Superconformal invariance property for three-point functions

Let's consider a combined scale/chiral transformation in superspace

$$x \xrightarrow{\Lambda} \Lambda x \,, \quad \theta \to \Lambda^{\frac{1}{2}} e^{\frac{i}{2}\Omega} \,, \quad \bar{\theta} \to \Lambda^{\frac{1}{2}} e^{-\frac{i}{2}\Omega}$$

Let's assume that a superfield $\mathcal{O}(z)$ has the following transformation property:

$$\mathcal{O}(z) \to \mathcal{O}'(z') = \Lambda^{-\Delta} e^{-i\left(\frac{N}{N-4}\right)\kappa\Omega} \mathcal{O}(z)$$

where we write:

$$\Delta = q + \bar{q} \,, \quad \kappa = q - \bar{q} \,.$$

Consider the infinitesimal form of the above, let $\Lambda = 1 + \lambda$

$$(1+\lambda)^{-\Delta} e^{-i\left(\frac{N}{N-4}\right)\kappa\Omega} \mathcal{O}(z)$$

$$= \mathcal{O}(z) - \lambda \Delta \mathcal{O}(z) - i\left(\frac{N}{N-4}\right)\kappa \Omega \mathcal{O}(z)$$

$$\Rightarrow \delta \mathcal{O}(z) = -\lambda(q+\bar{q})\mathcal{O}(z) - i\left(\frac{N}{N-4}\right)(q-\bar{q})\Omega\mathcal{O}(z)$$

$$= -2q\left\{\frac{1}{2}\left(\lambda + i\frac{N}{N-4}\Omega\right)\right\}\mathcal{O}(z)$$

$$- 2\bar{q}\left\{\frac{1}{2}\left(\lambda - i\frac{N}{N-4}\Omega\right)\right\}\mathcal{O}(z)$$

$$\Rightarrow \sigma(z) = \frac{1}{2}\left(\lambda + i\frac{N}{N-4}\Omega\right) \,, \quad \bar{\sigma} \ldots$$

Consistent with N-extended results in your paper.

Under the transformation above the two-point function transform as:

$$x_{\bar{i}j} \to \Lambda x_{\bar{i}j} \,, \quad x^2_{\bar{i}j} \to \Lambda^2 x^2_{\bar{i}j}$$

Now for the three-point covariants:

$$X_3 \longrightarrow \frac{1}{\Lambda} X_3 \,, \quad \Theta_3 \to \frac{1}{\Lambda^{1/2}} e^{-\frac{i}{2}\Omega} \Theta_3 \,, \quad \bar{\Theta}_3 \to \frac{e^{\frac{i}{2}\Omega}}{\Lambda^{1/2}} \bar{\Theta}_3$$

Let's do a consistency check against the infinitesimal transformation properties in the $N=2$ paper. Consider combined scale and $U(1)$ and eqn. (2.28):

$$\delta\Theta = -\frac{1}{N}\left((N-2)\sigma(z) + 2\bar{\sigma}(z)\right)\Theta$$

$$-\frac{1}{N}\left((N-2)\sigma(z) + 2\bar{\sigma}(z)\right)$$

$$= -\frac{1}{N}(N-2)\left\{\frac{1}{2}\left(\lambda + i\frac{N}{N-4}\Omega\right)\right\} - \frac{2}{N}\frac{1}{2}\left(\lambda - i\frac{N}{N-4}\Omega\right)$$

$$= -\frac{N-2}{N}\left(\lambda + \frac{iN}{N-4}\Omega\right) - \frac{1}{N}\left(\lambda - i\frac{N}{N-4}\Omega\right)$$

$$= \left\{-\frac{N-2}{2N} - \frac{1}{N}\right\}\lambda + \left\{-\frac{iN(N-2)}{2N(N-4)} + \frac{iN}{N(N-4)}\right\}\Omega$$

$$= -\frac{1}{2}\lambda - \frac{i}{2}\Omega$$

This is simply the infinitesimal form of .

Now consider the ansatz for the three-pt function, and the transformation above:

$$\langle O_1'(z_1')O_2'(z_2')O_3'(z_3')\rangle = \frac{1}{(x_{\bar{1}3}'^{2})^{\bar{q}_1}(x_{\bar{3}1}'^{2})^{q_1}(x_{\bar{2}3}'^{2})^{\bar{q}_2}(x_{\bar{3}2}'^{2})^{q_2}}\mathcal{H}(X_3', \Theta_3', \bar{\Theta}_3')$$

Now recall the homogeneity condition:

$$\mathcal{H}(\Delta\bar{\Delta}X, \Delta\Theta, \bar{\Delta}\bar{\Theta}) = \Delta^{2a}\bar{\Delta}^{2\bar{a}}\mathcal{H}(X,\Theta,\bar{\Theta})$$

For our transformation we have:

$$\Delta = \frac{e^{-\frac{i}{2}\Omega}}{\Omega^{1/2}},$$

and hence:

$$\langle O'(z_1')\dots\rangle = \underbrace{\Omega^{-2q_1-2\bar{q}_1-2q_2-2\bar{q}_2}}_{\text{prefactor}}\underbrace{\Omega^{-a-\bar{a}}\,e^{i(\bar{a}-a)\Omega}}_{\mathcal{H}\text{ homogenity}}$$

$$\times\frac{1}{(x_{\bar{1}3}^{2})^{\bar{q}_1}(x_{\bar{3}1}^{2})^{q_1}(x_{\bar{2}3}^{2})^{\bar{q}_2}(x_{\bar{3}2}^{2})^{q_2}}\mathcal{H}(X_3, \Theta_3, \bar{\Theta}_3)$$

$$= \Omega^{-2q_1-2\bar{q}_1-2q_2-2\bar{q}_2-a-\bar{a}}\,e^{i(\bar{a}-a)\Omega}$$

$$\times\langle O_1(z_1)O_2(z_2)O_3(z_3)\rangle$$

Using the transformation properties of the superfields we should obtain:

$$= \Omega^{-(q_1+\bar{q}_1)}\Omega^{-(q_2+\bar{q}_2)}\Omega^{-(q_3+\bar{q}_3)}e^{-i\frac{N}{N-4}\sum_{k=1}^{3}(q_k-\bar{q}_k)\Omega}$$

$$\times\langle O_1(z_1)O_2(z_2)O_3(z_3)\rangle$$

By comparison we obtain:

$$-2q_1 - 2\bar{q}_1 - 2q_2 - 2\bar{q}_2 - a - \bar{a} = -q_1 - \bar{q}_1 - q_2 - \bar{q}_2 - q_3 - \bar{q}_3$$

$$\Rightarrow \quad a + \bar{a} = q_3 + \bar{q}_3 - q_2 - \bar{q}_2 - q_1 - \bar{q}_1 \qquad (L)$$

$$= \Delta_3 - \Delta_2 - \Delta_1$$

This is the expected result as $a+\bar{a}$ is the homogeneity/weight of $\mathcal{H}$; it agrees with non-susy conventions.

Now for the R-charge factors, we have on LHS and RHS respectively:

LHS: $e^{i(\bar{a}-a)\Omega}$

RHS: $e^{-i\frac{N}{N-4}\sum_{k=1}^{3}(q_k - \bar{q}_k)\Omega}$

Hence we have:

$$a - \bar{a} = \frac{N}{N-4}\sum_{i=1}^{3}(q_i - \bar{q}_i)$$

Now we have the two conditions

$$a + \bar{a} = q_3 + \bar{q}_3 - q_2 - \bar{q}_2 - q_1 - \bar{q}_1$$

$$\left(1 - \frac{4}{N}\right)(a - \bar{a}) = q_1 - \bar{q}_1 + q_2 - \bar{q}_2 + q_3 - \bar{q}_3$$

We just add and subtract them:

$$\left(1-\frac{4}{N}\right)(a-\bar{a})+a+\bar{a}=2q_3-2\bar{q}_2-2\bar{q}_1$$

$$\Rightarrow -\frac{1}{2}\left\{\left(1-\frac{4}{N}\right)(a-\bar{a})+a+\bar{a}\right\}=\bar{q}_1+\bar{q}_2-q_3$$

$$\Rightarrow -\frac{1}{2}\left\{\left(2-\frac{4}{N}\right)a+\frac{4}{N}\bar{a}\right\}=\bar{q}_1+\bar{q}_2-q_3$$

$$\Rightarrow \left(\frac{2}{N}-1\right)a-\frac{2}{N}\bar{a}=\bar{q}_1+\bar{q}_2-q_3$$

Similarly:

$$\left(1-\frac{4}{N}\right)(a-\bar{a})-a-\bar{a}=2q_1+2q_2-2\bar{q}_3$$

$$\Rightarrow -\frac{4}{N}a-\left(2-\frac{4}{N}\right)\bar{a}=2q_1+2q_2-2\bar{q}_3$$

$$\Rightarrow \left(\frac{2}{N}-1\right)\bar{a}-\frac{2}{N}a=q_1+q_2-\bar{q}_3$$

So the right constraints are:

$$\left.\begin{aligned}\left(\frac{2}{N}-1\right)a-\frac{2}{N}\bar{a}&=\bar{q}_1+\bar{q}_2-q_3\\ \left(\frac{2}{N}-1\right)\bar{a}-\frac{2}{N}a&=q_1+q_2-\bar{q}_3\end{aligned}\right\}\ N\neq 4$$

$N=1 \Rightarrow a-2\bar{a}=\dots\ ,\ \bar{a}-2a=\dots$

$N=2 \Rightarrow -\bar{a}=\dots\ ,\ -a=\dots$

and so on...