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Zero Coupon Yield Curve Construction Methods in the European Markets

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Abstract

In this study, four frequently used yield curve construction methods are evaluated on a set of metrics with the aim of determining which method is most suitable for estimating yield curves from European zero rates. The included curve construction methods are Nelson-Siegel, Nelson-Siegel-Svensson, cubic spline interpolation and forward monotone convex spline interpolation. We let the methods construct yield curves on multiple sets of zero yields with different origins. It is found that while the interpolation methods show greater ability to adapt to variable market conditions as well as hedge arbitrary fixed income claims, they are outperformed by the parametric methods regarding the smoothness of the resulting yield curve as well as their sensitivity to noise and perturbations in the input rates. This apart from the Nelson-Siegel method's problem of capturing the behavior of underlying rates with a high curvature. The Nelson-Siegel-Svensson method did also exhibit instability issues when exposed to perturbations in the input rates. The Nelson-Siegel method and the forward monotone convex spline interpolation method emerge as most favorable in their respective categories. The ultimate selection between the two methods must however take the application at hand into consideration due to their fundamentally different characteristics.

Keywords: Zero coupon, Yield curve construction, Forward monotone convex spline, Cubic spline, Nelson Siegel, Nelson Siegel Svensson, Hedge, Interpolation, Parameterization, Evaluation, Principal component

Sammanfattning

Svensk titel: Metoder för att konstruera nollkupongkurvor på de europeiska marknaderna

I denna studie utvärderas fyra välanvända metoder för att konstruera yieldkurvor på ett antal punkter. Detta med syftet att utröna vilken metod som är bäst lämpad för att estimera yieldkurvor på Europeiska nollkupongräntor. Metoderna som utvärderas är Nelson-Siegel, Nelson-Siegel-Svensson, cubic spline-interpolering samt forward monotone convex spline-interpolering. Vi låter metoderna estimera yieldkurvor på flera sammansättningar nollkupongräntor med olika ursprung. Vi ser att interpoleringsmetoderna uppvisar en större flexibilitet vad gäller att anpassa sig till förändrade marknadsförutsättningar samt att replikera godtyckliga ränteportföljer. När det gäller jämnhet av yieldkurvan och känsligheten för brus och störningar i de marknadsräntor som kurvan konstrueras utifrån så presterar de parametriska metoderna däremot avsevärt bättre. Detta bortsett från att Nelson-Siegel-metoden hade problem att fånga beteendet hos nollkupongräntor med hög kurvatur. Vidare hade Nelson-Siegel-Svensson-metoden problem med instabilitet när de underliggande marknadsräntorna utsattes för störningar. Nelson-Siegel-metoden samt forward monotone convex spline-interpolering visade sig vara bäst lämpade för att konstruera yieldkurvor på de Europeiska marknaderna av de utvärderade metoderna. Vilken metod av de två som slutligen bör användas behöver bedömas från fall till fall grundat i vilken tillämpning som avses.

Nyckelord: Nollkupong, Yieldkurva, Forward monotone convex spline, Kubisk spline, Nelson Siegel, Nelson Siegel Svensson, Interpolering, Parameterisering, Utvärdering, Principialkomponent, Hedge

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1 Introduction

1.1 Background

Yield curves are fundamental entities both within fixed income investment analysis in general as well as within the field of mathematical finance. While the curve illustrates the relationship between interest rates and time to maturity of debt instruments, one might also extract important information about the market's expectations from the curve, and from these derive independent valuations and risk measures on fixed income instruments. Continuous yield curves are however not directly observable in the market as the selection of bonds are finite, whereupon the curve must be constructed from discrete data. This hardly poses any problem if no restrictions are set for the properties of the curve and the market is liquid enough to provide an adequate amount of data points. In practice, this is however seldom the case due to for example a number of properties required for deriving e.g. economically viable forward rate curves and arbitrage free valuations.

Luckily for the quantitative analyst, methods for constructing yield curves from sparse market data are readily available in both devkits such as `MATLAB` toolboxes and Python packages as well as in financial tools as Bloomberg Terminal and Refinitiv Eikon. In addition to this, these methods are often backed by academia to some extent and it is not uncommon that these methods are designed specifically to have certain sought properties. However, there exists no extensive analysis and in particular comparison of the performance of the readily used methods on European data in current times.

1.2 Aim

By conducting this project together with E. Öhman J:or Fonder AB, the aim is to enrich the knowledge base with empirical observations about the suitability of the included methods, and most importantly identify areas where the methods are lacking in accuracy, desired mathematical properties and economic viability. The study will do so by evaluating yield curves constructed from European market data on a set of criteria, which in turn are selected in order to reward both sought mathematical and financial properties.

1.3 Research questions

The aim of the study presented above is formalized in the following research questions, with respect to which the empirical findings will be discussed:

- Which yield curve construction method outputs the most viable yield curve for the European markets, given certain requirements on financial and mathematical validity?
- How robust is this method when facing variable market conditions such as a high-yield bond universe?

1.4 Delimitations

The study has surveyed current research within mathematical finance in order to encompass frequently used methods for yield curve construction, which has resulted in a selection of four methods with different advantages and disadvantages to be included in the study. While there are countless other methods with highly valued properties as well, this study will be restricted to these four methods in order to provide a sufficient depth of analysis within the time frame and scope of a master thesis project.

The data used for analysis will, as the title of the thesis hints of, consist of European zero coupon yields in order to capture the performance of the methods in the European market. Note that the yield curves are constructed directly from zero yields provided by the valuation methodologies of Bloomberg and Refinitiv and are not bootstrapped from coupon bonds manually, this in order to allocate as much time as the project allows on the method evaluation.

1.5 Disposition

The thesis can be broken down into two distinct parts - a theoretical one and an empirical one. First, relevant background theory is presented in order to define both the notation and yield curve construction methods used throughout the thesis. Thereafter, a summary of a selection of previous studies is given in order to give context on both the yield curve construction methods selected for this thesis as well as the evaluation criteria used later on. Moving on to the empirical part of the thesis, the methodology used for the evaluation is presented together with an overview of what data will be used. Thereafter, the empirical results of the evaluation are presented and discussed with respect to the research questions posed above. Finally, some concluding remarks are given and the findings are summarized for the reader.

2 Background

2.1 The term structure of interest rates

The term structure of interest rates has 4 canonical representations, all of which convey the same information about the market as they uniquely determine each other. The four representations are the zero coupon yield curve $y(t,T)$, the forward rate curve $f(t,T)$, the discount curve $p(t,T)$ and the par yield curve $w(t,T)$. [12]

While this thesis focuses on the zero coupon yield curve and the forward rate curve, a brief background on all representations is given below.

2.1.1 Zero coupon yield

A zero coupon bond (ZCB) with maturity T (also called a T-bond) is a financial contract entailing its owner to a payment of P units of currency at time T. By convention we let $P = 1$ without a loss of generality. Then, the price of the bond at time t is given by $p(t,T)$ and is called the discount function. [5]

Furthermore, the continuously compounded zero coupon yield $y(t,T)$ is given by

$$y(t,T) = -\frac{\log p(t,T)}{T-t} \tag{1}$$

2.1.2 Zero coupon yield curve

The zero coupon yield curve is defined as the function

$$T \to y(t,T)$$

where t is fixed at present time [5]. The yield curve relates the yield of ZCBs with different maturities contracted at time t, all with the same creditworthiness and preferably issued by the same actor. It is easy to see that the discount function and yield curve uniquely determine each other.

2.1.3 Instantaneous forward rate curve

The instantaneous forward rate curve $f(t,T)$, i.e. the price at time t of a momentarily loan at time T, can be calculated from equation 1 as

$$f(t,T) = -\frac{\partial \log y(t,T)}{\partial T} \tag{2}$$

with t fixed at the time of contracting.

2.1.4 Par yield curve

The par yield curve is slightly different than the three representations of the term structure of interest rates presented above and is not used directly in this thesis. It is defined as the yield to maturity on a synthetic security which is priced at par, which implies that its coupon rate equals the yield to maturity. Mathematically we have

$$w(t,T) = \frac{1 - p(t,T)}{w(t,T) \int_t^T p(t,s)\, ds} \tag{3}$$

again for a fixed t at present time.

2.1.5 Discrete forward rate

The discrete continuously compounded forward rate can be derived from the zero coupon bond yield by a no-arbitrage argument. Introduce the time point S such that $t \leq S \leq T$. Now, suppose that one invests one unit of currency in an S-bond at time t. At time S, one receives $e^{y(t,S)*(S-t)}$ which is reinvested in a T-bond immediately. At time T, the total return is

$$e^{y(t,S)*(S-t)} * e^{y(S,T)*(T-S)} = e^{y(t,S)*(S-t)+y(S,T)*(T-S)}$$

In order to avoid arbitrage, this must equal the return of investing in a T-bond at time t. That is,

$$e^{y(t,S)*(S-t)+y(S,T)*(T-S)} = e^{y(t,T)*(T-t)}$$

which results in an expression for the discrete continuously compounded forward rate $F(t,S,T)$ between S and T, contracted at t:

$$F(t,S,T) \coloneqq y(S,T) = \frac{y(t,T)*(T-t) - y(t,S)*(S-t)}{T-S} \tag{4}$$

2.1.6 Further relationships

The discrete continuously compounded forward rate can be obtained equally well as an average of the instantaneous forward rates over an interval:

$$F(t,S,T) = \frac{1}{T-S} \int_S^T f(t,x)\, dx \tag{5}$$

Furthermore, the zero coupon yield curve can be obtained from Equation 5 by having $S = t$ be fixed at present time:

$$y(t, T) = \frac{1}{T - t} \int_t^T f(t, x)\, dx \tag{6}$$

While the instantaneous forward rate is not directly observable in the market, the reader's intuition might be helped by thinking of the instantaneous forward rate as the marginal return obtained by extending the maturity of one's investment marginally. Those with a background in economics might find it helpful to relate the instantaneous forward rate and spot rate with the marginal and average cost of production as they relate in exactly the same way. [17]

2.2 Yield curve construction

As continuous zero coupon yield curves are not directly observable in the market they have to be constructed from market data. That is, given a set of yield-tenor-pairs $\{(y_i, T_i)\}_{i=0}^{n}$ of zero yields bootstrapped from market data at time t, the yield curve is constructed by finding an appropriate mapping $T \rightarrow y(t, T)$ for $T_0 \leq T \leq T_n$. There are several methods for constructing the yield curve from zero rates, which can be categorized into parametric methods and spline-based interpolation methods [3].

2.2.1 Nelson-Siegel

Parametric methods for constructing yield curves suggest a parameterization of a typical curve together with methods of solving for the specified parameters. Here, the idea is that a parameterization is able to fully capture the typical shape of yield curves and thereby be generally applicable. Nelson & Siegel (1987) proposed a parametric method for constructing yield curves with the aim of introducing a simple, yet flexible enough method to capture a selection of different behaviors visible in yield curves. The behaviors referenced by the authors are monotonicity, humping and taking on S-shapes. The authors observed that these behaviors are often related to solutions to differential equations, whereupon they began investigating the performance of the model whose instantaneous forward rates take the form of a homogeneous solution to a second order ordinary differential equation (ODE) with real and distinct roots. [16] That is,

$$f(t, T) = \beta_0 + \beta_1 e^{-\frac{T-t}{\tau_1}} + \beta_2 e^{-\frac{T-t}{\tau_2}}$$

where τ_1 and τ_2 are constants and β_0, β_1 and β_2 are determined by initial conditions of the differential equation in question. The associated yield curve as a function of T is given by integration according to Equation 6.

While the model presented above indeed exhibits promising traits in terms of ability to adapt to the sought behaviors of term structures, Nelson & Siegel (1987) found evidence of the resulting curves being overparameterized. One clear example is that given several different values for τ_1 and τ_2, it is possible to find values of β_0, β_1 and β_2 that make the resulting models nearly identical. The model which is ultimately suggested by the authors again lets the expression for the instantaneous forward rate follow the homogeneous solution to a second order ODE, this time with real and equal roots [16]:

$$f(t, T) = \beta_0 + \beta_1 e^{-\frac{T-t}{\tau}} + \beta_2 \frac{T-t}{\tau} e^{-\frac{T-t}{\tau}} \tag{7}$$

The resulting yield curve is again obtained by integration and takes the form

$$y(t,T) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{T-t}{\tau}}}{\frac{T-t}{\tau}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{T-t}{\tau}}}{\frac{T-t}{\tau}} - e^{-\frac{T-t}{\tau}} \right) \tag{8}$$

The shape of the yield curve is uniquely determined by the four parameters β_0, β_1, β_2 and τ, where β_0 and τ must be positive [16]. The parameters are found either by iterative techniques such as the grid-search algorithm, or found by nonlinear optimization techniques such as the BFGS algorithm. One soon realizes that τ is required to be positive for the exponential decay term to in fact be a decay term. As to why β_0 is required to be positive, this is merely a requirement posed by the authors and has to do with the long standing belief that long term interest rates always are positive.

As mentioned previously, the method aims to capture some fundamental characteristics of the term structure of interest rates. With this in mind, the constant term can be viewed as the horizontal asymptote of zero rates as time approaches infinity since the other two terms decrease as T increases. The second term, i.e. the one whose coefficient is β_1, represents an exponential time decay whose speed depends on the size of τ. The final term with coefficient β_2 produces a hump or a trough, depending on the sign of the coefficient. The placement of the hump is governed by the size of τ. While β_0 alone governs the long term behavior of the curve, the short term rates approach $\beta_0 + \beta_1$ as T approaches t. Figure 2.1 illustrates the different components which make up the Nelson-Siegel parameterization.

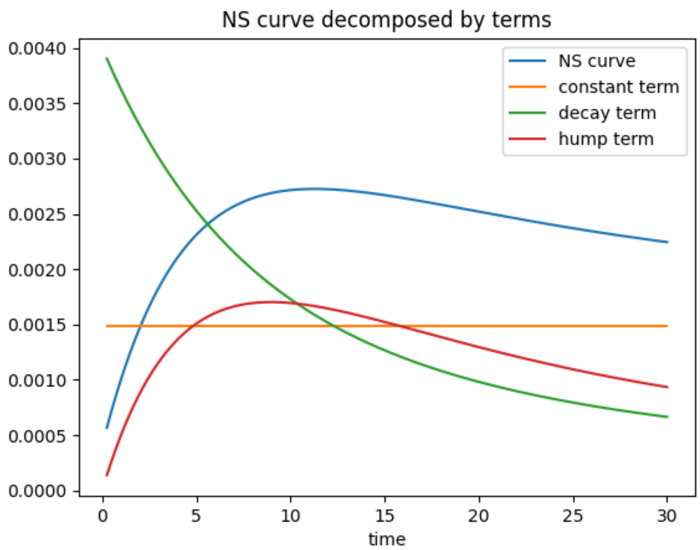


Figure 2.1: The Nelson-Siegel model and its different terms plotted together. All values of the curve are fictional and each term must be scaled by its corresponding β in order for the sum of the terms to equal the NS curve.

Today, the method is widely known as the Nelson-Siegel method and is to some extent still in use [3, 12]. One strength the authors claim of the method is that by capturing the general shape of the yield curve rather than approximating the curve locally, one should be able to use yields estimated by the curve outside of the range of tenors used to construct the curve [16]. This of course assumes that the zero yields do not behave unexpectedly outside of the input set, whereupon one should use caution when estimating out-of-sample.

2.2.2 Nelson-Siegel-Svensson

In an effort to improve the fit of the Nelson-Siegel method to market data as well as improving the method's flexibility to varying market conditions, Svensson (1994) proposed an extension of the Nelson-Siegel method by adding a fourth term to the expression of forward rates. A second hump-term governed by the new parameters β_3 and τ_2 was introduced, yielding the following expression for the instantaneous forward rate [17]:

$$f(t,T) = \beta_0 + \beta_1 e^{-\frac{T-t}{\tau_1}} + \beta_2 \frac{T-t}{\tau_1} e^{-\frac{T-t}{\tau_1}} + \beta_3 \frac{T-t}{\tau_2} e^{-\frac{T-t}{\tau_2}} \tag{9}$$

Following the same reasoning as with the Nelson-Siegel model, the yield curve is obtained from the instantaneous forward rate by integrating according to Equation 6, resulting in the following expression:

$$\begin{aligned} y(t,T) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{T-t}{\tau_1}}}{\frac{T-t}{\tau_1}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{T-t}{\tau_1}}}{\frac{T-t}{\tau_1}} - e^{-\frac{T-t}{\tau_1}} \right) \\ + \beta_3 \left(\frac{1 - e^{-\frac{T-t}{\tau_2}}}{\frac{T-t}{\tau_2}} - e^{-\frac{T-t}{\tau_2}} \right) \end{aligned} \tag{10}$$

Widely known as the Nelson-Siegel-Svensson method of yield curve construction, the resulting yield curve is uniquely determined by the six parameters β_0, β_1, β_2, β_3, τ_1 and τ_2. It is required that β_0, τ_1 and τ_2 all are positive, this with a similar motivation as for the Nelson-Siegel method. In his paper, Svensson discussed the applications for the proposed extension of the Nelson-Siegel model. The main takeaway is that while the original model provides a satisfactory fit to market data in most cases, its fit might be unsatisfactory when the term structure takes more exotic shapes not foreseen by Nelson & Siegel. In these cases, the extended model provides a closer fit due to the second hump- or trough term, allowing it to incorporate unusual kinks in the resulting yield curve [17].

Figure 2.2 illustrates the different terms of the Nelson-Siegel-Svensson curve. When comparing this to Figure 2.1, the extension made by Svensson to the original Nelson-Siegel model becomes clear.

The extended model is fairly frequently used for modeling yield curves among central banks [3, 12], whereupon it is a natural candidate model to include in this thesis.

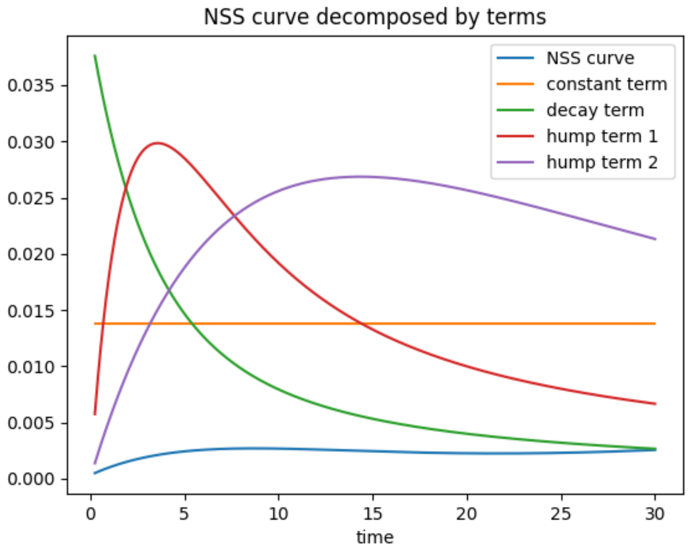


Figure 2.2: The Nelson-Siegel-Svensson model and its different terms plotted together. All values of the curve are fictional and each term must be scaled by its corresponding β in order for the sum of the terms to equal the NSS curve.

2.2.3 Cubic Spline

Interpolation methods strive to populate inner regions in a given set of yield-tenor-pairs with data points of a specified resolution in time, i.e. fill in the blanks between discrete knot points in order to form a curve. The intuitive way of doing this is to draw straight lines between the knot points, and while the resulting yield curve might suffice for certain applications, it fails to encompass fundamental financial properties such as a continuous forward rate curve. [14] While linear interpolation is not included in this study, it serves an educational purpose to compare to the upcoming methods.

An improvement to linear interpolation is cubic spline interpolation which always produces continuous forward rate curves. The idea behind the method is to utilize splines as interpolants between knot points, i.e. to fit piecewise cubic polynomials to each interval of knot points. However, as Hagan & West (2006) point out, there is no guarantee that the resulting forward rates are strictly positive when using cubic spline interpolation to produce the yield curve. [8]

The general method works according to the following [13]. Assume that the yield-tenor-pairs (y_i, T_i) and (y_{i+1}, T_{i+1}) are joined by a cubic polynomial, also known as a spline, on the form

$$Y_i(t) = a_i t^3 + b_i t^2 + c_i t + d_i,\ i \in [1, n-1]$$

We require that the spline intersects the knot points on the edges of its interval, that is

$$Y_i(T_i) = y_i,\ i \in [1, n-1]$$
$$Y_i(T_{i+1}) = y_{i+1},\ i \in [1, n-1]$$

So far, we have $2(n-1)$ equations for the $n-1$ splines, all with 4 unknowns for a total of $4(n-1) = 4n-4$ unknowns. Next, we wish for the splines to join smoothly at the knot points so we require that the first and second derivatives remain continuous between adjacent splines:

$$Y_i'(T_{i+1}) = Y_{i+1}'(T_{i+1}),\ i \in [1, n-2]$$
$$Y_i''(T_{i+1}) = Y_{i+1}''(T_{i+1}),\ i \in [1, n-2]$$

By doing this, we can guarantee the continuity of both the first and second derivative of the interpolated curve, which translates to a continuous forward rate curve. This restriction amounts to $2(n-2)$ equations for a total of $4n-6$, while the unknowns amount to $4n-4$, presenting a need for 2 additional equations in order to compute all coefficients for the splines. While these two equations can be chosen arbitrarily, they are preferably chosen as boundary conditions for the endpoints at T_1 and T_n. A common boundary condition is to require the second derivatives to equal zero, so

$$Y_1''(T_1) = 0$$
$$Y_{n-1}''(T_n) = 0$$

This produces what is called a natural cubic spline, where the entry and exit points of the splines in the interpolation interval consist of straight lines. The resulting system of equations becomes tridiagonal and it is thereby rather easy to solve for the sought coefficients. [4]

When using cubic spline interpolation in finance and especially for the construction of yield curves, the above mentioned boundary condition might however not result in the best possible properties of the interpolated curve. If one instead of imposing restrictions on the second derivative in the far endpoint rather requires the first derivative to equal zero, one obtains what is known as the financial cubic spline. [8] That is,

$$Y_1''(T_1) = 0$$
$$Y_{n-1}'(T_n) = 0$$

By doing this, the interpolant has a horizontal asymptote as time to maturity exceeds the interpolation interval, which alike the Nelson-Siegel method enables out of sample estimation of the yield curve under the assumption that no unforeseen behavior of the zero rates occurs beyond the interpolation interval. At the beginning of the interpolation interval, the interpolant is linear as with the natural cubic spline. [8]

2.2.4 Forward Monotone Convex Spline

In their 2006 paper, Hagan & West proposed a new interpolation method constructed specifically for yield curve interpolation, meaning that appropriate properties of the interpolant from a financial perspective were considered. Most notably, the method preserves the geometry of the inputs. In particular, the interpolated curve is guaranteed to be convex and the splines locally monotone between knot points if the input rates have the corresponding properties in the discrete case. Moreover, if all input rates are positive, the interpolated spot curve is guaranteed to be positive as well. [8]

The method is based on the work conducted by Hyman in 1983 related to monotonicity preserving interpolation methods [10]. The method developed by Hyman was however not designed with yield curve construction in mind, but rather developed for engineering applications. As phrased by Hagan & West, the method is completely unaware that it is dealing with a financial problem, and therefore there is nothing ensuring that the forward rates are positive and continuous. Thus, the proposed method is rather similar to the one proposed by Hyman (1983), however explicitly ensures that the forward rates are positive and continuous whenever the discrete ones are [8, 9].

In contrast to the cubic spline interpolation method, the forward monotone convex spline interpolation method fits splines to a set of instantaneous forward rates. The resulting forward curve is thereafter converted to a spot yield curve by integration as per Equation 6.

In order to proceed with the method, we remind ourselves that our given input $\{(y_i, T_i)\}_{i=0}^{n}$ is a set of tuples containing zero yields with tenor T_i, bootstrapped from market data at time t. As the forward monotone convex spline requires discrete forward rates $\{f_i^d\}_{i=0}^{n}$ as inputs, we construct these from the input rates per Equation 4 with $S = T_i$ and $T = T_{i+1}$. Note that this step is unnecessary if one wishes to use forward rates available in the market, e.g. FRA rates.

Now, instead of having the rate f_i^d belong only to the time point T_i, we assign it to the entirety of the interval (T_{i-1}, T_i). Furthermore, we calculate a new set of instantaneous forward rates $\{f_i\}_{i=0}^{n}$ at the knot points which is used by the interpolation algorithm moving forward. The rate $f(T_i) = f_i$ is calculated by having it lie on the line connecting the midpoints of the adjacent intervals [9]:

$$f_i = \frac{T_i - T_{i-1}}{T_{i+1} - T_{i-1}} f_{i+1}^d + \frac{T_{i+1} - T_i}{T_{i+1} - T_{i-1}} f_i^d \quad , \qquad i \in [1, n-1] \tag{11}$$

$$f_0 = f_1^d - \frac{1}{2}(f_1 - f_1^d) \tag{12}$$

$$f_n = f_n^d - \frac{1}{2}(f_{n-1} - f_n^d) \tag{13}$$

whereas the rates $f(T_0) = f_0$ and $f(T_n) = f_n$ are calculated so that the derivatives $f'(T_0) = f'(T_n) = 0$.

Hagan & West stress that an overnight rate might be available in certain emerging markets, whereby this rate should be preferred over Equation 12 for f_0 [8]. This is however not relevant for the interpolation conducted in this thesis. Furthermore, we can see that if the discrete forward rates f_i^d are positive, the instantaneous forward rates f_i are also positive for $i \in [1, n-1]$.

Proceeding with the interpolation, we seek a spline f on the interval $[T_{i-1}, T_i]$, $i \in [1, n]$ satisfying the following conditions:

i $\frac{1}{T_i - T_{i-1}} \int_{T_{i-1}}^{T_i} f(t)\, dt = f_i^d$

ii $f(T_i) = f_i$

iii $f(T_{i-1}) = f_{i-1}$

We make the anzats

$$\begin{aligned} f(\tau) &= K + Lx(\tau) + Mx(\tau)^2, \\ x(\tau) &= \frac{\tau - T_{i-1}}{T_i - T_{i-1}} \end{aligned}$$

for $\tau \in [T_{i-1}, T_i]$. By combining the three criteria presented above, we solve for the three unknowns K, L, and M, resulting in the system of equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} K \\ L \\ M \end{bmatrix} = \begin{bmatrix} f(T_{i-1}) \\ f(T_i) \\ 0 \end{bmatrix} \tag{14}$$

with the solution

$$f(\tau) = f_{i-1} - \left(4f_{i-1} + 2f_i - 6f_i^d\right) x(\tau) + \left(3f_{i-1} + 3f_i - 6f_i^d\right) x(\tau)^2 \tag{15}$$

and simplifying to

$$f(\tau) = \left(1 - 4x(\tau) + 3x(\tau)^2\right) f_{i-1} + \left(-2x(\tau) + 3x(\tau)^2\right) f_i \\ + \left(6x(\tau) - 6x(\tau)^2\right) f_i^d \tag{16}$$

Thereby, the basic interpolator is complete [8, 9]. If one wishes to extrapolate the yield curve beyond T_n, we have $f(T) = f(T_n)\,\forall\, T \geq T_n$. Finally, one simply applies Equation 6 in order to obtain the sought spot rates.

In their 2006 paper, Hagan & West present a handful of adjustments of the interpolant one can implement in order to post additional requirements on the interpolant. Recall that the instantaneous forward rates are guaranteed positive if the discrete forward rates are. An especially convenient feature presented in the paper is to require all instantaneous forward rates to be positive regardless of the sign of the discrete forward rates. This property is derived by a long series of arguments on regions of stability for the normalized interpolant, the details of which are not discussed in this thesis, but the curious reader is referred to the source paper [8]. In particular, the property is obtained by modifying Equations 11, 12 & 13 according to:

$$f_i \rightarrow bound(0, f_i, 2min(f_i^d, f_{i+1}^d))\,\forall\, i \in [1, n-1] \tag{17}$$

$$f_0 \rightarrow bound(0, f_0, 2f_1^d) \tag{18}$$

$$f_n \rightarrow bound(0, f_n, 2f_n^d) \tag{19}$$

where the $bound()$ function is defined as

$$bound(a, x, b) = min(max(a, x), b) = \begin{cases} a & \text{if } x \leq a \\ x & \text{if } a < x < b \\ b & \text{if } x \geq b \end{cases}$$

Note that should one have an overnight rate for f_0, Equation 18 is not to be implemented.

2.3 Principal Component Analysis

Principal component analysis (PCA) is a powerful tool for reducing the dimensionality of large data sets while preserving the features that explains most of the variance in the underlying data. This is done by transforming a data set consisting of linearly correlated features to its so called principal components (PC), which constitute an orthogonal basis for the space spanned by the original data. The principal components are ordered such that the first PC explains the most variance of the data, the second PC explains the second most variance, and so forth. By keeping only the first m principal components, one thereby reduces the dimensionality of the data set while retaining as much explanatory power as possible. [2, 11]

PCA can be applied on data sets in vastly different settings, and while one cannot present any general interpretation of the principal components, it is possible to find interpretations in specific cases which will become clear further on in this thesis.

Now, let $X_1, X_2, ..., X_n$ be the features of interest, each with T observations. The feature values must be mean centered across each observation. These form the matrix X of Equation 20 and constitute the data set on which the PCA is performed.

$$X = \big[X_1, X_2, \ldots, X_n\big] = \begin{bmatrix} x_{1,1} & \ldots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{T,1} & \ldots & x_{T,n} \end{bmatrix} \tag{20}$$

The principal components P_i are now obtained by the linear combination with the largest variance of the feature vectors

$$P_i = \phi_{1,i}X_1 + \phi_{2,i}X_2 + ... + \phi_{n,i}X_n \tag{21}$$

Here, the vector of coefficients $\phi_i = [\phi_{1,i}, \phi_{2,i}, ..., \phi_{n,i}]^T$ is called the loading vector of the i:th principal component and it is required that the sum of its squared elements equals one. The i:th principal component P_i is thereby determined by the loading vector ϕ_i which is obtained by solving the maximization problem

$$\max_{\phi_i} \left\{ \frac{1}{T} \sum_{k=1}^{T} \left(\sum_{j=1}^{n} \phi_{j,i}\, x_{k,j} \right)^2 \right\} \text{ subject to } \sum_{j=1}^{n} \phi_{j,i}^2 = 1 \tag{22}$$

The method of obtaining the principal components can be simplified using matrix notation. First, let V be the $n \times n$ covariance matrix of X. Furthermore, let W be the $n \times n$ matrix of eigenvectors of V, sorted by the corresponding eigenvalues from largest to smallest. One can show that the columns of W are in fact the loading vectors ϕ_i defined above using linear algebra. Now, the principal components P_i of X are obtained as the columns of P by

$$P = XW \tag{23}$$

The variance explained by the i:th principal component is now given by the eigenvalue λ_i of the i:th eigenvector in W. The fraction p_i of the total variance that is explained by P_i is given by

$$p_i = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \tag{24}$$

From the eigenvalues of W, one can select the first m columns of P and W based on what fraction of the total variance one requires be explained and construct the $n \times m$ matrix W^* and the $T \times m$ matrix P^*. These can thereafter be used to reconstruct a projection X^* of the original data on the m first principal components by

$$X^* = P^* W^{*T} = X W^* W^{*T} \tag{25}$$

3 Previous studies

As yield curve construction methods have been researched extensively during quite some time, there exists a large amount of literature regarding model proposals, properties one might seek in a yield curve and methods for evaluating the viability and robustness of a yield curve. In this section, notable studies are presented together with their key findings in order to summarize the academic foundation this thesis aims to build upon.

3.1 Nelson & Siegel (1987)

Nelson & Siegel proposed their well-known parametric yield curve method in 1987. While the details of the parameterization is explained in Section 2, the reasoning behind the method is left unmentioned. The authors state that the need of a parsimonious parameterization of the yield curve was first stressed by Milton Friedman (1977) [7], who argued that a parameterization using a handful of parameters might have higher informative value than more complex models, including interpolation methods. In particular with regards to education, the relationships of the term structure is most easily explained using compact models. With this in mind, Nelson & Siegel sought to find a simple, yet thorough parameterization of the yield curve able to capture three general shapes associated with the curve: monotonic, S-shaped and humped.

When evaluating their proposed method, the authors find that since a general parameterization captures the broad shape of the term structure rather than every last detail, it has high potential to outperform interpolation methods regarding out of sample estimation. This is briefly verified in the paper by pricing long term bonds fairly accurate using yield curves obtained by fitting the model to treasury bills. Finally, the applications for the model according to the authors is illustrative purposes, testing of theories on the shape of the yield curve and modeling of demand functions rather than for example pricing of interest rate derivatives or bonds. [16]

3.2 Hagan & West (2006 & 2008)

While the 2006 paper by Hagan & West has already been partially reviewed in Section 2, in particular while introducing the forward monotone convex spline interpolation method, the paper covers more than just the proposal of the method. In fact, the aim of the paper is to evaluate different interpolation methods against each other, during which the authors highlight both strengths and shortcomings of the different methods. The authors claim that an ideal method of constructing yield curves does not exist, but will always depend on what requirements one must pose of the resulting curve and must thereby be decided on a case-by-case basis. The paper includes a large amount of interpolation methods, but since many differ only by technicalities, it suffices to present the broad groups of methods evaluated in the paper:

- Linear interpolation
- Constant interpolation
- Cubic splines
- Quartic splines

One shortcoming of the above mentioned paper is that the evaluation only encompasses interpolation methods for curve construction, not parametric methods of any kind. This is one area that this thesis aims to expand upon by including one of the most promising interpolation method as well as readily used parametric methods.

In addition to the above mentioned evaluation, both the 2006 & 2008 papers discuss what properties of yield curves one might look for when evaluating methods against each other. The suggested properties are of both qualitative and quantitative nature, for example positivity, continuity and monotonicity of the related forward rate curve. Hagan & West argue that a curve construction method should be local in the sense that a change in an input should have fairly local effects on both the forward rate curve and yield curve. Related to this is the stability of the method when an input is perturbed - the sensitivity of the method in question cannot be too high in order to be applicable on real world data where noise is unavoidable. [8, 9]

3.3 du Preez & Maré (2013)

Following the issues with readily available curve construction methods, which were noted by Hagan & West amongst others, du Preez & Maré propose yet another yield curve interpolation method that ensures positive and continuous forward rates [6]. The work is based upon a shape-preserving cubic Hermite spline interpolation method and applies the method to the logarithm of the discount curve.

In the paper, weaknesses with readily available methods for curve construction are presented, many similar to those noted by Hagan & West. du Preez & Maré do however discuss problems with parametrizations of the yield curve as well, concluding that the fit of such models to market prices of securities is typically low. The authors argue that this yields unnecessarily high volatilities if one must rely on the curve for security pricing, whereupon financial institutions active in the securities market seldom rely solely on such models.

In the discussion of shortcomings of existing models, du Preez and Maré include the forward monotone convex spline interpolation method suggested by Hagan & West. Some weaknesses are presented, most importantly that the method under particular conditions might yield a discontinuous forward rate curve. The authors claim that spline interpolation methods often face a balancing act between continuity and monotonicity, where the forward monotone convex spline interpolation method prefers the latter.[6]

du Preez & Maré introduce yet another spline interpolation method for constructing yield curves with desirable properties, the monotone preserving $r(t)t$ method, which works rather similarly to the forward monotone convex spline method but prefers continuity of the forward rate curve over monotonicity [6]. The authors stress that while their proposed method indeed aims to solve a stated problem with the method by Hagan & West, it too has weaknesses, and what method the end user should implement remains a subjective matter. This method is not included in this thesis because it is deemed too similar to the forward monotone convex spline method. In addition, its inclusion would extend the scope of the thesis significantly and thereby reduce the level of detail one can provide of the other methods.

3.4 Muthoni (2015)

In Muthoni (2015), with a methodology much alike the one of this thesis, the aim is to find the best zero coupon yield curve construction method for Nairobi Securities Exchange [14]. The study evaluates the performance of the parametric Nelson-Siegel method and an extension of the monotone preserving $r(t)t$ method first introduced by du Preez & Maré [6].

The extended monotone preserving $r(t)t$ method was suggested by Muthoni, Onyango & Ongati in an effort to ensure that the resulting yield curve is differentiable at the knot points, something the original method fails to do. This is achieved by introducing the Hyman monotonicity constraint [10]. [15]

The model selection is based on the requirements posed on the resulting yield curve, namely that it should be differentiable at all points, that its associated forward rate curve is continuous and positive and that its associated discount curve is decreasing in time. Statistical measures such as R^2, RMSE & RMSPE are used together with a smoothness measure first suggested by Adams and van Deventer in 1994 [1] in order to determine which method is most suitable for yield curve construction in Nairobi Securities Exchange given their set of requirements.

The evaluation provides some useful insights regarding the model performance of parametric and spline based models one can expect in this paper. First, it is found that the spline interpolation method has a somewhat higher accuracy than the parametric model. When moving on to the smoothness measure, it is however apparent that the parametric model outperforms the spline interpolation method by roughly a factor of 2. It is argued that this property is especially important as the smoothness of the curve indicates differentiability and continuity. Moving on, a visual comparison of the resulting yield curves and forward rate curves highlights unrealistic behavior of the forward rate curve associated with the spline based model. Finally, it is found that while the parametric model yields a strictly decreasing discount curve, the curve of the spline based model starts increasing in the long term, again exhibiting unrealistic behavior. In conclusion, the Nelson-Siegel model is preferred over the monotone preserving $r(t)t$ method.

Besides the evaluation presented above, one notable finding in the paper is the observation that cubic spline interpolation methods seem to exhibit a general lack of locality when the input rates are exposed for a perturbation, creating sinusoidal waves which propagate throughout the resulting curve.

4 Method

The aim of this section is to paint a clear picture of how the evaluation will be conducted, what methods are included and why they have been selected as well as which properties are deemed desirable in a yield curve. Thereafter, the formalized evaluation criteria are presented and motivated, and finally the data used for the evaluation is presented.

4.1 Method selection and implementation

In the literature survey preceding this thesis, effort was made to gain an understanding of which methods are most prevalent both in notable research papers as well as in practical applications. By evaluating a selection of readily available methods with promising characteristics, this thesis hopes to lay some groundwork for actors in the field of financial mathematics seeking a suitable method for modeling the yield curve.

As noted by the Bank for International Settlements, methods for constructing yield curves can be broadly categorized into spline-based interpolation methods and parametric methods [3]. As has been made clear in Sections 2 & 3, there exists promising methods in both categories, whereupon it is desirable to include candidates from both groups in the evaluation.

While several previous studies have focused on improving specific traits of the methods in question, often related to the behavior and properties of the corresponding forward rate curves, no strict requirements are imposed on the forward rates when conducting the method selection herein. This thesis rather strives to include promising and well known models in a comprehensive analysis and utmost present a thorough discussion about in what areas specific models excel. This because while specific traits are indeed important, and sometimes necessary for particular applications, their importance depends heavily on the application and must thereby be valued on a case-by-case basis.

The methods are to be applied on zero yields obtained directly from Bloomberg and Refinitiv in order to estimate the full yield curve on which the evaluation metrics are calculated. The zero yields are taken as given as to direct full attention to the evaluation of the methods and not on the bootstrapping on zero yields. With the amount of research which has been put into the creation of zero yields from coupon yields, it is deemed unnecessary to attempt to outperform the readily available methods.

The methods to be evaluated are:

- Nelson-Siegel (NS) method,
- the Nelson-Siegel-Svensson (NSS) method,
- the cubic spline (CS) method,
- and the forward monotone convex spline (FMCS) method.

The Nelson-Siegel method is a rather obvious candidate due to the large influence the method has had over the years, and to some extent still has today. While its use has to some extent been decreasing in favor of the Nelson-Siegel-Svensson extension, it remains the core model and is therefore of interest to evaluate. The extended model by Svensson is rather prevalent among central banks around the world [12], and while it is not expected to be any groundbreaking differences between the two methods, it is of interest to evaluate how they both perform on the European market in current times.

The fitting of both the Nelson-Siegel method and the Nelson-Siegel-Svensson method is done by the NelsonSiegelSvensson package in Python 3.7 (CPython). The package uses the iterative Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm provided in Scipy in order to minimize the sum of squared residuals of the curve with respect to τ. In each iteration of the algorithm, all β-coefficients are found using ordinary least squares regression.

Moving on, a large number of interpolation methods rely on polynomial splines, and as cubic splines always produce continuous forward rate curves, they constitute the foundation for several more advanced methods. Cubic spline interpolation is therefore included in the analysis in order to give account of what properties and behaviors one might expect from this family of methods.

Finally, while also being a polynomial spline interpolation method, the forward monotone convex spline method introduced by Hagan & West in 2006 is based on quadratic splines. The method was developed with the aim to mitigate known shortcomings among interpolation methods, mainly by adapting the algorithm to a financial setting and imposing adjustments which guarantee sought properties, however at an increased complexity and reduced insight in the model behavior of the end user. As discussed in Section 3, du Preez & Maré have extended the method further by addressing known issues, for example where the forward rate curve may become discontinuous under certain circumstances. This has however introduced other weaknesses as the authors themselves mention, indicating that certain properties cannot coexist all at once but rather pose a balancing act. With all this in mind, the forward monotone convex spline interpolation method is deemed a suitable method to accompany the three other methods in order to both give some diversity regarding different families of methods, but also to represent more modern methods whose properties have been tailored to yield curve construction. This method is the most recently proposed one (2006 versus 1975, 1987 & 1994), and was in addition proposed by practitioners of the trade rather than academics [8].

Both interpolation methods are implemented from scratch in Python 3.7 (CPython).

4.2 Evaluation

When evaluating the characteristics of the yield curves, we aim to quantify desirable quantities in order for a fair comparison to be made. The composition of evaluation metrics has been made in order to capture a spectrum of properties and quantities of the yield curve. While some of the metrics provide numerical answers to which methods are preferable to others, other metrics have to be interpreted graphically in order to provide a result. We stress that none of the measures presented below give objective indications of the goodness of a method, but merely an indication of the performance of a method in the area captured by the measure.

4.2.1 Smoothness measure

In an effort to quantify the smoothness of a yield curve, the measure first suggested by Adams & van Deventer for finding the yield curve with a forward rate curve of maximum smoothness is used [1]. The smoothness measure Z, of which a lower value is better, is given by

$$Z = \frac{1}{2}\sum_{t=1.5}^{t_n}\left(\left[f(t) - f\left(t-\frac{1}{2}\right)\right] - \left[f\left(t-\frac{1}{2}\right) - f(t-1)\right]\right)^2 \tag{26}$$

4.2.2 Localness of perturbations in input rates

One feature of interest among yield curve construction methods is how widespread effects a perturbation in one isolated input rate to the method has on the output, further on referenced to as the localness of the method.

Hagan & West made efforts to quantify the localness in their 2006 comparison of interpolation methods, this by deriving localness indices l and u for each method. The localness indices were defined as the smallest indices for which a perturbation in the input rate with tenure t_i had no effect on the interpolant outside the interval $[t_{i-l}, t_{i+u}]$. [8]

The localness indices are used as a quantified measure of localness for all methods included in the evaluation. There is however reason to suspect that no nontrivial information is provided by the indices for these methods, whereupon the localness of each method is illustrated graphically by increasing a single rate across the range of tenors and observing both how the yield curve reacts to the change and how the change is dispersed across the range of tenors.

4.2.3 Sensitivity of perturbations in input rates

Dealing with financial markets, one cannot expect to work with textbook data all of the time. One must rather expect to have a considerable amount of noise in the data, whereupon it is of interest to know how much the result may change given a perturbation in the input rates. Strictly speaking, it is of interest to determine the sensitivity of each method to a perturbation in an input rate.

Let $\|M(y(t,T))\|$ and $\|M(f(t,T)\|$ denote the sensitivity of a given method M and its corresponding output, the yield curve and forward rate curve respectively.

Now, let the measures equal the measure proposed by Hagan & West in their 2006 paper [8]:

$$\|M(y(t,T))\| = \sup_T \max_i \left| \frac{\partial\, y(t,T)}{\partial\, y_i} \right| \tag{27}$$

$$\|M(f(t,T))\| = \sup_T \max_i \left| \frac{\partial f(t,T)}{\partial\, y_i} \right| \tag{28}$$

Here $y(t,T)$ is the yield curve constructed by the method M on the set of input rates $\{y_i\}_{i=0}^n$.

In this thesis, the sensitivity measures are calculated numerically by calculating the largest of all maximum differences between the original curve, yield or forward, and the $2n$ modified curves which are obtained by notching each input yield up and down by one basis point. The differences are calculated at each discretization time step. Finally, the resulting difference is scaled by 10^4 to compensate for the fact that one basis point is 10^{-4}.

4.2.4 Localness of hedges

Let us assume that the instruments whose zero rates are used as inputs to the yield curve construction methods are available for hedging an arbitrary portfolio. The portfolio is assumed to consist of fixed income instruments with a number of known, risky cash flows. Note that by requiring the cash flows to be known, certain substitutions and decompositions might be necessary, for example by deconstructing the two legs of an interest rate swap into a floating rate note and an ordinary coupon bond. Furthermore, only risky cash flows are considered, whereupon the riskless floating rate note can be discarded [8].

The aim of this evaluation method is to construct a hedge for the portfolio using the instruments whose zero rates were originally used as inputs. The hedge, which is a duration hedge, aims to render the risky portfolio unaffected by small changes in any one of the input rates, or equivalently having the total duration of both the hedging portfolio and the risky portfolio be zero. It is desired that the yield curve, when used to hedge a portfolio with the properties described above, produces hedge weights in the underlying instruments with maturities close to the cash flow dates of the portfolio. In the evaluation to come, the portfolio being hedged consists of a synthetic zero coupon bond with 4.5 years to maturity. Intuitively, when using a yield curve constructed from zero yields whose tenors include 4 years and 5 years, the hedge weights should ideally be distributed somewhat equally between the 4 year instrument and the 5 year instrument. While the ideal distribution between the two depends on how the zero curve looks like between the knot points, no sensible hedge can deviate vastly from this allocation during regular market conditions. The evaluation will take into consideration how significant leakage each method has into instruments with neighboring tenors as well as the magnitude of such leakage.

This method of evaluating yield curve construction methods was described in detail by Hagan & West in their 2006 paper on interpolation methods [8], where it was found that the forward monotone convex spline interpolation method produces reasonable hedges in a number of scenarios. This gives a clue of what to expect from the evaluation to come.

Proceeding to calculating the hedge weights, two classic and somewhat similar approaches are used: waves and bumping. The aim of both methods is to produce a hedging portfolio **q** which renders the risky portfolio unaffected by small changes in any one of the input zero rates. Beginning with bumping, we first select a curve construction method and construct a yield curve $y(t, T)$ from some set of zero rates. Then, new curves $y_j(t, T),\ j \in [1, n]$ are created by bumping the j:th rate up by one basis point and constructing a new curve on the modified data set. We then compute the price $\bar{v}$ of the portfolio being hedged using the original curve $y(t, T)$, as well as prices v_j using the curve $y_j(t, T)$. The prices are thereafter used to construct the vector **v** of differences:

$$\mathbf{v} = \begin{bmatrix} v_1 - \bar{v} \\ v_2 - \bar{v} \\ \vdots \\ v_{n-1} - \bar{v} \\ v_n - \bar{v} \end{bmatrix}$$

We move on to calculating a similar matrix of price differences for the instruments whose zero rates are used as inputs to the curve. Let $p_{i,j},\ i, j \in [1, n]$ denote the price of input instrument i under curve j, and let $p_i,\ i \in [1, n]$ denote the price under the original curve. Now construct the matrix P where each element $P_{i,j}$ is the change in price of the i:th instrument under curve j:

$$P_{i,j} = p_{i,j} - p_i$$

In an ideal setting a yield curve would reprice all of its inputs exactly, which is assumed in this calculation. It is only the instrument corresponding to the bumped rate that will not reprice, thus making the matrix diagonal. While knowing beforehand that this assumption does not hold for any of the included methods, it is nonetheless a common assumption to make when calculating these hedges [8]. In addition, the non-diagonal elements which are set to zero in the upcoming calculations are extremely small, both in a relative and an absolute fashion.

Finally, the hedge weight vector **q** is obtained as the solution to the system

$$P\mathbf{q} = \mathbf{v}$$

The portfolio **q** is a perfect hedge for the original portfolio exactly when the yield curve changes because one input rate is bumped by one basis point. Because a change of one basis point is comparatively small, the resulting price change of the original portfolio can be assumed to be linear, whereupon one can hope that the portfolio $\mathbf{q}$ constitutes a suitable hedge for all small changes in the input rates. [8]

The waves procedure for calculating the hedging portfolio is similar to the bumping procedure described above in many aspects. However, when constructing curve j, it is no longer only the j:th input which is bumped, but its adjacent rates as well in a triangular fashion. That is, rates $j-1$ and $j+1$ are bumped by one fourth of a basis point while rate j is bumped by one half of a basis point. On the boundaries, the rate outside of the input set is simply disregarded. Past the creation of the n curves using the modified rates, the waves procedure is identical to the bumping procedure. One finds that the matrix P is not diagonal but upper triangular, but all calculations remain the same.

4.2.5 Effects of global perturbations on the resulting curves

While one can study the effect an isolated change in a rate has on the yield curve, which is described in Section 4.2.2, this only tells something about how the methods react to erroneous data containing outliers. It is also interesting to investigate how the methods react to fundamental changes in the characteristics of the underlying rates, for example due to a fundamental change in the market outlook.

With the help of principal component analysis one can extract the key features which govern the underlying shape of the yield curve. Among economists it is generally believed that the shape of the curve is governed by three global factors: the long term steady rates, i.e. the asymptote of the curve for really long maturities, the term premium or the slope of the curve, which tells how much compensation an investor receives when choosing investments with longer maturities rather than short ones, and the curvature of the yield curve. Back in 1977, Friedman came to the conclusion that the general level together with the slope are the most relevant factors for describing a yield curve [7]. In an idealized world, assuming that the beliefs stated above are correct, one expects the three most influential principal components to explain all of the variance in the data. After extracting these, one is able to introduce global perturbations by modifying the principal components. This is equivalent with fundamentally changing the shape of the zero rates, which can be interpreted as a change in the market's expectations on future economic development.

When applying the PCA on a time series of zero rates, we fulfill the requirement of zero-centered data by subtracting the vector of means across the rows of the data matrix X. These mean values must later on be added to the rates reconstructed from the chosen PCs. There are implementations of PCA on interest rates which argue for the use of the correlation matrix rather than the covariance matrix. This does however require one to standardize the rates into Z-scores by dividing by the standard deviation in addition to subtracting the mean. In our setup, this would yield the same result as using the covariance matrix without standardization, hence the covariance matrix is used.

4.3 Data

While the data is not what is being evaluated in this study, several data sets are sourced in order to capture the performance of the methods on different markets. All metrics presented above require data per a specific date except for when analyzing the effects of global perturbations, which requires a time series in order to perform the PCA. The data collected per one specific date is obtained from Bloomberg whereas the time series data is collected from Refinitiv.

The data retrieved from Bloomberg is per 2022-03-02 and consists of the source yield-tenor-pairs $\{(y_i, T_i)\}_{i=1}^n$ of the curves presented below. n is given in the parentheses.

1. I21 SEK Sweden Sovereign Curve (15)
2. I16 EUR German Sovereign Curve (15)
3. BS558 EUR EU Composite B- BVAL (16)
4. BS627 EUR EU Composite BB+, BB, BB- BVAL (16)
5. BS166 EUR EU Composite BBB+, BBB, BBB- BVAL (16)

Here, the yield is the zero yield provided by Bloomberg's BVAL methodology. The tenors of all curves range from $3M$ to $30Y$.

Curve 1 is the Swedish government yield curve consisting of treasury bills and bonds. Curve 2 is the German counterpart. Curve 3 is a composite yield curve of all B- rated bonds issued in the EU, where the derivation of the composite yield is made according to Bloomberg's BVAL methodology. Curve 4 is a similar composite of all BB+, BB and BB- rated bonds whereas curve 5 encompasses all BBB+, BBB and BBB- rated bonds.

The time series data fetched from Refinitiv ranges from 2010-01-01 to 2022-04-01 and consists of weekly zero coupon bid yields for the following curves

6. SEK Sweden Government Curve
7. EUR German Government Curve
8. EUR Eurozone Eurobond Curve

Each time series entry consists of 18 yield-tenor pairs with maturities ranging from $1M$ to $25Y$. Due to lacking data quality for the earliest dates, the data sets are truncated by the removal of rows containing NaN values. A visualization of all three data sets can be found in Appendix C. Below is the Swedish data set plotted in 3D.

Curve 6 and 7 are the Swedish and German government yield curves, here calculated by Refinitiv rather than Bloomberg. Curve 8 is the zero coupon curve of Eurobonds emitted by the Eurozone members.

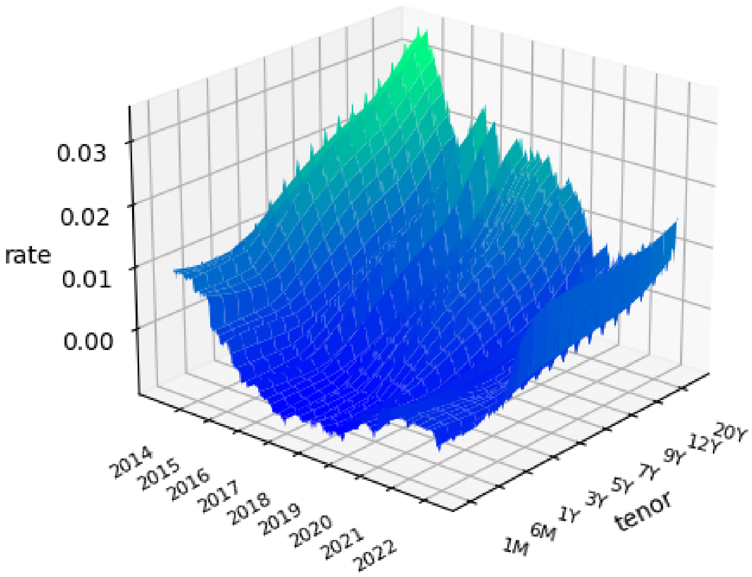


Figure 4.1: SEK Sweden Government Curve

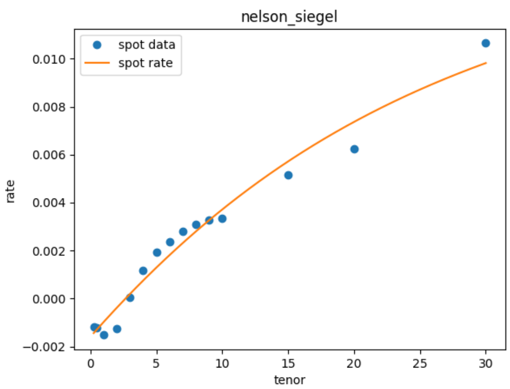
The bid yield is selected in favor of ask-, mid and last yield since it on some philosophical level represents an available trade offer in the market. This while the mid yield is a purely theoretical quote, and no trade is guaranteed to have settled at the mid yield. The ask yield is on the contrary the quote which a market participant is willing to settle for, but one is not able to trade at the ask yield at will. Finally, while the last yield is indeed the latest settlement yield of a trade, one is in general not able to engage in a trade at the particular quote ex post, much like the close yield. This leaves the bid yield as the only sensible option for an actor looking to trade in the market.

5 Results

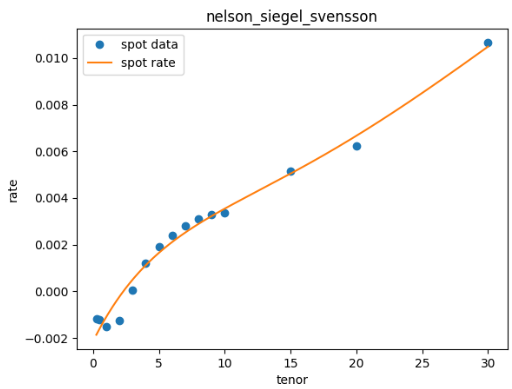
Herein, the results from both the yield curve construction as well as the following evaluation are presented. As all four methods are used to construct the yield curves for the five first data sets presented in Section 4.3, a total of 20 different yield curves were constructed.

5.1 Resulting yield curves

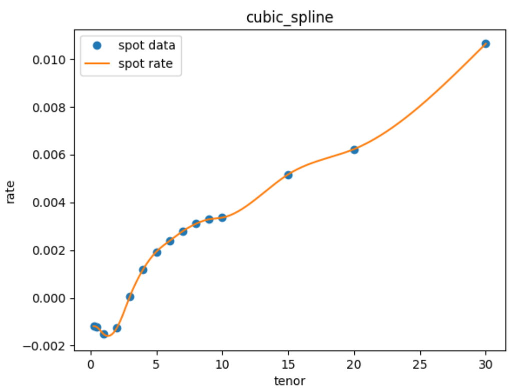
5.1.1 I21 SEK Sweden Sovereign Curve



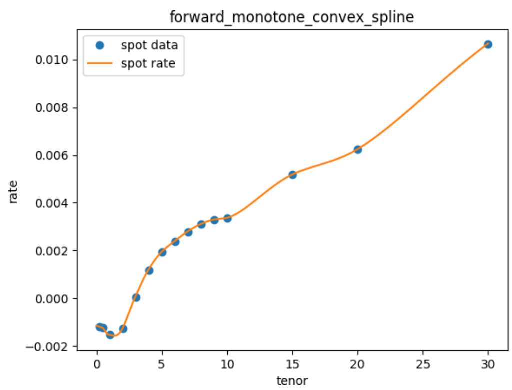
(a) Yield curve constructed using the Nelson-Siegel method described in Section 2.2.1.



(b) Yield curve constructed using the Nelson-Siegel-Svensson method described in Section 2.2.2.



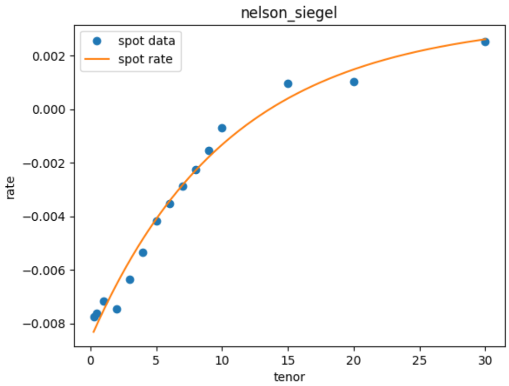
(c) Yield curve constructed using the financial cubic spline interpolation method described in Section 2.2.3.



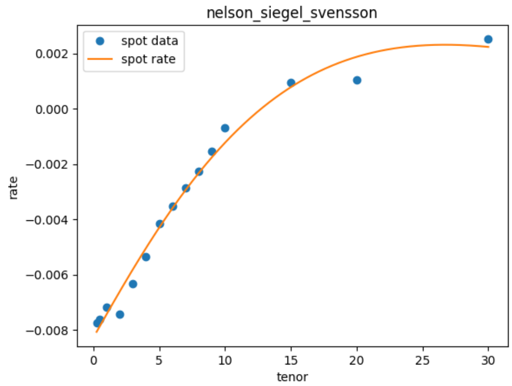
(d) Yield curve constructed using the forward monotone convex interpolation method described in Section 2.2.4.

Figure 5.1: Yield curves constructed from zero yields of Swedish government bills and bonds per 2022-03-02.

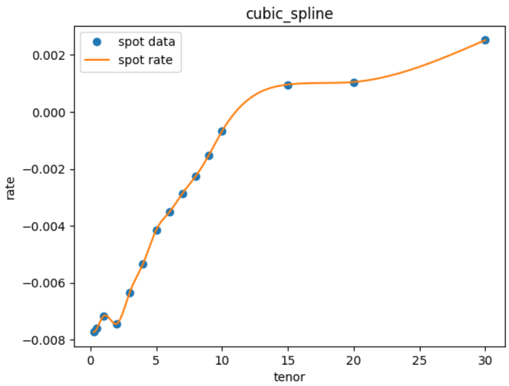
5.1.2 I16 EUR German Sovereign Curve



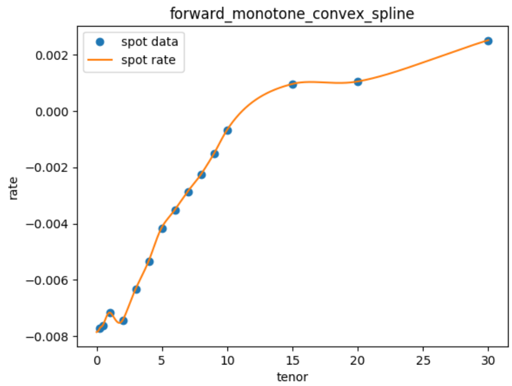
(a) Yield curve constructed using the Nelson-Siegel method described in Section 2.2.1.



(b) Yield curve constructed using the Nelson-Siegel-Svensson method described in Section 2.2.2.



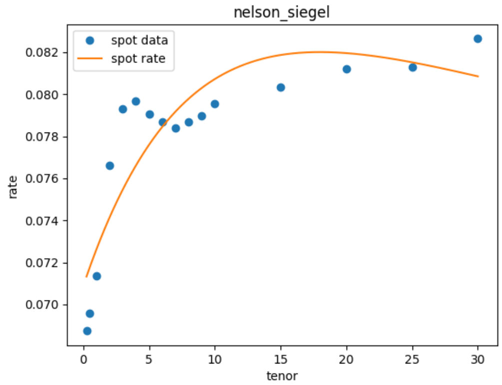
(c) Yield curve constructed using the financial cubic spline interpolation method described in Section 2.2.3.



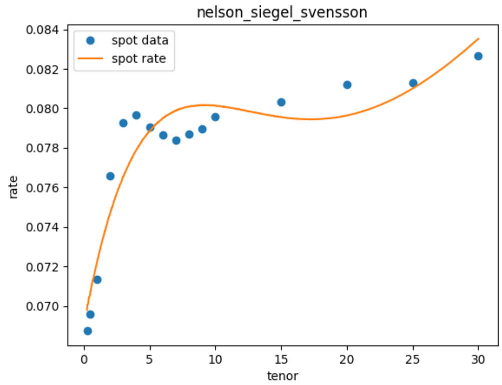
(d) Yield curve constructed using the forward monotone convex interpolation method described in Section 2.2.4.

Figure 5.2: Yield curves constructed from zero yields of German government bills and bonds per 2022-03-02.

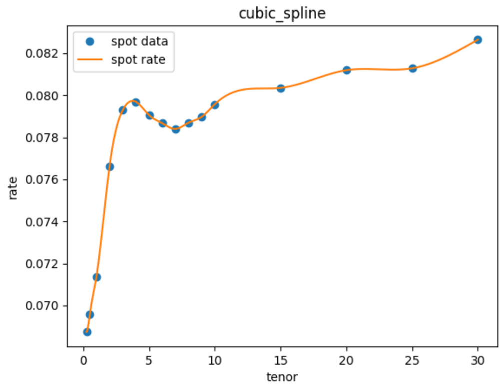
5.1.3 BS558 EUR EU Composite B- BVAL



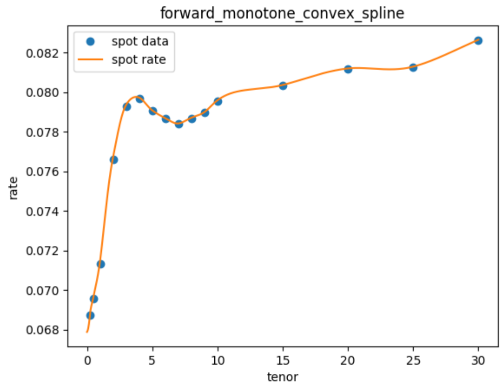
(a) Yield curve constructed using the Nelson-Siegel method described in Section 2.2.1.



(b) Yield curve constructed using the Nelson-Siegel-Svensson method described in Section 2.2.2.



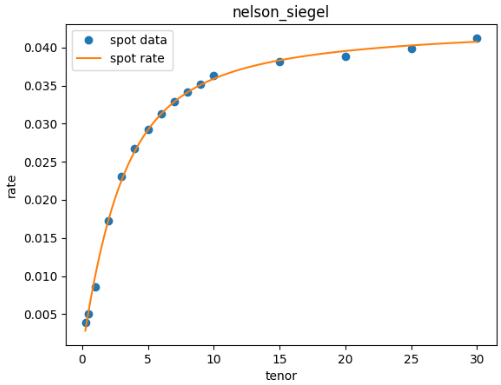
(c) Yield curve constructed using the financial cubic spline interpolation method described in Section 2.2.3.



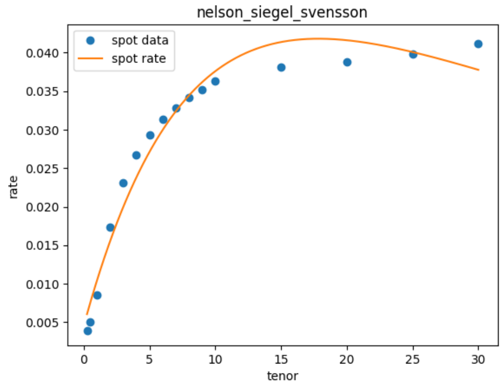
(d) Yield curve constructed using the forward monotone convex interpolation method described in Section 2.2.4.

Figure 5.3: Yield curves constructed from the BVAL composite zero yields of all B- rated bonds issued in the EU per 2022-03-02.

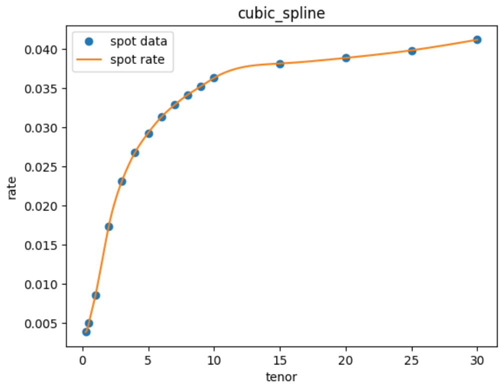
5.1.4 BS627 EUR EU Composite BB+, BB, BB- BVAL



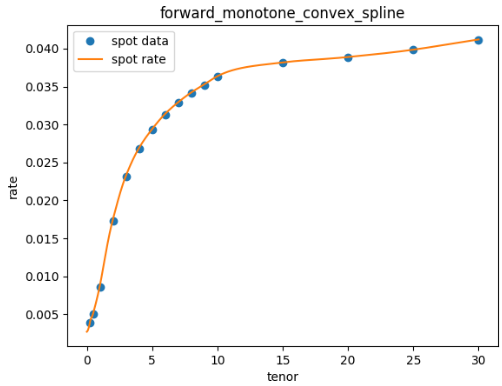
(a) Yield curve constructed using the Nelson-Siegel method described in Section 2.2.1.



(b) Yield curve constructed using the Nelson-Siegel-Svensson method described in Section 2.2.2.



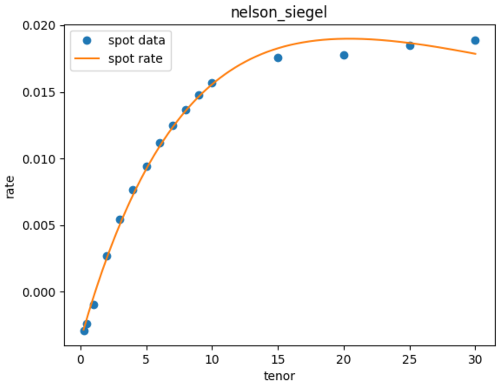
(c) Yield curve constructed using the financial cubic spline interpolation method described in Section 2.2.3.



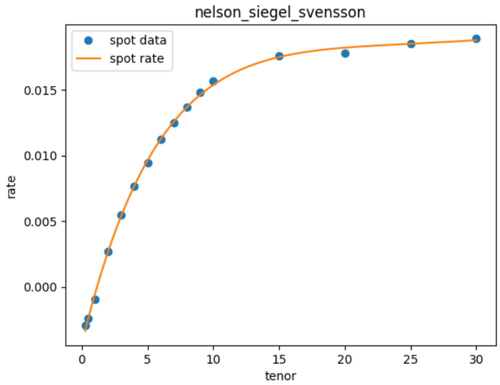
(d) Yield curve constructed using the forward monotone convex interpolation method described in Section 2.2.4.

Figure 5.4: Yield curves constructed from the BVAL composite zero yields of all BB+, BB and BB- rated bonds issued in the EU per 2022-03-02.

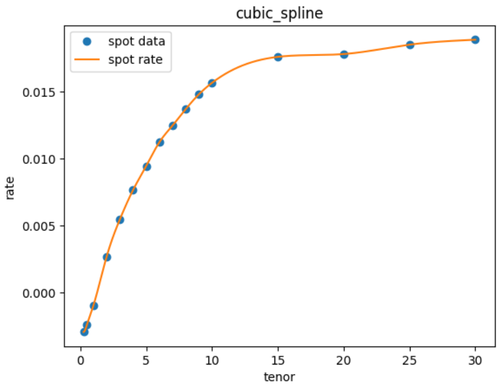
5.1.5 BS116 EUR EU Composite BBB+, BBB, BBB- BVAL



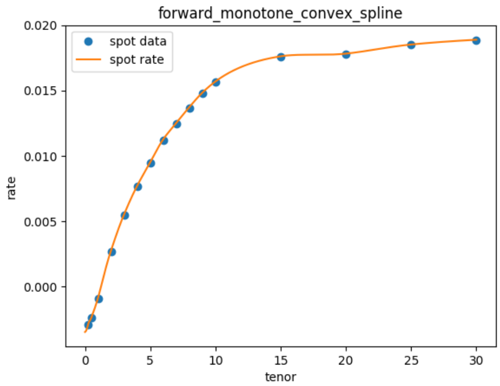
(a) Yield curve constructed using the Nelson-Siegel method described in Section 2.2.1.



(b) Yield curve constructed using the Nelson-Siegel-Svensson method described in Section 2.2.2.



(c) Yield curve constructed using the financial cubic spline interpolation method described in Section 2.2.3.



(d) Yield curve constructed using the forward monotone convex interpolation method described in Section 2.2.4.

Figure 5.5: Yield curves constructed from the BVAL composite zero yields of all BBB+, BBB and BBB- rated bonds issued in the EU per 2022-03-02.

5.2 Smoothness

The smoothness measure Z was calculated for all configurations of yield curve construction methods and data sets. As the measure should ideally equal zero, a lower measure is preferred to a higher one. The findings are presented in Table 5.1.

Table 5.1: The smoothness measure Z given by Equation 26, calculated for yield curves constructed by all methods on the different data sets.

	NS	NSS	CS	FMCS
SWE Government	$7.561 * 10^{-9}$	$3.620 * 10^{-8}$	$3.722 * 10^{-6}$	$6.607 * 10^{-6}$
GER Government	$2.694 * 10^{-8}$	$1.994 * 10^{-8}$	$1.827 * 10^{-5}$	$1.914 * 10^{-5}$
B- Composite	$9.885 * 10^{-8}$	$5.963 * 10^{-7}$	$3.909 * 10^{-5}$	$5.651 * 10^{-5}$
BB Composite	$4.808 * 10^{-6}$	$1.126 * 10^{-6}$	$2.218 * 10^{-5}$	$4.106 * 10^{-5}$
BBB Composite	$3.317 * 10^{-7}$	$4.919 * 10^{-7}$	$7.283 * 10^{-6}$	$1.011 * 10^{-5}$

5.3 Localness of the methods

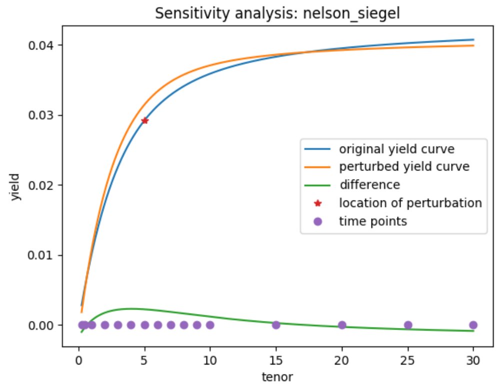
The localness indices presented in Section 4.2.2 are found numerically by introducing perturbations in the data sets and observing the dispersion of the resulting perturbation in the yield curve. For a data set $\{(y_i, T_i)\}_{i=0}^{n}$, the localness indices for each method are presented in Table 5.2

Table 5.2: The localness indices l and u introduced in Section 4.2.2 presented for all methods.

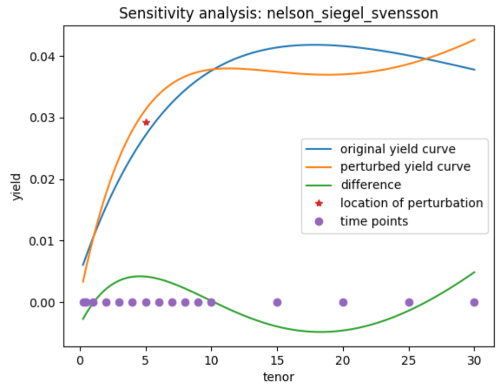
Method	l	u
NS	i	$n - i$
NSS	i	$n - i$
CS	i	$n - i$
FMCS	2	2

As hypothesized in Section 4.2.2, all methods but forward monotone convex spline exhibit an unbounded dispersion of the perturbation across tenors, whereby the localness index does not convey much useful information. The dispersion of perturbations across tenors do however differ between the methods, albeit the dispersion is unbounded, whereupon some illustrative graphs on the matter are presented in Figure 5.6.

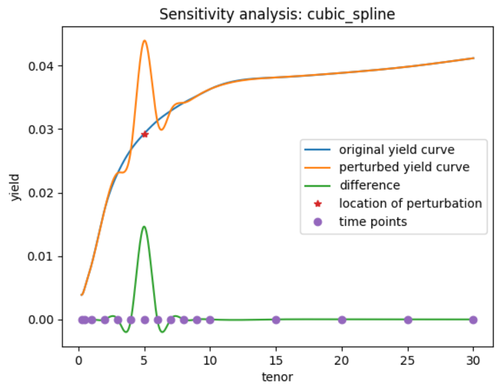
The examples presented in Figure 5.6 are conducted on the BB composite data set. A perturbation is introduced by increasing the 5 year rate by 50%, after which the yield curve is reconstructed and the difference to the original yield curve calculated.



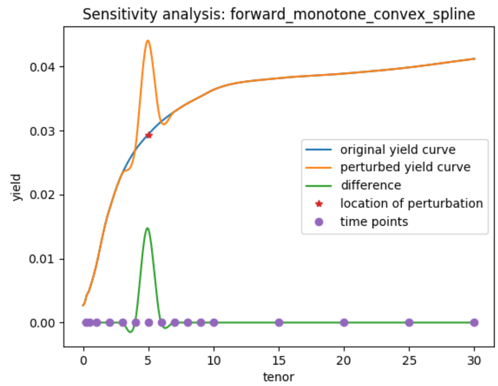
(a) The dispersion of a perturbation for a yield curve constructed using the Nelson-Siegel method.



(b) The dispersion of a perturbation for a yield curve constructed using the Nelson-Siegel-Svensson method.



(c) The dispersion of a perturbation for a yield curve constructed using the cubic spline method.



(d) The dispersion of a perturbation for a yield curve constructed using the forward monotone convex spline method.

Figure 5.6: The dispersion of a perturbation in the form of an increase in the underlying zero rate at $5Y$ of 50% for the different methods. The input rate being perturbed is indicated with a red star, while the time points of all input rates are indicated by purple dots.

Note that the parametric methods exhibit rather different dispersion behaviors than the interpolation methods. These characteristics are discussed further in Section 6.

5.4 Sensitivity of the methods

The sensitivity of each method is evaluated based on the sensitivity measures introduced in Section 4.2.3. These give a measure of the sensitivity of the method for noise in the input data, and quantify what throughput the methods have to spot rates and forward rates. If calculated analytically, the measures do not depend on any data set. However, as the measures are calculated numerically in this evaluation, values are reported for each data set. The measures are presented in Table 5.3 and Table 5.4.

Table 5.3: The sensitivity measure $\|M(y(t,T))\|$ of the spot rates defined by Equation 27, calculated for all methods using the different data sets.

	NS	NSS	CS	FMCS
SWE Government	0.755	7.791	1.671	2.008
GER Government	0.755	0.946	1.671	2.008
B- Composite	0.537	25.624	1.667	2.008
BB Composite	1.328	41.346	1.667	2.008
BBB Composite	0.537	8.839	1.667	2.008

Table 5.4: The sensitivity measure $\|M(f(t,T))\|$ of the forward rates defined by Equation 28, calculated for all methods using the different data sets.

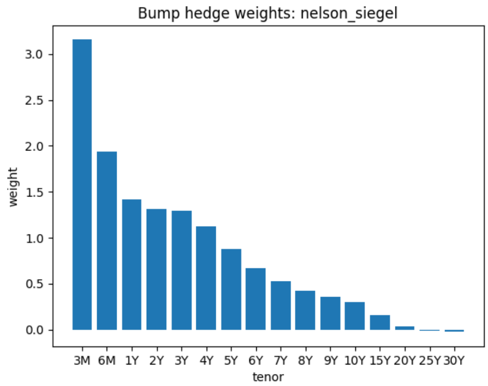
	NS	NSS	CS	FMCS
SWE Government	1.873	18.190	13.837	12.499
GER Government	1.873	3.844	13.837	12.499
B- Composite	1.378	60.697	13.832	12.499
BB Composite	1.832	385.202	13.832	12.499
BBB Composite	1.378	20.841	13.832	12.499

5.5 Localness of the resulting hedges

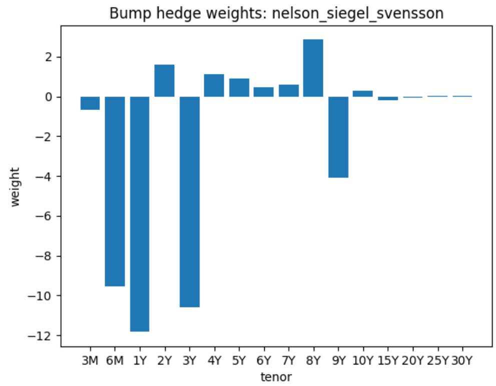
Using both the bump method and the wave method described in Section 4.2.4 to create a hedging portfolio for an arbitrary risky portfolio, we obtain two suggested hedges for each method and each data set, totalling 40 different hedges to be compared. For cubic spline interpolation and forward monotone convex spline interpolation, the hedge weights are nearly identically distributed between all data sets. The same is true for the Nelson-Siegel method, except regarding the BB composite data set. The Nelson-Siegel-Svensson method does however output vastly different hedge weight distributions for all data sets.

Below, only the hedges constructed using the BBB composite zero yields are presented for all methods. The curious reader is reminded that all hedges obtained for the NSS method are presented in Appendix A and Appendix B, and lay a suitable foundation for a discussion of the stability of the method when exposed to varying input data.

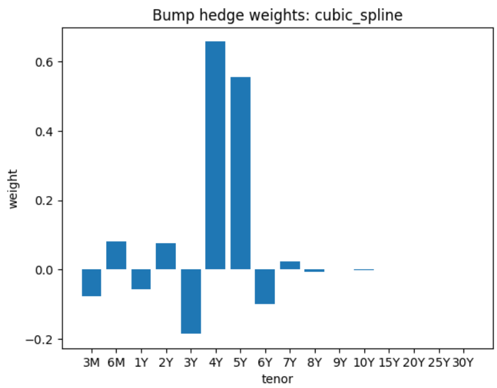
Again, the portfolio being hedged only consists of a synthetic zero coupon bond with 4.5 years to maturity and therefore it is expected to see weights of 50% in the 4 and 5 year bonds in the ideal case.



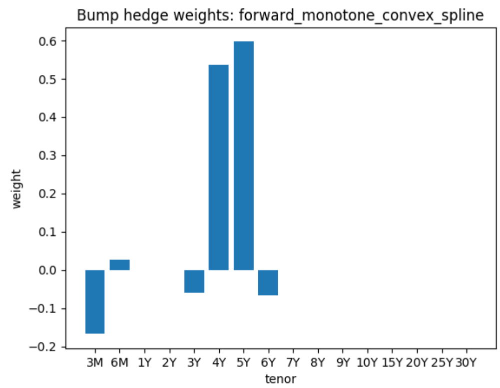
(a) Hedge weights obtained by applying the bump method on the yield curve obtained by the Nelson-Siegel method. The total duration of the hedging portfolio is 37.62 years.



(b) Hedge weights obtained by applying the bump method on the yield curve obtained by the Nelson-Siegel-Svensson method. The total duration of the hedging portfolio is -43.22 years.

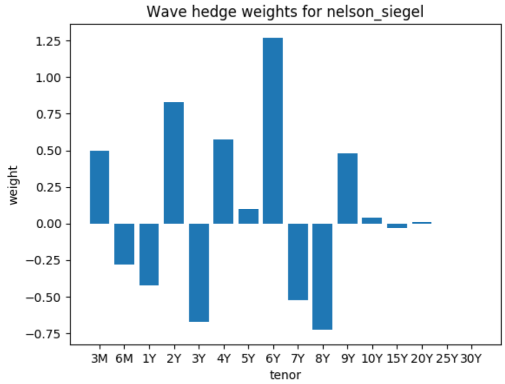


(c) Hedge weights obtained by applying the bump method on the yield curve obtained by the cubic spline method. The total duration of the hedging portfolio is 4.50 years.

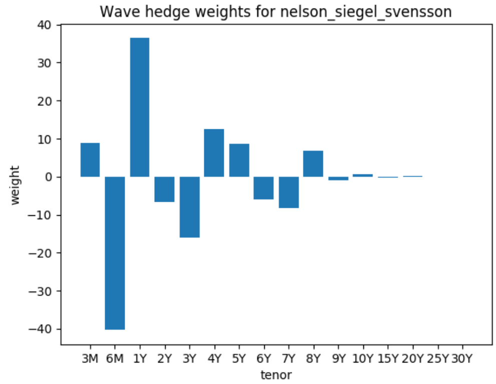


(d) Hedge weights obtained by applying the bump method on the yield curve obtained by the forward monotone convex spline method. The total duration of the hedging portfolio is 4.52 years.

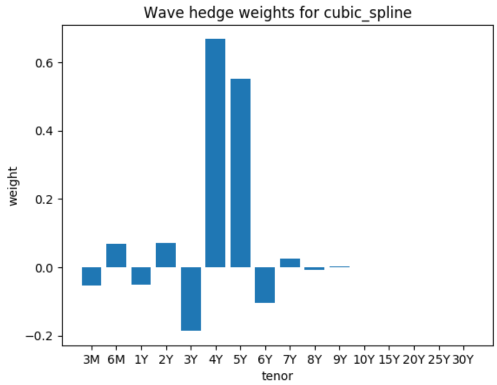
Figure 5.7: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the bump method described in Section 4.2.4. The bump method is using yield curves constructed on the BBB composite data set as inputs.



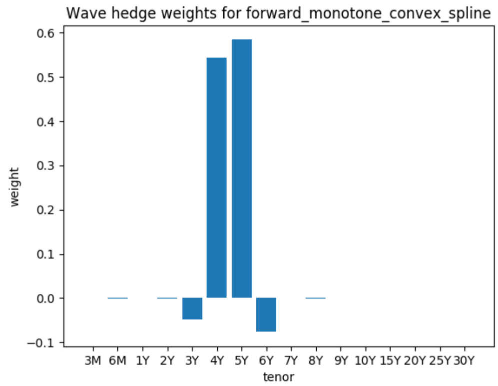
(a) Hedge weights obtained by applying the wave method on the yield curve obtained by the Nelson-Siegel method. The total duration of the hedging portfolio is 4.50 years.



(b) Hedge weights obtained by applying the wave method on the yield curve obtained by the Nelson-Siegel-Svensson method. The total duration of the hedging portfolio is 4.50 years.



(c) Hedge weights obtained by applying the wave method on the yield curve obtained by the cubic spline method. The total duration of the hedging portfolio is 4.50 years.



(d) Hedge weights obtained by applying the wave method on the yield curve obtained by the forward monotone convex spline method. The total duration of the hedging portfolio is 4.50 years.

Figure 5.8: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the wave method described in Section 4.2.4. The wave method is using yield curves constructed on the BBB composite data set as inputs.

5.6 Global perturbations of the principal components

It is clear from Table 5.5 that the three most influential principal components capture a large amount of the variance in all data sets.

Table 5.5: The portion of variance explained by the three most influential PCs.

	PC1	PC2	PC3	Total
SWE Government	88.14%	10.90%	0.78%	99.82%
GER Government	97.10%	2.08%	0.69%	99.87%
Eurobonds	97.10%	2.16%	0.61%	99.88%

Let us remind ourselves of the hypothesis that PC1 corresponds to the height of the curve, PC2 to the term premium or slope of the curve and PC3 to its general curvature. One way of illustrating this is to calculate an estimation of the term premium and compare to the second principal component. Figure 5.9 illustrates this by estimating the term premium as the difference between the $25Y$ and $3M$ rates.

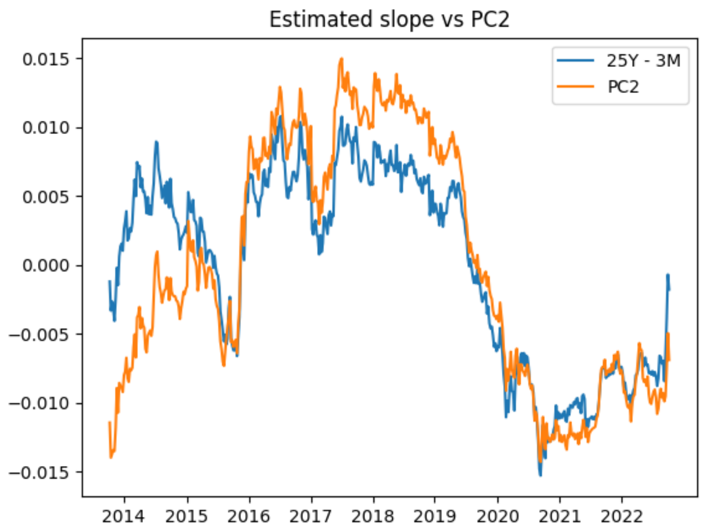
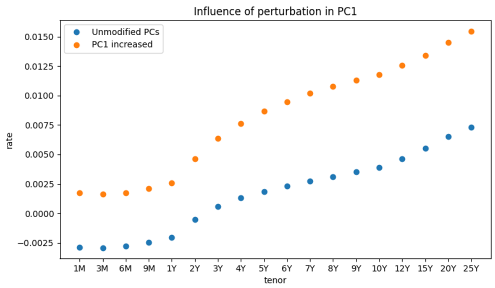
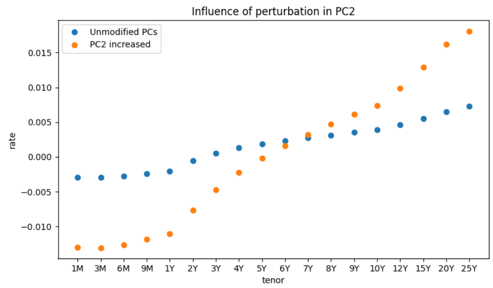


Figure 5.9: The second principal component plotted against the term premium estimated as the difference between the $25Y$ and $3M$ rates in the Swedish data set. The correlation between the two series is 88.575%.

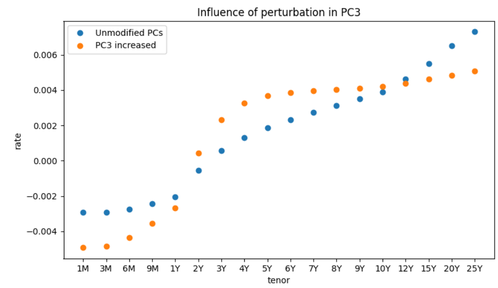
Moving on to introducing global perturbations in the data sets, we have three principal components with unique interpretations. In Figure 5.10, the effects of a perturbation in each of the three PCs on the reconstructed zero yields are illustrated to help with the intuition moving forward. Note that the principal components are disturbed by factors with different signs, this is due to the PCs having different directions. If one regards the PCs as orthogonal axes spanning a subspace of the original data, one realizes that the direction (positive or negative) of each PC is not important.



(a) PC1 (height) decreased by a factor 4



(b) PC2 (slope) decreased by a factor 4



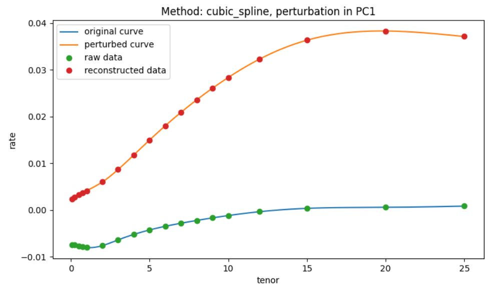
(c) PC3 (curvature) increased by a factor 4

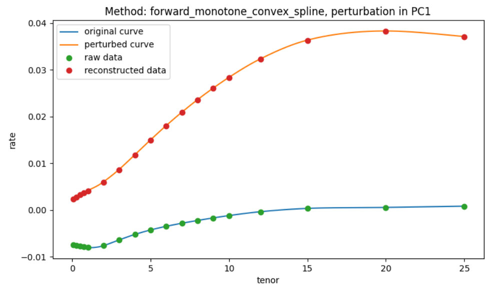
Figure 5.10: An illustration of the effects perturbations in the three PCs have on the reconstructed zero rates. Here, the zero rates per 2022-03-04 are plotted.

Moving on to using perturbations in the PCs to investigate the responses of the methods to new market scenarios. We now fit each method to both the original data per 2022-03-04 as well as the reconstructed data per the same date, where only the three first PCs are kept and one PC is disturbed.

The interpolation methods perform rather well when exposed to the global perturbations and in particular no sinusoidal behavior is noted for the cubic spline method in any configuration. Figure 5.11 presents examples of the behavior of the interpolation methods for all three varieties of perturbations.

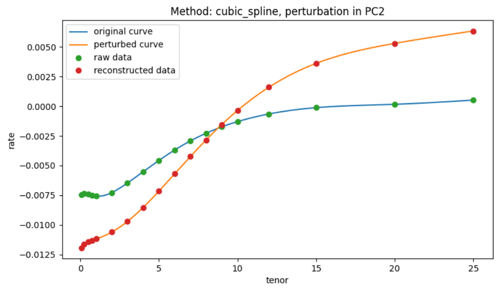
When evaluating the interpolation methods on the same perturbations, no noteworthy difference between the two can be observed.

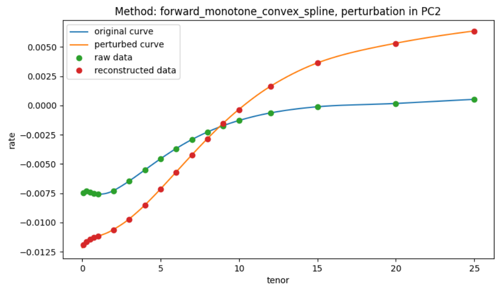




(a) Cubic spline reaction to a perturbation in PC1 on the German data set.

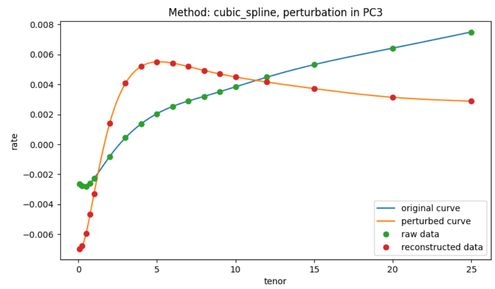
(b) Forward monotone convex spline reaction to a perturbation in PC1 on the German data set.

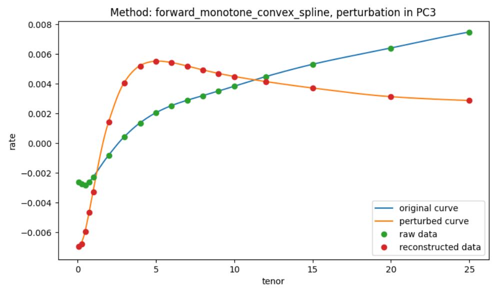




(c) Cubic spline reaction to a perturbation in PC2 on the Eurobond data set.

(d) Forward monotone convex spline reaction to a perturbation in PC2 on the Eurobond data set.



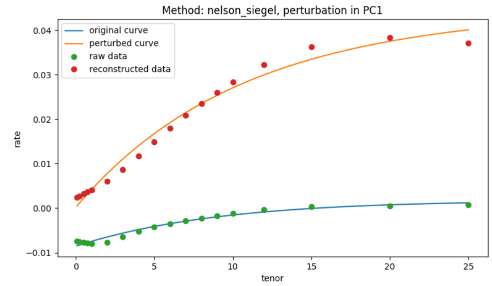


(e) Cubic spline reaction to a perturbation in PC3 on the Swedish data set.

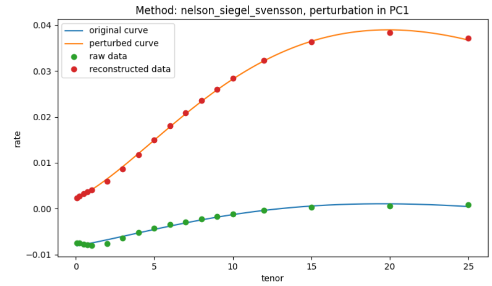
(f) Forward monotone convex spline reaction to a perturbation in PC3 on the Swedish data set.

Figure 5.11: An example of the behavior of the interpolation methods when exposed to global perturbations. Here, the zero rates per 2022-03-04 are plotted.

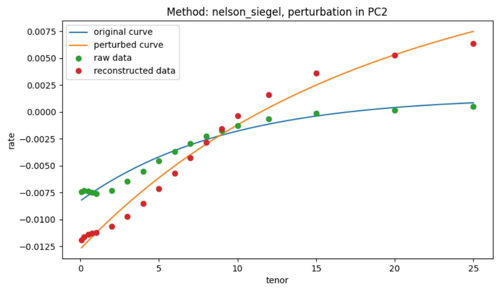
The parametric methods do however exhibit more variation in their reactions to the global perturbations. While changes in PC1 and PC2 are generally well handled, changes in curvature (PC3) pose a greater problem for the Nelson-Siegel method in particular. Examples of this are presented in Figure 5.12.



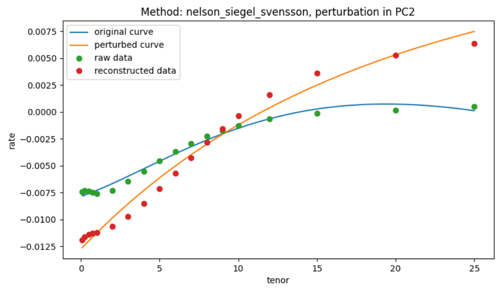
(a) Nelson-Siegel reaction to a perturbation in PC1 on the German data set. $R^2 : 0.9887$



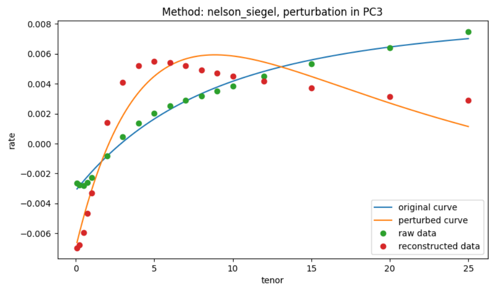
(b) Nelson-Siegel-Svensson reaction to a perturbation in PC1 on the German data set. $R^2 : 0.9997$



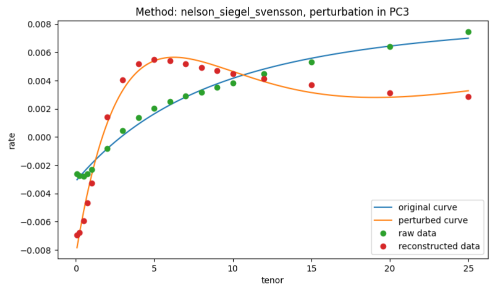
(c) Nelson-Siegel reaction to a perturbation in PC2 on the Eurobond data set. $R^2 : 0.9892$



(d) Nelson-Siegel-Svensson reaction to a perturbation in PC2 on the Eurobond data set. $R^2 : 0.9892$



(e) Nelson-Siegel reaction to a perturbation in PC3 on the Swedish data set. $R^2 : 0.9435$



(f) Nelson-Siegel-Svensson reaction to a perturbation in PC3 on the Swedish data set. $R^2 : 0.9901$

Figure 5.12: An example of the behavior of the parametric methods when exposed to global perturbations. Here, the zero rates per 2022-03-04 are plotted.

6 Discussion

Methods for constructing yield curves are available in abundance and the existing methods are continuously enhanced by both the private sector as well as by financial mathematicians. Therefore, it comes to no surprise that one can obtain outstanding performance in certain aspects by selecting methods specialized therein. While this might be interesting for those whose applications are narrow and require yield curves with strict properties, the more general applications hardly need to have such strict criteria imposed on the method. For those working with the more general applications, the question of interest should rather be what methods are most suitable for obtaining satisfactory performance in a broad range of aspects.

By studying Figures 5.1 through 5.5, it becomes rather apparent how the yield curves constructed by parametric methods differ from those constructed by interpolation methods. Naturally, one should look for an interpolation method if the yield curve is forced to pass through all knot points whereas a parametric curve rather fits to the general shape that the zero rates span.

What is especially apparent in Figure 5.3 is that the increased ability of the interpolation methods to fit underlying data lacking in smoothness enables them to capture sharp changes in the zero rates. As is visible in Figures 5.3 through 5.5, the zero rates exhibit a more digital behavior when moving down the rating chart from investment grade bonds to high yield bonds, which suggests that interpolation methods might have advantages to parametric methods in a high-yield bond universe.

The smoothness measure presented in Table 5.1 clearly speaks in favor of the parametric methods over the interpolation methods. The Nelson-Siegel and Nelson-Siegel-Svensson methods obtain a measure one to three orders of magnitude smaller than the cubic spline and forward monotone convex spline methods, which is not entirely unexpected as the yield curve of a parametric method follows the general shape of the underlying data, but is not required to pass through the knot points. It is further noted that the two parametric methods outperform each other on different data sets, making it difficult to draw accurate conclusions about how the smoothness of their resulting yield curves compare using this measure. Finally, the cubic spline method consistently obtains a lower value than the forward monotone convex spline method. One reason for this is that while there are more requirements posed on the interpolant of the forward monotone convex spline method, no requirements are posed on the derivatives of the interpolant at the knot points. On the contrary, the cubic spline method requires continuity of both the first and second derivative at the knot points.

The localness indices l and u are, as hypothesized in Section 4.2.2, not of much use for the evaluation in this thesis. In their 2006 study, Hagan & West [8] come to a similar conclusion. Even though their study encompasses a large number of methods, most methods disperse perturbations in the input rates across the whole range of tenors. They find that except for the forward monotone convex spline interpolation method, it is a feature of simple interpolation methods to keep perturbations local to their point of origin.

This being said, the dispersion behaviors of the methods differ greatly, which is illustrated in Figure 5.6. Again, the different behaviors between parametric methods and interpolation methods is clear. The parametric methods generally adjust their parameters to take the perturbation into account without changing the general shape of the yield curve too much, however certain perturbations cause the parameters to change vastly. These different behaviors are depicted in Figure 5.6a and Figure 5.6b, where the Nelson-Siegel method adjusts slightly to incorporate the perturbation into the optimization of its parameters, whereas the Nelson-Siegel-Svensson method changes its shape entirely. Intuitively, this is a result of the different ”shape”-terms mentioned in Section 2.2.1 fitting to another subset of the input rates. Both interpolation methods generally deal with the perturbation by producing a spike in the yield curve at the location of the disturbance. In the case of the forward monotone convex spline method, the spike vanishes completely within the range of 2 knot points, as the localness indices tell. The cubic spline method behaves similarly when studying Figure 5.6c, however as one might conclude from the localness indices the dispersion is unbounded. During the work on this thesis, perturbations have been found which produce what is best described by a sinusoidal throughput in the yield curve which is increasing in t.

Summarizing the findings above, it has been found that while the parametric methods generally react to perturbations in the input rates with smaller changes in the yield curve in absolute terms, a perturbation is guaranteed to influence the yield curve across the whole range of tenors. There are however special cases for which the curve changes its fundamental shape. On the other hand, the interpolation methods exhibit a strong local response to the perturbation which mainly influences the adjacent tenors of the curve.

The findings above are further supported by the sensitivity measures, which quantify the magnitude of the throughput of a given change in the input rates. First note that the sensitivity of the interpolation methods are very similar across all data sets. This is not unexpected as the analytical measure is independent of the data set, however results are presented for all data sets as the parametric methods exhibit unstable behavior when varying the underlying data. Indeed, the Nelson-Siegel-Svensson exhibits erratic behavior for the B- and BB composite data sets. This is, as explained above, a result of the different ”shape”-terms fitting to different subsets of the input rates when modifying the inputs slightly, leading to the large measures seen in Table 5.3 and Table 5.4. This behavior has to the best of my knowledge not been observed in prior research on the method. Nelson & Siegel (1987) did however notice that their original method quickly became overparameterized when adding extra parameters, something Svensson might have taken too lightly when proposing his extended method.

Moving on, both the cubic spline method and the forward monotone convex spline method have sensitivity measures of approximately 2 for the spot rates and 12-14 for the forward rates. This while the parametric methods have measures less than 1 for the spot rates and 1-4 for the forward rates. That is, when the methods do not exhibit unstable behavior as a result of the bumping of input rates which occur during the numerical calculation of said measures. This supports the conclusion that noise in the input data generally has small effects on the yield curve and forward rate curve at any given tenor for the parametric methods, but as observed before, noise at any given tenor tends to influence the whole range of tenors in the output. The interpolation methods do, on the other hand, produce larger errors in the yield curve and forward rate curve, but these are generally confined to a smaller range of tenors. The forward monotone convex spline method takes things a step further by strictly limiting the influence of the noise to plus and minus two knot points.

The ability to hedge arbitrary claims, albeit with known cash flows, is rather fundamental in fixed income portfolio management, and it is therefore something reasonable to expect a yield curve construction method to handle well. Whereas one soon realizes that the simple claim hedged in this thesis hardly constitutes a fair proxy for claims hedged in real world portfolio management scenarios, the hedge weights presented graphically in Figure 5.7 and Figure 5.8 nonetheless tell much about the suitability of each method for hedging applications.

Once again, an initial observation from studying the hedge weights is the vastly different behaviors between the two classes of methods. Three observations can be made, first that the parametric methods exhibit rather different behaviors between the bump and wave method while the interpolation methods exhibit nearly identical behavior for both hedging methods. Second, while the hedges obtained for the cubic spline method and the forward monotone convex spline method are rather similar, the hedges obtained for the Nelson-Siegel method and the Nelson-Siegel-Svensson method are very different from each other. Third, when studying the duration of the resulting hedging portfolio, the interpolation methods land in the ballpark of the expected 4.5 years, but the duration of the hedges obtained with the parametric methods varies significantly.

Some conclusions one may draw from the distribution of hedge weights are that the parametric methods do not deal with the hedging methodology used in this evaluation in a good manner. Even though the Nelson-Siegel method yielded a hedge with a reasonable duration in one scenario presented in Figure 5.8a, there is some significant leakage of hedge weights outside of the expected instruments of $4Y$ and $5Y$. As can be seen more clearly by studying all hedging results in Appendix A, more specifically in Figure A.1, all bump hedges using this method either results in an unreasonable duration of the hedging portfolios or a significant leakage of hedge weights. When using the wave method together with the Nelson-Siegel method, the hedging portfolio have reasonable durations, however the leakage into unreasonable instruments is still significant. The hedges obtained using the Nelson-Siegel-Svensson method manifests these conclusions further with similar results. An interesting observation is however that the wave hedges have correct durations to a larger extent compared to the bump hedges, however still with significant hedge leakage.

The interpolation methods on the other hand produce hedges with reasonable durations in all configurations. In fact, the hedges do not differ notably between the different data sets. The difference rather lies in that the wave hedges exhibit a smaller leakage compared to the bump hedges, and most notably that the forward monotone convex spline method consistently yields a smaller leakage than the cubic spline method. While the cubic spline method exhibits what is best described as a sinusoidal pattern, the only unexpected hedging position for the forward monotone convex spline method is a short position in the $3M$ bond, which only occurs for the bump hedge. None of the interpolation methods yield notable hedging positions in bonds with longer tenors than $5Y$, which aligns with the expectations. Between these methods, one is safe to say that the forward monotone convex spline yields a more satisfactory result as the sinusoidal leakage pattern of the cubic spline method is not present.

When studying how the different curve construction methods handle global perturbations, it is of no doubt that the interpolation methods show greater flexibility for dealing with variable market conditions compared to the parametric methods. While the latter handle parallel shifts of the zero yields rather well, more complex shifts in the yields proved more difficult to accommodate for the Nelson-Siegel method as Figure 5.12e shows. In particular a shift in PC3, corresponding to an increased curvature of the zero rates, caused a decrease in the R^2 of the curve fitted to the perturbed rates to 0.9435. Here, the additional hump term of the Nelson-Siegel-Svensson allows the yield curve to follow the zero rates more closely with $R^2 = 0.9901$, showing that the effort made by Svensson to increase the flexibility of the method had the sought effect [17]. The interpolation methods have an R^2 equal to 1 by design as the splines are connected at the knot points, so while their advantage to the parametric methods in this regard is not unexpected, the interpolation methods are still susceptible to erratic behavior when facing perturbations. This holds true in particular for the cubic spline method as noted by Hagan & West (2006) and Muhtoni (2015) [8, 14]. Such erratic behavior could however not be induced by modifying the principal components of the three time series available in this study.

To reiterate the research questions, we seek to find which yield curve construction method outputs the most viable yield curve for the European markets, given the requirements on both financial and mathematical properties posed in Section 4. The requirements are mainly set to capture properties frequently required for financial applications, as well as properties on stability and smoothness with the goal of ensuring a nice behavior of the resulting yield curve and its stability. The research question above is extended by asking how robust the method is when facing variable market conditions, which has been investigated by analyzing method performance across data sets with ratings ranging from AAA to B- as well as by decomposing time series of rates fundamental properties using PCA and studying the responses of the methods to perturbations in these.

What quickly emerges from the evaluation results is that the four methods all have different strengths and weaknesses. This comes as no surprise as the methods are inherently different and are designed for different purposes. While the parametric methods are superior regarding the smoothness of the yield curves as well as having low sensitivities to perturbations in the input rates, they do not provide satisfactory hedges using neither the bump nor wave procedure. In addition, especially the Nelson-Siegel-Svensson method exhibits rather serious instability issues when facing noise in the input rates. The interpolation methods produce yield curves which are not as smooth nor as unaffected by noise in the input rates. They are however much more suitable for calculating hedging portfolios, and do so rather well. In this regard, the forward monotone convex spline method performs especially well. Furthermore, they show a greater ability to adapt to variable market conditions than the parametric methods. The two classes of methods also react differently to noise in the input data, where the parametric methods tend to distribute perturbations across the entire range of tenors while the interpolation methods tend to keep the effects local. Here, the forward monotone convex spline method outshines the cubic spline method in the sense that the effects of a perturbation are strictly limited to plus and minus two knot points.

Weighing in all evaluated factors, the forward monotone convex spline method outperforms the cubic spline method in the sense that it ensures that input noise is contained close to its origin and that it is more suitable for hedging purposes. It does not suffer from the sinusoidal behavior the cubic spline method exhibits in certain scenarios either, as noted by Muthoni (2015) [14]. In addition, the method is designed to guarantee positivity, convexity and monotonicity of the forward rates given that the properties hold for the input rates, something the cubic spline method does not. The main area where the cubic spline method excels is that its sensitivity measure is somewhat smaller for the spot rates, but this is not something major enough to outweigh its shortcomings.

Between the Nelson-Siegel method and the Nelson-Siegel-Svensson method, the case is not as clear cut. Both methods outperform on different data sets, see for example Figure 5.4 where the Nelson-Siegel method constructs the more reasonable yield curve of the two, whereas the opposite holds in Figure 5.5. The Nelson-Siegel-Svensson method has shown problems with instability during the evaluation, which leads the metrics to strongly prefer the Nelson-Siegel method. This holds for the smoothness measure, the analysis of localness of perturbations and the sensitivity measures for both spot rates and forward rates. When looking at global perturbations, the additional flexibility of Svensson's extension does however prove valuable. As neither one of the methods are suitable for constructing hedging portfolios, this metric has to be left without regard between the two methods. Taking these factors into account, the Nelson-Siegel method is clearly the preferred method of the two when imposing the aforementioned requirements, especially on stability and robustness. As the Nelson-Siegel-Svensson method is an extension of the Nelson-Siegel method, and this method sometimes turns unstable when varying the input rates, one can suspect that additional extensions of the core method would further worsen its robustness.

Regarding the choice between the Nelson-Siegel method and the forward monotone convex spline interpolation method, things get more complicated.

While the interpolation method has the versatility of easily adapting to different types of data sets and being able to hedge arbitrary claims using the bump and wave procedures, the parametric method performs better on the smoothness and sensitivity criteria. Another advantage of the parametric method is its rather intuitive representation, see Equation 7 and Figure 2.1. This compared to the set of splines constituting the interpolation method, see Equation 16. This advantage was even highlighted by Nelson & Siegel themselves [16].

The methods handle perturbations in the input rates rather different, where the behavior of both methods might be favorable in different applications. By limiting the dispersion of perturbations, the forward monotone convex spline method ensures that noise in one input yield never disturbs the yield curve at unrelated tenors, thereby ensuring that subsequent valuations using the curve depend on a smaller set of factors. The local effect of a perturbation is however much larger than for the Nelson-Siegel method, which is unfavorable if one wishes to view the larger trend of the curve. In a practical setting, one might want to adjust the curve manually by adjusting the input rates, this with the goal to correct for perceived errors in the market's bond valuations. For such an application, the parametric method would be far more suitable as the general shape of the curve is preserved, as opposed to the interpolation method whose resulting curve changes by a local kink.

7 Conclusions

In this paper, four frequently used yield curve construction methods have been evaluated on a set of criteria. The method selection has been conducted with the goal of including promising and readily used methods from different families of curve construction methods. Furthermore, the evaluation criteria are designed and selected to encompass both mathematical and financial features often required in yield curves.

The results clearly illustrate the differences between the evaluated methods, and especially the differences between parametric methods and interpolation methods. While the extension made by Svensson (1994) to the Nelson-Siegel method shows greater flexibility than the core Nelson-Siegel method, it exhibits unstable behavior when exposed to noisy input rates and varying data sets. The Nelson-Siegel method is therefore deemed more suitable for general use on European markets given the evaluation criteria. The instability exhibited by the Nelson-Siegel-Svensson method has to the best of my knowledge not been identified in previous research, whereupon users of this method should be aware of the effects noisy data might have on the resulting yield curve.

Moreover, the parametric methods outperform the interpolation methods when it comes to smoothness of the yield curve and its sensitivity to noise in the input rates. The interpolation methods do on the other hand show a greater ability to adapt to varying market conditions as well as to produce valid hedges for arbitrary fixed income claims. The cubic spline method exhibited the sinusoidal behavior suggested by prior research both when being exposed to a perturbation in the input data as well as when hedging a zero coupon bond, whereupon the forward monotone convex spline method is preferable among the interpolation methods.

The ultimate decision on which of the two resulting methods to use, that is the Nelson-Siegel method and the forward monotone convex spline method, depends on the application at hand. Both methods have their clear advantages, partly due to their fundamentally different designs, which must be weighed against each other based on their respective importance to the application in question. To assist in this assessment, the features and properties of both methods have been highlighted and discussed throughout this thesis.

7.1 Future work

There are several openings for extending the analysis performed in this project. First and foremost, the method selection was performed in order to encompass several different types of yield curve construction methods. This was done in order to provide an orientation of the performance of different categories of methods in European markets in current times. The rather different behaviors of the methods has been well illustrated, and a natural expansion of this study would be to perform a similar evaluation within each category. That is, among parametric methods and interpolation methods respectively.

Second, as one soon realizes, the result of an evaluation heavily depends on the metrics by which the methods are evaluated. The selection of the metrics used in this evaluation has been described in Section 4, but are naturally to be refined according to what features are of interest in the context the study is conducted. Previous studies have ranged from providing mathematical derivations of properties among methods and a subsequent comparison of these to statistical measures as root mean squared error and similar measures of the goodness of fit.

Third, at the times during which the field of mathematical finance aimed much attention to yield curve construction methods, mainly the late twentieth century, it was widely believed that interest rates were always positive in a well functioning market. Several yield curve construction methods therefore require this for their suggested properties to hold. This has come to change in the recent decades, whereupon the requirement has been relaxed in some, but not all modern curve construction methods. As late as in Muhtoni (2015) we see the equivalent claim that the discount function has to be monotonically increasing as a criterion of a method implemented correctly. Both the forward monotone convex spline method introduced in 2006 by Hagan & West and the monotone preserving $r(t)t$ method introduced in 2013 by du Preez & Maré explicitly require a monotonically increasing discount function for all claimed properties to hold, however du Preez & Maré (2013) mention that the requirement can be relaxed in practice without too large consequences. With this in mind, an expanded study could focus on what consequences this leads to in practice, and evaluate methods which have relaxed this criterion against those who still pose the requirement in a current market setting.

Finally, the instability of the Nelson-Siegel-Svensson method is highly interesting to investigate further. While the method is generally well behaved when applied on ”nice” data, its sensitivity to noise in the input rates must be considered when considering the method for different applications.

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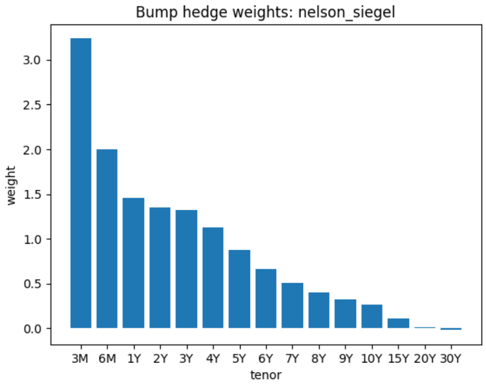
Appendix

A Localness of bump hedges 54

B Localness of wave hedges 58

C Time series 3D plots 62

A Localness of bump hedges



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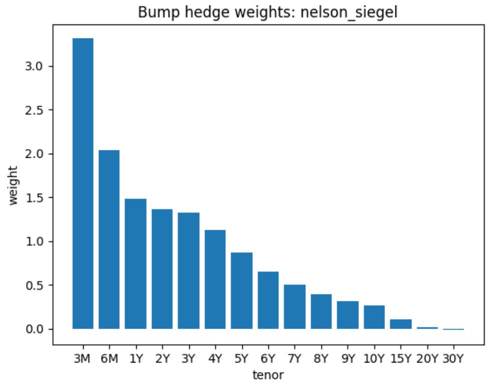

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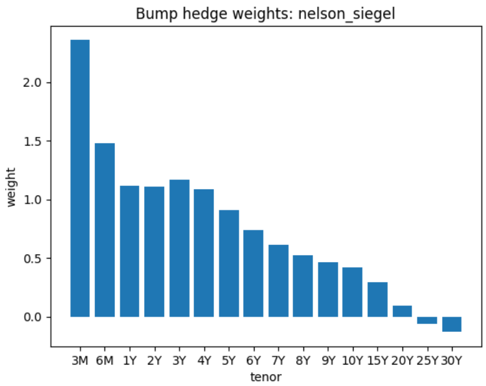

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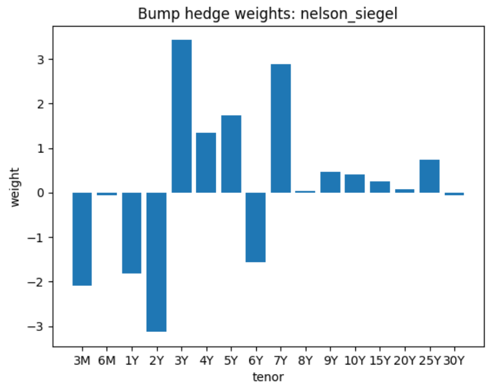

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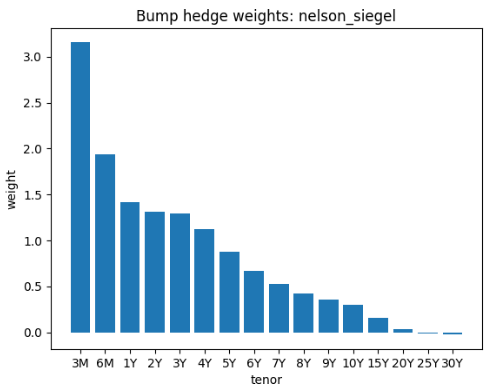

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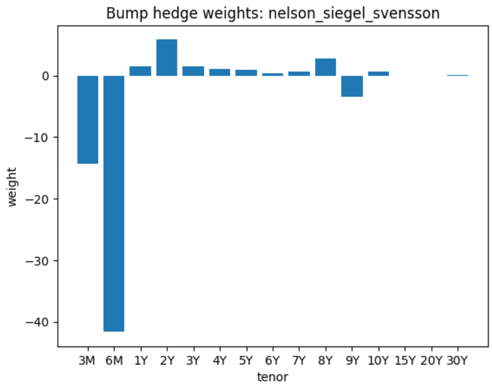
Figure A.1: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the bump method. The bump method is using yield curves constructed by the Nelson-Siegel method with varying underlying data sets.



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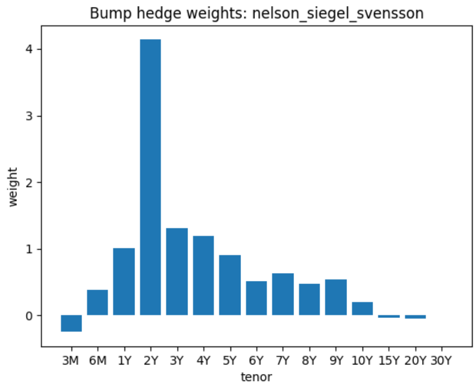

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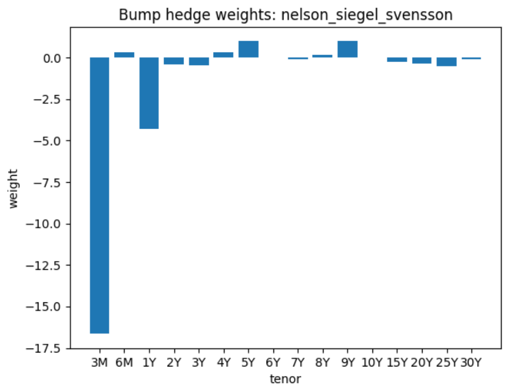

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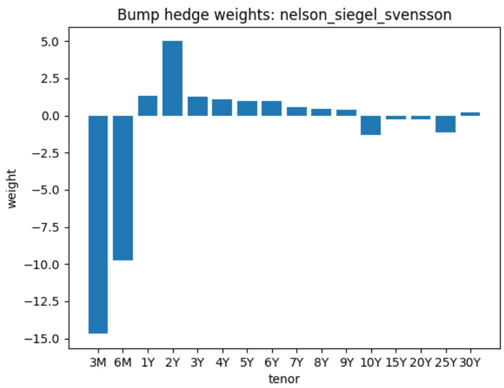

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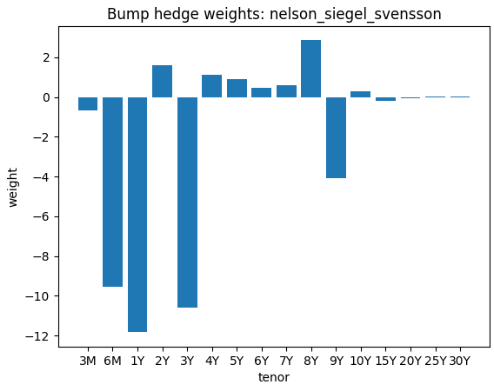

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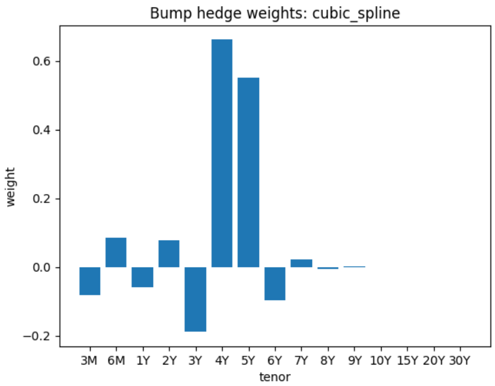
Figure A.2: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the bump method. The bump method is using yield curves constructed by the Nelson-Siegel-Svensson method with varying underlying data sets.



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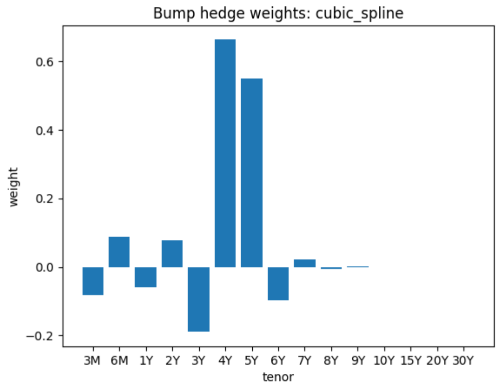

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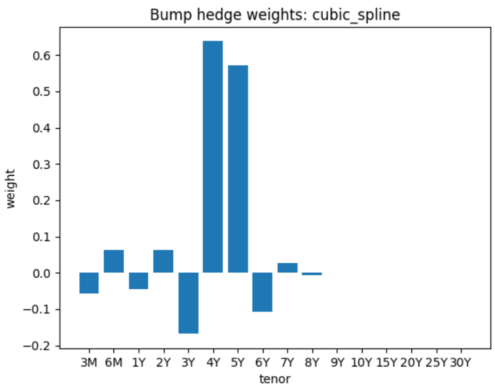

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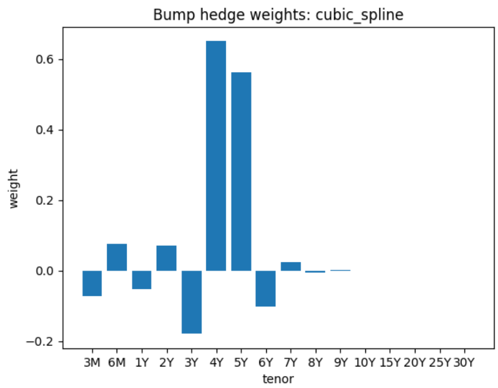

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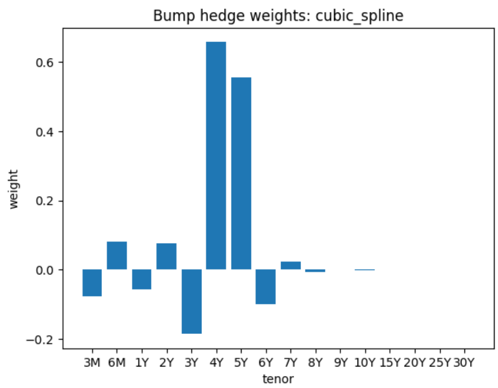

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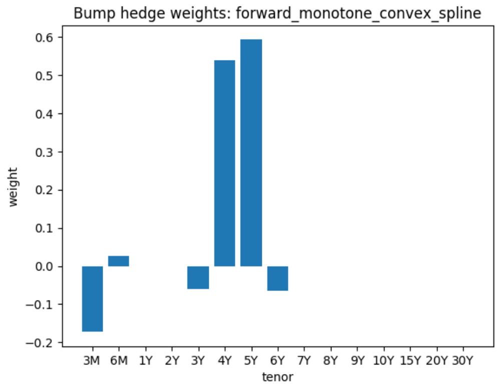
Figure A.3: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the bump method. The bump method is using yield curves constructed by the cubic spline interpolation method with varying underlying data sets.



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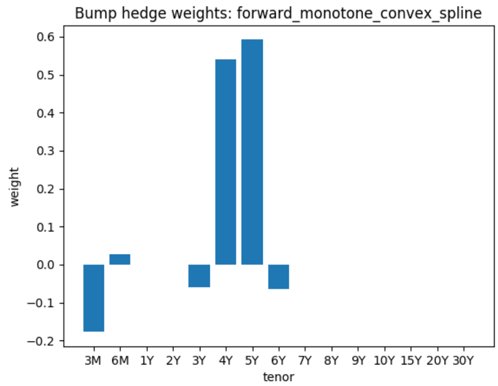

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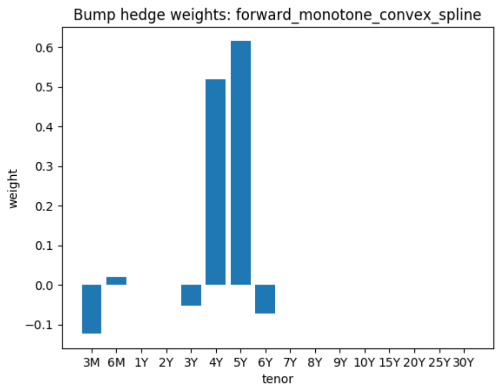
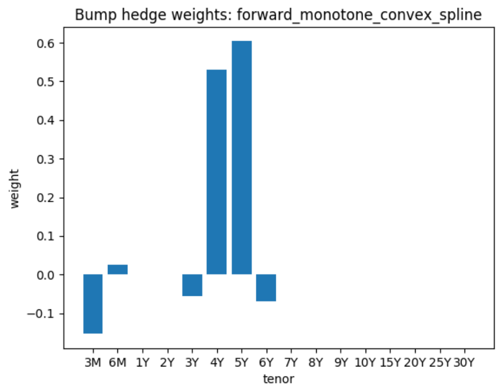
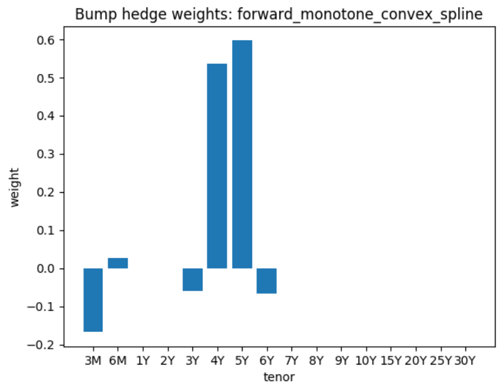


(c) BS558 EUR EU Composite B- BVAL



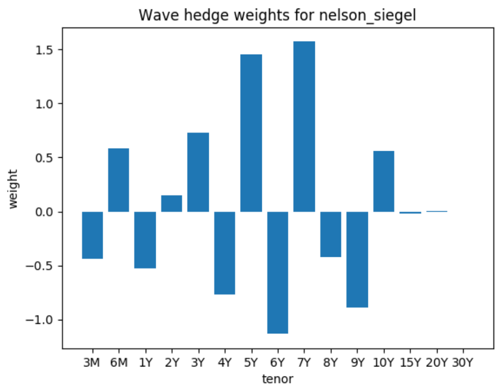
(d) BS627 EUR EU Composite BB+, BB, BB- BVAL



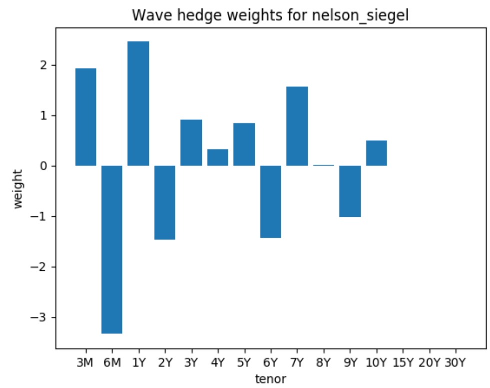
(e) BS166 EUR EU Composite BBB+, BBB, BBB- BVAL

Figure A.4: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the bump method. The bump method is using yield curves constructed by the forward monotone convex spline interpolation method with varying underlying data sets.

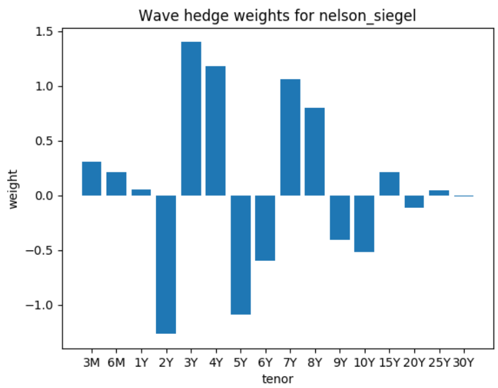
B Localness of wave hedges



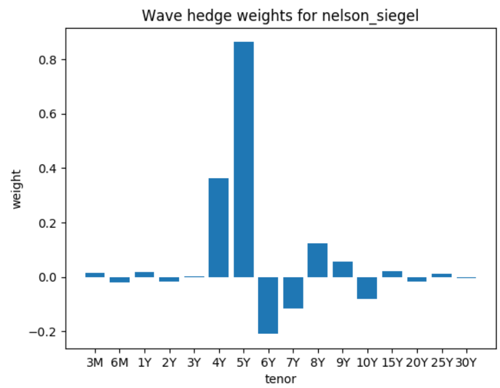
(a) I21 SEK Sweden Sovereign Curve



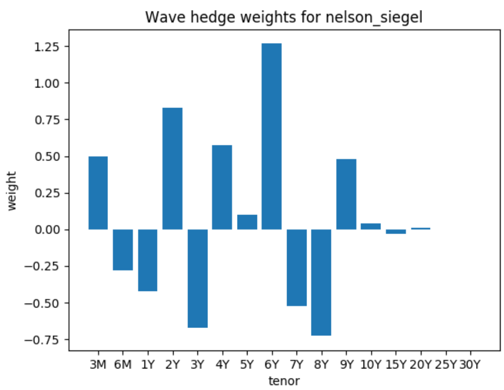
(b) I16 EUR German Sovereign Curve



(c) BS558 EUR EU Composite B- BVAL

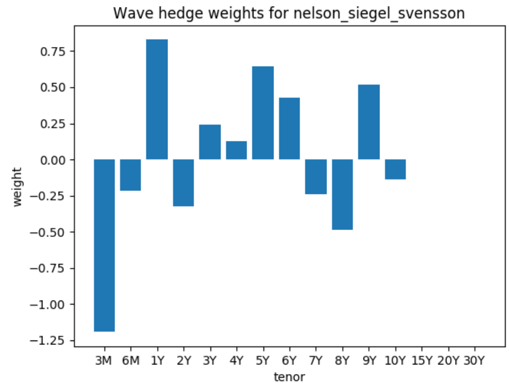


(d) BS627 EUR EU Composite BB+, BB, BB- BVAL

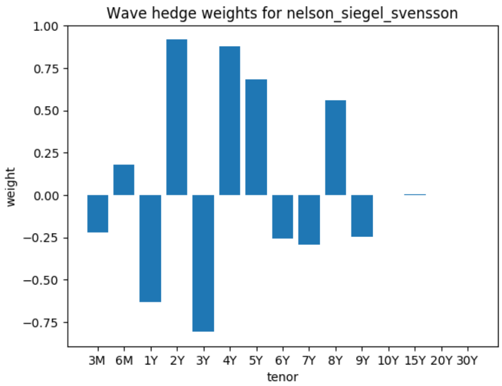


(e) BS166 EUR EU Composite BBB+, BBB, BBB- BVAL

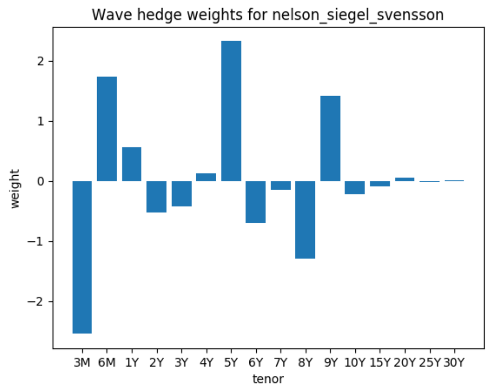
Figure B.1: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the wave method. The wave method is using yield curves constructed by the Nelson-Siegel method with varying underlying data sets.



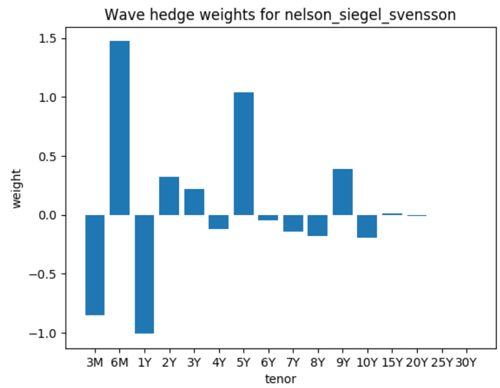
(a) I21 SEK Sweden Sovereign Curve



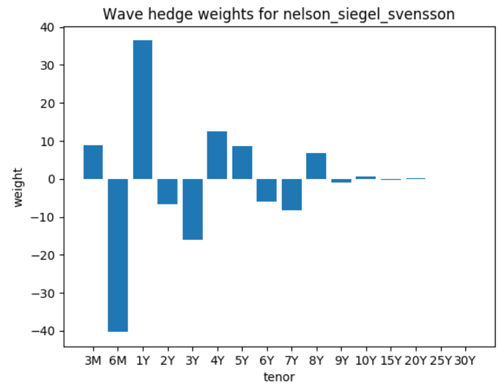
(b) I16 EUR German Sovereign Curve



(c) BS558 EUR EU Composite B- BVAL

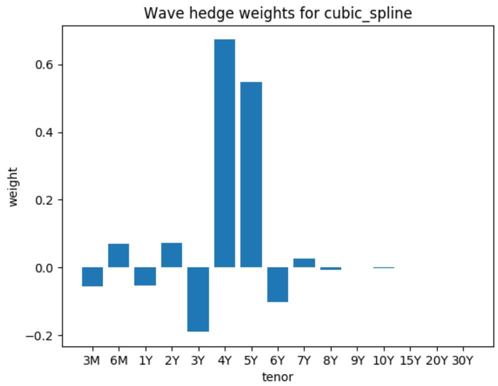


(d) BS627 EUR EU Composite BB+, BB, BB- BVAL

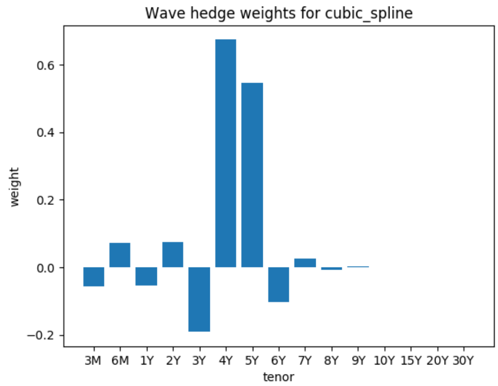


(e) BS166 EUR EU Composite BBB+, BBB, BBB- BVAL

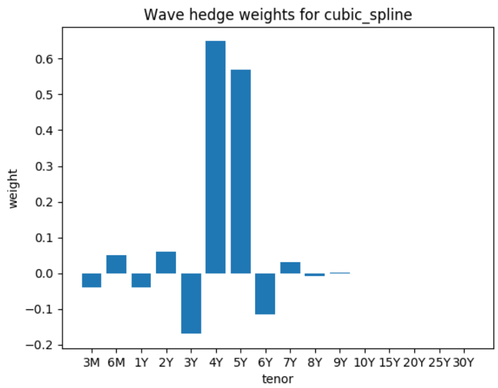
Figure B.2: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the wave method. The wave method is using yield curves constructed by the Nelson-Siegel-Svensson method with varying underlying data sets.



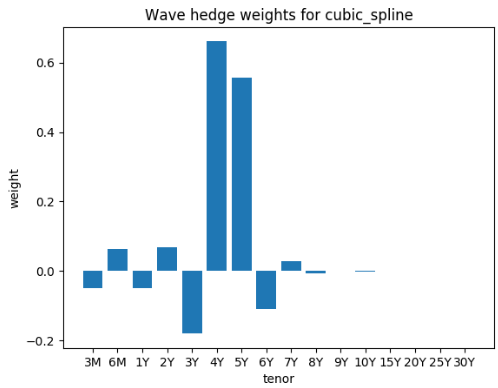
(a) I21 SEK Sweden Sovereign Curve



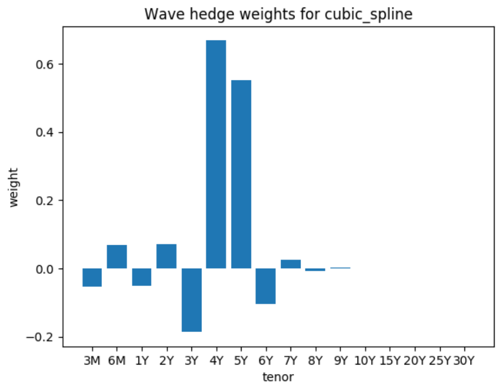
(b) I16 EUR German Sovereign Curve



(c) BS558 EUR EU Composite B- BVAL

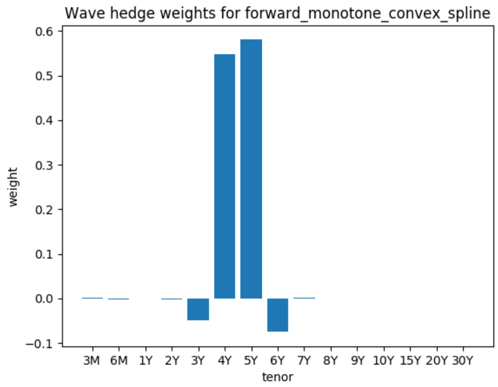


(d) BS627 EUR EU Composite BB+, BB, BB- BVAL

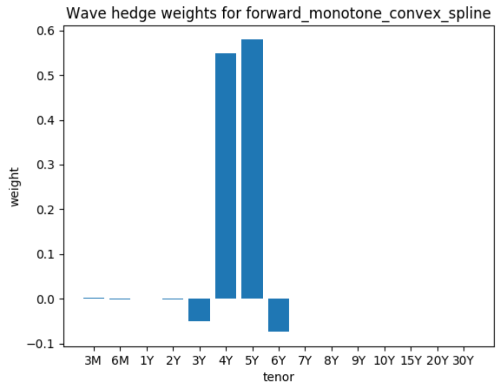


(e) BS166 EUR EU Composite BBB+, BBB, BBB- BVAL

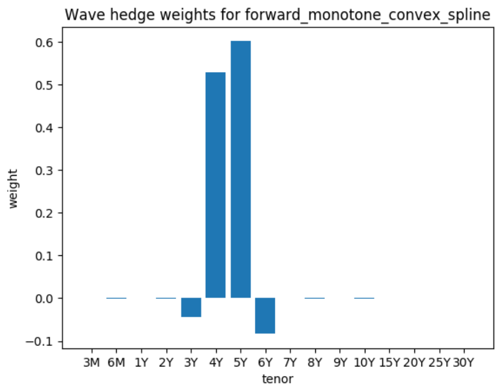
Figure B.3: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the wave method. The wave method is using yield curves constructed by the cubic spline interpolation method with varying underlying data sets.



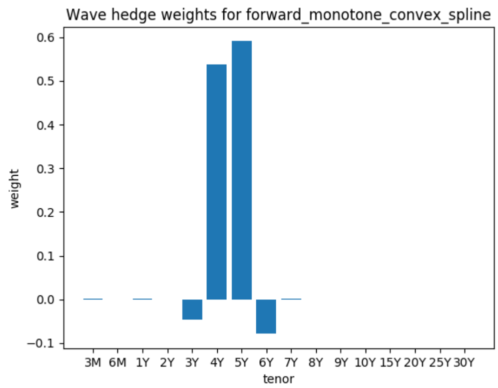
(a) I21 SEK Sweden Sovereign Curve



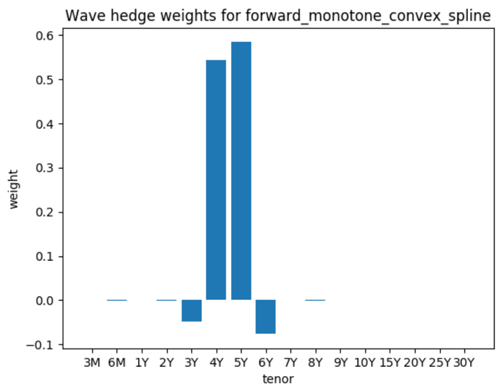
(b) I16 EUR German Sovereign Curve



(c) BS558 EUR EU Composite B- BVAL



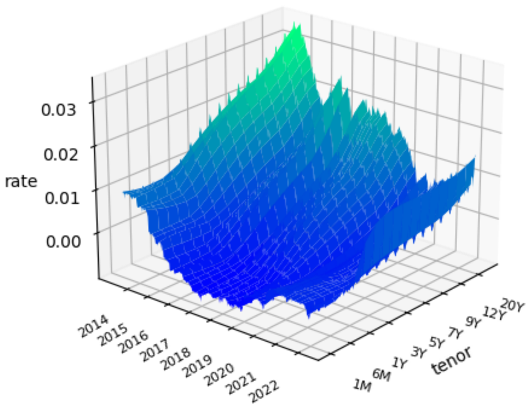
(d) BS627 EUR EU Composite BB+, BB, BB- BVAL



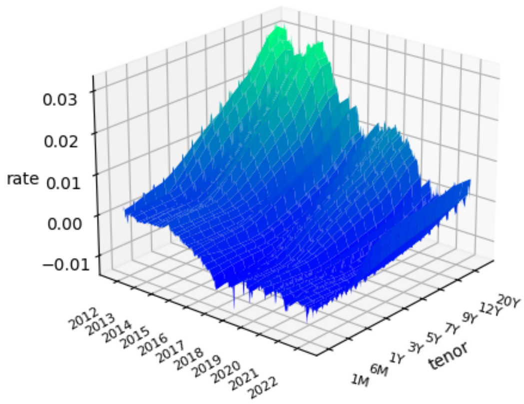
(e) BS166 EUR EU Composite BBB+, BBB, BBB- BVAL

Figure B.4: Bar charts of the hedge weights obtained when hedging a synthetic zero coupon bond with 4.5 years to maturity using the wave method. The wave method is using yield curves constructed by the forward monotone convex spline interpolation method with varying underlying data sets.

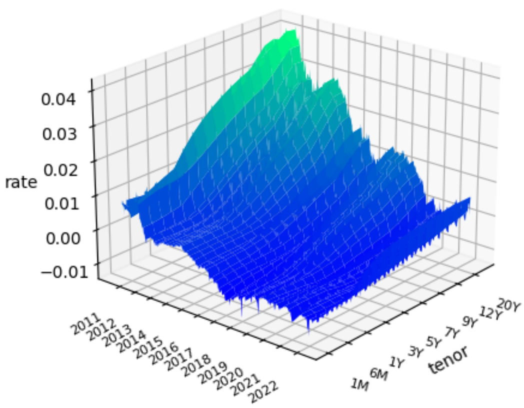
C Time series 3D plots



(a) SEK Sweden Government Curve



(b) EUR German Government Curve



(c) EUR Eurozone Eurobond Curve

Figure C.1: 3D plots of the raw time series data used for the principal component analysis.

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