# Revised calculation of $\mathcal{H}^I$ and $\tilde{\mathcal{H}}$ for arbitrary superfields

28/06/24

We consider the three-point function

$$\langle O_{A_1}(z_1)\, O'_{A_2}(z_2)\, O''_{A_3}(z_3)\rangle$$

where:

$$\left.\begin{aligned} O_{A_1}(z_1) &\equiv O_{\alpha(i_1)\dot\alpha(j_1)}(z_1) \\ O'_{A_2}(z_2) &\equiv O'_{\beta(i_2)\dot\beta(j_2)}(z_2) \\ O''_{A_3}(z_3) &\equiv O''_{\gamma(i_3)\dot\gamma(j_3)}(z_3) \end{aligned}\right\}$$

arbitrary superfield representations, most general case.

The general ansatz is:

$$\langle O_{A_1}(z_1)\, O'_{A_2}(z_2)\, O''_{A_3}(z_3)\rangle = \frac{\mathcal{I}_{A_1}{}^{\bar{A}_1'}(x_{1\bar{3}}, x_{\bar{1}3})\, \mathcal{I}_{A_2}{}^{\bar{A}_2'}(x_{2\bar{3}}, x_{\bar{2}3})}{(x_{1\bar{3}}{}^2)^{q_1}(x_{\bar{1}3}{}^2)^{\bar{q}_1}(x_{2\bar{3}}{}^2)^{q_2}(x_{\bar{2}3}{}^2)^{\bar{q}_2}}\, \mathcal{H}_{\bar{A}_1'\bar{A}_2'A_3}(X_3, \Theta_3, \bar{\Theta}_3)$$

Here we note that $A = \{\alpha(i), \dot\alpha(j)\} \Rightarrow \bar{A} = \{\alpha(j), \dot\alpha(i)\}$, and we have used the shorthand notation:

$$\begin{aligned} \mathcal{I}_A{}^{\bar{A}}(x_{1\bar{3}}, x_{\bar{1}3}) &\equiv \mathcal{I}_{\alpha(i)}{}^{\dot\alpha'(i)}(x_{1\bar{3}})\, \bar{\mathcal{I}}_{\dot\alpha(j)}{}^{\alpha'(j)}(x_{\bar{1}3}), \\ \bar{\mathcal{I}}_{\bar{A}}{}^{A}(x_{\bar{1}3}, x_{1\bar{3}}) &\equiv \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\alpha'(i)}(x_{\bar{1}3})\, \mathcal{I}_{\alpha(j)}{}^{\dot\alpha'(j)}(x_{1\bar{3}}). \end{aligned}$$

conjugate to eachother.

These operators act on $\mathcal{H}$ to the right

which have the following properties:

$$\begin{aligned} \mathcal{I}_A{}^{\bar{A}}(x_{1\bar{3}}, x_{\bar{1}3})\, \bar{\mathcal{I}}_{\bar{A}}{}^{B}(x_{\bar{1}3}, x_{1\bar{3}}) &= \mathcal{I}_{\alpha(i)}{}^{\dot\alpha'(i)}(x_{1\bar{3}})\, \bar{\mathcal{I}}_{\dot\alpha(j)}{}^{\alpha'(j)}(x_{\bar{1}3})\, \bar{\mathcal{I}}_{\dot\alpha'(i)}{}^{\beta(i)}(x_{\bar{1}3})\, \mathcal{I}_{\alpha'(j)}{}^{\dot\beta(j)}(x_{1\bar{3}}) \\ &= \delta_{\alpha(i)}{}^{\beta(i)}\, \delta_{\dot\alpha(j)}{}^{\dot\beta(j)} \\ &\equiv \delta_A{}^B \end{aligned}$$

$$\begin{aligned} \bar{\mathcal{I}}_{\bar{A}}{}^{A}(x_{\bar{1}3}, x_{1\bar{3}})\, \mathcal{I}_A{}^{\bar{B}}(x_{1\bar{3}}, x_{\bar{1}3}) &= \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\alpha'(i)}(x_{\bar{1}3})\, \mathcal{I}_{\alpha(j)}{}^{\dot\alpha'(j)}(x_{1\bar{3}})\, \mathcal{I}_{\alpha'(i)}{}^{\dot\beta(i)}(x_{1\bar{3}})\, \bar{\mathcal{I}}_{\dot\alpha'(j)}{}^{\beta(j)}(x_{\bar{1}3}) \\ &= \delta_{\dot\alpha(i)}{}^{\dot\beta(i)}\, \delta_{\alpha(j)}{}^{\beta(j)} \\ &\equiv \delta_{\bar{A}}{}^{\bar{B}} \end{aligned}$$

It will become apparent that this notation is very beneficial for calculations.

For the three-point building blocks, it will become clear that the following shorthand is useful:

$$\mathcal{I}_A{}^{\bar{A}}(\bar{X},X) \equiv \mathcal{I}_{\alpha(i)}{}^{\dot{\alpha}'(i)}(\bar{X})\,\bar{\mathcal{I}}_{\dot{\alpha}(j)}{}^{\alpha'(j)}(X)\,.$$
$$\bar{\mathcal{I}}_{\bar{A}}{}^{A}(X,\bar{X}) \equiv \bar{\mathcal{I}}_{\dot{\alpha}(i)}{}^{\alpha'(i)}(X)\,\mathcal{I}_{\alpha(j)}{}^{\dot{\alpha}'(j)}(\bar{X})\,.$$

conjugate to eachother.

act on $\mathcal{H}$ to the right

which have the following properties:

$$\begin{aligned}\mathcal{I}_A{}^{\bar{A}}(\bar{X},X)\,\bar{\mathcal{I}}_{\bar{A}}{}^{B}(\bar{X},X) &= \mathcal{I}_{\alpha(i)}{}^{\dot{\alpha}'(i)}(\bar{X})\,\bar{\mathcal{I}}_{\dot{\alpha}(j)}{}^{\alpha'(j)}(X)\,\bar{\mathcal{I}}_{\dot{\alpha}'(i)}{}^{\beta(i)}(\bar{X})\,\mathcal{I}_{\alpha'(j)}{}^{\dot{\beta}(j)}(X)\\ &= \delta_{\alpha(i)}{}^{\beta(i)}\delta_{\dot{\alpha}(j)}{}^{\dot{\beta}(j)}\\ &\equiv \delta_A{}^B\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{I}}_{\bar{A}}{}^{A}(X,\bar{X})\,\mathcal{I}_A{}^{\bar{B}}(X,\bar{X}) &= \bar{\mathcal{I}}_{\dot{\alpha}(i)}{}^{\alpha'(i)}(X)\,\mathcal{I}_{\alpha(j)}{}^{\dot{\alpha}'(j)}(\bar{X})\,\mathcal{I}_{\alpha'(i)}{}^{\dot{\beta}(i)}(X)\,\bar{\mathcal{I}}_{\dot{\alpha}'(j)}{}^{\beta(j)}(\bar{X})\\ &= \delta_{\dot{\alpha}(i)}{}^{\dot{\beta}(i)}\delta_{\alpha(j)}{}^{\beta(j)}\\ &\equiv \delta_{\bar{A}}{}^{\bar{B}}\end{aligned}$$

The objects $\mathcal{I}_A{}^{\bar{A}}(\bar{X},X)$, $\bar{\mathcal{I}}_{\bar{A}}{}^{A}(X,\bar{X})$ also act on $\mathcal{H}$ to the right, as we will soon see.

So overall we have the fundamental objects

$$\left.\begin{array}{l}\mathcal{I}_\alpha{}^{\dot{\alpha}}(x_{1\bar{3}}),\ \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(x_{\bar{1}3})\\ \mathcal{I}_\alpha{}^{\dot{\alpha}}(\bar{X}),\ \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\end{array}\right\}$$

acts on $\mathcal{H}$ to the right

and the shorthand:

$$\left.\begin{array}{l}\mathcal{I}_A{}^{\bar{A}}(x_{1\bar{3}},x_{\bar{1}3}),\ \bar{\mathcal{I}}_{\bar{A}}{}^{A}(x_{\bar{1}3},x_{1\bar{3}})\\ \mathcal{I}_A{}^{\bar{A}}(\bar{X},X),\ \bar{\mathcal{I}}_{\bar{A}}{}^{A}(X,\bar{X})\end{array}\right\}$$

Acts on $\mathcal{H}$ to the right and combines the fundamental inversions for a particular $(i,j)$ superfield rep. into a single object.

Let's test to make sure it's consistent in terms of the ansatz setup.

$$\langle \mathcal{O}_{A_1}(z_1)\mathcal{O}'_{A_2}(z_2)\mathcal{O}''_{A_3}(z_3)\rangle = \frac{\mathcal{I}_{A_1}{}^{\bar{A}_1'}(x_{1\bar{3}},x_{\bar{1}3})\,\mathcal{I}_{A_2}{}^{\bar{A}_2'}(x_{2\bar{3}},x_{\bar{2}3})}{(x_{1\bar{3}}^2)^{q_1}(x_{\bar{1}3}^2)^{\bar{q}_1}(x_{2\bar{3}}^2)^{q_2}(x_{\bar{2}3}^2)^{\bar{q}_2}}\,\mathcal{H}_{\bar{A}_1'\bar{A}_2'A_3}(X_3,\Theta_3,\bar{\Theta}_3)$$

which, using the objects defined above, is equivalent to the following:

$$\langle O_{\alpha(i_1)\dot\alpha(j_1)}(z_1)\, O'_{\beta(i_2)\dot\beta(j_2)}(z_2)\, O''_{\gamma(i_3)\dot\gamma(j_3)}(z_3)\rangle$$

$$= \frac{\mathcal{I}_{\alpha(i_1)}{}^{\dot\alpha'(i_1)}(x_{1\bar3})\,\bar{\mathcal{I}}_{\dot\alpha(j_1)}{}^{\alpha'(j_1)}(x_{\bar13})\,\mathcal{I}_{\beta(i_2)}{}^{\dot\beta'(i_2)}(x_{2\bar3})\,\bar{\mathcal{I}}_{\dot\beta(j_2)}{}^{\alpha'(j_2)}(x_{\bar23})}{(x_{1\bar3}^2)^{q_1}(x_{\bar13}^2)^{\bar q_1}(x_{2\bar3}^2)^{q_2}(x_{\bar23}^2)^{\bar q_2}}$$

$$\times \mathcal{H}_{\alpha'(j_1)\dot\alpha'(i_1)|\beta'(j_2)\dot\beta'(i_2)|\gamma(i_3)\dot\gamma(j_3)}(X_3,\Theta_3,\bar\Theta_3)$$

So far, the shorthand is consistent.

Let's now revisit the calculation for $\mathcal{H}^I$. First let us define the action of superinversion on $\mathcal{H}$ as follows:

$$\mathcal{I}_{A_1}{}^{\bar A_1}(\bar X,X)\,\mathcal{I}_{A_2}{}^{\bar A_2}(\bar X,X)\,\bar{\mathcal{I}}_{\bar A_3}{}^{A_3}(X,\bar X)\,\mathcal{H}_{\bar A_1\bar A_2A_3}(X,\Theta,\bar\Theta)$$

$$\equiv \mathcal{I}_{\alpha(i_1)}{}^{\dot\alpha'(i_1)}(\bar X)\,\bar{\mathcal{I}}_{\dot\alpha(j_1)}{}^{\alpha'(j_1)}(X)\,\bar{\mathcal{I}}_{\dot\beta(i_2)}{}^{\beta'(i_2)}(X)\,\mathcal{I}_{\beta(j_2)}{}^{\dot\beta'(j_2)}(\bar X)$$

$$\times\, \bar{\mathcal{I}}_{\dot\gamma(i_3)}{}^{\gamma'(i_3)}(X)\,\mathcal{I}_{\gamma(j_3)}{}^{\dot\gamma'(j_3)}(X)\,\mathcal{H}_{\alpha'(j_1)\dot\alpha'(i_1)|\beta'(j_2)|\dot\beta'(i_2)\gamma'(i_3)\dot\gamma'(j_3)}(X,\Theta,\bar\Theta)$$

We know that $\mathcal{H}$ can generically be expanded in the form:

$$\mathcal{H}(X,\Theta,\bar\Theta) = \mathcal{F}(X) + \Theta\bar\Theta G(X) + \Theta^2 A_1(X) + \bar\Theta^2 A_2(X) + \Theta^2\bar\Theta^2 A_3(X)$$

Hence, $\mathcal{H}$ can be constructed from the tensorial building blocks

$$\varepsilon_{\alpha\beta},\ \varepsilon_{\dot\alpha\dot\beta},\ X_{\alpha\dot\alpha},\ \Theta_\alpha,\ \bar\Theta_{\dot\alpha},\ (X\cdot\bar\Theta)_\alpha,\ (X\cdot\Theta)_{\dot\alpha}$$

$$\Theta^2,\ \bar\Theta^2,\ J=\Theta^\alpha\bar\Theta^{\dot\alpha}X_{\alpha\dot\alpha},\ \Theta^2\bar\Theta^2 \propto \frac{J^2}{X^2}$$

However, we note that for conserved supercurrents $A_1 = A_2 = A_3 = 0$. We do not need to assume this in what follows.

Recall that the building blocks $X$, $\Theta$, $\bar\Theta$ have the following transformation properties:

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha'}(x)\,\mathcal{I}_{\alpha}{}^{\dot\alpha'}(\bar{x})\,X_{\alpha'\dot\alpha'} = -\frac{X}{\bar{X}}\bar{X}_{\alpha\dot\alpha} = \bar{X}^{I}{}_{\alpha\dot\alpha}$$

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha'}(x)\,\mathcal{I}_{\alpha}{}^{\dot\alpha'}(\bar{x})\,\bar{X}_{\alpha'\dot\alpha'} = -\frac{\bar{X}}{X}X_{\alpha\dot\alpha} = X^{I}$$

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x)\,\Theta_{\alpha} = -\frac{1}{X}(X\cdot\Theta)_{\dot\alpha} = \bar{\Theta}^{I}_{\dot\alpha}$$

$$\mathcal{I}_{\alpha}{}^{\dot\alpha}(\bar{x})\,\bar{\Theta}_{\dot\alpha} = -\frac{1}{\bar{X}}(\bar{X}\cdot\bar{\Theta})_{\alpha} = \Theta^{I}_{\alpha}$$

all inversion operators have the same form as page 1.

For the other building blocks we obtain

$$\mathcal{I}_{\alpha}{}^{\dot\alpha}(\bar{x})\,\mathcal{I}_{\beta}{}^{\dot\beta}(\bar{x})\,\varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{\alpha\beta}$$

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x)\,\bar{\mathcal{I}}_{\dot\beta}{}^{\beta}(x)\,\varepsilon_{\alpha\beta} = -\varepsilon_{\dot\alpha\dot\beta}$$

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x)(X\cdot\Theta)_{\alpha} = -X\bar{\Theta}_{\dot\alpha} = -\frac{X}{\bar{X}}\bar{X}_{\dot\alpha}{}^{\alpha}\Theta^{I}_{\alpha} = -(\bar{X}^{I}\cdot\Theta^{I})_{\dot\alpha}$$

$$\mathcal{I}_{\alpha}{}^{\dot\alpha}(\bar{x})(X\cdot\Theta)_{\dot\alpha} = \frac{X}{\bar{X}}\bar{X}_{\alpha\dot\alpha}\bar{\Theta}^{I\dot\alpha} = -(\bar{X}^{I}\cdot\bar{\Theta}^{I})_{\alpha}$$

$$\begin{aligned}
\Theta^{\alpha}\bar{\Theta}^{\dot\alpha}X_{\alpha\dot\alpha} &= \bar{\mathcal{I}}^{\alpha}{}_{\dot\beta}(x)\,\bar{\Theta}^{I\dot\beta}\,\mathcal{I}^{\dot\alpha}{}_{\beta}(\bar{x})\,\Theta^{I\alpha}X_{\alpha\dot\alpha} \\
&= \bar{\Theta}^{I\dot\beta}\,\Theta^{I\alpha}\,\bar{X}^{I}_{\alpha\dot\alpha}
\end{aligned}$$

$$\begin{aligned}
\Theta^{2} = \Theta^{\alpha}\Theta_{\alpha} &= \bar{\mathcal{I}}^{\alpha}{}_{\dot\beta}(x)\,\bar{\Theta}^{I\dot\beta}\,\bar{\mathcal{I}}_{\alpha\dot\gamma}(x)\,\bar{\Theta}^{I\dot\gamma} \\
&= -\varepsilon_{\dot\beta\dot\gamma}\bar{\Theta}^{I\dot\beta}\bar{\Theta}^{I\dot\gamma} \\
&= -(\bar{\Theta}^{I})^{2}
\end{aligned}$$

$$\bar{\Theta}^{2} = -(\Theta^{I})^{2}$$

$$\Theta^{2}\bar{\Theta}^{2} = (\bar{\Theta}^{I})^{2}(\Theta^{I})^{2}$$

Hence we obtain the replacements:

$$\left.\begin{aligned} X \xrightarrow{\mathcal{I}} \bar{X}^{\mathcal{I}}, \quad \bar{X} \xrightarrow{\mathcal{I}} X^{\mathcal{I}}, \quad \theta_\alpha \xrightarrow{\mathcal{I}} \bar{\theta}^{\mathcal{I}}_{\dot\alpha}, \quad \bar{\theta}_{\dot\alpha} \longrightarrow \theta^{\mathcal{I}}_{\alpha} \\ J = \theta\cdot\bar{\theta}\cdot X \xrightarrow{\mathcal{I}} \bar{\theta}^{\mathcal{I}}\theta^{\mathcal{I}}\bar{X}^{\mathcal{I}} := \bar{J}^{\mathcal{I}} \\ \theta^2\bar{\theta}^2 \longrightarrow (\bar{\theta}^{\mathcal{I}})^2(\theta^{\mathcal{I}})^2 \end{aligned}\right\} \text{even}$$

$$\left.\begin{aligned} \varepsilon \xrightarrow{\mathcal{I}} -\bar{\varepsilon}, \quad \bar{\varepsilon} \xrightarrow{\mathcal{I}} -\varepsilon \\ (X\cdot\theta)_{\dot\alpha} \xrightarrow{\mathcal{I}} -(\bar{X}^{\mathcal{I}}\cdot\bar{\theta}^{\mathcal{I}})_\alpha, \quad (X\cdot\bar{\theta})_\alpha \xrightarrow{\mathcal{I}} -(\bar{X}^{\mathcal{I}}\cdot\theta^{\mathcal{I}})_{\dot\alpha} \\ \theta^2 \xrightarrow{\mathcal{I}} -(\bar{\theta}^{\mathcal{I}})^2, \quad \bar{\theta}^2 \xrightarrow{\mathcal{I}} -(\theta^{\mathcal{I}})^2 \end{aligned}\right\} \text{odd}$$

Hence, the action of $\mathcal{I}_{X,\bar{X}}$ is essentially that of conjugation, except expressed in terms of inverted variables.

Now let's consider the following general ansatz for $\mathcal{H}$:

$$\mathcal{H}(X,\theta,\bar{\theta}) = \sum_i A_i \mathcal{H}_i(X,\theta,\bar{\theta})$$

where $A_i = a_i + ib_i$ are complex coefficients. Hence we may write:

$$\begin{aligned} \mathcal{H}(X,\theta,\bar{\theta}) &= \sum_i a_i \mathcal{H}_i(X,\theta,\bar{\theta}) + i\sum_i b_i \mathcal{H}_i(X,\theta,\bar{\theta}) \\ &= \mathcal{H}_R(X,\theta,\bar{\theta}) + i\mathcal{H}_I(X,\theta,\bar{\theta}) \end{aligned}$$

where $\mathcal{H}_R$ and $\mathcal{H}_I$ are linear combinations of structures with real coefficients.

This implies

$$\mathcal{H}_R = \sum_i a_i \mathcal{H}_i , \qquad \mathcal{H}_I = \sum_i b_i \mathcal{H}_i .$$

In what follows we will assume we are working with the linear combinations $\mathcal{H}_R$ and $\mathcal{H}_I$ (which we will not distinguish and simply call $\mathcal{H}$), as they have desirable conjugation properties.

The resulting action of $\mathcal{I}$ on $\mathcal{H}$ is then:

$$\begin{aligned}\mathcal{H}^{\mathcal{I}}_{A_1 A_2 \bar{A}_3}(X,\theta,\bar{\theta}) &= \mathcal{I}_{A_1}{}^{\bar{A}_1}(\bar{X},X)\, \mathcal{I}_{A_2}{}^{\bar{A}_2}(\bar{X},X)\, \bar{\mathcal{I}}_{\bar{A}_3}{}^{A_3}(X,\bar{X})\, \mathcal{H}_{\bar{A}_1\bar{A}_2 A_3}(X,\theta,\bar{\theta}) \\ &= \sum_i a_i\, \mathcal{I}_{A_1}{}^{\bar{A}_1}(\bar{X},X)\, \mathcal{I}_{A_2}{}^{\bar{A}_2}(\bar{X},X)\, \bar{\mathcal{I}}_{\bar{A}_3}{}^{A_3}(X,\bar{X})\, \mathcal{H}_{i\, \bar{A}_1\bar{A}_2 A_3}(X,\theta,\bar{\theta}) \\ &= \pm \sum_i a_i \left[\mathcal{H}_i(\bar{X}^{\mathcal{I}}, \bar{\theta}^{\mathcal{I}}, \theta^{\mathcal{I}})\right]_{A_1 A_2 \bar{A}_3}\end{aligned}$$

However, recall the definition for $\bar{\mathcal{H}}$:

$$\bar{\mathcal{H}}_{AB\bar{C}}(X,\theta,\bar{\theta}) := \left[\mathcal{H}(\bar{X},\bar{\theta},\theta)\right]_{AB\bar{C}}$$

i.e. $\bar{\mathcal{H}}$ is equal to $\mathcal{H}$ with replacements $X \to \bar{X}$, $\theta \to \bar{\theta}$, $\bar{\theta} \to \theta$, and has conjugated index structure. Hence, we obtain

$$\begin{aligned}\mathcal{H}^{\mathcal{I}}_{A_1 A_2 \bar{A}_3}(X,\theta,\bar{\theta}) &= \pm \sum_i a_i \bar{\mathcal{H}}_{i\, A_1 A_2 \bar{A}_3}(X^{\mathcal{I}}, \theta^{\mathcal{I}}, \bar{\theta}^{\mathcal{I}}) \\ &= \pm \bar{\mathcal{H}}_{A_1 A_2 \bar{A}_3}(X^{\mathcal{I}}, \theta^{\mathcal{I}}, \bar{\theta}^{\mathcal{I}})\end{aligned}$$

Hence, it is natural to partition our solutions into $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$ contributions which satisfy the condition

$$\mathcal{H}^{\mathcal{I}(\pm)}_{A_1 A_2 \bar{A}_3}(X,\theta,\bar{\theta}) = \pm \bar{\mathcal{H}}^{(\pm)}_{A_1 A_2 \bar{A}_3}(X^{\mathcal{I}}, \theta^{\mathcal{I}}, \bar{\theta}^{\mathcal{I}})$$

Where $\mathcal{H}^{(+)}$ has an overall even number of the odd building blocks, while $\mathcal{H}^{(-)}$ has an overall odd number of them.

Hence, the action of $\mathcal{I}_{X,\bar{X}}$ on $\mathcal{H}$ results in a "pseudo-conjugate" of $\mathcal{H}$ except expressed in terms of the inverted variables. The sign corresponds to how many of the "odd" tensorial building blocks appear in a given tensor structure.

One could then consider, for real superfields, the possibility of forming linear combinations

$$\mathcal{H}(X,\Theta,\bar{\Theta}) = \mathcal{H}^{E(+)}(X,\Theta,\bar{\Theta}) + \mathcal{H}^{E(-)}(X,\Theta,\bar{\Theta}) + \mathcal{H}^{O(+)}(X,\Theta,\bar{\Theta}) + \mathcal{H}^{O(-)}(X,\Theta,\bar{\Theta})$$

such that:

$$\mathcal{H}^{E/O}_{A_1A_2A_3}(X^I,\Theta^I,\bar{\Theta}^I) = \pm \mathcal{H}^{E/O\,I}_{A_1A_2A_3}(X,\Theta,\bar{\Theta})$$

Where we have used the fact that $A_i = \bar{A}_i$ (self conjugate reps).

Hence, we can drop the $I$ label and for each sector we get:

$$\mathcal{H}^{E(\pm)}(X,\Theta,\bar{\Theta}) = \pm \bar{\mathcal{H}}^{E(\pm)}(X,\Theta,\bar{\Theta})$$
$$\mathcal{H}^{O(\pm)}(X,\Theta,\bar{\Theta}) = \mp \bar{\mathcal{H}}^{O(\pm)}(X,\Theta,\bar{\Theta})$$

So, structures which are even under super inversion are of type '+' and hermitian, and type '−' and anti-hermitian. Similarly, structures which are odd under superinversion are of type '+' anti-hermitian, and type '−' hermitian.

However, in 4D SCFT the structures $\mathcal{H}^{(-)}$ cannot appear due to the following reasoning:

$k_i, \bar{k}_i, q_i, \bar{q}_i$ = no. of $P_i, \bar{P}_i$, $S_i, \bar{S}_i$.

$$i_1 + i_2 + i_3 - j_1 - j_2 - j_3 = 0$$

$$= 2(k_1 + k_2 + k_3) + p_1 + p_2 + p_3 + q_1 + q_2 + q_3$$
$$- 2(\bar{k}_1 + \bar{k}_2 + \bar{k}_3) - \bar{p}_1 - \bar{p}_2 - \bar{p}_3 - \bar{q}_1 - \bar{q}_2 - \bar{q}_3 = 0$$

$$= 2(k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3) + p_1 + p_2 + p_3 - \bar{p}_1 - \bar{p}_2 - \bar{p}_3$$
$$+ q_1 + q_2 + q_3 - \bar{q}_1 - \bar{q}_2 - \bar{q}_3 = 0$$

The number odd building blocks is

$$N^0 = k_1 + k_2 + k_3 + \bar{k}_1 + \bar{k}_2 + \bar{k}_3 + q_1 + q_2 + q_3 + \bar{q}_1 + \bar{q}_2 + \bar{q}_3$$

Consider $O(\Theta) = 0$, then $p_i = \bar{p}_i = q_i = \bar{q}_i = 0$

$$2(k_1 + k_2 + k_3 - \bar{k}_1 - \bar{k}_2 - \bar{k}_3) = 0$$

$$\Rightarrow k_1 + k_2 + k_3 = \bar{k}_1 + \bar{k}_2 + \bar{k}_3$$

Hence

$$N^0 = 2(k_1 + k_2 + k_3) = \text{even},$$

so for vector-like currents, at $O(\Theta\bar{\Theta}) = 0$, we have $N^0$ = even and so the solution is of type $\mathcal{H}^{(+)}$.

Now consider $O(\theta\bar\theta)$. The following holds:

$$p_1+p_2+p_3+\bar q_1+\bar q_2+\bar q_3+o=1\,,$$
$$\bar p_1+\bar p_2+\bar p_3+q_1+q_2+q_3+o=1\,,$$

and so

$$q_1+q_2+q_3=1-o-\bar p_1-\bar p_2-\bar p_3$$
$$\bar q_1+\bar q_2+\bar q_3=1-o-p_1-p_2-p_3$$

$$\Rightarrow\quad q_1+q_2+q_3-\bar q_1-\bar q_2-\bar q_3=p_1+p_2+p_3-\bar p_1-\bar p_2-\bar p_3$$

Hence

$$i_1+i_2+i_3-j_1-j_2-j_3$$
$$=2(k_1+k_2+k_3-\bar k_1-\bar k_2-\bar k_3)+p_1+p_2+p_3-\bar p_1-\bar p_2-\bar p_3$$
$$+q_1+q_2+q_3-\bar q_1-\bar q_2-\bar q_3=0$$

$$=2(k_1+k_2+k_3-\bar k_1-\bar k_2-\bar k_3+q_1+q_2+q_3-\bar q_1-\bar q_2-\bar q_3)=0$$

$$\Rightarrow\quad k_1+k_2+k_3-\bar k_1-\bar k_2-\bar k_3+q_1+q_2+q_3-\bar q_1-\bar q_2-\bar q_3=0$$

$$\Rightarrow\quad k_1+k_2+k_3+q_1+q_2+q_3=\bar k_1+\bar k_2+\bar k_3+\bar q_1+\bar q_2+\bar q_3$$

Hence at $O(\theta\bar\theta)$

$$N^0=k_1+k_2+k_3+\bar k_1+\bar k_2+\bar k_3+q_1+q_2+q_3+\bar q_1+\bar q_2+\bar q_3$$
$$=2(k_1+k_2+k_3+q_1+q_2+q_3)=\text{even}$$

So at $O(\theta\bar\theta)$ the solution is also of type $\mathcal{H}^{(+)}$!

Hence, for vector-like supercurrents we only obtain structures of type $\mathcal{H}^{(+)}$.

Now we know that we can write the ansatz as

$$\mathcal{H} = \mathcal{H}_R + i\mathcal{H}_I$$

now let's assume that we have decomposed $\mathcal{H}_R$ and $\mathcal{H}_I$ into a linearly independent basis of structures which are even/odd under the action of superinversion:

$$\mathcal{H}_R = \mathcal{H}_R^{E(+)} + \mathcal{H}_R^{O(+)} = \sum_{i=1}^{N_E} a_i^E \mathcal{H}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} a_i^O \mathcal{H}_{(i)}^{O(+)}$$

$$\mathcal{H}_I = \mathcal{H}_I^{E(+)} + \mathcal{H}_I^{O(+)} = \sum_{i=1}^{N_E} b_i^E \mathcal{H}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} b_i^O \mathcal{H}_{(i)}^{O(+)}$$

Here we recall that $\mathcal{H}_i^{E(+)}$ are hermitian while $\mathcal{H}_i^{O(+)}$ are anti-hermitian according to the superinversion criteria.

If the superfields in the three-point function are all real, then we must impose the following condition on $\mathcal{H}$:

$$\mathcal{H}(X,\Theta,\bar{\Theta}) = \bar{\mathcal{H}}(X,\Theta,\bar{\Theta})$$

This condition gives:

$$\sum_{i=1}^{N_E} a_i^E \mathcal{H}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} a_i^O \mathcal{H}_{(i)}^{O(+)} + i\left(\sum_{i=1}^{N_E} b_i^E \mathcal{H}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} b_i^O \mathcal{H}_{(i)}^{O(+)}\right)$$

$$= \sum_{i=1}^{N_E} a_i^E \bar{\mathcal{H}}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} a_i^O \bar{\mathcal{H}}_{(i)}^{O(+)} - i\left(\sum_{i=1}^{N_E} b_i^E \bar{\mathcal{H}}_{(i)}^{E(+)} + \sum_{i=1}^{N_O} b_i^O \bar{\mathcal{H}}_{(i)}^{O(+)}\right)$$

and hence

$$\sum_{i=1}^{N_O} a_i^O \mathcal{H}_{(i)}^{O(+)} + i \sum_{i=1}^{N_E} b_i^E \mathcal{H}_{(i)}^{(+)} = 0 .$$

which implies $a_i^0 = b_i^E = 0 \ \forall i$.

Therefore, given a linearly independent basis of structures which are even/odd under superinversion, after imposing the reality condition, $\mathcal{H}$ is of the form

$$\mathcal{H} = \mathcal{H}^{(+)} + i\mathcal{H}^{(-)}$$
$$= \sum_{i=1}^{N_E} a_i^E \mathcal{H}_i^{E(+)} + i \sum_{i=1}^{N_O} b_i \mathcal{H}_i^{O(+)} ,$$

$$\mathcal{H}_i^{E(+)} = \bar{\mathcal{H}}_i^{E(+)} , \quad \mathcal{H}_i^{O(+)} = -\bar{\mathcal{H}}_i^{O(+)} ,$$

where $\mathcal{H}_i^{E(+)}$ are even under superinversion, while $\mathcal{H}_i^{O(+)}$ are odd under superinversion.
If the reality condition is not imposed we obtain double this number. This is <u>exactly</u> what occurs in 4D CFT, and we can already see this in some of our results.

Now let's tackle the computation of $\hat{H}$ using these new insights. Recall that the following identities hold:

$$x_{1\bar{3}\alpha}{}^{\dot\beta}\, x_{\bar{1}3\dot\alpha}{}^{\beta}\, X_{3\beta\dot\beta} = -\frac{\bar{X}_{1\alpha\dot\alpha}}{\bar{X}_1^2}\,, \qquad x_{13\alpha}{}^{\dot\beta}\, x_{\bar{1}\bar{3}\dot\alpha}{}^{\beta}\, \bar{X}_{3\beta\dot\beta} = -\frac{X_{1\alpha\dot\alpha}}{X_1^2} \qquad \text{(a)}$$

$$\frac{x_{1\bar{3}}^2}{x_{\bar{1}3}^2}\, x_{\bar{1}3\dot\alpha}{}^{\alpha}\, \Theta_{3\alpha} = -\frac{1}{X_1^2} X_{1\alpha\dot\alpha}\Theta_1^{\alpha} \qquad \frac{x_{\bar{1}3}^2}{x_{1\bar{3}}^2}\, x_{1\bar{3}\alpha}{}^{\dot\alpha}\, \bar{\Theta}_{3\dot\alpha} = -\frac{1}{\bar{X}_1^2} \bar{X}_{1\alpha\dot\alpha}\bar{\Theta}_1^{\dot\alpha} \qquad \text{(b)}$$

Let's rewrite them so the two-point functions are normalised and expressed in terms of inversion operators defined at the start.

(a) Indices suppressed:

$$\hat{x}_{1\bar{3}}\,\hat{x}_{\bar{1}3}\, X_3 = -\frac{1}{x_{1\bar{3}}\, x_{\bar{1}3}\, \bar{X}_1^2}\,\bar{X}_1 = \frac{1}{x_{1\bar{3}}\, x_{\bar{1}3}\, X_1 \bar{X}_1}\,\bar{X}_1^{I}$$

$$\hat{x}_{1\bar{3}}\,\hat{x}_{\bar{1}3}\, \bar{X}_3 = -\frac{1}{x_{1\bar{3}}\, x_{\bar{1}3}\, X_1^2}\,X_1 = \frac{1}{x_{1\bar{3}}\, x_{\bar{1}3}\, X_1 \bar{X}_1}\,X_1^{I}$$

where we recall:

$$X^{I}_{\alpha\dot\alpha} = -\left(\frac{\bar{X}^2}{X^2}\right)^{1/2} X_{\alpha\dot\alpha}\,, \qquad \bar{X}^{I}_{\alpha\dot\alpha} = -\left(\frac{X^2}{\bar{X}^2}\right)^{1/2} \bar{X}_{\alpha\dot\alpha}$$

Now let $I_{\alpha}{}^{\dot\alpha}(x_{1\bar{3}}) = \hat{x}_{1\bar{3}\alpha}{}^{\dot\alpha}$, $\bar{I}_{\dot\alpha}{}^{\alpha}(x_{\bar{1}3}) = \hat{x}_{\bar{1}3\dot\alpha}{}^{\alpha}$, we obtain

$$I_{\alpha}{}^{\dot\alpha'}(x_{1\bar{3}})\, \bar{I}_{\dot\alpha}{}^{\alpha'}(x_{\bar{1}3})\, X_{3\alpha'\dot\alpha'} = \frac{1}{x_{\bar{1}3}\, x_{1\bar{3}}\, X_1\bar{X}_1}\, \bar{X}^{I}_{1\alpha\dot\alpha}$$

$$I_{\alpha}{}^{\dot\alpha'}(x_{1\bar{3}})\, \bar{I}_{\dot\alpha}{}^{\alpha'}(x_{\bar{1}3})\, \bar{X}_{3\alpha'\dot\alpha'} = \frac{1}{x_{\bar{1}3}\, x_{1\bar{3}}\, X_1\bar{X}_1}\, X^{I}_{1\alpha\dot\alpha}$$

Now for (b):

$$\frac{x_{1\bar{3}}^2}{x_{\bar{1}3}^2} x_{\bar{1}3\,\dot{\alpha}}{}^{\alpha}\, \theta_{3\alpha} = -\frac{1}{X_1^2} X_{1\alpha\dot{\alpha}}\, \theta_1^{\alpha} = \frac{1}{X_1} \bar{\theta}^{I}_{1\dot{\alpha}}$$

where we recall

$$\bar{\theta}^{I}_{\dot{\alpha}} = \frac{1}{X} X_{\dot{\alpha}}{}^{\alpha}\, \theta_{\alpha}$$

$$= \underbrace{\bar{I}_{\dot{\alpha}}{}^{\alpha}(X)}_{\text{we defined this earlier}} \theta_{\alpha}$$

Rearranging, we obtain:

$$\bar{I}_{\dot{\alpha}}{}^{\alpha}(x_{\bar{1}3})\, \theta_{3\alpha} = \frac{x_{\bar{1}3}}{x_{1\bar{3}}^2} \frac{1}{X_1} \bar{\theta}^{I}_{1\dot{\alpha}} = c\, \bar{\theta}^{I}_{1\dot{\alpha}}$$

where $c = \frac{x_{\bar{1}3}}{x_{1\bar{3}}^2} \frac{1}{X_1}$. Similarly, we obtain

$$\frac{x_{\bar{1}3}^2}{x_{1\bar{3}}^2} x_{1\bar{3}\,\alpha}{}^{\dot{\alpha}}\, \bar{\theta}_{3\dot{\alpha}} = -\frac{1}{\bar{X}_1^2} \bar{X}_{1\alpha\dot{\alpha}}\, \bar{\theta}_1^{\dot{\alpha}} = \frac{1}{\bar{X}_1} \theta^{I}_{1\alpha},$$

where we recall

$$\theta^{I}_{\alpha} = \frac{1}{\bar{X}} \bar{X}_{\alpha}{}^{\dot{\alpha}}\, \bar{\theta}_{\dot{\alpha}}$$

$$= \underbrace{I_{\alpha}{}^{\dot{\alpha}}(\bar{X})}\, \bar{\theta}_{\dot{\alpha}} \longrightarrow \text{we also defined this earlier}$$

Rearranging, we obtain:

$$I_{\alpha}{}^{\dot{\alpha}}(x_{1\bar{3}})\, \bar{\theta}_{3\dot{\alpha}} = \frac{x_{1\bar{3}}}{x_{\bar{1}3}^2} \frac{1}{\bar{X}_1} \theta^{I}_{1\alpha} = \bar{c}\, \theta^{I}_{1\alpha}$$

where $\bar{c} = \frac{x_{1\bar{3}}}{x_{\bar{1}3}^2} \frac{1}{\bar{X}_1}$.

We also note that:

$$c\bar{c} = \left(\frac{x_{\bar{1}3}}{x_{1\bar{3}}^2}\frac{1}{X_1}\right)\left(\frac{x_{1\bar{3}}}{x_{\bar{1}3}^2}\frac{1}{\bar{X}_1}\right) = \frac{1}{x_{\bar{1}3}x_{1\bar{3}}X_1\bar{X}_1}$$

Hence, overall we have:

$$\mathcal{I}_\alpha{}^{\dot{\alpha}'}(x_{1\bar{3}})\,\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{\bar{1}3})\,X_{3\alpha'\dot{\alpha}'} = c\bar{c}\,\bar{X}^{\mathcal{I}}_{1\,\alpha\dot{\alpha}}$$

$$\mathcal{I}_\alpha{}^{\dot{\alpha}'}(x_{1\bar{3}})\,\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha'}(x_{\bar{1}3})\,\bar{X}_{3\alpha'\dot{\alpha}'} = c\bar{c}\,X^{\mathcal{I}}_{1\,\alpha\dot{\alpha}}$$

$$\bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(x_{\bar{1}3})\,\Theta_{3\alpha} = c\,\bar{\Theta}^{\mathcal{I}}_{1\dot{\alpha}}$$

$$\mathcal{I}_{\alpha}{}^{\dot{\alpha}}(x_{1\bar{3}})\,\bar{\Theta}_{3\dot{\alpha}} = \bar{c}\,\Theta^{\mathcal{I}}_{1\alpha}$$

where

all inversion operators match the defns on page 1.

$$X^{\mathcal{I}}_{\alpha\dot{\alpha}} = -\left(\frac{\bar{X}^2}{X^2}\right)^{1/2} X_{\alpha\dot{\alpha}}\,, \quad \bar{X}^{\mathcal{I}}_{\alpha\dot{\alpha}} = -\left(\frac{X^2}{\bar{X}^2}\right)^{1/2}\bar{X}_{\alpha\dot{\alpha}}\,,$$

$$\Theta^{\mathcal{I}}_\alpha = \mathcal{I}_\alpha{}^{\dot{\alpha}}(\bar{X})\,\bar{\Theta}_{\dot{\alpha}}\,, \quad \bar{\Theta}^{\mathcal{I}}_{\dot{\alpha}} = \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\,\Theta_\alpha\,,$$

So we may write:

$$X_3 \xrightarrow{\mathcal{I}_{\bar{1}3,1\bar{3}}} c\bar{c}\,\bar{X}^{\mathcal{I}}_1\,, \qquad \bar{X}_3 \xrightarrow{\mathcal{I}_{\bar{1}3,1\bar{3}}} c\bar{c}\,X^{\mathcal{I}}_1\,,$$

$$\Theta_3 \xrightarrow{\mathcal{I}_{\bar{1}3,1\bar{3}}} c\,\bar{\Theta}^{\mathcal{I}}_1\,, \qquad \bar{\Theta}_3 \xrightarrow{\mathcal{I}_{\bar{1}3,1\bar{3}}} \bar{c}\,\Theta^{\mathcal{I}}_1\,,$$

Note that we must also consider the objects

$$(X_3\cdot\bar{\Theta}_3)_\alpha = X_{3\alpha\dot{\alpha}}\bar{\Theta}^{\dot{\alpha}}\,, \quad (X_3\cdot\Theta_3)_{\dot{\alpha}} = X_{3\alpha\dot{\alpha}}\Theta_3^\alpha\,,$$

$$\Theta_3^2\,,\quad \bar{\Theta}_3^2\,,\quad J_3 = \Theta_3^\alpha\bar{\Theta}_3^{\dot{\alpha}}X_{3\alpha\dot{\alpha}}\,,\quad \Theta_3^2\bar{\Theta}_3^2$$

However:

$$\bar{J}_3 = \Theta_3^\alpha \bar{\Theta}_3^{\dot\alpha} X_{3\alpha\dot\alpha} = c\bar{c}\, \bar{\mathcal{I}}^{\alpha}{}_{\dot\beta}(x_{\bar{1}3}) \bar{\Theta}_1^{I\dot\beta} \mathcal{I}^{\dot\alpha}{}_{\beta}(x_{1\bar{3}}) \Theta_1^{I\beta}$$
$$\times \left( \mathcal{I}_\alpha{}^{\dot\gamma}(x_{\bar{1}3}) \bar{\mathcal{I}}_{\dot\alpha}{}^{\gamma}(x_{1\bar{3}}) \bar{X}^I_{1,\gamma\dot\gamma}\, c\bar{c} \right)$$

$$= c^2\bar{c}^2 \left\{ \bar{\mathcal{I}}^{\alpha}{}_{\dot\beta}(x_{\bar{1}3}) \mathcal{I}_\alpha{}^{\dot\gamma}(x_{\bar{1}3}) \mathcal{I}^{\dot\alpha}{}_{\beta}(x_{1\bar{3}}) \bar{\mathcal{I}}_{\dot\alpha}{}^{\gamma}(x_{1\bar{3}}) \right\} \bar{\Theta}_1^{I\dot\beta} \Theta_1^{I\beta} \bar{X}^I_{\gamma\dot\gamma}$$

$$= c^2\bar{c}^2 \delta_{\dot\beta}{}^{\dot\gamma} \delta_\beta{}^\gamma \bar{\Theta}_1^{I\dot\beta} \Theta_1^{I\beta} \bar{X}^I_{1,\gamma\dot\gamma}$$

$$= c^2\bar{c}^2 \bar{\Theta}_1^{I\dot\alpha} \Theta_1^{I\alpha} \bar{X}^I_{1,\alpha\dot\alpha}$$

$$= \left(c\bar{\Theta}_1^{I\dot\alpha}\right)\left(\bar{c}\,\Theta_1^{I\alpha}\right)\left(c\bar{c}\,\bar{X}^I_{1,\alpha\dot\alpha}\right) = c^2\bar{c}^2 \bar{J}_1^I$$

Similarly

$$\Theta_3^2 = -c^2(\bar{\Theta}_1^I)^2, \qquad \bar{\Theta}_3^2 = -\bar{c}^2(\Theta_1^I)^2$$

$$\Longrightarrow \quad \Theta_3^2\bar{\Theta}_3^2 = c^2\bar{c}^2(\bar{\Theta}_1^I)^2(\Theta_1^I)^2$$

which is equivalent to the replacement:

$$\Theta_3 \to c\bar{\Theta}_1^I, \quad \bar{\Theta}_3 \to \bar{c}\,\Theta_1^I, \quad X_3 \longrightarrow c\bar{c}\,\bar{X}_1^I.$$

Again, similar to conjugation except the result is expressed in terms of $\bar{X}_1^I, \Theta_1^I, \bar{\Theta}_1^I$ and the "odd" building blocks obtain a sign.

Now for the other building blocks

$$\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x_{\bar{1}3})(X_3\cdot\bar{\Theta}_3)_\alpha = \bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x_{\bar{1}3}) X_{3\alpha\dot\delta} \bar{\Theta}_3^{\dot\delta}$$
$$= -\bar{\mathcal{I}}_{\dot\alpha}{}^{\alpha}(x_{\bar{1}3}) X_{3\alpha\dot\delta} \mathcal{I}_\gamma{}^{\dot\delta}(x_{1\bar{3}}) \mathcal{I}^{\gamma}{}_{\dot\beta}(x_{1\bar{3}}) \bar{\Theta}_3^{\dot\beta}$$
$$= -\left(c\bar{c}\,\bar{X}^I_{1,\gamma\dot\alpha}\right)\left(\bar{c}\,\Theta_1^{I\gamma}\right)$$
$$= -c\bar{c}^2 (\bar{X}_1^I\cdot\Theta_1^I)_{\dot\alpha}$$

Similarly, one can show:

$$\mathcal{I}_\alpha{}^{\dot\alpha}(x_{1\bar 3})(X_3\cdot\Theta_3)_{\dot\alpha} = \mathcal{I}_\alpha{}^{\dot\alpha}(x_{1\bar 3})X_{3\gamma\dot\alpha}\Theta_3^\gamma$$
$$= -\mathcal{I}_\alpha{}^{\dot\alpha}(x_{1\bar 3})X_{3\gamma\dot\alpha}\mathcal{I}_{\dot\beta}{}^{\gamma}(x_{\bar 1 3})\mathcal{I}_\delta{}^{\dot\beta}(x_{\bar 1 3})\Theta_3^\delta$$
$$= -\left(c\bar c\bar X^{\mathcal{I}}_{1\alpha\dot\beta}\right)\left(c\bar\Theta_1^{\mathcal{I}\dot\beta}\right)$$
$$= -c^2\bar c\left(\bar X_1^{\mathcal{I}}\cdot\bar\Theta_1^{\mathcal{I}}\right)_\alpha$$

In all cases

$$X_3 \to c\bar c\bar X_1^{\mathcal{I}},\quad \bar X_3 \to c\bar c X_1^{\mathcal{I}},$$
$$\Theta_3 \to c\bar\Theta_1^{\mathcal{I}},\quad \bar\Theta_3 \to \bar c\,\Theta_1^{\mathcal{I}}.$$

However, there is an overall sign change for these objects, as we will regard them as parity odd.

To summarise:

$$\left.\begin{aligned} &X_3 \longrightarrow c\bar c\bar X_1^{\mathcal{I}},\quad \bar X_3\longrightarrow c\bar c X_1^{\mathcal{I}},\\ &\Theta_3 \longrightarrow c\bar\Theta_1^{\mathcal{I}},\quad \bar\Theta_3 \longrightarrow \bar c\,\Theta_1^{\mathcal{I}},\\ &\Theta_3\cdot\bar\Theta_3\cdot X_3 \longrightarrow c^2\bar c^2\,\bar\Theta_1^{\mathcal{I}}\,\Theta_1^{\mathcal{I}}\,\bar X_1^{\mathcal{I}}\\ &\Theta_3^2\bar\Theta_3^2 \longrightarrow c^2\bar c^2(\bar\Theta_1^{\mathcal{I}})^2(\Theta_1^{\mathcal{I}})^2 \end{aligned}\right\}\ \text{even}$$

$$\left.\begin{aligned} &\varepsilon \xrightarrow{\mathcal{I}_{\bar 1 \bar 3\bar 1 3}} -\bar\varepsilon,\quad \bar\varepsilon \xrightarrow{\mathcal{I}_{\bar 1 \bar 3\bar 1 3}} -\varepsilon\\ &X_3\cdot\Theta_3 \longrightarrow -c^2\bar c\,\bar X_1^{\mathcal{I}}\cdot\bar\Theta_1^{\mathcal{I}},\quad X_3\cdot\bar\Theta_3 \longrightarrow -c\bar c^2 \bar X_1^{\mathcal{I}}\cdot\Theta_1^{\mathcal{I}}\\ &\Theta_3^2 \longrightarrow -c^2(\bar\Theta_1^{\mathcal{I}})^2,\quad \bar\Theta_3^2\longrightarrow -\bar c^2(\Theta_1^{\mathcal{I}})^2 \end{aligned}\right\}\ \text{odd}$$

The above transformations demonstrate that the action of $\mathcal{I}_{1\bar{3},\bar{1}3}$ is equivalent to the action of $\mathcal{I}_{X,\bar{X}}$ combined with a scale transformation.

Hence, for the action of $\mathcal{I}_{1\bar{3},\bar{1}3}$ on $\mathcal{H}$ we obtain the following general result.

$$\mathcal{I}_{A}{}^{\bar{A}}(x_{1\bar{3}},x_{\bar{1}3})\,\mathcal{I}_{B}{}^{\bar{B}}(x_{1\bar{3}},x_{\bar{1}3})\,\bar{\mathcal{I}}_{\bar{C}}{}^{C}(x_{\bar{1}3},x_{1\bar{3}})\,\mathcal{H}_{\bar{A}\bar{B}C}(X_3,\Theta_3,\bar{\Theta}_3)$$

$$\equiv \mathcal{I}_{\alpha(i_1)}{}^{\dot{\alpha}'(i_1)}(x_{1\bar{3}})\,\bar{\mathcal{I}}_{\dot{\alpha}(j_1)}{}^{\alpha'(j_1)}(x_{\bar{1}3})\,\mathcal{I}_{\beta(i_2)}{}^{\dot{\beta}'(i_2)}(x_{1\bar{3}})\,\bar{\mathcal{I}}_{\dot{\beta}(j_2)}{}^{\beta'(j_2)}(x_{\bar{1}3})$$
$$\times\,\bar{\mathcal{I}}_{\dot{\gamma}(i_3)}{}^{\gamma'(i_3)}(x_{\bar{1}3})\,\mathcal{I}_{\gamma(j_3)}{}^{\dot{\gamma}'(j_3)}(x_{1\bar{3}})\,\mathcal{H}_{\alpha'(j_1)\dot{\alpha}'(i_1)\beta'(j_2)\dot{\beta}'(i_2)\gamma'(i_3)\dot{\gamma}'(j_3)}(X_3,\Theta_3,\bar{\Theta}_3)$$

$$= \pm\left[\mathcal{H}(c\bar{c}\bar{X}_1^{I},\, c\bar{\Theta}^{I},\, \bar{c}\Theta^{I})\right]_{AB\bar{C}} \quad \text{Homogeneity condition.}$$

$$= \pm\, c^{2a}\,\bar{c}^{2\bar{a}}\left[\mathcal{H}(\bar{X}_1^{I},\bar{\Theta}^{I},\Theta^{I})\right]_{AB\bar{C}}$$

$$= \pm\left(\frac{x_{\bar{1}3}}{x_{1\bar{3}}^{2}}\frac{1}{X_1}\right)^{2a}\left(\frac{x_{1\bar{3}}}{x_{\bar{1}3}^{2}}\frac{1}{\bar{X}_1}\right)^{2\bar{a}}\left[\mathcal{H}(\bar{X}_1^{I},\bar{\Theta}^{I},\Theta^{I})\right]_{AB\bar{C}}$$

Hence, we may write:

$$= \pm\,\frac{x_{\bar{1}3}^{2(a-2\bar{a})}\,x_{1\bar{3}}^{2(\bar{a}-2a)}}{X_1^{2a}\,\bar{X}_1^{2\bar{a}}}\,\bar{\mathcal{H}}(X_1^{I},\Theta_1^{I},\bar{\Theta}_1^{I})$$

$$= \frac{x_{\bar{1}3}^{2(a-2\bar{a})}\,x_{1\bar{3}}^{2(\bar{a}-2a)}}{X_1^{2a}\,\bar{X}_1^{2\bar{a}}}\,\mathcal{H}^{I}(X_1^{I},\Theta_1^{I},\bar{\Theta}_1^{I})$$

where again we assume without loss of generality that $\mathcal{H}$ is a linear combination of tensor structures with real coefficients.

This makes it very explicit that $\mathcal{H}^I$ has structure very similar to $\bar{\mathcal{H}}$, and that the inversion in superspace is similar to a "pseudo-conjugation".

The general formula for the action of $\mathcal{I}_{1\bar{3},\bar{1}3}$ on $\mathcal{H}_{\bar{A}_1\bar{A}_2A_3}(X_3,\Theta_3,\bar{\Theta}_3)$ is therefore

$$\mathcal{I}_{A_1}{}^{\bar{A}_1}(x_{1\bar{3}},x_{\bar{1}3})\,\mathcal{I}_{A_2}{}^{\bar{A}_2}(x_{1\bar{3}},x_{\bar{1}3})\,\bar{\mathcal{I}}_{\bar{A}_3}{}^{A_3}(x_{\bar{1}3},x_{1\bar{3}})\,\mathcal{H}_{\bar{A}_1\bar{A}_2A_3}(X_3,\Theta_3,\bar{\Theta}_3)$$

$$= \pm\frac{x_{\bar{1}3}^{2(a-2\bar{a})}\,x_{1\bar{3}}^{2(\bar{a}-2a)}}{X_1^{2a}\,\bar{X}_1^{2\bar{a}}}\,\bar{\mathcal{H}}_{A_1A_2\bar{A}_3}(X_1^I,\Theta_1^I,\bar{\Theta}_1^I)$$

$$= \frac{x_{\bar{1}3}^{2(a-2\bar{a})}\,x_{1\bar{3}}^{2(\bar{a}-2a)}}{X_1^{2a}\,\bar{X}_1^{2\bar{a}}}\,\mathcal{H}^I_{A_1A_2\bar{A}_3}(X_1^I,\Theta_1^I,\bar{\Theta}_1^I)$$

I think this neatly resolves any ambiguities regarding the superinversion and the computation of $\mathcal{H}^I$. It's clearly a type of pseudo-conjugation. Let's also note that at no point in the calculations did we have to introduce homogeneous degree 0 building blocks. The result is also completely general as we did not assume that we are working with conserved supercurrents.

Let's now derive the $\mathcal{H}$ tilde transformation using the updated notation. We will see that it is a bit more compact and generalises well to arbitrary superfield reps.

We consider the three-point function

$$\langle \mathcal{O}_{A_1}(z_1)\, \mathcal{O}'_{A_2}(z_2)\, \mathcal{O}''_{A_3}(z_3) \rangle$$

where we recall:

$$\left.\begin{aligned} \mathcal{O}_{A_1}(z_1) &\equiv \mathcal{O}_{\alpha(i_1)\dot\alpha(j_1)}(z_1) \\ \mathcal{O}'_{A_2}(z_2) &\equiv \mathcal{O}_{\beta(i_2)\dot\beta(j_2)}(z_2) \\ \mathcal{O}''_{A_3}(z_3) &\equiv \mathcal{O}_{\gamma(i_3)\dot\gamma(j_3)}(z_3) \end{aligned}\right\}$$

arbitrary superfield representations, most general case.

Now consider the two different formulations:

$$\langle \mathcal{O}_{A_1}(z_1)\, \mathcal{O}'_{A_2}(z_2)\, \mathcal{O}''_{A_3}(z_3) \rangle = \frac{\mathcal{I}_{A_1}{}^{\bar A_1'}(x_{1\bar 3}, x_{\bar 1 3})\, \mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 3}, x_{\bar 2 3})}{(x_{1\bar 3}^2)^{q_1}(x_{\bar 1 3}^2)^{\bar q_1}(x_{2\bar 3}^2)^{q_2}(x_{\bar 2 3}^2)^{\bar q_2}}\, \mathcal{H}_{\bar A_1' \bar A_2' A_3}(X_3, \Theta_3, \bar\Theta_3)$$

$$\langle \mathcal{O}''_{A_3}(z_3)\, \mathcal{O}'_{A_2}(z_2)\, \mathcal{O}_{A_1}(z_1) \rangle = \frac{\mathcal{I}_{A_3}{}^{\bar A_3'}(x_{3\bar 1}, x_{\bar 3 1})\, \mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 1}, x_{\bar 2 1})}{(x_{3\bar 1}^2)^{q_3}(x_{\bar 3 1}^2)^{\bar q_3}(x_{2\bar 1}^2)^{q_2}(x_{\bar 2 1}^2)^{\bar q_2}}\, \hat{\mathcal{H}}_{\bar A_3' \bar A_2' A_1}(X_1, \Theta_1, \bar\Theta_1)$$

One can explicitly check that the ansatz with $\hat{\mathcal{H}}$ has the right index structure.

We equate the two ansatz to obtain:

$$\mathcal{I}_{A_3}{}^{\bar A_3'}(x_{3\bar 1}, x_{\bar 3 1})\, \mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 1}, x_{\bar 2 1})\, \hat{\mathcal{H}}_{\bar A_3' \bar A_2' A_1}(X_1, \Theta_1, \bar\Theta_1)$$

$$= \frac{(x_{3\bar 1}^2)^{q_3}(x_{\bar 3 1}^2)^{\bar q_3}(x_{2\bar 1}^2)^{q_2}(x_{\bar 2 1}^2)^{\bar q_2}}{(x_{1\bar 3}^2)^{q_1}(x_{\bar 1 3}^2)^{\bar q_1}(x_{2\bar 3}^2)^{q_2}(x_{\bar 2 3}^2)^{\bar q_2}}\, \mathcal{I}_{A_1}{}^{\bar A_1'}(x_{1\bar 3}, x_{\bar 1 3})\, \mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 3}, x_{\bar 2 3})\, \mathcal{H}_{\bar A_1' \bar A_2' A_3}(X_3, \Theta_3, \bar\Theta_3)$$

Note that:

$$\frac{(x_{3\bar 1}^2)^{q_3}(x_{\bar 3 1}^2)^{\bar q_3}(x_{2\bar 1}^2)^{q_2}(x_{\bar 2 1}^2)^{\bar q_2}}{(x_{1\bar 3}^2)^{q_1}(x_{\bar 1 3}^2)^{\bar q_1}(x_{2\bar 3}^2)^{q_2}(x_{\bar 2 3}^2)^{\bar q_2}} = (x_{1\bar 3}^2)^{\bar q_3 - q_1}(x_{\bar 1 3}^2)^{q_3 - \bar q_1}(\bar X_1^2 x_{1\bar 3}^2)^{-q_2}(X_1^2 x_{\bar 1 3})^{-\bar q_2}$$

$$= \frac{x_{1\bar 3}^{-2(\bar a - 2a)}\, x_{\bar 1 3}^{-2(a - 2\bar a)}}{\bar X_1^{2q_2} X_1^{2\bar q_2}}$$

Hence:

$$\mathcal{I}_{A_3}{}^{\bar A_3'}(x_{3\bar 1},x_{\bar 3 1})\,\mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 1},x_{\bar 2 1})\,\hat{\mathcal{H}}_{\bar A_3'\bar A_2' A_1}(X_1,\Theta_1,\bar\Theta_1)$$

$$= \frac{x_{1\bar 3}^{-2(\bar a-2a)}\, x_{\bar 1 3}^{-2(a-2\bar a)}}{\bar X_1^{2q_2} X_1^{2\bar q_2}}\, \mathcal{I}_{A_1}{}^{\bar A_1'}(x_{1\bar 3},x_{\bar 1 3})\, \mathcal{I}_{A_2}{}^{\bar A_2'}(x_{2\bar 3},x_{\bar 2 3})\, \mathcal{H}_{\bar A_1'\bar A_2' A_3}(X_3,\Theta_3,\bar\Theta_3)$$

Now multiply from the left by the appropriate inverse inversion operators:

$$\hat{\mathcal{H}}_{\bar A_3\bar A_2 A_1}(X_1,\Theta_1,\bar\Theta_1) = (-1)^{i_2+j_2+i_3+j_3}\, \frac{x_{1\bar 3}^{-2(\bar a-2a)}\, x_{\bar 1 3}^{-2(a-2\bar a)}}{\bar X_1^{2q_2} X_1^{2\bar q_2}}\, \mathcal{I}_{A_1}{}^{\bar A_1'}(x_{1\bar 3},x_{\bar 1 3})\, \bar{\mathcal{I}}_{\bar A_2}{}^{B}(x_{\bar 1 2},x_{1\bar 2})\, \mathcal{I}_{B}{}^{\bar A_2'}(x_{2\bar 3},x_{\bar 2 3})$$
$$\times\, \bar{\mathcal{I}}_{\bar A_3}{}^{A_3'}(x_{\bar 1 3},x_{1\bar 3})\, \mathcal{H}_{\bar A_1'\bar A_2' A_3'}(X_3,\Theta_3,\bar\Theta_3)$$

Now use the relation:

$$\bar{\mathcal{I}}_{\bar A}{}^{B}(x_{\bar 1 2},x_{1\bar 2})\cdot\mathcal{I}_{B}{}^{A'}(x_{2\bar 3},x_{\bar 2 3})$$
$$\equiv \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\beta(i)}(x_{\bar 1 2})\,\mathcal{I}_{\alpha(j)}{}^{\dot\beta(j)}(x_{1\bar 2})\,\mathcal{I}_{\beta(i)}{}^{\dot\alpha'(i)}(x_{2\bar 3})\,\bar{\mathcal{I}}_{\dot\beta(j)}{}^{\alpha'(j)}(x_{\bar 2 3})$$
$$= \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\beta(i)}(x_{\bar 1 2})\,\mathcal{I}_{\beta(i)}{}^{\dot\alpha'(i)}(x_{2\bar 3})\,\mathcal{I}_{\alpha(j)}{}^{\dot\beta(j)}(x_{1\bar 2})\,\bar{\mathcal{I}}_{\dot\beta(j)}{}^{\alpha'(j)}(x_{\bar 2 3})$$
$$= \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\beta(i)}(\bar X_1)\,\mathcal{I}_{\beta(i)}{}^{\dot\alpha'(i)}(x_{1\bar 3})\,\mathcal{I}_{\alpha(j)}{}^{\dot\beta(j)}(X_1)\,\bar{\mathcal{I}}_{\dot\beta(j)}{}^{\alpha'(j)}(x_{\bar 1 3})$$
$$= \bar{\mathcal{I}}_{\dot\alpha(i)}{}^{\beta(i)}(\bar X_1)\,\mathcal{I}_{\alpha(j)}{}^{\dot\beta(j)}(X_1)\,\mathcal{I}_{\beta(i)}{}^{\dot\alpha'(i)}(x_{1\bar 3})\,\bar{\mathcal{I}}_{\dot\beta(j)}{}^{\alpha'(j)}(x_{\bar 1 3})$$
$$\equiv \bar{\mathcal{I}}_{\bar A}{}^{B}(\bar X_1,X_1)\,\mathcal{I}_{B}{}^{A'}(x_{1\bar 3},x_{\bar 1 3})$$

identity is in the paper.

comparing to objects at the start where I define the inversion operators, this has swapped dependence on $X$ and $\bar X$. Not a problem, as we will see it gives the right result.

We then obtain

$$\hat{\mathcal{H}}_{\bar A_3\bar A_2 A_1}(X_1,\Theta_1,\bar\Theta_1) = (-1)^{i_2+j_2+i_3+j_3}\, \frac{x_{1\bar 3}^{-2(\bar a-2a)}\, x_{\bar 1 3}^{-2(a-2\bar a)}}{\bar X_1^{2q_2} X_1^{2\bar q_2}}\, \mathcal{I}_{A_1}{}^{\bar A_1'}(x_{1\bar 3},x_{\bar 1 3})\, \bar{\mathcal{I}}_{\bar A_2}{}^{B}(\bar X_1,X_1)\, \mathcal{I}_{B}{}^{\bar A_2'}(x_{1\bar 3},x_{\bar 1 3})$$
$$\times\, \bar{\mathcal{I}}_{\bar A_3}{}^{A_3'}(x_{\bar 1 3},x_{1\bar 3})\, \mathcal{H}_{\bar A_1'\bar A_2' A_3'}(X_3,\Theta_3,\bar\Theta_3)$$

$$= (-1)^{i_2+j_2+i_3+j_3} \frac{x_{1\bar 3}^{-2(\bar a-2a)}\, x_{\bar 1 3}^{-2(a-2\bar a)}}{\bar X_1^{2q_2} X_1^{2\bar q_2}} \, \bar{\mathcal I}_{\bar A_2}{}^{B}(\bar X_1, X_1)$$

$$\times\, \mathcal I_{A_1}{}^{\bar A_1'}(x_{1\bar 3}, x_{\bar 1 3})\, \mathcal I_{B}{}^{\bar A_2'}(x_{1\bar 3}, x_{\bar 1 3})\, \bar{\mathcal I}_{\bar A_3}{}^{A_3'}(x_{\bar 1 3}, x_{1\bar 3})\, \mathcal H_{\bar A_1' \bar A_2' A_3'}(X_3, \Theta_3, \bar\Theta_3)$$

Using the result for $\mathcal H \to \mathcal H^{I}$, (see ✱ on page 8)

$$\mathcal I_{A_1}{}^{\bar A_1'}(x_{1\bar 3}, x_{\bar 1 3})\, \mathcal I_{A_2}{}^{\bar A_2'}(x_{1\bar 3}, x_{\bar 1 3})\, \bar{\mathcal I}_{\bar A_3}{}^{A_3'}(x_{\bar 1 3}, x_{1\bar 3})\, \mathcal H_{\bar A_1' \bar A_2' A_3'}(X_3, \Theta_3, \bar\Theta_3)$$

$$= \frac{x_{\bar 1 3}^{2(a-2\bar a)}\, x_{1\bar 3}^{2(\bar a-2a)}}{X_1^{2a}\, \bar X_1^{2\bar a}}\, \mathcal H^{I}_{A_1 A_2 \bar A_3}(X_1, \Theta_1, \bar\Theta_1)$$

where $\mathcal H^{I}_{A_1 A_2 \bar A_3}(X, \Theta, \bar\Theta) := \pm\, \bar{\mathcal H}_{A_1 A_2 \bar A_3}(X^{I}, \Theta^{I}, \bar\Theta^{I})$. We now substitute this above

$$\hat{\mathcal H}_{\bar A_3 \bar A_2 A_1}(X_1, \Theta_1, \bar\Theta_1) = \frac{\cancel{x_{1\bar 3}^{-2(\bar a-2a)}}\, \cancel{x_{\bar 1 3}^{-2(a-2\bar a)}}}{\bar X_1^{2q_2} X_1^{2\bar q_2}} \, \frac{\cancel{x_{\bar 1 3}^{2(a-2\bar a)}}\, \cancel{x_{1\bar 3}^{2(\bar a-2a)}}}{X_1^{2a}\, \bar X_1^{2\bar a}} (-1)^{i_2+j_2+i_3+j_3}$$

$$\cdot\, \bar{\mathcal I}_{\bar A_2}{}^{A_2'}(\bar X_1, X_1)\, \mathcal H^{I}_{A_1 A_2' \bar A_3}(X_1, \Theta_1, \bar\Theta_1)$$

$$\hat{\mathcal H}_{\bar A_3 \bar A_2 A_1}(X_1, \Theta_1, \bar\Theta_1) = \frac{(-1)^{i_2+j_2+i_3+j_3}}{X_1^{2(a+\bar q_2)}\, \bar X_1^{2(\bar a+q_2)}}\, \bar{\mathcal I}_{\bar A_2}{}^{A_2'}(\bar X_1, X_1)\, \mathcal H^{I}_{A_1 A_2' \bar A_3}(X_1, \Theta_1, \bar\Theta_1)$$

Hence, we may write:

$$\hat{\mathcal H}_{\bar A_3 \bar A_2 A_1}(X, \Theta, \bar\Theta) = \frac{(-1)^{i_2+j_2+i_3+j_3}}{X^{2(a+\bar q_2)}\, \bar X^{2(\bar a+q_2)}}\, \bar{\mathcal I}_{\bar A_2}{}^{A_2}(\bar X, X)\, \mathcal H^{I}_{A_1 A_2 \bar A_3}(X, \Theta, \bar\Theta)$$

which is very similar to the result obtained by Osborn. Note that it holds for arbitrary superfields.

When expanded in terms of the fundamental inversion operators defined at the start:

$$\hat{\mathcal{H}}_{\gamma(j_3)\dot{\gamma}(i_3)\beta(j_2)\dot{\beta}(i_2)\alpha(i_1)\dot{\alpha}(j_1)}(X,\Theta,\bar{\Theta}) = \frac{(-1)^{i_2+j_2+i_3+j_3}}{X^{2(a+\bar{q}_2)}\bar{X}^{2(\bar{a}+q_2)}}\bar{\mathcal{I}}_{\dot{\beta}(i_2)}{}^{\beta'(i_2)}(\bar{X})\,\mathcal{I}_{\beta(j_2)}{}^{\dot{\beta}'(j_2)}(X)$$

$$\times\,\mathcal{H}^{\mathcal{I}}_{\alpha(i_1)\dot{\alpha}(j_1)\beta'(i_2)\dot{\beta}'(j_2)\gamma(j_3)\dot{\gamma}(i_3)}(X,\Theta,\bar{\Theta})$$

and we note here that $\mathcal{H}^{\mathcal{I}}$ has index structure conjugate to the $\mathcal{H}$ which we started with.

Note that this also agrees with the formula Jess derived, so it is fully consistent, however it is now generalised to fields in arbitrary reps.

Also, we have to $O(\Theta\bar{\Theta})$

$$\frac{1}{X^{2(a+\bar{q}_2)}\bar{X}^{2(\bar{a}+q_2)}} = \frac{1}{X^{2(a+\bar{q}_2)}X^{2(\bar{a}+q_2)}}\left(1-4i(\bar{a}+q_2)J+\dots\right)$$

$$= \frac{1}{X^{2(a+\bar{a}+q_2+\bar{q}_2)}}\left(1-4i(\bar{a}+q_2)J\right)$$

where

$$a+\bar{a}+q_2+\bar{q}_2 = \Delta_3-\Delta_2-\Delta_1+\Delta_2$$

$$= \Delta_3-\Delta_1$$

Hence, we have:

$$\frac{1}{X^{2(a+\bar{q}_2)}\bar{X}^{2(\bar{a}+q_2)}} = (X^2)^{\Delta_1-\Delta_3}\left(1-4i(\bar{a}+q_2)J\right)$$

This matches the transformation for $\hat{\mathcal{H}}$ in the 4D CFT case upon setting $\Theta=\bar{\Theta}=0$.

So what are the proposed changes?

1) There are two fundamental inversion operators

$$\mathcal{I}_{\alpha}{}^{\dot{\alpha}}(\cdot)\,, \quad \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(\cdot)$$

For two-point functions we always use

$$\mathcal{I}_{\alpha}{}^{\dot{\alpha}}(x_{i\bar{j}})\,, \qquad \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(x_{\bar{i}j}) \quad \text{(same as before)}$$

For three-point functions we (almost) always use

$$\mathcal{I}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\,, \quad \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X) \quad \text{(swapped bars on } \mathcal{I}\text{?)}$$

which have inverses on page 1, 2. However, when we compute $\mathcal{H}$, the operators

$$\mathcal{I}_{\alpha}{}^{\dot{\alpha}}(X)\,, \quad \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(\bar{X})$$

appear, but this is the only time this occurs. This also happens in Jessica's original calculation, so it's not unusual.

2) We introduce the shorthand notation on page 1, 2 to streamline calculations.

3) We use the definitions

$$X^{I}_{\alpha\dot{\alpha}} = -\left(\frac{\bar{X}^2}{X^2}\right)^{1/2} X_{\alpha\dot{\alpha}}\,, \quad \bar{X}^{I}_{\alpha\dot{\alpha}} = -\left(\frac{X^2}{\bar{X}^2}\right)^{1/2} \bar{X}_{\alpha\dot{\alpha}}\,,$$

$$\Theta^{I}_{\alpha} = \mathcal{I}_{\alpha}{}^{\dot{\alpha}}(\bar{X})\,\bar{\Theta}_{\dot{\alpha}}\,, \quad \bar{\Theta}^{I}_{\dot{\alpha}} = \bar{\mathcal{I}}_{\dot{\alpha}}{}^{\alpha}(X)\,\Theta_{\alpha}$$

which makes the "conjugated" structure of $\mathcal{H}^{I}$ much more explicit and allows us to properly define the notion of structures being even/odd under superinversion.