Pricing Interest Rate Futures Options with Futures-Style Margining

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INTRODUCTION

Buyers of conventional options are required to pay sellers the full amount of the option premium, and sellers are required to post margin. An alternative method is to drop the initial payment of the option premium and require futures-style margining for the buyer (long) and the seller (short). Both the long and the short are required to put up margin and the accounts are marked to the market each day as settlement prices for the options change. The long position still has the right to exercise the option.[1] Futures-style margining for options is currently used at the London International Financial Futures Exchange (LIFFE). In the United States, it has been proposed by several exchanges, but it has not been approved or implemented.

Discussions of options with futures-style margining are contained in Duffie (1988, ch. 8) and Lieu (1990). Duffie has noted that options with futures-style margining are effectively futures contracts on options. Using this principle, Lieu has developed extensions of Black's model for pricing futures options with futures-style margining. The pricing model and results are derived by using the arbitrage methods of Black and Scholes (1973) with fixed interest rates. For this reason, Lieu notes that the model cannot be applied to options on interest rate futures. This study shows that the results derived by Lieu in the Black–Scholes framework apply in a general equilibrium setting with stochastic interest rates and can, therefore, be applied to all types of futures options including options on interest rate futures. In the following section, the general results are derived by using the framework of Cox, Ingersoll, and Ross (1985a,b). It is then shown how to price the interest rate futures options by modifying several existing models.

PRICING FUTURES OPTIONS IN A GENERAL EQUILIBRIUM MODEL WITH STOCHASTIC INTEREST RATES AND FUTURES-STYLE MARGINING

The framework for this analysis is the model of Cox, Ingersoll, and Ross [CIR, (1985a)], in which no restrictions are placed on the utility function. A set of state variables, y_t, which follow diffusion processes are assumed and the instantaneous interest rate, $r(y_t)$, and the spot price, $S(y_t)$ are determined. One of the state variables may be a spot price.

[1] If the option is American, it can be exercised on or before the expiration date. If the option is European, it can be exercised on the expiration date only.

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In their analysis of futures and forward prices, CIR (1981) show that the futures price should be equal to the value of an asset that pays, at delivery, a cash flow equal to

$$\exp\left\{\int_t^{T_f} r(y_u)\,du\right\} S\left(y_{T_f}\right)$$

where t is current time and T_f is the futures delivery date.[2] In the CIR continuous time model, the equilibrium futures price is determined as follows:

$$\begin{aligned} f(y_t, t, T_f) &= \hat{E}_t\left[\exp\left\{-\int_t^{T_f} r(y_u)\,du\right\}\exp\left\{\int_t^{T_f} r(y_u)\,du\right\} S\left(y_{T_f}\right)\right] \\ &= \hat{E}_t\left[S\left(y_{T_f}\right)\right] \end{aligned}$$

where $\hat{E}_t()$ is the risk adjusted expectation operator, conditional on information at time t. The risk adjustment, as described in CIR (1985a), is performed by subtracting the risk premium from the mean in the diffusion process for each state variable y_{it}, before evaluating the expectation. In this model, the futures price is the risk adjusted expectation of the spot price at delivery.

Now consider a European call on this futures with futures-style margining for the call option. As noted by Duffie and Lieu, this futures option is equivalent to a futures contract on an option, and the option is an option on a futures. The payoff for the standard European call option is $\max\left[0, f\left(y_T, T, T_f\right) - K\right]$ and, in the CIR model, the option is valued by taking the following risk adjusted expectation:

$$\hat{E}_t\left[\exp\left\{-\int_t^{T} r(y_u)\,du\right\}\max\left[0, f(y_T, T, T_f) - K\right]\right]$$

where K is the strike price and T is the expiration date for this futures option. To price a futures contract on this option, the risk adjusted expectation of the value of the option at expiration is expressed as:

$$C(y_t, t, t_y) = \hat{E}_t\{\max\left[0, f(y_T, T, T_f) - K\right]\}$$

The principal difference between standard options and options with futures-style margining is the discount factor. There is no discount factor in the pricing of the futures option with futures-style margining because the long is not required to make the initial cash payment for the option premium.

It is now easy to show that the European futures option should sell for more than the intrinsic value of an otherwise equivalent American futures option. The payoff function for the option is a convex function of the futures price; by applying Jensen's inequality, one obtains:

$$C(y_t, t, T) \geq \max\left[0, \hat{E}_t(f(y_T, T, T_f) - K)\right] = \max\left[0, f(y_t, t, T_f) - K\right]$$

The last term on the right is the intrinsic value of the corresponding American futures option, and for any time t prior to expiration the European call should sell for more than

[2] In this analysis, the marking to market for futures contracts is done continuously. In actual futures markets, the marking to market is usually done at the end of each trading day, but during the week of the stock market crash of October 1987, several exchanges implemented extra settlements during the middle of the trading day.

the intrinsic value if there is some probability that the option could be out-of-the-money at expiration. Because American calls sell for at least as much as European calls, the price of the American call will exceed its intrinsic value prior to expiration and it will not be exercised early by rational investors. The price of the American call will therefore be equal to the price of the European call, and one can use a European model to price the American call. The same results apply for put options because the payoff function for the put is also a convex function of the futures price.

$$P(y_t, t, T) = \hat{E}_t\left\{\max\left[0, K - f\left(y_{T_f}, T, T_f\right)\right]\right\}$$
$$P(y_t, t, T) \geq \max[0, K - \hat{E}_t f((y_T, T, T_f))] = \max[0, K - f(y_t, t, T_f)] \quad (2)$$

These two results demonstrate that Lieu's properties B and C for futures calls and futures puts also hold in models with stochastic interest rates and can be applied to interest rate futures options. The put–call parity relation derived by Lieu also holds in this model:

$$\begin{aligned} C(y_t, t, T) - P(y_t, t, T) &= \hat{E}_t\{\max[0, f(y_T, T, T_f) - K] - \max[0, K - f(y_T, T, T_f)]\} \\ &= \hat{E}_t\{f(y_T, T, T_f) - K\} = f(y_t, t, T_f) - K \end{aligned} \quad (3)$$

This put–call parity relation also applies for American puts and calls because the American options will not be exercised early.[3]

SPECIFIC MODELS FOR INTEREST RATE FUTURES OPTIONS

The options on interest rate futures which are traded at the LIFFE are options with futures-style margining. This section shows how to modify existing models to price options on interest rate futures and bond futures. Our first example is a modification of Black's model for pricing Eurodollar (ED) futures options. The final settlement price for ED futures is determined by taking an average of the London Interbank Offer Rate (LIBOR) on the delivery date and subtracting the rate from 100. Prior to delivery, traders calculate the futures rate, which is 100 minus the futures price. Conventional ED futures options are frequently priced by applying Black's model: the futures rate is assumed to have a lognormal distribution and the short term interest rate is assumed to be fixed.[4] The resulting formula for the conventional futures call is

$$C^*(R, t) = e^{-r(T-t)}[(100 - K)N(d_2) - RN(d_1)]$$

where $N(d_1)$ is the standard normal distribution function,

$$d_1 = \frac{\ln\left(\frac{R}{100 - K}\right) + \frac{1}{2}\sigma^2(T - t)}{\sqrt{\sigma^2(T - t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

σ is the volatility for the futures rate. This futures call option is equivalent to a put on the futures rate.[5]

For the ED futures option with futures-style margining, the assumption that the futures rate is lognormally distributed is applied. The assumption that the short term interest

[3]From the put–call parity relation, one can easily verify that Lieu's properties E and F also hold in this general equilibrium model with stochastic interest rates.

[4]For a description of this application, see Hull (1989, ch. 10).

[5]The futures put is equivalent to a call on the futures rate.

rate is fixed is not needed. Let $f(t)$ be the futures price and $R(t)$ be the futures rate: $R(t) = 100 - f(t)$. Assume that R is a geometric Brownian motion process: let $R = e^x$ and x be determined by the diffusion process: $dx = \mu(x)dt + \sigma dz$. With the $\mu(x)$ term, some drift in the futures rate is allowed, but it will have no effect on the option price. By applying the Ito lemma:

$$dR = \left(\mu(\ln R) + \frac{1}{2}\sigma^2\right)Rdt + \sigma R dz$$

$$df = -dR$$

To price the option a zero investment hedged portfolio is formed, which includes a position in the futures contract and an offsetting position in the futures option: $V = wf + C(R,t)$. By setting $w = C_R$, which is less than zero, the risk is eliminated. In equilibrium, a zero investment portfolio with no risk should have a return equal to zero and this condition produces the following partial differential equation (P.D.E) for the option pricing function:

$$C_t + \frac{1}{2}\sigma^2 R^2 C_{RR} = 0$$

The solution is[6]

$$C(R,t) = (10 - K)N(d_2) - RN(d_1) \tag{4}$$

where

$$d_1 = \frac{\ln\left(\frac{R}{100 - K}\right) + \frac{1}{2}\sigma^2(T - t)}{\sqrt{\sigma^2(T - t)}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

The effect of the drift term in the futures rate drops out of the model when the risk-free hedge is formed. The model is similar to Black's model for ED futures options, but there is no discounting. The model is somewhat ad hoc because a process for the futures rate is assumed which may not be consistent with a broader notion of equilibrium in financial markets.

This version of Black's model for ED futures options does not incorporate any of the potential effects from mean reversion in short term interest rates. An alternative approach is to assume a stochastic process for LIBOR and then derive the futures price and the price for the futures option. To do this, assume that LIBOR is equal to e^y and that y is a mean reverting diffusion process:

$$dy = \kappa(\theta - y)dt + \sigma dz$$

where κ is the rate of mean reversion and θ is the long run average.[7] Let $\lambda\sigma^2$ be the risk premium for the state variable y. Because the futures price is the risk-adjusted expectation of the spot price at delivery, the futures rate is the risk-adjusted expectation

[6] It is easy to verify that this solution satisfies the P.D.E. and the boundary condition that $C(R, t = T) = \max[0, (100 - R) - K]$.

[7] The parameters κ, θ, and σ should be greater than zero.

of LIBOR at delivery:

$$R(t,T_f) = \hat{E}_t[\exp\{y(T_f)\}]$$
$$= \exp\left\{e^{-\kappa(T_f-t)}y(t) + \left(\theta - \frac{\lambda\sigma^2}{\kappa}\right)(1 - e^{-\kappa(T_f-t)}) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T_f-t)})\right\}$$

By applying Ito's lemma, the diffusion for the futures rate is

$$dR = \mu(t)Rdt + \sigma e^{-\kappa(T_f-t)}Rdz$$

Again, a zero investment hedged portfolio is formed, applying the equilibrium condition that this portfolio should have a zero return. The resulting P.D.E. is

$$C_t + 1/2\sigma^2 e^{-2\kappa(T_f-t)}R^2C_{RR} = 0$$

The solution for this option pricing function is[8]

$$C(R,t) = (100 - K)N(d_2) - RN(d_1) \tag{5}$$

where

$$d_1 = \left(\frac{\ln\left(\frac{R}{100 - K}\right) + \frac{1}{2}V}{\sqrt{V}}\right) \qquad d_2 = d_1 - \sqrt{V}$$

and

$$V = \int_t^T \sigma^2 e^{-2\kappa(T_f-s)}\,ds = \frac{\sigma^2}{2\kappa}e^{-2\kappa(T_f-T)}(1 - e^{-2\kappa(T-t)})$$

In this model, options with longer maturities have lower average volatilities, $V/(T - t)$, because of the mean reversion. If $\kappa = 0$, then $V = \sigma^2(T - t)$, and the average volatilities are equal across different maturities.

Finally, the CIR one factor model of the term structure is used to price options on bond futures. Consider first an option on a discount bond futures, such as the option on Treasury-bill futures. In this model, the instantaneous interest rate is driven by the following diffusion process:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\,dz$$

and the risk premium for the interest rate is λr. CIR (1985b) show that the price of the discount bond that pays \$1 at maturity is

$$D(r,t,s) = A(t,s)e^{-B(t,s)r}$$

where

$$A(t,s) = \left[\frac{2\gamma e^{(\kappa+\lambda+\gamma)(s-t)/2}}{2\gamma + (\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1)}\right]^{\frac{2\kappa\theta}{\sigma^2}}$$

$$B(t,s) = \frac{2(e^{\gamma(s-t)} - 1)}{2\gamma + (\kappa + \lambda + \gamma)(e^{\gamma(s-t)} - 1)}$$

and

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}$$

[8]It is also easy to verify that this solution satisfies the P.D.E. and the boundary condition.

The price of a futures contract on the discount bond is determined by taking the risk-adjusted expectation of the discount bond price at delivery:

$$f(r,t;T_f,s) = 100A^*(t,T_f,s)e^{-B^*(t,T_f,s)r}$$

where

$$A^*(t,T_f,s) = A(T_f,s)\left(\frac{2(\kappa+\lambda)}{2(\kappa+\lambda)+\sigma^2B(T_f,s)(1-e^{-(\kappa+\lambda)(T_f-t)})}\right)^{\frac{2\kappa\theta}{\sigma^2}}$$

and

$$B^*(t,T_f,s) = \left(\frac{2(\kappa+\lambda)e^{-(\kappa+\lambda)(T_f-t)}B(T_f,s)}{2(\kappa+\lambda)+\sigma^2B(T_f,s)(1-e^{-(\kappa+\lambda)(T_f-t)})}\right)$$

The futures price for a bond with \$100 of par value is used. By setting up a zero investment hedged portfolio of the futures and the futures option, the following P.D.E. for the futures call is derived:

$$C_t + (\kappa\theta - \kappa r - \lambda r)C_r + \frac{1}{2}\sigma^2 rC_{rr} = 0$$

This P.D.E. is similar to the P.D.E. in CIR (1985b), but the term $-rC$ is omitted because there is no initial investment required for this option. By following CIR, one can find the solution by taking the following risk-adjusted expectation:

$$C(r,t) = \hat{E}_t(\max[0, f(r,T;T_f,s) - K])$$

The solution is[9]

$$C(r,t) = f(r,t;T_f,s)\chi^2\left(z_1;\frac{4\kappa\theta}{\sigma^2},\lambda_1^*\right) - K\chi^2\left(z_2;\frac{4\kappa\theta}{\sigma^2},\lambda_2^*\right) \tag{6}$$

where $\chi^2(z;\nu,\lambda^*)$ is the noncentral chi-square distribution function with degrees of freedom, ν, and noncentrality parameter, λ^*, and

$$z_2 = \frac{4(\kappa+\lambda)r^*}{\sigma^2(1-e^{-(\kappa+\lambda)(T-t)})} \qquad \lambda_2^* = \frac{4(\kappa+\lambda)re^{-(\kappa+\lambda)(T-t)}}{\sigma^2(1-e^{-(\kappa+\lambda)(T-t)})}$$

$$z_1 = z_2 + 2B^*(T,T_f,s)r^* \qquad \lambda_1^* = \frac{2(\kappa+\lambda)\lambda_2^*}{2(\kappa+\lambda)+\sigma^2B^*(T,T_f,s)(1-e^{-(\kappa+\lambda)(T-t)})}$$

$$r^* = \frac{1}{B^*(T,T_f,s)}\ln\left(\frac{100A^*(T,T_f,s)}{K}\right)$$

The parameters in these noncentral chi-square distribution functions are different from the parameters used in the CIR model for call options on discount bonds. As before, there is no discounting in the model.

Jamshidian (1989) has shown that an option on a coupon bond can be valued as a portfolio of options on discount bonds in one factor models. This same result can be applied to price an option on a coupon bond futures in the CIR one factor model of the term structure. The coupon bond futures is priced by summing the futures prices for discount bonds:

$$f(r,t) = \sum_{i=1}^{N} a_iA^*(t,T_f,s_i)e^{-B^*(t,T_f,s_i)r}$$

[9]The verification of this pricing solution is contained in an appendix that can be obtained from the authors upon request.

where a_i is equal to the coupon payment for periods $i = 1, \ldots, (N - 1)$, and a_N is equal to the coupon plus par value. The portfolio of options is constructed by first determining r^* such that

$$\sum_{i=1}^{N} a_i A^*(t, T_f, s_i) e^{-B^*(t,T_f,s_i)r^*} = K$$

where K is now the strike price for the coupon bond futures option. The strike price for each discount bond option in the portfolio becomes

$$K_i = a_i A^*(t, T_f, s_i) e^{-B^*(t,T_f,s_i)r^*}$$

and the option is written on a discount bond with par value equal to a_i. The price of the coupon bond futures option is equal to the summation of the prices of these discount bond futures options. Jamshidian's result works in the one factor model because all of the calls will be exercised if $r(T) < r^*$. None of the calls will be exercised if $r(T) > r^*$.[10]

Black's model, without mean reversion, is now applied to ED futures option traded at the LIFFE and the Chicago Mercantile Exchange (CME). The following price quotes for at-the-money call options are taken from *The Wall Street Journal* for January 25, 1990.

		Futures Settlement Price	Strike Price	Option Settlement Price
LIFFE	June	91.67	91.5	0.43
	Sept.	91.60	91.5	0.48
CME	June	91.63	91.5	0.40
	Sept.	91.56	91.5	0.45

The model for futures options with futures-style margining is used to price the two options on the LIFFE, calculating an implied volatility of 0.159163 or 15.9% for the futures rate. The model prices are 0.4273 for the June call and 0.4821 for the September call.[11] This implied volatility is used to value the American options at the CME. The European prices from Black's model are 0.3929 for the June call and 0.4391 for the September call. The American values, calculated from the binomial model, are 0.3967 for the June call and 0.4457 for the September call. When the American values from the binomial model are rounded off, they match exactly the prices for the two calls at the CME. There are small differences in the futures settlements prices between the two markets because of the time difference between London and Chicago. For a direct comparison of the option prices, the prices for futures options with futures-style margining are recalculated using the futures prices at the CME. The prices are summarized as follows:

	European Model Price	American Model Price	Model Price with Futures-Style Margining
June	0.3929	0.3967	0.4053
Sept.	0.4391	0.4457	0.4619

[10]A similar result holds for puts on coupon bond futures.

[11]The March calls are not used because their implied volatilities are much lower.

The prices for the ED options with futures-style margining are higher and this difference is reflected in the market prices.

SUMMARY

This study demonstrates that Lieu's previous results on futures options with futures-style margining also hold in a general equilibrium model with stochastic interest rates. It is shown the results can be applied to interest rate futures options. The most useful result is that futures options with futures-style margining should not be exercised early because their prices should exceed the intrinsic value prior to expiration. It follows that these American futures options will have the same prices as comparable European futures options, and one can price the American futures options with a European pricing model. By using the general equilibrium model of CIR, this result is demonstrated in a model with stochastic interest rates. Lieu's previous results are extended to interest rate futures options. Pricing the interest rate futures options with futures-style margining is simplified because there is no discount factor in the solution. One simply evaluates the risk-adjusted expectation of the cash flow on the option at expiration. This study also shows how to modify several existing models for interest rate futures options to allow for futures-style margining. The prices for these futures options will be higher than the prices on futures options that do not have futures-style margining.

Bibliography

Abramowitz, M. and Stegun, I. A. (eds.) (1972): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards, Applied Mathematics Series 55, Washington, DC: U.S. Government Printing Office.

Black, F. (1976, September): "The Pricing of Commodity Contracts," *Journal of Financial Economics,* 3:167–179.

Black, F. and Scholes, M. (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy,* 8 637–659.

Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1981, December): "The Relationship Between Forward Prices and Futures Prices," *Journal of Financial Economics,* 9:321–346.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985a, March): "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica,* 53:363–384.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985b, March): "A Theory of the Term Structure of Interest Rates," *Econometrica,* 53:385–408.

Duffie, D. (1989): *Futures Markets,* Englewood Cliffs, NJ: Prentice-Hall.

Hull, J. (1989): *Options, Futures, and Other Derivative Securities,* Englewood Cliffs, NJ: Prentice-Hall.

Jamshidian, F. (1989, March): "An Exact Bond Option Formula," *Journal of Finance,* 44:205–209.

Karlin, S., and Taylor, H. M. (1981): *A Second Course in Stochastic Processes,* New York: Academic Press.

Lieu, D., (1990, August): "Option Pricing with Futures-Style Margining," *The Journal of Futures Markets,* 10:327–338.