Computation of $\tilde{\mathcal{H}}$: monomials

1) $\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\beta}(X)\bar{\mathcal{I}}_{\alpha}{}^{\dot{\beta}}(\bar{X})\hat{X}_{\beta\dot{\beta}} = \hat{\tilde{X}}^{I}_{\alpha\dot{\alpha}} = -\hat{\bar{X}}_{\alpha\dot{\alpha}}$

2) $\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\tilde{\theta}_{\alpha} = \hat{\tilde{\theta}}^{I}_{\dot{\alpha}}$, $\bar{\mathcal{I}}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\hat{\tilde{\bar{\theta}}}_{\dot{\alpha}} = \hat{\tilde{\theta}}^{I}_{\alpha}$

3) $\bar{\mathcal{I}}_{\alpha}{}^{\dot{\alpha}}(\bar{X})(\hat{X}\cdot\hat{\bar{\theta}})_{\dot{\alpha}} = -(\tilde{\hat{X}}^{I}\cdot\tilde{\hat{\theta}}^{I})_{\alpha}$

$\mathcal{I}_{\dot{\alpha}}{}^{\alpha}(X)(\hat{X}\cdot\hat{\tilde{\bar{\theta}}})_{\alpha} = -(\tilde{\hat{X}}^{I}\cdot\hat{\tilde{\bar{\theta}}}^{I})_{\dot{\alpha}}$

4) $\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\tilde{\mathcal{I}}_{\dot{\beta}}{}^{\beta}(X)\Sigma_{\alpha\beta} = -\Sigma_{\dot{\alpha}\dot{\beta}}$

$\bar{\mathcal{I}}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\bar{\mathcal{I}}_{\beta}{}^{\dot{\beta}}(\bar{X})\Sigma_{\dot{\alpha}\dot{\beta}} = -\Sigma_{\dot{\alpha}\dot{\beta}}$

The building blocks are

P_i, $\tilde{P}_i$, Q_i, $\tilde{Q}_i$, $\mathcal{Z}_i$, R_i, $\tilde{R}_i$, S_i, $\tilde{S}_i$, J

For the $P_i, \tilde{P}_i$ building blocks we use 4):

$$\bar{u}^{\dot{\alpha}}\bar{v}^{\dot{\beta}}\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\,\mathcal{I}_{\dot{\beta}}{}^{\beta}(X)\frac{\partial}{\partial u^{\alpha}}\frac{\partial}{\partial v^{\beta}}\left(\Sigma_{\delta\sigma}u^{\delta}v^{\sigma}\right)$$

$$= \bar{u}^{\dot{\alpha}}\bar{v}^{\dot{\beta}}\left(-\Sigma_{\dot{\alpha}\dot{\beta}}\right) = -\tilde{P}_3$$

$$\Rightarrow P_i \xrightarrow{\tilde{\mathcal{I}}} -\tilde{P}_i \quad , \quad \tilde{P}_i \xrightarrow{\bar{\mathcal{I}}} -P_i$$

For Q:

$$\bar{u}^{\dot{\alpha}}v^{\alpha}\,\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\beta}(X)\bar{\mathcal{I}}_{\alpha}{}^{\dot{\beta}}(\bar{X})\frac{\partial}{\partial u^{\beta}}\frac{\partial}{\partial\bar{v}^{\dot{\beta}}}\big(\underbrace{\hat{X}_{\sigma\dot{\sigma}}u^{\sigma}\bar{v}^{\dot{\sigma}}}_{Q_3}\big)$$

$$= \bar{u}^{\dot{\alpha}}v^{\alpha}\hat{\tilde{X}}^{I}_{\alpha\dot{\alpha}} = -\hat{\bar{X}}_{\alpha\dot{\alpha}}v^{\alpha}\bar{u}^{\dot{\alpha}} = -\frac{\bar{X}_{\alpha\dot{\alpha}}}{(\bar{X}^2)^{1/2}}v^{\alpha}\bar{u}^{\dot{\alpha}}$$

$$= -\bar{Q}_3$$

$$= -\left(X_{\alpha\dot\alpha} - 4i\theta_\alpha\bar\theta_{\dot\alpha}\right)\left(\frac{1}{(X^2)^{1/2}} - 2i\frac{\theta^\alpha\bar\theta^{\dot\alpha}X_{\alpha\dot\alpha}}{(X^2)^{3/2}}\right)v^\alpha\bar u^{\dot\alpha}$$

$$= -\frac{1}{(X^2)^{1/2}}\left(X_{\alpha\dot\alpha} - 4i\theta_\alpha\bar\theta_{\dot\alpha}\right)\left(1 - 2i\hat\theta^\alpha\hat{\bar\theta}^{\dot\alpha}\hat X_{\alpha\dot\alpha}\right)v^\alpha\bar u^{\dot\alpha}$$

$$= -\left(\hat X_{\alpha\dot\alpha} - 4i\hat\theta_\alpha\hat{\bar\theta}_{\dot\alpha}\right)\Big(1 - 2i\underbrace{\hat\theta^\sigma\hat{\bar\theta}^{\dot\sigma}\hat X_{\sigma\dot\sigma}}_{J}\Big)v^\alpha\bar u^{\dot\alpha}$$

$$= -\left(\hat X_{\alpha\dot\alpha} - 4i\hat\theta_\alpha\hat{\bar\theta}_{\dot\alpha} - 2i\hat X_{\alpha\dot\alpha}J - 8\theta_\alpha\bar\theta_{\dot\alpha}J\right)v^\alpha\bar u^{\dot\alpha}$$

$$= -\tilde{Q}_3 + 4iR_2\tilde{R}_1 + 2i\tilde{Q}_3 J - \cancel{8R_2\tilde{R}_1 J}$$

$O(\theta^2\bar\theta^2)$ does not appear in $\tilde{R}$

$$= -\tilde{Q}_3(1-2iJ) + 4iR_2\tilde{R}_1$$

Hence under $\mathcal{I}$: (up to $O(\theta\bar\theta)$)

$$Q_3 \xrightarrow{\mathcal{I}} Q_3^{\mathcal{I}} = -\bar{Q}_3 = -\tilde{Q}_3(1-2iJ) + 4iR_2\tilde{R}_1$$

$$\Rightarrow \quad Q_1 \xrightarrow{\mathcal{I}} Q_1^{\mathcal{I}} = -\bar{Q}_1 = -\tilde{Q}_1(1-2iJ) + 4iR_3\tilde{R}_2$$

$$Q_2 \xrightarrow{\mathcal{I}} Q_2^{\mathcal{I}} = -\bar{Q}_2 = -\tilde{Q}_2(1-2iJ) + 4iR_1\tilde{R}_3$$

Similarly for $\tilde{Q}$ we have:

$$\tilde{Q}_3 \longrightarrow \tilde{Q}_3^{\mathcal{I}} = -Q_3(1-2iJ) - 4i\tilde{R}_2R_1$$

$$\tilde{Q}_1 \longrightarrow \tilde{Q}_1^{\mathcal{I}} = -Q_1(1-2iJ) - 4i\tilde{R}_3R_2$$

$$\tilde{Q}_2 \longrightarrow \tilde{Q}_2^{\mathcal{I}} = -Q_2(1-2iJ) - 4i\tilde{R}_1R_3$$

For the action of $\mathcal{I}\mathcal{I}$ on $\mathcal{Z}_i$:

$$\mathcal{Z}_i \longrightarrow \mathcal{Z}_i^{\mathcal{I}} = -\bar{\mathcal{Z}}_i = -\mathcal{Z}_i(1-2iJ) + 4iR_i\tilde{R}_i$$

For powers $Q_3^n \longrightarrow (Q_3^{\mathcal{I}})^n$

$$(Q_3^{\mathcal{I}})^n = (-1)^n(\tilde{Q}_3 - 2iJ\hat{\tilde{Q}}_3 - 4iR_2\tilde{R}_1)^n$$

$$= (-1)^n(\tilde{Q}_3^n - 2in\hat{\tilde{Q}}_3^n J - 4in\tilde{Q}_3^n R_2\tilde{R}_1)$$

Now consider the action of $\mathcal{I}$ on R_i

$$\bar{u}^{\dot{\alpha}}\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\frac{\partial}{\partial u^{\alpha}}\left(\varepsilon_{\delta\sigma}u^{\delta}\hat{\Theta}^{\sigma}\right)$$

$$= \bar{u}^{\dot{\alpha}}\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\hat{\Theta}_{\alpha}$$

$$= \bar{u}^{\dot{\alpha}}\hat{\Theta}^{\mathcal{I}}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{u}^{\dot{\alpha}}\hat{\Theta}^{\mathcal{I}\dot{\beta}}$$

$$= \tilde{R}_1^{\mathcal{I}} = -\tilde{S}_1$$

$$u^{\alpha}\bar{\tilde{\mathcal{I}}}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\frac{\partial}{\partial\bar{u}^{\dot{\alpha}}}\left(\varepsilon_{\dot{\delta}\dot{\sigma}}\bar{u}^{\dot{\delta}}\hat{\tilde{\Theta}}^{\dot{\sigma}}\right)$$

$$= u^{\alpha}\bar{\tilde{\mathcal{I}}}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\hat{\tilde{\Theta}}_{\dot{\alpha}} = R_1^{\mathcal{I}}$$

$$= -S_1 \quad (\text{neglecting } \tilde{\Theta}^2\Theta \text{ terms})$$

For the S_i building blocks

$$\bar{u}^{\dot{\alpha}}\tilde{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\frac{\partial}{\partial u^{\alpha}}\left(u^{\delta}\hat{X}_{\delta\dot{\delta}}\hat{\tilde{\Theta}}^{\dot{\delta}}\right)$$

$$= \bar{u}^{\dot{\alpha}}\left(-\hat{\tilde{X}}^{\mathcal{I}}_{\alpha\dot{\alpha}}\hat{\tilde{\Theta}}^{\mathcal{I}\alpha}\right) = -\tilde{S}^{\mathcal{I}}$$

$$= \bar{u}^{\dot{\alpha}}\hat{X}_{\dot{\alpha}}{}^{\alpha}\hat{X}_{\alpha\dot{\delta}}\hat{\tilde{\Theta}}^{\dot{\delta}}$$

$$= -\bar{u}^{\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\delta}}\hat{\tilde{\Theta}}^{\dot{\delta}} = -\tilde{R}_1$$

Hence, under $\mathcal{I}$:

$$P_i \xrightarrow{\mathcal{I}} -\tilde{P}_i \quad , \quad \tilde{P}_i \xrightarrow{\mathcal{I}} -P_i$$

$$Q_1 \xrightarrow{\mathcal{I}} Q_1^{\mathcal{I}} = -\bar{Q}_1 = -\tilde{Q}_1(1-2iJ) + 4i R_3 \tilde{R}_2$$
$$Q_2 \xrightarrow{\mathcal{I}} Q_2^{\mathcal{I}} = -\bar{Q}_2 = -\tilde{Q}_2(1-2iJ) + 4i R_1 \tilde{R}_3$$
$$Q_3 \xrightarrow{\mathcal{I}} Q_3^{\mathcal{I}} = -\bar{Q}_3 = -\tilde{Q}_3(1-2iJ) + 4i R_2 \tilde{R}_1$$

$$\tilde{Q}_1 \xrightarrow{\mathcal{I}} \tilde{Q}_1^{\mathcal{I}} = -Q_1(1-2iJ) - 4i \tilde{R}_3 R_2$$
$$\tilde{Q}_2 \xrightarrow{\mathcal{I}} \tilde{Q}_2^{\mathcal{I}} = -Q_2(1-2iJ) - 4i \tilde{R}_1 R_3$$
$$\tilde{Q}_3 \xrightarrow{\mathcal{I}} \tilde{Q}_3^{\mathcal{I}} = -Q_3(1-2iJ) - 4i \tilde{R}_2 R_1$$

$$\mathcal{Z}_i \longrightarrow \mathcal{Z}_i^{\mathcal{I}} = -\tilde{\mathcal{Z}}_i = -\mathcal{Z}_i(1-2iJ) + 4i R_i \tilde{R}_i$$

$$R_i \longrightarrow \tilde{R}_i^{\mathcal{I}} = -\tilde{S}_i \quad , \quad \tilde{R}_i \longrightarrow R_i^{\mathcal{I}} = -S_i$$
$$S_i \longrightarrow -\tilde{S}_i^{\mathcal{I}} = -\tilde{R}_i \quad , \quad \tilde{S}_i \longrightarrow -S_i^{\mathcal{I}} = -R_i$$

$$J \longrightarrow J^{\mathcal{I}} = J$$

Now consider the action of $\mathcal{I}(\bar v, v)$, $\tilde{\mathcal{I}}(v, \bar v)$ on the monomials. $\mathcal{I}$ and $\tilde{\mathcal{I}}$ act only on:

$$P_1,\ \tilde P_1,\ P_3, \tilde P_3,\ Q_1,\ \tilde Q_1,\ Q_3, \tilde Q_3,\ \mathcal{L}_2,\ R_2, \tilde R_2,\ S_2,\ \tilde S_2$$

Let's compute:

$$\begin{aligned}
\mathcal{I}(X; \bar v, v) P_1 &= \bar v^{\dot\alpha} \hat X_{\dot\alpha}{}^{\alpha} \frac{\partial}{\partial v^\alpha} \left(\varepsilon_{\sigma\delta} v^\sigma \omega^\delta \right) \\
&= \bar v^{\dot\alpha} \hat X_{\dot\alpha}{}^{\alpha} \varepsilon_{\alpha\delta} \omega^\delta \\
&= - \hat X_{\alpha\dot\alpha} \omega^\alpha \bar v^{\dot\alpha} = - \tilde Q_1
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{I}}(\bar X; v, \bar v) \tilde P_1 &= v^\alpha \hat{\bar X}_\alpha{}^{\dot\alpha} \frac{\partial}{\partial \bar v^{\dot\alpha}} \left(\varepsilon_{\dot\sigma\dot\delta} \bar v^{\dot\sigma} \bar\omega^{\dot\delta} \right) \\
&= v^\alpha \hat{\bar X}_\alpha{}^{\dot\alpha} \varepsilon_{\dot\alpha\dot\delta} \bar\omega^{\dot\delta} \\
&= - \hat{\bar X}_{\alpha\dot\alpha} v^\alpha \bar\omega^{\dot\alpha} \\
&= - \frac{1}{(\bar X^2)^{1/2}} \left(X_{\alpha\dot\alpha} - 4i \theta_\alpha \bar\theta_{\dot\alpha} \right) v^\alpha \bar\omega^{\dot\alpha} \\
&= - \frac{1}{(X^2)^{1/2}} (1 - 2iJ) \left(X_{\alpha\dot\alpha} - 4i \theta_\alpha \bar\theta_{\dot\alpha} \right) v^\alpha \bar\omega^{\dot\alpha} \\
&= - \left\{ \hat X_{\alpha\dot\alpha} - 2i \hat X_{\alpha\dot\alpha} J - 4i \hat\theta_\alpha \hat{\bar\theta}_{\dot\alpha} \right\} v^\alpha \bar\omega^{\dot\alpha} \\
&= - Q_1 (1 - 2iJ) + 4i R_2 \tilde R_3
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}(X; \bar v, v) P_3 &= \bar v^{\dot\alpha} \hat X_{\dot\alpha}{}^{\alpha} \frac{\partial}{\partial v^\alpha} \left(\varepsilon_{\sigma\delta} u^\sigma v^\delta \right) \\
&= \bar v^{\dot\alpha} \hat X_{\dot\alpha}{}^{\alpha} \varepsilon_{\sigma\alpha} u^\sigma \\
&= \hat X_{\alpha\dot\alpha} u^\alpha \bar v^{\dot\alpha} = Q_3
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{I}}(\bar{X}; v, \bar{v})\tilde{P}_3 &= v^\alpha \hat{\bar{X}}_\alpha{}^{\dot\alpha} \frac{\partial}{\partial \bar{v}^{\dot\alpha}}\left(\Sigma_{\dot\sigma\dot\delta}\bar{u}^{\dot\sigma}\bar{v}^{\dot\delta}\right) \\
&= v^\alpha \hat{\bar{X}}_\alpha{}^{\dot\alpha}\Sigma_{\dot\sigma\dot\alpha}\bar{u}^{\dot\sigma} \\
&= \hat{\bar{X}}_{\alpha\dot\alpha} v^\alpha \bar{u}^{\dot\alpha} \\
&= \frac{1}{(\bar{X}^2)^{1/2}}\left(X_{\alpha\dot\alpha} - 4i\theta_\alpha\bar{\theta}_{\dot\alpha}\right)v^\alpha\bar{u}^{\dot\alpha} \\
&= \frac{1}{(X^2)^{1/2}}(1-2iJ)\left(X_{\alpha\dot\alpha} - 4i\theta_\alpha\bar{\theta}_{\dot\alpha}\right)v^\alpha\bar{u}^{\dot\alpha} \\
&= \left\{\hat{X}_{\alpha\dot\alpha} - 2i\hat{X}_{\alpha\dot\alpha}J - 4i\hat{\theta}_\alpha\hat{\bar{\theta}}_{\dot\alpha}\right\}v^\alpha\bar{u}^{\dot\alpha} \\
&= \tilde{Q}_3(1-2iJ) - 4iR_2\tilde{R}_1
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}(X; \bar{v}, v)Q_1 &= \bar{v}^{\dot\alpha}\hat{X}_{\dot\alpha}{}^{\alpha}\frac{\partial}{\partial v^\alpha}\left(\hat{X}_{\sigma\dot\sigma}v^\sigma\bar{\omega}^{\dot\sigma}\right) \\
&= \bar{v}^{\dot\alpha}\hat{X}_{\dot\alpha}{}^{\alpha}\hat{X}_{\alpha\dot\sigma}\bar{\omega}^{\dot\sigma} \\
&= -\bar{v}^{\dot\alpha}\Sigma_{\dot\alpha\dot\sigma}\bar{\omega}^{\dot\sigma} = -\tilde{P}_1
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{I}}(\bar{X}; v, \bar{v})\tilde{Q}_1 &= v^\alpha\hat{\bar{X}}_\alpha{}^{\dot\alpha}\frac{\partial}{\partial\bar{v}^{\dot\alpha}}\left(\hat{X}_{\sigma\dot\sigma}\bar{v}^{\dot\sigma}\omega^\sigma\right) \\
&= v^\alpha\hat{\bar{X}}_\alpha{}^{\dot\alpha}\hat{X}_{\sigma\dot\alpha}\omega^\sigma \\
&= \frac{1}{(\bar{X}^2)^{1/2}}\left(X_\alpha{}^{\dot\alpha} - 4i\theta_\alpha\bar{\theta}^{\dot\alpha}\right)\hat{X}_{\sigma\dot\alpha}v^\alpha\omega^\sigma \\
&= \frac{1}{(X^2)^{1/2}}(1-2iJ)\left(X_\alpha{}^{\dot\alpha} - 4i\theta_\alpha\bar{\theta}^{\dot\alpha}\right)\hat{X}_{\sigma\dot\alpha}v^\alpha\omega^\sigma \\
&= -P_1(1-2iJ) - 4iR_2S_3
\end{aligned}
$$

$$\mathcal{I}(x;\bar v,v)\tilde{Q}_3 = \bar v^{\dot\alpha}\hat{\tilde X}_{\dot\alpha}{}^{\alpha}\frac{\partial}{\partial v^\alpha}\left(\hat{X}_{\sigma\dot\sigma}\bar u^{\dot\sigma}v^\sigma\right)$$

$$= \bar v^{\dot\alpha}\hat{\tilde X}_{\dot\alpha}{}^{\alpha}\hat{X}_{\alpha\dot\sigma}\bar u^{\dot\sigma}$$

$$= -\bar v^{\dot\alpha}\varepsilon_{\dot\alpha\dot\sigma}\bar u^{\dot\sigma} = \tilde P_3$$

$$\bar{\mathcal{I}}(\bar x;v,\bar v)Q_3 = v^\alpha\hat{\bar{\tilde X}}_\alpha{}^{\dot\alpha}\frac{\partial}{\partial\bar v^{\dot\alpha}}\left(\hat{\bar X}_{\sigma\dot\sigma}u^\sigma\bar v^{\dot\sigma}\right)$$

$$= v^\alpha\hat{\bar{\tilde X}}_\alpha{}^{\dot\alpha}\hat{\bar X}_{\sigma\dot\alpha}u^\sigma$$

$$= \frac{1}{(\bar X^2)^{1/2}}\left(X_\alpha{}^{\dot\alpha}-4i\theta_\alpha\bar\theta^{\dot\alpha}\right)\hat{\bar X}_{\sigma\dot\alpha}v^\alpha u^\sigma$$

$$= \frac{1}{(X^2)^{1/2}}(1-2iJ)\left(X_\alpha{}^{\dot\alpha}-4i\theta_\alpha\bar\theta^{\dot\alpha}\right)\hat{\bar X}_{\sigma\dot\alpha}v^\alpha u^\sigma$$

$$= P_3(1-2iJ)-4iR_2S_1$$

$$\mathcal{I}(x;\bar v,v)\bar{\mathcal{I}}(\bar x;v,\bar v)\mathcal{Z}_2 = -\bar{\mathcal{Z}}_2 = \ldots \quad \text{(see above)}$$

Transformation for $R_2,\tilde R_2,S_2,\tilde S_2$ have already been worked out above:

$$R_2\to\tilde R_2^I=-\tilde S_2\;,\quad \tilde R_2\to R_2^I=-S_2$$

$$S_2\to-\tilde S_2=\tilde R_2\;,\quad \tilde S_2\to-S_2=R_2$$

Hence, under $\tilde{\mathcal{I}}^{(2)}$, the transformation rules for the monomials are as follows:

Hence, under $\tilde{\mathcal{I}}^{(2)}$:

$$P_1 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -\tilde{Q}_1 \,, \quad \tilde{P}_1 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -Q_1(1-2iJ) + 4i R_2 \tilde{R}_3$$

$$P_3 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} Q_3 \,, \quad \tilde{P}_3 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} \tilde{Q}_3(1-2iJ) - 4i R_2 \tilde{R}_1$$

$$Q_1 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -\tilde{P}_1 \,, \quad \tilde{Q}_1 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -P_1(1-2iJ) - 4i R_2 S_3$$

$$Q_3 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} P_3(1-2iJ) - 4i R_2 S_1 \,, \quad \tilde{Q}_3 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} \tilde{P}_3$$

$$Z_2 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -\tilde{Z}_2 = -Z_2(1-2iJ) + 4i R_2 \tilde{R}_2$$

$$R_2 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -\tilde{S}_2 \,, \quad \tilde{R}_2 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -S_2$$

$$S_2 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -\tilde{R}_2 \,, \quad \tilde{S}_2 \xrightarrow{\tilde{\mathcal{I}}^{(2)}} -R_2$$

Keep in mind this is only up to $O(\theta\bar{\theta})$.

Now the task is to combine the rules above in sequence, i.e.

$$\mathcal{H}(X,\theta,\bar{\theta}) \xrightarrow{\mathcal{I}} \mathcal{H}^{\mathcal{I}}(X,\theta,\bar{\theta}) \equiv \mathcal{H}(X^{\mathcal{I}},\theta^{\mathcal{I}},\bar{\theta}^{\mathcal{I}}) \xrightarrow{\tilde{\mathcal{I}}^{(2)}} \tilde{\mathcal{H}}(X,\theta,\bar{\theta})$$

This might be easier to carry out computationally.