Reality constraints

- In terms of the ansatz : $H_{\cdot\cdot}^{\cdot\cdot}(X, \Theta, \bar{\Theta}) = \left(H_{\cdot\cdot}^{\cdot\cdot}(X, \Theta, \bar{\Theta})\right)^*$

$$\Rightarrow \begin{cases} X_{\alpha\dot{\alpha}} \to \bar{X}_{\alpha\dot{\alpha}} = X_{\alpha\dot{\alpha}} - 4i\,\Theta_\alpha \bar{\Theta}_{\dot{\alpha}} \\ \Theta_\alpha \to \bar{\Theta}_{\dot{\alpha}} \\ \bar{\Theta}_{\dot{\alpha}} \to \Theta_\alpha \end{cases}$$

- In the previous work, we have shown that in general $H(X, \Theta, \bar{\Theta})$ always takes the following form:

$$H(X, \Theta, \bar{\Theta}) = F(X) - \frac{1}{2} P^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X),$$

with $\quad P_{\alpha\dot{\alpha}} = -4\,\Theta_\alpha \bar{\Theta}_{\dot{\alpha}}$

In terms of F and G, the reality constraint can be written as

$$H(X, \Theta, \bar{\Theta}) = H(\bar{X}, \bar{\Theta}, \Theta)$$

$$\begin{aligned} \Rightarrow \quad F(X) - \frac{1}{2} P^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X) &= F(\bar{X}) - \frac{1}{2} P^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(\bar{X}) \\ &= F(X + iP) - \frac{1}{2} P^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X + iP) \\ &= F(X) + iP^m \partial_m F(X) + \frac{i^2}{2} P^m P^n \partial_m \partial_n F(X) \\ &\quad - \frac{1}{2} P^{\dot{\alpha}\alpha}\left[G_{\alpha\dot{\alpha}}(X) - \frac{i}{2} P^{\dot{\beta}\beta} \partial_{\beta\dot{\beta}} G_{\alpha\dot{\alpha}}(X) \right] \\ &= F(X; \bar{a}_i) - \frac{i}{2} P^{\dot{\alpha}\alpha} \partial_{\alpha\dot{\alpha}} F(X; \bar{a}_i) \\ &\quad - \frac{1}{2} P^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X; \bar{b}_i) \\ &\quad - \frac{1}{8} P^2 \left[\Box F(X; \bar{a}_i) + i\,\partial^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X; \bar{b}_i) \right] \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} &F(X; a_i) = F(X; \bar{a}_i) \Rightarrow a_i \text{ real} \\ &G_{\alpha\dot{\alpha}}(X; b_i) = G_{\alpha\dot{\alpha}}(X; \bar{b}_i) + i\partial_{\alpha\dot{\alpha}} F(X; a_i) \end{aligned}}$$

<u>Note</u> : (boxed) reality is automatically satisfied, provided: the above condition

$$\Box F(X; \bar{a}_i) + i\,\partial^{\dot{\alpha}\alpha} G_{\alpha\dot{\alpha}}(X; \bar{b}_i) = 0$$

is satisfied & conservation at z_1 and z_2 hold.

- Let's write down the ~~transfo~~ transformation rules of the building blocks under reality transformations

$$X \to \bar{X}\,; \quad \Theta \to \bar{\Theta}$$
and
$$u_\alpha \to \bar{u}_{\dot\alpha}, \quad v_\alpha \to \bar{v}_{\dot\alpha}, \quad w_\alpha \to \bar{w}_{\dot\alpha}$$

The building blocks are :

$$P_i\,,\ \tilde{P}_i\,,\ Q_i\,,\ \tilde{Q}_i\,,\ Z_i\,,\ R_i\,,\ \tilde{R}_i\,,\ S_i\,,\ \tilde{S}_i\,,\ J$$

More explicitly :

$$P_1 = \epsilon_{\alpha\beta} v^\alpha w^\beta\,, \qquad P_2 = \epsilon_{\alpha\beta} w^\alpha u^\beta\,, \qquad P_3 = \epsilon_{\alpha\beta} u^\alpha v^\beta$$
$$\tilde{P}_1 = \epsilon_{\dot\alpha\dot\beta} \bar{v}^{\dot\alpha} \bar{w}^{\dot\beta}\,, \qquad \tilde{P}_2 = \epsilon_{\dot\alpha\dot\beta} \bar{w}^{\dot\alpha} \bar{u}^{\dot\beta}\,, \qquad \tilde{P}_3 = \epsilon_{\dot\alpha\dot\beta} \bar{u}^{\dot\alpha} \bar{v}^{\dot\beta}$$

$$Q_1 = \hat{X}_{\alpha\dot\alpha} v^\alpha \bar{w}^{\dot\alpha}\,, \qquad Q_2 = \hat{X}_{\alpha\dot\alpha} w^\alpha \bar{u}^{\dot\alpha}\,, \qquad Q_3 = \hat{X}_{\alpha\dot\alpha} u^\alpha \bar{v}^{\dot\alpha}$$
$$\tilde{Q}_1 = \hat{X}_{\alpha\dot\alpha} w^\alpha \bar{v}^{\dot\alpha}\,, \qquad \tilde{Q}_2 = \hat{X}_{\alpha\dot\alpha} u^\alpha \bar{w}^{\dot\alpha}\,, \qquad \tilde{Q}_3 = \hat{X}_{\alpha\dot\alpha} v^\alpha \bar{u}^{\dot\alpha}$$

$$Z_1 = \hat{X}_{\alpha\dot\alpha} u^\alpha \bar{u}^{\dot\alpha}\,, \qquad Z_2 = \hat{X}_{\alpha\dot\alpha} v^\alpha \bar{v}^{\dot\alpha}\,, \qquad Z_3 = \hat{X}_{\alpha\dot\alpha} w^\alpha \bar{w}^{\dot\alpha}$$

Fermionic :
$$\hat{\Theta}_\alpha = \frac{\Theta_\alpha}{X^{1/2}}\,, \qquad \hat{\bar{\Theta}}_{\dot\alpha} = \frac{\bar{\Theta}_{\dot\alpha}}{X^{1/2}}$$

$$R_1 = \epsilon_{\alpha\beta} u^\alpha \hat{\Theta}^\beta\,, \qquad R_2 = \epsilon_{\alpha\beta} v^\alpha \hat{\Theta}^\beta\,, \qquad R_3 = \epsilon_{\alpha\beta} w^\alpha \hat{\Theta}^\beta\,,$$
$$\tilde{R}_1 = \epsilon_{\dot\alpha\dot\beta} \bar{u}^{\dot\alpha} \hat{\bar{\Theta}}^{\dot\beta}\,, \qquad \tilde{R}_2 = \epsilon_{\dot\alpha\dot\beta} \bar{v}^{\dot\alpha} \hat{\bar{\Theta}}^{\dot\beta}\,, \qquad \tilde{R}_3 = \epsilon_{\dot\alpha\dot\beta} \bar{w}^{\dot\alpha} \hat{\bar{\Theta}}^{\dot\beta}$$

$$S_1 = \hat{X}_{\alpha\dot\alpha} u^\alpha \hat{\bar{\Theta}}^{\dot\alpha}\,, \qquad S_2 = \hat{X}_{\alpha\dot\alpha} v^\alpha \hat{\bar{\Theta}}^{\dot\alpha}\,, \qquad S_3 = \hat{X}_{\alpha\dot\alpha} w^\alpha \hat{\bar{\Theta}}^{\dot\alpha}$$
$$\tilde{S}_1 = \hat{X}_{\alpha\dot\alpha} \bar{u}^{\dot\alpha} \hat{\Theta}^\alpha\,, \qquad \tilde{S}_2 = \hat{X}_{\alpha\dot\alpha} \bar{v}^{\dot\alpha} \hat{\Theta}^\alpha\,, \qquad \tilde{S}_3 = \hat{X}_{\alpha\dot\alpha} \bar{w}^{\dot\alpha} \hat{\Theta}^\alpha$$

$$J = \hat{X}_{\alpha\dot\alpha} \hat{\Theta}^\alpha \hat{\bar{\Theta}}^{\dot\alpha}$$

Note that
$$\hat{\bar{X}}_{\alpha\dot\alpha} = \frac{\bar{X}_{\alpha\dot\alpha}}{\bar{X}} = (\text{keeping terms up to } \mathcal{O}(\theta\bar\theta) \text{ only})$$
$$= \left(X_{\alpha\dot\alpha} - 4i\,\Theta_\alpha \bar{\Theta}_{\dot\alpha}\right)\left(\frac{1}{X} - 2i\,\frac{\Theta^\beta \bar{\Theta}^{\dot\beta} X_{\beta\dot\beta}}{X^3}\right)$$
$$\therefore \quad \boxed{\hat{\bar{X}}_{\alpha\dot\alpha} = \hat{X}_{\alpha\dot\alpha} - 4i\,\hat{\Theta}_\alpha \hat{\bar{\Theta}}_{\dot\alpha} - 2i\,J\,\hat{X}_{\alpha\dot\alpha}\,.}$$

So : each monomial will transform to its conjugate :

$$P_i \to \tilde{P}_i \quad ; \quad \tilde{P}_i \to P_i$$

$$\begin{aligned} Q_1 \to \hat{\bar{X}}_{\alpha\dot{\alpha}} w^\alpha \bar{v}^{\dot{\alpha}} &= \left(\hat{X}_{\alpha\dot{\alpha}} - 4i\, \hat{\Theta}_\alpha \hat{\tilde{\Theta}}_{\dot{\alpha}} - 2iJ\, \hat{X}_{\alpha\dot{\alpha}} \right) w^\alpha \bar{v}^{\dot{\alpha}} \\ &= \tilde{Q}_1 - 4i R_3 \tilde{R}_2 - 2iJ \tilde{Q}_1 \\ &= (1-2iJ)\tilde{Q}_1 - 4i R_3 \tilde{R}_2 \end{aligned}$$

$$Q_2 \to (1-2iJ)\tilde{Q}_2 - 4i R_1 \tilde{R}_3$$

$$Q_3 \to (1-2iJ)\tilde{Q}_3 - 4i R_2 \tilde{R}_1$$

$$\tilde{Q}_1 \to \hat{\bar{X}}_{\alpha\dot{\alpha}} v^\alpha \bar{w}^{\dot{\alpha}} = (1-2iJ) Q_1 - 4i R_2 \tilde{R}_3$$

$$\tilde{Q}_2 \to (1-2iJ) Q_2 - 4i R_3 \tilde{R}_1$$

$$\tilde{Q}_3 \to (1-2iJ) Q_3 - 4i R_1 \tilde{R}_2$$

$$Z_i \to (1-2iJ) Z_i - 4i R_i \tilde{R}_i$$

$$R_i \to \tilde{R}_i \quad ; \quad \tilde{R}_i \to R_i$$

$$S_i \to \tilde{S}_i \quad ; \quad \tilde{S}_i \to S_i$$

$$J \to J$$