

Superconformal invariance property for three-point functions

Let's consider a combined scale/chiral transformation in superspace

$$\mathcal{N} \xrightarrow{\lambda} \mathcal{N}\mathcal{X}, \quad \mathcal{O} \rightarrow \mathcal{N}^{\frac{1}{2}} e^{\frac{i}{2}\kappa\mathcal{S}^2}, \quad \bar{\mathcal{O}} \rightarrow \mathcal{N}^{\frac{1}{2}} e^{-\frac{i}{2}\bar{\kappa}\bar{\mathcal{S}}^2}$$

Let's assume that a superfield $\mathcal{O}(z)$ has the following transformation property:

$$\mathcal{O}(z) \rightarrow \mathcal{O}'(z') = \mathcal{N}^{-\Delta} e^{-i\left(\frac{N}{N-4}\right)\kappa\mathcal{S}^2} \mathcal{O}(z)$$

where we write:

$$\Delta = q + \bar{q}, \quad \kappa = q - \bar{q}.$$

Consider the infinitesimal form of the above, let $\mathcal{N} = 1 + \lambda$

$$\begin{aligned} & (1+\lambda)^{-\Delta} e^{-i\left(\frac{N}{N-4}\right)\kappa\mathcal{S}^2} \mathcal{O}(z) \\ &= \mathcal{O}(z) - \lambda \Delta \mathcal{O}(z) - i\left(\frac{N}{N-4}\right) \kappa \mathcal{S}^2 \mathcal{O}(z) \\ \Rightarrow \delta \mathcal{O}(z) &= -\lambda(q + \bar{q}) \mathcal{O}(z) - i\left(\frac{N}{N-4}\right)(q - \bar{q}) \mathcal{S}^2 \mathcal{O}(z) \\ &= -2q \left\{ \frac{1}{2} \left(\lambda + i\frac{N}{N-4} \mathcal{S}^2 \right) \right\} \mathcal{O}(z) \\ &\quad - 2\bar{q} \left\{ \frac{1}{2} \left(\lambda - i\frac{N}{N-4} \mathcal{S}^2 \right) \right\} \mathcal{O}(z) \\ \Rightarrow \mathcal{O}(z) &= \frac{1}{2} \left(\lambda + i\frac{N}{N-4} \mathcal{S}^2 \right), \quad \bar{\mathcal{O}} \dots \end{aligned}$$

Consistent with N -extended results in your paper.

Under the transformation above the two-point function transform as:

$$x_{\bar{i}j} \rightarrow \lambda x_{\bar{i}j}, \quad x_{\bar{i}j}^2 \rightarrow \lambda^2 x_{\bar{i}j}^2$$

Now for the three-point covariants:

$$X_3 \rightarrow \frac{1}{\lambda} X_3, \quad \Theta_3 \rightarrow \frac{1}{\lambda^{1/2}} e^{-\frac{i}{2}\Omega} \Theta_3, \quad \bar{\Theta}_3 \rightarrow \frac{e^{\frac{i}{2}\Omega}}{\lambda^{1/2}} \bar{\Theta}_3$$

Let's do a consistency check against the infinitesimal transformation properties in the $N=2$ paper. Consider combined scale and $U(1)$ and eqn. (2.28):

$$\begin{aligned} S\Theta &= -\frac{1}{N} ((N-2)\phi(z) + 2\bar{\phi}(z))\Theta \\ &\quad - \frac{1}{N} ((N-2)\phi(z) + 2\bar{\phi}(z)) \\ &= -\frac{1}{N} (N-2) \left\{ \frac{1}{2} \left(\lambda + i \frac{N}{N-4} \Omega \right) \right\} - \frac{2}{N} \frac{1}{2} \left(\lambda - i \frac{N}{N-4} \Omega \right) \\ &= -\frac{N-2}{N} \left(\lambda + i \frac{N}{N-4} \Omega \right) - \frac{1}{N} \left(\lambda - i \frac{N}{N-4} \Omega \right) \\ &= \left\{ -\frac{N-2}{2N} - \frac{1}{N} \right\} \lambda + \left\{ -\frac{iN(N-2)}{2N(N-4)} + \frac{iN}{N(N-4)} \right\} \Omega \\ &= -\frac{1}{2} \lambda - \frac{i}{2} \Omega \end{aligned}$$

This is simply the infinitesimal form of .

Now consider the ansatz for the three-pt function, and the transformation above:

$$\langle O'_1(z'_1) O'_2(z'_2) O'_3(z'_3) \rangle = \frac{1}{(x'^2_{13})^{q_1} (x'^2_{31})^{q_2} (x'^2_{23})^{q_3}} \mathcal{H}(X'_3, \theta'_3, \bar{\theta}'_3)$$

Now recall the homogeneity condition:

$$\mathcal{H}(\Delta \bar{X}, \Delta \theta, \bar{\Delta} \bar{\theta}) = \Delta^{2a} \bar{\Delta}^{\bar{a}} \mathcal{H}(X, \theta, \bar{\theta})$$

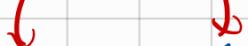
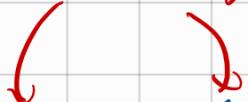
For our transformation we have:

$$\Delta = \frac{e^{-\frac{i}{2}\Omega}}{N^{1/2}},$$

and hence:

$$\begin{aligned} \langle O'_1(z'_1) \dots \rangle &= N^{-2q_1 - 2\bar{q}_1 - 2q_2 - 2\bar{q}_2} N^{-a - \bar{a}} e^{i(\bar{a} - a)\Omega} \\ &\quad \times \frac{1}{(x'^2_{13})^{q_1} (x'^2_{31})^{q_2} (x'^2_{23})^{q_3}} \mathcal{H}(X'_3, \theta'_3, \bar{\theta}'_3) \\ &= N^{-2q_1 - 2\bar{q}_1 - 2q_2 - 2\bar{q}_2 - a - \bar{a}} e^{i(\bar{a} - a)\Omega} \\ &\quad \times \langle O_1(z_1) O_2(z_2) O_3(z_3) \rangle \end{aligned}$$

\mathcal{H} homogeneity



Using the transformation properties of the superfields we should obtain:

$$\begin{aligned} &= N^{-(q_1 + \bar{q}_1)} N^{-(q_2 + \bar{q}_2)} N^{-(q_3 + \bar{q}_3)} e^{-i \frac{N}{N-4} \sum_{k=1}^3 (q_k - \bar{q}_k) \Omega} \\ &\quad \times \langle O_1(z_1) O_2(z_2) O_3(z_3) \rangle \end{aligned}$$

By comparison we obtain:

$$\begin{aligned} -2q_1 - 2\bar{q}_1 - 2q_2 - 2\bar{q}_2 - a - \bar{a} &= -q_1 - \bar{q}_1 - q_2 - \bar{q}_2 - q_3 - \bar{q}_3 \\ \Rightarrow a + \bar{a} &= q_3 + \bar{q}_3 - q_2 - \bar{q}_2 - q_1 - \bar{q}_1 \quad (1) \\ &= \Delta_3 - \Delta_2 - \Delta_1 \end{aligned}$$

This is the expected result as $a + \bar{a}$ is the homogeneity/weight of fl; it agrees with non-susy conventions.

Now for the R-charge factors, we have on LHS and RHS respectively:

$$\begin{aligned} \text{LHS: } & e^{i(\bar{a}-a)\Omega} \\ \text{RHS: } & e^{-i\frac{N}{N-4}\sum_{k=1}^3(q_k - \bar{q}_k)\Omega} \end{aligned}$$

Hence we have:

$$a - \bar{a} = \frac{N}{N-4} \sum_{i=1}^3 (q_i - \bar{q}_i)$$

Now we have the two conditions

$$a + \bar{a} = q_3 + \bar{q}_3 - q_2 - \bar{q}_2 - q_1 - \bar{q}_1$$

$$(1 - \frac{4}{N})(a - \bar{a}) = q_1 - \bar{q}_1 + q_2 - \bar{q}_2 + q_3 - \bar{q}_3$$

We just add and subtract them:

$$\left(1 - \frac{4}{N}\right)(a - \bar{a}) + a + \bar{a} = 2q_3 - 2\bar{q}_2 - 2\bar{q}_1$$

$$\Rightarrow -\frac{1}{2} \left\{ \left(1 - \frac{4}{N}\right)(a - \bar{a}) + a + \bar{a} \right\} = \bar{q}_1 + \bar{q}_2 - q_3$$

$$\Rightarrow -\frac{1}{2} \left\{ \left(2 - \frac{4}{N}\right)a + \frac{4}{N}\bar{a} \right\} = \bar{q}_1 + \bar{q}_2 - q_3$$

$$\Rightarrow \left(\frac{2}{N} - 1\right)a - \frac{2}{N}\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3$$

Similarly:

$$\left(1 - \frac{4}{N}\right)(a - \bar{a}) - a - \bar{a} = 2q_1 + 2q_2 - 2\bar{q}_3$$

$$\Rightarrow -\frac{4}{N}a - \left(2 - \frac{4}{N}\right)\bar{a} = 2q_1 + 2q_2 - 2\bar{q}_3$$

$$\Rightarrow \left(\frac{2}{N} - 1\right)\bar{a} - \frac{2}{N}a = q_1 + q_2 - \bar{q}_3$$

So the right constraints are:

$$\begin{cases} \left(\frac{2}{N} - 1\right)a - \frac{2}{N}\bar{a} = \bar{q}_1 + \bar{q}_2 - q_3 \\ \left(\frac{2}{N} - 1\right)\bar{a} - \frac{2}{N}a = q_1 + q_2 - \bar{q}_3 \end{cases} \quad N \neq 4$$

$$N=1 \Rightarrow a - 2\bar{a} = \dots, \bar{a} - 2a = \dots$$

$$N=2 \Rightarrow -\bar{a} = \dots, -a = \dots$$

and so on ...