Test Case 3 Find $u(x, 4)$ and $u(x, 40)$ where

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,$$

$$u(x, 0) = \begin{cases} 1 & \text{for } |x| < \frac{1}{3}, \\ 0 & \text{for } \frac{1}{3} < |x| \leq 1, \end{cases}$$

which is linear advection of a square wave. Use 600 evenly spaced grid points and $\lambda = \Delta t / \Delta x = 0.8$. This example illustrates convergence as $\Delta x \to 0$ and $\Delta t \to 0$. It also illustrates how dissipation, dispersion, and other numerical artifacts accumulate with large times for discontinuous solutions.

Test Case 4 Find $u(x, 0.6)$ where

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0,$$

$$u(x, 0) = \begin{cases} 1 & \text{for } |x| < \frac{1}{3}, \\ 0 & \text{for } \frac{1}{3} < |x| \leq 1. \end{cases}$$

Use 40 evenly spaced grid points and $\lambda = \Delta t/\Delta x = 0.8$. The scalar conservation law is Burgers' equation and the initial condition is a square wave. This problem was solved in Example 4.3. To review briefly, the jump from zero to one at $x = -1/3$ creates an expansion fan, while the jump from one to zero at $x = 1/3$ creates a shock. The unique sonic point for Burgers' equation is $u^* = 0$. Although the exact solution never crosses the sonic point, a numerical method with spurious overshoots or oscillations may cross the sonic point once or even several times, causing a dramatic error if the numerical method has sonic point problems.

Test Case 5 Find $u(x, 0.3)$ where

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0,$$

$$u(x, 0) = \begin{cases} 1 & \text{for } |x| < \frac{1}{3}, \\ -1 & \text{for } \frac{1}{3} < |x| \leq 1. \end{cases}$$

Use 40 evenly spaced grid points and $\lambda = \Delta t/\Delta x = 0.8$. The scalar conservation law is Burgers' equation and the initial condition is a square wave. This problem was solved in Example 4.4. To review briefly, the jump from minus one to one at $x = -1/3$ creates a sonic expansion fan, while the jump from one to minus one at $x = 1/3$ creates a steady shock. Notice that both the expansion fan and the shock symmetrically span the sonic point $u^* = 0$.

17.1 Lax–Friedrichs Method

This section concerns the Lax–Friedrichs method, discovered in 1954. To derive the Lax–Friedrichs method, first consider FTCS: forward time central space.

$$u_i^{n+1} = u_i^n - \frac{\lambda}{2}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right). \tag{11.16}$$