Calibration of the SABR model in illiquid markets

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1 What is the skew?

European options or fully margined (SAFEX) American options are priced and often hedged using the Black-Scholes or SAFEX Black model. In these models there is a one-to-one relation between the price of the option and the volatility parameter σ, and option prices are often quoted by stating the implied volatility σ_{imp}, the unique value of the volatility which yields the option price when used in the formula. In the classical Black-Scholes-Merton world, volatility is a constant. But in reality, options with different strikes require different volatilities to match their market prices. This is the market skew or smile.

2 Local volatility models

The development of local volatility models in (Dupire 1994), (Dupire 1997), (Derman & Kani 1994), (Derman, Kani & Chriss 1996) and (Derman & Kani 1998) was a major advance in handling smiles and skews. Another crucial thread of development is the stochastic volatility approach, for which the reader is referred to (Hull & White 1987), (Heston 1993), (Lewis 2000), (Fouque, Papanicolaou & Sircar 2000), (Lipton 2003), and finally (Hagan, Kumar, Lesniewski & Woodward 2002), which is the model we will consider here.

Local volatility models are self-consistent, arbitrage-free, and can be calibrated to precisely match observed market smiles and skews. Possibly they are often preferred to the stochastic volatility models for computational reasons: the local volatility models are tree models; to price with stochastic volatility models typically means Monte Carlo. However, it has recently been observed (Hagan et al. 2002) that the dynamic behaviour of smiles and skews predicted by local volatility models is exactly opposite the behaviour observed in the marketplace: local volatility models predict that the skew moves in the opposite direction to the market level, in reality, it moves in the same direction. This leads to extremely poor hedging results within these models, and the hedges are often worse than the naive Black model hedges, because these naive hedges are in fact consistent with the smile moving in the same direction as the market.

3 Stochastic volatility models

Stochastic volatility models are in general characterised by the use of two driving correlated Brownian motions, one which determines the diffusion of the underlying process, and the other determines the diffusion of the volatility process. For example, the model of (Hull & White 1987) can be summarised as follows:

$$dF = \phi F dt + \sigma F dW_1 \tag{1}$$

$$d\sigma^2 = \mu\sigma^2 dt + \xi\sigma^2 dW_2 \tag{2}$$

$$dW_1 dW_2 = \rho\, dt \tag{3}$$

where ϕ, μ and ξ are time and state dependent functions, and dW_1 and dW_2 are correlated Brownian motions.

Similarly, the model of (Heston 1993) proceeds with the pair of driving equations

$$dF = \mu F dt + \sigma F dW_1 \tag{4}$$

$$d\sigma = -\beta\sigma dt + \delta dW_2 \tag{5}$$

$$dW_1 dW_2 = \rho\, dt \tag{6}$$

where this time μ, β and δ are constants.

As another example, the models of (Fouque et al. 2000) are variations on the following initial set-up:

$$dF = \mu F dt + \sigma F dW_1 \tag{7}$$

$$dy = \alpha(m - y)dt + \beta dW_2 \tag{8}$$

$$dW_1 dW_2 = \rho\, dt \tag{9}$$

where this time α, m and β are constants, and for example $y = \ln \sigma$. Here, the process for y is a mean reverting Ornstein-Uhlenbeck process.

The model we consider here is known as the stochastic $\alpha\beta\rho$ model, or SABR model. Here

$$dF = \alpha F^{\beta}\, dW_1 \tag{10}$$

$$d\alpha = v\alpha\, dW_2 \tag{11}$$

$$dW_1\, dW_2 = \rho\, dt \tag{12}$$

where the factors F and α are stochastic, and the parameters β, ρ and v are constants.

- α is a 'volatility-like' parameter: not equal to the volatility, but there will be a functional relationship between this parameter and the at the money volatility, as we shall see in due course.
- v is the volatility of volatility, a model feature which acknowledges that volatility obeys well known clustering in time.
- $\beta \in [0, 1]$ determines the relationship between futures spot and at the money volatility: $\beta \approx 1$ indicates that the user believes that if the market were to move up or down in an orderly fashion, the at the money volatility

level would not be significantly affected (lognormal like). $\beta << 1$ indicates that if the market were to move then at the money volatility would move in the opposite direction (normal like).

4 The option pricing formula

A desirable feature of any local or stochastic volatility model is that the model will reproduce the prices of the vanilla instruments that were used as inputs to the calibration of the model. Material failure to do so will make the model not arbitrage free and render it almost useless.

A significant feature of the SABR model is that the prices of vanilla instruments can be recovered from the model in closed form (up to the accuracy of a series expansion). This is dealt with in detail in (Hagan et al. 2002, Appendix B). Essentially it is shown there that the price of a vanilla option under the SABR model is given by the appropriate Black formula, provided the correct implied volatility is used. For given α, β, ρ, v and τ, this volatility is given by:

$$\sigma(X,F) = \frac{\alpha\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FX)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta v\alpha}{(FX)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}v^2\right)\tau\right)}{(FX)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\frac{F}{X}+\frac{(1-\beta)^4}{1920}\ln^4\frac{F}{X}\right]}\frac{z}{\chi(z)} \tag{13}$$

$$z = \frac{v}{\alpha}(FX)^{(1-\beta)/2}\ln\frac{F}{X} \tag{14}$$

$$\chi(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right) \tag{15}$$

Although the formula appears fearsome, it is closed form, so practically instantaneous. This formula of course can be viewed as a functional form for the volatility skew, and so, when this volatility skew is observable, we have some sort of error minimisation problem, which, subject to the caveats raised in (Hagan et al. 2002), is quite elementary. The thesis here is the same calibration problem in the absence of an observable skew, in which case, we need a model to infer the parametric form of the skew given a history of traded data.

5 The market we consider for this analysis

We consider the equity futures market traded at the South African Futures Exchange. For details of the operation of this market the reader is referred to (SAFEX 2005), (West 2005, Chapter 10).

This market is characterised by an illiquidity that is gross compared to other markets. We will focus on the TOP40 (the index of the biggest shares, as determined by free float market capitalisation and liquidity) futures options contracts. Contracts exist for expiry in March, June, September and December of each year. Amongst these, the following March contract is the most liquid, along with the nearest contract. Nevertheless, the March contract only becomes liquid in anything like a meaningful manner about two years before expiry.

One point that needs to be noted here is that futures options are American and fully margined, that is, the buyer of options does not pay an outright premium for the option, but is subject to margin flow, being the difference in the mark to market values on a daily basis. It can be shown (see (West 2005, Chapter 10)) that the appropriate option pricing formula in this setting is

$$V_C = FN(d_1) - XN(d_2) \tag{16}$$

$$V_P = XN(-d_2) - FN(-d_1) \tag{17}$$

$$d_{1,2} = \frac{\ln \frac{F}{X} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{18}$$

$$\tau = T - t \tag{19}$$

It can be shown that it is sub-optimal to exercise either calls or puts early, and so one should not be surprised that the option pricing formula are 'Black like' even though the option is American. Furthermore, the fact that the options are fully margined has the attractive consequence that the risk free rate does not appear in the pricing formulae. This is indeed fortunate as the South African yield curve itself is subject to a paucity of data compared to many markets, and hence may require some art in construction, which will typically be proprietary.

6 The β parameter

Market smiles can generally be fit more or less equally well with any specific choice of β. In particular, β cannot be determined by fitting a market smile since this would clearly amount to "fitting the noise". Selecting β from "aesthetic" or other a priori considerations usually results in $\beta = 1$ (stochastic lognormal), $\beta = 0$ (stochastic normal), or $\beta = \frac{1}{2}$ (stochastic models for interest rates c.f. (Cox, Ingersoll & Ross 1985).

We will not discuss the choice of β any further here; sufficient to say that statistical procedures (log-log regression) which are suggested in (Hagan et al. 2002) find a value of $\beta \approx 0.7$ is fairly reliable and stable in the South African market.

7 The α parameter

Note as in (Hagan et al. 2002) that if $F = X$ then the z and $\chi(z)$ terms are removed from the equation, as then $\frac{z}{\chi(z)} = 1$ in the sense of a limit[1], and so

$$\sigma_{\text{atm}} = \frac{\alpha\left(1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{F^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\upsilon\alpha}{F^{1-\beta}} + \frac{2-3\rho^2}{24}\upsilon^2\right)\tau\right)}{F^{1-\beta}} \tag{20}$$

To obtain α from σ_{atm}, we invert (20). Doing so, we easily see that α is a root of the cubic

$$\frac{(1-\beta)^2\tau}{24F^{2-2\beta}}\alpha^3 + \frac{\rho\beta\upsilon\tau}{4F^{1-\beta}}\alpha^2 + \left(1 + \frac{2-3\rho^2}{24}\upsilon^2\tau\right)\alpha - \sigma_{\text{atm}}F^{1-\beta} = 0 \tag{21}$$

where we are assuming that we have already solved for ρ and υ. For typical parameter inputs, this cubic has only one real root, but it is perfectly possible

[1] Some care needs to be taken with machine precision issues here. One can have that $z \approx 0$ and $\chi(z) = 0$ to double precision. This needs to be trapped, and the limit result invoked, again putting $\frac{z}{\chi(z)} = 1$.

for it to have three real roots, in which case we seek the smallest positive root[2]. One wants a rapid algorithm to find α to double precision, as then when in code finding the skew volatility for an option which is in fact at-the-money, one recovers the at-the-money volatility exactly. We use the Tartaglia method (as published by Cardano in the 16th century!) to find the desired real root. For this, we use the implementation and code in (Press, Teukolsky, Vetterling & Flannery 1992, §5.6). See (Weisstein 1999–2004) for a synopsis of the history of these root finding methods.

8 Calibration to existing market data

The values of ρ and v need to be fitted. As already discussed, the value of F and the value of σ_{atm} are inputs, and given these and the values of ρ and v, α is no longer a required input parameter.

As already discussed, we fix in advance the value of β. Then, for any in-

[2]When there are three real roots, they are of the order of -1000, 1 and +1000. So we take the root of order 1.

put pair (ρ, v), we determine an error expression $\text{err}_{\rho,v}$, which per trade is the distance between

- the currency cost that the trade was done at;
- the currency cost that the trade would have been done at if all the legs of the trade had been done on the SABR skew with that ρ and v.

The total error across all trades is some sum of these errors. The trades that have been observed in the market may be weighted for age, for example, by using an exponential decay factor: the further in the past the trade is, the less contribution it makes to the optimisation. Trades which are far in the past might simply be ignored.

Then, we seek the minimum of these error expressions $\text{err}_{\rho,v}$ amongst all pairs (ρ, v), for which we use the Nelder-Mead simplex search. See (Press et al. 1992, §10,4). The Nelder-Mead algorithm is a non-analytic search method that is very robust. Note that the error expression is essentially non-differentiable because it implicitly involves the root of a cubic (in other words, the differentiation involved would be horrendous).

As one can see in Figure 1 - this result is typical - the choice of parameters is fairly robust, with the minimum found at the bottom of a shallow valley.

As pointed out in (Hagan et al. 2002), the idea is that the parameter selection change infrequently (perhaps only once or twice a month) whereas the input values of F and σ_{atm} change as frequently as they are observed. This is in order to ensure hedge efficiency.

9 How well does the model fit the data?

9.1 Comparing the model to the recorded trade history and to other skews

The appropriate mechanism to compare this skew to other skews is via the ability to reproduce accurately the pricing performance. Comparing the actual volatilities in each skew is meaningless if a significant number of structures have traded, as opposed to outright options. Given a record of trades, we can compare

- The prices that were recorded in the market.

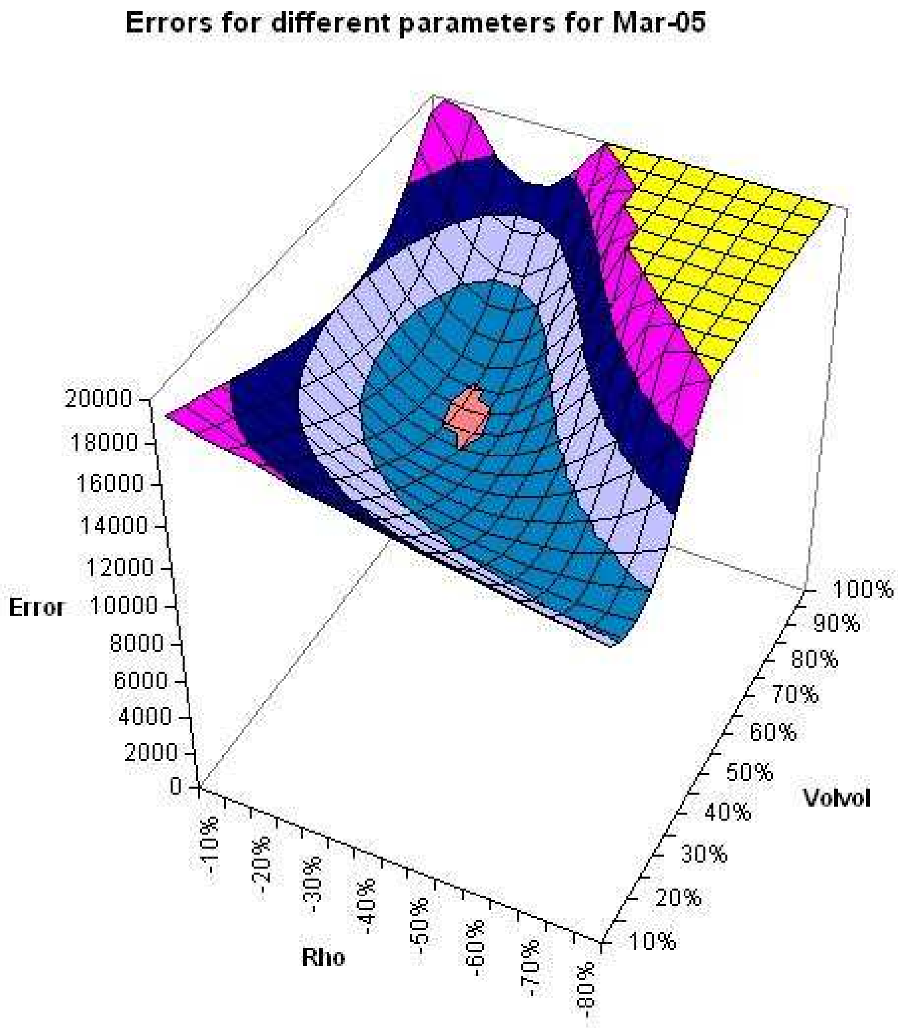

Figure 1: The error quantities for ρ and v.

- The prices that the model estimates.
- The prices that any other competing model of the skew estimates.

See Figure 2.

9.2 Trade sets

An issue which often arises, is that certain strategies (e.g. bull or bear spreads, butterflies, condors) trade for a pair, triple or quadruple of volatilities which may appear off market. In reality it is the price of the strategy that is trading and so a relevant set of volatilities may be found which is closer to the market than may at first appear.

To achieve this, we first determine the price of the strategy P, implemented at time t.

We would like to re-map this to the identical strategy, with the same price, but booked at different volatilities. These volatilities are found to be as close as

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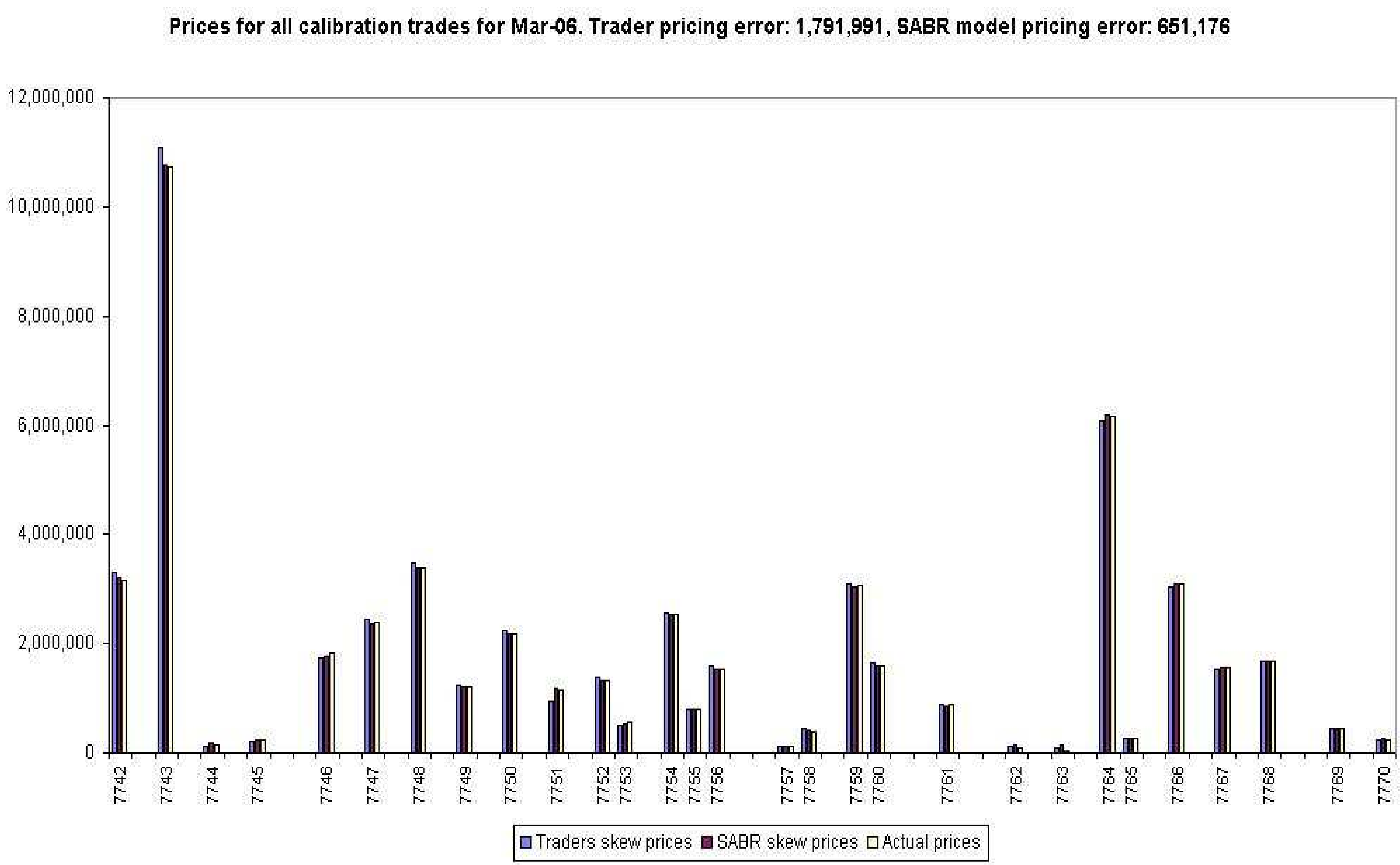

possible to the volatilities on the skew curve. Thus, we would like to minimise

$$f\left(\sigma_1^{\text{best}},\ \sigma_2^{\text{best}},\ \ldots,\ \sigma_n^{\text{best}}\right) = \sum_{i=1}^{n} |Q_i| \mathcal{V}\left(\sigma_i^{\text{mod}}\right)\left(\sigma_i^{\text{best}} - \sigma_i^{\text{mod}}\right)^2 \tag{22}$$

where 'best' denoted fitted volatilities and 'mod' denotes volatilities from the SABR model, subject to

$$P = P\left(\sigma_1^{\text{best}},\ \sigma_2^{\text{best}},\ \ldots,\ \sigma_n^{\text{best}}\right) \tag{23}$$

This minimisation is done using the method of Lagrange multipliers. We have a system of $n+1$ non-linear equations in ($\sigma_1^{\text{best}},\ \sigma_2^{\text{best}},\ \ldots,\ \sigma_n^{\text{best}}$, and λ) and so we use the multidimensional Newton-Raphson method for this part of the problem. By the 'economic' nature of the problem it is fairly clear that the zero is unique and that pathology will not arise in the use of the Newton-Raphson method. Convergence is very rapid.[3] Thus, for any strategy booked, equivalent volatilities can be fitted which are most compatible with the SABR model selected.

[3]Note that the matrix will almost certainly not be of size greater than 5×5.

We can now graph the volatilities that were booked, and compare them to the remodelled volatilities. See Figure 3.

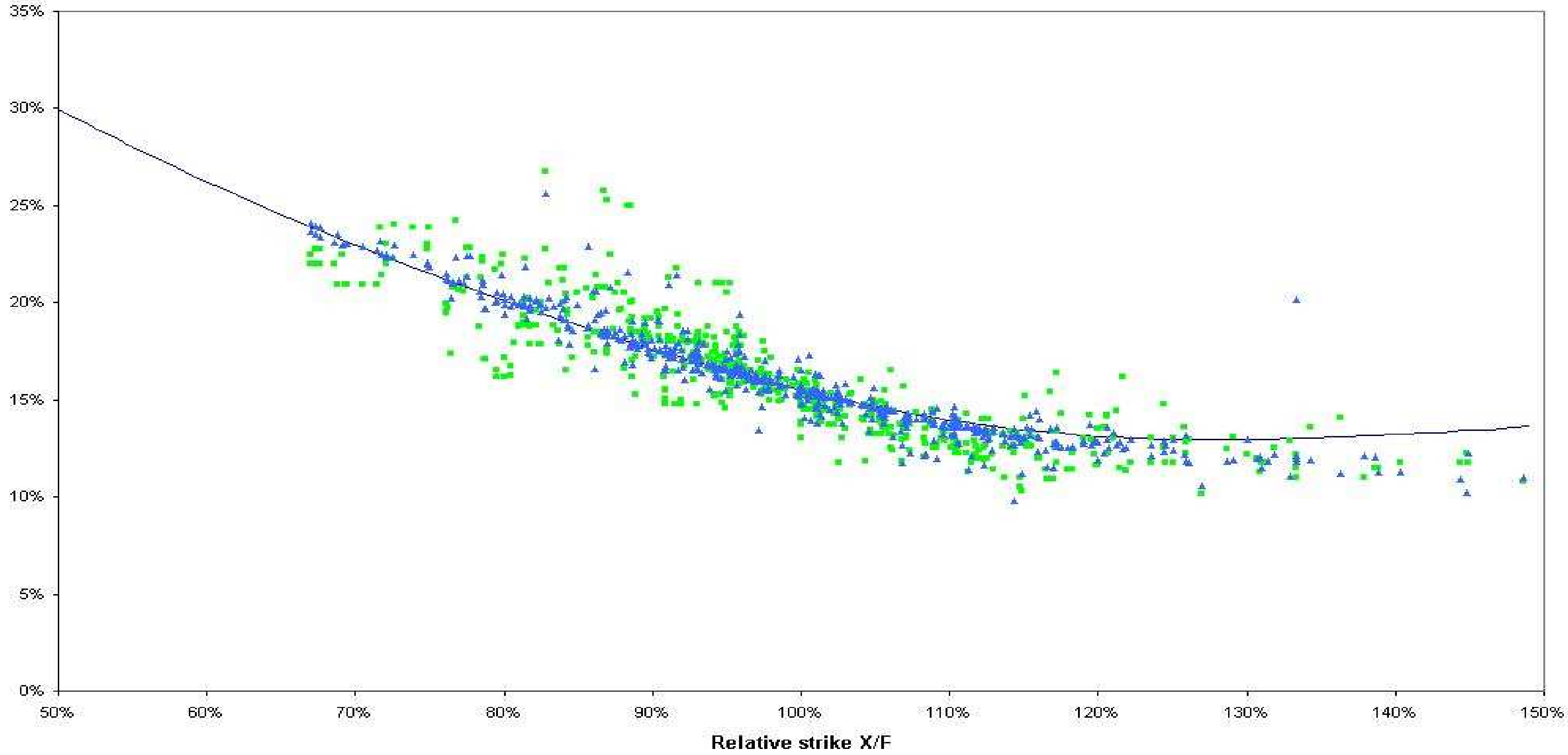

Figure 3: The SABR model for March 2005 expiry, with traded (quoted) volatilities (squares), and with strategies recalibrated to a fitted skew (triangles), and the fitted skew itself (solid line). Historical trades have simply been shifted by the difference in the then and current at-the-money volatility.

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