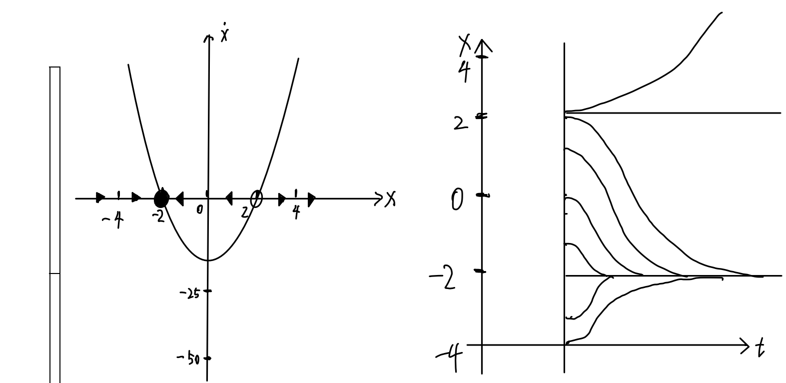
Task 1: pages 36-40 ex. 2.2.1, 2.2.3, 2.2.8, 2.4.1, 2.4.3;
Task 2: pages 79-83 ex. 3.1.1, 3.2.3, 3.4.1, 3.4.5, 3.4.6;
Task 3: pages 141-144 ex. 5.1.7, 5.2.3, 5.2.5, 5.3.4;
Task 4: pages 182-184 ex. 6.3.1, 6.3.3, 6.4.2.

2.2 Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of x(t) for different initial conditions. Then try for a few minutes to obtain the analytical solution for x(t); if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1 $\dot{x} = 4x^2 - 16$



$\therefore x^* = -2$ (stable)

$x^* = 2$ (unstable).

$\dot{x} = 4x^2 - 16.$

$$\frac{dx}{dt} = 4x^2 - 16$$

$$dt = \frac{1}{4x^2 - 16} dx$$

$$\int dt = \int \frac{1}{4x^2-16} dx$$

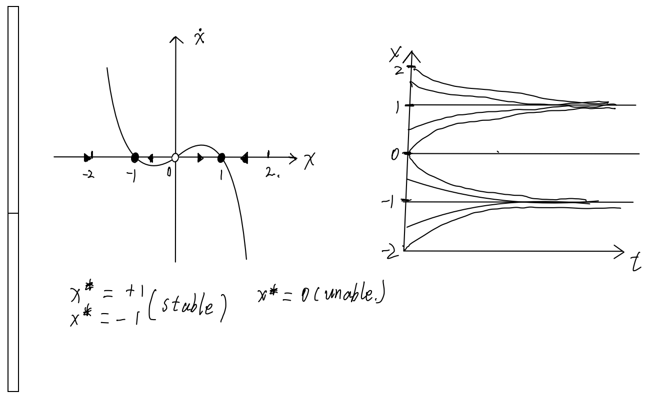
$$4t + C = \frac{1}{4} \ln\left(\frac{x-2}{x+2}\right)$$

$$\therefore x(t) = \frac{-2(1 + Ce^{16t})}{1 - Ce^{16t}}$$

$$C = \frac{x-2}{x+2}.$$

(a) 2.2.1

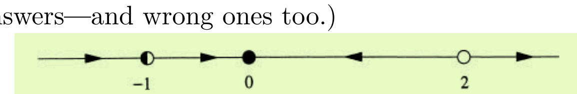
2.2.3 $\dot{x} = x - x^3$



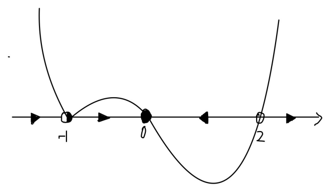
$$\dot{x} = x - x^3$$
$$\int dt = \int \frac{1}{x - x^3} dx$$
$$\int dt = \int \frac{1}{x(1-x^2)} dx$$
$$t + c_1 = \int \frac{1}{x} dx + \frac{1}{2}\int \frac{1}{x+1} dx - \frac{1}{2}\int \frac{1}{x-1} dx$$
$$t + c_1 = -\frac{1}{2}\ln(x^2-1) + \ln(x)$$
$$2t + 2c_1 = \ln\left(\frac{x^2}{1-x^2}\right)$$
$$e^{2t+2c_1} = \frac{x^2}{1-x^2}$$
$$\therefore x(t) = \pm \frac{c_1 e^t}{\sqrt{c_1^2 e^{2t} + 1}}$$

(b) 2.2.3

2.2.8 (Working backwards, from flows to equations) Given an equation , we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too.)

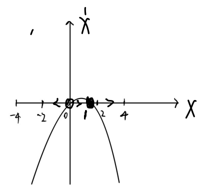


$$(x + 1)^2 x(x - 2)$$



2.4 Linear Stability Analysis Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because f (x*) = 0, use a graphical argument to

2.4.1 $\dot{x} = x(1 - x)$



from the graph get $x_1^* = 0$ (unstable) $x_2^* = 1$ (stable)

Let $\dot{x} = x(1-x) = 0$

$x_1^* = 0 \quad x_2^* = 1$

Due to $f'(x) = 1 - 2x$

$\therefore f'(x_1^*) = 1 - 2x_1^*$

$= 1$

$\therefore f'(x_2^*) = 1 - 2x_2^*$

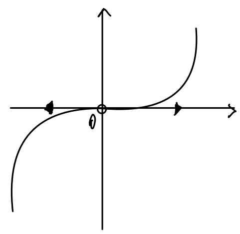
$= -1$

$\therefore x_1^* = 0$ (unstable)

$x_2^* = 1$ (stable)

(c) 2.4.1

2.4.3 $\dot{x} = tan(x)$



from the graph $x^* = 0$ (unstable).

Let $\dot{x} = \tan(x) = 0$.

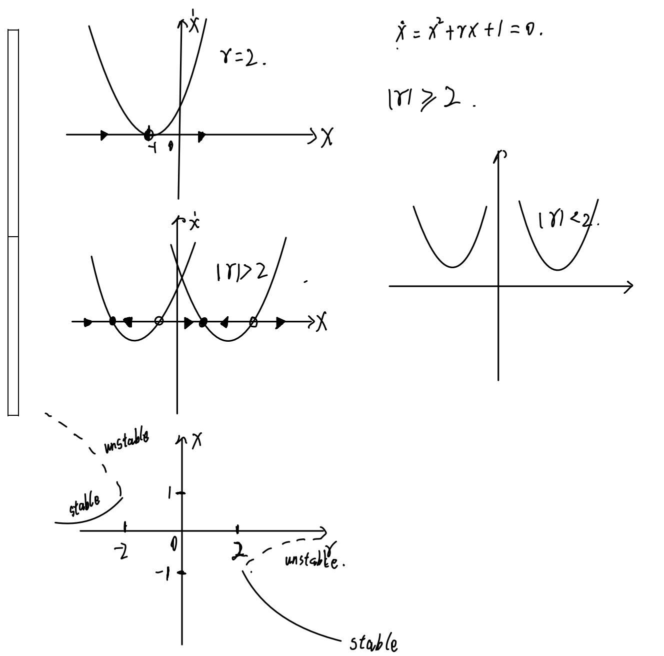
get $x^* = 0$.

Due to $f'(x^*) = \sec^2(x) = 1$.

$\therefore$ $x^* = 0$ is unstable.

(d) 2.4.3

3.1.1 $\dot{x} = 1 + rx + x^2$

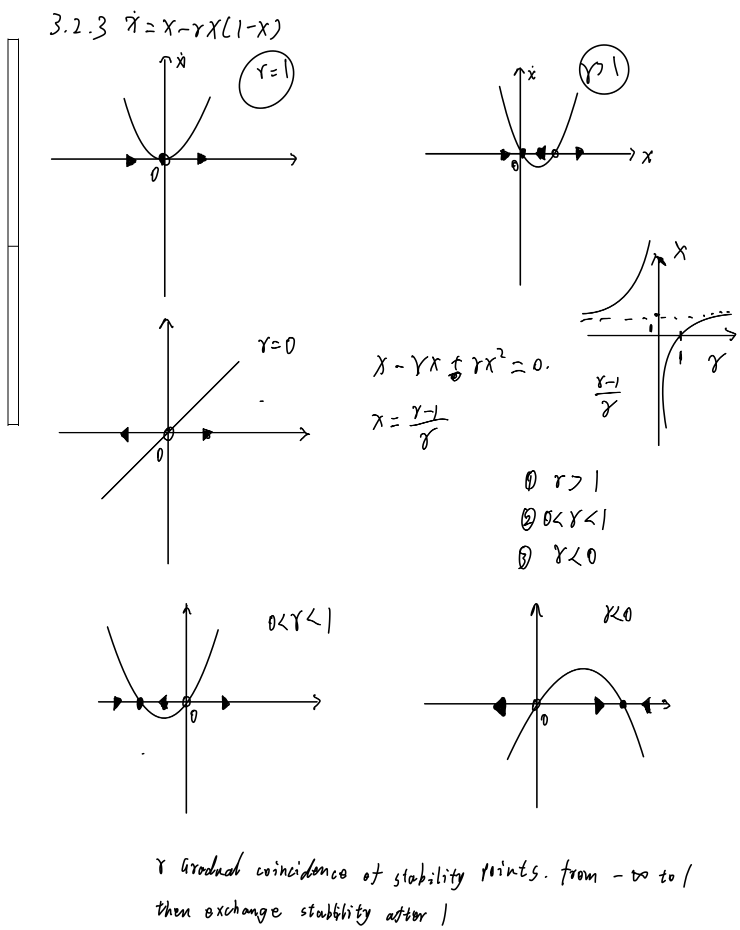


(e) 3.1.1

3.2 Transcritical Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x* vs. r.

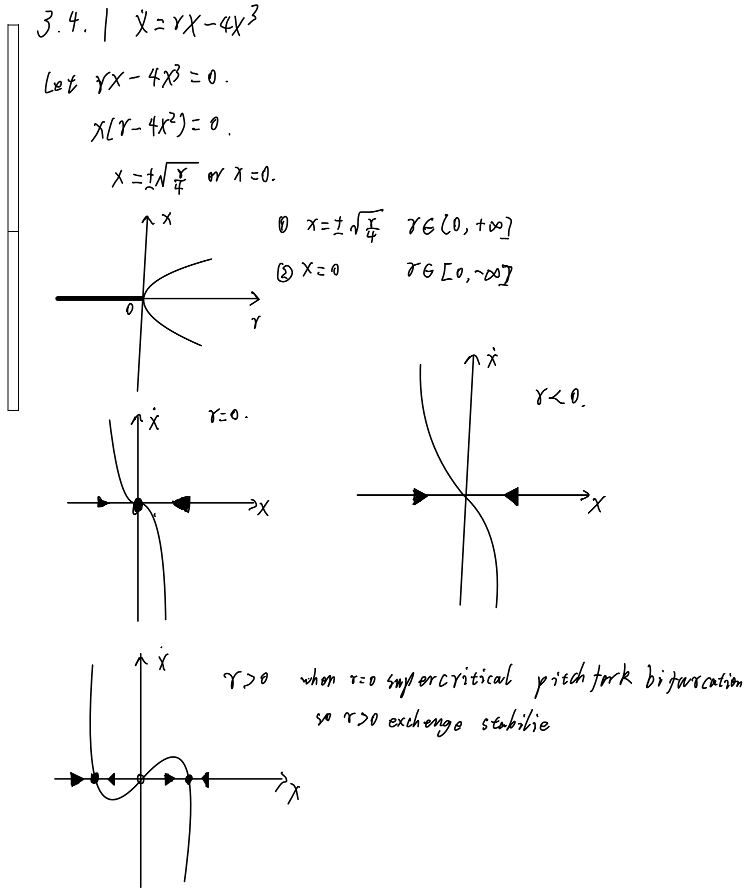
3.2.3 $\dot{x} = xrx(1x)$



(f) 3.2.3

3.4 Pitchfork Bifurcation In the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x* vs. r.

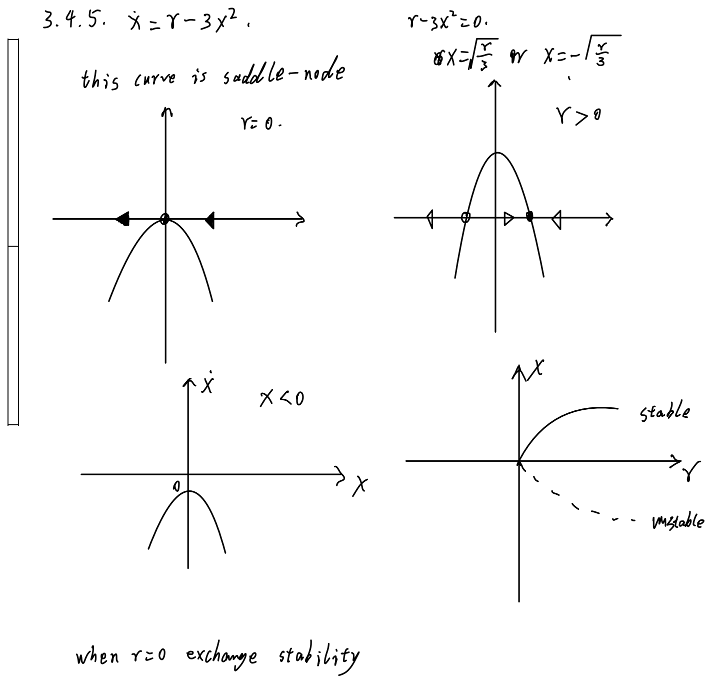
3.4.1 $\dot{x} = rx - 4x^3$



(g) 3.4.1

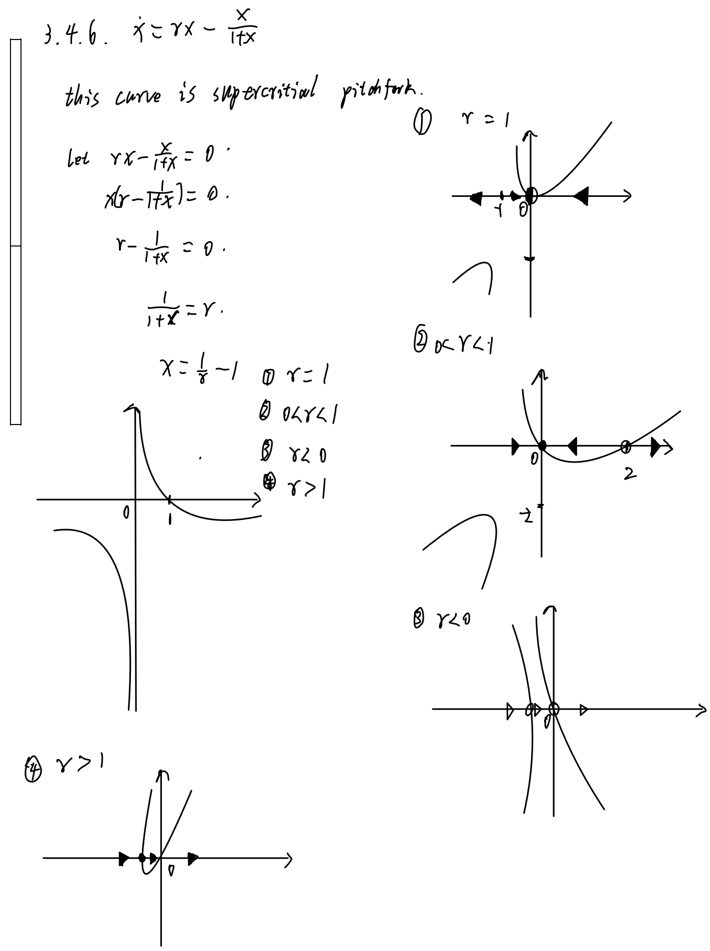
The next exercises are designed to test your ability to distinguish among the various types of bifurcations—it's easy to confuse them! In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x * vs. r.

3.4.5 $\dot{x} = r - 3x^2$



(h) 3.4.5

3.4.6 $\dot{x} = rx - x/(1+x)$



(i) 3.4.6

5

Sketch the vector field for the following systems. Indicate the length and direction of the vectors with reasonable accuracy. Sketch some typical trajectories.

5.1.7 $\dot{x} = x, \dot{y} = x + y$

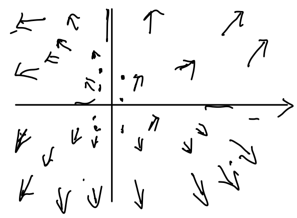
5.1.7

$\dot{x} = x$

$\dot{y} = x + y$

$x(t) = C_1 e^t$

$y(t) = C_1 e^t t + C_2 e^t$



(j) 5.1.7

5.2.3 $\dot{x} = y, \dot{y} = 2x - 3y$

5.2.3. $\dot{x} = y$

$\dot{y} = -2x - 3y.$

$A_x = \begin{pmatrix} 0, & 1 \\ -2 & -3 \end{pmatrix}$

$\tau = -3.$

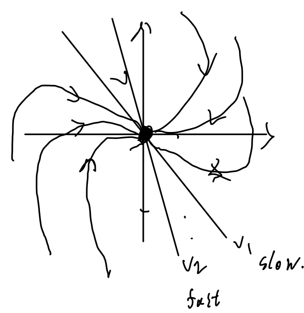
$\Delta = 2$

$\lambda_{1,2} = \dfrac{-3 \pm \sqrt{(-3)^2 - (4 \times 2)}}{2}$

$= \dfrac{-3 \pm 1}{2}$

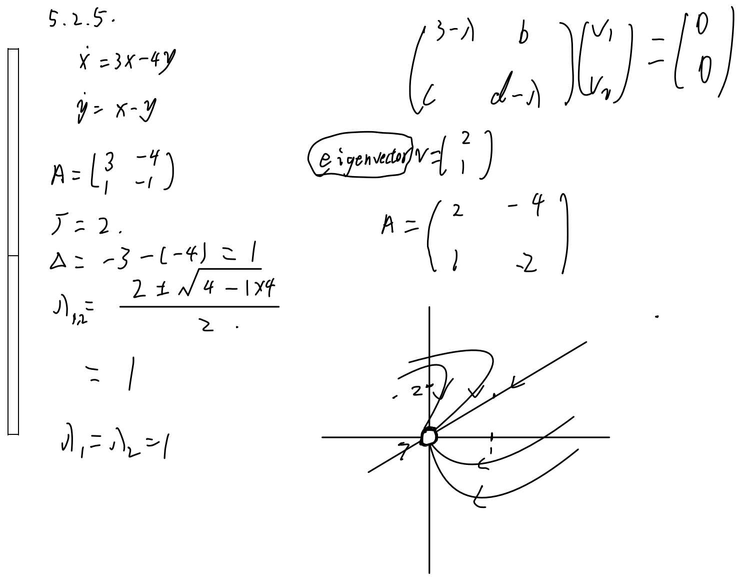
$\lambda_2 = -2 \quad \lambda_1 = -1$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$



(k) 5.2.3

5.2.5 $\dot{x} = 3x - 4y, \dot{y} = x - y$



(l) 5.2.5

5.3.4 $analyze \dot{R} = aJ, \dot{J} = bR$

5.3.4 Analyze $\dot{R} = aJ$ $\dot{J} = bR$.

$\dot{R} = aJ$

$\dot{J} = bR$.

$A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$

$\tau = 0$

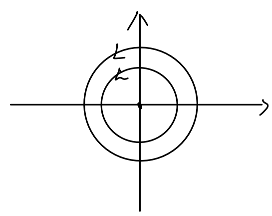
$\Delta = -ab > 0$.

$\tau^2 - 4\Delta = -4ab$

$\lambda_{1,2} = \dfrac{0 \pm \sqrt{0^2 - 4ab}}{2} = \pm\sqrt{ab}$

$\because \alpha = \tau/2$ $\quad w = \sqrt{4\Delta - \tau^2}/2$.

$\therefore \alpha = 0$.



(m) 5.3.4

6.3 For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

6.3.1 $\dot{x} = x - y, \dot{y} = x^2 - 4$

6.3.1 $\dot{x} = x - y$ $\dot{y} = x^2 - 4$

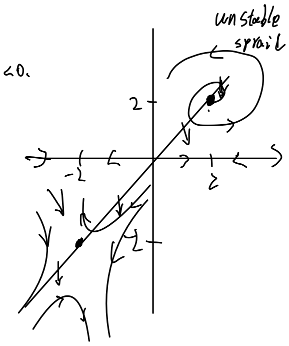
let $\dot{x} = 0$, $\dot{y} = 0$ get $y = x = \pm 2$.

$A = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$

$A_1 = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$ $\Delta = 4$ $\tau^2 - 4\Delta < 0$. $\tau = 1$

$A_2 = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$ $\Delta = -4$ $\tau = 1$

$\lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2}$



(n) 6.3.1

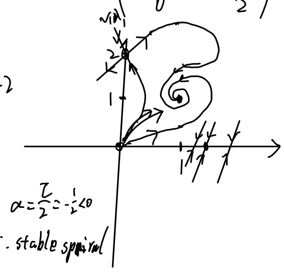
6.4.2 $\dot{x} = x(3 - 2x - y), \dot{y} = y(2 - x - y)$

6.4.2. $\dot{x} = x(3-2x-y) = 0$, get $(0,2)$, $(1,1)$ $(\frac{3}{2},0)$ $(0,0)$,

$\dot{y} = y(2-x-y) = 0$,

$$A = \begin{pmatrix} -2x-y+3 & -x \\ -y & -x-2y+2 \end{pmatrix} \qquad (0,0)\ A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \ \lambda = 3, 2$$

$$(0,2)\ A = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \ \lambda = 1, -2$$



$$(1,1)\ A = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \quad \tau = -1 \quad \Delta = 1 \quad \alpha = \frac{\tau}{2} = -\frac{1}{2} < 0$$

$\therefore$ stable spiral

$$(\tfrac{3}{2},0)\ A = \begin{pmatrix} 0 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \quad \tau = -\frac{1}{2} \quad \Delta = 0$$

$\lambda = 1, 0.$

(o) 6.4.2