A real-time zero-coupon yield curve cubic spline in Excel*

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Abstract

This paper details a method for estimating a zero-coupon yield curve using a set of securities data. The approach uses a McCullough cubic spline and can be estimated using restricted least squares in Excel which provides a considerable advantage over other more advanced, but not necessarily more accurate models.

1. Introduction

The zero-coupon yield curve (or spot curve) could be considered the single most important tool for the fixed-income portfolio manager and investment analyst. Dozens of methods and models exist for calculating the spot curve, most requiring either well-prepared data, copious amounts of computer programming, or significant computing power. This paper presents an alternative approach which provides a good trade-off between sophistication and ease of implementation

*This suggested approach should only be used for asset valuation in a portfolio management context, and not for market-making applications.

The motivations for creating a spot curve can be numerous. Security valuation is the most common, where interpolation of spot rates for specific dates is necessary. For example, suppose an investment dealer is offering for sale a particular US agency security, newly issued. Utilizing a spot curve for agency securities, it is possible to calculate a theoretical price for the security that is consistent with currently traded securities. Measurement of market expectations can also be the motivation, particularly for a central bank, and so a zero-coupon yield curve allows for the calculation the market expected path of short rates. For portfolio managers, active or indexed, often the question arises: "which bond do I buy?" A spot curve is the first step towards a process for differentiating between several similar securities, with the goal of finding relative value.

The process described in this paper is a practical implementation of an old well accepted method for fitting a spline to the yield curve, which can be implemented in a spreadsheet and provide real-time estimates of the spot curve. This model should prove almost instantaneous on a normal speed desktop computer and thus has a reasonable balance between speed and sophistication.

First, it is necessary to establish some terminology. A zero-coupon yield (also called a spot rate) is the discount rate (typically default-free) at which one can equate a present value from a future cash flow for any maturity. It is often associated in practice as the yield to maturity on a non-coupon paying bond, such as a US Treasury STRIP. A spot curve is a functional estimate of zero-coupon yields. It is a mechanism that allows one to calculate the zero-coupon yield for any specified maturity. A spline is a functional estimation of a curve using pieces (usually polynomials) joined together to give more flexibility.

While there are many approaches to modelling the yield curve, the category discussed here involves smoothing splines which are merely intent on accurately describing current spot rates, without having any equilibrium or arbitrage conditions imposed. There are various models of spot curves available to the practitioner. The more popular approaches are the following:

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- McCullough (1975) – Estimated a cubic spline, or series 3rd order polynomial regressions joined at knotpoints, modelling the discount function
- Fisher, Nychka, Zervos (1995) – (FNZ) Estimated a smoothing basis spline with one smoothness parameter over the entire forward curve.
- Waggoner (1997) – Improvement on the FNZ smoothed spline but with varying degrees of smoothness across the curve.
- Nelson and Siegel (1987)– Parametric exponential specification of the discount function which favours parsimony over goodness of fit. Svensson (1995) extends this model.

A description of these approaches can be found in Bliss (1996) and Anderson and Sleath (2000). Bliss(1996) tested the McCullough, FNZ, and Nelson & Siegel (N&S) models for accuracy of pricing in-sample and out of sample and determined that, despite the sophistication of the FNZ, the McCullough approach provided more accurate pricing than both FNZ and N&S. If the McCullough model then can be implemented in an efficient manner, it could be said that the result would be among the most accurate and efficient (even considering the age of the model).

This paper describes a method for estimating the discount function from any set of bond pricing data, and describes a method for calculating this in Excel, using, if necessary, real-time data. A technically advanced method of splining the yield curve using the McCullough specification is described, allowing for very elaborate yield curve shapes, while still allowing for the implementation in Excel with the use of a restricted least squares algorithm.

2. The spot curve as a discount function

Imagine an exercise where the cash flows from a set of 30 bonds, with maturities from 1-30 years, are deposited into buckets (one bucket for each year). A regression of the prices of these bonds (a 30 x 1 matrix) on the cash flows divided into buckets (a 30 x 30 matrix) would yield 30 coefficients. Each coefficient would be the discount factor for cash flows in that bucket. This approach requires one coefficient (discount factor) for each bucket. Since this method equates the same discount factor for cash flows in January as in December, for practical purposes, it would be necessary to have a bucket for every cash flow. This means as many as 900 buckets (and coefficients) would be required to devise an accurate discount curve for our set of 30 bonds (with semi-annual payments all on different dates). The financial literature on the topic has focused on devising a method to accurately model the discount curve with a small number of coefficients. McCullough (1975) specified an approach which estimated the discount function as a polynomial, described next.

The present value of a bond worth 100 dollars at maturity can be described as:

$$P = cD_1 + cD_2 + cD_3 + ... + (c+100)D_T \quad (1)$$

where

P is price of the bond

c is the coupon

D is the discount factor relevant to the specific cash flow. i.e. $D_3 = \frac{1}{(1+i_3)^3}$

Assuming that the discount function (the discount factor for each maturity, t) is decreasing, smooth and concave, it could be modelled as:

$$D(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 \quad (2)$$

where b_0, b_1, b_2 and b_3 are estimated coefficients.

The present value of a bond described by equation (1) can be compactly written as:

$$P = \sum_{t=1}^{T} C(t)D(t) \quad (3)$$

Substituting equation (2) into equation (3) yields:

$$P = \sum_{t=1}^{T} c(t)(b_0 + b_1 t + b_2 t^2 + b_3 t^3) \quad (4)$$

Rearranging equation (4),

$$P = b_0\left[\sum_{t=1}^{T} c(t)\right] + b_1\left[\sum_{t=1}^{T} tc(t)\right] + b_2\left[\sum_{t=1}^{T} t^2 c(t)\right] + b_3\left[\sum_{t=1}^{T} t^3 c(t)\right] \quad (5)$$

Given data for at least five bonds, equation (5) can be estimated via ordinary least squares. The resulting function can return the discount factor for any maturity. If a yield curve is required instead of a discount function, a simple transformation can be done for the required maturity.

For small data sets, or relatively smooth yield curves, the above process may provide enough flexibility for the occasional user. However, for less parsimonious or smooth yield curves, a more complex function will usually be required. A perfect fit will be achieved by increasing the order of the polynomial used in equation (5) until there are almost as many polynomial terms as there are bonds. Of course, this is not practical for more than ten bonds. As an alternative, it is possible to join together several low order polynomials to create very dynamic curves, known as splines. Shea (1984) showed how a spline could be specified as a case of restricted least squares, which is akin to ordinary least squares (or linear regression) but with restrictions placed on the variables. In section four, four cubic polynomials are joined together to create a very robust and flexible spot curve. In the next section, the simple version of the discount function as specified in equation (5) is described.

3. Estimating the basic McCullough polynomial in Excel

3.1 Data setup

In this exercise, data for US Treasuries were used consisting of the current date (settle date), coupon, maturity, and price. The dirty price of the bond was produced by summing the accrued interest, calculated in Excel, and the market price. Non-callable notes and bonds with a maximum maturity of 10-years and a minimum maturity of 1-month were used. In practice, the data can be from a live market-based feed, allowing the possibility of updating the curve as prices change.

3.2 Calculation of c × t

For each bond, the sum of the cash flow times the maturity of that cash flow for t, t^2 and t^3 must be calculated as specified in equation (5). This step is the most demanding part of the entire exercise. While it is possible to calculate this on the spreadsheet, it is much more feasible to use a small amount of visual basic for the calculation.

Assuming there are n bonds, there will be a total of 4×n data points with four observations for each bond. The first observation, corresponding to the first element in equation (5) is just the sum of all of the cash flows for the bond, being the sum of all coupons plus the principal at maturity. For a two-year bond paying an annual 5% coupon, the value would be 2x5 + 100 = 110. (Semi-annual coupon payments would be 4x2.5 + 100 = 110) The second term for equation 5 would be the sum of the product of each cash flow value times the maturity of that cash flow. For a two-year semi-annual interest-paying bond (again 5%), the calculation would be the sum of 2.5×½ + 2.5 × 1 + 2.5 × 1½ + 2.5 × 2 + 100 × 2. The third element of equation (5) is similar in calculation to the second, but instead of using the maturity for each cash flow, the maturity squared is used. Again, for our two-year 5% semi-annual bond, the calculation would be the sum of $2.5 \times ½^2 + 2.5 \times 1^2 + 2.5 \times 1½^2 + 2.5 \times 2^2 + 100 \times 2^2$.

Finally, the fourth element is the same method as the third, except that the maturity of each cash flow is raised to the third power (i.e. $2.5 \times ½^3 + 2.5 \times 1^3 + 2.5 \times 1½^3 + 2.5 \times 2^3 + 100 \times 2^3$).

While these examples are simple and have nice round maturities, in practice, coupons are rarely arriving in exactly six months from now, but more likely at some odd fraction of a year (e.g. .35 or .76 years) away from now. The McCullough method is just as simple in this circumstance. The fractional part of the year is used as the maturity of the cash flow. So, the data point for the **fourth element** of equation (5) for a bond maturing in 1.68 years paying 5% semi-annually, would be $2.5 \times 0.18^3 + 2.5 \times 0.68^3 + 2.5 \times 1.18^3 + 2.5 \times 1.68^3 + 100 \times 1.68^3$.

Appendix 1 contains a detailed example of the calculations and results for an exercise in constructing the polynomial.

3.3 Estimating the coefficients for the discount function

Excel provides a multitude of methods for calculating the coefficients for linear regressions of various types. A non-exhaustive list would include LINEST, LOGEST, SLOPE, TREND, the Data Analysis Regression tool, and a toolbox of Matrix routines. The Matrix approach is the most difficult, but is also the most robust and will be the method chosen here, for reasons that will become obvious in the next section. As is detailed in most statistical textbooks, regression coefficients in a matrix form can be estimated as:

$$\hat{\beta} = (X^T X)^{-1} X^T Y \qquad (6)$$

where $\hat{\beta}$ is a column vector of the estimated coefficients, X is the matrix of independent variables, and Y is the column vector of dependent variables (^{T} denotes a transposition of a matrix and $^{-1}$ denotes the inversion of a matrix). The X variables for this exercise will be the four columns of calculated c × t totals, as specified in section 3.2. The Y variable will be a column of dirty prices for each bond (market quoted price plus accrued interest). The three matrix functions in Excel that are needed are: MINVERSE, TRANSPOSE, and MMULT. So, an example from a spreadsheet that would return a column vector of coefficients for a regression would look like the following:

```
=MMULT(MINVERSE(MMULT(TRANSPOSE((X_)),(X_))),MMULT(TRANSPOSE((X_)),(Y_)))
```

where X_ is a defined range containing a 10 × 4 matrix of c × t values, and Y_ is a 10 × 1 column vector containing the dirty bond prices. As this is a native Excel worksheet function, the coefficients are updated each time there is a change in the price of any bond (so long as re-calculation is automatic).

The resulting coefficients are simply plugged back into equation (2) in order to obtain the discount factor for any maturity (t). A zero coupon yield can then be computed from the discount factor as follows:

$$r = \left(\left(\frac{1}{DF} \right)^{1/2t} - 1 \right) \times 2 \tag{7}$$

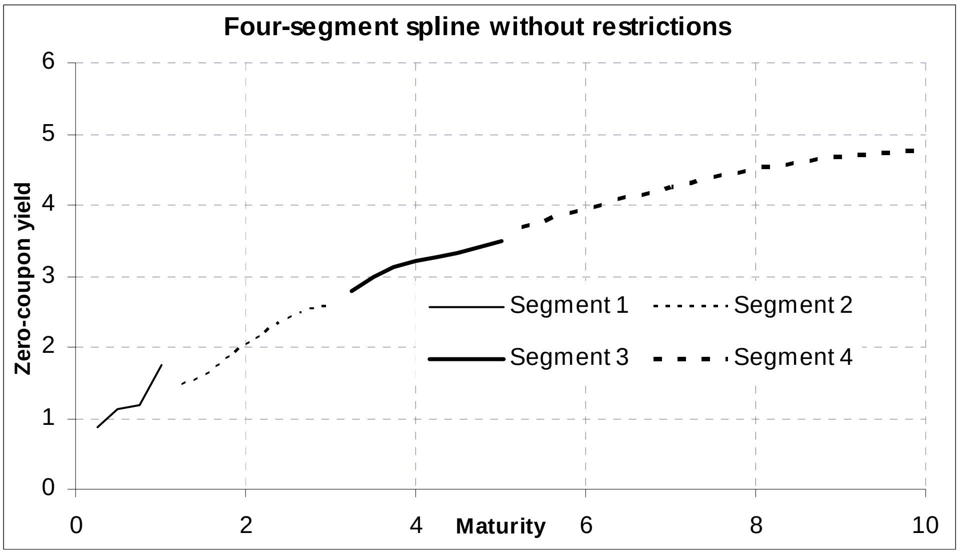
where r is the zero-coupon yield using street convention calculation (semi-annually compounded for STRIPS), DF is the discount factor, as calculated from equation (2) using the estimated coefficients, and t is the maturity of the discount factor. Alternatively, a simpler calculation of the zero-coupon yield with continual compounding is:

$$r = {-\ln(DF)} \big/ t \tag{8}$$

where ln() denotes the natural logarithm operator. In the next section, estimation of equation (5) is expanded to produce a full spline, consisting of several cubic polynomials.

4. Estimating a spline as a case of restricted least squares

While the simple polynomial approximation of the spot curve is acceptable for a very smooth curve, in practice the shapes can be very erratic, particularly in the shorter end of the curve. The required curve must then be able to handle more shapes than could typically be delivered by a third-order polynomial like the one shown in sections 2 and 3, and so we turn to the use of a (basis) spline. As mentioned in the previous section, a spline is a set of polynomials joined together to form a yield curve that can have a great many possible shapes. These segments are joined together at “knot points”. In the model described below, the spline is of the discount function (as described in the McCullough model) with four segments, each segment being a 3rd order polynomial. This essentially means that the discount function will be a function of t, t^2 and t^3 with an intercept, requiring the estimation of four coefficients for each segment, giving a total of 16 coefficients. Ordinary least squares cannot be used because there is no guarantee that the last discount factor in segment one will be very close to the first discount factor in segment two. In other words, the smoothness of the entire curve cannot be assured.

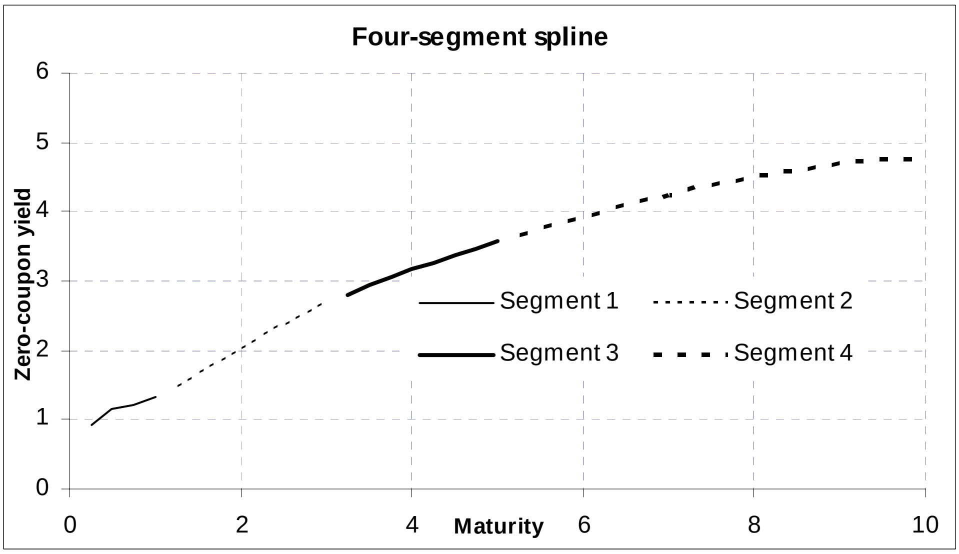


To overcome this challenge, restricted least squares can be used. The restrictions are fairly intuitive. At the knot points, we would like to have the level, first derivative and second derivative constant. This will ensure a smooth function and no visible jumps at the knot points. As detailed in Shea (1984), restricted least squares for a spline can be estimated in matrix form. This allows a fairly powerful estimation to be made with any spreadsheet that can handle matrices. Restricted least squares, where the restrictions are specified in matrix form would be specified as a modification of equation (6) as follows:

$$\hat{B}_R = \hat{\beta} + [(X^T X)^{-1} R^T][R(X^T X)^{-1} R^T]^{-1}(r - R\hat{\beta}) \tag{9}$$

where

$\hat{\beta}_R$ is a column vector of coefficients for the restricted regression.



We can define d as representing the maturity of each knot point, giving five knot points for a four segment spline, the first representing the minimum maturity (i.e. 1 month).

Formally, the spline model, a modification of equation (5) can be specified as:

$$D_t = \begin{matrix} b_{1,1}C + b_{2,1}Ct + b_{3,1}Ct^2 + b_{4,1}Ct^3 + & \text{for all cash flows where } d_1 < t < d_2 \\ b_{1,2}C + b_{2,2}Ct + b_{3,2}Ct^2 + b_{4,2}Ct^3 + & \text{for all cash flows where } d_2 < t < d_3 \\ b_{1,3}C + b_{2,3}Ct + b_{3,3}Ct^2 + b_{4,3}Ct^3 + & \text{for all cash flows where } d_3 < t < d_4 \\ b_{1,4}C + b_{2,4}Ct + b_{3,4}Ct^2 + b_{4,4}Ct^3 + & \text{for all cash flows where } d_4 < t < d_5 \end{matrix} \tag{10}$$

Where D_t is the discount factor, or present value of $1 at the horizon t, C is the cash flow of the bond, either coupon or principal, and b is a set of estimated coefficients.

There are four restrictions that should be specified in the restricted model as established by Shea (1984):

1) $b_{1j} + b_{2j}d_{j+1} + b_{3j}d_{j+1}^2 + b_{4j}d_{j+1}^3 = b_{1j+1} + b_{2j+1}d_{j+1} + b_{3j+1}d_{j+1}^2 + b_{4j+1}d_{j+1}^3$ (11)

2) $b_{2j} + 2b_{3j}d_{j+1} + 3b_{4j}d_{j+1}^2 = b_{2j+1} + 2b_{3j+1}d_{j+1} + 3b_{4j+1}d_{j+1}^2$ (12)

3) $2b_{3j} + 6b_{4j}d_{j+1} = 2b_{3j+1} + 6b_{4j+1}d_{j+1}$ (13)

4) $b_{11} = 1$ (14)

for j=1, . . . k-3 where d is the knot point, or upper bound of the segment of the spline. So, for example, if the knot points, including the upper and lower bounds for the spline were 0, 2, 5, 7, and 10 years, $d_1 = 0$, $d_2 = 2$, $d_3 = 5$, $d_4 = 7$, and $d_5 = 10$.

The first restriction, equation (11), establishes that the level of the discount factors should be equal at each knot point. Equation (12) sets the first derivative, or rate of change, constant at the knot point, and restriction three keeps the second derivative constant. It may also be feasible to have the fourth restriction where the discount factor for t=0 is forced to be 1 (i.e. the present value of one dollar today should be one dollar).

In matrix form, four restrictions modelled by equation (11) to (14) can be represented in matrix form as given below:

$$R = \begin{bmatrix} -1 & -d_2 & -d_2^2 & -d_2^3 & 1 & d_2 & d_2^2 & d_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -d_3 & -d_3^2 & -d_3^3 & 1 & d_3 & d_3^2 & d_3^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -d_4 & -d_4^2 & -d_4^3 & 1 & d_4 & d_4^2 & d_4^3 \\ 0 & -1 & -2d_2 & -3d_2^2 & 0 & 1 & 2d_2 & 3d_2^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2d_3 & -3d_3^2 & 0 & 1 & 2d_3 & 3d_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2d_4 & -3d_4^2 & 0 & 1 & 2d_4 & 3d_4^2 \\ 0 & 0 & -2 & -6d_2 & 0 & 0 & 2 & 6d_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -6d_3 & 0 & 0 & 2 & 6d_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -6d_4 & 0 & 0 & 2 & 6d_4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (15)$$

Here, d_2, d_3, and d_4 are the knot points where the four segments join, typically denominated in years.

5. Implementing the spline in Excel

Calculation of the four segment spline in Excel is done in a similar fashion to the simple polynomial, with a few exceptions. Instead of four columns of data containing the $\sum C$, $\sum Ct$, $\sum Ct^2$ and $\sum Ct^3$, there need to be 16. Four sets for each set of cash flows falling within the particular segment of the spline are needed. Assuming the three knot points were chosen at 1.5 years, 3 years, and 6 years, the first four columns would calculate equation (2) for all cash flows that fall between zero and 1.5 years. The fifth through eighth columns calculate equation (2) for all cash flows that fall between 1.5 and three years, and so on. So, the shortest maturity bond would have only four data observations, and twelve zeros. The longest maturity bond (which would have cash flows for every year) would have non-zero numbers for all sixteen observations. The Y matrix, again would be an n × 1 vector of dirty prices (market price + accrued interest), where n is the total number of bonds. The X matrix is the n × 16 set of data calculated as described above.

The next step is to estimate equation (9) in matrix form. This requires several “intermediate” matrixes, the most important being the matrix of restrictions (R) which is detailed in equation (15). This would be represented as a 10x16 (r × c) matrix in Excel. An example using the 1.5, 3, and 6-year knot points can be found in Appendix 2. In addition, the model must be estimated in its unrestricted form, as equation (9) includes $\hat{\beta}$ in the right-hand side of the equation. This requires simply estimating equation (6) using the same process as described in section 1, but using the 16 column set of X data as detailed in this section, along with the Y column vector of dirty prices. The final intermediate matrix is r, which is a 10 × 1 matrix of zeros, with the exception of the 10^{th} element being a 1. (e.g. r = [0 0 0 0 0 0 0 0 0 1]). The matrix r is set to zero as all of the restrictions are bound to equal zero, with the exception of the 10^{th} restriction where the intercept of the discount function is bound to the value of 1. (i.e. the present value of one dollar today is one dollar.

The 16 × 1 matrix of coefficients for the restricted least squares, equation (9), is estimated in several steps, and provides a smooth functional estimate of the discount function. The present value of any cash flow can then be calculated for any maturity. To calculate this, a variant of equation (2) is needed:

$$D_t = \begin{array}{ll} b_{1,1} + b_{2,1}t + b_{3,1}t^2 + b_{4,1}t^3 + & \text{if cash flow falls within } \mathrm{d}_1 < \mathrm{t} < \mathrm{d}_2 \\ b_{1,2} + b_{2,2}t + b_{3,2}t^2 + b_{4,2}t^3 + & \text{if cash flow falls within } \mathrm{d}_2 < \mathrm{t} < \mathrm{d}_3 \\ b_{1,3} + b_{2,3}t + b_{3,3}t^2 + b_{4,3}t^3 + & \text{if cash flow falls within } \mathrm{d}_3 < \mathrm{t} < \mathrm{d}_4 \\ b_{1,4} + b_{2,4}t + b_{3,4}t^2 + b_{4,4}t^3 + & \text{if cash flow falls within } \mathrm{d}_4 < \mathrm{t} < \mathrm{d}_5 \end{array} \qquad (16)$$

5.2 Knot points

Choice of the knot points can be an important issue, and unfortunately, there is not an easy answer as to the best method. An optimization method could be used, but because the "objective function" might not be very smooth, a very advanced search algorithm such as simulated annealing or a genetic searching would likely be required as described in Ramaswamy (2000). For the less technically inclined, intuition might be a sufficient guide. If there is reason to believe in some market segmentation, this may provide suggestions as to the appropriate knot points. An obvious starting point would be the separation point between the money market and the bond market. Another possible point would be either at the start or end maturity of a cheapest-to-deliver basket of bonds. McCullough suggests dividing placing the knot points so that there are an equal number of bonds with maturities in each segment.

5.3 Using the coefficients

The estimated coefficients can be used for a multitude of tasks, the most common being security valuation. By multiplying each cash flow of a security by the relevant discount factor for that horizon, a theoretical value can be computed. Alternatively, the discount function could be used to calculate the spot curve (equations (7) and (8)), and corresponding forward rates. This might be useful for an option valuation model such as Black-Derman-Toy. Finally, there is the possibility of calculating a par yield curve. A par bond is simply a bond where the price is 100, and the coupon and yield are the same. This might be useful for calculating a constant maturity par bond yield (such as those in the Federal Reserve H.15 release). Par yields are calculated as:

$$Par\ Y_T = \frac{(1 - DF_T)}{\frac{1}{F}\sum_{t=1}^{T} DF_t} \tag{17}$$

Where T is the number of coupons, F is the frequency of coupon payments per year, and DF is the relevant discount factor for a coupon paid at t.

6. Concluding remarks

An investment analyst or portfolio manager often needs to value a security in a real-time pricing context. The requirement of a sophisticated spot curve that updates in short order can be satisfied with the use of the restricted-least-squares four segment spline. It is not the most sophisticated algorithm available, but it is likely a strong contender for a model that optimally balances speed with sophistication. In addition, the simpler McCullough method can be used for estimating simpler curves, where only a few securities may be available.

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Appendix 1: An example of the McCullough polynomial approximation of the zero-coupon yield curve

Bond:	*Maturity*	*Price*	*Coupon*	*Price + AI*	$\sum_{t=1}^{T} c(t)$	$\sum_{t=1}^{T} tc(t)$	$\sum_{t=1}^{T} t^2 c(t)$	$\sum_{t=1}^{T} t^3 c(t)$	*Discount factor*	*Zero-coupon yield*
1	31.12.2003	100.2461	3.25	101.53	101.625	11.1294	1.218821	0.133478	0.99876	1.13
2	15.02.2004	100.875	4.75	102.15	102.375	24.1047	5.675582	1.336345	0.99786	0.91
3	31.05.2004	101.1484	3.25	102.70	103.25	53.4565	28.07797	14.75796	0.99493	0.97
4	15.08.2004	103.5469	6	105.16	106	76.2820	55.61949	40.72759	0.99213	1.08
5	31.10.2004	100.8555	2.125	100.99	102.125	95.9249	90.36799	85.25127	0.98874	1.20
6	31.12.2004	100.4805	1.75	101.17	102.625	112.7522	124.9673	138.7337	0.98567	1.30
7	15.05.2005	107.6758	6.75	107.81	110.125	158.0695	230.8553	339.5298	0.97778	1.52
8	15.02.2006	108.082	5.625	109.60	114.0625	241.0541	528.8348	1171.763	0.95755	1.94
9	15.07.2006	112.418	7	114.89	121	294.1492	757.3779	1977.531	0.94466	2.15
10	15.02.2007	111.9453	6.25	113.63	121.875	361.5674	1134.981	3617.175	0.92444	2.43

Coefficients:

B0	0.9993535
B1	-0.0045940
B2	-0.0075599
B3	0.0005666

Appendix 2: An example of the R matrix of restrictions for the 1.5, 3, and 6-year knot points

-1	-0.8	-0.64	-0.512	1	0.8	0.64	0.512	0	0	0	0	0	0	0	0
0	0	0	0	-1	-3	-9	-27	1	3	9	27	0	0	0	0
0	0	0	0	0	0	0	0	-1	-5	-25	-125	1	5	25	125
0	-1	-1.6	-1.92	0	1	1.6	1.92	0	0	0	0	0	0	0	0
0	0	0	0	0	-1	-6	-27	0	1	6	27	0	0	0	0
0	0	0	0	0	0	0	0	0	-1	-10	-75	0	1	10	75
0	0	-2	-4.8	0	0	2	4.8	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-2	-18	0	0	2	18	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-2	-30	0	0	2	30
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0