Implications of (super)inversion covariance

Consider 4D CFT. We have the inversion operators

$$\mathcal{I}_{\alpha\dot\alpha}(X) = \hat{X}_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}\hat{X}_m \,, \quad \hat{X}^m = \frac{X^m}{(X^2)^{1/2}} .$$

here, X^m is the three-point building block e.g

$$X_3^m = \frac{x_{13}^m}{x_{13}^2} - \frac{x_{23}^m}{x_{23}^2} \,,$$

$$\Rightarrow \quad X_{3\alpha\dot\alpha} = (x_{13}^{-1})_{\alpha\dot\beta}\, x_{12}^{\dot\beta\gamma}\, (x_{32}^{-1})_{\gamma\dot\alpha} \,.$$

The action of $\mathcal{I}_{\alpha\dot\alpha}(X)$ on X is:

$$\mathcal{I}_\alpha{}^{\dot\alpha'}(X)\,\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha'}(X)\, X_{\alpha'\dot\alpha'} = -X_{\alpha\dot\alpha} = X^{I}_{\alpha\dot\alpha}$$

We also have

$$\mathcal{I}_\alpha{}^{\dot\alpha}(X)\,\mathcal{I}_\beta{}^{\dot\beta}(X)\,\varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{\alpha\beta} \,,$$

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(X)\,\bar{\mathcal{I}}_{\dot\beta}{}^{\beta}(X)\,\varepsilon_{\alpha\beta} = -\varepsilon_{\dot\alpha\dot\beta} \,,$$

Hence the action of $\mathcal{I}$ is

$$\left.\varepsilon \xrightarrow{\mathcal{I}} -\bar\varepsilon \,, \quad \bar\varepsilon \xrightarrow{\mathcal{I}} -\varepsilon \quad \right\} \text{odd}$$

$$\left. X \xrightarrow{\mathcal{I}} X^{I} \quad \right\} \text{even}$$

Now consider a tensor

$$\mathcal{H}_{A_1A_2A_3}(X) \equiv \mathcal{H}_{\alpha(j_1)\dot\alpha(i_1)\beta(j_2)\dot\beta(i_2)\gamma(i_3)\dot\gamma(j_3)}(X)$$

where $A_1 = \{\alpha(i_1), \dot\alpha(j_1)\}$, $A_2 = \{\beta(i_2), \dot\beta(j_2)\}$
$A_3 = \{\gamma(i_3), \dot\gamma(j_3)\}$. $\mathcal{H}$ is a linear combination of the tensor product of the structures

$X, \Sigma, \bar{\Sigma}$, with complex coefficient c_i as follows:

basis of structures

$$\mathcal{H}(X) = \sum_i c_i \mathcal{H}_i(X)$$
$$= \sum_i a_i \mathcal{H}_i(X) + i \sum_i b_i \mathcal{H}_i(X)$$
$$= \mathcal{H}^A(X) + i\mathcal{H}^B(X)$$

when $\mathcal{H}^A(X)$ and $\mathcal{H}^B(X)$ are linear combinations of tensor structures with real coefficients.

Now define the conjugation operation $*$

$$\Sigma \xrightarrow{*} \bar{\Sigma}, \quad \bar{\Sigma} \xrightarrow{*} \Sigma,$$
$$X \xrightarrow{*} \bar{X} = X$$

$$\mathcal{H}_{(i)\bar{A}_1\bar{A}_2A_3}(\Sigma, \bar{\Sigma}, X) \xrightarrow{*} \bar{\mathcal{H}}_{(i)A_1A_2\bar{A}_3}(\Sigma, \bar{\Sigma}, X)$$
$$= \mathcal{H}_{(i)A_1A_2\bar{A}_3}(\bar{\Sigma}, \Sigma, \bar{X})$$
$$= \mathcal{H}_{(i)A_1A_2\bar{A}_3}(\bar{\Sigma}, \Sigma, X)$$

structures in our basis

Hence, for $\mathcal{H}^A$ and $\mathcal{H}^B$:

$$\mathcal{H}^A(\Sigma, \bar{\Sigma}, X) \xrightarrow{*} \bar{\mathcal{H}}^A(\Sigma, \bar{\Sigma}, X) = \sum a_i \bar{\mathcal{H}}_{(i)}(\Sigma, \bar{\Sigma}, X)$$
$$= \sum a_i \mathcal{H}_{(i)}(\bar{\Sigma}, \Sigma, X)$$

Hence $\mathcal{H} = \mathcal{H}^A + i\mathcal{H}^B$ conjugates to:

$$\mathcal{H} = \mathcal{H}^A + i\mathcal{H}^B \xrightarrow{*} \bar{\mathcal{H}} = \bar{\mathcal{H}}^A - i\bar{\mathcal{H}}^B$$

Now consider the action of inversion on $\mathcal{H}_{\bar{A}_1\bar{A}_2A_3}(X) \equiv \mathcal{H}_{\bar{A}_1\bar{A}_2A_3}(\Sigma,\bar{\Sigma},X)$, defined by

$$\mathcal{I}_{\alpha(i_1)}{}^{\dot{\alpha}'(i_1)}(X)\,\bar{\mathcal{I}}_{\dot{\alpha}(j_1)}{}^{\alpha'(j_1)}(X)\,\mathcal{I}_{\beta(i_2)}{}^{\dot{\beta}'(i_2)}(X)\,\bar{\mathcal{I}}_{\dot{\beta}(j_2)}{}^{\beta'(j_2)}(X)$$
$$\mathcal{I}_{\gamma(j_3)}{}^{\dot{\gamma}'(j_3)}(X)\,\bar{\mathcal{I}}_{\dot{\gamma}(i_3)}{}^{\gamma'(i_3)}(X)\,\mathcal{H}_{\alpha'(j_1)\dot{\alpha}'(i_1)\beta'(j_2)\dot{\beta}'(i_2)\gamma'(i_3)\dot{\gamma}'(j_3)}(X)$$

Due to the action of $\mathcal{I}$ on the building blocks, we may partition our solution into $\mathcal{H}^{(\pm)}(X)$: $+$ for even $\Sigma,\bar{\Sigma}$, $-$ for odd $\Sigma,\bar{\Sigma}$.

If we consider the real linear combinations $\mathcal{H}^A$ and $\mathcal{H}^B$ (just call $\mathcal{H}$ for now), we divide them into $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$ and get the following action:

$$\mathcal{I}_{\alpha(i_1)}{}^{\dot{\alpha}'(i_1)}(X)\,\bar{\mathcal{I}}_{\dot{\alpha}(j_1)}{}^{\alpha'(j_1)}(X)\,\mathcal{I}_{\beta(i_2)}{}^{\dot{\beta}'(i_2)}(X)\,\bar{\mathcal{I}}_{\dot{\beta}(j_2)}{}^{\beta'(j_2)}(X)$$
$$\mathcal{I}_{\gamma(j_3)}{}^{\dot{\gamma}'(j_3)}(X)\,\bar{\mathcal{I}}_{\dot{\gamma}(i_3)}{}^{\gamma'(i_3)}(X)\,\mathcal{H}^{(\pm)}_{\alpha'(j_1)\dot{\alpha}'(i_1)\beta'(j_2)\dot{\beta}'(i_2)\gamma'(i_3)\dot{\gamma}'(j_3)}(X)$$

$$= \mathcal{I}_{\alpha(i_1)}{}^{\dot{\alpha}'(i_1)}(X)\,\bar{\mathcal{I}}_{\dot{\alpha}(j_1)}{}^{\alpha'(j_1)}(X)\,\mathcal{I}_{\beta(i_2)}{}^{\dot{\beta}'(i_2)}(X)\,\bar{\mathcal{I}}_{\dot{\beta}(j_2)}{}^{\beta'(j_2)}(X)$$
$$\mathcal{I}_{\gamma(j_3)}{}^{\dot{\gamma}'(j_3)}(X)\,\bar{\mathcal{I}}_{\dot{\gamma}(i_3)}{}^{\gamma'(i_3)}(X)\sum_i a_i\,\mathcal{H}^{(\pm)}_{\alpha'(j_1)\dot{\alpha}'(i_1)\beta'(j_2)\dot{\beta}'(i_2)\gamma'(j_3)\dot{\gamma}'(i_3)}(\Sigma,\bar{\Sigma},X)$$

$$= \pm\sum a_i\,\mathcal{H}^{(\pm)}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(i_2)\dot{\beta}(j_2)\gamma(i_3)\dot{\gamma}(j_3)\gamma(i_3)}(\bar{\Sigma},\Sigma,X^{\mathcal{I}})$$

$$= \pm\,\bar{\mathcal{H}}^{(\pm)}_{\alpha(i_1)\dot{\alpha}(j_1)\beta(i_2)\dot{\beta}(j_2)\gamma(i_3)\dot{\gamma}(j_3)\gamma(i_3)}(\Sigma,\bar{\Sigma},X^{\mathcal{I}})$$

Hence:

$$\mathcal{H}^{\mathcal{I}(\pm)}_{A_1A_2\bar{A}_3}(X) = \pm\,\bar{\mathcal{H}}^{(\pm)}_{A_1A_2\bar{A}_3}(X^{\mathcal{I}})$$

Hence, for $\mathcal{H} = \mathcal{H}^A + i\mathcal{H}^B$

$$= \mathcal{H}^{A(+)} + \mathcal{H}^{A(-)} + i\left(\mathcal{H}^{B(+)} + \mathcal{H}^{B(-)}\right)$$

where:

$$\mathcal{H}^{A(\pm)}(X) = \sum_i a_i^{(\pm)} \mathcal{H}_i^{(\pm)}(X),$$
$$\mathcal{H}^{B(\pm)}(X) = \sum_i b_i^{(\pm)} \mathcal{H}_i^{(\pm)}(X),$$

where $\mathcal{H}_i^{(\pm)}$ is a linearly independent basis of even/odd structures respectively, we have:

$$\mathcal{H}^I(X) = \mathcal{H}^{AI}(X) + i\mathcal{H}^{BI}(X)$$
$$= \bar{\mathcal{H}}^{A(+)}(X^I) - \bar{\mathcal{H}}^{A(-)}(X^I)$$
$$+ i\left(\bar{\mathcal{H}}^{B(+)}(X^I) - \bar{\mathcal{H}}^{B(-)}(X^I)\right)$$

Now suppose we are working with vector-like fields (in general they are complex). Then for $\mathcal{H}_{\bar{A}_1\bar{A}_2A_3}(X)$ we have $A_1 = \bar{A}_1$, $A_2 = \bar{A}_2$, $A_3 = \bar{A}_3$. Hence, there exists a map from

$$\mathcal{H}^I_{A_1A_2A_3}(X) = \mathcal{I}_{A_1}{}^{A_1'}(X)\,\mathcal{I}_{A_2}{}^{A_2'}(X)\,\mathcal{I}_{A_3}{}^{A_3'}(X)\,\mathcal{H}_{A_1'A_2'A_3'}(X)$$

onto the same rep as $\mathcal{H}$. We may freely impose the condition:

$$\mathcal{H}_{A_1A_2A_3}(X^I) = \pm\mathcal{H}^I_{A_1A_2A_3}(X)$$
$$= \mathcal{I}_{A_1}{}^{A_1'}(X)\,\mathcal{I}_{A_2}{}^{A_2'}(X)\,\mathcal{I}_{A_3}{}^{A_3'}(X)\,\mathcal{H}_{A_1'A_2'A_3'}(X)$$

This is essentially an inversion pseudo covariance condition. It is (2.14) of O&P except for the case where R is an inversion.

If we impose this condition on $\mathcal{H} = \mathcal{H}^A + i\mathcal{H}^B$ we obtain:

$$
\begin{aligned}
\mathcal{H}(X^I) &= \mathcal{H}^{A(+)}(X^I) + \mathcal{H}^{A(-)}(X^I) \\
&\qquad + i\left(\mathcal{H}^{B(+)}(X^I) + \mathcal{H}^{B(-)}(X^I)\right) \\
&= \mathcal{H}^I(X) \\
&= \bar{\mathcal{H}}^{A(+)}(X^I) - \bar{\mathcal{H}}^{A(-)}(X^I) \\
&\qquad + i\left(\bar{\mathcal{H}}^{B(+)}(X^I) - \bar{\mathcal{H}}^{B(-)}(X^I)\right)
\end{aligned}
$$

Which implies:

$$
\begin{aligned}
&\mathcal{H}^{A(+)} - \bar{\mathcal{H}}^{A(+)} + \mathcal{H}^{A(-)} + \bar{\mathcal{H}}^{A(-)} \\
&\quad = i\left(\mathcal{H}^{B(+)} - \bar{\mathcal{H}}^{B(+)} + \mathcal{H}^{B(-)} + \bar{\mathcal{H}}^{B(-)}\right)
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
&\mathcal{H}^{A(+)} - \bar{\mathcal{H}}^{A(+)} + \mathcal{H}^{A(-)} + \bar{\mathcal{H}}^{A(-)} = 0, \\
&\mathcal{H}^{B(+)} - \bar{\mathcal{H}}^{B(+)} + \mathcal{H}^{B(-)} + \bar{\mathcal{H}}^{B(-)} = 0.
\end{aligned}
$$

Here we let

$$
\mathcal{H}^{A(\pm)} = \sum_i a_i \mathcal{H}_i^{E(\pm)}, \quad \mathcal{H}^{B(\pm)} = \sum_i a_i \mathcal{H}_i^{E(\pm)}
$$

Where $\mathcal{H}^{E(\pm)}$ denotes linearly independent basis

Note, in the case of vector-like supercurrents, we have $k_i, \bar{k}_i$ = no. of $P_i \bar{P}_i$.

$$\Delta S = \frac{1}{2}(i_1 + i_2 + i_3 - j_1 - j_2 - j_3) = 0$$

$$\Delta S = k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3 = 0$$

Hence $k_1 + k_2 + k_3 + \bar{k}_1 + \bar{k}_2 + \bar{k}_3$ = even. Hence, we have $\mathcal{H}^{E(-)} = 0$. Hence we just have the conditions

$$\begin{aligned} \mathcal{H}^{A(+)} - \bar{\mathcal{H}}^{A(+)} &= 0, \\ \mathcal{H}^{B(+)} - \bar{\mathcal{H}}^{B(+)} &= 0. \end{aligned} \implies \begin{aligned} \mathcal{H}^{A(+)} &= \bar{\mathcal{H}}^{A(+)} \\ \mathcal{H}^{B(+)} &= \bar{\mathcal{H}}^{B(+)} \end{aligned}$$

But recall

$$\mathcal{H}^{A(+)} = \sum a_i^{(+)} \mathcal{H}_i^{(+)}, \quad \bar{\mathcal{H}}^{A(+)} = \sum a_i^{(+)} \bar{\mathcal{H}}_i^{(+)}$$

Similarly for $\mathcal{H}^{B(+)}$. The constraints above will generically result in relations among the $a_i^{(+)}$, and relations among the $b_i^{(+)}$. Solving these conditions is equivalent to just forming a new basis $\mathcal{H}_i'^{(+)}$ where each structure is manifestly Hermitian. We will call this basis $\mathcal{H}_i^{E(+)}$ "even". We will assume it is linearly independent.

Hence we may write

$$\mathcal{H}^E = \mathcal{H}^{A(+)} + i\mathcal{H}^{B(+)} = \sum_{i=1}^{N_E} (a_i^{E(+)} + i b_i^{E(+)}) \mathcal{H}_i^{E(+)}$$

$$= \sum_{i=1}^{N_E} \underbrace{c_i^{E(+)}}_{\text{complex}} \mathcal{H}_i^{E(+)}$$

Where $\mathcal{H}_i^{E(+)} = \bar{\mathcal{H}}_i^{E(+)}$ is a basis of linearly independent structures which we define as parity even.

Similarly, consider

$$\mathcal{H}_{A_1A_2A_3}(X^I) = -\mathcal{H}^I_{A_1A_2A_3}(X)$$

If we impose this condition on $\mathcal{H} = \mathcal{H}^A + i\mathcal{H}^B$ we obtain (assuming $\mathcal{H}^{A(-)} = \mathcal{H}^{B(-)} = 0$).

$$\begin{aligned}
\mathcal{H}(X^I) &= \mathcal{H}^{A(+)}(X^I) + i\mathcal{H}^{B(+)}(X^I) \\
&= -\mathcal{H}^I(X) \\
&= -\bar{\mathcal{H}}^{A(+)}(X^I) - i\bar{\mathcal{H}}^{B(+)}(X^I)
\end{aligned}$$

Which implies:

$$\Rightarrow \quad \mathcal{H}^{A(+)} + \bar{\mathcal{H}}^{A(+)} = 0$$
$$\mathcal{H}^{B(+)} + \bar{\mathcal{H}}^{B(+)} = 0$$

and hence, for the parity-odd cut, using the same arguments we can change our basis $\mathcal{H}_i^{(+)}$ into another basis $\mathcal{H}_i^{O(+)}$ which is manifestly anti-hermitian so the above conditions are automatically satisfied.

$$\mathcal{H}_i^{O(+)} = -\bar{\mathcal{H}}_i^{O(+)}$$

We will denote this basis $\mathcal{H}_i^{O(+)}$ as parity odd.
Hence we may write

$$\mathcal{H}^O = \mathcal{H}^{A(+)} + i\mathcal{H}^{B(+)} = \sum_{i=1}^{N_O} (a_i^{O(+)} + ib_i^{O(+)})\mathcal{H}_i^{O(+)}$$

$$= \sum_{i=1}^{N_O} \underbrace{c_i^{O(+)}}_{\text{complex}} \mathcal{H}_i^{O(+)}$$

Where $\mathcal{H}_i^{O(+)} = -\bar{\mathcal{H}}_i^{O(+)}$ is a basis of linearly independent structures which we define as parity odd. We will assume we can construct this basis so it is linearly independent.

Hence, in general, for the tensor $\mathcal{H}$ which satisfies the inversion pseudo covariance condition, we may write it in the form:

$$\mathcal{H} = \mathcal{H}^E + \mathcal{H}^O$$

$$= \sum_{i=1}^{N_E} c_i^{E(+)}\mathcal{H}_i^{E(+)} + \sum_{i=1}^{N_O} c_i^{O(+)}\mathcal{H}_i^{O(+)}$$

where $\mathcal{H}_i^{E(+)} = \bar{\mathcal{H}}_i^{E(+)}$, $\mathcal{H}_i^{O(+)} = -\bar{\mathcal{H}}_i^{O(+)}$, and c_i^E, c_i^O are complex parameteer.

Since $\mathcal{H}^{E(+)}$ and $\mathcal{H}^{O(+)}$ are linearly independent of eachother, imposing conservation and or point switch symmetries results in relations among the $a_i^{E/O(+)}$, and relations among the $b_i^{E/O(+)}$, reducing the overall number of even/odd coefficients.

This is a very general proof of why our 4D CFT solutions look the way they do. You can see it in all examples!

What I have demonstrated in 4D N=1 SCFT is that we can partition our structures in the same way based on covariance under superinversion.
The proof is identical!

In SUSY case: $k_i, \bar{k}_i, q_i, \bar{q}_i$ = no. of $P_i, \bar{P}_i$ $S_i, \bar{S}_i$.

$$i_1 + i_2 + i_3 - j_1 - j_2 - j_3 = 0$$

$$= 2(k_1 + k_2 + k_3) + p_1 + p_2 + p_3 + q_1 + q_2 + q_3$$
$$- 2(\bar{k}_1 + \bar{k}_2 + \bar{k}_3) - \bar{p}_1 - \bar{p}_2 - \bar{p}_3 - \bar{q}_1 - \bar{q}_2 - \bar{q}_3 = 0$$

$$= 2(k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3) + p_1 + p_2 + p_3 - \bar{p}_1 - \bar{p}_2 - \bar{p}_3$$
$$+ q_1 + q_2 + q_3 - \bar{q}_1 - \bar{q}_2 - \bar{q}_3 = 0$$

The number odd building blocks is

$$N^0 = k_1 + k_2 + k_3 + \bar{k}_1 + \bar{k}_2 + \bar{k}_3 + q_1 + q_2 + q_3 + \bar{q}_1 + \bar{q}_2 + \bar{q}_3$$

Consider $O(\theta) = 0$, then $p_i = \bar{p}_i = q_i = \bar{q}_i = 0$

$$2(k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3) = 0$$

$$\Rightarrow k_1 + k_2 + k_3 = \bar{k}_1 + \bar{k}_2 + \bar{k}_3$$

Hence

$$N^0 = 2(k_1 + k_2 + k_3) = \text{even},$$

so for vector-like currents, at $O(\theta\bar{\theta}) = 0$, we have N^0 = even and so the solution is of type $\mathcal{H}^{(+)}$.

Now consider $O(\theta\bar{\theta})$ then the following holds

$$p_1 + p_2 + p_3 + \bar{q}_1 + \bar{q}_2 + \bar{q}_3 + \sigma = 1$$
$$\bar{p}_1 + \bar{p}_2 + \bar{p}_3 + q_1 + q_2 + q_3 + \sigma = 1$$

So
$$q_1 + q_2 + q_3 = 1 - \sigma - \bar{p}_1 - \bar{p}_2 - \bar{p}_3$$
$$\bar{q}_1 + \bar{q}_2 + \bar{q}_3 = 1 - \sigma - p_1 - p_2 - p_3$$

$$\Rightarrow \quad q_1+q_2+q_3-\bar{q}_1-\bar{q}_2-\bar{q}_3 = p_1+p_2+p_3-\bar{p}_1-\bar{p}_2-\bar{p}_3$$

Hence

$$i_1+i_2+i_3-j_1-j_2-j_3$$

$$= 2(k_1+k_2+k_3-\bar{k}_1-\bar{k}_2-\bar{k}_3)+p_1+p_2+p_3-\bar{p}_1-\bar{p}_2-\bar{p}_3$$
$$+q_1+q_2+q_3-\bar{q}_1-\bar{q}_2-\bar{q}_3 = 0$$

$$= 2(k_1+k_2+k_3-\bar{k}_1-\bar{k}_2-\bar{k}_3+q_1+q_2+q_3-\bar{q}_1-\bar{q}_2-\bar{q}_3) = 0$$

$$\Rightarrow \quad k_1+k_2+k_3-\bar{k}_1-\bar{k}_2-\bar{k}_3+q_1+q_2+q_3-\bar{q}_1-\bar{q}_2-\bar{q}_3 = 0$$

$$\Rightarrow \quad k_1+k_2+k_3+q_1+q_2+q_3 = \bar{k}_1+\bar{k}_2+\bar{k}_3+\bar{q}_1+\bar{q}_2+\bar{q}_3$$

Hence at $O(\theta\bar{\theta})$

$$N^0 = k_1+k_2+k_3+\bar{k}_1+\bar{k}_2+\bar{k}_3+q_1+q_2+q_3+\bar{q}_1+\bar{q}_2+\bar{q}_3$$
$$= 2(k_1+k_2+k_3+q_1+q_2+q_3) = \text{even}$$

So at $O(\theta\bar{\theta})$ the solution is also of type $\mathcal{H}^{(+)}$!

So the same story is happening in the SUSY case.

What about in 3D CFT?

In 3D CFT we define the inversion operators

$$\mathcal{I}_{\alpha\beta}(X) = \hat{X}_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\hat{X}_m \,, \quad \hat{X}^m = \frac{X^m}{(X^2)^{1/2}} \,.$$

here, X^m is the three-point building block e.g

$$X_3^m = \frac{x_{13}^m}{x_{13}^2} - \frac{x_{23}^m}{x_{23}^2} \,,$$

$$\Rightarrow \quad X_{3\,\alpha\beta} = (\gamma^m)_{\alpha\beta} X_3^m$$

The action of $\mathcal{I}_{\alpha\beta}(X)$ on X is:

$$\mathcal{I}_\alpha{}^{\alpha'}(X)\,\mathcal{I}_\beta{}^{\beta'}(X)\,X_{\alpha'\beta'} = -X_{\alpha\beta} = X^{\mathcal{I}}_{\alpha\beta}$$

We also have

$$\mathcal{I}_\alpha{}^{\alpha'}(X)\,\mathcal{I}_\beta{}^{\beta'}(X)\,\varepsilon_{\alpha'\beta'} = -\varepsilon_{\alpha\beta}$$

Hence the action of $\mathcal{I}$ is

$$\left.\varepsilon \xrightarrow{\mathcal{I}} -\varepsilon \ \right\} \text{odd}$$

$$\left.X \xrightarrow{\mathcal{I}} X^{\mathcal{I}} \ \right\} \text{even}$$

Now consider a tensor

$$\mathcal{H}_{A_1A_2A_3}(X) \equiv \mathcal{H}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X)$$

where $\mathcal{H}_{A_1A_2A_3}(X)$ is a linear combination of the tensor products of ε, X.

Now define the action of inversion on $\mathcal{H}$ as follows

$$\mathcal{I}_{\alpha(2s_1)}{}^{\alpha'(2s_1)}(X)\,\mathcal{I}_{\beta(2s_2)}{}^{\beta'(2s_2)}(X)\,\mathcal{I}_{\gamma(2s_3)}{}^{\gamma'(2s_3)}(X)\,\mathcal{H}_{\alpha'(2s_1)\beta'(2s_2)\gamma'(2s_3)}(X)$$
$$\equiv \mathcal{H}^{I}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X)$$

We may decompose $\mathcal{H}$ as follows:

$$\mathcal{H} = \mathcal{H}^{(+)} + \mathcal{H}^{(-)}$$

where $\mathcal{H}^{(+)}$ contains even no. of ε and $\mathcal{H}^{(-)}$ contains odd no. of ε. We will assume

$$\mathcal{H}^{(+)}(X) = \sum a_i \mathcal{H}^{(+)}_i(X),$$
$$\mathcal{H}^{(-)}(X) = \sum b_i \mathcal{H}^{(-)}_i(X),$$

where $\mathcal{H}^{(\pm)}_i$ are a basis of linearly independent even/odd structures. The action of $\mathcal{I}$ on $\mathcal{H}^{(\pm)}_i$ is

$$\mathcal{I}_{\alpha(2s_1)}{}^{\alpha'(2s_1)}(X)\,\mathcal{I}_{\beta(2s_2)}{}^{\beta'(2s_2)}(X)\,\mathcal{I}_{\gamma(2s_3)}{}^{\gamma'(2s_3)}(X)\,\mathcal{H}^{(\pm)}_{\alpha'(2s_1)\beta'(2s_2)\gamma'(2s_3)}(X)$$
$$= \pm\,\mathcal{H}^{(\pm)}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X^I)$$
$$= \mathcal{H}^{I\,(\pm)}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X^I)$$

Now let's impose the following condition on $\mathcal{H}$:

$$\mathcal{H}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X^I) = \pm\,\mathcal{H}^{I}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X)$$

In the + case we get

$$\mathcal{H}(X^I) = \mathcal{H}^{(+)}(X^I) + \mathcal{H}^{(-)}(X^I)$$
$$= \mathcal{H}^{(+)}(X^I) - \mathcal{H}^{(-)}(X^I)$$
$$\Rightarrow \mathcal{H}^{(-)}(X) = 0$$

while for − we get :

$$\mathcal{H}(X^I) = \mathcal{H}^{(+)}(X^I) + \mathcal{H}^{(-)}(X^I)$$
$$= -\left(\mathcal{H}^{(+)}(X^I) - \mathcal{H}^{(-)}(X^I)\right)$$
$$\Rightarrow \mathcal{H}^{(+)}(X) = 0.$$

Hence to identify the parity-even/odd solutions we can impose the condition:

$$\mathcal{H}_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X^I) = \pm \mathcal{H}^I_{\alpha(2s_1)\beta(2s_2)\gamma(2s_3)}(X)$$

which is a type of pseudo-covariance condition.

Quick note on 4D N=1 superinversion

We have the chiral coordinates:

$$x^a_{(\pm)} = x^a \pm i\theta\sigma^a\bar{\theta}$$

$$\Rightarrow x_{(\pm)\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} \pm i(\sigma^a)_{\alpha\dot{\alpha}}(\sigma^a)_{\beta\dot{\beta}}\theta^\beta\bar{\theta}^{\dot{\beta}}$$

$$= x_{\alpha\dot{\alpha}} \mp 2i\theta_\alpha\bar{\theta}_{\dot{\alpha}}$$

These objects have inverses

$$x_\pm^{-1} = -\frac{x_\pm}{x_\pm^2}$$

We then define superinversion (see Osborn):

$$(x^I_\pm)_{\alpha\dot{\alpha}} = \frac{x_\mp}{x_\mp^2} = \mathcal{I}_\alpha{}^{\dot{\alpha}'}(x_\pm)\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_\mp)\, x_{\pm\alpha'\dot{\alpha}'}$$

$$\bar{\theta}^I_{\dot{\alpha}} = -\frac{x_{+\alpha\dot{\alpha}}}{x_+^2}\theta^\alpha \quad , \quad \theta^I_\alpha = -\frac{x_{-\alpha\dot{\alpha}}}{x_-^2}\bar{\theta}^{\dot{\alpha}}$$

$$= \frac{1}{x_+}\bar{\mathcal{I}}_{\dot{\alpha}}{}^\alpha(x_+)\theta_\alpha \qquad = \frac{1}{x_-}\mathcal{I}_\alpha{}^{\dot{\alpha}}(x_-)\bar{\theta}_{\dot{\alpha}}$$

removed factor of i on the transform of the $\theta, \bar{\theta}$ (I believe it is equivalent?)

We want super-inversion operation on the three-pt blocks to be somewhat equivalent so it forms a representation of the underlying coordinate inversion. Promote as follows:

$$x_+ \to X \ , \ x_- \to \bar{X} \ , \ \theta \to \Theta \ , \ \bar{\theta} \to \bar{\Theta}$$

Now the transformation more closely resembles the coordinate inversion. And in this regard we see $\mathcal{I}(\bar{X})$ acting on Θ would not form a representation of the coordinate inversion. Maybe this is the logic?