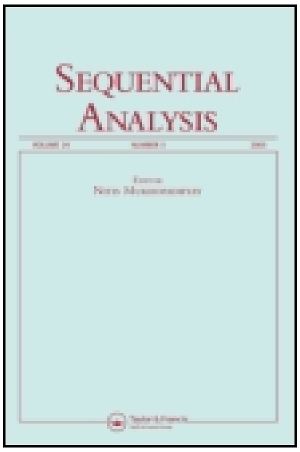

Sequential Analysis: Design Methods and Applications
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lsqa20

SEQUENTIAL METHODS FOR DETECTING CHANGES IN THE VARIANCE OF ECONOMIC TIME SERIES

Stefan Schipper [a] & Wolfgang Schmid [a]

[a] Department of Statistics, Europe-University, P.O. Box 1786, Frankfurt (Oder), 15207, Germany

Version of record first published: 24 Aug 2007.

To cite this article: Stefan Schipper & Wolfgang Schmid (2001): SEQUENTIAL METHODS FOR DETECTING CHANGES IN THE VARIANCE OF ECONOMIC TIME SERIES, Sequential Analysis: Design Methods and Applications, 20:4, 235-262

To link to this article: http://dx.doi.org/10.1081/SQA-100107647

SEQUENTIAL ANALYSIS, 20(4), 235–262 (2001)

SEQUENTIAL METHODS FOR DETECTING CHANGES IN THE VARIANCE OF ECONOMIC TIME SERIES

Stefan Schipper and Wolfgang Schmid

Europe-University, Department of Statistics,
P.O. Box 1786, 15207 Frankfurt (Oder), Germany

ABSTRACT

In this paper EWMA charts and CUSUM charts are introduced for detecting changes in the variance of a GARCH process. The moments of the EWMA statistics are calculated. They permit a better understanding of the underlying control procedure. In an extensive simulation study all control schemes are compared with each other. "Optimal" smoothing parameters and "optimal" reference values are tabulated. It is shown how these charts can be applied to monitor stock market returns.

Key Words: Control charts; Variance changes; Garch processes; Statistical process control; Time series analysis

1 INTRODUCTION

One of the main aims of statistical process control is to monitor whether an observed process coincides with a supposed target process. Each new sample has to be examined. Can it reasonably be explained by the probability distribution of the target process or not? The procedure is a sequential one, the observations are investigated step by step. Such problems can be found in all areas of science.

Several authors showed that control charts for independent variables like for instance the Shewhart, the EWMA (exponentially weighted moving average), and the CUSUM (cumulative sum) scheme cannot be directly applied to time series (e.g. Montgomery and Mastrangelo (1991)). It is necessary to take the structure of the time series into account. In the last ten years several control charts for time series have been introduced. In principle it can be distinguished between residual charts and modified control charts. Residual charts make use of a transformation of the data. The aim is to derive statistics (residuals) which are again independent variables since then it is possible to apply well-known methods to these quantities (e.g. Alwan and Roberts (1988), Harris and Ross (1991)). Modified control schemes are based on similar statistics as the classical procedures for independent samples. Therefore they can be regarded as direct extensions (e.g. Yashchin (1993), Schmid (1997), Kramer and Schmid (1998)). In most studies the target process was assumed to be an autoregressive moving average (ARMA) process.

In this paper the target process is mainly a GARCH (generalized autoregressive conditional heteroskedasticity) process. These processes have been widely discussed in economical literature (see Bollerslev et al. (1992), Gourieroux (1997)). Contrary to ARMA processes their conditional variance is not constant. For that reason they are able to describe a behaviour which can frequently be observed in economics — periods of large fluctuations alternate with relatively quiet phases. Control charts for GARCH processes were introduced by Severin and Schmid (1999). They dealt with the detection of shifts in the mean.

Here, we focus on detecting changes in the variance. This problem is of great interest in practice since the variance measures the risk of an asset. In Figure 1 the returns of the Siemens share are plotted from November 2, 1995 to October 10, 1997. The returns are calculated as $R_t = 100 \ln(K_t/K_{t-1})$ where K_t denotes the value of the share at time t. On November 7, 1996 the price goes down from 80.2 DM to 73.27 DM. It is interesting that after November 7, 1996 the variance is greater than before. Figure 2 illustrates the historical sample variances when the window width is 20 days. The horizontal lines are the medians of the sample variances determined by the observations within a period of "equal" variance. This figure shows that there are shifts in the variance of the process. Change-points in the variance are also described by Hsu (1977, 1979) and Chen and

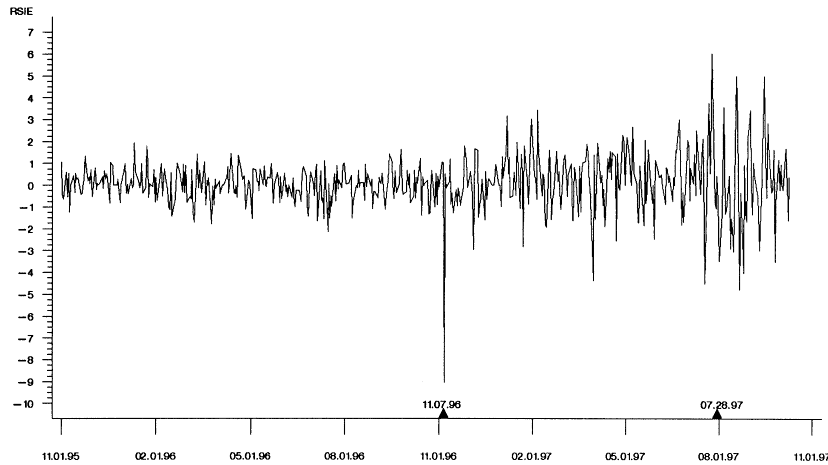

Figure 1. Returns of the Siemens share (in %) from November 2, 1995 to October 10, 1997.

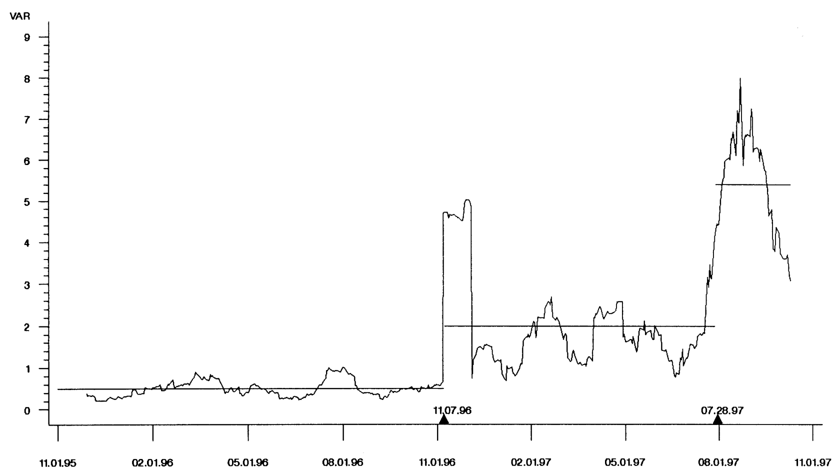

Figure 2. 20-day historical sample variances (in $\%^2$) of the Siemens share from November 2, 1995 to October 10, 1997.

Gupta (1997) who analyzed the U.S. stock market returns during the period 1971–1974. Contrary to their papers we choose a sequential approach. Moreover, the underlying variables are not assumed to be independent and normally distributed.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

In Section 2 a brief introduction into GARCH processes is given. It is explained how in our paper the target process and the observed process are related with each other.

In Section 3 several control charts for the detection of changes in the variance of a GARCH process are proposed. The idea of exponentially weighted averaging is applied to the squared observations, the logarithm of the squared observations, the conditional variance and the residuals. Several characteristic quantities of the EWMA statistics are calculated (Theorems 1 to 3). They permit a better understanding of the underlying control procedure. For these statistics the cumulative sums are calculated, too. It is studied how the designs of the charts have to be chosen. Statements about the choice of the reference values are given.

In an extensive simulation study these control schemes are compared with each other (Section 4). We present the results for two GARCH processes which reflect typical situations. As a measure for the performance of a control scheme the average run length (ARL) is taken. The run length of a control scheme is equal to the first sample at which it is concluded to be out-of-control. The ARL is defined as the expectation of the run length. All charts are calibrated such that the in-control ARL is the same if no change is present. Their behaviour is analyzed with respect to a scale deviation. In principle it turns out that no chart is the best overall. However, the EWMA attempt based on the conditional variance provides in nearly all cases the minimal out-of-control ARL. A small smoothing parameter should be chosen, for instance $\lambda = 0.1$.

In Section 5 these results are applied to the returns of the Deutsche Bank share. A brief description of some properties of GARCH processes, tables of the critical values, and the proofs of the theorems are given in the appendix.

2 MODELING STRUCTURAL DEVIATIONS

We distinguish between the target process $\{Y_t\}$ and the observed process $\{X_t\}$. Suppose that sequentially data $x_1, x_2, \ldots$ are taken from a quantity of interest. The data are realizations of the observed process $\{X_t\}$. Each observation is examined whether it can reasonably be explained by the distribution law of $\{Y_t\}$ or not. While the distribution of $\{X_t\}$ is unknown the distribution of $\{Y_t\}$ is assumed to be known. For instance it can be obtained by fitting a suitable process to historical data of the characteristic. Here, in most cases the target process is assumed to be a GARCH process (see Section 2.1).

Analyzing the returns of stocks it can be seen that frequently outliers and shifts in the variance appear. Because outliers can be explained by an increase of the variance we focus on the detection of changes in the variance of $\{Y_t\}$. It is natural to consider the variance since this quantity reflects the risk of an asset (e.g. Ingersoll (1987)). We propose control charts which are especially able to detect such types of deviations. In order to compare these charts it is necessary to introduce a relationship between both processes. This is done in Section 2.2.

2.1 The Target Process

Engle (1982) and Bollerslev (1986) introduced the concept of GARCH models. These processes are widely applied to model financial time series (cf. Bollerslev et al. (1992), Gourieroux (1997)). Their great advantage is the ability to describe variability over time. Their variance is constant, but not the conditional variance. The conditional variance of a GARCH process depends on previous observations. It follows an ARMA process. This is a great advantage in relation to pure ARMA modeling.

A stochastic process $\{Y_t\}$ is called a GARCH(p, q) process if

$$Y_t = \varepsilon_t \, \eta_t \tag{1}$$

with $\eta_t > 0$ and

$$\eta_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \, Y_{t-i}^2 + \sum_{j=1}^{p} \beta_j \, \eta_{t-j}^2 \tag{2}$$

for $t \in \mathbb{Z}$. It is assumed that $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ for all i, j. Moreover, the random variables $\{\varepsilon_t\}$ are supposed to be independent with $E(\varepsilon_t) = 0$ and $\mathrm{Var}\,(\varepsilon_t) = 1$.

Bollerslev (1986) proved that $\{Y_t\}$ is weakly stationary if the coefficients satisfy the condition

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1. \tag{3}$$

It follows that $E(Y_t) = 0$, $\mathrm{Cov}\,(Y_t, Y_s) = 0$ for $t \neq s$ and

$$\mathrm{Var}\,(Y_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j} \; = \; \gamma_0 \, .$$

In the following we will always assume that the GARCH process is weakly stationary. Several properties of GARCH processes are presented in the appendix. They are of importance for the further analysis.

2.2 Modeling the Observed Process by a Scale Variation

The variance measures the risk of an asset. For that reason it is a fundamental quantity for a portfolio manager. As the returns of an asset react very sensibly on new information it is not surprising that changes in the variance of stock market returns can frequently be observed.

First let $\{Y_t\}$ be an arbitrary process with mean μ_0 and variance γ_0. The observed process $\{X_t\}$ is generated from the target process by a scale transform, more precisely

$$X_t = \begin{cases} Y_t & \text{for } 1 \leq t < \tau \\ \mu_0 + \Delta(Y_t - \mu_0) & \text{for } t \geq \tau \end{cases} \tag{4}$$

with $\Delta \geq 1$ and $\tau \in \mathbb{N}$. Thus a change in the scale appears at position τ if $\Delta > 1$. We say that the process $\{X_t\}$ is out-of-control. If $\Delta = 1$ it is called in-control. It holds for the observed process that $E(X_t) = \mu_0$, $\mathrm{Var}\,(X_t) = \gamma_0$ for $t < \tau$, and $\mathrm{Var}\,(X_t) = \Delta^2\gamma_0$ for $t \geq \tau$.

3 MODIFIED CONTROL CHARTS FOR THE VARIANCE

Control charts are an important tool in statistical process control. They can be used to detect structural deviations from a target process. Up to now most authors considered control charts for the mean of a time series. Papers about variance changes for correlated data can rarely be found. Here we propose several control schemes for the variance of a time series. As in the last section $\{X_t\}$ denotes the observed process. Let $\{Y_t\}$ be the target process with mean μ_0 and variance γ_0. μ_0 and γ_0 are assumed to be known. γ_0 is called the target value.

3.1 EWMA Type Charts

EWMA charts have been widely discussed in literature, however, nearly exclusively assuming independent samples. A detailed comparison of simultaneous EWMA charts for the mean and the variance is given by Reynolds and Stoumbos (2000). EWMA charts for detecting changes in the mean of a stationary process were considered by Schmid (1997), Schöne and Schmid (1997) and Severin and Schmid (1999).

3.1.1 A Chart Based on Squared Observations

Applying the EWMA recursion to the squared observations we get

$$Z_t = (1-\lambda)Z_{t-1} + \lambda\,(X_t - \mu_0)^2 \tag{5}$$

for $t \geq 1$, $\lambda \in (0, 1]$. As a starting value for Z_0 we choose the in-control value of $\mathrm{Var}\,(X_t)$ which is equal to γ_0. This chart was discussed by MacGregor and Harris (1993) for an ARMA(1,1) process. Contrary to their study we deal with more general processes. Especially we cover the field of GARCH processes. Moreover, we analyze the run length behaviour.

Solving the difference Equation (5) leads to

$$\begin{aligned} Z_t &= \lambda \sum_{i=0}^{t-1}(1-\lambda)^i (X_{t-i}-\mu_0)^2 \;+\; (1-\lambda)^t\,\gamma_0 \\ &= \lambda \sum_{i=0}^{t-1}(1-\lambda)^i ((X_{t-i}-\mu_0)^2 - \mathrm{Var}\,(X_{t-i})) \\ &\quad + \gamma_0 \;+\; \gamma_0\,(\Delta^2-1)\,(1-(1-\lambda)^{t-\tau+1})\;I_{\{\tau,\,\tau+1,..\}}(t) \end{aligned}$$

Theorem 1 a) The expected value of Z_t is

$$\begin{aligned} E(Z_t) &= \gamma_0 + \gamma_0\,(\Delta^2-1)\,(1-(1-\lambda)^{t-\tau+1})\;I_{\{\tau,\,\tau+1,..\}}(t) \\ &\to \Delta^2\,\gamma_0 \quad \text{for} \quad t\to\infty. \end{aligned}$$

b) Let $\{(Y_t-\mu_0)^2\}$ be weakly stationary with autocovariance function $\{\delta_h\}$. Then it follows for $\tau = 1$ that

$$\mathrm{Var}\,(Z_t) \;=\; \Delta^4\,\lambda^2 \sum_{v=-(t-1)}^{t-1} \delta_v \sum_{i=\max\{0,\,-v\}}^{\min\{t-1-v,\,t-1\}} (1-\lambda)^{2i+v}\,.$$

If additionally $\{\delta_h\}$ is absolutely summable then

$$\lim_{t\to\infty} \mathrm{Var}\,(Z_t) = \Delta^4\,\frac{\lambda}{2-\lambda}\sum_{v=-\infty}^{\infty}\delta_v\,(1-\lambda)^{|v|} \tag{6}$$

$$= \Delta^4\,\lambda^2 \int_{-\pi}^{\pi} f_{Y^2}(x)\,\frac{1}{1-2\,(1-\lambda)\,\cos x + (1-\lambda)^2}\,dx\,. \tag{7}$$

f_{Y^2} denotes the spectral density of the weakly stationary process $\{(Y_t-\mu_0)^2\}$.

c) Let $\{Y_t\}$ be a GARCH(1,1) process with $\varepsilon_t \sim F$ and $E(\varepsilon_t^4) = \kappa < \infty$ for all t. Suppose that $\kappa\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$ then we get in the case $\tau = 1$

$$\lim_{t\to\infty} \operatorname{Var}(Z_t) = (\kappa - 1)\, \Delta^4\, \frac{\lambda}{2-\lambda}\, \gamma_0^2 \times \frac{1 - \beta_1^2 - 2\alpha_1\beta_1 + (1-\lambda)(\alpha_1 - \beta_1 + \beta_1^2(\alpha_1 + \beta_1))}{(1-(1-\lambda)(\alpha_1+\beta_1))(1-\beta_1^2 - 2\alpha_1\beta_1 - \kappa\alpha_1^2)}.$$

Remark. 1. In the in-control state $\Delta = 1$ we have $E(Z_t) = \gamma_0$.

2. Now let the target process $\{Y_t\}$ be a two-sided moving average with mean μ_0, i.e. $Y_t = \mu_0 + \sum_{i=-\infty}^{\infty} a_i\, \varepsilon_{t-i}$ with $\{a_i\}$ absolutely summable. Let $\{\varepsilon_t\}$ be independent and normally distributed with mean 0 and variance σ_ε^2. Then $\delta_h = \gamma_0^2 + 2\gamma_h^2$ (see Brockwell and Davis (1991, p. 227)) where $\{\gamma_h\}$ denotes the autocovariance function of $\{Y_t\}$. For instance, for a weakly stationary ARMA(1,1) process $Y_t = \alpha Y_{t-1} + \varepsilon_t + \beta\varepsilon_{t-1}$ we get that

$$\lim_{t\to\infty} \operatorname{Var}(Z_t) = \Delta^4\, \frac{1}{2-\lambda} \left(2\gamma_0^2 + \lambda\gamma_0^2 + 4\gamma_1^2\, \frac{\lambda(1-\lambda)}{1-(1-\lambda)\alpha^2} \right)$$

with

$$\gamma_0 = \sigma_\varepsilon^2\, \frac{1 + 2\alpha\beta + \beta^2}{1-\alpha^2} \qquad \text{and} \qquad \gamma_1 = \sigma_\varepsilon^2\, \frac{(1+\alpha\beta)(\alpha+\beta)}{1-\alpha^2} \tag{8}$$

since

$$\delta_h = \begin{cases} 3\gamma_0^2 & \text{for } h = 0 \\ \gamma_0^2 + 2\alpha^{2(h-1)}\gamma_1^2 & \text{for } h \geq 1. \end{cases}$$

3. For arbitrary GARCH(p,q) processes the integral in Equation (7) can be easily determined by numerical integration for given values of $\alpha_0, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_p$ and λ.

For applications it is easier to apply the asymptotic variance instead of the exact one. In that case it is sufficient to calculate the control limits once. It is concluded that the process $\{X_t\}$ is out-of-control at time t if

$$(Z_t - E_1(Z_t)) \Big/ \sqrt{\lim_{t\to\infty} \operatorname{Var}_1(Z_t)} \; > \; c \tag{9}$$

where $E_1(Z_t)$ and $\operatorname{Var}_1(Z_t)$ are the in-control values ($\Delta = 1$) of $E(Z_t)$ and $\operatorname{Var}(Z_t)$, respectively. $c > 0$ is a constant. The upper control limit is given by

$$\mathrm{UCL} = \gamma_0 + c\sqrt{\lim_{t\to\infty} \operatorname{Var}_1(Z_t)}. \tag{10}$$

Putting $\lambda = 1$ this scheme reduces to the Shewhart chart. The run length of the EWMA chart is $N = \inf\{t \in \mathbb{N}\colon Z_t > \mathrm{UCL}\}$. c is chosen such that the in-control ARL $E_1(N)$ is equal to a specified value. Sometimes it is easier to write $\mathrm{UCL} = c^\star \gamma_0$. If $\{Y_t\}$ is a GARCH process then the quantities c and $c^\star$ do not depend on α_0. Note that the ARL is not a function of α_0, too. These results can be obtained by the same arguments as given in Severin and Schmid (1999).

All EWMA charts treated in this paper are based on similar decision rules as in (9). However, the EWMA recursion is applied to distinct statistics. The control limits vary. In each case c and $c^\star$ have to be calculated again. The determination of the average run lengths turns out to be complicated. For independent variables it can be calculated numerically by solving an integral equation (see Crowder (1987)). For GARCH processes no explicit formula is known. Therefore we use simulations.

3.1.2 A Chart Based on the Logarithm of the Squared Observations

Let us look closer at the model introduced in Equation (4). If we square both sides of the equation and then take the logarithm we get

$$\ln \frac{(X_t - \mu_0)^2}{\gamma_0} = \ln \frac{(Y_t - \mu_0)^2}{\gamma_0} + 2(\ln \Delta)\, I_{\{\tau, \tau+1, ..\}}(t). \tag{11}$$

A change point model for the location is present. We apply the EWMA recursion to the left side of equation (11)

$$Z_t = (1-\lambda) Z_{t-1} + \lambda \ln \frac{(X_t - \mu_0)^2}{\gamma_0}$$

for $t \geq 1$ with $Z_0 = E(\ln (Y_t - \mu_0)^2/\gamma_0) = \gamma_0^\star$. As no explicit formula for the distribution of a GARCH process is known up to now the quantity $\gamma_0^\star$ has to be determined by simulations. For independent variables this chart was discussed by Crowder and Hamilton (1992).

From the arguments given above we conclude that

$$\begin{aligned} E(Z_t) &= \gamma_0^\star + 2\ln\Delta(1-(1-\lambda)^{t-\tau+1})\, I_{\{\tau,\tau+1,..\}}(t) \\ &\to \gamma_0^\star + 2\ln\Delta = E\left(\ln \frac{(X_t-\mu_0)^2}{\gamma_0}\right) \end{aligned}$$

for $t \to \infty$. In the in-control state we have $E(Z_t) = \gamma_0^\star$.

Suppose that $\{\ln(Y_t - \mu_0)^2\}$ is a weakly stationary process. If $\{\delta_v^\star\}$ denotes its autocovariance function then the asymptotic variance of $\mathrm{Var}(Z_t)$ can be determined with Formula (6) but with $\delta_v^\star$ instead of δ_v. However, the calculation of $\delta_v^\star$ turns out to be complicated.

Thus a signal is given at time t if $Z_t > \gamma_0^\star + c\sqrt{\lim_{t\to\infty}\mathrm{Var}_1(Z_t)} = c^\star$. For GARCH processes c and $c^\star$ do not depend on α_0

3.1.3 A Chart Based on the Conditional Variance

In this section let $\{Y_t\}$ be a weakly stationary GARCH process with existing fourth moments. For $t \geq 1$ let $\hat{\eta}_t^2$ be the best linear predictor of Y_t^2 based on $Y_{t-1}^2, \ldots, Y_1^2, 1$ (see Appendix 7.1). Then $\hat{\eta}_t^2$ is a linear function of $Y_{t-1}^2, \ldots, Y_1^2, 1$, i.e.

$$\hat{\eta}_t^2 = a_{tt} + \sum_{i=1}^{t-1} a_{ti} Y_{t-i}^2 \tag{12}$$

with certain coefficients $a_{t1}, \ldots, a_{tt}$. Because $\mathrm{Cov}(Y_t^2 - \hat{\eta}_t^2, 1) = 0$ we obtain that $E(\hat{\eta}_t^2) = E(Y_t^2) = \gamma_0$ and thus $a_{tt} = \gamma_0(1 - \sum_{i=1}^{t-1} a_{ti})$. We consider the best linear predictor of X_t^2 in terms of the observed values $X_{t-1}^2, \ldots, X_1^2, 1$. Unfortunately it cannot be determined in practice since this quantity depends on Δ. Therefore it is calculated assuming that $\Delta = 1$ (in-control). This expression is denoted by $\hat{\sigma}_t^2$. Consequently we have

$$\begin{aligned}\hat{\sigma}_t^2 &= a_{tt} + \sum_{i=1}^{t-1} a_{ti} X_{t-i}^2 \qquad (13)\\ &= \begin{cases} \Delta^2\,\hat{\eta}_t^2 - (\Delta^2 - 1)(a_{tt} + \sum_{i=t-\tau+1}^{t-1} a_{ti}\, Y_{t-i}^2) & \text{for } t > \tau \\ \hat{\eta}_t^2 & \text{for } t \leq \tau \end{cases}.\end{aligned}$$

As $\{Y_t^2\}$ is an ARMA process (cf. Appendix 7.1) we may apply the results given in Section 5 of Brockwell and Davis (1991). Using these methods we can determine $\hat{\eta}_t^2$ and $\hat{\sigma}_t^2$ recursively. For instance we get for a GARCH(1,1) process that for $t \geq 2$

$$\hat{\sigma}_t^2 = \gamma_0 + (\alpha_1 + \beta_1)(X_{t-1}^2 - \gamma_0) - \beta_1\,(X_{t-1}^2 - \hat{\sigma}_{t-1}^2)/r_{t-1}\,, \tag{14}$$

$$r_t = 1 + \beta_1^2 - \beta_1^2/r_{t-1}\,, \tag{15}$$

and

$$\hat{\sigma}_1^2 = \gamma_0, \qquad r_1 = 1 + \frac{\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2}\,. \tag{16}$$

In the next theorem we calculate the first two moments of $\hat{\sigma}_t^2$. It shows that for $\Delta > 1$ the quantity $\hat{\sigma}_t^2$ underestimates the variance in the out-of-control state while for $\Delta < 1$ it is overestimated. It is important that the expectation of the quantities given in Equations (13) and (14) can be calculated without demanding that the fourth moments of $\{Y_t\}$ exist. However, these quantities are only equal to the best predictor if the fourth moments are finite.

Theorem 2 a) Let $\{Y_t\}$ be a weakly stationary GARCH process. Then it holds that

$$E\left(\hat{\sigma}_t^2\right) = \begin{cases} \Delta^2\,\gamma_0 - (\Delta^2 - 1)\left(a_{tt} + \gamma_0 \sum_{i=t-\tau+1}^{t-1} a_{ti}\right) & \text{for } t > \tau \\ \gamma_0 & \text{for } t \le \tau \end{cases}.$$

If the fourth moments of $\{Y_t\}$ exist then it follows that for $\tau = 1$

$$\operatorname{Cov}\left(\hat{\sigma}_t^2, \hat{\sigma}_s^2\right) = \begin{cases} \Delta^4 \operatorname{Cov}(\hat{\eta}_t^2, \hat{\eta}_s^2) & \text{for } t, s \ge 2 \\ 0 & \text{for } t = 1 \vee s = 1 \end{cases}.$$

b) Let $\{Y_t\}$ be a weakly stationary GARCH(1,1) process. Then it holds that for $t > \tau$

$$E\left(\hat{\sigma}_t^2\right) - \Delta^2\,\gamma_0 = \frac{\beta_1}{r_{t-1}}\,(E(\hat{\sigma}_{t-1}^2) - \Delta^2\,\gamma_0) - \alpha_0(\Delta^2 - 1). \tag{17}$$

Moreover,

$$\lim_{t\to\infty} E\left(\hat{\sigma}_t^2\right) - \operatorname{Var}(X_t) = -(\Delta^2 - 1)\,\frac{\alpha_0}{1 - \beta_1}. \tag{18}$$

Let $\{Y_t^2\}$ be weakly stationary, too. Let $\sigma^2 = \operatorname{Var}(Y_t^2 - \eta_t^2)$. It holds that for $\tau = 1$ and $2 \le s \le t$

$$\operatorname{Cov}\left(\hat{\sigma}_t^2, \hat{\sigma}_s^2\right) = \Delta^4\,\sigma^2\,(\alpha_1 + \beta_1)^{t-s}(r_1 - r_s)\,.$$

We propose a control scheme based on the observed conditional variance $\hat{\sigma}_t^2$. Of course our decision at position t is based on $\hat{\sigma}_{t+1}^2$ since we want to judge the observations at time t.

The EWMA statistic for the conditional variance is

$$Z_t = (1 - \lambda)Z_{t-1} + \lambda\,\hat{\sigma}_{t+1}^2$$

for $t \ge 1$, $\lambda \in (0, 1]$. As a starting value for Z_0 we choose the in-control value of $E(\hat{\sigma}_{t+1}^2)$ which is equal to γ_0.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Theorem 3 a) Let $\{Y_t\}$ be a weakly stationary GARCH process. Then it holds that for $\tau = 1$

$$E(Z_t) = \Delta^2\gamma_0 - (\Delta^2 - 1)\gamma_0(1-\lambda)^t - \lambda(\Delta^2 - 1)\sum_{i=0}^{t-1}(1-\lambda)^i a_{t+1-i,\,t+1-i}$$
$$\to \Delta^2\gamma_0 - (\Delta^2 - 1)\,a_\infty \quad \text{for} \quad t \to \infty$$

provided that $a_\infty = \lim_{i\to\infty} a_{ii}$ exists. For a GARCH(1,1) process it follows that $a_\infty = \alpha_0/(1-\beta_1)$.

b) Let $\{Y_t\}$ be a GARCH(1,1) process with $\varepsilon_t \sim F$ and $E(\varepsilon_t^4) = \kappa < \infty$ for all t. If $\kappa\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1$ then it holds for $\tau = 1$ that

$$\lim_{t\to\infty} \mathrm{Var}\,(Z_t) = (\kappa - 1)\,\Delta^4\,\frac{\lambda}{2-\lambda}\,\gamma_0^2\,\frac{\alpha_1^2}{1-\beta_1^2-2\alpha_1\beta_1-\kappa\alpha_1^2} \times \frac{1+(1-\lambda)(\alpha_1+\beta_1)}{1-(1-\lambda)(\alpha_1+\beta_1)}\,.$$

It is interesting that for a GARCH(1,1) process the asymptotic variance of the EWMA recursion based on the conditional variance is smaller than the asymptotic variance of the EWMA statistic based on the squared observations. This result is obtained by comparing Theorem 1c with Theorem 3b.

For applications, the decision rule based on the asymptotic variance must be favored. If

$$Z_t > \gamma_0 + c\sqrt{\lim_{t\to\infty}\mathrm{Var}_1(Z_t)} = c^\star\gamma_0$$

it is decided that the process is out-of-control at time t. The critical values c, $c^\star$, and the ARL do not depend on α_0.

We want to compare the chart based on the conditional variance with the chart based on squared observations. Which chart will be better at all? To get more insight we consider the standardized statistics for both charts. The asymptotic mean is given by

$$\lim_{t\to\infty} E\left(\frac{Z_t - E_1(Z_t)}{\sqrt{\lim_{t\to\infty}\mathrm{Var}_1(Z_t)}}\right). \tag{19}$$

Using Theorem 1c and 3b it can easily be shown for a GARCH(1,1) process that for the chart based on the conditional variance (19) is larger than for the chart based on squared observations provided that $\Delta > 1$. Of course, in the in-control state (19) is always equal to zero. This shows that if we take the same value of c for both schemes then it can be expected that the EWMA chart for the conditional variance provides better results since the influence on the asymptotic mean of the standardized quantity is larger.

3.1.4 A Residual Chart

Here the underlying process is assumed to be a GARCH process. The EWMA statistic for the variance of the residuals is as follows

$$Z_t = (1-\lambda)\,Z_{t-1} + \lambda\,\frac{X_t^2}{\hat{\sigma}_t^2}$$

for $t \geq 1$, $\lambda \in (0,1]$. As a starting value for Z_0 we use $E(\varepsilon_t^2) = 1$.

The residuals $X_t^2/\hat{\sigma}_t^2$, $t \geq 1$ are not independent variables. If no change arises $\hat{\sigma}_t^2$ behaves approximately like η_t^2 if t is large. The residuals are "asymptotically" independent. Because results about the exact mean and the exact variance are difficult to obtain we recommend the use of the decision rule based on the asymptotic considerations. The process is concluded to be out-of-control at time t if $Z_t > c^\star$. $c^\star$ as well as the ARL do not depend on α_0.

3.2 CUSUM Type Charts

The CUSUM chart was introduced by Page (1954). Like the EWMA chart it is a control chart with memory. At each time point the decision is based not only on the present observation, but on previous observations, too. The influence of former values decreases exponentially for the EWMA chart. The CUSUM chart gives all former observations the same weight.

CUSUM charts can be derived by the log likelihood ratio (see Siegmund (1985, Ch. II.6)). For instance, assuming that the variables $\{Y_t\}$ are independent and normally distributed with expectation μ_0 and variance γ_0 the log likelihood ratio for model (4) (τ known) is equal to

$$\frac{1-1/\Delta^2}{2\,\gamma_0}\,(S_n - n\,\gamma_0\,K(\Delta) - (S_{\tau-1} - (\tau-1)\,\gamma_0\,K(\Delta))) \tag{20}$$

for $n \geq \tau$ with $S_n = \sum_{i=1}^{n}(X_i - \mu_0)^2$ and $K(\Delta) = 2 \ln \Delta^{1/(1-1/\Delta^2)}$. It is concluded that the process is out-of-control at time t if $S_t^+ = S_t - t\,\gamma_0\,K(\Delta) - \min_{0\leq v\leq t}(S_v - v\,\gamma_0\,K(\Delta)) \geq c\,\gamma_0$. S_t^+ can be calculated recursively, viz

$$S_t^+ = \max\{0, S_{t-1}^+ + (X_t - \mu_0)^2 - K(\Delta)\,\gamma_0\} \tag{21}$$

for $t \geq 1$ with $S_0^+ = 0$. It is interesting that in (21) a multiple of the expectation of $(X_t - \mu_0)^2$ is subtracted while for CUSUM mean charts a multiple of the standard deviation is subtracted. For scale deviations the formula of the reference value is complicater than for mean shifts where K is equal to the half of the expected shift. Because in our paper Δ is greater or equal to 1 it follows that $K(\Delta) \geq 1$, too. Unfortunately Δ is unknown in practice. The formula $K(\Delta)$ can be used to calculate a reference value for a fixed value of Δ against which the analyst wants to be protected.

By making use of these results we introduce CUSUM type charts for a variance change in a stationary process. We apply recursion (21) to various statistics. Note that this derivation was done assuming independent and normally distributed variables. Both is not valid for an GARCH process. This procedure is chosen since the exact likelihood ratio for a GARCH process is an extremely complicated function. Up to now even no explicit formula for the univariate marginal distribution of a GARCH process is known.

In the following let $\{Y_t\}$ be weakly stationary with mean μ_0 and variance γ_0.

3.2.1 A Chart Based on Squared Observations

First we apply (21) to the squared observations

$$S_t^+ = \max\{0, S_{t-1}^+ + (X_t - \mu_0)^2 - K\,\gamma_0\}$$

for $t \geq 1$ with $S_0^+ = 0$. K is an arbitrary nonnegative number. It is concluded that the process is out-of-control at time t if $S_t^+ \geq c^*\gamma_0$. Again, $c^\star$ is determined such that the in-control ARL takes a given value. Therefore we fix $K \geq 0$ and calculate $c^\star$ such that the in-control ARL is equal to a specified value. If $\{Y_t\}$ is a GARCH process $c^\star$ and the ARL do not depend on α_0. This is an advantage for the tabulation of the values (cf. Section 3.1.1).

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

3.2.2 A Chart Based on the Logarithm of the Squared Observations

Here we apply the classical CUSUM recursion of Page (1954) for the detection of a shift in the mean of normal variables to $\ln((X_t - \mu_0)^2/\gamma_0)$. Thus the CUSUM statistic is

$$S_t^+ = \max\left\{0, S_{t-1}^+ + \ln\frac{(X_t - \mu_0)^2}{\gamma_0} - K\sqrt{\mathrm{Var}\,(\ln(Y_1 - \mu_0)^2)}\right\}$$

for $t \geq 1$ with $S_0^+ = 0$ and $K \geq 0$. Note that $\mathrm{Var}\,(\ln((Y_1 - \mu_0)^2/\gamma_0)) = \mathrm{Var}\,(\ln(Y_1 - \mu_0)^2)$. If $S_t^+ \geq c\sqrt{\mathrm{Var}\,(\ln(Y_1 - \mu_0)^2)} = c^\star$ an alarm is given at time t. For a GARCH model $c^\star$ and the ARL do not depend on α_0.

3.2.3 A Chart Based on the Conditional Variance

Let $\{Y_t\}$ be a weakly stationary GARCH process. The CUSUM statistic for the conditional variance is defined as

$$S_t^+ = \max\left\{0, S_{t-1}^+ + \hat{\sigma}_{t+1}^2 - K\,\gamma_0\right\}$$

for $t \geq 1$, $K \geq 0$. As a starting value for S_0^+ we choose $S_0^+ = 0$. If $S_t^+ \geq c^*\,\gamma_0$ it is decided that the process is out-of-control at time t. $c^\star$ and the ARL are not functions in α_0.

3.2.4 A Residual Chart

Let $\{Y_t\}$ be a weakly stationary GARCH process. Applying the above CUSUM recursion to the residuals this leads to

$$S_t^+ = \max\left\{0, S_{t-1}^+ + \frac{X_t^2}{\hat{\sigma}_t^2} - K\right\}$$

for $t \geq 1$ where $S_0^+ = 0$ and $K \geq 0$. If $S_t^+ \geq c^\star$ then the process is out-of-control at time t. Again, $c^\star$ and the ARL do not depend on α_0.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

4 COMPARISON STUDY

In Section 3 several control schemes for detecting a change in the variance of a time series were introduced. Now the question arises which of these schemes is the best one. Here we present the results of an extensive simulation study. We confine ourselves to charts based on the asymptotic variance. As a measure for the performance the average run length is used. All charts were calibrated such that the in-control ARL is always the same. A discussion about the choice of the in-control ARL for financial time series can be found in Severin and Schmid (1999). Following their suggestions we choose the value 60 which reflects roughly three months at the stock exchange. The out-of-control behaviour of all schemes is analyzed with respect to model (4). In each case the average run length was determined within a simulation study based on 100 000 repetitions.

The target process is taken as a GARCH(1,1) process. In finance we are usually faced with the situation that $\alpha_1 + \beta_1$ is close to 1 and that α_0 is small (cf. Section 5). This is the reason why we select the parameter constellations

$$\text{process I:} \quad \alpha_0 = 0.1, \quad \alpha_1 = 0.05, \quad \beta_1 = 0.9,$$

$$\text{process II:} \quad \alpha_0 = 1.0, \quad \alpha_1 = 0.25, \quad \beta_1 = 0.7.$$

The variance of the first process is $\gamma_0 = 2$ and for the second it is equal to $\gamma_0 = 20$. $\{\varepsilon_t\}$ is assumed to be standard normally distributed.

The EWMA and the CUSUM charts depend on an additional parameter. For that reason the comparison of these charts is complicate. In our study we consider the smoothing parameters $\lambda \in \{0.1, 0.25, 0.5, 0.75, 1.0\}$ for the EWMA charts and the reference values $K \in \{0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0\}$ for the CUSUM charts. The critical values $c^\star$ of all charts are listed in Appendix 7.3 (see Table 5 and Table 6). The out-of-control average run lengths of the EWMA charts are given in Table 2 (process I and II) and those of the CUSUM charts in Table 2 (process I) and Table 3 (process II). The bold typed values indicate on the parameters which lead to the smallest out-of-control ARL for a fixed value of Δ. The statistics given in the tables refer to the control charts introduced in Section 3. For instance, in Table 1 x_t^2 stands for the EWMA chart based on squared observations, etc. Due to numerical instabilities the tables of the CUSUM charts are incomplete. The missing values are denoted by the symbol $\star$. For these constellations we were unable to determine critical values. In these cases it turned out that larger values of the run length appear more frequent. Because the specified in-control ARL 60 is relatively small the ARL reacts extremely sensibly on small variations of the critical value.

Table 1. Out-of-Control ARLs of Several EWMA Charts for Process I and Process II (First Entry: x_t^2 Chart, Second Entry: $\ln x_t^2$ Chart, Third Entry: $\hat{\sigma}_{t+1}^2$ Chart, Fourth Entry: Residual Chart)

	Process I					Process II				
$\Delta\|\lambda$	0.1	0.25	0.5	0.75	1.0	0.1	0.25	0.5	0.75	1.0
1.00	60.30	59.86	60.11	60.29	60.27	60.76	59.90	59.83	60.41	60.34
	60.04	60.27	60.06	59.75	59.73	60.19	60.04	59.79	59.50	59.58
	59.72	60.15	59.95	60.25	60.07	59.97	60.42	60.19	60.24	59.98
	59.87	60.08	59.97	59.59	59.96	59.93	59.95	59.99	59.47	59.94
1.10	33.59	35.09	35.82	36.32	36.27	42.44	43.62	44.19	43.96	43.64
	37.10	37.78	37.36	36.54	36.54	44.41	45.42	45.04	44.52	43.98
	31.98	32.83	33.11	32.88	32.56	**41.46**	43.78	43.77	43.80	43.51
	36.08	38.68	40.00	40.80	41.65	44.69	46.46	47.62	47.38	47.80
1.20	20.92	22.58	23.58	23.72	24.32	30.57	32.64	33.47	33.37	33.44
	25.63	25.98	25.40	24.61	24.41	34.31	35.34	35.34	34.35	33.18
	19.68	20.36	20.26	20.24	19.95	**29.82**	32.39	32.80	32.69	32.53
	23.76	26.55	28.45	29.43	30.26	34.90	37.31	38.55	38.95	39.26
1.30	14.31	15.66	16.48	16.93	17.25	22.66	25.41	25.76	25.76	25.51
	19.18	19.07	18.47	17.54	17.22	27.18	28.27	28.07	27.14	25.71
	13.33	13.96	13.74	13.59	13.40	**22.01**	24.85	25.02	24.98	24.87
	16.74	18.91	20.79	21.87	22.66	27.97	30.82	32.40	32.62	33.19
1.40	10.58	11.58	12.15	12.57	12.94	17.38	19.89	20.50	20.34	20.37
	15.16	14.98	14.08	13.20	12.92	22.16	23.11	22.73	21.81	20.31
	9.81	10.34	10.01	9.90	9.85	**16.66**	19.39	19.70	19.71	19.55
	12.42	14.35	15.83	16.82	17.41	22.96	26.03	27.47	28.21	28.77
1.50	8.23	9.00	9.56	9.84	10.12	13.47	16.12	16.69	16.56	16.38
	12.47	12.13	11.25	10.39	10.03	18.56	19.34	18.98	18.02	16.59
	7.77	8.08	7.88	7.72	**7.51**	**12.70**	15.32	15.75	15.62	15.51
	9.77	11.15	12.44	13.33	13.84	19.23	22.17	23.93	24.20	25.00
2.00	3.93	4.18	4.36	4.50	4.56	5.38	6.90	7.27	7.28	7.28
	6.86	6.28	5.46	4.80	4.57	9.75	9.91	9.15	8.27	7.35
	3.92	3.99	3.77	3.64	**3.56**	**4.91**	6.37	6.62	6.56	6.43
	4.46	4.81	5.25	5.60	5.91	9.90	11.91	13.44	14.22	14.59
2.50	2.75	2.86	2.93	2.99	3.01	3.16	4.03	4.25	4.35	4.31
	5.05	4.51	3.73	3.21	3.04	6.69	6.51	5.77	4.96	4.35
	2.83	2.83	2.66	2.56	**2.51**	**2.86**	3.67	3.82	3.80	3.71
	3.04	3.15	3.29	3.42	3.57	6.27	7.57	8.76	9.32	9.72
3.00	2.24	2.31	2.32	2.34	2.38	2.34	2.89	3.03	3.10	3.09
	4.16	3.67	2.94	2.52	2.38	5.26	4.95	4.20	3.51	3.10
	2.32	2.32	2.19	2.11	**2.07**	**2.10**	2.63	2.70	2.68	2.66
	2.40	2.44	2.51	2.58	2.66	4.43	5.26	6.08	6.64	7.04

Table 2. Out-of-Control ARLs of Several CUSUM Charts for Process I (First Entry: x_t^2 Chart, Second Entry: $\ln x_t^2$ Chart, Third Entry: $\hat{\sigma}_{t+1}^2$ Chart, Fourth Entry: Residual Chart)

$\Delta \| K$	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	"Optimal" K
1.00	60.11	60.15	60.06	60.30	60.39	60.16	59.92	60.16	
	59.96	59.96	⋆	⋆	⋆	⋆	⋆	⋆	
	60.01	60.10	59.92	59.90	⋆	⋆	⋆	⋆	
	60.05	59.99	59.83	60.05	60.11	60.22	60.01	59.64	
1.10	47.41	42.83	36.96	35.40	35.44	35.62	36.03	35.94	$K = 1.10$: 35.43
	36.13	36.32	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.10$: 36.88
	53.91	51.41	43.82	33.83	⋆	⋆	⋆	⋆	$K = 1.10$: **32.85**
	51.78	48.42	42.47	38.63	38.92	39.63	40.03	40.21	$K = 1.10$: 38.68
1.20	38.66	32.78	25.58	23.43	23.17	23.21	23.51	23.72	$K = 1.19$: 23.03
	23.81	23.99	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.18$: 24.08
	48.76	44.70	34.56	21.71	⋆	⋆	⋆	⋆	$K = 1.19$: **19.95**
	45.59	40.62	32.29	26.91	26.78	27.63	28.15	28.44	$K = 1.19$: 26.74
1.30	32.39	26.07	19.02	16.66	16.24	16.42	16.51	16.79	$K = 1.29$: 16.19
	16.79	17.01	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.26$: 16.81
	44.34	39.50	28.65	15.36	⋆	⋆	⋆	⋆	($K = 1.19$: **13.52**)
	40.72	34.82	25.76	19.90	19.61	20.14	20.62	20.96	$K = 1.29$: 19.70
1.40	27.61	21.59	14.94	12.56	12.18	12.26	12.39	12.38	$K = 1.37$: 12.18
	12.55	12.70	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.34$: 12.52
	40.59	35.28	24.50	11.79	⋆	⋆	⋆	⋆	($K = 1.19$: **9.78**)
	36.84	30.33	20.98	15.45	14.87	15.24	15.68	15.96	$K = 1.37$: 14.94
1.50	23.90	18.13	12.24	10.08	9.60	9.56	9.65	9.73	$K = 1.66$: 9.58
	9.84	9.90	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.41$: 9.81
	37.23	31.87	21.50	9.50	⋆	⋆	⋆	⋆	($K = 1.19$: **7.53**)
	33.54	26.76	17.70	12.47	11.69	11.86	12.23	12.54	$K = 1.66$: 12.04
2.00	13.52	9.75	6.29	4.97	4.58	4.43	4.42	4.43	$K = 1.85$: 4.42
	4.52	4.49	⋆	⋆	⋆	⋆	⋆	⋆	$K = 0.69$: 4.57
	26.12	21.29	13.59	5.11	⋆	⋆	⋆	⋆	($K = 1.19$: **3.56**)
	22.56	15.93	9.12	5.97	5.31	5.19	5.23	5.31	$K = 1.85$: 5.27
2.50	9.04	6.47	4.27	3.40	3.12	3.04	2.99	2.99	$K = 2.18$: 2.98
	3.05	2.99	⋆	⋆	⋆	⋆	⋆	⋆	($K = 0.74$: 3.03)
	19.95	15.99	10.09	3.71	⋆	⋆	⋆	⋆	($K = 1.19$: **2.51**)
	16.13	10.52	5.85	3.92	3.47	3.37	3.34	3.33	$K = 2.18$: 3.36
3.00	6.64	4.81	3.28	2.68	2.49	2.41	2.38	2.37	$K = 2.47$: 2.36
	2.42	2.36	⋆	⋆	⋆	⋆	⋆	⋆	($K = 0.74$: 2.38)
	16.05	12.80	8.09	3.02	⋆	⋆	⋆	⋆	($K = 1.19$: **2.05**)
	11.91	7.48	4.28	3.01	2.70	2.60	2.57	2.55	$K = 2.47$: 2.56

Table 3. Out-of-Control ARLs of Several CUSUM Charts for Process II (First Entry: x_t^2 Chart, Second Entry: $\ln x_t^2$ Chart, Third Entry: $\hat{\sigma}_{t+1}^2$ Chart, Fourth Entry: Residual Chart)

$\Delta\|K$	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	"Optimal" K
1.00	60.26	60.12	59.65	60.48	60.28	60.41	60.16	60.00	
	60.02	60.02	⋆	⋆	⋆	⋆	⋆	⋆	
	60.00	60.33	59.90	59.67	60.14	60.43	⋆	⋆	
	60.03	59.94	59.72	59.96	60.01	59.59	60.07	60.10	
1.10	47.12	44.34	44.32	44.30	44.07	44.10	44.20	44.28	$K=1.10$: 44.28
	44.05	44.03	⋆	⋆	⋆	⋆	⋆	⋆	$K=0.10$: 44.39
	50.10	44.40	44.18	44.08	43.86	**43.68**	⋆	⋆	$K=1.10$: 43.92
	55.05	52.77	48.42	46.44	46.82	46.92	47.30	47.12	$K=1.10$: 46.26
1.20	38.33	34.00	33.69	33.67	33.53	33.41	33.26	33.26	$K=1.19$: 33.57
	33.28	33.20	⋆	⋆	⋆	⋆	⋆	⋆	$K=0.18$: 33.51
	42.59	34.41	33.19	33.06	32.37	**32.75**	⋆	⋆	$K=1.19$: 32.97
	51.16	47.49	40.96	37.43	38.07	38.20	38.48	38.71	$K=1.19$: 37.76
1.30	31.82	26.94	26.38	26.34	26.18	25.83	25.93	25.84	$K=1.29$: 26.11
	25.90	25.72	⋆	⋆	⋆	⋆	⋆	⋆	$K=0.26$: 25.89
	36.81	27.68	26.04	25.50	25.17	25.18	⋆	⋆	$K=1.29$: **25.14**
	47.99	43.33	35.60	31.14	31.46	31.72	32.31	32.06	$K=1.29$: 31.47
1.40	27.30	21.79	21.05	20.98	20.76	20.57	20.54	20.38	$K=1.37$: 20.62
	20.52	20.43	⋆	⋆	⋆	⋆	⋆	⋆	$K=0.34$: 20.44
	32.38	22.94	20.65	20.20	19.86	**19.55**	⋆	⋆	$K=1.37$: 19.75
	45.43	40.25	31.61	26.40	26.70	27.08	27.53	27.58	$K=1.37$: 26.78
1.50	23.62	17.98	17.13	17.00	16.75	16.61	16.53	16.51	$K=1.66$: 16.64
	16.60	16.54	⋆	⋆	⋆	⋆	⋆	⋆	$K=0.41$: 16.51
	28.72	19.28	16.87	16.26	15.87	**15.65**	⋆	⋆	$K=1.66$: **15.65**
	43.27	37.58	28.50	22.94	22.85	23.40	23.70	23.92	$K=1.66$: 23.70
2.00	13.60	9.15	8.00	7.64	7.41	7.39	7.30	**7.27**	$K=1.85$: 7.37
	7.37	7.31	⋆	⋆	⋆	⋆	⋆	⋆	($K=0.55$: 7.30)
	17.71	10.45	7.83	7.13	6.59	**6.50**	⋆	⋆	($K=1.66$: **6.50**)
	35.80	29.01	19.24	13.41	12.81	13.25	13.58	13.63	$K=1.85$: 13.62
2.50	9.15	5.85	4.93	4.59	4.45	4.38	4.36	4.31	$K=2.18$: 4.30
	4.36	4.33	⋆	⋆	⋆	⋆	⋆	⋆	($K=0.55$: 4.33)
	12.50	7.06	4.91	4.21	3.85	3.75	⋆	⋆	($K=1.66$: **3.74**)
	31.01	23.77	14.39	9.08	8.34	8.44	8.73	8.86	$K=2.18$: 9.08
3.00	6.79	4.31	3.56	3.30	3.18	3.13	3.08	3.09	$K=2.47$: 3.07
	3.11	3.08	⋆	⋆	⋆	⋆	⋆	⋆	($K=0.55$: 3.08)
	9.55	5.36	3.63	3.00	2.74	**2.64**	⋆	⋆	($K=1.66$: 2.65)
	27.43	20.00	11.14	6.64	5.89	5.94	6.11	6.27	$K=2.47$: 6.52

The results for the EWMA charts are very interesting. It is a well-known fact that for the detection of large deviations in the mean Shewhart type charts are more suitable, whereas for smaller shifts EWMA and CUSUM schemes must be preferred. Our tables illustrate that this behaviour cannot always be observed in the presence of variance changes. For the schemes based on the squared observations and the residuals the smallest smoothing parameter ($\lambda = 0.1$) always leads to the smallest out-of-control ARL (cf. Table 1). It is not surprising that another type of behaviour can be found for the chart based on the logarithm since this chart is derived assuming a change point model (see Section 3.1.2). But it is unexpected that $\lambda = 1.0$, i.e. the Shewhart scheme, possesses the smallest ARLs. Only for the EWMA scheme applied to the conditional variance no unique recommendation can be made. For process I and $\Delta \geq 1.5$ the Shewhart chart beats the chart with memory, for all other constellations the EWMA scheme with $\lambda = 0.1$ is the best one. Next we want to compare the EWMA schemes taking optimal smoothing parameters. It shows that the chart based on the conditional variance always provides the smallest out-of-control ARL, followed by the chart based on squared observations. For process II the logarithm chart is better than the residual chart. But for process I the residual method dominates the logarithm approach.

In Table 2 (process I) and Table 3 (process II) the results of the CUSUM charts are listed. In the last column the reference values are given which were obtained by the log likelihood ratio for independent and normally distributed variables, i.e. $K(\Delta) = \ln\,\Delta$ for the logarithm chart and $K(\Delta) = 2\ln\Delta^{1/(1-1/\Delta^2)}$ for the other schemes. In this context Δ has to be known. Sometimes the values of K stand in parentheses. This means that we were unable to determine the ARL for $K(\Delta)$. Then we list the largest reference value ($\leq K(\Delta)$) for which we could compute the ARL. The values of the last column give a good hint how the reference value should be chosen. For this choice the out-of-control ARL lies close to the minimum ARL. This is a remarkable result since it shows that $K(\Delta)$ gives a good hint how the reference value should be chosen. A comparison of the ARLs shows that the best results were always obtained for the chart based on the conditional variance. On the second place we find the charts based on the squared observations and the logarithm. For process I the chart for the squared observations turns out to be better than the logarithm chart, while for process II the logarithm chart has the smaller out-of-control ARL for the optimal design. Comparing the results of the EWMA and CUSUM charts we see that mostly the EWMA scheme based on the conditional variance ($\lambda = 0.1$) is the best chart. Only for process I it is sometimes beaten by the CUSUM approach. Therefore we recommend to apply the EWMA chart with smoothing parameter $\lambda = 0.1$. It nearly always provides the minimum out-of-control ARL. Furthermore, it can easily be applied in practice.

5 APPLICATION

In this section we study the daily returns of the shares of the Deutsche Bank. It is described how the above control charts can be applied.

To determine a target process we use the data from May 22, 1996 to May 21, 1997. The SAS routine AUTOREG was applied to fit a GARCH process to the data. We receive the following model

$$Y_t = \varepsilon_t \sqrt{0.0629 + 0.228\, Y_{t-1}^2 + 0.757\, \eta_{t-1}^2} \tag{22}$$

with $\varepsilon_t \sim t_6$ (normalized t-distribution with 6 degrees of freedom). It is not surprising that $\{\varepsilon_t\}$ is not normally distributed. This behaviour was already mentioned by other authors.

Next the daily returns of the Deutsche Bank are investigated from May 22, 1997 to October 10, 1997. The data are analyzed sequentially. The aim is to detect deviations from the fitted model. Here we show the results for the Shewhart chart based on squared observations and for the EWMA chart based on squared observations with smoothing parameter $\lambda = 0.1$. The critical values were determined by simulations such that the in-control ARL is equal to 60. The results of both charts are plotted in Figure 3 and 4. The dashed lines are the control limits $c^\star \gamma_0$.

The application of an EWMA scheme turns out to be more difficult than the application of a Shewhart chart. If, for instance, an observation is

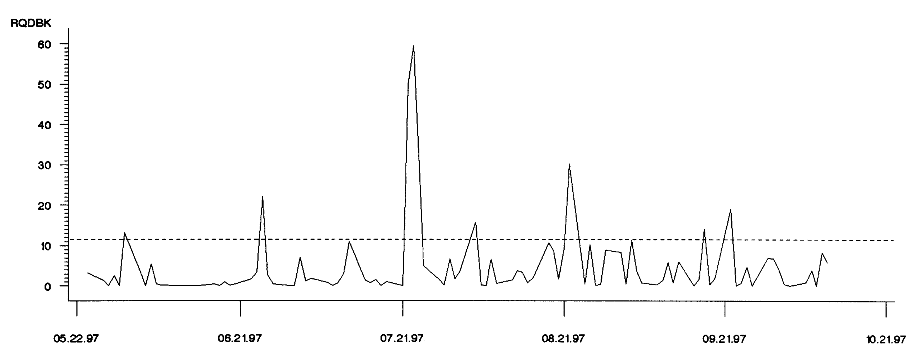

Figure 3. Shewhart chart based on the squared returns applied to the Deutsche Bank share from May 22, 1997 to October 10, 1997.

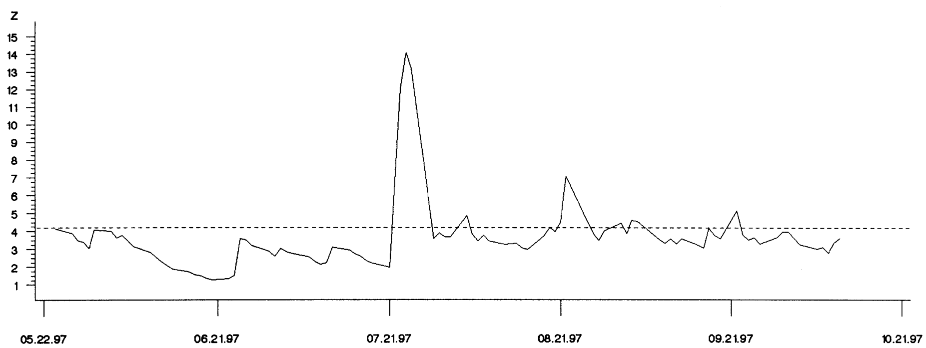

Figure 4. EWMA chart ($\lambda = 0.1$) based on the squared observations applied to the Deutsche Bank share from May 22, 1997 to October 10, 1997 (with restart).

signaled to be out-of-control at time t then the problem arises whether we use this observation for the calculation of the EWMA statistic at time $t+1$. Then we would have an EWMA scheme with head-start and the assumptions for the calibration of the scheme are no longer fulfilled. Here we replaced the outlying observation by the target value. The EWMA scheme was restarted. This procedure can be justified by the mean-reverting behaviour of stocks.

On May 30, 1997 the Shewhart chart signals the first out-of-control situation. From the EWMA chart this value is not classified to be suspicious. This observation has a natural explanation. It was caused by the gold dispute between the German government and the Federal Bank. The second signal on June 25, 1997 is again given by the Shewhart scheme. The EWMA statistic increases but it is still below the control limit. After good Wall-Street results the Dax reached a new record on June 25, 1997. On July 21, 1997 the Vereinsbank and the Hypo-Bank announced their merger after the stock exchange had closed. This caused wild takeover imaginations in the financial sector on July 22, 1997. Both the EWMA chart and the Shewhart chart trigger an alarm. On July 23, 1997 the Deutsche Bank announced that their half year results show a profit of 27.5% after tax. In this period both charts were out-of-control for several times.

After that time the values of the EWMA statistic lie at a higher level. The control limit is exceeded repeatedly (see Table 4).

It is the task of the fund manager to decide whether a more risky paper should still belong to the portfolio or not. On the one hand higher profits can be made, on the other hand the loss may increase. A control chart provides a hedge tool against structural deviations.

Table 4. Changes Detected by the Shewhart Chart Based on Squared Observations (S) and the EWMA Scheme (With Restart) Applied to the Squared Observations (E: $\lambda = 0.1$)

Date	S	E
5.30.97	X	
6.25.97	X	
7.22.97	X	X
7.23.97	X	X
7.24.97	X	X
7.25.97		X
8.04.97	X	X
8.19.97		X
8.21.97		X
8.22.97	X	X
8.26.97		X
8.29.97		X
9.01.97		X
9.03.97		X
9.04.97	X	X
9.17.97	X	
9.22.97	X	X

6 CONCLUSIONS

In this paper several control schemes for the variance of a time series are introduced. As a measure for the performance of a control chart the average run length is used. Within our comparison study the best results are obtained for the EWMA chart based on the conditional variance. A good choice of the smoothing parameter overall is $\lambda = 0.1$.

It is shown that control charts can be very effectively used to monitor structural deviations in the volatility of a financial time series. Thus these results provide a very useful tool for risk management of securities.

7 APPENDIX

7.1 Some Properties of GARCH Processes

Let $\{Y_t\}$ be a weakly stationary GARCH process. It is interesting that $\{Y_t^2\}$ is an ARMA(max {p,q},p) process with mean γ_0 (cf. Bollerslev (1986)) since

$$Y_t^2 - \gamma_0 = \sum_{i=1}^{q} \alpha_i (Y_{t-i}^2 - \gamma_0) + \sum_{j=1}^{p} \beta_j (Y_{t-j}^2 - \gamma_0) + v_t - \sum_{j=1}^{p} \beta_j v_{t-j} \,,$$

with $v_t = Y_t^2 - \eta_t^2 = (\varepsilon_t^2 - 1)\,\eta_t^2$. We observe that $E(v_t) = 0$. If the fourth moments of $\{Y_t\}$ are finite then $\mathrm{Cov}\,(v_t, v_s) = 0$ for $t \neq s$. Moreover, Theorem 3.1.1 of Brockwell and Davis (1991) implies that $\{Y_t^2\}$ is weakly stationary, too. Because of (3) the polynomial $1 - \sum_{i=1}^{q} \alpha_i z^i - \sum_{j=1}^{p} \beta_j z^j$ has no zero within the unit circle. Bollerslev (1986) derived a formula for the fourth moment of a GARCH(1,1) process assuming $\{\varepsilon_t\}$ to be normally distributed. We generalize his result to arbitrary distributions. Suppose that $\varepsilon_t \sim F$ and that $E(\varepsilon_t^4) = \kappa < \infty$ then the fourth moment of a GARCH(1,1) process exists if $\kappa\,\alpha_1^2 + 2\,\alpha_1\,\beta_1 + \beta_1^2 < 1$ and

$$E(Y_t^4) \;=\; \kappa \,\frac{\alpha_0^2 (1 + \alpha_1 + \beta_1)}{(1 - \alpha_1 - \beta_1)(1 - \beta_1^2 - 2\alpha_1\beta_1 - \kappa\,\alpha_1^2)} \,. \tag{23}$$

This can be derived by using that $\sigma^2 := \mathrm{Var}\,(v_t) = (\kappa - 1)\,E(Y_t^4)/\kappa$, $r_1 = \mathrm{Var}\,(Y_1^2)/\sigma^2$ (see below) and the representation (16) of r_1 for a GARCH(1,1) process.

Now, it is important to note that η_t^2 is equal to the best linear predictor (in the L^2 sense) of Y_t^2 in terms of $Y_{t-1}^2, Y_{t-2}^2, \ldots$. This predictor makes use of the knowledge of an infinite past. For $t \geq 1$ let $\hat{\eta}_t^2$ be the best linear predictor of Y_t^2 based on the actually observed values $Y_{t-1}^2, \ldots, Y_1^2, 1$. We consider the squared residuals $r_t = E(Y_t^2 - \hat{\eta}_t^2)^2/\sigma^2$. Using the projection theorem (see Brockwell and Davis (1991)) it follows that

$$\sigma^2\, r_t \;=\; E(Y_t^2 - \gamma_0)^2 - E(\hat{\eta}_t^2 - \gamma_0)^2 = \mathrm{Var}\,(Y_t^2) - \mathrm{Var}\,(\hat{\eta}_t^2) \,.$$

Hence $\mathrm{Var}\,(\hat{\eta}_t^2) = \sigma^2 (r_1 - r_t)$ since $\mathrm{Var}\,(Y_t) = \sigma^2\, r_1$. This shows that $\{\hat{\eta}_t^2\}$ is not a weakly stationary process. However, $\{r_t\}$ converges to 1 if t converges to infinity (Brockwell and Davis (1991, p. 176)). Thus $\{\hat{\eta}_t^2\}$ is asymptotically stationary.

7.2 Proofs of the Theorems

Proof of Theorem 1 b) Straightforward calculations show that

$$\begin{aligned} \mathrm{Var}\,(Z_t) &= \lambda^2\, \Delta^4 \sum_{i,j=0}^{t-1} (1-\lambda)^{i+j}\, \mathrm{Cov}\,((Y_{t-i} - \mu_0)^2, (Y_{t-j} - \mu_0)^2) \\ &= \lambda^2\, \Delta^4 \sum_{v=-(t-1)}^{t-1} \delta_v \sum_{i=\max\{0,\,-v\}}^{\min\{t-1-v,\,t-1\}} (1-\lambda)^{2i+v} \,. \end{aligned}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

c) Since $\{Y_t^2\}$ is an ARMA(1,1) process we apply the formulas for the autocovariances (cf. (8)). We get $\delta_h = (\alpha_1 + \beta_1)^{h-1}\,\delta_1$ for $h \geq 1$ and

$$\delta_0 \;=\; \frac{1 - 2\alpha_1\beta_1 - \beta_1^2}{1 - (\alpha_1 + \beta_1)^2}\sigma^2, \qquad \delta_1 \;=\; \frac{\alpha_1\,(1 - \alpha_1\beta_1 - \beta_1^2)}{1 - (\alpha_1 + \beta_1)^2}\sigma^2$$

with $\sigma^2 = \mathrm{Var}\,(v_t) = (\kappa - 1)\,E(Y_1^4)/\kappa$. The remainder is obtained immediately.

Proof of Theorem 2 a) This part follows immediately since $E(\hat\eta_t^2) = \gamma_0$.

b) Solving Equation (17) leads to

$$E(\hat\sigma_t^2) - \Delta^2\gamma_0 = -\alpha_0(\Delta^2 - 1)\,I_t \;+\; II_t$$

with

$$I_t = 1 + \frac{\beta_1}{r_{t-1}} + \frac{\beta_1^2}{r_{t-1}r_{t-2}} + \cdots + \frac{\beta_1^{t-\tau-1}}{r_{t-1}\ldots r_{\tau+1}}\,,$$
$$II_t = -\gamma_0(\Delta^2 - 1)\;\frac{\beta_1^{t-\tau}}{r_{t-1}\ldots r_\tau}\;.$$

Since $\lim_{t\to\infty} r_t = 1$ we obtain that $\lim_{t\to\infty} II_t = 0$. Moreover, $r_t \leq r_{t-1}$ implies that $\{I_t\}$ is nondecreasing. Because $r_t \geq 1$ the sequence $\{I_t\}$ is bounded. Therefore the limit of $\{E(\hat\sigma_t^2)\}$ exists. Taking the limit on both sides of Equation (17) leads to Equation (18).

Let $2 \leq s < t$ then we get

$$\begin{aligned}
\mathrm{Cov}(\hat\eta_s^2, \hat\eta_t^2) &= (\alpha_1 + \beta_1)\,E((\hat\eta_s^2 - \gamma_0)(Y_{t-1}^2 - \gamma_0)) \\
&\quad - \beta_1\,E((\hat\eta_s^2 - \gamma_0)(Y_{t-1}^2 - \hat\eta_{t-1}^2))/r_{t-1} \\
&= (\alpha_1 + \beta_1)\,\mathrm{Cov}(\hat\eta_s^2, Y_{t-1}^2) \quad \text{and} \\
\mathrm{Cov}(\hat\eta_s^2, Y_t^2) &= E((\hat\eta_s^2 - \gamma_0)((\alpha_1 + \beta_1)(Y_{t-1}^2 - \gamma_0) + v_t - \beta_1 v_{t-1})) \\
&= (\alpha_1 + \beta_1)\,\mathrm{Cov}(\hat\eta_s^2, Y_{t-1}^2) \;=\; (\alpha_1 + \beta_1)^{t-s}\,\mathrm{Cov}(\hat\eta_s^2, Y_s^2) \\
&= (\alpha_1 + \beta_1)^{t-s} E((\hat\eta_s^2 - \gamma_0)(Y_s^2 - \gamma_0)) \;=\; (\alpha_1 + \beta_1)^{t-s} E(\hat\eta_s^2 - \gamma_0)^2 \\
&= \sigma^2\,(\alpha_1 + \beta_1)^{t-s}(r_1 - r_s).
\end{aligned}$$

The last equation holds since we demanded that $\{Y_t^2\}$ is weakly stationary and thus the projection theorem can be applied (see Appendix 7.1). The result follows with part a).

Proof of Theorem 3 a) We observe that

$$\lambda\sum_{i=0}^{t-1}(1-\lambda)^i\,a_{t+1-i,t+1-i} = \lambda\sum_{i=2}^{t+1}(1-\lambda)^{t+1-i}\,(a_{ii} - a_\infty) \;+\; a_\infty(1 - (1-\lambda)^t)\,.$$

Since $a_{ii} \to a_\infty$ if i tends to infinity we get that $\lim_{i\to\infty} \sum_{i=0}^{t-1}(1-\lambda)^i \times a_{t+1-i,\,t+1-i} = a_\infty$. For a GARCH(1,1) process we get $a_{ii} = \alpha_0 + \beta_1\, a_{i-1,\,i-1}/\, r_{i-1}$. Therefore (cf. proof of Theorem 2b) $a_\infty = \alpha_0/(1-\beta_1)$.

b) It follows with Theorem 2 that

$$\begin{aligned}\mathrm{Var}(Z_t) &= \lambda^2\,\Delta^4\,\sigma^2 \sum_{i=0}^{t-1}(1-\lambda)^{2i}(r_1 - r_{t+1-i}) \\ &\quad + 2\lambda^2\,\Delta^4\,\sigma^2 \sum_{j=1}^{t-1}\sum_{i=0}^{j-1}(1-\lambda)^{i+j}(\alpha_1+\beta_1)^{j-i}(r_1 - r_{t+1-j}) \\ &= \lambda^2\,\Delta^4\,\sigma^2 \sum_{i=0}^{t-1}(1-\lambda)^{2i}(r_1 - r_{t+1-i}) + 2\lambda^2\,\Delta^4\sigma^2\,\frac{\alpha_1+\beta_1}{\alpha_1+\beta_1-(1-\lambda)} \\ &\quad \times \sum_{i=0}^{t-1}(1-\lambda)^j(r_1 - r_{t+1-j})((\alpha_1+\beta_1)^j - (1-\lambda)^j)\end{aligned}$$

with σ^2 as in Appendix 7.1. With the same arguments as given in the proof of Theorem 2b we get that $\lim_{t\to\infty} \sum_{j=0}^{t-1}(1-\lambda)^j\,(\alpha_1+\beta_1)^j\, r_{t+1-j} = 1/(1-(1-\lambda)(\alpha_1+\beta_1))$. Consequently,

$$\lim_{t\to\infty} \mathrm{Var}\,(Z_t) = \lambda\,\Delta^4\,\sigma^2\,\frac{r_1-1}{2-\lambda}\left(1 + 2\,\frac{(1-\lambda)(\alpha_1+\beta_1)}{1-(1-\lambda)(\alpha_1+\beta_1)}\right)$$

and thus the result follows.

7.3 Tables of the Critical Values

Table 5. Critical Values $c^\star$ for Several EWMA Charts (Upper Entry: Process I, Lower Entry: Process II)

λ	0.10	0.25	0.50	0.75	1.00
x_t^2	1.421	2.096	3.144	4.179	5.245
	1.116	1.602	2.336	3.018	3.698
$\ln x_t^2$	−0.641	−0.027	0.611	1.107	1.657
	−0.959	−0.263	0.398	0.858	1.309
$\hat{\sigma}_{t+1}^2$	1.044	1.104	1.154	1.186	1.220
	1.002	1.164	1.360	1.514	1.664
Residual chart	1.494	2.223	3.378	4.535	5.736
	1.496	2.229	3.396	4.561	5.774

Table 6. Critical Values $c^{\star}$ for Several CUSUM Charts (Upper Entry: Process I, Lower Entry: Process II)

K	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
x_t^2	39.795	24.355	12.235	7.505	5.72	4.778	4.163	3.699
	20.285	9.085	5.600	4.088	3.207	2.650	2.220	1.866
$\ln x_t^2$	1.293	0.565	⋆	⋆	⋆	⋆	⋆	⋆
	0.811	0.139	⋆	⋆	⋆	⋆	⋆	⋆
$\hat{\sigma}_{t+1}^2$	43.255	27.85	11.455	1.074	⋆	⋆	⋆	⋆
	25.785	9.305	3.581	1.520	0.589	0.177	⋆	⋆
Residual chart	43.39	28.83	15.68	8.678	6.438	5.404	4.727	4.229
	43.48	28.96	15.82	8.777	6.511	5.452	4.772	4.280

REFERENCES

1. Alwan, L.C. and Roberts, H.V. (1988). Time-series modeling for statistical process control. *Journal of Business and Economic Statistics*, **6**, 87–95.
2. Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics*, **31**, 307–327.
3. Bollerslev, T., Chou, R.Y. and Kroner, K.F. (1992). ARCH modelling in finance: a selective review of the theory and empirical evidence. *Journal of Econometrics*, **52**, 5–59.
4. Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, Springer, New York.
5. Chen, J. and Gupta, A.K. (1997). Testing and locating variance changepoints with application to stock prices. *Journal of the American Statistical Association*, **92**, 739–747.
6. Crowder, S.V. (1987). A simple method for studying run-length distributions of exponentially weighted moving average charts. *Technometrics*, **27**, 401–407.
7. Crowder, S.V. and Hamilton, M.D. (1992). An EWMA for monitoring a process standard deviation. *Journal of Quality Technology*, **24**, 12–21.
8. Engle, R.F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**, 987–1007.
9. Gourieroux,Ch.(1997).*ARCHModelsandFinancialApplications*,Springer, New York.
10. Harris, T.J. and Ross, W.H. (1991). Statistical process control procedures for correlated observations. *Canadian Journal of Chemical Engineering*, **69**, 48–57.

11. Hsu, D.A. (1977). Tests for variance shifts at an unknown time point. *Applied Statistics*, **26**, 279–284.
12. Hsu, D.A. (1979). Detecting shifts of parameters in gamma sequences with applications to stock price and air traffic flow analysis. *Journal of the American Statistical Association*, **74**, 31–40.
13. Ingersoll, J.E. (1987). *Theory of Financial Decision Making*, Rowman & Littlefield.
14. Kramer, H. and Schmid, W. (1998). On the average delay of control schemes, In: *Advances in Stochastic Models for Reliability*, *Quality and Safety*, Kahle, W., von Collani, E., Franz, J., and Jensen, U. (eds.), Birkhäuser, Boston, 341–360.
15. Page, E.S. (1954). Continuous inspection schemes. *Biometrika*, **41**, 100–114.
16. MacGregor, J.F. and Harris, T.J. (1993). The exponentially weighted moving variance. *Journal of Quality Technology*, **25**, 106–118.
17. Montgomery, D.C. and Mastrangelo, C.M. (1991). Some statistical process control methods for autocorrelated data. *Journal of Quality Technology*, **23**, 179–204.
18. Reynolds, M. and Stoumbos, Z. (2000). Individual control charts for monitoring the mean and the variance of processes subject to step changes and drifts, to appear In: *Nonlinear Analysis: Theory*, *Methods and Applications*.
19. Schmid, W. (1997). On EWMA charts for time series, In: *Frontiers in Statistical Quality Control* **5**, Lenz, H.J. and Wilrich, P.Th. (eds.), Physica-Verlag, Heidelberg, 115–137.
20. Schmid, W. and Schöne, S. (1997). Some properties of the EWMA control chart in the presence of autocorrelation. *Annals of Statistics*, **25**, 1277–1283.
21. Severin, T. and Schmid, W. (1999). Monitoring changes in GARCH processes. *Allgemeines Statistisches Archiv*, **83**, 281–307.
22. Siegmund, D. (1985). *Sequential Analysis*, Springer, New York.
23. Yashchin, E. (1993). Performance of CUSUM control schemes for serially correlated observations. *Technometrics*, **35**, 35–52.

Received October 2000
Revised June 2001
Recommended by M. Reynolds