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Financial Derivatives (University of Manchester)

BLOOMBERG VOLATILITY CUBE

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BACKGROUND

The purpose of the Volatility Cube application is to assist in a pricing of the non-liquid and exotic interest rate derivatives. Such pricing must be consistent with the market prices of the liquid instruments that serve as an input. This document covers methodology used by the application. This application is available through Bloomberg **VCUB** function.

MARKET DATA

Two main categories of liquid quotes are available for use as an input to the Volatility Cube application: interest rate caps and swaptions. As a rule, for a given maturity, the cap quotes are available for the ATM strikes as well as for a set of fixed absolute strikes. The swaptions quotes are typically available for the ATM strikes and, for the most liquid markets, for a set of relative (to the ATM) strikes.

Generally, the input data may be reduced (as described below) to a set of a individual European swaptions described by the following characteristics:

T – *expiry* or *term*: time to the expiration of the option

τ – *tenor*: the length of the underlying swap

K – *strike*: swaption strike

f_{fix} – payment frequency of the fixed leg of the swap
f_{float} – payment frequency of the floating leg of the swap

Considering that the swap payment frequencies are typically standard for the given market (with the few exceptions described below) we can think of the input data as a set of points in a 3-dimensional space with the dimensions of expiry, tenor and strike.

The following subsections briefly describe the structure of the Volatility Cube input data.

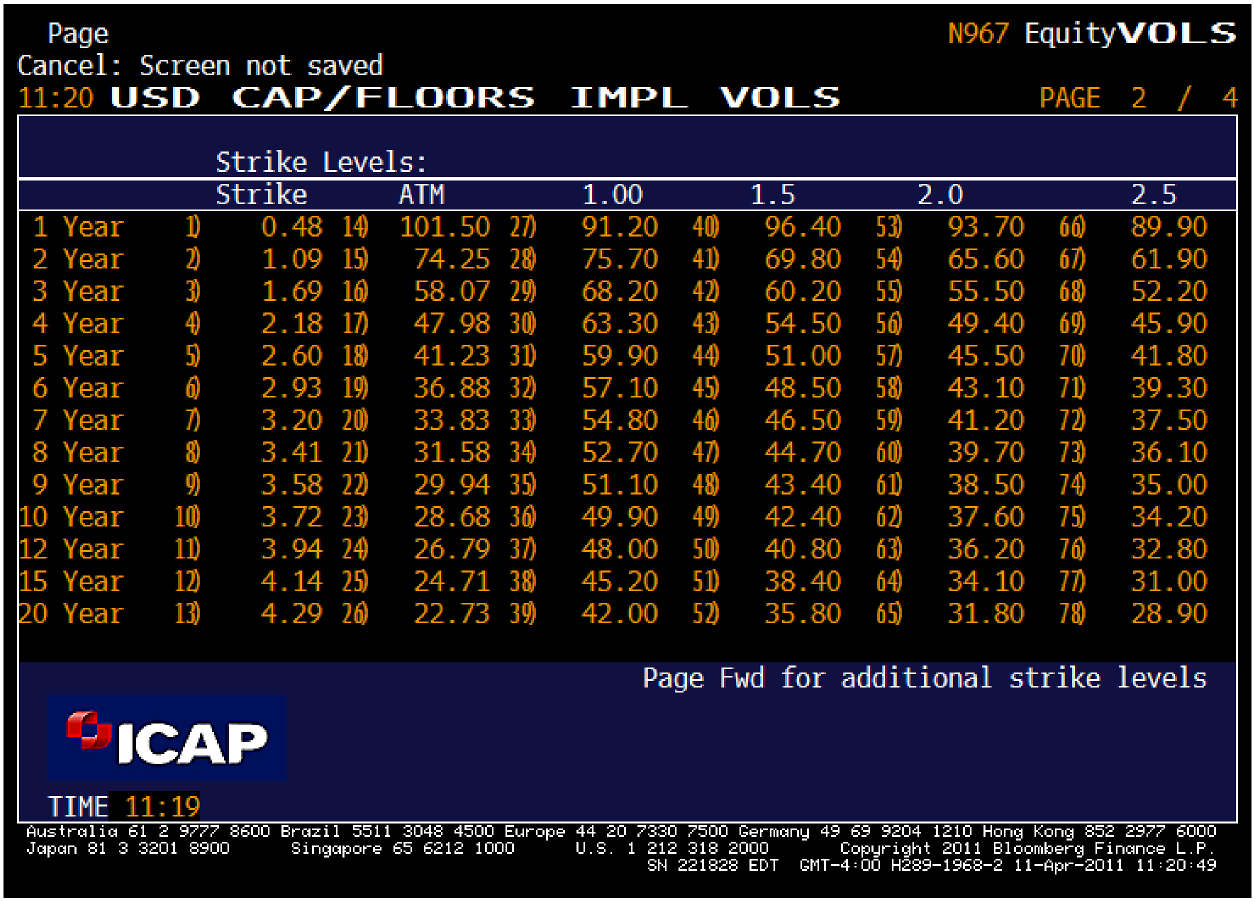
Interest Rate Caps (Fixed Strikes)

An interest rate cap comprises of a sequential series of interest rate options called caplets. All the caplets have the same index (such as LIBOR) as an underlying and share the same strike. The beginning of the index accrual period of a caplet coincides with the option expiry (subject to notification period). The caplet payoff is payable at the end of the corresponding index accrual period.[1] The time of the end of the last accrual period is called *maturity* of the cap. For a given (quoted) price of the cap its implied volatility (Black) is defined as the one that being applied (according to Black model) to every caplet in the cap delivers the given price of the cap.

In most markets the interest rate caps quotes are available for a fixed set of maturities and a fixed set of strikes. Thus, the quotes can be represented by the nodes of the rectangular grid of expiry/strike. This fact is relevant as the alignment of caps with different expiries by a strike allows for a simplified procedure of deconstructing the implied volatilities of individual caplets from the input implied volatilities of the caps (a.k.a. "caps stripping"). The details of the stripping procedure are described later in the document.

The following screenshot demonstrates the fragment of the cap volatility data (both fixed strikes and ATM) provided, in this case, by ICAP.

[1] In most markets the caplet starting immediately is not included in the cap. For example, in USD market the accrual period of the first caplet in the standard cap starts 3 months and ends 6 months after the settlement date.



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11:20 USD CAP/FLOORS IMPL VOLS PAGE 2 / 4

Strike Levels:

		Strike		ATM		1.00		1.5		2.0		2.5
1 Year	1)	0.48	14)	101.50	27)	91.20	40)	96.40	53)	93.70	66)	89.90
2 Year	2)	1.09	15)	74.25	28)	75.70	41)	69.80	54)	65.60	67)	61.90
3 Year	3)	1.69	16)	58.07	29)	68.20	42)	60.20	55)	55.50	68)	52.20
4 Year	4)	2.18	17)	47.98	30)	63.30	43)	54.50	56)	49.40	69)	45.90
5 Year	5)	2.60	18)	41.23	31)	59.90	44)	51.00	57)	45.50	70)	41.80
6 Year	6)	2.93	19)	36.88	32)	57.10	45)	48.50	58)	43.10	71)	39.30
7 Year	7)	3.20	20)	33.83	33)	54.80	46)	46.50	59)	41.20	72)	37.50
8 Year	8)	3.41	21)	31.58	34)	52.70	47)	44.70	60)	39.70	73)	36.10
9 Year	9)	3.58	22)	29.94	35)	51.10	48)	43.40	61)	38.50	74)	35.00
10 Year	10)	3.72	23)	28.68	36)	49.90	49)	42.40	62)	37.60	75)	34.20
12 Year	11)	3.94	24)	26.79	37)	48.00	50)	40.80	63)	36.20	76)	32.80
15 Year	12)	4.14	25)	24.71	38)	45.20	51)	38.40	64)	34.10	77)	31.00
20 Year	13)	4.29	26)	22.73	39)	42.00	52)	35.80	65)	31.80	78)	28.90

Page Fwd for additional strike levels

ICAP

TIME 11:19

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2011 Bloomberg Finance L.P.
SN 221828 EDT GMT-4:00 H289-1968-2 11-Apr-2011 11:20:49

Fig 1. Implied volatilities (Black) of the caps

Interest Rate Caps (At-The-Money)

The at-the-money caps are defined as caps with the strike equal to the corresponding forward swap rate starting at the beginning of the first caplet in the cap[2] and ending at the cap's maturity. Naturally, only one ATM cap quote is available for those maturities for which they are provided. Aside from providing an extra data point, the ATM caps are generally more liquid. Therefore this valuable data should be, despite certain complications to the stripping procedure (see below), incorporated into the Volatility Cube input data set.

For some less liquid markets the ATM cap quotes are the only available cap quotes.

Swaptions (At-The-Money)

At-the-money swaptions are normally provided for a set of selected expiries and tenors. The following screenshot demonstrates the fragment of the ATM swaptions data provided by ICAP.

Backpage N967 Corp VOLS
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18:16 USD SWAPTION VOLS – LOG VOL PAGE 1 / 2
ICAP - ATM SWAPTIONS - IBOR BASED

Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1M	72.2	74.5	60.8	49.3	43.4	37.2	33.2	30.4	28.4
3M	76.3	67.7	56.2	48.9	43.7	38.2	34.5	31.8	29.9
6M	85.7	66.9	54.8	46.6	41.5	36.9	33.7	31.5	29.9
1Y	77.3	58.2	48.2	41.2	37.2	33.9	31.6	29.9	28.6
2Y	49.7	41.4	36.4	33.0	30.8	29.2	28.0	27.0	26.2
3Y	36.7	32.5	30.0	28.3	27.0	26.1	25.4	24.7	24.2
4Y	29.5	27.7	26.4	25.5	24.8	24.2	23.6	23.2	22.7
5Y	25.9	25.0	24.2	23.6	23.2	22.7	22.3	21.9	21.6
7Y	22.1	21.8	21.5	21.2	20.9	20.5	20.3	20.0	19.8
10Y	19.4	19.2	18.9	18.7	18.5	18.4	18.2	18.1	18.0
15Y	16.8	16.8	16.7	16.7	16.6	16.5	16.5	16.4	16.3
20Y	16.0	15.9	15.6	15.4	15.2	15.1	15.0	15.0	14.9
25Y	15.9	15.9	15.6	15.3	15.0	15.0	15.0	15.0	14.9
30Y	15.9	15.8	15.6	15.4	15.1	15.2	15.2	15.3	15.3

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2011 Bloomberg Finance L.P.
SN 221828 EDT GMT-4:00 H289-1968-3 18-Apr-2011 18:16:25

Fig 2. Implied volatilities (Black) of the ATM swaptions

[2] Note that (as mentioned above) the first caplet in the standard cap starts one caplet period after the settlement date. For example in USD market it starts 3 months after the settlement date.

Swaptions (Out-The-Money)

Currently the OTM swaptions quotes are rarely available in the less liquid markets. In addition, even for the major markets the availability of the OTM swaption quotes for a particular Bloomberg client varies depending on the client’s data subscription. Hence the input of the OTM swaption data into the Volatility Cube cannot be taken for granted.

When available, the OTM data is typically provided on a fixed grid of off-the-money strikes specified as an offset relative to the ATM strike (aka pickup/giveup).

Premium Quotes

The market standard for the swaption quoting seems to be increasingly switching from quoting the equivalent Black volatilities to quoting directly swaption prices (aka premiums). For both ATM and OTM swaptions, the premium quotes are typically available for the same expiry/tenor/strike points as are the volatility quotes described above.

Normally, for the strikes below the ATM the quoted price is the price of the put (receiver) option, for the strikes above ATM – price of the call (payer) option and for the ATM strike – price of the ATM straddle (defined as a combination of a call and a put on the same underlying with the same strike).

Forward Premium Quotes

An alternative way to quote the option premium is to express it in terms of a price of a forward contract on this option with the same expiration as the one of the option. One possible rationale behind such quotation is to avoid an uncertainty in discounting due to the increased basis spread.

The following screenshot gives an example of the EUR ATM swaptions quoted by their forward premiums:

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10:41 EUR Swaption Fwd Prem OIS PAGE 1 / 2
ICAP

Term	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y
1M Opt	13.5	37.5	57.5	76.5	92.5	105.0	119.0	132.0	143.0
2M Opt	20.5	53.0	81.0	107.0	131.0	151.0	170.0	188.0	205.0
3M Opt	26.0	65.5	99.5	132.0	163.0	188.0	210.0	232.0	254.0
6M Opt	42.0	96.5	144.0	190.0	232.0	270.0	305.0	339.0	372.0
9M Opt	58.0	122.0	179.0	234.0	285.0	331.0	377.0	418.0	460.0
1Y Opt	72.0	146.0	211.0	272.0	333.0	388.0	439.0	486.0	533.0
18M Opt	93.5	183.0	263.0	337.0	408.0	478.0	544.0	604.0	662.0
2Y Opt	110.5	210.0	301.0	384.0	467.0	546.0	620.0	690.0	759.0
3Y Opt	137.0	251.0	362.0	462.0	559.0	655.0	746.0	831.0	913.0
4Y Opt	150.5	280.0	399.0	512.0	620.0	728.0	832.0	930.0	1022.0
5Y Opt	158.5	297.0	427.0	550.0	665.0	779.0	892.0	999.0	1102.0
7Y Opt	170.5	325.0	470.0	611.0	741.0	867.0	990.0	1111.0	1231.0
10Y Opt	186.5	361.0	525.0	681.0	830.0	976.0	1113.0	1250.0	1382.0
15Y Opt	213.0	420.0	609.0	791.0	961.0	1133.0	1295.0	1455.0	1612.0
20Y Opt	230.5	455.0	664.0	864.0	1055.0	1243.0	1425.0	1604.0	1774.0
25Y Opt	238.5	469.0	686.0	894.0	1098.0	1295.0	1484.0	1669.0	1853.0
30Y Opt	245.0	474.0	693.0	902.0	1104.0	1305.0	1496.0	1686.0	1875.0

Fig 5. ATM swaptions premiums

CAP STRIPPING

Basic Algorithm

As mentioned before, the cap stripping is the process of extracting the volatilities of the individual caplets implied by the quotes of the caps that consist of them. Consider the set of the caps with the same strike. As each cap includes all the caplets from the cap with lesser maturities, the difference of the prices of two given caps is equal to the sum of the prices of all the caplets between the maturities of these caps.

In other words, let $\{T_i\}$ be the available cap maturities, $\{t_j\}$ – starting dates of all the caplets of which the caps consist, $\{P_i\}$ – the prices of the caps and $\{p_j\}$ – prices of the caplets ($1 \le i \le N_{caps}$ and $1 \le j \le N_{caplets}$). Then:

$$P_{i+1}(K) - P_i(K) = \sum_{t_j = T_i}^{t_j < T_{i+1}} p_j(K) \tag{1}$$

This, in conjunction with the extra assumption as for the shape of the term structure of the caplets allows us to derive all the prices/volatilities of the individual caplets. One example of such assumption is the caplet Black volatility remaining constant in between the sequential cap maturities. This method of piecewise-constant stripping is currently used in the Volatility Cube.

> Note: despite the obvious simplification and shortcomings (such as discontinuity of the caplet volatility term structure) this assumption is quite robust as even the simplest forms of continuous interpolation can easily lead to instability of the method.

Assuming piecewise-constant term structure of caplets' volatility, rewriting (1), the problem of cap stripping can be expressed as follows: for each quoted strike K, find the set of caplets' implied volatilities $\{\sigma_i\}$ such that:

$$P_{i+1}(K) - P_i(K) = \sum_{t_j = T_i}^{t_j < T_{i+1}} p_j(K, \sigma_i) \tag{2}$$

Then, for each quoted strike K, system (2) can be solved in an obvious manner:

1. start from the first available cap maturity ($i = 1$)
2. at each step, knowing σ_j for each $j \le i$ (if any) find next σ_{i+1} that satisfies (2)
3. increment i, repeat step 2 until the last cap maturity is reached

After this procedure - called "*cap stripping*"- is applied to each fixed strike for which the cap quotes are available we obtain full set of caplets for those strikes, therefore forming a caplet volatility grid.

Incorporating ATM Caps

The abovementioned difficulty of incorporating the ATM cap quotes should become clear at this point. Indeed, as each ATM cap has a different strike (at least for non-flat curve) each following cap does not contain the caplets from the previous cap as a subset. Also, as ATM strikes generally do not match any of the fixed strikes for which off-the-money caps are provided, the latter also cannot be directly used for ATM cap stripping.

This can be resolved by using one of the smile models (such as described above). Once we have determined the way of producing the volatility values for the strikes that are not an immediate part of the input data set we can use this technique in order to derive the caplet volatilities needed for ATM cap stripping. Then the only additional restriction we impose on the stripping algorithm is that at each step we require the caps with shorter maturities to be stripped for every strike and the full smile model to be built based on the obtained results.

The algorithm is as follows:

1. strip the fixed-strike caps grid in a way described above
2. build a smile model for each available caplet maturity based on the grid data obtained in step 1
3. start from the first available ATM cap maturity (i = 1)
4. at each step for given ATM strike K_{ATM} for each $j \le i$ (if any) retrieve $\sigma_j(K_{ATM})$ from the caplet smile models obtained in step 2
5. knowing $\sigma_j(K_{ATM})$ for each $j \le i$ (if any) find next $\sigma_{i+1}(K_{ATM})$ that satisfies (2)
6. increment i, repeat steps 4-5 until the last ATM cap maturity is reached

EUR Caps Quotation

In certain markets the frequency of the cap payments (as well as the reference index they are based upon) varies based on the cap's maturity. Most notably, in Euro market the traded caps are 3m-based below 2y maturity and 6m-based after that as illustrated by the following chart:

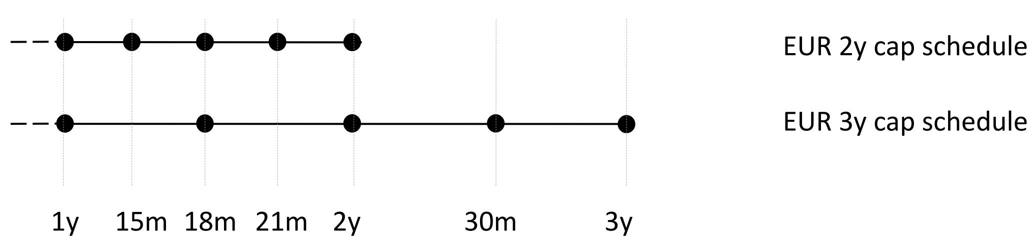


Fig 8. Quoted EUR caps frequency

In this case the value of the caplets with shorter maturities can no longer be directly used for stripping. In order to obtain the needed volatility/value of the 6m caplets we must adopt some model based on the available 3m caplets.

First, consider a market with no significant basis spread. Strictly speaking, in order to derive the implied volatilities of 6m-based caplets from the volatilities of 3m-based caplets one needs to know the joint distribution of the 3m forward rates (or, if a standard lognormal assumption is made, the correlations of these rates). Short of that we are forced to settle for a simplified model. The technique (due to Hagan) employed by the VolCube is based on the interpolation/extrapolation of implied volatility in the tenor dimension with subsequent payment frequency conversion. In this case this involves flat extrapolation of 3m caplet volatility into the (fictitious) 6m swaption volatility (3m-based) and subsequent strike/volatility conversion to the 6m caplet volatility (as described in the Appendix C).

When substantial basis spread is present in the market we can no longer claim that the rates and volatilities corresponding to the instruments with different payment frequencies may (even in theory) be derived from one another. In this case a separate basis model is required (see below). Note that the frequency correction described above is not applied in this case.

Assuming the rules of strike/volatility conversion are given by the chosen basis model, we can perform the stripping as follows:

1. For all available strikes $\{K_j^{3m}\}$ strip the 3m-based caps with maturities up to T^*=2y in the way described above
2. Convert the strikes and the obtained implied volatilities of 3m-based caplets $\{\sigma_i^{3m}(K_j^{3m})\}$ into the corresponding strikes and volatilities of 6m-based caplets $\{\sigma_i^{6m}(K_j^{6m})\}$
3. Using obtained $\{\sigma_{ij}^{6m}\}$ build the complete 6m-based caplet volatility model for expiries up to T^*
4. Proceed with stripping 6m-based caps with maturities above T*. Assume the prices of the 6m-based caplets with expiries up to T^* be determined by the model build in the previous step.
5. Continue with the stripping until the last cap maturity is reached

VOLATILITY MODEL

As it was mentioned above, simplistically, the market volatility data can be interpreted as a function of three variables: expiry, strike and tenor. We consider behavior of the model in each of these dimensions starting from the strike.

Smile Models

Due to its usual shape, the implied volatility taken as one-dimensional function of strike is commonly called smile or skew. As quoted values are available only for a selected set of strikes an extra assumption – a smile model – is needed to fill in the gaps. Such smile model may be as simple as an interpolation/extrapolation or as complex as a fully-specified model of interest rate dynamics. Currently the Volatility Cube supports four smile models: piecewise linear (PWL), CEV, SABR and PWL/SABR model.

Piecewise Linear (PWL) Model

PWL smile model makes the simplest possible smile assumption based on the input volatilities by interpolating between the available volatility data points. The interpolation is linear in lognormal (Black) space.

In addition, the smile extension beyond the available strike data points needs to be performed as the strike coverage provided by the input data may be not sufficient. Obviously, the flat extrapolation may lead to undesirably shaped smiles as demonstrated below:

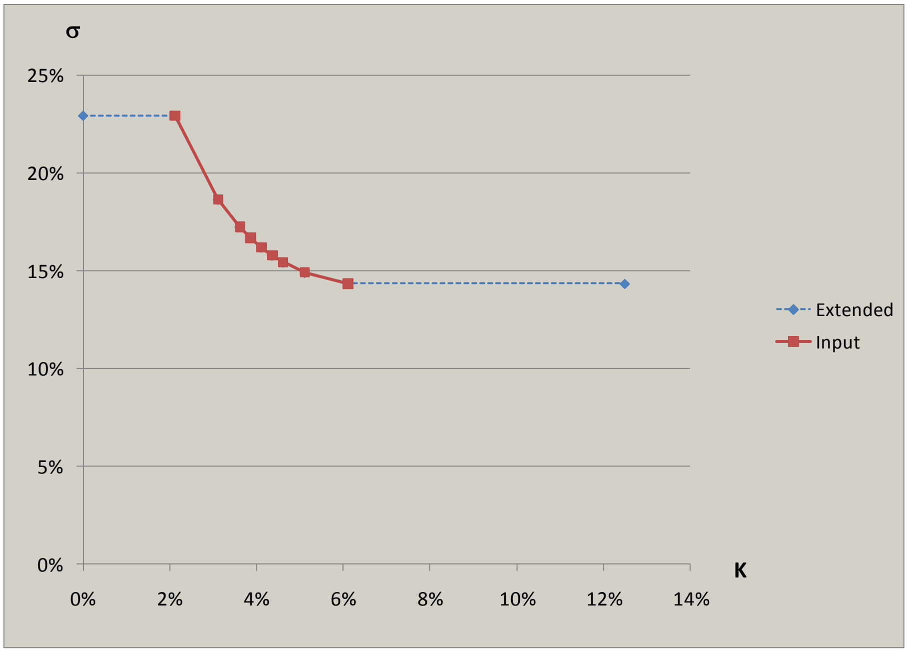


Fig 5. Flat smile extrapolation

Another possibility is to use the last two data points as basis for a linear extrapolation as seen below:

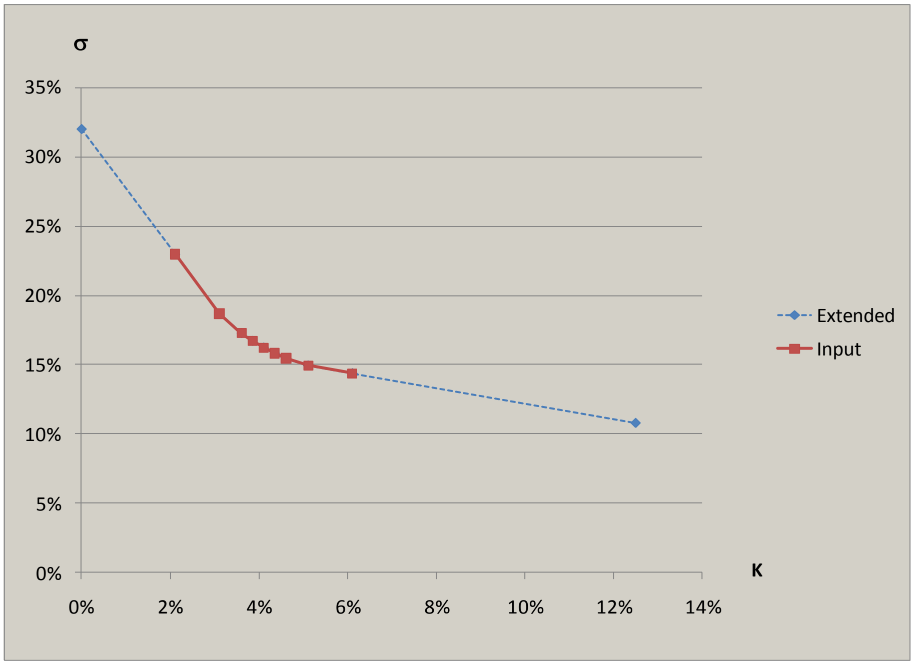


Fig 6. Linear smile extrapolation

While more acceptable in most cases this technique may produce negative volatility values for large strikes. Even if the volatility is forcefully capped by zero this behavior is still undesirable. In order to avoid these complications we apply a combination of linear and flat extrapolation as demonstrated by the following chart (See more in Appendix A).

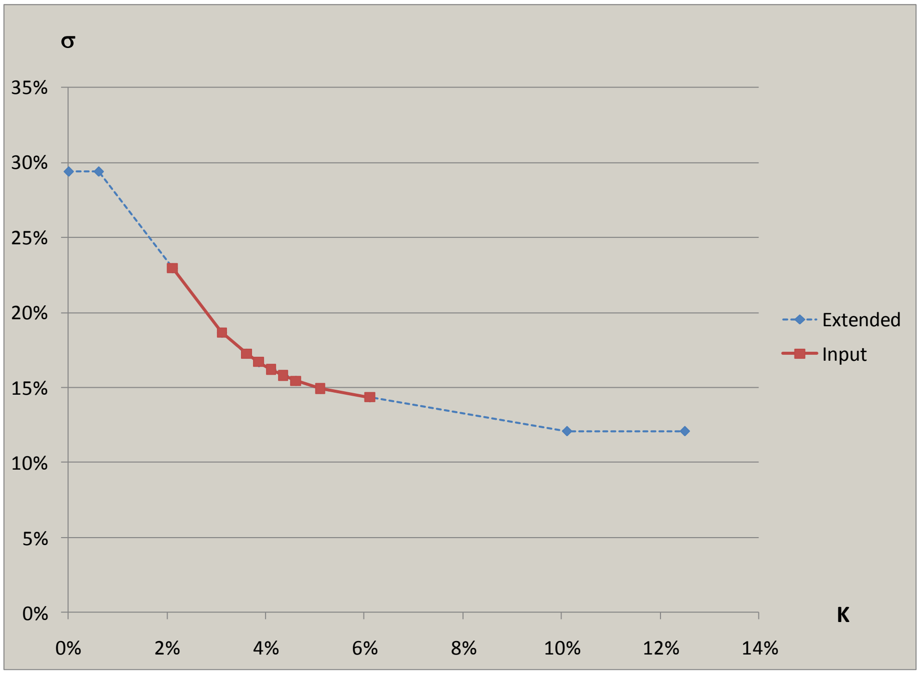


Fig 7. Linear/flat smile extrapolation

The issue of smile extrapolation is addressed by using SABR/CEV smile model.

SABR Smile Model

SABR model was proposed by Hagan at al. [1] and is extensively described in the literature.

Briefly, the general dynamics of the underlying index or swap rate *F(t)* in the SABR model is assumed to be:

$$dF = \alpha F^{\beta} W_1, \quad F(0) = f$$
$$d\alpha = \nu \alpha W_2 \qquad (3)$$

where W_1 and W_2 are correlated:

$$dW_1 dW_2 = \rho dt$$

Here α, β, ν and ρ are the parameters of a SABR-generated smile. In practice, however, there is large degree of redundancy between β and ρ and for the purposes of parameter stability one of these two parameters is often assumed fixed (or otherwise predetermined). The present implementation of the Volatility Cube assumes $\rho = 0$.

An example of a SABR-generated smile is displayed below:

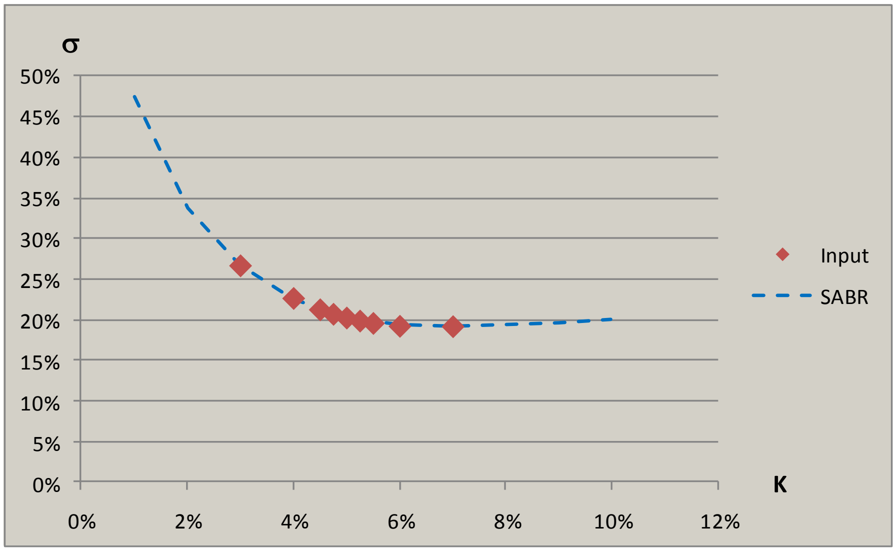


Fig 7. SABR smile example

While capturing the essential features of the smile structure and providing satisfactory smile extrapolation, SABR model is not an interpolation and, generally, cannot exactly reproduce the input data points. On the other hand, as we have seen a single SABR-produced smile has not more than four degrees of freedom. In our case it is even less: we only use three degrees of freedom – as mentioned before we fix the value of ρ. Hence for each smile we must have at least three datapoints in order for the model to be well-defined.

In the current version of the Volatility Cube we require the model to match exactly the input ATM volatility. Subject to this constraint, the model parameters are best-fitted to the input data in the least-RMS sense.

See Appendix C for more details on SABR and associated conversion between the SABR model parameters and the equivalent implied volatility.

CEV Smile Model

While separately developed, CEV model can be viewed as a special case of SABR (with zero volatility stochasticity). Using the same notation as in the previous section, the general CEV dynamics is given by:

$$dF = \alpha F^{\beta} W, \qquad F(0) = f \tag{4}$$

See Appendix C for more details on CEV and associated volatility conversion.

PWL/SABR Mixed Model

As mentioned above, both PWL and SABR smile models have disadvantages: PWL model does not prove a satisfactory extrapolation beyond the available data and SABR is incapable of exactly recovering the input data points. Mixed PWL/SABR model attempts to address both these issues.

The smile is constructed in the following manner:

1. PWL interpolation is constructed from the input data
2. SABR model is fitted to the same input data
3. On both ends of the PWL interpolation, two offsets are separately computed as differences between the end point of the PWL interpolation and the value of the SABR model in this point

Upon construction, the model returns the value of the PWL interpolation for the points within input data range and shifted SABR values for the points beyond it. By construction, the shift guarantees that the smile is continuous in every point.

Expiry/Tenor Volatility Interpolation

As described above, the choice of the smile model determines the interpolation technique in the strike dimension. Given the set of smiles constructed for the set of expiry/tenor points for which data is available we determine the volatility for the rest of the expiry/tenor/strike points by interpolation.

Caplet Volatility Term Structure

The currently used algorithm of cap stripping (see above) assumes that the caplet volatility between the available cap maturities remains constant. This suggests the discontinuous piecewise-constant term structure of caplet volatility. Nevertheless we can improve the shape of the term structure without any changes to the cap stripping technique. In particular we can avoid the discontinuity of the volatility function and the price jumps potentially resulting from it.

Indeed, let $\{T_i\}$ be the times of available cap maturities and $\{t_i^j\}$ be the corresponding caplet start dates so that for each i the time interval $[T_i, T_{i+1}]$ is fully covered by the caplets with starting times t_i^j while $1 \le j \le j_{max}(i)$. Then for each i the cap stripping algorithm must assume the constant volatility for all the caplets with the starting times between $t_i^{j_1}$ and $t_i^{j_{max}(i)}$. It does not however make any assumptions as for the nature of the volatility function between $t_i^{j_{max}(i)}$ and T_{i+1} (as the caplets with these starting times are not a part of any input cap). We can take advantage of this fact and interpolate (e.g. linearly) the volatility function between these points thus eliminating the discontinuity between these two points. The following picture illustrates this method as applied to 3m-based caplets.

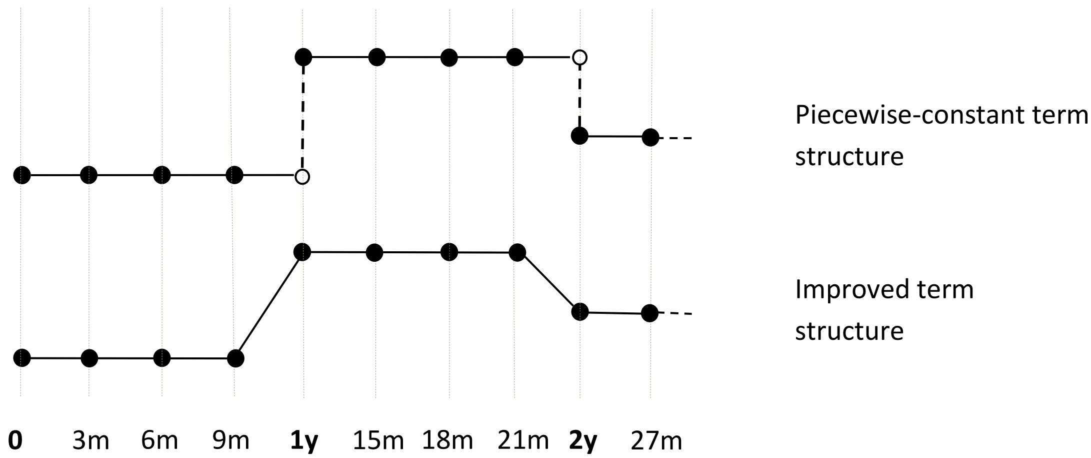


Fig 8. Caplet term structure example

Swaption Expiry/Tenor Volatility Interpolation

Let the ATM strike for a given expiry/tenor point be F_{ij}. Assume that we need to obtain a volatility value for the swaption with the expiry T, tenor τ and strike K. Let the corresponding swap rate be F. As it was described above, the swaption smile data is available for the expiry/tenors pairs that lie in the nodes of the rectangular grid.

The following picture illustrates this using USD market as an example. As we see, the swaption smile data (marked by dots) is available for expiries 1m, 3m, 6m, 9m, 1y, etc. and for tenors 1y, 2,y, 3y, 4y, 5y, etc. Note that due to the proper treatment of the missing data points the smile data is present for each note in the grid.

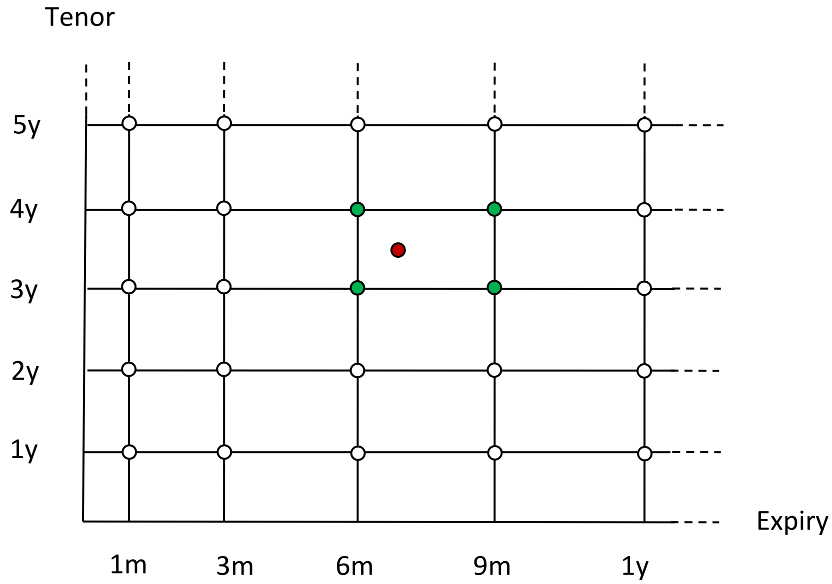


Fig 9. Swaption expiry/tenor interpolation

Suppose we need to compute the volatility for the expiry T^*, tenor τ^* and strike K^*. First, for a given expiry/tenor point $\{T^*, \tau^*\}$ we determine four bracketing points of the grid for which the smile data is available: $\{T_i, \tau_j\}$, $\{T_i, \tau_{j+1}\}$, $\{T_{i+1}, \tau_j\}$ and $\{T_{i+1}, \tau_{j+1}\}$. The smiles at these points will be used for the volatility interpolation. The strikes K_{ij} for each of these smiles are to be chosen on the moneyness basis so that the K_{ij} - F_{ij} = $K^* - F$. The volatility $\sigma(T^*, \tau^*, K)$ is then determined by a bilinear interpolation between the four volatility values $\sigma(T_i, \tau_j, K_{ij})$, $\sigma(T_i, \tau_{j+1}, K_{i,j+1})$, $\sigma(T_{i+1}, \tau_j, K_{i+1,j})$ and $\sigma(T_{i+1}, \tau_{j+1}, K_{i+1,j+1})$.

For example, in the figure above the required point (marked by the red dot) corresponds to T^*=7m, τ^*=3.6y. In this case the bracketing points (marked by the green dots) are {6m,3y}, {9m, 3y}, {6m, 4y} and {9m, 4y}.

This method can be used without modification in the case when four bracketing points all correspond either to swaptions or to the caplets. This is not the case for certain tenors. The following picture demonstrates a typical the data availability using USD data as an example. In this example, the swaption data is available for tenors above 1y and the standard caplets data (obtained through cap stripping) corresponds to standard LIBOR tenor of 3m. We treat the tenors between 3m and 1y as belonging to the caplets of non-standard tenor (e.g. such as 6m LIBOR caplets).

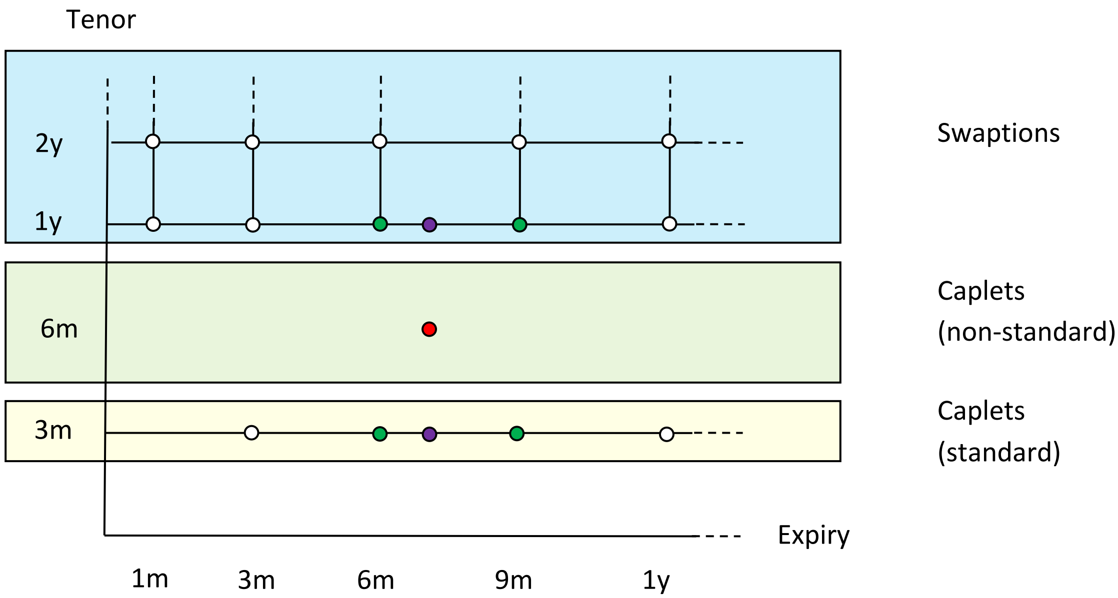


Fig 10. Non-standard caplets interpolation

As we have seen above the volatilities both for swaptions (tenor > 1y) and for standard caplets (tenor=3m) can be obtained by direct interpolation of the available smile data. Two modifications are required in order to produce the volatilities of the non-standard caplets.

First, generally speaking the swaption and caplet data may be provided on two different expiry/tenor grids. To account for that, we perform bracketing and interpolation in two steps. First, we determine bracketing points (marked by green dots) *separately* for the swaption and caplet grids. Then interpolation in expiry dimension is performed separately using these points: the resulting interpolated points are marked by purple dots. After that, the interpolation in the tenor dimension occurs between these two interpolated points (as before, the requested expiry/tenor point is marked by the red dot).

Secondly, the swaptions and caplets have different (fixed leg) payment frequency and daycounting conventions. As a result, their corresponding implied volatilities should not be directly compared and a correction is required (see Appendix C). As we assume the volatility for the tenors shorter than the shortest available swaption tenor to correspond to the non-standard caplets we must operate in the caplet space and convert the swaption volatilities to the equivalent caplet volatility conforming to the caplet payment frequency and daycounting convention.

In the example shown by Fig 10, upon interpolating swaption volatility in expiry dimension we convert the obtained implied swaption volatility (marked the upper purple dot) to the equivalent caplet implied volatility (as described in Appendix C). After that, the interpolation in the tenor dimension occurs.

This algorithm is independent from the smile model and is applied uniformly to any smile model selected (such as SABR).

Swaption Expiry/Tenor Volatility Extrapolation

Both for swaptions and caplets, beyond the available data the volatility function is extrapolated flat in expiry and tenor dimensions.

Swaptions Smile Construction

Using OTM Swaptions

When OTM swaption quotes are available, the construction of the swaption smile is simply reduced to fitting a selected smile model (see above) to the available data. However as we have seen the OTM quotes are typically provided only for a selected number of term/tenor points. Generally, the ATM swaption quotes are available for much larger set. The combination of these two datasets can be done in the several ways. The following describes the implementation currently used by the application.

One way to provide the OTM volatilities needed to produce a smile model for each ATM swaption is to directly interpolate between those OTM data points that are available.

As we have seen, the set of the traded OTM swaptions is selected based on the “moneyness” of the swaptions understood as the difference between the swaption strike and the ATM strike (i.e. the forward swap rate for the same term/tenor as the swaption in question). The same is true for the shape of the smile which depends less on the absolute value of the strike but rather on the relative “moneyness” of the given option. Therefore it is natural to interpolate in the relative rather than absolute strike space. Currently the Volatility Cube defaults to the simplest piecewise linear interpolation.

Upon producing the OTM volatilities for every term/tenor point for which the ATM volatility is available we proceed with the construction of the individual smiles as described above.

Smile "Lifting"

As it was described above in the section dedicated to the market data, the availability of the OTM swaption volatility quotes to the Volatility Cube application may vary greatly depending on the market or the user privileging.

In absence of reliable OTM swaption quotes the swaption smile cannot be uniquely derived from the input data and is subject to modeling. One of the possible models is a so called "smile lifting". Based on market observations, it is postulated that the shape of the swaption smile corresponding to a certain expiry and tenor has much stronger dependency on expiry then on tenor. Hence, according to this assumption, it is reasonable to model the swaption smile shapes based on the (available) smiles of the caplets with the corresponding expiries.

More specifically, consider:

T	-expiry
τ	- swaption tenor
$\sigma_c(K)$	- caplet smile (for given expiry)
$\sigma_s(K)$	- swaption smile (for given expiry and tenor)
K_c^{ATM}	- ATM caplet strike (for given expiry)
K_s^{ATM}	- ATM swaption strike (for given expiry and tenor)
$\sigma_c(K_c^{ATM})$	- ATM caplet volatility
$\sigma_s(K_s^{ATM})$	- ATM swaption volatility

Then, the swaption smile may be modeled after a (known) caplet smile as follows:

$$\sigma_s(K_s) = \sigma_s(K_s^{ATM}) \frac{\sigma_c\left(\widetilde{K}_c\right)}{\sigma_c(K_c^{ATM})}, \quad \widetilde{K}_c = K_c^{ATM} \frac{K_s}{K_s^{ATM}} \tag{5}$$

Missing Data Handling

We have assumed so far that the data required for the grid construction (such as cap or ATM swaption volatility) is available in the nodes of this grid. This may not be the case and certain datapoints may be missing from the application input. In the current version of the application the missing datapoints are filled with the linear interpolation.

Basis Model

Increased spreads between the indices of a different tenor such as 3m and 6m LIBOR (a.k.a. *basis spread*) have added complexity to the interest rate volatility modeling. Indeed, in absence of basis the rates and their volatilities can be (at least in theory) derived from one another. It is no longer the case in presence of a significant basis spread in all major markets.

Therefore while providing the implied volatilities for the instruments with non-standard payment frequency (such as 6m caps in USD) we, strictly speaking, can no longer rely on the technical conversion as described in Appendix C. A separate set of assumptions – a basis model – is needed.

In the current version (v2.0) we implement a simple Black volatility correction that attempts to address this problem. Suppose that we want to obtain the implied Black volatility σ_1 of the swaption with expiry T, tenor τ, strike K_1 and the (non-standard) payment frequency of the underlying swap β_1. Then, assuming that the volatility σ_0 corresponding to the (standard) frequency β_0 is known for each strike, we set:

$$\sigma_1(T,\tau,K_1;\beta_1) = \sigma_0(T,\tau,K_0;\beta_2)\frac{F_1(T,\tau;\beta_1)}{F_0(T,\tau;\beta_0)} \tag{6}$$

where:

T	- time of expiry of the swaption (same for both swaptions)
τ	- tenor of the underlying swap (same for both swaptions)
$\beta_{0,1}$	- payment frequencies of the underlying swaps
$\sigma_{0,1}$	- implied Black volatilities (corresponding to $\beta_{0,1}$)
$F_{0,1}$	- forward swap rates (corresponding to $\beta_{0,1}$)
$K_{0,1}$	- swaption strikes (see below)

As we see, the Black volatility is assumed to be inversely proportionate to the corresponding forward swap rate. This is (roughly) consistent with the assumption of invariant spread between the rates corresponding to different frequency. Another way to look at it is to say that we assume similar normalized (Bachelier) implied volatilities of the at-the-money swaptions corresponding to different payment frequencies.

Note that strikes K_0 and K_1 used in this formula are not necessarily assumed to be the same. Indeed, if we believe that formula (6) should relate the volatilities of the swaptions with the similar moneyness (e.g. at-the-money) rather than with the similar absolute strike, we must relate the strikes K_0 and K_1 accordingly. One natural way to do so is to define moneyness as $\mu = K\text{-}F$, thus setting: $K_1 = K_0 + (F_1\text{-}F_0)$. We refer to such strike transformation as "moneyness adjustment".

Currently the VolCube 2.0 provides for the following conversion options for the volatility of the swaptions with non-standard payment frequency:

1. "Scaling forwards" conversion (as described above)
2. Frequency adjustment (assuming no basis spread – see Appendix C)
3. No adjustment

As described above, for each of these options we may relate the volatilities with either same strikes or the strikes with the same moneyness. This option is controlled by the “Moneyness adjustment” checkbox.

On top of the conversion options described, the VolCube application allows for a user-specified correction to the volatilities of swaptions with non-standard frequency. This correction can be applied in the following linear form: σ’ = σ*factor + spread.

The following screenshot shows the configuration screen for the conversion options described above.

Fig 11. Volatility conversion options

NORMAL VOLATILITY CALIBRATION SPACE

In recent years, developed economies have started showing signs of deflation, with negative interest rate scenarios becoming more plausible. The interest rate derivatives markets in these currencies are also quoting cap and floor options with negative strikes. When either the forward rate or the strike becomes negative, the standard Black implied volatility cube cannot be used anymore. In this case, the market would typically quote a normal volatility, a shifted lognormal volatility, or directly a swaption premium.

In order to handle this scenario, the VCUB function on Bloomberg provides an option to calibrate normal volatility quotes, as shown in the following screenshot.

Fig 12. Normal / Black Volatility calibration option

When the normal calibration space option is chosen in VCUB, the market quotes, whether premiums or lognormal volatilities, are first translated into normal volatilities. Then the volatility cube construction, including interpolation and extrapolation, will be conducted in the space of normal volatilities. For example, the PWL model would mean that the normal volatility is piecewise linear. Likewise, the SABR model parameters are calibrated to the implied normal volatilities. All the other methods described earlier, such as the cap stripping, smile model, expiry/tenor volatility interpolation and swaption smile construction, are unchanged.

Some minor differences exist when calibrating in the normal volatility space instead of the lognormal volatility space:

a) Basis model: In the Black model, the lognormal implied volatility is scaled as in equation (6) to retain the same ATM normal volatility across swaptions of different payment frequencies. In the normal volatility space, as one is already dealing with normal volatilities, the ATM normal volatilities are simply held unchanged across all payment frequencies. The two models will give different basis adjustments for non-ATM swaptions.
b) Smile lifting model: In the Black model, the swaption volatility smile shape is the same as lognormal caplet implied volatility smile, as given by equation (5). The normal model uses the same approach for the swaption volatility smile shape by lifting the normal caplet implied volatility smile.
c) The interpolation / extrapolation methods are the same for both the lognormal and the normal models, although actual extrapolated values of normal volatilities will, of course, differ from extrapolated values of lognormal volatilities.
d) In the Black model, since the forward rate and strike are bounded below by zero, it is not possible to price an option with a negative strike. In the normal model, this limitation is removed from the PWL smile model. However, the SABR smile model still has a natural boundary at zero (for nonzero beta). This will be addressed in a forthcoming release with a shifted SABR model.

SHIFTED SABR SMILE MODELS

Since the regular SABR model is limited to positive rate range, it cannot be used generally for negative rate scenario. Even if we change to normal volatility framework for VCUB, except using the piecewise linear or some simple extrapolation method, we still cannot directly use the VCUB for strikes at negative range.

In order to take the advantage of SABR model for smile fitting efficiency, and in the meantime also make it useful for negative rate options, we will extend standard SABR into Shifted-SABR model. How much is the shift is user imputable, as shown in the following screenshot.

Fig 13. Shifted-SABR model with the user input shift

The shifted-SABR model is just a simple extension for regular SABR model originally proposed by Hagan at al. [1] which is extensively described in the literature.

Briefly, the general dynamics of the underlying index or swap rate *F(t)* in the shifted SABR model is assumed to be:

$$dF = \alpha (F + shift)^{\beta} W_1, F(0) = f$$

$$d\alpha = \nu \alpha W_2$$

where W_1 and W_2 are correlated:

$$dW_1 dW_2 = \rho dt$$

If we replace $\widetilde{F} = F + shift$ then we get exactly SABR dynamics for $\widetilde{F}$, then only difference is that both forwards and strikes are shifted from the original market. Then previous SABR fitting techniques automatically works. Then we can use this model to get the implied volatility up to –*shift* (if we ignore the approximation error etc).

APPENDIX A – METHODOLOGY COMPARISON: V1.0 VS. V2.0

In this section we describe the changes in the methodology between versions 1.0 and 2.0 of the Volatility Cube model and the corresponding differences in the expected model output.

1. Improved Piecewise-linear Smile Extrapolation

In most case it is necessary for the piecewise-linear (PWL) smile model to extend the volatility smile outside the available data interval. In order to preserve the smile shape the extrapolation is performed (when possible) in a piecewise-linear fashion using the last two datapoints available from each end.

In the absence of such extension (in other words, with the flat extrapolation), the smile shape will be greatly distorted in the area that may be used both directly (by application requesting volatility for the low and high strikes) and indirectly (for example when the swaption smiles are modeled based on the caplet smiles). The chart below shows an example of such not-extended smile:

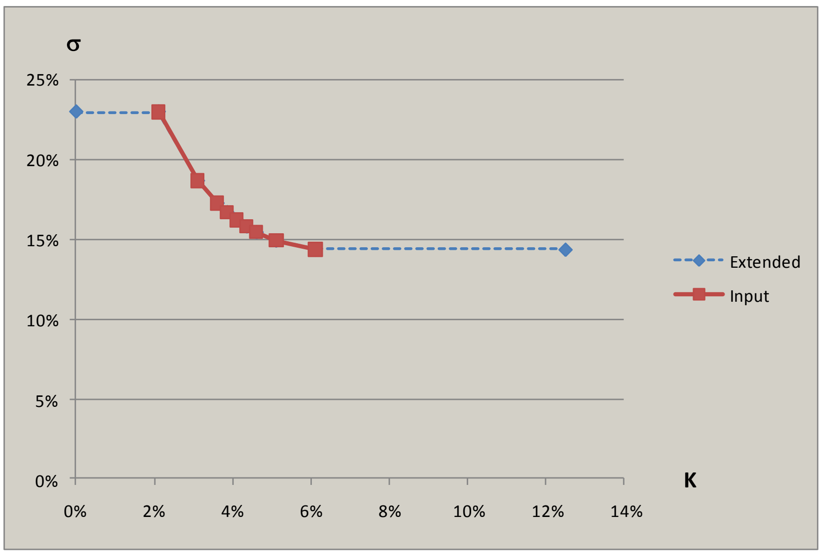


Fig A.1. Flat extrapolation of a smile

In addition, in order to prevent the appearance of very large and very small (potentially even negative) output volatility values is practical to limit the piecewise-linear extrapolation for very small and very large strikes. The volatility is safely assumed constant beyond these limits – as shown on the chart below.

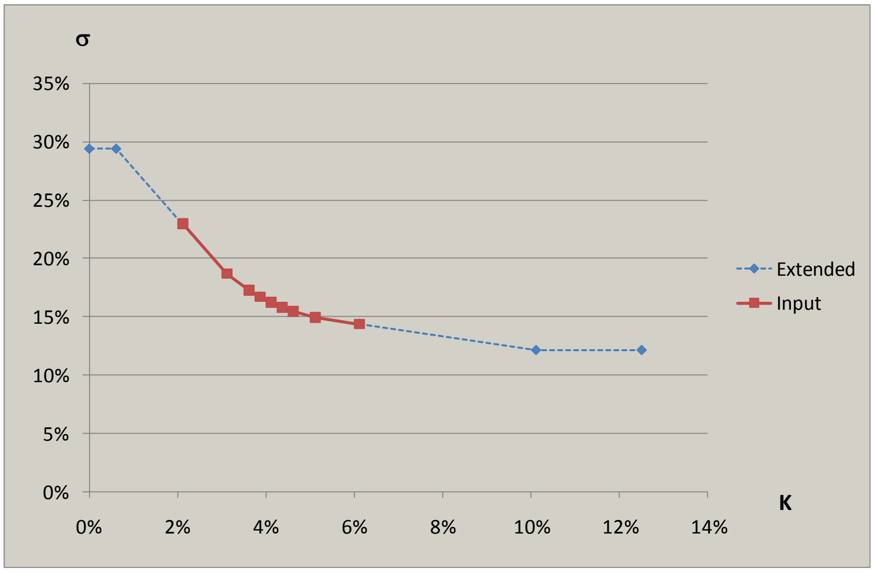
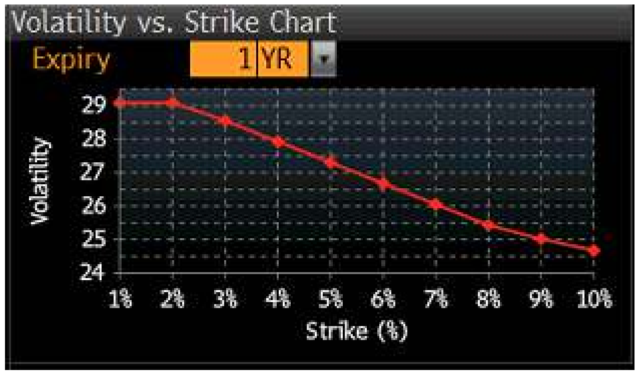


Fig A.2. Volatility Cube smile extrapolation

In the version 2.0 of the Volatility Cube we improve the handling of extrapolation as compared to the model version 1.0: in certain cases, during the process of modeling the swaption smiles based on the caplet smiles (a.k.a. "smile lifting") the Volatility Cube model version 1.0 was limiting the piecewise-extrapolation earlier than necessary. The following example (1y10y swaption smile, default Bloomberg USD cube as of May 15, 11) demonstrates this improvement.

v1.0:



v2.0:

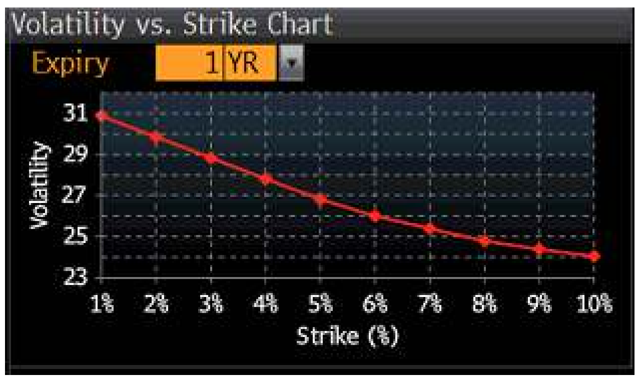
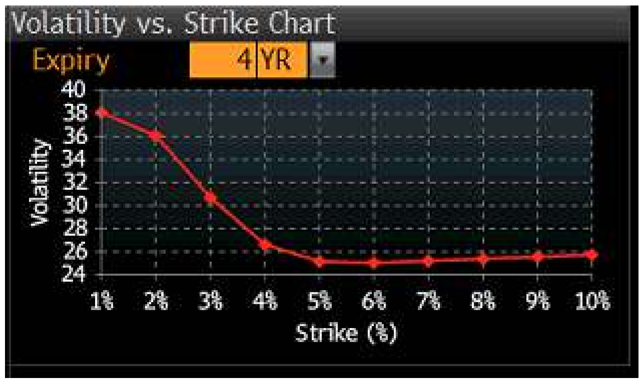


Fig A.3. Smile extrapolation improvement

2. Missing OTM Smile Extrapolation

In the following example (4y5y swaption smile, USD Golden Copy NY 5PM cube, as of May 15, 11) demonstrates a similar improvement in extrapolating missing OTM smiles beyond the range that is directly available through the interpolation. More specifically, when OTM data is missing for certain expiry/tenor point (in this case 4y5y) the smile can be constructed by using the available ATM volatility and interpolating the smile shape between the neighboring OTM smiles (in this case 1y5y and 5y5y) for which the data is available. In both model versions 1.0 and 2.0 the interpolation works in a similar way. However, as demonstrated by the following screenshots, the model version 2.0 implements better extrapolation technique[3].

v1.0:



v2.0:

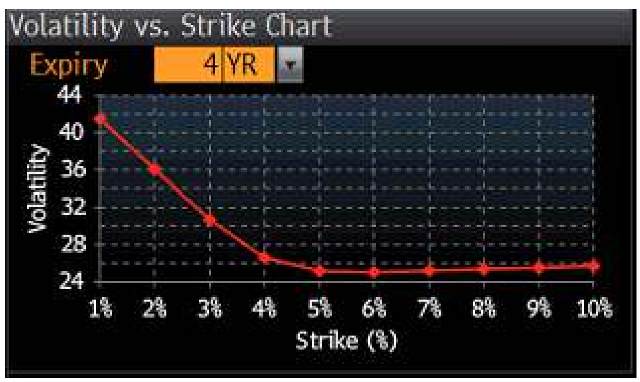


Fig A.4. Missing smile extrapolation improvement

[3] In particular, when filling the missing data the model version 1.0 assumes the volatility data in the bracketing expiry/tenor points to be always available. When this is not the case – e.g. when desired moneyness corresponds to a negative strike for some expiry/tenor – the algorithm is forced to use the data for the smallest available strike which leads to the distortion of the smile shape. The model version 2.0 uses better algorithm that does not rely on the full availability of the bracketing points.

The next chart provides more details of the described improvement. As we see, in the area of extrapolation (below -200 bp off the money), the smile produced by the model version 2.0 (purple) captures the overall shape of the input data (blue and red) much better than the smile produced by the model version 1.0 (green).

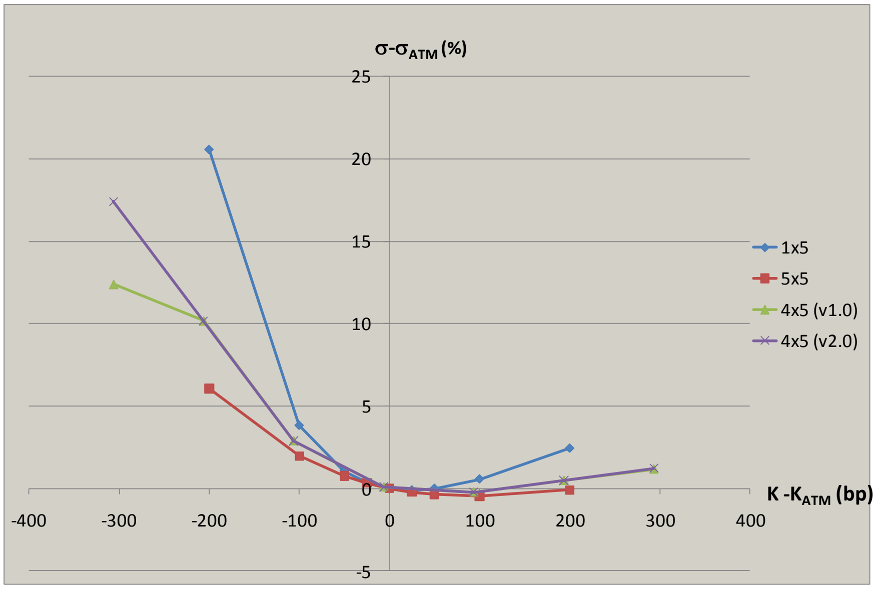


Fig A.5. Missing smile extrapolation example

3. SABR Fitting

The SABR model fitting has been improved in the version 2.0. Most notably, the version 2.0 provides better fit for the ATM volatility (when provided) and best-fits the rest of the datapoints while the model version 1.0 provides a uniform fit treating the ATM datapoint just like any other. Therefore we do not expect a complete match between the SABR smiles produced by the model version 1.0 and 2.0.

The following chart and table demonstrate the SABR fitting to the ill-shaped input data. The quotes are for 2m2y OTM swaptions, CFIR data source as of 3/1/11. The blue dots represent the input data, red line is SABR fitting by the model version 1.0 and the green line is SABR fitting by the model version 2.0.

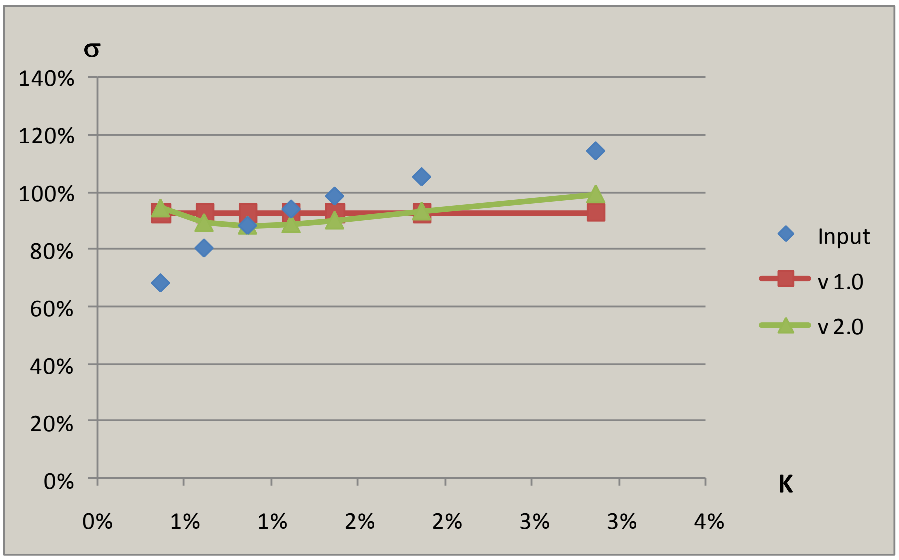


Fig A.6. SABR fitting improvement

	K	input	v 1.0	v 2.0
-50	0.3687	67.98	92.67	94.3202
-25	0.6187	80.21	92.53	89.1842
0	**0.8687**	**88.13**	**92.5**	**88.2478**
25	1.1187	93.9	92.51	88.7421
50	1.3687	98.42	92.55	90.0432
100	1.8687	105.24	92.64	93.2357
200	2.8687	114.33	92.83	99.3056

	v 1.0	v 2.0
ATM error	4.37	0.1178
mean error	11.82	10.85651
RMS error	14.33215	13.31469

Table A.6. SABR fitting improvement

APPENDIX B – SABR/CEV VOLATILITY CONVERSION

The following conversion formulas are due to Hagan et al [1].

Consider a swaption with the expiry τ, strike K, and the corresponding forward rate *F*. Let $\sigma_B(K)$ and $\sigma_N(K)$ be its implied volatility according to lognormal (Black) and normal (Bachelier) model correspondingly. Then, the volatilities are linked in the following manner:

$$\frac{1}{\sigma_N(K)} = \frac{1}{\sigma_B(K)} \frac{\log(F/K)}{f-K} \left\{ 1 + \tfrac{1}{24}\left(1 - \tfrac{1}{120} y^2\right) \sigma_B^2 \tau + \tfrac{1}{5760} \sigma_B^4 \tau^2 + \dots \right\} \qquad \text{(C.1)}$$

where:

$$y = \log \frac{F}{K}$$

Near the money we should make the replacement:

$$\frac{f-K}{\log(f/K)} \approx |fK|^{1/2} \left\{ 1 + \tfrac{1}{24} y^2 + \tfrac{1}{1920} y^4 \right\} \qquad \text{if } |(f-K)/f| < 0.01 \qquad \text{(C.2)}$$

Consider SABR model with volatility α, exponent β, correlation ρ and vol of vol ν. The equivalent normal (Bachelier) volatility σ_N of the option with the strike *K* is:

$$\frac{1}{\sigma_N(K)} \approx \frac{1}{\alpha} \frac{f^{1-\beta} - K^{1-\beta}}{(1-\beta)(f-K)} \cdot \frac{x(z)}{z} \cdot \left\{ \frac{1 + \frac{\beta(2-\beta)}{24} \frac{1 - \frac{2-2\beta+\beta^2}{120} y^2}{1 + \frac{(1-\beta)^2}{12} y^2} \frac{\alpha^2 \tau}{(fK)^{1-\beta}} + \frac{\beta(2-\beta)}{80} \left((1-\beta)^2 + \tfrac{1}{72} \beta(2-\beta) \right) \frac{\alpha^4 \tau^2}{(fK)^{2-2\beta}}}{1 + \frac{\beta\rho}{4} \frac{\alpha \nu \tau}{(fK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24} \nu^2 \tau} \right\} \qquad \text{(C.3)}$$

where

$$z = \frac{\nu}{\alpha} (fK)^{\frac{1-\beta}{2}} \log \frac{F}{K}, \qquad y = \log \frac{F}{K}$$

and $x(z)$ is defined as:

$$x(z) = \log \left(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right)$$

For ATM options this reduces to:

$$\frac{1}{\sigma_N^0} \approx \frac{1}{\alpha f^{\beta}} \left\{ \frac{1 + \frac{\beta(2-\beta)\alpha^2\tau}{24 f^{2-2\beta}} + \frac{\beta(2-\beta)}{80}\left((1-\beta)^2 + \frac{1}{72}\beta(2-\beta)\right)\frac{\alpha^2\tau^4}{f^{4-4\beta}}}{1 + \frac{\beta\rho\alpha\nu\tau}{4 f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\tau} \right\} \quad \text{(C.4)}$$

Very near the money we should replace

$$\frac{f^{1-\beta} - K^{1-\beta}}{(1-\beta)(f-K)} \approx \frac{1}{(fK)^{\beta/2}} \frac{1 + \frac{(1-\beta)^2}{24} y^2}{1 + \frac{1}{24} y^2}$$

$$\frac{x(z)}{z} \approx 1 + \tfrac{1}{2}\rho z - \tfrac{1-3\rho^2}{6} z^2 + \ldots$$

if $|(f-K)/f| < 0.0001$ (C.5)

One can obtain the formulas for the conversion of CEV parameters into equivalent normal volatility from the above by assuming v = 0.

APPENDIX C – RATE AND VOLATILITY CONVERSION DUE TO THE PAYMENT FREQUENCY

In absence of the basis spread the rates and volatilities corresponding to the swaps/options that are based on different fixed leg payment frequencies can be derived from each other. So, in theory, the rates and volatilities corresponding to a certain payment frequency can be derived from those corresponding to a different frequency.

The following conversion formulas are due to Hagan [2].

Consider two FRA/swap rates R_{to} and R_{from} that correspond to payment frequencies q_{to} and q_{from}. Then:

$$R_{to} \approx \frac{q_{to}}{\lambda_{to}} \left\{ \left(1 + \frac{\lambda_{from} R_{from}}{q_{from}} \right)^{\frac{q_{from}}{q_{to}}} - 1 \right\} \tag{D.1}$$

Here λ_{to} and λ_{from} are the corresponding *daycount fractions*, defined so that λ = 365/360 if the daycount convention is Act/360 and λ = 1 for any other convention known to the author.

Consider the CEV model with volatility α and the exponent β. Then the volatilities corresponding to different payment frequencies are related as follows:

$$\alpha_{to} \approx \alpha_{from} \frac{\lambda_{from}}{\lambda_{to}} \frac{R_{from} - K_{from}}{R_{to} - K_{to}} \frac{R_{to}^{1-\beta} - K_{to}^{1-\beta}}{R_{from}^{1-\beta} - K_{from}^{1-\beta}} \frac{1 + \frac{\lambda_{to}(f_{to} + K_{to})}{2q_{to}}}{1 + \frac{\lambda_{to}(f_{from} + K_{from})}{2q_{from}}} \tag{D.2}$$

Here $x^{1-\beta}$ should be interpreted as $-|x|^{1-\beta}$ when $x < 0$.

Near the money this should be replaced by:

$$a_{to} \approx a_{from} \frac{\lambda_{from}}{\lambda_{to}} \left(\frac{R_{from} K_{from}}{R_{to} K_{to}} \right)^{\beta/2} \frac{1 + \frac{\lambda_{to}(f_{to} + K_{to})}{2q_{to}}}{1 + \frac{\lambda_{to}(f_{from} + K_{from})}{2q_{from}}} \tag{D.3}$$

For the lognormal (Black) volatility (β = 1) formula D.2 turns into:

$$\sigma_{to}^{B} \approx \sigma_{from}^{B} \frac{\lambda_{from}}{\lambda_{to}} \frac{R_{from} - K_{from}}{R_{to} - K_{to}} \frac{\log(R_{to} / K_{to})}{\log(R_{from} / K_{from})} \frac{1 + \frac{\lambda_{to}(R_{to} + K_{to})}{2q_{to}}}{1 + \frac{\lambda_{to}(R_{from} + K_{from})}{2q_{from}}} \tag{D.4}$$

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[3] Hagan , P. S., Konikov, M. (2003), `Volatility Cube: Construction and Use’, Bloomberg L.P. (internal) IDOC # 2019820