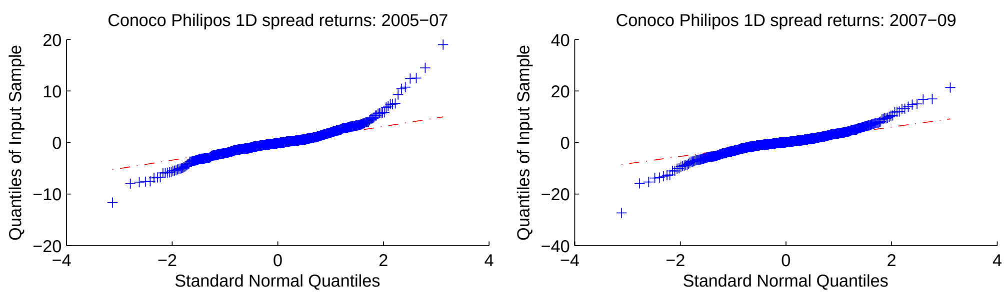

Figure 4: Quantile plots of 1-day spread returns of Conoco Philips vs normal distribution.

3.5 Absence of correlation between spread returns and changes in spread volatility

Property 5. *No significant correlation is observed between spread return and moves in (realized) volatility of spreads.*

Figure 8 shows that, for Conoco Philips, there is no significant linear relationship between the daily returns of the 20-day realized spread volatility and the CDS spread daily returns. Then for each obligor, we regress the realized volatility returns with different rolling windows against the spread returns and check whether the regression models have positive slopes. Table 7 shows that, for short rolling windows, only less than half of the obligors give positive slopes in the OLS regression at 95% confidence level. Although more obligors give positive slopes when the rolling window gets larger, Table 8 shows that the relationship between returns of the realized volatilities and the CDS spread returns is not linear, as the R^2 of the OLS regression is extremely small.

This property shows that asymmetric conditional volatility model such as GJR model [32] is not necessary for modeling the CDS spreads.

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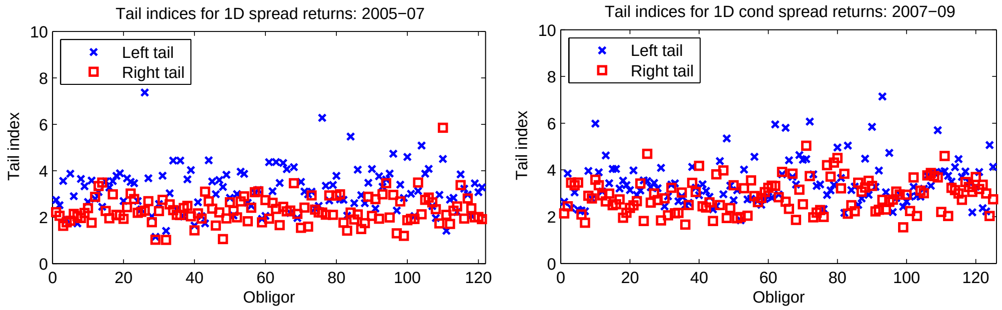

Figure 5: Hill estimators for the tail indices of CDS spread daily returns. The number of largest (smallest) observations for the estimation is equal to the maximum between 2.5% of the sample size.

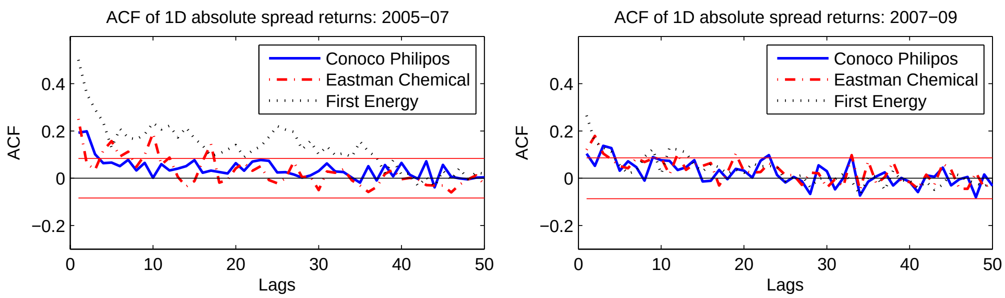

Figure 6: Sample autocorrelation function of daily absolute spread returns in 2005-07 and 2007-09. Obligors: Conoco Philips, Eastman Chemical and First Energy . The 95% confidence interval bounds are computed under the hypothesis that the time series is an i.i.d. noise.

3.6 Heavy-tailed conditional distributions

Heavy tails in unconditional distributions of spread returns can be due to the conditional heteroskedasticity observed above. So it is natural to investigate whether heavy tails persist after we correct for this heteroskedasticity. We estimate an ARMA-GARCH model as described in section 3.4 and study the resulting standardized residuals, the *conditional spread returns*.

Property 6 (Heavy-tailed conditional distribution)**.** *Even after accounting for heteroskedasticity and volatility clustering using an ARMA-GARCH model, the conditional distribution (of residuals) exhibits heavy tails. In 2005-07, conditional spread returns have heavier right tails with tail indices in the range of* $2 \sim 4$ *and lighter left tails with tail indices in the range of* $3 \sim 6$*. In 2007-09, conditional spread return tails are symmetric with tail indices in the range of* $3 \sim 5$*. We cannot reject the null hypothesis that the conditional spread returns have tail indices larger than or equal to 2, which suggests that conditional spread returns have finite variance.*

Although the heavy tails of the spread returns are partly contributed from conditional heteroscedasticity, quantile plots in Figure 9 and the Hill estimators for tail indices in Figure 10 show that the conditional spread returns also exhibit heavy tails. We estimate the confidence intervals for the tail indices using the asymptotic normality of the Hill estimators under i.i.d. assumptions for conditional spread returns [33]. From Figure 11, we observe that almost none of the confidence intervals contains the value 1, and most intervals contain values larger than or equal to 2. This suggests that the conditional spread returns have finite variances.

3.7 Positive serial dependence in extreme values

Property 7. *Extreme CDS spread returns exhibit positive serial dependence, especially for large upward movements and small time lags at 1 or 2 days.*

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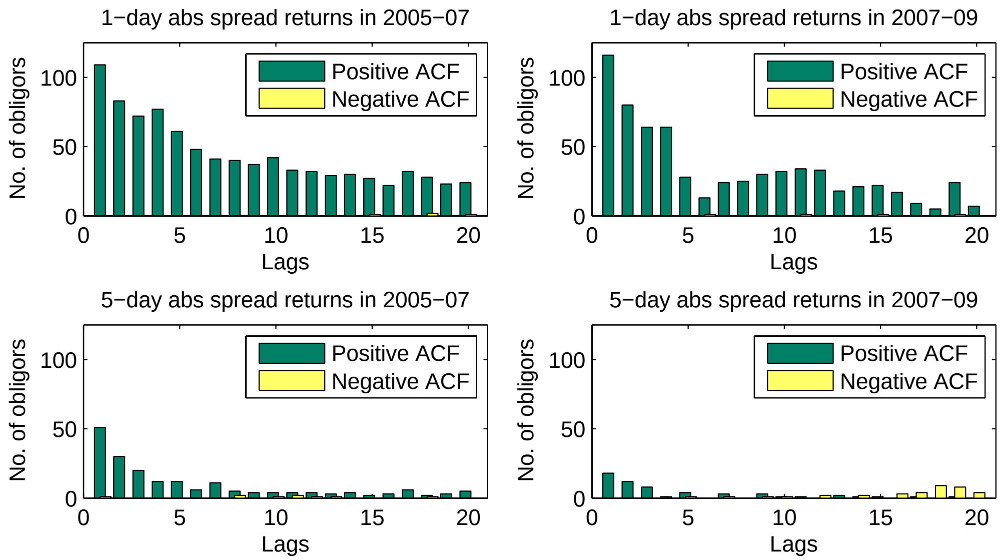

Figure 7: Number of obligors whose absolute spread returns have statistically significant positive and negative autocorrelations at each lag at 95% confidence level. The critical values for statistical testing is computed under the hypothesis that the time series is a Gaussian white noise.

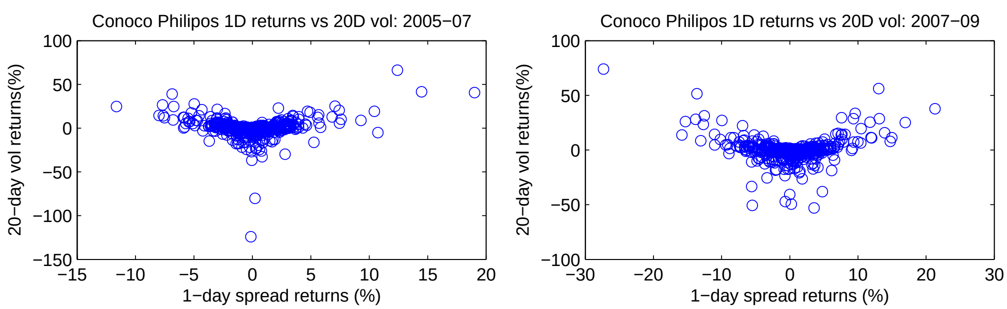

Figure 8: Daily spread returns vs 20-day realized volatility returns of Conoco Philips.

Given the presence of heavy tails, it is interesting to see whether extreme moves in CDS spreads are isolated occurrences or whether they exhibit any serial dependence. In order to examine the serial dependence of extreme values, Davis and Mikosch [20] introduce the *extremogram*, in the framework of multivariate regularly varying processes (see Resnick [55] for definitions).

Let $(X_t)_{t=0,1,\ldots}$ be a (strictly) stationary and regularly varying process with tail index α. Consider a sequence of "high quantiles" $a_m \to \infty$ as $m \to \infty$ such that the probability that an observation exceeds a_m is of order $1/m$: $P(|X_0| > a_m) \sim m^{-1}$. Davis and Mikosch [20] define the (right tail) *extremogram* $(\rho^+(k), k \geq 1)$ of X as, in the case $P(X_0 > a_m) > 0$, as

$$\rho^+(k) = \lim_{m\to\infty} P(X_k > a_m | X_0 > a_m) = \lim_{m\to\infty} \frac{P(X_0 > a_m, X_k > a_m)}{P(X_0 > a_m)}$$

$\rho^+(k) \in [0,1]$ behaves intuitively as a "tail autocorrelation" function: a large positive value of $\rho^+(k)$ indicates serial dependence in the large values X. Similarly, the left tail extremogram $(\rho^-(k), k \geq 1)$ is defined as

$$\rho^-(k) = \lim_{m\to\infty} P(X_k < -a_m | X_0 < -a_m) = \lim_{m\to\infty} \frac{P(X_0 < -a_m, X_k < -a_m)}{P(X_0 < -a_m)}.$$

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Period	Slope	Realized volatilities rolling window (in days)			
		5	10	20	50
2005-07	Positive	49	62	73	80
	Negative	11	9	12	13
2007-09	Positive	32	52	67	81
	Negative	1	4	7	9

Table 7: Number of obligors that appear to have positive or negative slopes at 95% confidence level in the regression of realized volatility daily returns against CDS spread daily returns. Realized volatilities are computing using 5, 10, 20, 50-day rolling windows.

Period	Realized volatilities rolling window (in days)			
	5	10	20	50
2005-07	0.016	0.031	0.043	0.062
2007-09	0.007	0.013	0.021	0.033

Table 8: Average R^2, across all sample obligors, of the OLS regression of realized volatility daily returns against CDS spread daily returns.

These extremograms may be estimated by their empirical counterparts:

$$\hat{\rho}^+(h) = \frac{\sum_{t=1}^{n-h} 1_{X_t > a_m, X_{t+h} > a_m}}{\sum_{t=1}^{n} 1_{X_t > a_m}}, \quad \hat{\rho}^-(h) = \frac{\sum_{t=1}^{n-h} 1_{X_t < -a_m, X_{t+h} < -a_m}}{\sum_{t=1}^{n} 1_{X_t < -a_m}}, \tag{1}$$

where n is the number of samples, and (a_m) is chosen such that $P(|X_0| > a_m) \sim m^{-1}$, $m \to \infty$ and $m/n = o(1)$. (1) are called the *empirical extremograms*.

We choose the threshold a_m to be the 95% (resp. 5%) quantile for the right (resp. left) tail extremograms. We construct the confidence bound in the empirical extremogram via a bootstrap method: we randomly permute the time series and compute an empirical extremogram for each shuffled sample. At each lag, we use the 95% percentile across the simulated empirical extremogram values to be the 95% confidence bound. We refer readers to [21] for details on computing the confidence bound by using bootstrap method.

Figure 12 shows the empirical extremograms (1) for the daily spread return of Eastman Chemical. The empirical extremogram appears to be significantly larger than the 95% confidence bounds for 1 day. As shown by Davis and Mikosch [20], this may come from the autocorrelation and conditional heteroscedasticity, which we have observed in section 3.2 and section 3.4.

Davis and Mikosch [20] also show that, under certain conditions, the empirical extremogram (1) follows a multivariate normal distribution asymptotically:

$$\sqrt{n/m}\left[\hat{\rho}^+(i) - \rho^+(i)\right]_{i=1,\dots,h} \xrightarrow{d} N(0, F\Sigma F'),$$

where F and Σ are matrices which depend on the law of (X_t). We refer readers to [20] for details of this central limit theorem. Using this asymptotic normality, we construct confidence intervals for the empirical extremograms. Figure 13 shows the number of obligors whose 95% confidence intervals do not contain zero, i.e. the corresponding empirical extremogram value is significantly positive. In both periods, more than half of the obligors appear to be serially dependent for large upward movements at small lags (1 and 2 days). On the other hand, serial dependence of large downward movements is not as common.

3.8 Co-movements in CDS spreads

Study of co-movements in CDS spreads provides strong evidence for dependence across obligors of CDS spread returns. Small co-movements may be studies using Pearson correlation coefficients, while large co-movements tend to display different dependence patterns and are better characterized using tail dependence coefficients[20, 55].

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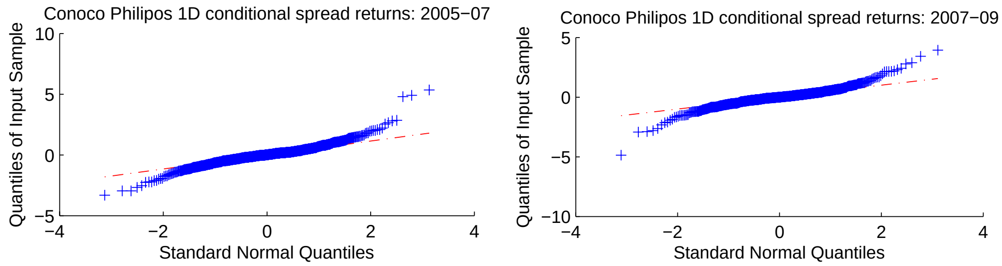

Figure 9: Quantile plots of Conoco Philipos conditional spread returns vs normal distribution.

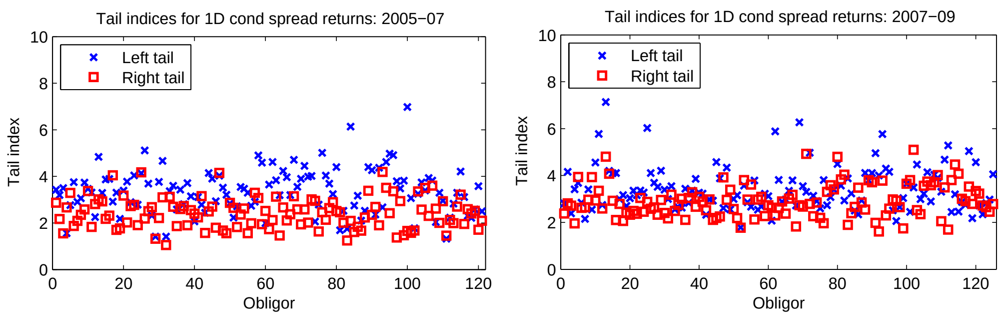

Figure 10: Hill estimators for the tail indices of conditional spread returns from ARMA-GARCH models fitted to daily spread returns. The number of largest (smallest) observations for the estimation is equal to the maximum between 2.5% of the sample size.

Property 8. *Cross-obligor correlations of CDS spread returns increase substantially from the range of* $0.0 \sim 0.2$ *in 2005-07 to the range of* $0.3 \sim 0.5$ *in 2007-09. Correlations are similar within each industrial sector and rating category.*

To display patterns observed in the correlation coefficients, we plot all the sample correlation coefficients in Figure 14 for each sample period. The cross-obligor correlations of spread returns have increased substantially during the subprime crisis. This suggests that after the series of bankruptcies and economic crisis since 2007, investors expect the shifting in credit quality, which can be represented by the changes of the CDS spreads to be more correlated across obligors.

If two obligors belong to the same group, e.g. rating category and industrial sector, it is common to expect that they are subjected to similar risk factors. Therefore, we may expect a higher correlation in spread returns between two obligors in the same group than two obligors in different groups. However, we do not observe this in Figure 15 and 16. This implies that a factor model for CDS spreads with rating or industry specific factors may not be necessary.

Property 9. *CDS spreads appear to have large co-movements. Upward co-movements are common in both 2005-07 and 2007-09, while downward co-movements are more common in 2007-09 than in 2005-07.*

In order to study whether the CDS spreads have large co-movements, we consider the conditional right tail probability $P(r_t^j > q^j | r_t^i > q^i)$ where (r_t^i, r_t^j) are the spread returns for obligor i and j at time t and (q^i, q^j) are some large constants. Similarly, we can consider the conditional left tail probability to study downward common jumps. We say the CDS spreads of obligor i and j are asymptotically independent if this probability is close to 0.

In Figure 17, we compute the natural estimators for the conditional left and right tail probabilities

$$\hat{p}^- = \frac{\sum_t 1_{r_t^i < h^i, r_t^j < h^j}}{\sum_t 1_{r_t^i < h^i}}, \quad \hat{p}^+ = \frac{\sum_t 1_{r_t^i > q^i, r_t^j > q^j}}{\sum_t 1_{r_t^i > q^i}} \tag{2}$$

where h^i and q^i are chosen to be the 5% and 95% quantile of obligor i's daily spread returns respectively.

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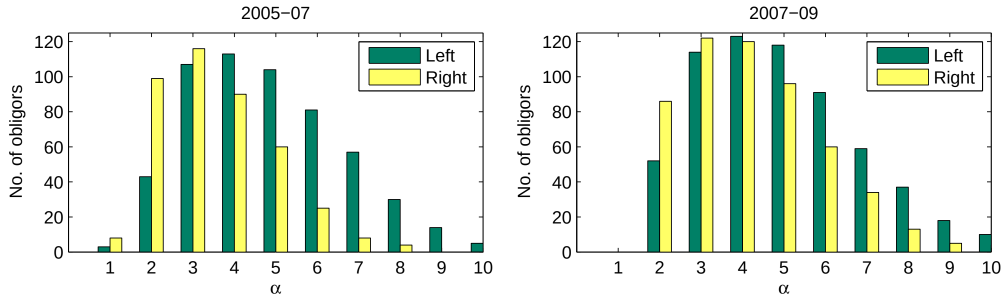

Figure 11: Number of obligors whose 95% confidence interval for the Hill estimator contains α. Data series: conditional spread returns from ARMA-GARCH models.

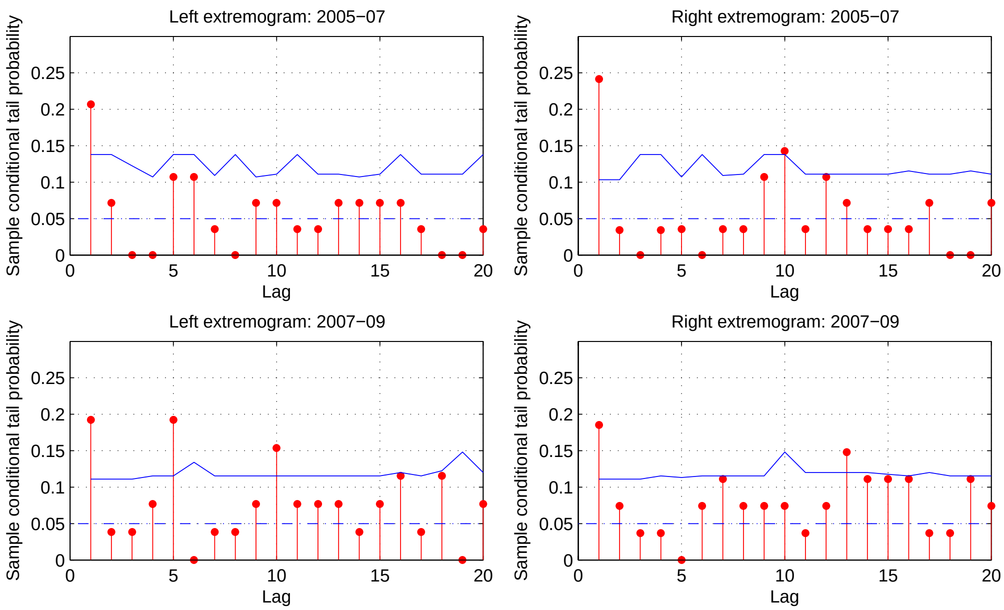

Figure 12: Empirical extremograms for daily spread returns of Eastman Chemical in 2005-07 (top) and 2007-09 (bottom). For the left tail estimates (resp. right tail estimates), threshold a_m is equal to 5% (resp. 95%) quantile of the sample. Blue solid line is the 95% confidence bound constructed from randomly permuted data series. Blue dotted line is equal to 5%, which is the theoretical value for an independent series.

Observe that for many cases, the conditional probabilities are significantly different from the independence case, especially in 2007-09.

In order to perform a more robust inference on the large co-movements, we follow Coles, Heffernan and Tawn [11] and study the right tail dependence measure

$$\chi = \lim_{u\to\infty} \frac{2\ln(P(U^j > u))}{\ln(P(U^i > u, U^j > u))} - 1 \tag{3}$$

where U^i is the Fréchet transformation of the spread return for obligor i. Values of $\chi > 0$, $\chi = 0$ and $\chi < 0$ correspond respectively to when the spread returns of the two obligors are positively associated in extremes, independent and negatively associated respectively. Under a broad set of conditions [45, 46] the following estimator of χ

$$\hat{\chi} = 2/\hat{\alpha} - 1$$

is consistent, where $\hat{\alpha}$ is the Hill estimator for the random variable $Z = \min(U^i, U^j)$. A similar analysis can be performed for the left tail dependence measure (see [38, Ch 7.2.]).

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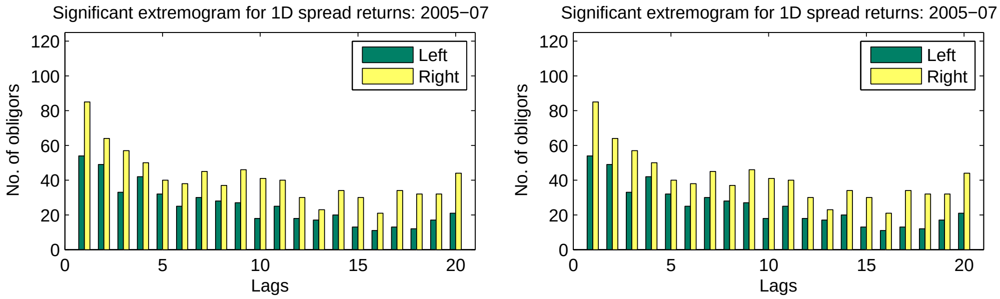

Figure 13: Number of obligors in which the 95% confidence intervals for the empirical extremograms do not contain zero. The confidence intervals are computed from asymptotic normality.

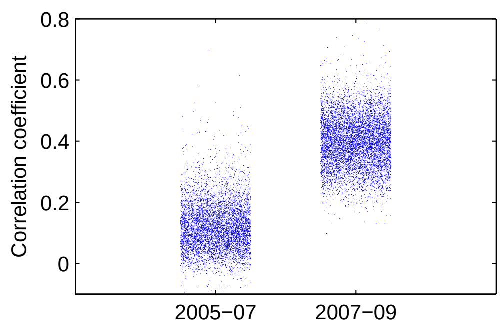

Figure 14: Jitter plot of cross-obligor correlations of daily spread returns.

At 95% confidence level, we find that about 73% of the obligor pairs reject the hypothesis that $\chi = 0$ for large upward co-movements in both 2005-07 and 2007-09. On the other hand, 92% of the obligor pairs reject that $\chi = 0$ for large downward co-movements in 2007-09 and there is 38% in 2005-07. This shows strong evidence that CDS spreads exhibit large co-movements, especially for upward jumps. On the other hand, downward co-movements are more common in the subprime crisis period. Note that the results on testing large co-movements for the conditional spread returns from ARMA-GARCH models are similar and we will not further discuss in this section.

3.9 Principal component analysis

Principal component analysis (PCA) provides insights into the co-movements of CDS spreads.

Property 10.

- *The first principal component represents parallel moves in the CDS spreads and accounts for 12% (resp. 40%) of the daily variance of CDS spreads in 2005-07 (resp. 2007-09).*
- *The main contribution to the variance comes from large idiosyncratic moves ("jumps").*

Table 9 shows the CDS spread return variance explained by the principal component factors. Notice that a relatively large number of factors is needed to explain substantial amount of the variance. This suggests that the main contribution of the variance comes from idiosyncratic variation, especially in 2005-07. In fact, these observations also hold for conditional spread returns, which are shown in Table 10.

Since our modeling approach in section 5 will involve specification of the condition spread returns, we will focus on it (instead of the unconditional one) in the remaining of this section. Nevertheless, we find that the conclusion drawn from the conditional spread return is very similar to the unconditional one. From Figure 18 and 19 we observe that the first factor loadings are all positive and the magnitudes are similar across the obligors. This shows that the main driver of the underlying risk is roughly equal to an equally weighted CDS portfolio which can be approximated by the credit indices such as CDX. Figure 18 shows some obvious

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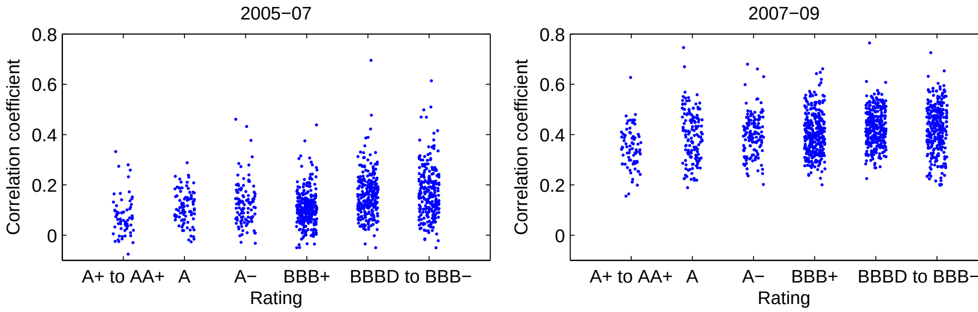

Figure 15: Jitter plots of cross-obligor correlations coefficients in different rating categories of daily spread returns.

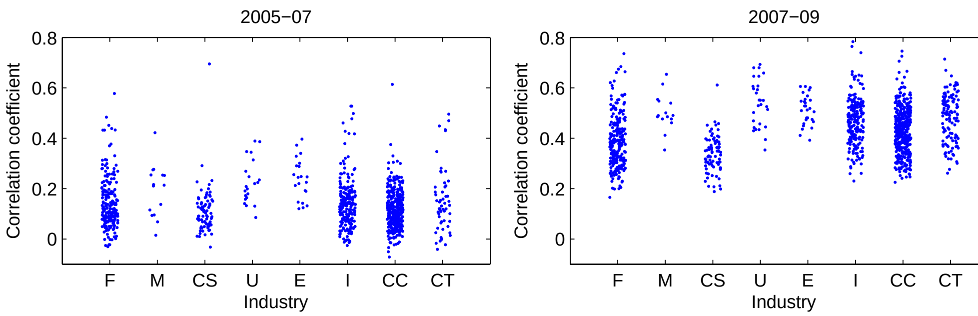

Figure 16: Jitter plots of cross-obligor correlations in different industrial sectors of daily spread returns. F: Financial, M: Materials, CS: Consumer Stable, U: Utilities, E: Energy, I: Industrial, CC: Consumer Cyclical, CT: Communication and Technology.

spikes in the factor time series especially in 2005-07. While the spikes are mostly from the second and higher order factors, it suggests that some extreme moves are due to idiosyncratic risk.

Period		No. of Principal Components							
		1	2	3	4	5	25	80	90
2005-07	% Explained	12.1%	5.3%	4.3%	3.8%	3.0%	1.1%	0.3%	0.2%
	% Cumulative	12.1%	17.4%	21.7%	25.5%	28.5%	61.3%	93.8%	96.3%
2007-09	% Explained	40.1%	3.6%	2.7%	2.3%	2.0%	0.7%	0.2%	0.2%
	% Cumulative	40.1%	43.7%	46.5%	48.8%	50.7%	71.1%	93.7%	95.8%

Table 9: Percentage of variance that can be explained by the principal components. 112 obligors in 2005-07 period and 123 obligors in 2007-09 period. Data series: unconditional CDS spread daily returns.

Property 11 (PC factor distribution)**.** *PC factors appear to have heavy tails with tail indices are in the range of* $2 \sim 5$ *for the first few factors and increase to the range of* $4 \sim 10$ *for the last few factors. The distribution of the first four PC factors are well represented by a Student-t distribution.*

Figure 20 shows the tail indices for the PC factors. In both periods, we observe that the 95% confidence intervals of all tail index estimators contain values equal to or larger than 2, which implies that the PC factors have finite variances.

According to the Bayesian information criteria (BIC) shown in Table 11, Student t distribution appears to be the best fit for the first few factors in both periods. On the other hand, normal distribution provides the worst fit while the first few factors appear to have heavy-tailed distributions.

Property 12 (Large moves in the first principal component)**.** *Jumps in the first principal component are not only related to credit events, but also related to other market information such as changes in interest rates and economic outlook.*

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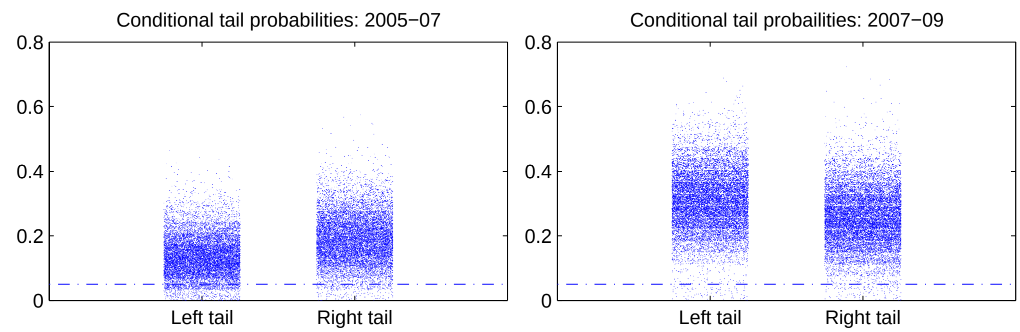

Figure 17: Estimators, $\hat{p}^-$ and $\hat{p}^+$, for the conditional probability of having a large CDS spread move in an obligor given that there is a large CDS spread move in other obligor. Each point represents the conditional tail probability for a pair of obligors. Blue dotted lines at 5% level represents the estimator value for independent series.

Period		Principal Component							
		1	2	3	4	5	25	80	90
2005-07	% Explained	12.1%	3.2%	2.9%	2.6%	2.1%	1.1%	0.4%	0.3%
	% Cumulative	12.1%	15.2%	18.2%	20.7%	22.8%	52.4%	90.9%	94.5%
2007-09	% Explained	37.3%	2.6%	1.8%	1.7%	1.6%	0.8%	0.3%	0.2%
	% Cumulative	37.3%	39.9%	41.8%	43.4%	45.0%	65.3%	92.0%	94.6%

Table 10: Percentage of variance that can be explained by the principal components. 112 obligors in 2005-07 period and 123 obligors in 2007-09 period. Data series: conditional spread daily returns from ARMA-GARCH models.

We investigate the causes of jumps in the first PC factor, which sequentially cause large co-movements in the CDS spreads. In particular, we look at the financial market on the day when the first PC factor exhibits large moves. Table 12 and 13 show the headlines of New York Times when the first PC factor has its largest and smallest moves. From the descriptions of the market conditions, we observe that the large moves are related to, not only credit events, such as Lehman bankruptcy on 15 September 2008 and the collapse of two Bear Stearns hedge funds on 20 June 2007, but also other market information, such as changes in interest rates on 18 March 2008 and economic outlook on 15 April 2005 and 15 October 2008.

While some of the previous works have focused on jumps in the CDS spreads when credit events occur [28], our results show that it is not sufficient to explain the common jumps in the CDS spreads. Moreover, as we will show in section 3.10, the first PC factor does not exhibit large movements during some of the important credit events such as Federal bailout of Fannie Mae and Freddie Mac and bankruptcy of General Motors. This leads to the question whether CDS spreads will in fact have significant movements during credit events, which will be further explored in the next section.

3.10 Spread movements during credit events

Property 13. *CDS spreads do not necessarily experience upward jumps during credit events.*

We study the spread returns during the credit events in 2008-09 shown in Table 14. On each event dates, we study two versions of “normalized” spread returns in Figure 21:

- Unconditional normalization: Daily spread returns normalized by the sample standard deviation computed from the data in 2007-09.
- Conditional normalization: Conditional spread returns from ARMA-GARCH fitted to the daily spread returns in 2007-09.

Notice that the patterns between the unconditional normalized spread returns and the conditional spread returns are similar during all credit events. On 15 September 2008 (Lehman bankruptcy) and 29 September 2008 (sale of Wachovia banking operations to Citigroup), most CDS spreads increase substantially. On the other hand, on 8 September 2008 (Federal bailout of Fannie Mae and Freddie Mac) and 1 July 2009 (General

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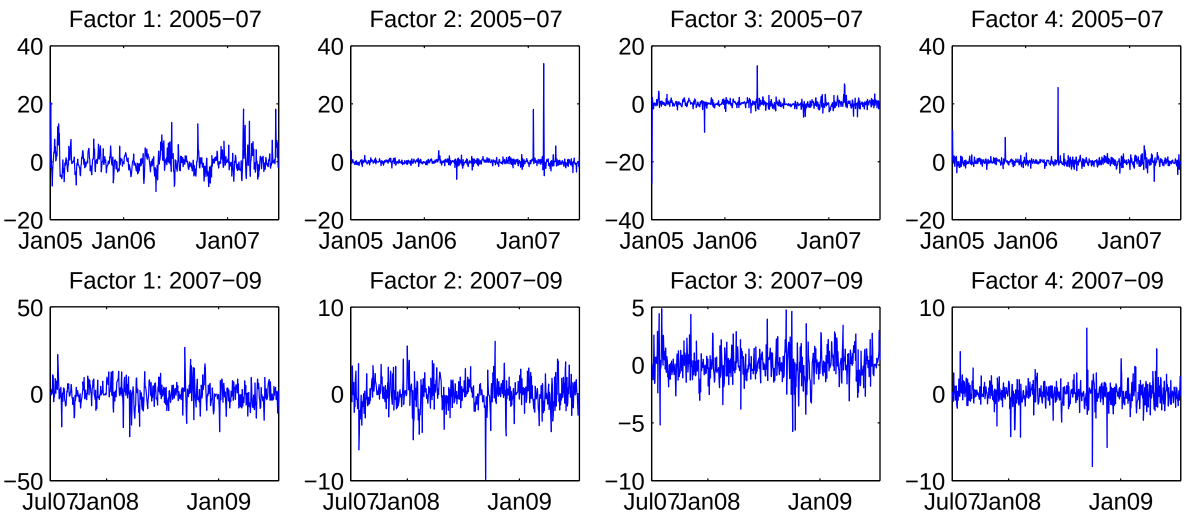

Figure 18: The first four principal component factors.

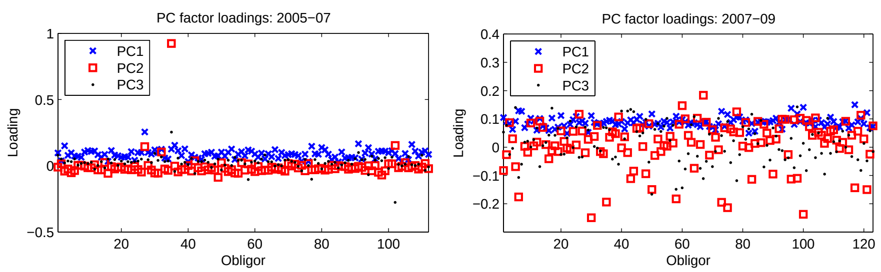

Figure 19: Loadings of the first three principal component factors.

motors bankruptcy), most CDS spreads decrease substantially. In general, we have no evidence that the CDS spreads will move in a particular direction during the credit events. Our observations are consistent with Cont and Kan [16] who study index tranche spreads on credit event dates and find no strong evidence that the index tranche spreads must have upward jumps during credit events.

One explanation of the absence of large changes in the CDS spreads during credit events is that the market has already anticipated the events which no longer appear as "shocks" to the investors. Indeed, Guo, Jarrow and Lin [34] distinguish the *recorded* default date which is defined as the actual announcement date of default, from the *economic* default date which is defined as the first date when the market prices the firms debt as if it has defaulted. An interesting topic will be on examining the changes of CDS spreads during the economic default dates vs the recorded default dates and we will leave it for future research.

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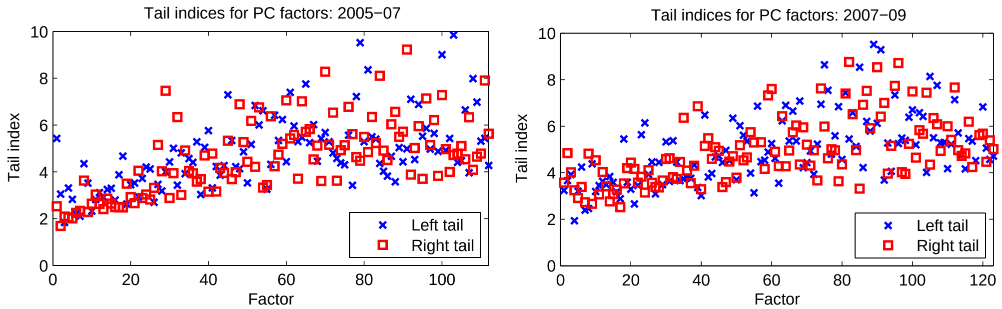

Figure 20: Hill estimators for the tail indices of the PC factors. The number of largest (smallest) observations for estimation is equal to 2.5% of the sample size.

PC factor	Normal	Student t	Double Exp	NIG	Stable
2005-07					
1	1498	1359*	1369	1359*	1372
2	2225	1390	1499	1431	1388*
3	1839	1350*	1418	1374	1356
4	1740	1300*	1338	1316	1311
5	1559	1370*	1399	1381	1379
2007-09					
1	1523	1495*	1512	1501	1506
2	1497	1449*	1453	1452	1466
3	1501	1461*	1477	1465	1476
4	1518	1417*	1430	1425	1425
5	1463	1366*	1370	1370	1380

Table 11: BIC of the PC factors fitted to (1) normal, (2) Student t, (3) double exponential, (4) normal inverse Gaussian and (5) stable distribution. Lowest BIC among the distributions is indicated by *.

Period	PC factor 1	Date	New York Times headlines
2005-07	5.96	15-Apr-05	"Stocks plunge to lowest point since election. I.B.M. earning a factor. Market slump continues amid uncertainty over economy's growth."
	5.09	20-Jun-07	"Bear Stearns staves off collapse of two hedge funds. Several lenders pull back from a larger liquidation, for now."
	4.36	20-Jun-06	"Timber becomes tool in effort to cut estate tax."
2007-09	4.45	15-Sep-08	"Bids to halt financial crisis reshape landscape of Wall Street: Merrill is sold; Failing to find buyer, Lehman set to file for bankruptcy."
	3.62	26-Jul-07	"Global stock markets tumble amid deepening credit fears."
	3.52	15-Oct-08	"After big rally, grim outlook still looms on profits and jobs."

Table 12: The three largest values of the first principal component factor (common factor) in 2005-07 and 2007-09.

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Period	PC factor 1	Date	New York Times headlines
2005-07	-2.49	21-Mar-07	"Fed weighs words about its next move. (The central bank left the overnight Federal funds rate at 5.25 percent, a level unchanged since last June.)"
	-2.30	26-Apr-06	"Second thoughts in Congress on oil tax breaks. New political pressure over record profits as gas prices soar."
	-2.27	28-Nov-05	"U.S. Declines a chance to criticize Yuan policy."
2007-09	-4.53	18-Mar-08	"Fed trims rates sharply, sending the markets up. Cut 3/4 of a point is less than expected signs of split on policy at Central Bank."
	-3.33	24-Mar-08	"JP Morgan in negotiations to raise Bear Stearns bid. Price per share would quintuple to $10 to appease firm's shareholders."
	-3.24	6-Jan-09	"Automakers fear a new normal of low sales. As prices rise, some see $2 gas."

Table 13: The three smallest values of the first principal component factor (common factor) in 2005-07 and 2007-09.

Date	Event	CDX.NA.IG	PC factor 1
14-Jan-08	BoA agreed to purchase Countrywide Financial	-	0.15
17-Mar-08	JPMorgan agreed to purchase Bear Stearns	-	0.39
1-Jul-08	BoA acquired Countrywide Financial	-	0.80
8-Sep-08	Federal takeover of Fannie Mae and Freddie Mac	S1-10	-2.15
15-Sep-08	Lehman Brothers filed for bankruptcy	-	4.45
17-Sep-08	Federal bailout of AIG	-	-0.69
26-Sep-08	Washington Mutual filed for bankruptcy	S1-10	0.26
29-Sep-08	Wachovia sold banking operations to Citigroup	-	0.86
3-Oct-08	Wells Fargo agreed to purchase Wachovia	-	-0.98
8-Dec-08	Tribune Company filed for bankruptcy	S6	-1.59
31-Mar-09	Idearc Inc filed for bankrupcty	S1-7	-0.35
29-May-09	Visteon Corporation filed for bankruptcy	S1	-1.14
1-Jun-09	General Motors filed for bankruptcy	-	-2.21
7-Jul-09	LEAR Corporation filed for bankruptcy	S4	0.19

Table 14: Highlights of credit events in 2008-09. From left to right: the event date or the next business day after the event; description of the events; the CDX.NA.IG series that the obligors belong to; first principal component factor values.

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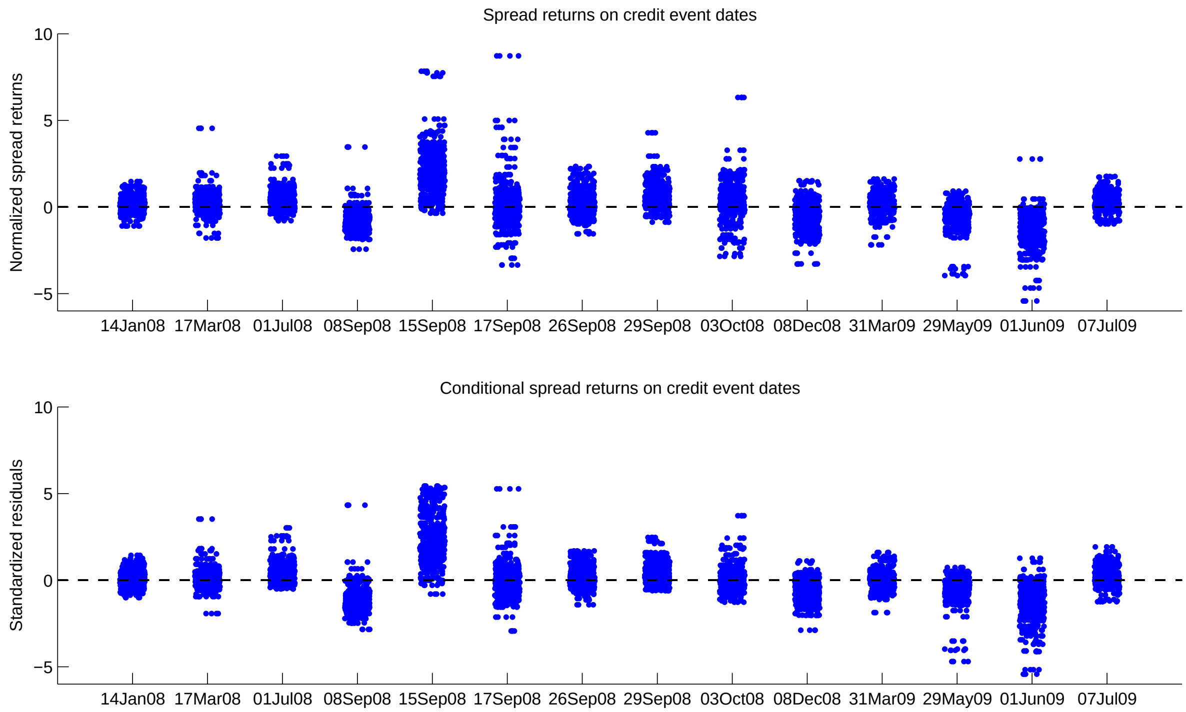

Figure 21: Jitter plots of daily spread returns normalized by the unconditional sample standard deviations (top) and the conditional daily spread returns from ARMA-GARCH models (bottom) during the credit events in Table 14. Each point represents the spread return/conditional spread return of an obligor on the corresponding event date.

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4 Comparison with affine jump-diffusion models

Affine jump-diffusion models [24, 51] are reduced-form models for default risk in which the default time of an obligor i is modeled as a random time with intensity

$$\lambda_t^i = X_t^i + a^i X_t^0 \tag{4}$$

where $(X_t^k), k = 0, i$ are independent affine jump-diffusion processes whose dynamics under a pricing measure $\mathbb{Q}$ is given by

$$dX_t^k = (\kappa_0^k + \kappa_1^{k,\mathbb{Q}} X_t^k)dt + \sigma^k \sqrt{X_t^k} dW_t^{k,\mathbb{Q}} + dJ_t^k \tag{5}$$

where $(W_t^{k,\mathbb{Q}})$ is a standard Brownian motion and (J_t^k) is a compound poisson process with jump intensity $\ell^{k,\mathbb{Q}}$ and jump sizes are exponentially distributed with mean $\mu^{k,\mathbb{Q}}$ under a risk-neutral pricing measure $\mathbb{Q}$. Choosing an affine risk premium as in [29], the risk-neutral intensity under the real-world statistical measure $\mathbb{P}$ is given by

$$\text{Under } \mathbb{P}: \quad dX_t^k = (\kappa_0^k + \kappa_1^{k,\mathbb{P}} X_t^k)dt + \sigma^k \sqrt{X_t^k} dW_t^{k,\mathbb{P}} + dJ_t^k. \tag{6}$$

where $(W_t^{k,\mathbb{P}})$ is a standard Brownian motion under $\mathbb{P}$ and (J_t^k) is now a compound Poisson process with jump intensity $\ell^{k,\mathbb{P}}$ and jump sizes are exponentially distributed with mean $\mu^{k,\mathbb{P}}$ under $\mathbb{P}$.

This model has been widely used in the literature such as pricing and hedging credit derivatives [24, 51, 25, 29, 10, 6], modeling default correlations [24, 51], portfolio selection [31] and counterparty risk modeling [18]. The main advantage of this model is having a closed-form expression for the default probability (see Appendix A and [44]). However, no work has been done to justify the use of affine jump-diffusion model in terms of the empirical properties of CDS spreads.

4.1 A simulation study

We simulate time series of 5-year CDS spreads under the affine jump-diffusion model and study their statistical properties. We use parameters estimated by Feldhütter [29] for CDX.NA.IG.6 CDS and CDO tranche spreads from 30 March 2006 to 30 September 2006 using a Markov Chain Monte Carlo method (MCMC). Parameters for (X_t^0) are equal to

$$\begin{aligned} & (\kappa_0^0, \kappa_1^{0,\mathbb{Q}}, \sigma^0, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}}, \kappa_1^{0,\mathbb{P}}, \ell^{0,\mathbb{P}}, \mu^{0,\mathbb{P}}) \\ = \; & (1.59 \times 10^{-5}, 0.46, 3.66 \times 10^{-2}, 3.18 \times 10^{-3}, 1.23, 0.44, 3.37 \times 10^{-3}, 0.0023) \end{aligned}$$

and parameters for (X_t^i) are equal to

$$\begin{aligned} & \kappa_1^{i,\mathbb{Q}} = \kappa_1^{0,\mathbb{Q}}, \quad \kappa_1^{i,\mathbb{P}} = \kappa_1^{0,\mathbb{P}}, \quad \sigma^i = \sqrt{a^i}\sigma^0, \quad \mu^{i,\mathbb{Q}} = \mu^{0,\mathbb{Q}}, \quad \mu^{i,\mathbb{P}} = \mu^{0,\mathbb{P}}, \\ & w = \frac{a^i \kappa_0^0}{a^i \kappa_0^0 + \kappa_0^i} = \frac{\ell^{0,\mathbb{Q}}}{\ell^{0,\mathbb{Q}} + \ell^{i,\mathbb{Q}}} = \frac{\ell^{0,\mathbb{P}}}{\ell^{0,\mathbb{P}} + \ell^{i,\mathbb{P}}} = 0.9742, \end{aligned}$$

and a^i is equal to the CDS spread of obligor i on 30 March 2006 divided by the average CDS spreads among all obligors on the same day[2]. For each CDX.NA.IG.6 obligor, we simulate 1000 daily observations of 5-year CDS spreads. Appendix B describes our simulation method based on Euler scheme.

Figure 22 shows a simulated time series of 5-year CDS spread and the corresponding daily spread returns. As we can see, there is an upward jump in the spread return on about day 610 which is contributed by the Poisson jump component in the default intensity (4).

4.2 Absence of serial dependence

Figure 23 and 24 show no significant serial correlations in the simulated spread returns. This illustrates that the simple "volatility term", $\sigma\sqrt{X_t^i}$, in the affine jump-diffusion process (6) is insufficient to produce the volatility clustering feature observed in the empirical data. For extreme values, the affine jump-diffusion model also underestimates the dependence where the extremograms are less than 0.1 at all lags comparing to the empirical values of $0.1 \sim 0.3$ in Figure 12.

[2]Each parameter is set to be the median of the estimated distribution by MCMC in [29].

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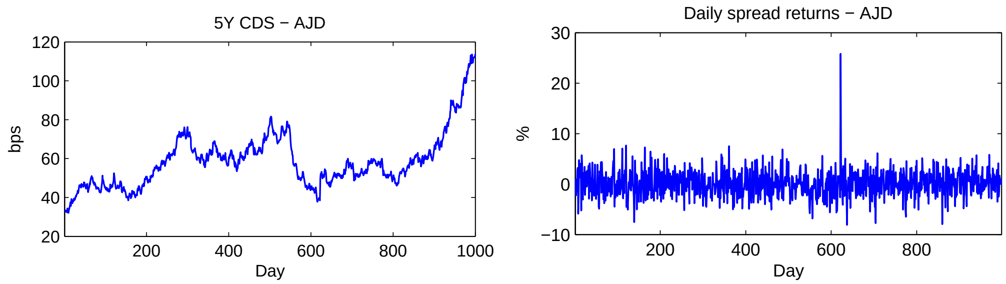

Figure 22: Simulated time series of CDS spreads and daily spread returns from the affine jump-diffusion model.

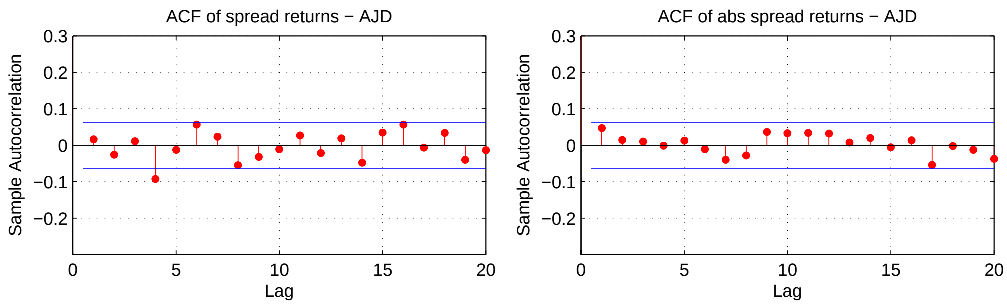

Figure 23: Sample autocorrelation function for simulated daily spread returns and absolute spread returns from the affine jump-diffusion model.

4.3 Distributional properties of CDS spread returns

Quantile plots in Figure 25 show that distribution of the simulated spread returns appear to be very close to the normal distribution except for occasional outliers in the right tail, resulting from upward jumps, as observed in Figure 22. More generally, we observe that the right tail indices are similar to the empirical observations which are in the range of $3 \sim 5$. However, the model substantially underestimates the left tail: left tail indices appear to be in the range of $5 \sim 7$. This shows that the AJD model does not produce the type of two-sided heavy-tailed distributions observed for spread returns, and underestimates the left tails of these returns.

4.4 Co-movements in CDS spread returns

Figure 26 shows that the affine jump-diffusion model, which is estimated by a MCMC method [29], significantly overestimates the probability of having co-movements in the CDS spreads. Comparing to the empirical observations in Figure 14 and Figure 17, the cross-obligor correlations and the probabilities of large co-movements are substantially higher than those from the historical data.

4.5 Principal components of CDS spread moves

Table 15 shows that the first PC factor of the simulated spread returns explains more than 93% of the variance, which is significantly larger than the empirical case.

	No. of Principal Components				
	1	2	3	4	5
% Explained	93.0%	7.0%	0.0%	0.0%	0.0%
% Cumulative	93.0%	100.0%	100.0%	100.0%	100.0%

Table 15: Percentage of variance that can be explained by the principal components. Data: simulated CDS speards based on affine jump-diffusion model.

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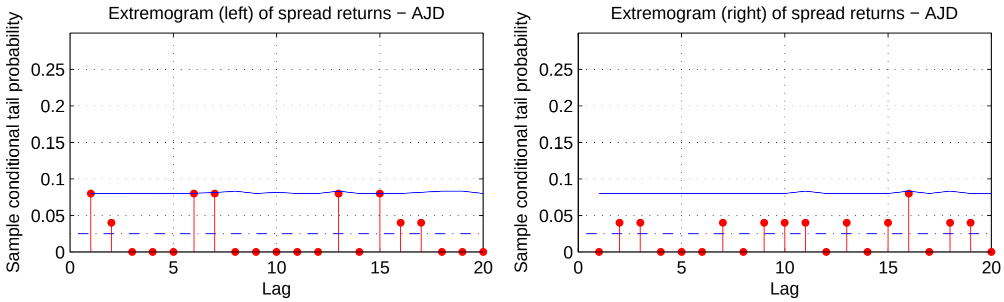

Figure 24: Empirical extremograms for simulated daily spread returns from the affine jump-diffusion model. For the left tail estimates (resp. right tail estimates), threshold a_m is equal to 5% (resp. 95%) quantile of the sample. Blue line is the 95% confidence bound constructed from randomly permuted data series. Blue dotted line is equal to 5%, which is the theoretical value for an independent series.

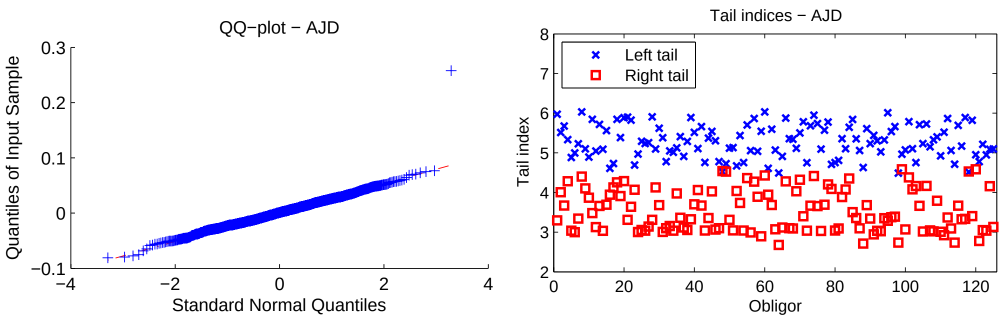

Figure 25: Quantile plots (left) and Hill estimators for the tail indices (right) of simulated CDS spread return from the affine jump-diffusion model.

4.6 Goodness-of-fit vs statistical properties

In the previous studies of the affine jump-diffusion model, Feldhütter [29], Azizpour, Giesecke and Kim [4] conclude that the model fits well to the CDS and CDO tranche spread time series in which the errors between model and market spreads are sufficiently small. However, our simulations, using parameters estimated by Feldhütter [29], show that the goodness-of-fit does not necessarily imply the ability to reproduce the stylized properties.

Indeed, our results show that models can pass goodness-of-fit tests while having statistical properties which are qualitatively different from the data. In section 6, we will show that models which cannot capture the important stylized properties can lead to worse performance than other realistic models when they are applied to the risk management of CDS portfolios.

5 A heavy-tailed multivariate time series model for CDS spreads

When computing Value-at-Risk (VaR) for credit portfolios and initial margin for trading accounts, one needs a model for the loss distribution of the credit derivatives. Based on the empirical properties observed in section 3, we propose a statistical model for CDS spreads.

5.1 A heavy-tailed multivariate AR-GARCH model

Our objective is to build a multivariate time series model for CDS spread returns, which satisfies the following properties, observed in section 3:

- Positive autocorrelations at small lags

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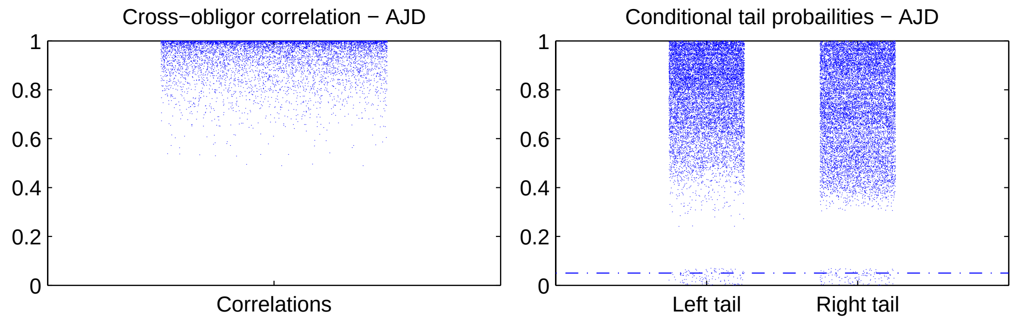

Figure 26: Left: Jitter plot of cross-obligor correlations. Right: Estimators, $\hat{p}^-$ and $\hat{p}^+$, for the conditional probability of having a large CDS spread move in an obligor given that there is a large CDS spread moves in other obligor. Each point represents the conditional tail probability between two obligors. Blue dotted lines at 5% level represents the estimator value for independent series. Data: Simulated spread returns from the affine jump-diffusion model.

- Volatility clustering and conditional heteroscedasticity
- Two-sided heavy-tailed distributions for both unconditional and conditional spread returns
- Heterogeneity of the tail indices for spread returns
- Large co-movements
- A heavy-tailed common factor that drives the parallel shift of CDS spreads

In order to capture positive autocorrelations and conditional heteroscedasticity, we model CDS spread returns as AR(1)-GARCH(1,1) processes where the spread return of, say obligor i, follows

$$
\begin{aligned}
r_t^i &= C^i + \phi^i r_{t-1}^i + \epsilon_t^i, && (7)\\
\epsilon_t^i &= \sigma_t^i Z_t^i && (8)
\end{aligned}
$$

for $t = 0, \Delta t, 2\Delta t, ...$ where (Z_t^i) is an i.i.d. sequence and σ_t^i is the conditional volatility which follows

$$(\sigma_t^i)^2 = K^i + G^i(\sigma_{t-1}^i)^2 + A^i(\epsilon_{t-1}^i)^2 \tag{9}$$

where $K^i > 0$, $G^i \geq 0$, $A^i \geq 0$, $G^i + A^i < 1$. We then assume the heavy-tailed conditional spread returns follow

$$Z_t^i = a^i V_t^0 + b^i V_t^i \tag{10}$$

where (V_t^0) and $(V_t^1, ..., V_t^n)$ are i.i.d. sequences which follow a Student t_{ν^0} distribution and a multivariate Student t distribution with degree of freedom $\nu^1 = \nu^i$ for all i respectively. In this case, Z_t^i, $i = 1, ..., n$, share the same degree of freedom equal to $\min(\nu^0, \nu^1)$. Although this model restricts to the case of homogeneous tail index for conditional spread returns, the simple correlation structure allows efficient simulation to estimate the CDS portfolio loss distribution.

In the rest of this paper, we call this model the *MAG* (**M**ultivariate **A**R-**G**ARCH) model.

5.2 Parameter estimation

The MAG model can be estimated by maximum likelihood, but this approach has two main problems. First, if we maximize the joint likelihood function by considering all CDS time series simultaneously, we need to solve a high dimensional optimization problem which is not necessarily well-posed. Second, if an additional CDS time series is added to the data set, we need to repeat the maximum likelihood estimation again.

In order to overcome these problems, we consider a *quasi maximum likelihood estimation* which allows to break down the estimation into smaller optimization problems for each univariate series. Furthermore, we use CDX.NA.IG on-the-run index as our common risk factor (V_t^0). This is consistent with our earlier analysis in section 3.9 that the first principal component factor is approximately an equally weighted CDS portfolio. The full estimation procedure is as follows:

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1. For each obligor i, Z_t^i is assumed to have a Student t distribution. For each CDS spread series, we estimate the AR(1)-GARCH(1,1) coefficients in (7)-(9) together with the degree of freedom by maximizing the likelihood function. Let $(\hat{Z}_t^i)$ be the standardized residuals for obligor i.

2. An AR(1)-GARCH(1,1) model with Student t distributed noise is fitted to the CDX.NA.IG index returns by maximum likelihood. The degree of freedom, ν^0, of the noise is estimated together with the AR-GARCH coefficients. Let $(\hat{V}_t^0)$ be the standardized residuals for the CDX.NA.IG index.

3. For each obligor i, $(\hat{Z}_t^i)$ is regressed against $(\hat{V}_t^0)$ by ordinary least squares (OLS) method with zero intercept coefficient. The estimator for a^i is set to be the slope coefficient of the regression. Let $(\hat{Y}_t^i)$ be the residuals from the OLS regression.

4. Assume that $(\hat{Y}_t^i)$ are i.i.d. samples of a scaled Student t distribution with degree of freedom $\tilde{\nu}^i$. $\tilde{\nu}^i$ is estimated by maximum likelihood.

5. Degree of freedom for the idiosyncratic risk factor V_t^i is set to be $\nu^i = \frac{1}{n}\sum_{j=1}^n \tilde{\nu}^j$ for all i. Estimator for b^i is set to be $\hat{b}^i = \sqrt{var(\hat{Y}^i)(\nu^i - 2)/\nu^i}$ and estimator for V_t^i is equal to $\hat{V}_t^i = \hat{Y}_t^i/\hat{b}^i$.

6. The correlation parameters for the multivariate t distribution of $(V_t^1, ..., V_t^n)$ is estimated by

$$\hat{\rho}_{i,j} = \sin\left(\frac{\pi}{2}\hat{\tau}_{i,j}\right), \tag{11}$$

   where $\hat{\tau}_{i,j}$, $i, j = 1, ..., n$ are the Kendall tau correlation coefficients of $(\hat{V}_t^i, \hat{V}_t^j)$.

In step 1, we assume that Z_t^i is a Student t variable instead of a weighted sum of two Student t variables. This allows us to consider each time series separately and reduce the high dimensional problem into several one dimensional problems. This approach leads to a quasi maximum likelihood estimation for the AR-GARCH coefficients but not an exact maximum likelihood estimation. Indeed, Newey and Steigerwald [52] show that estimators from quasi maximum likelihood estimation are consistent if either the conditional mean is identically zero, or the assumed and true error PDFs are symmetric about zero. Mathematical verification of consistency is beyond the scope of this paper. Instead, we will backtest our model in section 6. We refer readers to chapter 5 of [38] for more details on quasi maximum likelihood.

In step 2, we consider the standardized residuals of the credit index returns from a AR(1)-GARCH(1,1) model as the common risk factor observations $(\hat{V}_t^0)$. The reason is that we want to filter the positive autocorrelations and conditional heteroscedasticity in the index returns. It is not surprising that the credit index, which is an equally weighted portfolio of CDS, also exhibits similar stylized properties as in the constituent CDS series. Nevertheless, we will not further illustrate the stylized properties of the credit index.

Klüppelberg and Kuhn [39] show that the correlation estimator (11) is consistent and asymptotically normal. However, it does not guarantee that the resulting correlation matrix is positive definite. In those cases, we adjust the correlation matrix by using the eigenvalue shifting method proposed by Rousseeuw and Molenberghs [56]. In particular, we replace the negative eigenvalues of the correlation matrix by a small positive constant. Then, we scale the new matrix so that the diagonal values are equal to 1. We refer readers to [56] for the details of this algorithm.

Figure 27 shows the confidence intervals for the degree of freedom of each OLS residual series $(\hat{Y}_t^i)$, and the estimator for ν^1, which is the average degree of freedom. Estimator for ν^1 appears to be roughly equal to the intersection of the confidence intervals.

Table 5.2 shows the estimated model parameters for three selected obligors. We observe that the degree of freedom for the common factor (index) is larger than the one for the idiosyncratic factors in both sample periods.

5.3 Reproducing stylized properties of CDS spreads

The model estimated as above is observed to have the right qualitative properties in the sense that it matches well the stylized properties of CDS spreads listed in section 3.

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	C	ϕ	K	G	A	a	b	ν^0	ν
2005-07									
Conoco Philips	0.00	-0.02	0.00	0.73	0.23	0.08	0.44	3.39	2.68
	(0.00)	(0.04)	(0.00)	(0.08)	(0.11)	(0.02)	-	(0.51)	-
Eastman Chemical	0.00	0.14	0.00	0.67	0.29	0.13	0.43	3.39	2.68
	(0.00)	(0.04)	(0.00)	(0.08)	(0.14)	(0.02)	-	(0.51)	-
First Energy	0.00	0.04	0.00	0.86	0.14	0.07	0.49	3.39	2.68
	(0.00)	(0.04)	(0.00)	(0.03)	(0.04)	(0.03)	-	(0.51)	-
2007-09									
Conoco Philips	0.00	0.16	0.00	0.89	0.11	0.21	0.47	5.74	3.22
	(0.00)	(0.04)	(0.00)	(0.05)	(0.07)	(0.03)	-	(1.11)	-
Eastman Chemical	0.00	0.19	0.00	0.81	0.15	0.38	0.49	5.74	3.22
	(0.00)	(0.04)	(0.00)	(0.06)	(0.06)	(0.03)	-	(1.11)	-
First Energy	0.00	0.21	0.00	0.64	0.36	0.30	0.53	5.74	3.22
	(0.00)	(0.04)	(0.00)	(0.08)	(0.14)	(0.03)	-	(1.11)	-

Table 16: Estimated parameters for the MAG model. Values in brackets are standard errors of the estimators.

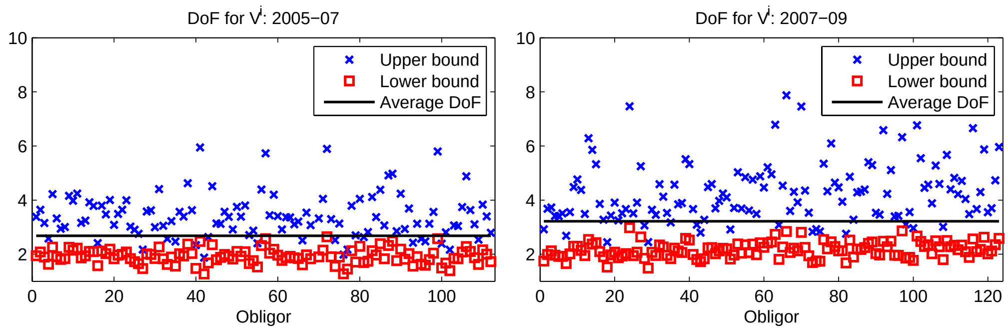

Figure 27: Upper and lower bounds of the 95% confidence intervals for the degree of freedom estimators of V_t^i. Black dotted line represents the average of the degree of freedom estimators across obligors.

We consider model parameters estimated from the 2007-09 sample and simulate CDS time series by using the MAG model. We illustrate univariate properties by using parameters of Eastman Chemical that are shown in Table 5.2. Figure 28, 29 and 30 show the autocorrelation functions, empirical extremograms, quantile plots and the tail indices[3] of the CDS spread returns simulated from the MAG model.

Unlikely the affine jump-diffusion model, the MAG model is able to reproduce the observed stylized properties: the simulated CDS spread returns exhibit positive serial dependence in spread returns, absolute spread returns and extreme spread returns, and they appear to have two-sided heavy-tailed distributions with tail indices in the range of $2 \sim 6$. Moreover, Figure 31 shows that the MAG model also reproduces realistic empirical probability of having co-movements in the CDS spreads.

[3]We also compute tail indices numerically by solving an integral equation for GARCH(1,1) model (see [22, 50]). The results are similar to the simulation study and they will not be shown in this paper.

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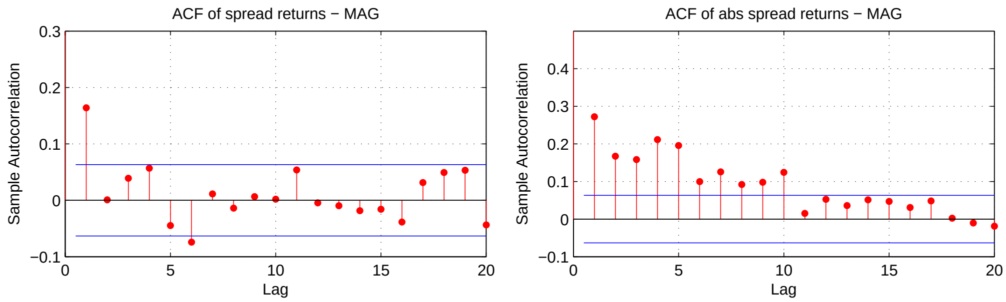

Figure 28: Sample autocorrelation function for simulated daily spread returns and absolute spread returns from the heavy-tailed multivariate AR-GARCH model (MAG).

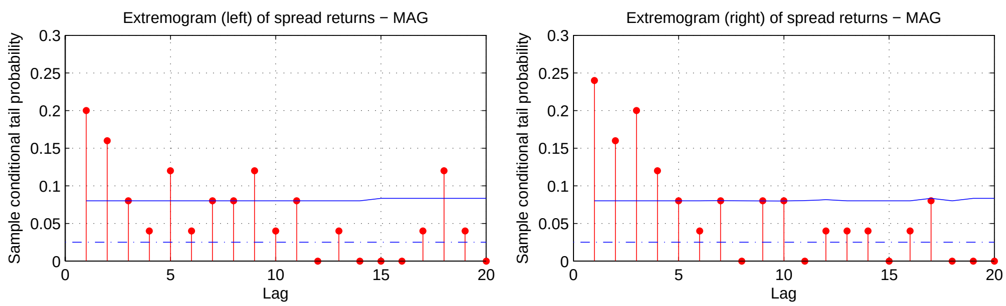

Figure 29: Empirical extremograms for simulated daily spread returns from the heavy-tailed multivariate AR-GARCH model (MAG). For the left tail estimates (resp. right tail estimates), threshold a_m is equal to 5% (resp. 95%) quantile of the sample. Blue line is the 95% confidence bound constructed from randomly permuted data series. Blue dotted line is equal to 5%, which is the theoretical value for an independent series.

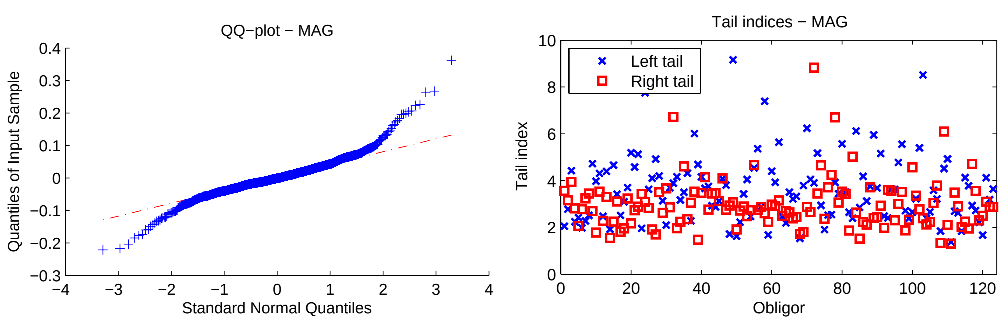

Figure 30: Quantile plots (left) and Hill estimators for the tail indices (right) of simulated CDS spread return from the heavy-tailed multivariate AR-GARCH model (MAG).

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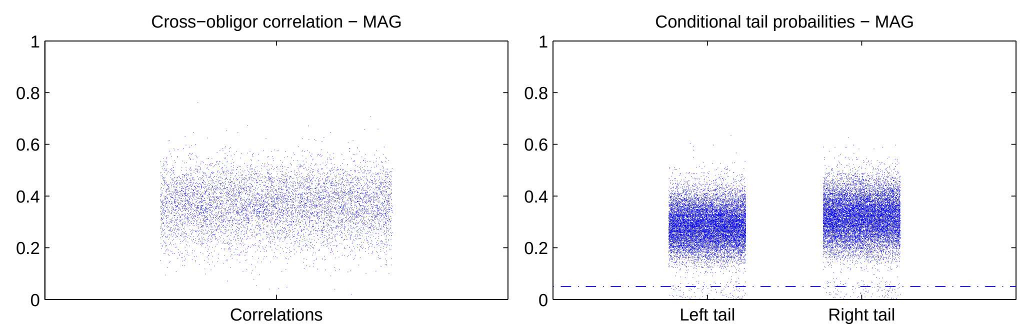

Figure 31: Left: Jitter plot of cross-obligor correlations. Right: Estimators, $\hat{p}^-$ and $\hat{p}^+$, for the conditional probability of having a large CDS spread move in an obligor given that there is a large move in the CDS spread of another obligor. Each point represents the conditional tail probability for a pair of obligors. Blue dotted lines at 5% level represents the estimator value for independent series. Data: Simulated spread returns from the heavy-tailed multivariate AR-GARCH model.

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6 Application: estimating loss distributions for CDS portfolios

Computation of the loss distributions, especially loss quantiles, for CDS portfolios is important in practice since it underlies risk measurement and the determination of margin requirements for the clearing of CDS contracts by central counterparties [14]. In this section, we use our heavy-tailed multivariate AR-GARCH model (MAG model) to estimate the loss quantiles for CDS portfolios, and compare its empirical performance with two other models.

We consider various examples of CDS portfolios with different long/short positions of various sizes. On each trading day, we estimate the 1% quantile for the daily loss, which corresponds the 99% 1-day Value-at-Risk (VaR), and compare these quantile levels to the realized daily loss across the sample. These VaR levels are not computed using a Gaussian model, which, as noted above, is not appropriate for modeling variations in CDS spreads, but using the heavy-tailed multifactor model described in section 5. If the model provides a good forecast of the quantiles of the loss distribution, then the realized loss will exceed the 99% VaR approximately 1% of the time. The proportion and timing of these exceedances allow to formally measure the accuracy of the model in predicting the tails of the portfolio loss distribution and to compare with other models.

6.1 Models for CDS portfolios

We compare three models in our empirical study: the affine jump-diffusion model [24], a random walk model proposed by Saita [57] and the MAG model that we proposed in section 5.

6.1.1 Affine jump-diffusion model

In section 4, we have shown that the affine jump-diffusion model is not able to reproduce the desired stylized properties, even though Feldhütter [29] and Azizpour et al [4] show that this model provides a good fit to the CDS and CDO data. We further examine this model by backtesting its performance on estimating the loss distributions for different CDS portfolios.

We estimate the model parameters by using a maximum likelihood method and principal component decomposition on the CDS spreads as follows:

1. We project daily changes of CDS spreads onto principal component factors.

2. We set the common risk factor values of the model equal to the first principal component factor values, and estimate the parameters describing the common risk factor by maximum likelihood method.

3. For each obligor, we estimate both statistical and risk-neutral parameters of the idiosyncratic risk factors from CDS spreads by maximum likelihood method.

4. We estimate risk-neutral parameters of the common risk factor by maximum likelihood method.

Details of the above procedure are given in Appendix C. Using these model parameters, we estimate VaR by Monte Carlo simulation (see Appendix B).

6.1.2 Random walk model

Saita [57] suggests that simple CDS spread returns follows a random walk

$$\frac{s^i_{t+\Delta t} - s^i_t}{s^i_t} = \sigma_{Z(i)}\sqrt{\Delta t}\left(\alpha^M_{Z(i)} N^M_t + \alpha^Z_{Z(i)} N^{Z(i)}_t + \sqrt{1 - (\alpha^Z_{Z(i)})^2 + (\alpha^M_{Z(i)})^2} N^i_t\right)$$

for $t = 0, \Delta t, 2\Delta t, ...$ where $Z(i)$ is obligor i's sector, $(N^M_t, N^{Z(i)}_t, N^i_t)$ are the market, sector and idiosyncratic factors which are independent among themselves as well as serially independent, $(\alpha^M_{Z(i)}, \alpha^Z_{Z(i)})$ are the factor loadings. For each time t, the factors are assumed to have zero means and unit variances. Obligors in the same sector are assumed to share the same spread volatility and factor loadings.

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The spread return volatilities are estimated by

$$\widehat{\sigma}_{Z(i)} = \sqrt{\frac{1}{n_Z} \sum_{t,i} \left(\frac{\Delta s_t^i}{s_t^i \sqrt{\Delta t}} \right)^2}$$

where n_Z is the number of terms in the summation which is over all data points in time and for all issuers that belong to sector Z. We estimate the loadings $(\alpha^M_{Z(i)}, \alpha^Z_{Z(i)})$ by OLS regression across all obligors.

In this model, the distribution of spread returns is not directly specified. Instead, Saita [57] computes VaR as follows. Consider a CDS portfolio with notional values $\mathbf{w} = (w_1, ..., w_n)$. Let $Spread01^i_t$ be the sensitivity of CDS i value with respect to 1 bp change in S^i_t, Ω be the estimated correlation matrix of the simple spread returns, and Σ be the diagonal matrix with diagonal element $\Sigma_{i,i}$ equal to the estimated volatility of the simple spread returns for obligor i. We define $R01^i_t = w_i\, S^i_t\, Spread01^i_t$ with $\mathbf{R01}_t = (R01^1_t, ..., R01^n_t)$. Then, the portfolio volatility (standard deviation of daily portfolio P&L) is equal to

$$\sigma_{\mathbf{w}} = \sqrt{\Delta t\ \mathbf{R01}_t\ \Sigma\ \Omega\ \Sigma\ \mathbf{R01}_t^{\mathrm{T}}}.$$

Then 99% VaR for portfolio $\mathbf{w}$ is set to

$$VaR = \beta\, \sigma_{\mathbf{w}},$$

where β is a scaling parameter. We estimate β as in [57]:

1. Select a type of CDS portfolio, e.g. equally weighted with n CDS in protection seller's positions.
2. Randomly create 100 portfolios of the selected type.
3. Compute the historical P&L and model-implied volatility for each portfolios.
4. For a given β, compute the historical exceedance ratio for this portfolio type:
$$\frac{1}{100} \sum_{k=1}^{100} \frac{\text{Number of days where loss of portfolio } k \text{ is larger than } \beta\sigma_{\mathbf{w}_k}}{\text{Total number of days}}, \tag{12}$$
where $\sigma_{\mathbf{w}_k}$ is the model-implied volatility of portfolio k.
5. If the historical exceedance ratio (12) is larger than 1%, we increase β. Similarly, if (12) is smaller than 1%, we decrease β.
6. Repeat step 4 – 5 until (12) is equal to 1%.

We repeat the estimation procedure of the scaling parameter β for each portfolio type.

6.1.3 Heavy-tailed multivariate AR-GARCH model

In section 5, we propose a heavy-tailed multivariate AR-GARCH model for the CDS spread returns. Since this model is built upon the observed stylized properties, it will be interesting to see whether it can also provide a good estimation for CDS portfolio loss distribution.

6.2 Backtesting

We evaluate the performance of the models in predicting loss quantiles for 9 types of CDS portfolios:

- Short-only: selling protection on 100, 40 or 10 obligors
- Long-only: buying protection on 100, 40 or 10 obligors
- Long-short: buying protection on 50, 20 or 5 obligors and selling protection on 50, 20 or 5 obligors

For each portfolio type, we consider 100 randomly chosen combinations of obligors and each position has the same notional value. Thus, we evaluate 900 CDS portfolios in total. In each sample period, we start our evaluation on the 250th trading day so that we have sufficient amount of data for model calibration. This gives 324 and 285 evaluation days in 2005-07 and 2007-09 respectively.

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6.2.1 Exceedance probabilities

Table 17 shows the percentage of days on which the portfolio loss exceeds the VaR estimates. In both 2005-07 and 2007-09, the affine jump diffusion model estimated using MCMC significantly overestimates the downside risk for short-only and long-short portfolios: there are no trading days that have losses exceed the VaR estimates for all short-only portfolios in 2005-07 and for almost all long-short portfolios in both periods. Moreover, the affine jump-diffusion model significantly underestimates the downside risk for the long-only portfolios. These observations are the results of upward jumps in the default intensity (6) in which the estimated jump sizes and frequencies appear to overestimate the potential losses for the short-only and long-short portfolios.

In 2005-07, the MAG model provide the best loss quantile prediction with exceedance probabilities almost equal to 1%, but the random walk model appears to overestimate the downside risk for all portfolio types. On average, the random walk model gives only 0.4% of the trading days that have losses exceed the VaR estimates, comparing to 1.0% given by the MAG model. In 2007-09, the MAG model still provides a good prediction of loss quantiles, especially for the long-short portfolios. However, the random walk model underestimates the downside risk of all portfolio types and gives, 2.6% of exceedances, compared to 1.0% for the MAG model.

In the Kupiec test [40], the exceedance probability for a portfolio is significantly different from 1% at 95% level if the number of exceedances is smaller than 1 or larger than 7 for the 2005-07 period which has 324 evaluation days, and small than 1 or larger than 6 for the 2007-09 period which has 285 evaluation days. Thus, the confidence interval for the number of exceedances are $[1, 7]$ and $[1, 6]$ respectively for 2005-07 and 2007-09.

Table 18 shows that the MAG model gives the fewest number of portfolios (11.3% and 2.4% in 2005-07 and 2007-09 respectively) whose number of exceedances is outside the Kupiec confidence interval. The random walk model performs better than the the affine jump diffusion model in 2005-07, but the two models are comparable in 2007-09. Overall, we observe that the MAG model provides better loss quantile prediction than the two other models.

	2005-07			2007-09		
	AJD	RW	MAG	AJD	RW	MAG
Short 100	0.0%	0.0%	1.3%	1.2%	3.2%	0.6%
Short 40	0.0%	0.2%	1.7%	0.3%	3.2%	0.7%
Short 10	0.0%	0.5%	1.8%	0.1%	3.1%	1.0%
Long 100	15.5%	0.2%	0.2%	3.1%	1.7%	1.3%
Long 40	2.2%	0.3%	0.4%	1.3%	2.0%	1.2%
Long 10	1.7%	0.3%	0.6%	0.4%	2.0%	1.1%
Long 50 Short 50	0.0%	0.8%	0.9%	0.0%	3.1%	1.1%
Long 20 Short 20	0.0%	0.7%	1.0%	0.0%	2.8%	1.0%
Long 5 Short 5	0.1%	0.7%	1.2%	0.1%	2.6%	1.0%
Average	2.2%	0.4%	1.0%	0.7%	2.6%	1.0%

Table 17: Empirical exceedance probabilities: percentage of trading days which have losses larger than the 99% VaR. Portfolio type: protection sellers (short), protection buyers (long). Models: affine jump-diffusion model (AJD), random walk model (RW), heavy-tailed multivariate AR-GARCH model (MAG).

6.2.2 Clustering of exceedances

Figure 32 and 33 show, at each date, the total number of portfolios whose losses exceeded the VaR estimates. We group the portfolios into three categories: short-only, long-only and long-short. Each category contains 300 different portfolios.

In 2005-07, exceedances of all portfolio types appear to be evenly distributed in time under both the random walk model and the MAG model. On the other hand, exceedances of the long-only portfolios cluster in the second half of 2007 under the affine jump-diffusion model. This is due to the fact that the CDS spread return distributions implied by the affine jump-diffusion model skew significantly to the right because of the upward jump components.

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	2005-07			2007-09		
	AJD	RW	MAG	AJD	RW	MAG
Short 100	100.0%	100.0%	0.0%	1.0%	100.0%	0.0%
Short 40	100.0%	52.0%	5.0%	46.0%	99.0%	0.0%
Short 10	89.0%	17.0%	22.0%	84.0%	92.0%	1.0%
Long 100	100.0%	22.0%	23.0%	93.0%	2.0%	0.0%
Long 40	40.0%	36.0%	27.0%	11.0%	27.0%	0.0%
Long 10	49.0%	43.0%	13.0%	68.0%	31.0%	4.0%
Long 50 Short 50	100.0%	7.0%	1.0%	100.0%	81.0%	2.0%
Long 20 Short 20	100.0%	14.0%	1.0%	97.0%	70.0%	7.0%
Long 5 Short 5	77.0%	21.0%	10.0%	85.0%	59.0%	8.0%
Average	83.9%	34.7%	11.3%	65.0%	62.3%	2.4%

Table 18: Test for accuracy of loss quantile estimation: percentage of portfolios that have less than 1 or more than 7 (resp. less than 1 or more than 6) exceedances in 2005-07 (resp. 2007-09), i.e. which reject the null hypothesis that the exceedance probability is equal to 1% under the Kupiec test at 95% confidence level.

In 2007-09, exceedances of the short-only portfolios cluster in late 2008 under the random walk model. This period begins with the bankruptcy of Lehman Brothers, then followed by a series of market shocks lead to upward jumps in the CDS spreads. From Figure 34, we can see that the random walk model is slow in reacting to those market shocks. On the other hand, the MAG model appears to adjust quickly to the volatile market in the late 2008 and the number of short-only portfolios that have losses exceed the VaR estimates reduces quickly after Lehman's bankruptcy.

In order to further examine whether exceedances are serially correlated, we perform a Ljung-Box test on the exceedance sequence $(1_{\{L_t>VaR_t\}})$ for each portfolio where L_t is the portfolio loss at time t and VaR_t is the VaR estimate. As suggested by Berkowitz, Christoffersen and Pelletier [5], we carry out the test for the first five lags. Table 19 shows that the MAG model gives very few portfolios that have autocorrelated exceedances (4.6% and 7.2% in 2005-07 and 2007-09 respectively). On the other hand, the random walk model and the affine jump-diffusion model give a significantly larger number of portfolios that have autocorrelated exceedances, with 52.6% and 13.9% respectively in 2007-09. This confirms our earlier observations from Figure 32 and 33.

	2005-07			2007-09		
	AJD	RW	MAG	AJD	RW	MAG
Short 100	0.0%	0.0%	0.0%	81.0%	100.0%	0.0%
Short 40	0.0%	3.0%	1.0%	18.0%	100.0%	1.0%
Short 10	2.0%	13.0%	11.0%	4.0%	98.0%	24.0%
Long 100	100.0%	0.0%	0.0%	6.0%	0.0%	0.0%
Long 40	60.0%	3.0%	2.0%	3.0%	1.0%	0.0%
Long 10	46.0%	2.0%	9.0%	13.0%	8.0%	5.0%
Long 50 Short 50	0.0%	21.0%	2.0%	0.0%	60.0%	11.0%
Long 20 Short 20	0.0%	17.0%	5.0%	0.0%	57.0%	15.0%
Long 5 Short 5	6.0%	18.0%	11.0%	0.0%	49.0%	9.0%
Average	23.8%	8.6%	4.6%	13.9%	52.6%	7.2%

Table 19: Test for autocorrelated exceedances: percentage of portfolios which reject the null hypothesis that the exceedance sequence $(1_{\{L_t>VaR_t\}})$ is serially uncorrelated, by using the Ljung-Box test at 95% confidence level.

6.2.3 Expected shortfall

Expected shortfall (ES) is defined as the expected loss given that the loss is larger than a given quantile (VaR). The *average relative shortfall deviation* (Table 20)

$$\frac{1}{N} \sum_{t:L_t>VaR_t} \frac{L_t - ES_t}{ES_t}, \tag{13}$$

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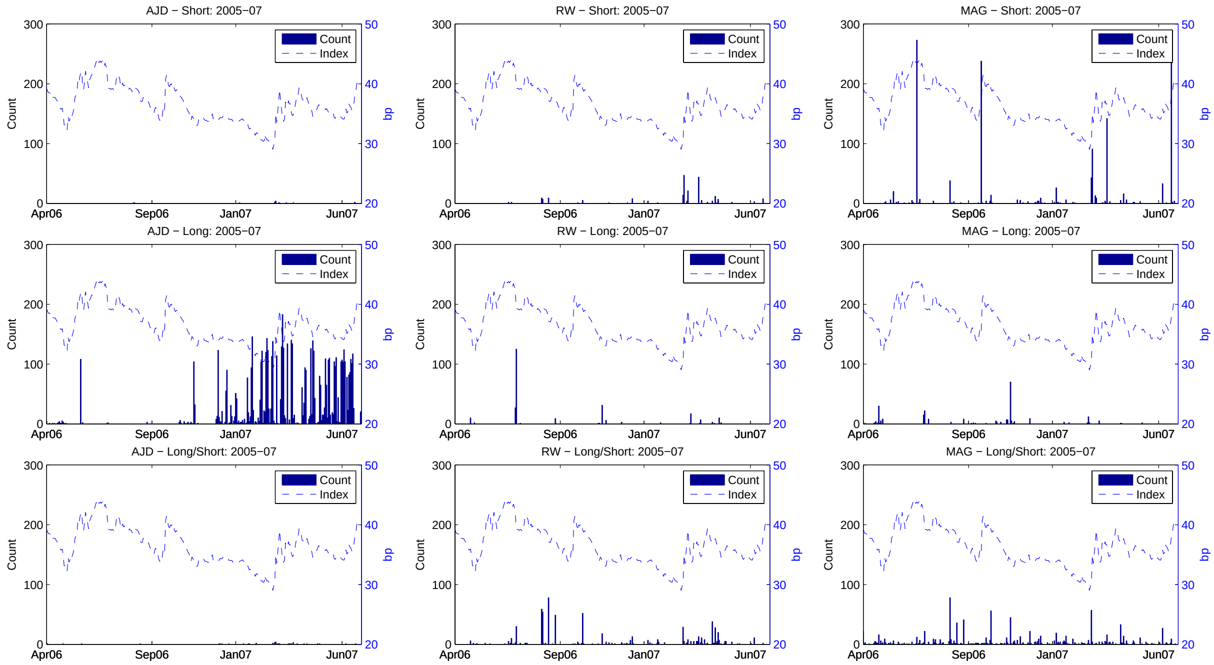

Figure 32: CDX.NA.IG on-the-run index and number of portfolios that have losses exceed 99% VaR in 2005-07. Portfolio types: protection sellers only (short), protection buyers only (long); half protection sellers half protection buyers (long-short).

where L_t is the portfolio loss at time t, VaR_t is the 99% VaR estimates, ES_t is the 99% expected shortfall estimates, and N is the number of trading days where the loss is larger than the VaR estimate of the previous day, measures how much the realized loss deviates from the expected shortfall. If the model provides a good estimate for the loss distribution, we expect the shortfall deviation (13) to be close to zero.

In 2005-07, the random walk model gives the smallest (in absolute value) relative shortfall deviation on average at -1.7%. On the other hand, the MAG model overestimates the expected shortfall for all portfolio types with an average relative shortfall deviation of -17.2%. This shows that the MAG model is more on the safe side in terms of loss prediction during exceedances.

In 2007-09, the random walk model significantly underestimates expected shortfall where the relative shortfall deviation increases to 21.5%. This means that extreme losses appear to be more severe than anticipated by the random walk model. In this period, the MAG model give the best expected shortfall estimates with shortfall deviations equal to 2.2% on average.

In order to examine whether the sample expected shortfall is statistically different from the model expected shortfall, we follow McNeil, Frey and Embrechts [48] and assume that the loss process follows

$$L_t = \mu_t + \sigma_t Z_t,$$

where μ_t and σ_t are measurable at time $t-1$, and (Z_t) is an i.i.d. sequence. Then, we define *expected shortfall residual* by

$$R_t = \frac{L_t - ES_t}{\sigma_t} 1_{L_t > VaR_t}, \tag{14}$$

where (R_t) is an i.i.d. sequence with zero mean. For each portfolio, we test whether the expected shortfall residual (14) has zero mean by computing the confidence intervals based on the bootstrap method [26, 48]. In order to have sufficient number of samples to obtain meaningful results[4], we combine the observations in the two sample periods. We omit those time series that has none or only one exceedance.

[4]For each sample period with roughly 300 evaluation days, the typical number of nonzero expected shortfall residual observations is only 3, if the exceedance probability is 1%.

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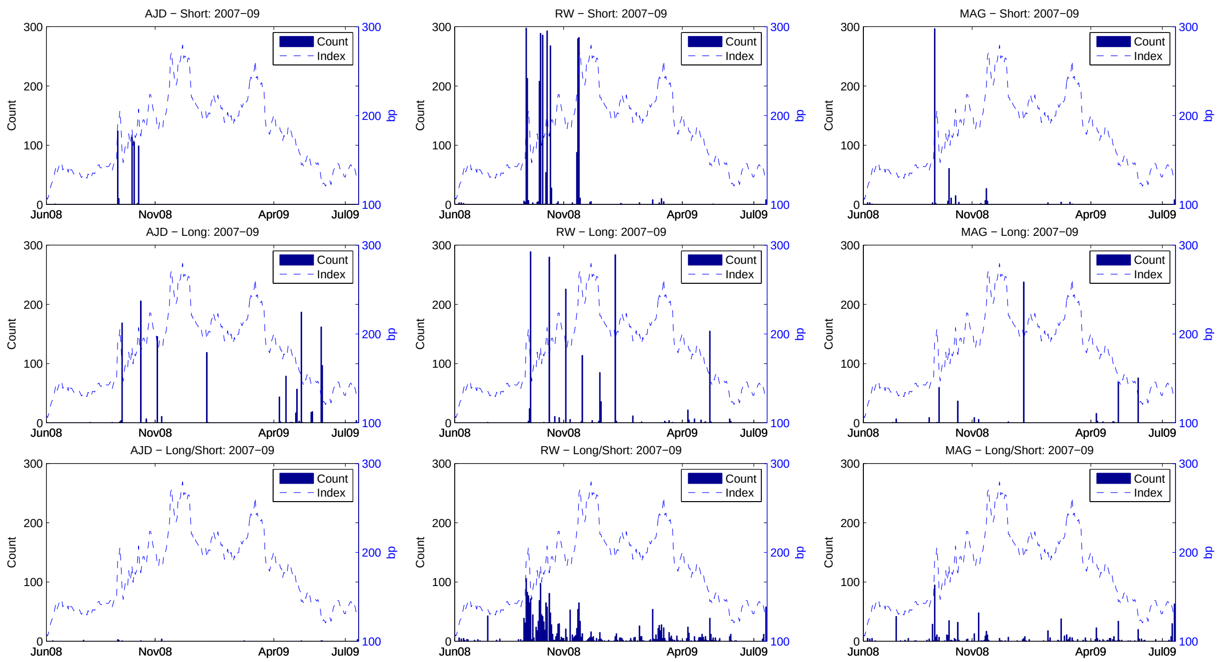

Figure 33: CDX.NA.IG on-the-run index and number of portfolios that have losses exceed 99% VaR in 2007-09. Portfolio types: protection sellers only (short), protection buyers only (long); half protection sellers half protection buyers (long-short).

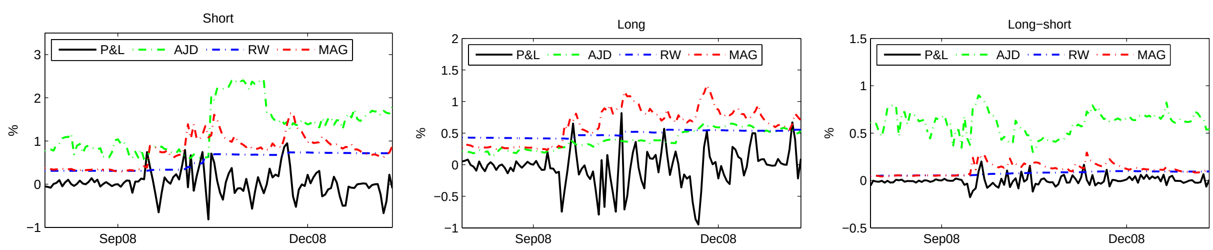

Figure 34: Daily P&L of a short-only, a long-only and a long-short portfolio, and the corresponding 99% VaR computed by the affine jump-diffusion model (AJD), the random walk model (RW) and the heavy-tailed multivariate AR-GARCH model (MAG) in 2008. Values are expressed as percentages of the total portfolio notional value.

Table 21 shows that, on average, the MAG model gives the fewest number of portfolios (41.6%) that can reject the hypothesis that the expected shortfall residual has zero mean, comparing to 47.9% of the affine jump-diffusion model and 45.2% of the random walk model.

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	2005-07			2007-09		
	AJD	RW	MAG	AJD	RW	MAG
Short 100	-	-	-18.6%	-1.9%	27.4%	28.1%
Short 40	-	-13.1%	-21.8%	-6.0%	28.8%	21.3%
Short 10	-13.2%	-5.7%	-15.9%	17.6%	28.4%	7.1%
Long 100	-14.0%	-7.6%	-25.8%	29.3%	14.2%	-15.0%
Long 40	-10.6%	4.2%	-15.8%	6.7%	12.0%	-12.7%
Long 10	39.8%	0.8%	-13.3%	19.2%	10.2%	-8.3%
Long 50 Short 50	-	-0.8%	-20.1%	-	24.7%	-1.3%
Long 20 Short 20	-	3.9%	-15.2%	-24.8%	27.2%	1.6%
Long 5 Short 5	31.1%	4.4%	-7.9%	48.1%	20.6%	-1.2%
Average	6.6%	-1.7%	-17.2%	11.0%	21.5%	2.2%

Table 20: Average relative shortfall deviation (13) for CDS portfolios.

	AJD	RW	MAG
Short 100	19.6%	100.0%	0.0%
Short 40	41.7%	73.0%	18.0%
Short 10	50.0%	56.0%	24.0%
Long 100	98.0%	8.0%	100.0%
Long 40	28.0%	32.0%	87.8%
Long 10	59.7%	25.0%	50.0%
Long 50 Short 50	-	36.0%	41.0%
Long 20 Short 20	-	47.0%	22.2%
Long 5 Short 5	38.5%	30.0%	31.0%
Average	47.9%	45.2%	41.6%

Table 21: Test for accuracy of expected shortfall estimation: percentage of portfolios which reject the null hypothesis that expected shortfall residual (14) has zero mean at 95% confidence level. Data: 2005-09.

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Appendix

A Moment generating function for affine jump diffusion process

Consider an affine jump diffusion process

$$dX_t = (\kappa_0 + \kappa_1 X_t)dt + \sigma\sqrt{X_t}dW_t + dJ_t$$

where (W_t) is a standard Brownian motion and (J_t) is a compound Poisson process with intensity ℓ and exponential jump size with mean μ under a risk-neutral measure $\mathbb{Q}$. The moment generating function of the cumulative affine jump-diffusion process is equal to

$$E^{\mathbb{Q}}\left[\exp\left(q\int_t^{t+T} X_u du\right)\Big|\mathcal{F}_t\right] = \exp\left(\alpha(q,T) + \beta(q,T)X_t\right)$$

where

$$\begin{aligned}
\alpha(q,T) &= \frac{2\kappa_0}{\sigma^2}\ln\left[\frac{2\gamma e^{(\gamma-\kappa_1)T/2}}{2\gamma + (\gamma-\kappa_1)(e^{\gamma T}-1)}\right] - \frac{2q\ell\mu}{\gamma+\kappa_1+2q\mu}T - \frac{2\ell\mu}{\sigma^2+2\mu\kappa_1+2q\mu^2}\ln\left[1+\frac{[\gamma-\kappa_1-2q\mu](e^{\gamma T}-1)}{2\gamma}\right], \\
\beta(q,T) &= \frac{2q(e^{\gamma T}-1)}{2\gamma+(\gamma-\kappa_1)(e^{\gamma T}-1)}, \\
\gamma &= \sqrt{\kappa_1^2 - 2q\sigma^2},
\end{aligned}$$

for $q < \frac{\kappa_1^2}{2\sigma^2}$.

Let τ^i be the random default time of an obligor i whose default intensity follows (4). Then, the survival probability under the affine jump-diffusion model is equal to

$$\begin{aligned}
\mathbb{Q}(\tau^i > T|\mathcal{G}_t) &= 1_{\tau^i>t}E^{\mathbb{Q}}\left[e^{-\int_t^T \lambda_u^i du}|\mathcal{F}_t\right] \\
&= 1_{\tau^i>t}E^{\mathbb{Q}}\left[e^{-\int_t^T X_u^i du}|\mathcal{F}_t\right]E^{\mathbb{Q}}\left[e^{-a^i\int_t^T X_u^0 du}|\mathcal{F}_t\right] \\
&= 1_{\tau^i>t}\exp\left(\alpha^i(-1,T) + \alpha^0(-a^i,T) + \beta^i(-1,T)X_t^i + \beta^0(-a^i,T)X_t^0\right)
\end{aligned}$$

where $(\mathcal{F}_t)$ is the filtration generated by $(W_t^{k,\mathbb{Q}})$ and (J_t^k) for $k=0,i$ and $(\mathcal{G}_t)$ is the filtration generating by all risk factors including the default time process, i.e. $\mathcal{G}_t = \mathcal{F}_t\bigvee\mathcal{H}_t$ where $(\mathcal{H}_t)$ is the filtration generated by the default indicator process $(1_{\tau^i>t})$.

B CDS spread simulation under affine jump-diffusion model

Let τ^i be the random default time of an obligor i. Assume that recovery rate R^i is deterministic. Given $\tau^i > t$, the fair CDS spread at time t is defined as

$$S_t^i = \frac{(1-R^i)\sum_{k=1}^M B(t,T_k)\mathbb{Q}(T_{k-1}<\tau^i\le T_k|\mathcal{G}_t)}{\sum_{k=1}^M (T_k-T_{k-1})B(t,T_k)\mathbb{Q}(\tau^i>T_k|\mathcal{G}_t)} \tag{15}$$

where $T_1,...,T_M$ are the payment dates and $\mathcal{G}_t$ represents all the available market information up to time t. In this case, modeling the dynamic of the CDS spread is equivalent to modeling the dynamic of the conditional survival probabilities $\mathbb{Q}(\tau^i > T|\mathcal{G}_t)$.

Let $\Theta_i^{\mathbb{Q}} = (\kappa_0^i, \kappa_1^{i,\mathbb{Q}}, \sigma^i, \ell^{i,\mathbb{Q}}, \mu^{i,\mathbb{Q}}, a^i)$ and $\Theta_i^{\mathbb{P}} = (\kappa_0^i, \kappa_1^{i,\mathbb{P}}, \sigma^i, \ell^{i,\mathbb{P}}, \mu^{i,\mathbb{P}})$, $\Theta_{i,j}^{\mathbb{Q}} = (\Theta_i^{\mathbb{Q}}, \Theta_j^{\mathbb{Q}})$, $\Theta_{i,j}^{\mathbb{P}} = (\Theta_i^{\mathbb{P}}, \Theta_j^{\mathbb{P}})$. From (15), we know that the CDS spreads under the affine jump-diffusion model can be written as

$$S_t^i = S^{ajd}(\Theta_{0,i}^{\mathbb{Q}}; t, X_t^0, X_t^i) \tag{16}$$

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where

$$
\begin{aligned}
S^{ajd}(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i) &= \frac{D(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i)}{P(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i)}, \\
D(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i) &= (1-R)\sum_{k=1}^{M} B(t,T_k)\left[\bar{F}(\Theta^{\mathbb{Q}}_{0,i};t,T_{k-1},x^0,x^i) - \bar{F}(\Theta^{\mathbb{Q}}_{0,i};t,T_k,x^0,x^i)\right], \\
P(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i) &= \sum_{k=1}^{M}(T_k-T_{k-1})B(t,T_k)\bar{F}(\Theta^{\mathbb{Q}}_{0,i};t,T_k,x^0,x^i), \\
\bar{F}(\Theta^{\mathbb{Q}}_{0,i};t,T,x^0,x^i) &= e^{\alpha^i(-1,T-t)+\alpha^0(-a^i,T-t)+\beta^i(-1,T-t)x^i+\beta^0(-a^i,T-t)x^0}
\end{aligned}
$$

for $x^0, x^i \geq 0$. Notice that the dynamic of the CDS spread process (S^i_t) is governed by the dynamic of the affine jump-diffusion processes (X^0_t) and (X^i_t).

Recall that (X^k_t) follows (6) under $\mathbb{P}$. By time discretization, we write

$$
\text{Under } \mathbb{P}: \quad X^k_{(j+1)\Delta t} = X^k_{j\Delta t} + (\kappa^k_0 + \kappa^{k,\mathbb{P}}_1 X^k_{j\Delta t})\Delta t + \sigma^k\sqrt{X^k_{j\Delta t}}\Delta W^{k,\mathbb{P}}_j + \Delta J^k_j \tag{17}
$$

where Δt is the discretization step which is set to be 1 day, $(\Delta W^{k,\mathbb{P}}_j)_{j=1,\ldots,J}$ are i.i.d. $N(0,\sqrt{\Delta t}^2)$ normal random variables and $(\Delta J^k_j)_{j=1,\ldots,J}$ are i.i.d. random variables where $\Delta J^k_j = 0$ with probability $1-\ell^{k,\mathbb{P}}\Delta t$ and ΔJ^k_j is exponentially distributed with mean $\mu^{k,\mathbb{P}}$ with probability $\ell^{k,\mathbb{P}}\Delta t$. Note that the dynamic of $(X^k_{j\Delta t})_{j=1,\ldots,J}$ depends on the parameters $\Theta^{\mathbb{P}}_k$.

We can simulate time series of CDS spreads for n obligors under the affine jump-diffusion models as follows:

1. Simulate time series $(X^k_{j\Delta t})_{j=1,\ldots,J}$ for $k=0,\ldots,n$ from (17).

2. For each time step j, compute CDS spread from (16).

C Estimation of the affine jump-diffusion model from CDS spread time series

In this section, we describe a method based on principal component analysis for estimating the parameters of the affine jump-diffusion model from time series of CDS spreads. THe method involves the following steps:

1. Decompose daily changes of CDS spreads into principal component factors.

2. Set $(x^0_{j\Delta t})_{j=0,\ldots,J}$ equal to the first principal component factor values and estimate $\Theta^{\mathbb{P}}_0$ by maximum likelihood method.

3. For each obligor i, estimate $(\Theta^{\mathbb{Q}}_i, \Theta^{\mathbb{P}}_i)$ to the CDS spreads with the given values of $(x^0_{j\Delta t})_{j=0,\ldots,J}$ and $(\widehat{\Theta}^{\mathbb{P}}_0, \kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}})$ by maximum likelihood method.

4. Calibrate $(\kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}})$ by maximum likelihood method with given values of $(x^0_{j\Delta t})_{j=0,\ldots,J}$, $\widehat{\Theta}^{\mathbb{P}}_0$ and $(\widehat{\Theta}^{\mathbb{Q}}_i, \widehat{\Theta}^{\mathbb{P}}_i)$, $i=1,\ldots,n$.

5. Repeat Step 3-4 until convergence.

Step 1: Decomposition of daily changes of CDS spreads

By the first order approximation of the Taylor series expansion, the pricing function (16) can be approximated by

$$
S^{ajd}(\Theta^{\mathbb{Q}}_{0,i};t,x^0,x^i) \approx S^{ajd}(\Theta^{\mathbb{Q}}_{0,i};t,0,0) + \partial_{x^0}S^{ajd}(\Theta^{\mathbb{Q}}_{0,i};t,0,0)\,x^0 + \partial_{x^i}S^{ajd}(\Theta^{\mathbb{Q}}_{0,i};t,0,0)\,x^i \tag{18}
$$

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where the partial derivatives have the closed-form expression:

$$\partial_{x^0} S^{ajd}(\Theta^{\mathbb{Q}}; t, x^0, x^i) = \frac{\partial_{x^0} D(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i)}{P(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i)} - S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i) \frac{\partial_{x^0} P(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i)}{P(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i)}, \tag{19}$$

$$\begin{aligned}\partial_{x^0} D(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i) = {} & (1-R) \sum_{k=1}^{M} B(t, T_k) \big[\beta(-a^i, T_{k-1} - t) \bar{F}(\Theta^{\mathbb{Q}}_{0,i}; t, T_{k-1}, x^0, x^i) \\ & - \beta(-a^i, T_k - t) \bar{F}(\Theta^{\mathbb{Q}}_{0,i}; t, T_k, x^0, x^i) \big],\end{aligned} \tag{20}$$

$$\partial_{x^0} P(\Theta^{\mathbb{Q}}_{0,i}; t, x^0, x^i) = \sum_{k=1}^{M} (T_k - T_{k-1}) B(t, T_k) \beta(-a^i, T_k - t) \bar{F}(\Theta^{\mathbb{Q}}_{0,i}; t, T_k, x^0, x^i), \tag{21}$$

and we have a similar expression for $\partial_{x^i} S^{ajd}(\Theta^{\mathbb{Q}}; t, x^0, x^i)$ by replacing a^i in (20) and (21) by 1.

Using this linear approximation, the change in CDS spreads under the affine jump-diffusion model is equal to

$$\Delta S^i_t \approx \partial_{x^0} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0)\, \Delta X^0_t + \partial_{x^i} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0)\, \Delta X^i_t \tag{22}$$

where Δ is the difference operator with $\Delta S^i_t = S^i_t - S^i_{t-\Delta t}$.

There are three advantages of this approximation. First, this approximation is linear in the values of x^0 and x^i which will become handy when we want to decompose the CDS spreads into different linear factors. Second, the approximation is also strictly increasing in the variables x^0 and x^i, which preserves the property of the exact function. Finally, the minimum possible CDS spread under this approximation is the same as the one computed under the exact function.

However, this approximation also shows the problem of under-determination of the state variables x^0 and x^i: at each time t, we have $n+1$ state variables to be determined, $(x^0_t, x^1_t, ..., x^n_t)$, but we only observe n market variables, CDS spreads $(s^1_t, ..., s^n_t)$. Because of the monotonicity and continuity of (16), a small increase in x^0_t can be compensated by a small decrease in $(x^1_t, ..., x^n_t)$ and give the same CDS spreads before the changes. One solution is to consider additional CDS spreads with different maturities. However, CDS with maturities other than 5 years are not very liquid which gives poor estimation of the parameters. We solve this problem by decomposing the CDS spreads into different factors using the principal component method.

Consider the data matrix containing the daily changes in the CDS spreads of all obligors. Using an eigenvector decomposition we obtain

$$\Delta s^i_{j\Delta t} = c^i_1 F^1_{j\Delta t} + \sum_{k=2}^{n} c^i_k F^k_{j\Delta t}$$

where c^i_k is the factor loading of obligor i to factor k and F^k_t is the k^{th} principal component factor value at time t. By comparing to (22), we identify the common risk factor in the affine jump-diffusion model $(\Delta x^0_{j\Delta t})_{j=1,...,J}$ with the first principal component

$$\partial_{x^0} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0) = c^i_1, \tag{23}$$

$$\Delta x^0_{j\Delta t} = F^1_{j\Delta t} \quad \text{for } j = 1, ..., J. \tag{24}$$

Step 2: Maximum likelihood estimation of $\Theta^{\mathbb{P}}_0$

Assume that x^0_0 is given. From (24), we obtain the observations of $(x^0_{j\Delta t})_{j=1,...,J}$. Recall that (X^0_t) follows (6) under $\mathbb{P}$. By time discretization, we have (17). For a time step where there is no jump, i.e. $\Delta J^0_j = 0$, $X^0_{(j+1)\Delta t}$ is normally distributed given the value of $X^0_{j\Delta t}$. If there is a jump, $X^0_{(j+1)\Delta t}$ is equal to the sum of a normal and an exponential random variable given the value of $X^0_{j\Delta t}$. Therefore, the density of $X^0_{(j+1)\Delta t}$

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given $X^0_{j\Delta t} = x_j$ is equal to

$$\begin{aligned} f(x_{j+1}|\Theta^{\mathbb{P}}_0, x_j) &= (1-\ell^{0,\mathbb{P}}\Delta t)\frac{1}{\sigma^0\sqrt{x_j\Delta t}}\phi\left(\frac{x_{j+1}-x_j-(\kappa^0_0+\kappa^{0,\mathbb{P}}_1 x_j)\Delta t}{\sigma^0\sqrt{x_j\Delta t}}\right) \\ &\quad + \frac{\ell^{0,\mathbb{P}}\Delta t}{\mu^{0,\mathbb{P}}}\exp\left(\frac{x_j+(\kappa^0_0+\kappa^{0,\mathbb{P}}_1 x_j)\Delta t}{\mu^{0,\mathbb{P}}} + \frac{x_j\Delta t(\sigma^0)^2}{2(\mu^{0,\mathbb{P}})^2} - \frac{x_{j+1}}{\mu^{0,\mathbb{P}}}\right) \\ &\quad \times\Phi\left(\frac{x_{j+1}-x_j-(\kappa^0_0+\kappa^{0,\mathbb{P}}_1 x_j)\Delta t - x_j\Delta t(\sigma^0)^2/\mu^{0,\mathbb{P}}}{\sqrt{x_j\Delta t}\sigma^0}\right) \end{aligned} \tag{25}$$

where $\phi(.)$ and $\Phi(.)$ are the density and cumulative distribution function of standard normal distribution respectively. Using this density function, we obtain estimators for x^0_0 and $\Theta^{\mathbb{P}}_0$ by maximizing the log-likelihood function:

$$\max_{\Theta^{\mathbb{P}}_0} \mathcal{L}^0(x^0_0, \Theta^{\mathbb{P}}_0)$$

where

$$\mathcal{L}^0(x^0_0, \Theta^{\mathbb{P}}_0) = \sum_{j=1}^{J} \ln f(x^0_{j\Delta t}|\Theta^{\mathbb{P}}_0, x^0_{(j-1)\Delta t})$$

is the log-likelihood function.

Step 3: Maximum likelihood estimation of $\Theta^{\mathbb{P}}_i$ and $\Theta^{\mathbb{Q}}_i$
Consider the time series $(s^i_{j\Delta t})_{j=0,\dots,J}$ of CDS spreads for obligor i. From Steps 1 and 2, we obtain a time series of risk factor values $(x^0_{j\Delta t})_{j=0,\dots,J}$ and a parameter estimate $\widehat{\Theta}^{\mathbb{P}}_0$. Assume that we also know the parameters $(\kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}})$. Then we estimate $(\Theta^{\mathbb{P}}_i, \Theta^{\mathbb{Q}}_i)$ as follows.

As in Step 2, we discretize the process (X^i_t) under $\mathbb{P}$ and compute the transition density (25) for $(X^i_{j\Delta t})$. From (18), we can write the log-likelihood function as for $(x^0_{j\Delta t})$ in Step 2. In addition, we account for the constraint (23) by adding a penalty term in the likelihood function:

$$\begin{aligned} \mathcal{L}^i(\Theta^{\mathbb{P}}_i, \Theta^{\mathbb{Q}}_i, \widehat{\Theta}^{\mathbb{P}}_0, \kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}}, w) &= \sum_{j=1}^{J} \ln f(x^i_{j\Delta t}|\Theta^{\mathbb{P}}_i, \widehat{\Theta}^{\mathbb{P}}_0, \Theta^{\mathbb{Q}}_{0,i}, x^i_{(j-1)\Delta t}, x^0_{(j-1)\Delta t}) \\ &\quad - \ln \partial_{x^i} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0) \\ &\quad - w|1 - \partial_{x^0} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0)/c^i_1| \end{aligned} \tag{26}$$

where $f(.|.)$ is the transition density (25), w is the weight on the penalty with respect to (23) which will be set to 10^5 and

$$x^i_t = \left(s^i_t - S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0) - \partial_{x^0} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0)\, x^0_t\right) / \partial_{x^i} S^{ajd}(\Theta^{\mathbb{Q}}_{0,i}; t, 0, 0).$$

Then, we obtain the estimates for $(\Theta^{\mathbb{P}}_i, \Theta^{\mathbb{Q}}_i)$ by solving

$$\max_{\Theta^{\mathbb{P}}_i, \Theta^{\mathbb{Q}}_i} \mathcal{L}^i(\Theta^{\mathbb{P}}_i, \Theta^{\mathbb{Q}}_i, \widehat{\Theta}^{\mathbb{P}}_0, \kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}}, w).$$

We repeat this estimation for each obligor and obtain parameter estimates $(\widehat{\Theta}^{\mathbb{P}}_i, \widehat{\Theta}^{\mathbb{Q}}_i)$ for $i = 1, ..., n$.

Step 4: Maximum likelihood estimation of $(\kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}})$
After obtaining $\widehat{\Theta}^{\mathbb{P}}_0$ and $(\widehat{\Theta}^{\mathbb{P}}_i, \widehat{\Theta}^{\mathbb{Q}}_i)$ for $i = 1, ..., n$, we estimate the remaining parameters $(\kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}})$ which govern the $\mathbb{Q}$-dynamic of (X^0_t) by maximizing the likelihood function

$$\max_{\kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}}} \sum_{i=1}^{n} \mathcal{L}^i(\widehat{\Theta}^{\mathbb{P}}_i, \widehat{\Theta}^{\mathbb{Q}}_i, \widehat{\Theta}^{\mathbb{P}}_0, \kappa^{0,\mathbb{Q}}_1, \ell^{0,\mathbb{Q}}, \mu^{0,\mathbb{Q}}, 0)$$

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where $\mathcal{L}^i(.)$ is the log-likelihood function (26). We set the penalty weight $w = 0$ to allow larger degree of freedom for the optimal solution.

Step 5: Iteration
In the final step, we iterate Step 3-4 until the solution converges. We find that it usually takes two to three iterations for convergence of the parameters within errors of 10^{-6}.

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