Cantor did, simply as a collection of elements. For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

• Notation

If S is a set, the notation $\boldsymbol{x \in S}$ means that x is an element of S. The notation $\boldsymbol{x \notin S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . .}** to refer to the set of all positive integers. (The symbol **. . .** is called an **ellipsis** and is read "and so forth.")

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1.2.1 Using the Set-Roster Notation

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Solution

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

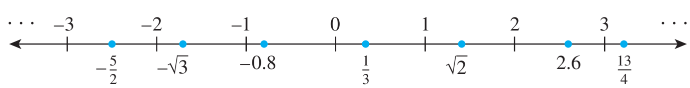
Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the table on the next page.

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

Note The **Z** is the first letter of the German word for integers, *Zahlen*. It stands for the *set* of all integers and should not be used as a shorthand for the word *integer*.

Addition of a superscript $+$ or $-$ or the letters *nonneg* indicates that only the positive or negative or nonnegative elements of the set, respectively, are to be included. Thus $\mathbf{R}^+$ denotes the set of positive real numbers, and $\mathbf{Z}^{nonneg}$ refers to the set of nonnegative integers: 0, 1, 2, 3, 4, and so forth. Some authors refer to the set of nonnegative integers as the set of **natural numbers** and denote it as **N**. Other authors call only the positive integers natural numbers. To prevent confusion, we simply avoid using the phrase *natural numbers* in this book.

The set of real numbers is usually pictured as the set of all points on a line, as shown below. The number 0 corresponds to a middle point, called the *origin*. A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. *Note that 0 is neither positive nor negative* Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes. The set of integers corresponds to a collection of points located at fixed intervals along the real number line. Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

Note We read the left-hand brace as "the set of all" and the vertical line as "such that." In all other mathematical contexts, however, we do not use a vertical line to denote the words "such that"; we abbreviate "such that" as "s. t." or "s. th." or "$\cdot \ni \cdot$."

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements *x* in *S* such that *P(x)* is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all such that

Occasionally we will write $\{x \mid P(x)\}$ without being specific about where the element x comes from. It turns out that unrestricted use of this notation can lead to genuine contradictions in set theory. We will discuss one of these in Section 6.4 and will be careful to use this notation purely as a convenience in cases where the set S could be specified if necessary.

Example 1.2.2 Using the Set-Builder Notation

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

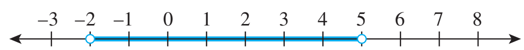
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

Solution

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between -2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between -2 and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$. ■

Subsets

A basic relation between sets is that of subset.

• Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

It follows from the definition of subset that for a set A not to be a subset of a set B means that there is at least one element of A that is not an element of B. Symbolically:

$A \nsubseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.