Example 1.2.3 Subsets

Let $A = \mathbf{Z}^+$, $B = \{n \in \mathbf{Z} \mid 0 \le n \le 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$
b. C is a proper subset of A
c. C and B have at least one element in common
d. $C \subseteq B$ e. $C \subseteq C$

Solution

a. False. Zero is not a positive integer. Thus zero is in B but zero is not in A, and so $B \not\subseteq A$.

b. True. Each element in C is a positive integer and, hence, is in A, but there are elements in A that are not in C. For instance, 1 is in A and not in C.

c. True. For example, 100 is in both C and B.

d. False. For example, 200 is in C but not in B.

e. True. Every element in C is in C. In general, the definition of subset implies that all sets are subsets of themselves.

Example 1.2.4 Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ b. $\{2\} \in \{1, 2, 3\}$ c. $2 \subseteq \{1, 2, 3\}$
d. $\{2\} \subseteq \{1, 2, 3\}$ e. $\{2\} \subseteq \{\{1\}, \{2\}\}$ f. $\{2\} \in \{\{1\}, \{2\}\}$

Solution Only (a), (d), and (f) are true.

For (b) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false.

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (c) is false.

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to either $\{1\}$ or $\{2\}$, and so (e) is false. ■

Cartesian Products

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Kazimierz Kuratowski (1896–1980)

With the introduction of Georg Cantor's set theory in the late nineteenth century, it began to seem possible to put mathematics on a firm logical foundation by developing all of its various branches from set theory and logic alone. A major stumbling block was how to use sets to define an ordered pair because the definition of a set is unaffected by the order in which its elements are listed. For example, $\{a, b\}$ and $\{b, a\}$ represent the same set, whereas in an ordered pair we want to be able to indicate which element comes first.

In 1914 crucial breakthroughs were made by Norbert Wiener (1894–1964), a young American who had recently received his Ph.D. from Harvard and the German mathematician Felix Hausdorff (1868–1942). Both gave definitions showing that an ordered pair can be defined as a certain type of set, but both definitions were somewhat awkward. Finally, in 1921, the Polish mathematician Kazimierz Kuratowski (1896–1980) published the following definition, which has since become standard. It says that an ordered pair is a set of the form

$$\{\{a\}, \{a, b\}\}.$$

This set has elements, $\{a\}$ and $\{a, b\}$. If $a \neq b$, then the two sets are distinct and a is in both sets whereas b is not. This allows us to distinguish between a and b and say that a is the first element of the ordered pair and b is the second element of the pair. If $a = b$, then we can simply say that a is both the first and the second element of the pair. In this case the set that defines the ordered pair becomes $\{\{a\}, \ \{a, a\}\}$, which equals $\{\{a\}\}$.

However, it was only long after ordered pairs had been used extensively in mathematics that mathematicians realized that it was possible to define them entirely in terms of sets, and, in any case, the set notation would be cumbersome to use on a regular basis. The usual notation for ordered pairs refers to $\{\{a\}, \{a, b\}\}$ more simply as (a, b).

• Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 1.2.5 Ordered Pairs

a. Is $(1, 2) = (2, 1)$?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of $(1, 1)$?

Solution

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2.1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But $1 \neq 2$, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair $(1, 1)$, the first and the second elements are both 1.

• Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 1.2.6 Cartesian Products

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

e. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Solution

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

Note This is why it makes sense to call a Cartesian product a product!

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B. $B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

e. $\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs (x, y) where both x and y are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point. The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.

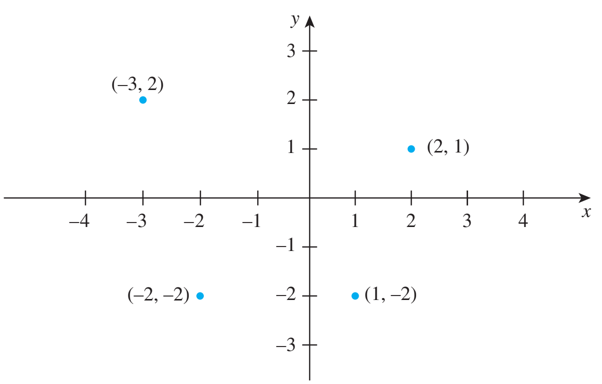

Figure 1.2.1: A Cartesian Plane

Test Yourself

1. When the elements of a set are given using the set-roster notation, the order in which they are listed ____.

2. The symbol $\mathbf{R}$ denotes ____.

3. The symbol $\mathbf{Z}$ denotes ____.

4. The symbol $\mathbf{Q}$ denotes ____.

5. The notation $\{x \mid P(x)\}$ is read ____.

6. For a set A to be a subset of a set B means that, ____.

7. Given sets A and B, the Cartesian product $A \times B$ is ____.