SWAPTION PRICING

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Abstract. Implementation details for the pricing of European swaptions in different frameworks are presented.

1. Introduction

This note describes the pricing of cash-settled and physical delivery European swaptions.

The framework of the pricing is a Black formula with implied volatility (like the SABR model approximation described in Hagan et al. [2002]). In this framework, for each forward, strike and tenor an implied volatility is provided. From the implied volatility, the price is computed through the Black formula.

The implied volatility is usually obtained for a set of standard vanilla swaptions. In this context, standard means constant strike for all swap lifetime and standard conventions for each currency. The possibility of having swaptions on non-standard swaps need to be taken into account and a modification to usual definition of annuity and forward is introduced to map the non-standard swaps into the implied volatility for standard swap framework.

The modifications are such that two swaptions with the same cash-flows but different conventions (and thus different reported fixed rate) will have the same price. For example a swaption on a swap with fixed leg convention ACT/360 and a rate of 3.60% will always have the same price as a swaption on a swap with fixed leg convention ACT/365 and a rate of 3.65%.

2. Notation

The analysis framework is a *multi-curves setting* as described in Henrard [2010a]. There is one discounting curve denoted $P^D(s,t)$ and one forward curve $P^j(s,t)$ where j is the relevant Ibor tenor.

2.1. Swap.

The swap underlying the swaption has a start date t_0, a tenor T, m payments per annum, and fixed leg payment dates $(t_i)_{1\leq i\leq n}$. The accrual fractions for each fixed period are $(\delta_i)_{1\leq i\leq n}$; the rates for each fixed period are $(K_i)_{1\leq i\leq n}$. The floating leg payment dates are $(\tilde{t}_i)_{1\leq i\leq \tilde{n}}$ and the fixing period start and end dates are (s_i) and (e_i).

2.2. Standard figures.

The *delivery annuity* (also called PVBP or level) is

$$A_t = \sum_{i=1}^{n} \delta_i P^D(t,t_i).$$

The swap rate is

$$S_t = \frac{\sum_{i=1}^{\tilde{n}} P^D(t,\tilde{t}_i)\left(\frac{P^j(t,s_i)}{P^j(t,\tilde{e}_i)}-1\right)}{A_t}.$$

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The cash-settled annuity is

$$G(S) = \sum_{i=1}^{n} \frac{\frac{1}{m}}{(1+\frac{1}{m}S)^i} = \frac{1}{S}\left(1 - \frac{1}{(1+\frac{1}{m}S)^n}\right)$$

To cover the case where the fixed rate is not the same for all coupon, a *strike equivalent* is introduced:

$$K = \frac{\text{PV}}{A_0} = \frac{\sum_{i=1}^{n} \delta_i K_i P^D(0,t_i)}{\sum_{i=1}^{n} \delta_i P^D(0,t_i)}.$$

Note that the strike equivalent is curve dependent.

2.3. **Convention-modified figures.** To ensure absence of arbitrage when the conventions are changed, modified versions of standard figures are introduced. Let C be the standard convention.

The convention C-modified delivery annuity is

$$A_t^C = \sum_{i=1}^{n} \delta_i^C P^D(t,t_i).$$

The convention-modified swap rate is

$$S_t^C = \frac{\sum_{i=1}^{\tilde{n}} P^D(t,\tilde{t}_i)\left(\frac{P^j(t,s_i)}{P^j(t,\tilde{e}_i)} - 1\right)}{A_t^C}.$$

The convention-modified strike equivalent is

$$K^C = \frac{\text{PV}}{A_0^C}.$$

2.4. **Swpation.** The swaption expiration time is denoted θ.

3. Black implied volatilities

In general, the Black implied volatility can be expiration θ, tenor T, forward S_0, and strike K and model parameters (p) dependent. The implied volatility is denoted $\sigma(\theta,T,S_0,K,p)$

Two Black implied volatility frameworks are considered: expiration/tenor dependent Black and expiry/tenor dependent SABR.

3.1. **Black.** The Black implied volatility is expiration/tenor dependent. It is not strike dependent.

3.2. **SABR.** The SABR parameters are expiration/tenor dependent. For a given set of SABR parameters, the smile is forward and strike dependent and given by one SABR volatility function (Hagan, HaganAlternative, Berestycki, Johnson, Paulot, ...).

4. Physical delivery swaptions

4.1. **Standard.** The standard price on 0 of a physical delivery swaption in a framework with Black implied volatility is

$$P = A_0 \text{Black}(S_0, K, \sigma(\theta,T,S_0,K,p)).$$

4.2. **Convention modified.** The price taking into account the fact that the implied volatilities are provided for standard-convention instruments is

$$P = A_0^C \text{Black}(S_0^C, K^C, \sigma(\theta,T,S_0^C,K^C,p)).$$

Note that the convention change in the swap rate and the strike equivalent is the inverse of the one in the annuity. The modification impact appears only through the volatility; if the smile is flat, the modification has no impact. If the swaption has a standard convention, there is no impact at all.

5. Cash-settled swaptions

The cash-settled swaptions can be viewed as exotic versions of the physical delivery ones (a function of the swap rate paid at a non-natural time). There are several ways to approach this feature. The first one is the standard market formula (a copy of the physical delivery formula). Other uses more sophisticated approximations.

In the physical annuity numeraire A_t, the generic formula of the cash-settled swaption value is

$$A_0 \mathrm{E}^A \left[\frac{P^D(\theta, t_0)}{A_\theta} G(S_\theta)(K - S_\theta)^+ \right]$$

5.1. Standard (market formula).

The standard price on 0 of a cash-settled swaption in a framework with Black implied volatility is

$$P = G(S_0)\mathrm{Black}(S_0, K, \sigma(\theta, T, S_0, K, p)).$$

This standard market formula is obtained by *copying* the physical delivery one and replacing the annuity. This formula is not arbitrage free as reported in Mercurio [2008] and further analysed in Henrard [2010b].

The cash-settled swaption settle against a published fixing and consequently are always standard conventions. There is no need to introduce the convention modified version.

5.2. Linear Terminal Swap Rate.

The idea presented in this section are from [Andersen and Piterbarg, 2010, Chapter 16].

In the cash-settled generic formula, the part which is not directly dependent on S is replaced by a function dependent on it (this is the *Terminal Swap Rate* part of the name)

$$\frac{P^D(\theta, t_0)}{A_\theta} \simeq \alpha(S_\theta).$$

In the *Linear* version, the terminal swap rate function is taken to be a linear function of the swap rate:

$$\alpha(s) = \alpha_0 s + \alpha_1.$$

A simple choice for the constant α_0 and α_1 is

$$\alpha_1 = \frac{1}{\sum_{i=1}^n \delta_i} \quad \text{and} \quad \alpha_0 = \frac{1}{S_0} \left(\frac{P^D(0, t_0)}{A_0} - \alpha_2 \right).$$

This choice is proposed in the above-mentioned reference in Equation (16.56) where there is an explanation on why this is a reasonable choice. It is called a *simplified approach*.

Note that the linear TSR is such that $\alpha(S_0) = P^D(0, t_0) A_0^{-1}$ (the approximation is exact in 0 and ATM).

Using this approximation, the cash-settled swaption can be priced in a way similar to a CMS by replication:

$$A_0 \left(k(K)\, \mathrm{Swpt}(S_0, K) + \int_K^{+\infty} (k''(x)(x - K) + 2k'(x))\, \mathrm{Swpt}(S_0, x) dx \right)$$

with $k(x) = (\alpha_1 x + \alpha_2) G(x)$ and $\mathrm{Swpt}(S_0, x)$ the price of a physical delivery swaption with strike x when the forward is S_0.

6. Sensitivities

The sensitivities computed are the curve sensitivities and sensitivities to the model parameters. To obtain them, we use the derivatives of the Black formula with respect to the forward $D_S\mathrm{Black}$ and the volatility $D_\sigma\mathrm{Black}$.

6.1. **Black.** The curve sensitivity is obtained by computing the derivative of the annuity A (A_0^C or $G(S_0)$) and the forward S_0^C with respect to the curve, and composing with the derivative of the Black formula with respect to the forward:

$$D_{r_i}P = D_{r_i}A\text{Black}(S_0^C, K^C, \sigma(\theta, T)) + AD_S\text{Black}(S_0^C, K^C, \sigma(\theta, T))D_{r_i}S_0^C.$$

The parameter sensitivity is the sensitivity to the Black volatilities (also called vega):

$$D_\sigma P = AD_\sigma\text{Black}(S_0^C, K^C, \sigma(\theta, T)).$$

6.2. **SABR.** For the curve sensitivity with respect to Black, we have to take into account the sensitivity of the volatility to the forward:

$$\begin{aligned} D_{r_i}P &= D_{r_i}A\text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p)) \\ &\quad + AD_S\text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p))D_{r_i}S_0^C \\ &\quad + AD_\sigma\text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p))D_S\sigma(\theta, T, S_0^C, K^C, p)D_{r_i}S_0^C. \end{aligned}$$

The parameter sensitivity is the sensitivity to the SABR parameters p_i:

$$D_{p_i}P = AD_\sigma\text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p))D_{p_i}\sigma(\theta, T, S_0^C, K^C, p).$$

7. Extrapolation

For an implied volatility smile, the standard approximation methods do not produce very good results far away from the money. Moreover, the model calibration close to the money does not necessarily provide relevant information for far away strikes. When only vanilla options are priced, these problems are relatively minor as the time value of those far away from the money options is small. This is not the case for those instruments where the pricing depends on the full smile. In the swaption world, they include the CMS swap and cap/floor (pre and post-fixed).

For those reasons, it is useful to have a framework using an implied volatility method up to a certain level and using an (arbitrage-free) extrapolation beyond with the flexibility to adjust the smile tail weight. Such an extrapolation based on Benaim et al. [2008] is described in the note OpenGamma Research [2011].

8. Implementation

The European physical delivery swaptions are implemented in the class `SwaptionPhysicalFixedIbor`. The pricing with the Black volatility is implemented in the method `SwaptionPhysicalFixedIborBlackSwaptionMethod`. The pricing with SABR implied volatility is implemented in the method `SwaptionPhysicalFixedIborSABRMethod`. The pricing with SABR implied volatility and extrapolation is implemented in the method `SwaptionPhysicalFixedIborSABRExtrapolationRightMethod`.

The European cash-settled swaptions are implemented in the class `SwaptionCashFixedIbor`. The pricing with the Black volatility is implemented in the method `SwaptionCashlFixedIborBlackSwaptionMethod`. The pricing with SABR implied volatility is implemented in the method `SwaptionCashFixedIborSABRMethod`. The pricing with SABR implied volatility and extrapolation is implemented in the method `SwaptionCashFixedIborSABRExtrapolationRightMethod`. The Linear Terminal Swap Rate method is implemented in `SwaptionCashFixedIborLinearTSRMethod`.

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