Introduction to quantum mechanics of the single system

한정연

Textbook Link

Lecture By Prof. John Watrous

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0. Overview

Quantum Mechanics 양자역학

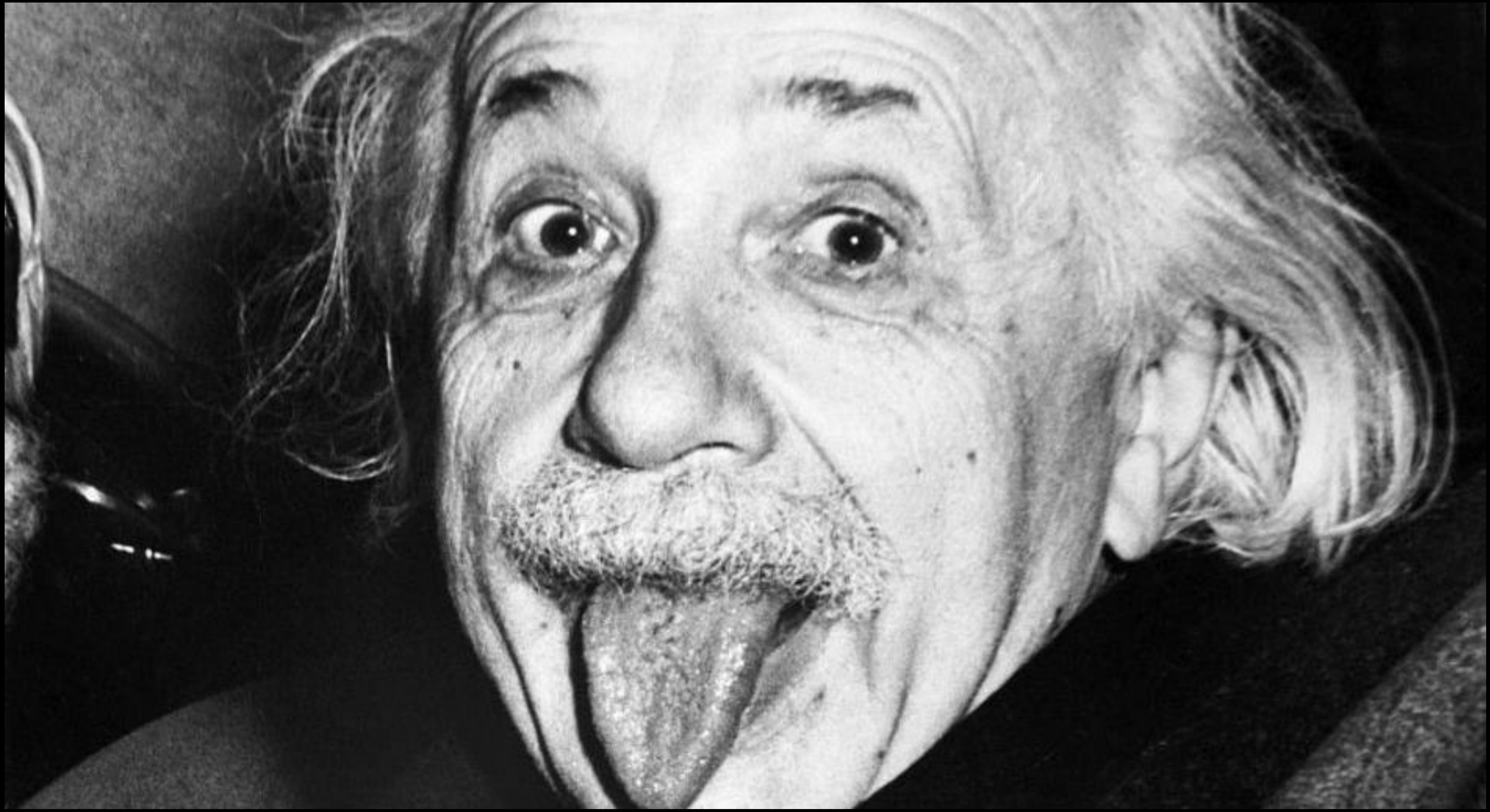
$\cup$

Quantum Information 양자정보

0. Overview

양자역학 = 원자/소립자 스케일의 물리

Probability Storm ????!!??!!



신은 주사위 놀이를 하지 않는다.

- Albert Einstein -

0. Overview

Note

- 물리적인 내용 및 Origin도 소개할 예정

- 하지만, 양자정보의 수학적 기술에 초점을 맞춰서 보시길 바랍니다 ☺

- 물리적 내용에 관심이 더 있으시다면 ‘양자역학’ 강의를 추천합니다.

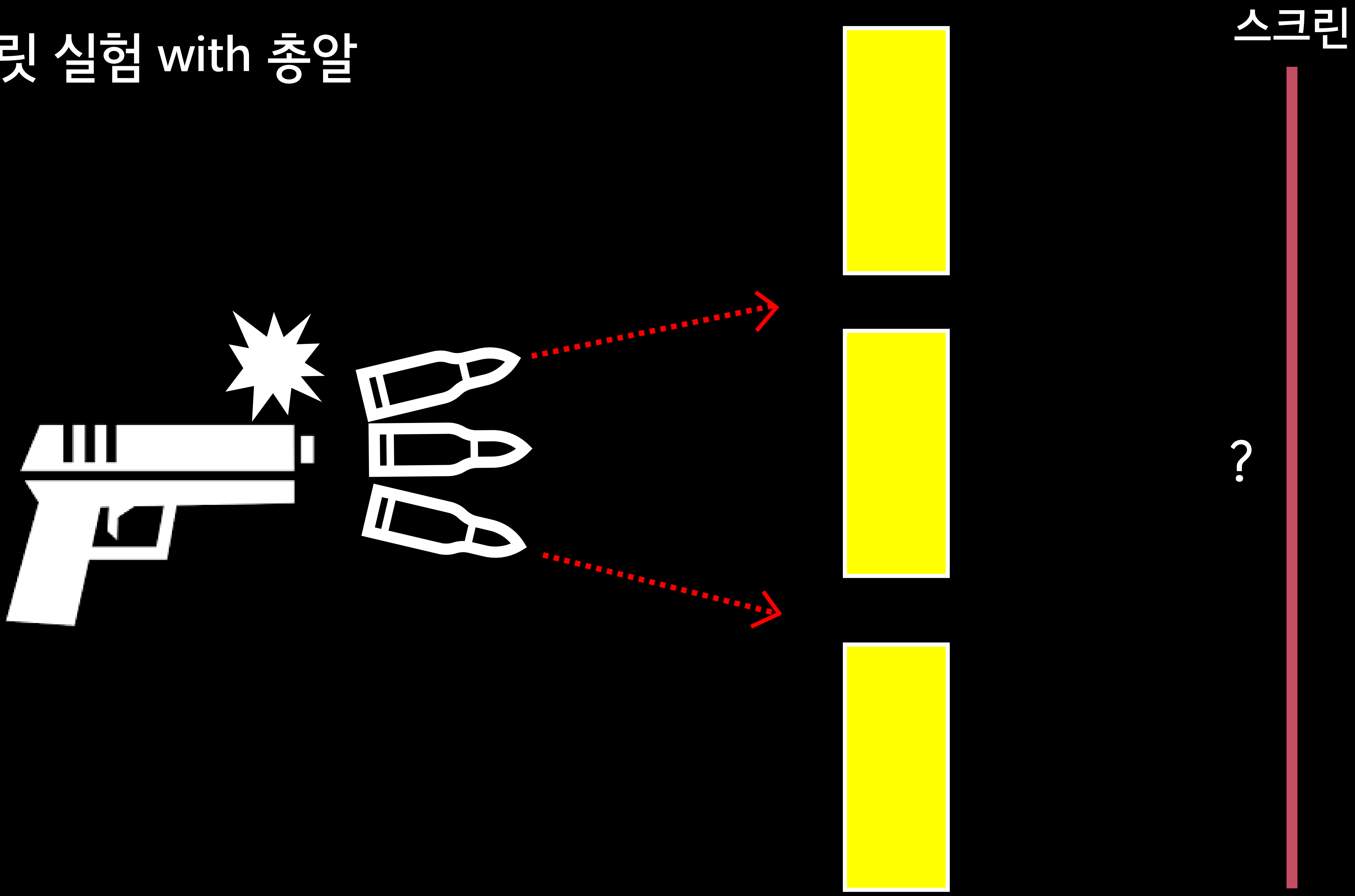
0. Overview

Keywords

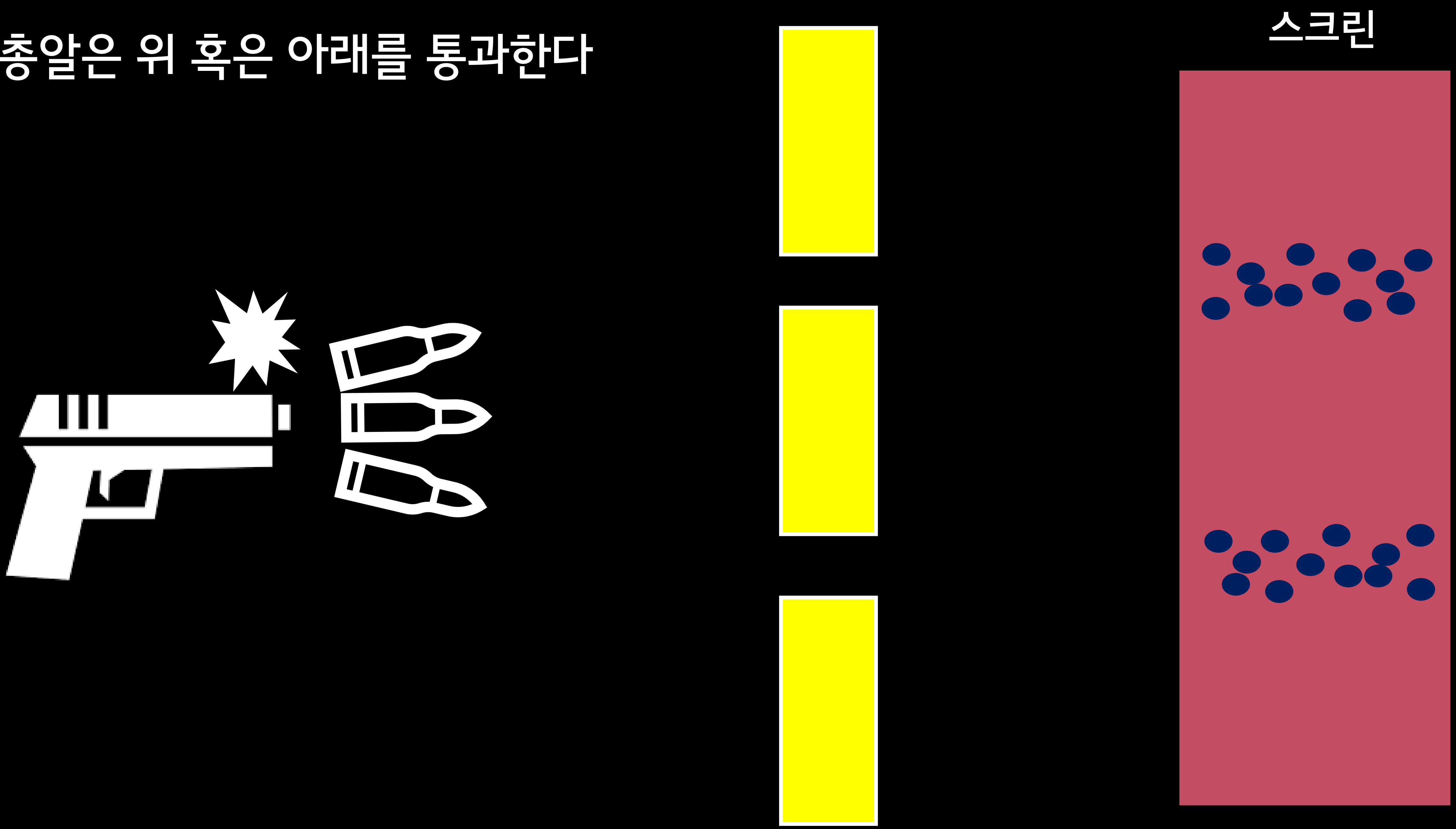
- Quantum State (Vector) ➔ Qubit
- Measurement
- Unitary Operation

1. 고전 확률론

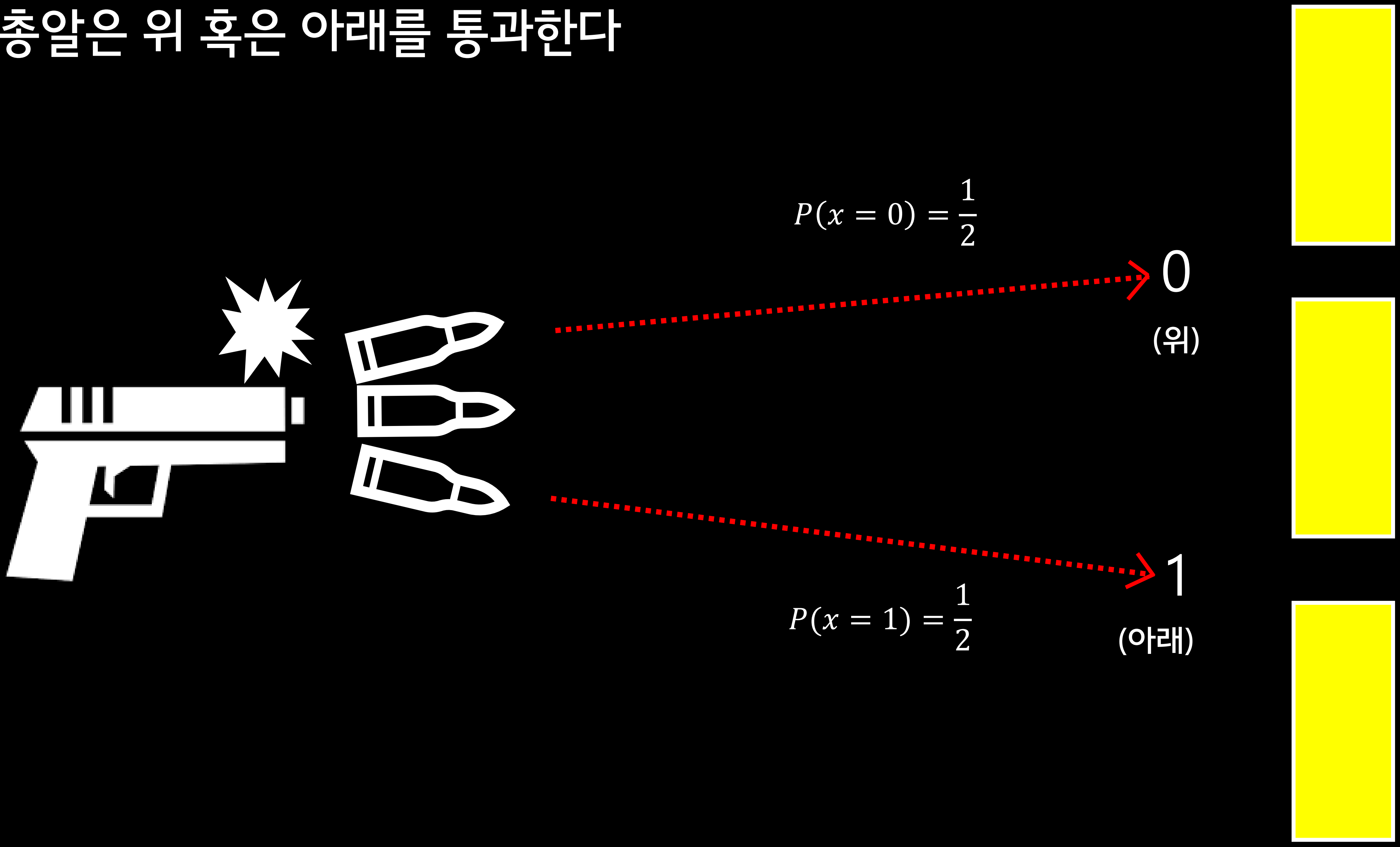
이중슬릿 실험 with 총알



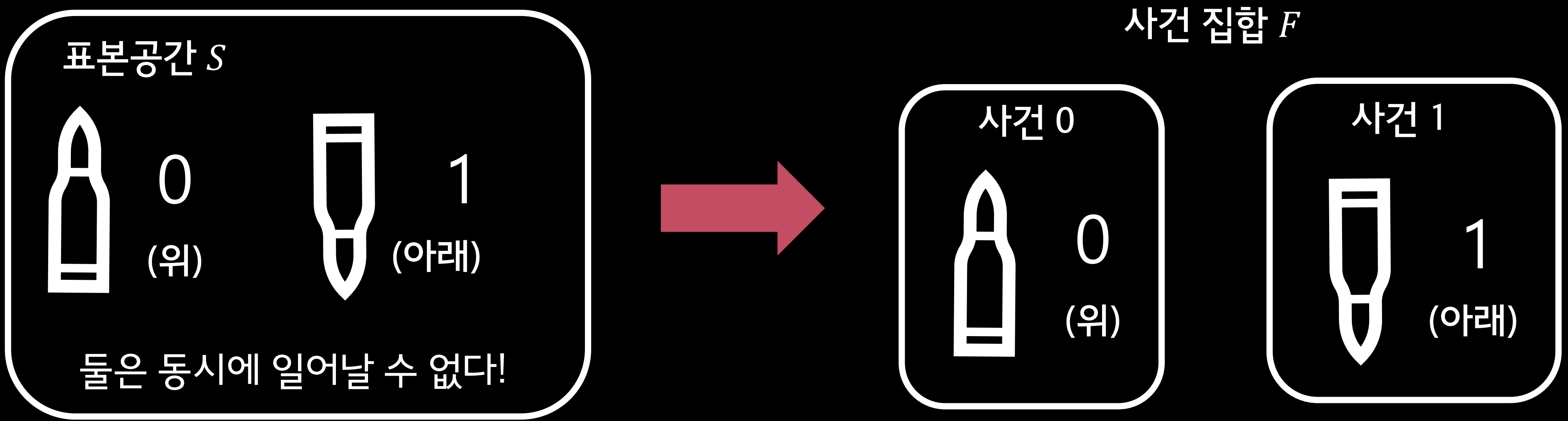
총알은 위 혹은 아래를 통과한다



총알은 위 혹은 아래를 통과한다



확률의 공리적 정의 (A. Kolmogorov)



1 $P(S) = 1$ $\Rightarrow$ $P(x=0) + P(x=1) = 1$

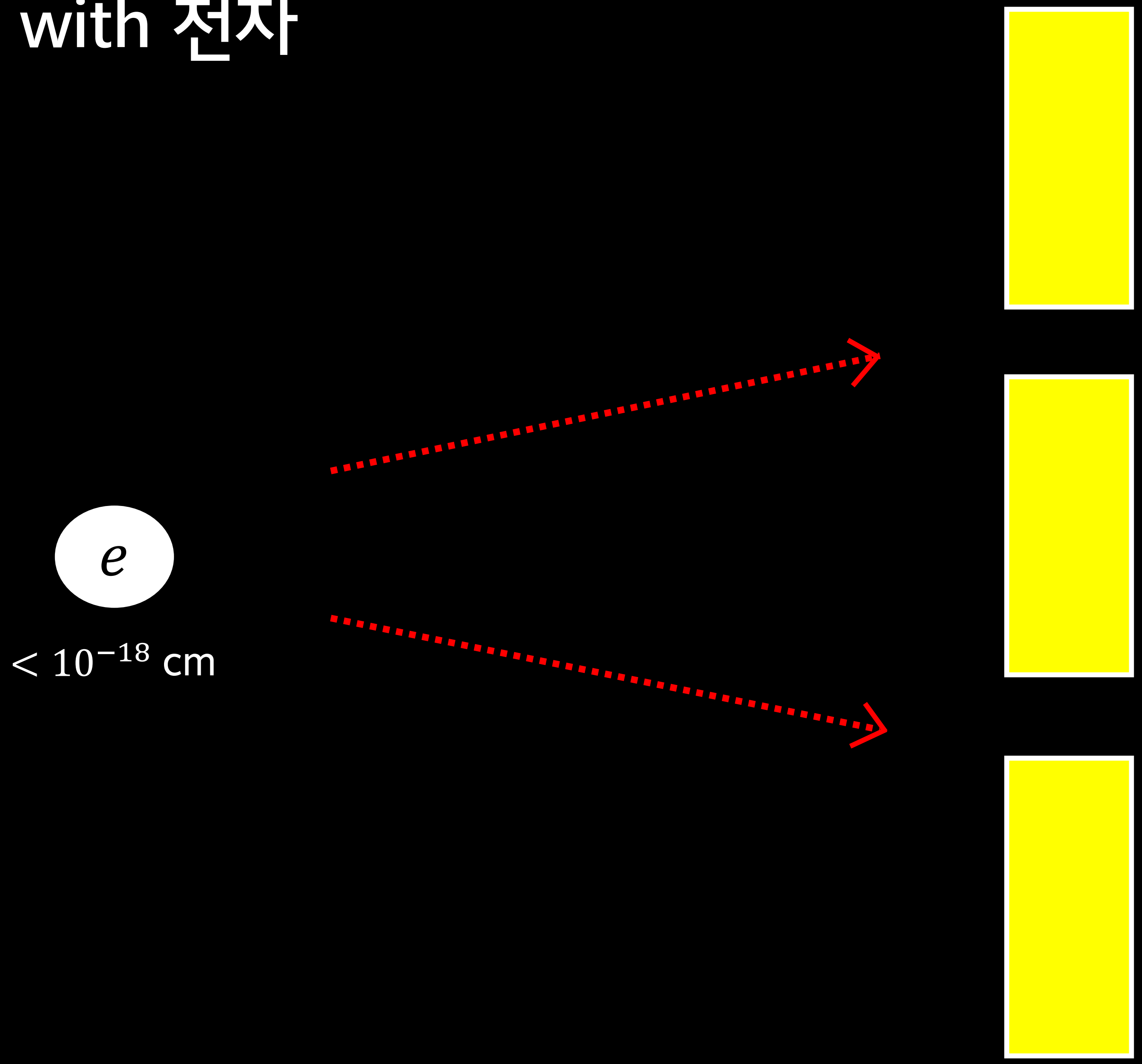
2 $\forall\, x \in F, 0 \leq P(x) \leq 1$ $\Rightarrow$ $P(x=0) = 1/2,\quad P(x=1) = 1/2$

3 $P(x^c) = 1 - P(x),\quad P(x = A \cup B) = P(x = A) + P(x = B) - P(x = A \cap B)$

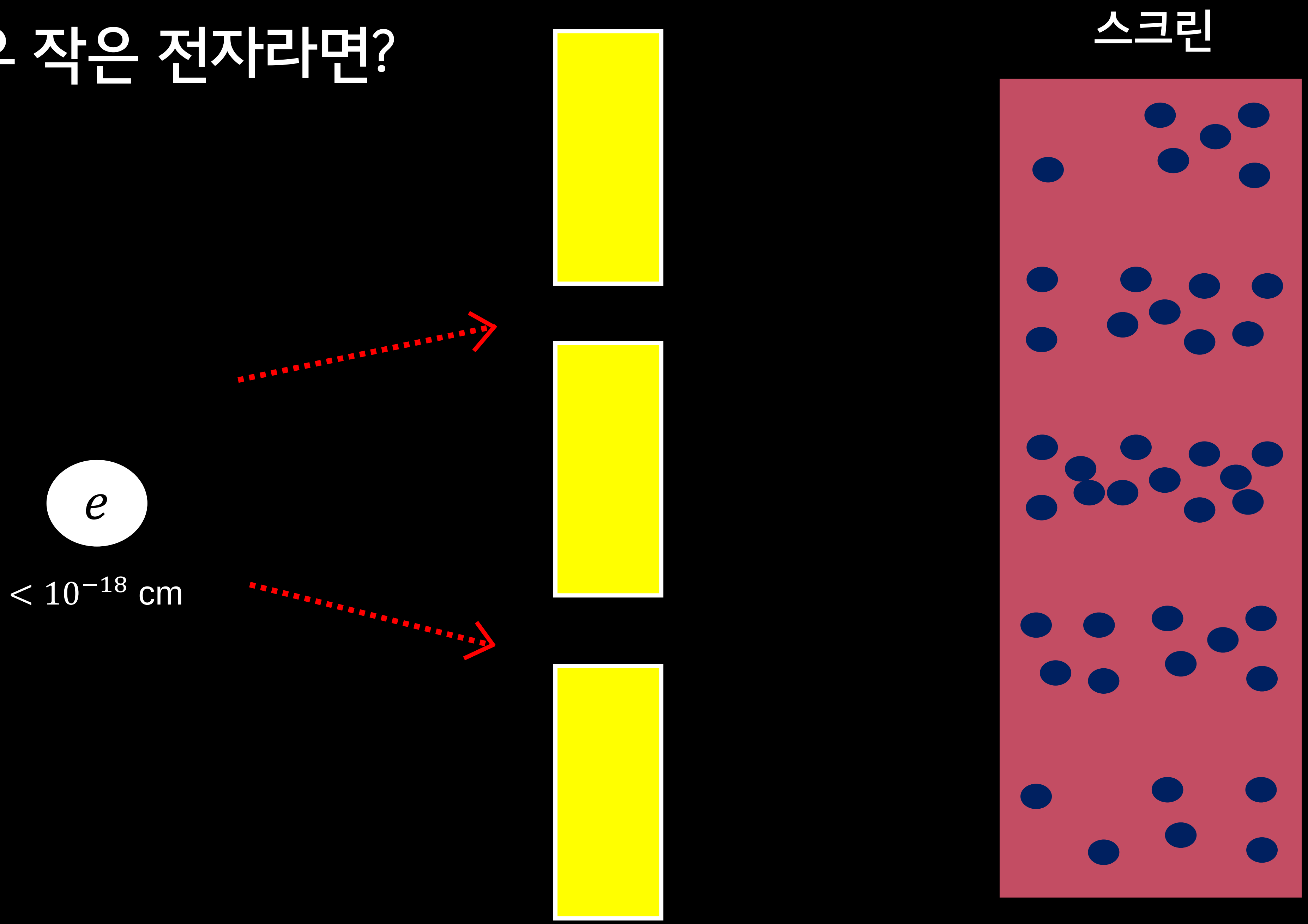
$\Rightarrow$ $P(x=0) = 1 - P(x=1),\quad P(x = 0 \cup 1) = P(x=0) + P(x=1)$

2. Quantum Bit (Qubit) and State

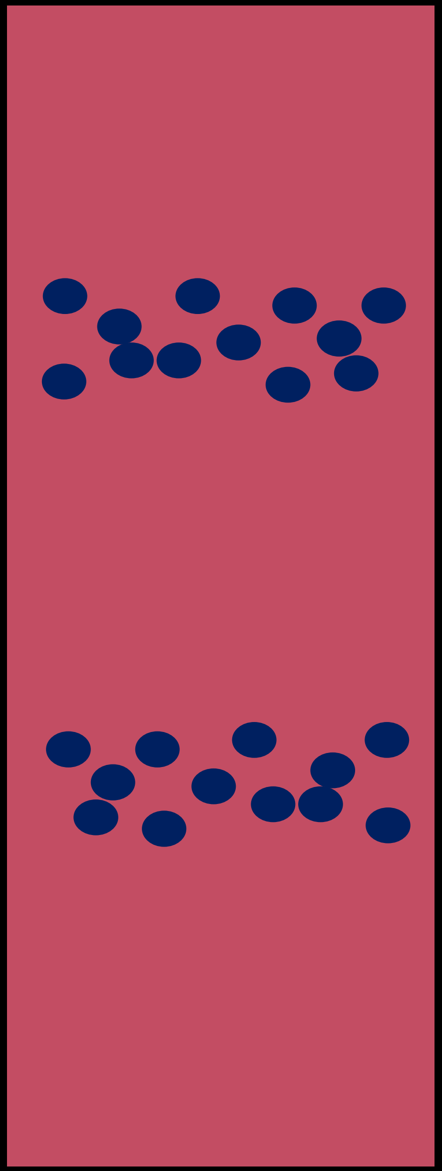
이중슬릿 실험 with 전자



총이 아닌 매우 작은 전자라면?

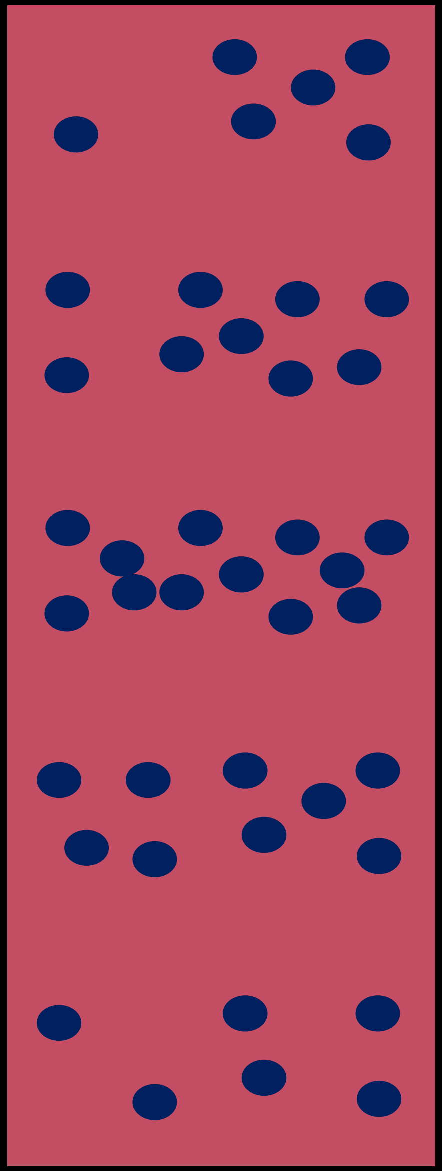


스크린 (총알)

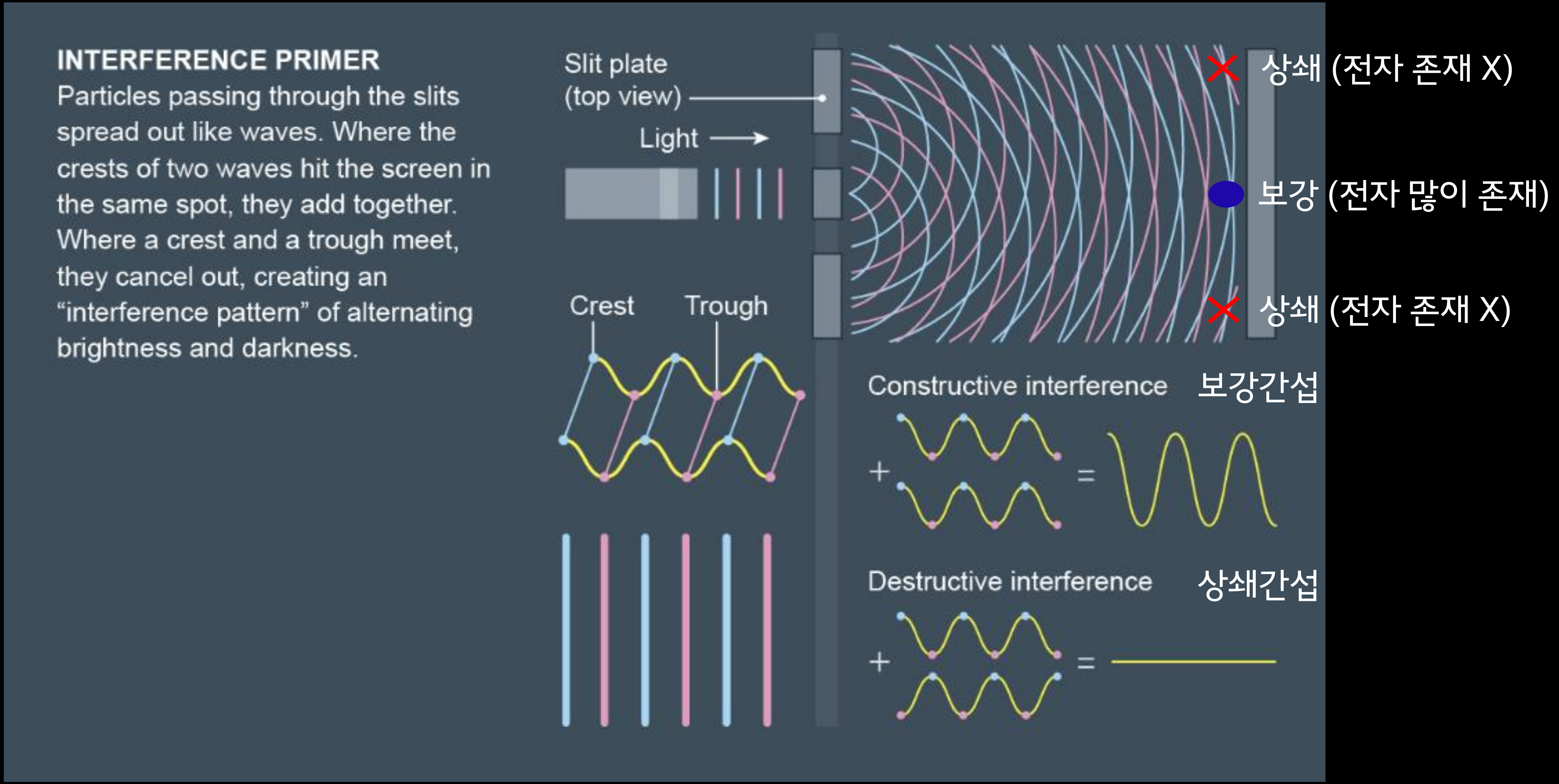


차이가 뭐길래?

스크린 (전자)



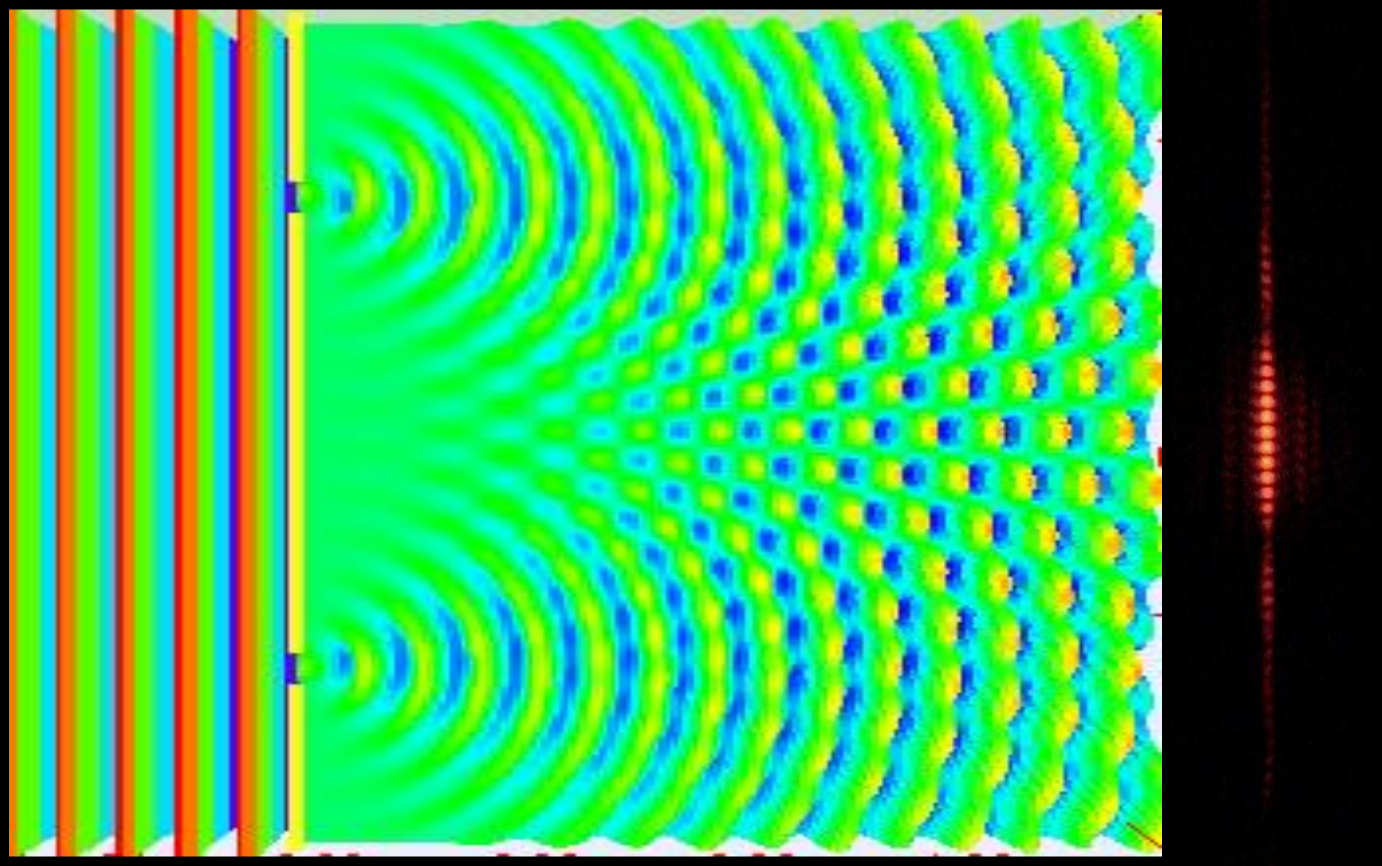
전자 = 입자이자 파동



Credit from Nick Bockelman

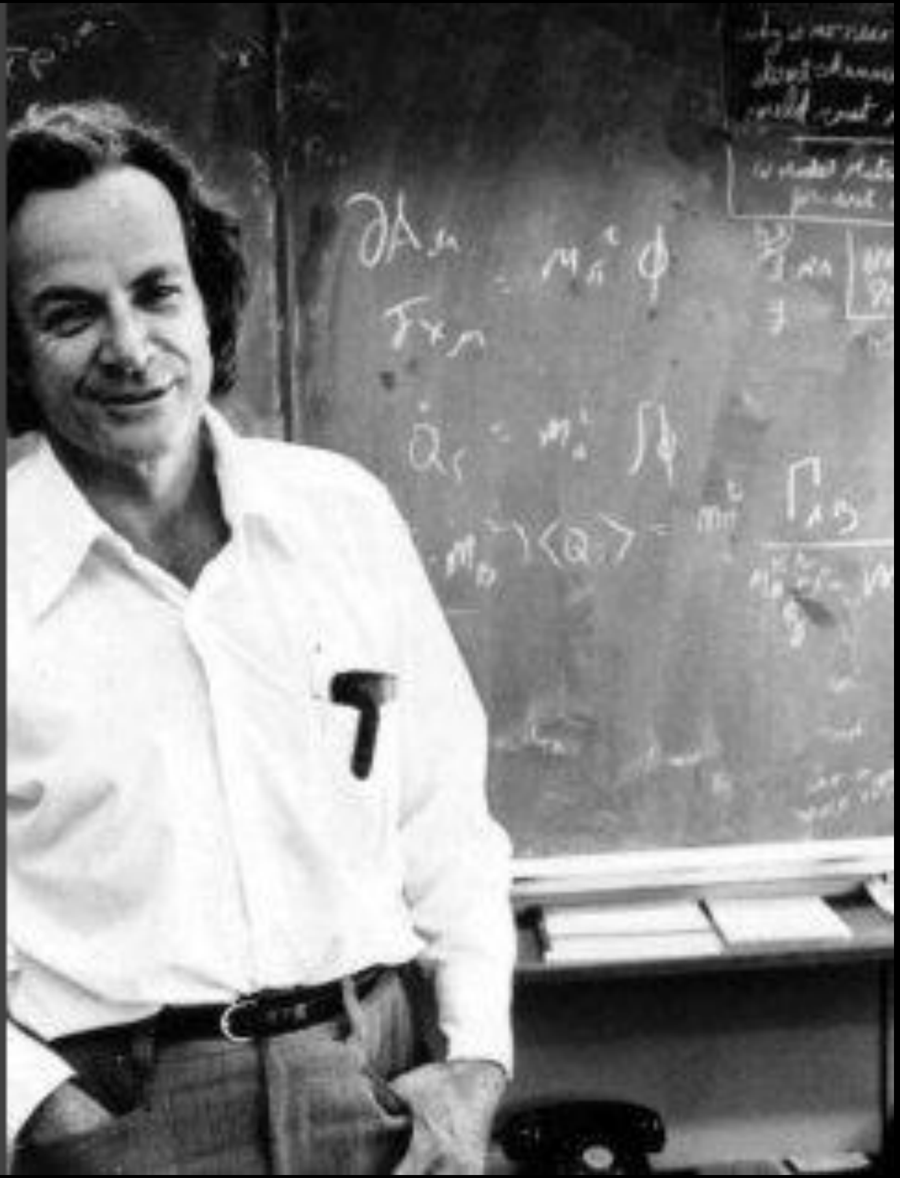
양자역학이 어려운 이유

답: 전자는 파동이 되어 위와 아래를 동시에 통과한다.



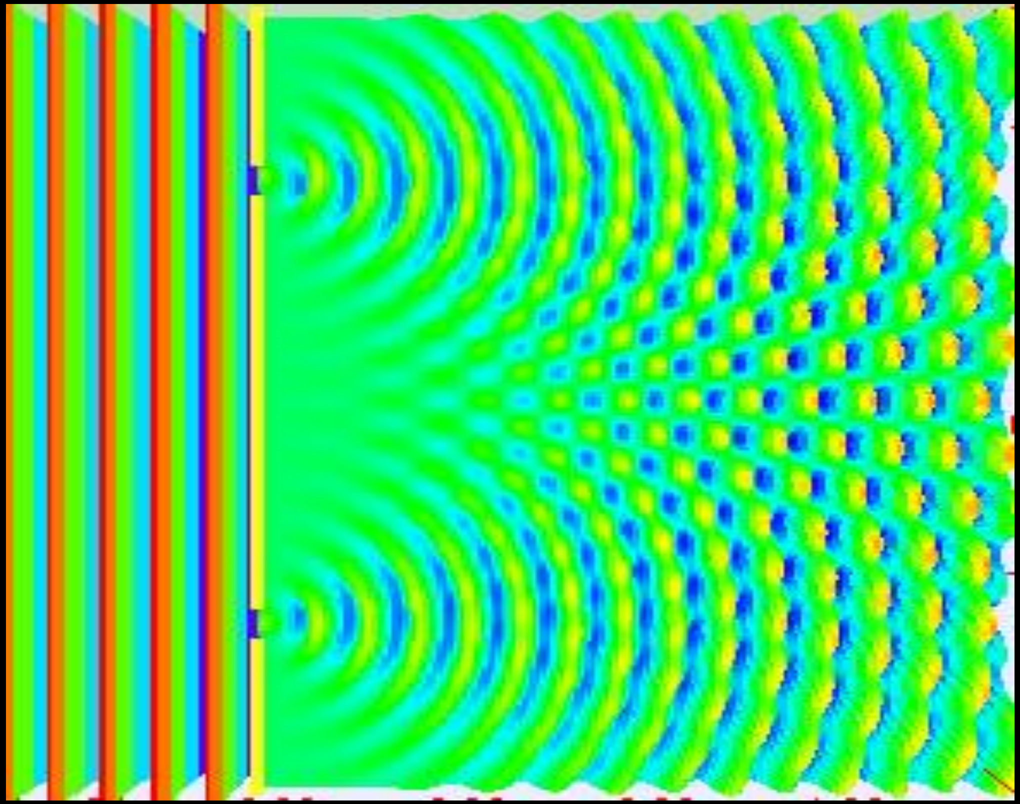
I think I can safely say that nobody understands quantum mechanics.

Richard Feynman



양자역학에서 양자정보로

중첩 원리 (Superposition Principle)



양자역학

위와 아래로 동시에 간다.

양자정보

0과 1을 동시에 표현한다.

Classical Bit

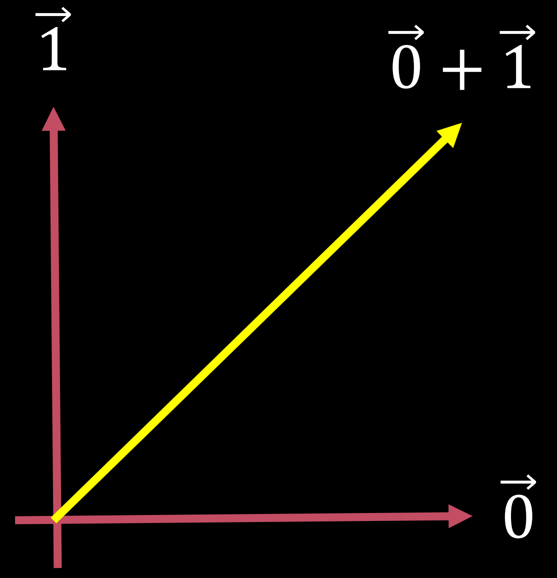
$$0 + 1$$

$$=$$

$$1$$

Quantum Bit (Qubit)

$$\neq \quad \vec{0} + \vec{1}$$



Qubit (Dirac's Notation Bra-Ket Notation)

	2차원 벡터 공간의 Basis		2차원 Basis Ket (Dirac's Notation)
위로가는 상태 (0)	$\vec{0}$	➡	$\vert 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
아래로가는 상태 (1)	$\vec{1}$		$\vert 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

이중슬릿의 상태 ➔ 위와 아래로 동시에 가는 상태

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

정규화 상수

Qubit의 일반적 정의 (Dirac's Bra-Ket Notation)

$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$$

1. $\alpha, \beta \in \mathbb{C}$ (복소수)
2. $|\alpha|^2 = \alpha^* \times \alpha \geq 0, |\beta|^2 = \beta \times \beta^* \geq 0$
3. $|\alpha|^2 + |\beta|^2 = 1$

If $|\alpha|^2 + |\beta|^2 = N(\neq 1)$ ➡ $\frac{|\psi\rangle}{\sqrt{N}} = \frac{\alpha}{\sqrt{N}}|0\rangle + \frac{\beta}{\sqrt{N}}|1\rangle$

➔ Qubit!

Note

허수: $i = \sqrt{-1}$

복소수: 실수와 허수의 합 (x, y는 실수)

$$\alpha = x + yi$$

복소수 켤레: 허수의 부호를 바꿈

$$\alpha^* = x - yi$$

복소수 크기

$$|\alpha|^2 = \alpha \times \alpha^* = x^2 + y^2$$

Exercise

$$|\psi\rangle = \frac{1}{3}|0\rangle + \frac{2}{3}|1\rangle = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

1. 올바른 Qubit 인가?

2. 아니라면 어떻게 변환해야 하는가?

Exercise

$$|\psi\rangle = \frac{1}{3}|0\rangle + \frac{2}{3}|1\rangle = \frac{1}{3}\binom{1}{2}$$

1. 올바른 Qubit 인가?

X, 정규화 계수 $\left|\frac{1}{3}\right|^2 + \left|\frac{2}{3}\right|^2 = \frac{5}{9} \neq 1$

2. 아니라면 어떻게 변환해야 하는가?

$|\psi\rangle$을 정규화 계수의 root, $\sqrt{5/9}$로 나눠야 함!

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle = \frac{1}{\sqrt{5}}\binom{1}{2}}$$

Exercise

$$|\psi\rangle = (1+2i)|0\rangle - 2|1\rangle = \begin{pmatrix} 1+2i \\ -2 \end{pmatrix}$$

1. 올바른 Qubit 인가?

2. 아니라면 어떻게 변환해야 하는가?

Exercise

$$|\psi\rangle = (1+2i)|0\rangle - 2|1\rangle = \begin{pmatrix} 1+2i \\ -2 \end{pmatrix}$$

1. 올바른 Qubit 인가?

X, 정규화 계수 $|1+2i|^2 + |-2|^2 = 9 \neq 1$

2. 아니라면 어떻게 변환해야 하는가?

$|\psi\rangle$을 정규화 계수의 root, $\sqrt{9} = 3$으로 나눠야 함!

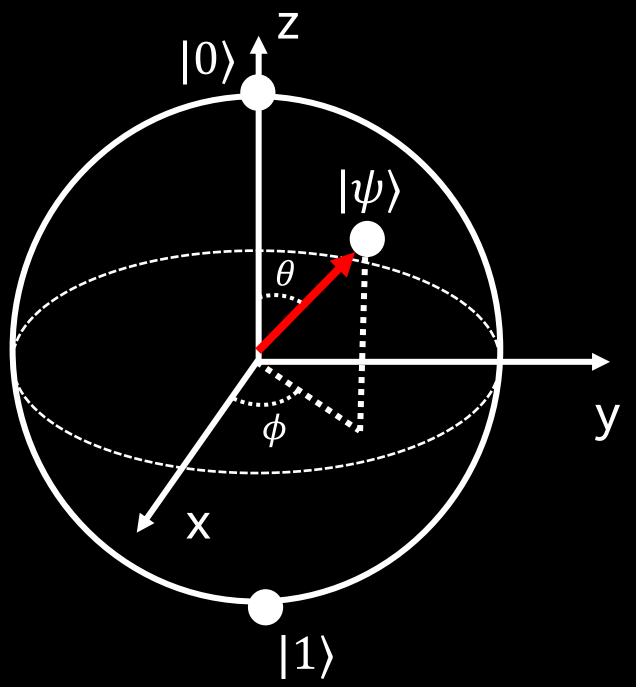
$$\boxed{|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle = \frac{1}{3}\begin{pmatrix} 1+2i \\ -2 \end{pmatrix}}$$

Qubit on the Bloch sphere (3차원 구)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$0 \le \theta \le \pi$

$0 \le \phi \le 2\pi$



알아두면 좋은 양자 상태들

양자상태	θ	ϕ	참고
$\mathbf{|0\rangle}$	0	0	북극 (z축)
$\mathbf{|1\rangle}$	π	0	남극 (z축)
$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{|0\rangle + |1\rangle})$	$\pi/2$	0	x축
$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{|0\rangle - |1\rangle})$	$\pi/2$	π	x축
$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{|0\rangle + \mathit{i}|1\rangle})$	$\pi/2$	$\pi/2$	y축
$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{|0\rangle - \mathit{i}|1\rangle})$	$\pi/2$	$3\pi/2$	y축

Q. $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ 같은 상태는 어떻게 구현? → 잠시후!

Conjugate quantum state (Bra state)

Quantum State (Vector) or Bra

$$\boxed{\langle\psi| = \langle 0|\alpha^* + \langle 1|\beta^* = (\alpha^* \; \beta^*)}$$

$$\boxed{\langle\psi| := (|\psi\rangle^*)^T := |\psi\rangle^+}$$

$$\langle 0| = (1 \; 0)$$

$$\langle 1| = (0 \; 1)$$

벡터의 내적

$$\langle\psi|\psi\rangle = (\alpha^* \; \beta^*)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1$$

$$\langle 0|0\rangle = (1 \; 0)\begin{pmatrix}1\\ 0\end{pmatrix} = 1$$

$$\langle 1|1\rangle = (0 \; 1)\begin{pmatrix}0\\ 1\end{pmatrix} = 1$$

$$\langle 0|1\rangle = (1 \; 0)\begin{pmatrix}0\\ 1\end{pmatrix} = 0$$

Exercise

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle = \frac{1}{3}\begin{pmatrix} 1+2i \\ -2 \end{pmatrix}$$

$$\Downarrow$$

$$\langle\psi| = \ ?$$

Exercise

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle = \frac{1}{3}\begin{pmatrix} 1+2i \\ -2 \end{pmatrix}$$

$$\Downarrow$$

$$\langle\psi| = \langle 0|\frac{1-2i}{3} - \langle 1|\frac{2}{3} = \frac{1}{3}\begin{pmatrix} 1-2i & -2 \end{pmatrix}$$

Inner Product vs Outer Product

- 양자상태 $|\psi\rangle$ (Ket state), $\langle\psi|$ (Bra state)가 주어졌을 때,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \qquad \langle\psi| = \langle 0|\alpha^* + \langle 1|\beta^* = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$$

- Inner Product: 양자 상태의 내적 ➔ 숫자 (scalar)를 줘야 함

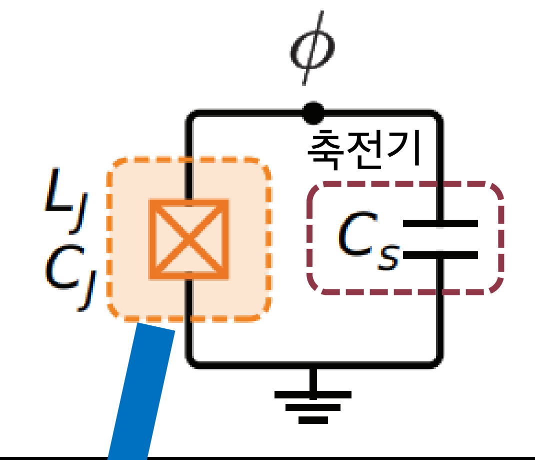
$$\langle\psi|\psi\rangle = \alpha \times \alpha^* + \beta \times \beta^* = 1$$

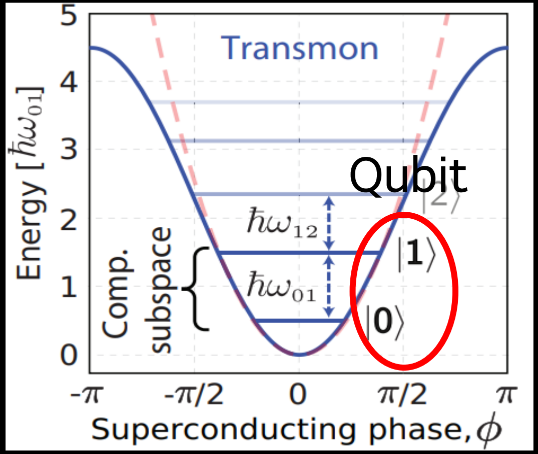
- Outer Product: 양자 상태의 외적 ➔ 행렬을 줘야 함

$$|\psi\rangle\langle\psi| = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} = \begin{pmatrix}\alpha \times \alpha^* & \alpha \times \beta^* \\ \beta \times \alpha^* & \beta \times \beta^*\end{pmatrix} = \begin{pmatrix}|\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2\end{pmatrix}$$

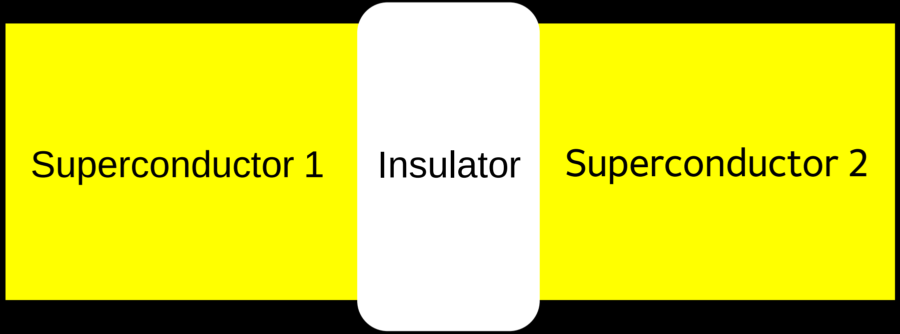
부록: Qubit의 예시

IBM: 초전도 Transmon Qubit

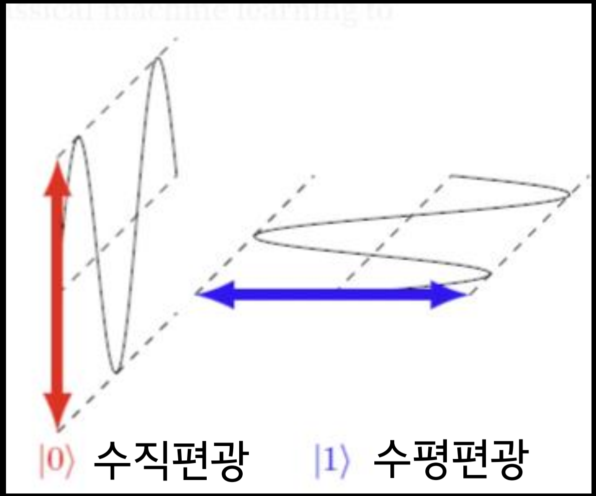




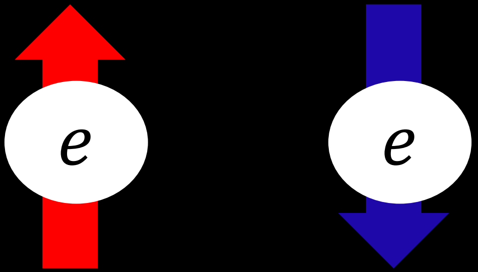
Josephson junction (초전도체 + 절연체)



Photonic Qubit (빛의 편광)



Electronic spin Qubit (전자 스핀)



Reference: Applied Physics Reviews **6**, 021318 (2019)

Quantum state (Dirac's Bra-Ket Notation)

$$\boxed{|\psi\rangle = \sum_{n=0} c_n\,|n\rangle = c_0|0\rangle + c_1|1\rangle + \ \ldots}$$

1. $\forall n, c_n \in \mathbb{C}$

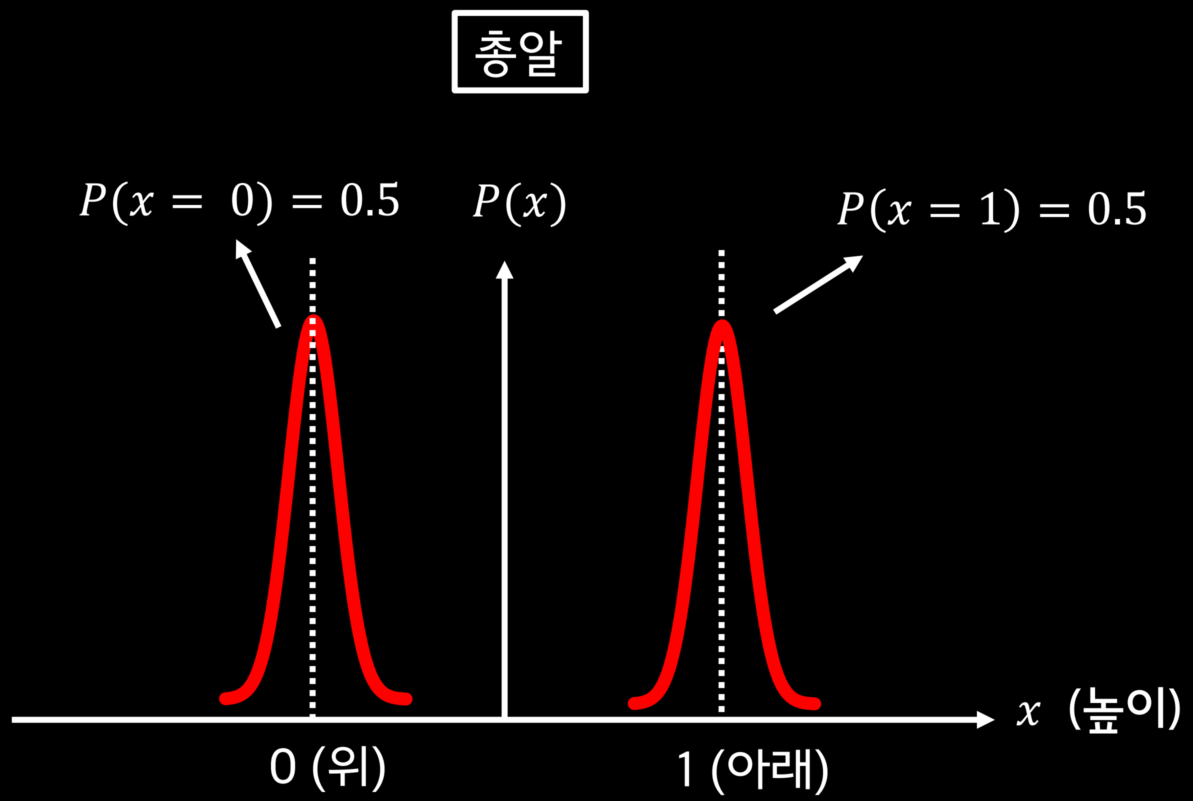
2. $\forall n, |c_n|^2 \geq 0$

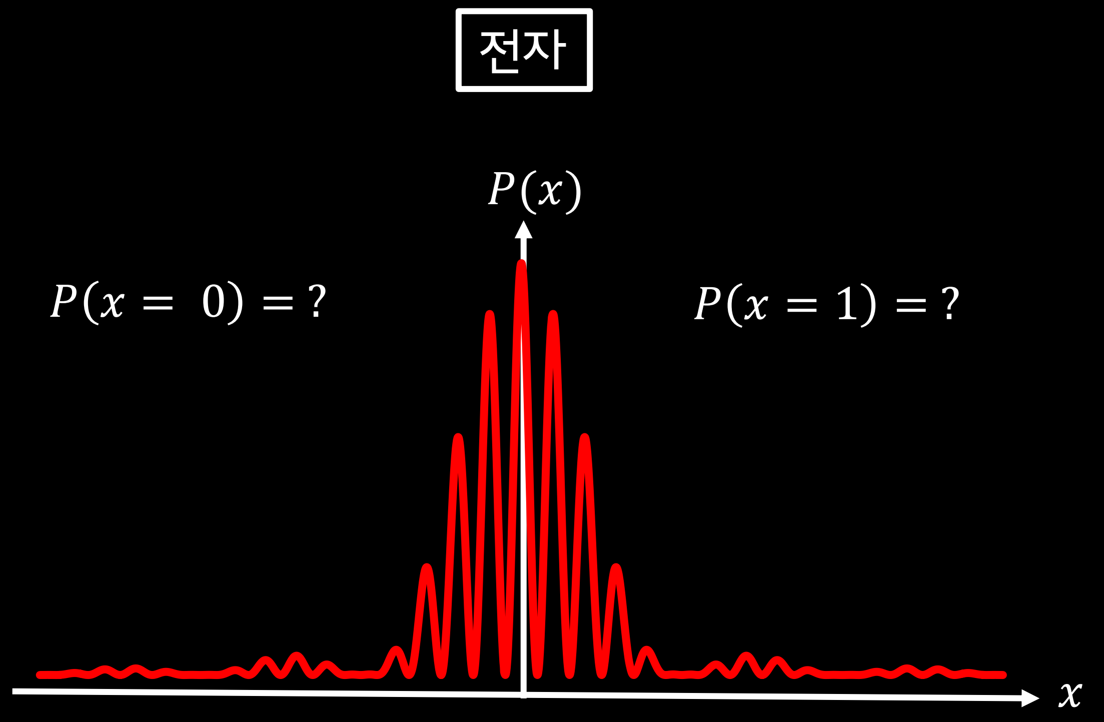
3. $\langle\psi|\psi\rangle = \sum_{n=0}|c_n|^2 = 1$

Ex. $\frac{1}{\sqrt{385}}\sum_{k=0}^{9}(k+1)\,|k\rangle = \frac{1}{\sqrt{385}}\begin{pmatrix}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\end{pmatrix}$

3. 관측 및 확률론

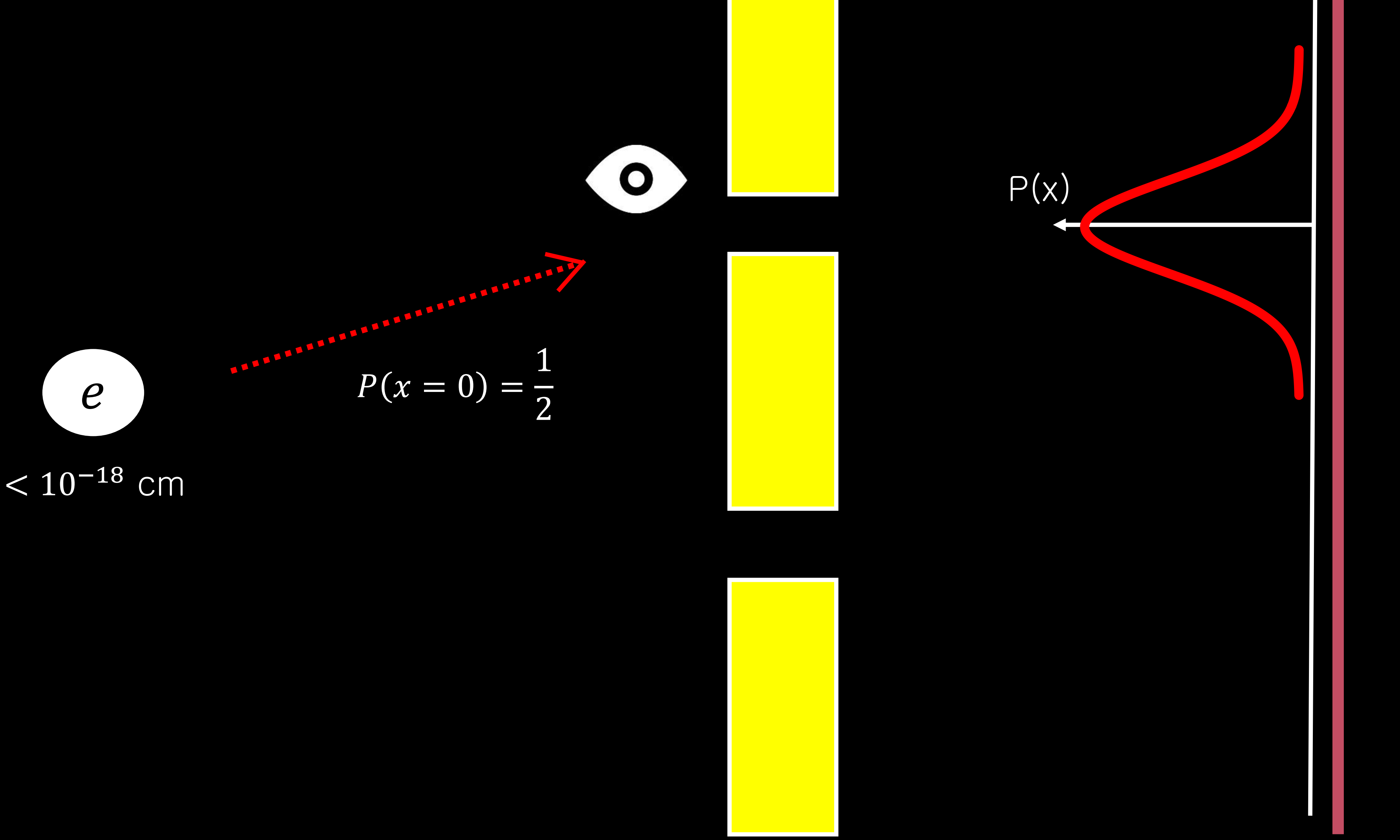
확률 분포



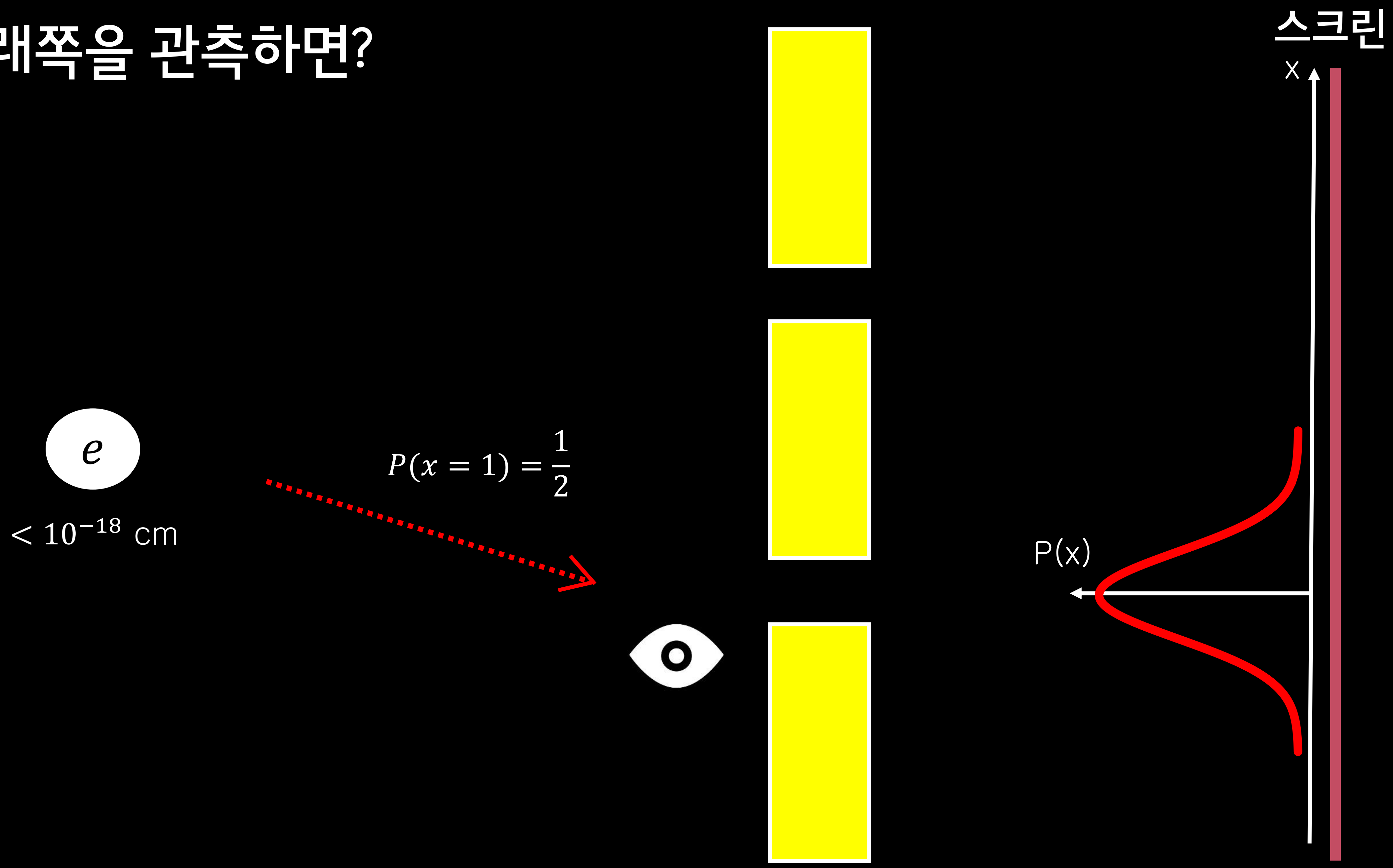


전자는 진짜 양쪽에 둘 다 존재하는 건가?? ➔ 관측

위 쪽을 관측하면??



아래쪽을 관측하면?



Measurement Formulation

Qubit

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

0을 (위쪽) 관측

$\frac{1}{2}$ 의 확률로 $|0\rangle$

$\frac{1}{2}$ 의 확률로 아무것도 관측 못 함

1을 (아래쪽) 관측

$\frac{1}{2}$ 의 확률로 아무것도 관측 못 함

$\frac{1}{2}$ 의 확률로 $|1\rangle$

Measurement Formulation

Qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

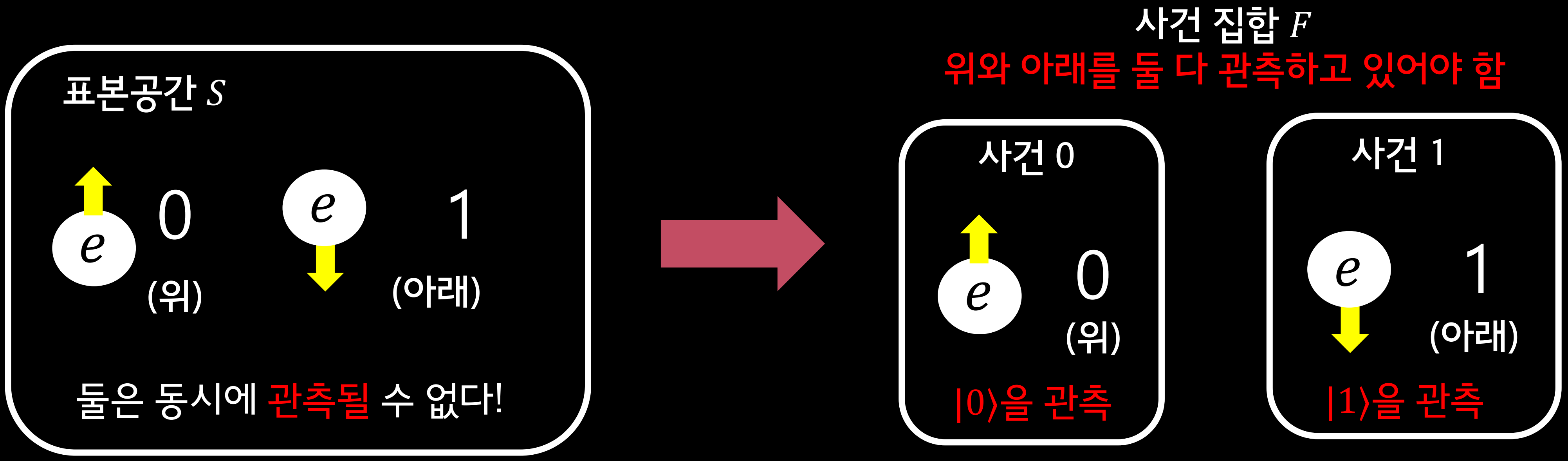
0을 관측할 확률

$$P(x=0) = |\langle 0|\psi\rangle|^2 = \langle 0|\psi\rangle \times \langle \psi|0\rangle = |\alpha|^2 \quad (|0\rangle\text{의 계수})$$

1을 관측할 확률

$$P(x=1) = |\langle 1|\psi\rangle|^2 = \langle 1|\psi\rangle \times \langle \psi|1\rangle = |\beta|^2 \quad (|1\rangle\text{의 계수})$$

확률의 공리적 정의 (A. Kolmogorov)



1 $P(S) = 1$ $\Rightarrow$ $P(x=0) + P(x=1) = 1$ $\Rightarrow$ Qubit을 정규화 해야 하는 이유!

2 $\forall\, x \in F, 0 \leq P(x) \leq 1$ $\Rightarrow$ $P(x=0) = 1/2,\quad P(x=1) = 1/2$

3 $P(x^c) = 1 - P(x),\qquad P(x = A \cup B) = P(x = A) + P(x = B) - P(x = A \cap B)$

$\Rightarrow$ $P(x=0) = 1 - P(x=1),\quad P(x = 0 \cup 1) = P(x=0) + P(x=1)$

Exercise

이중슬릿

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$P(x=0) = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$P(x=1) = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

Exercise

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

$$P(x=0) = \,?$$

$$P(x=1) = \,?$$

Exercise

$$|\psi\rangle = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

$$P(x=0) = \left(\frac{1+2i}{3}\right) \times \left(\frac{1-2i}{3}\right) = \frac{5}{9}$$

$$P(x=1) = \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) = \frac{4}{9}$$

4. Unitary 행렬 및 Quantum operations

Schrödinger's Equation

Qubit을 포함한 양자상태가 시간에 따라서 어떻게 바뀌는지 기술하는 미분방정식

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

시간에 대한 미분

양자상태가 시간의 함수가 됨

Hamiltonian (해밀토니안)
→ 양자상태를 움직이는 Operator

미분방정식 해 (해밀토니안은 시간에 따라 바뀌지 않는다 가정)

$$|\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi(0)\rangle$$

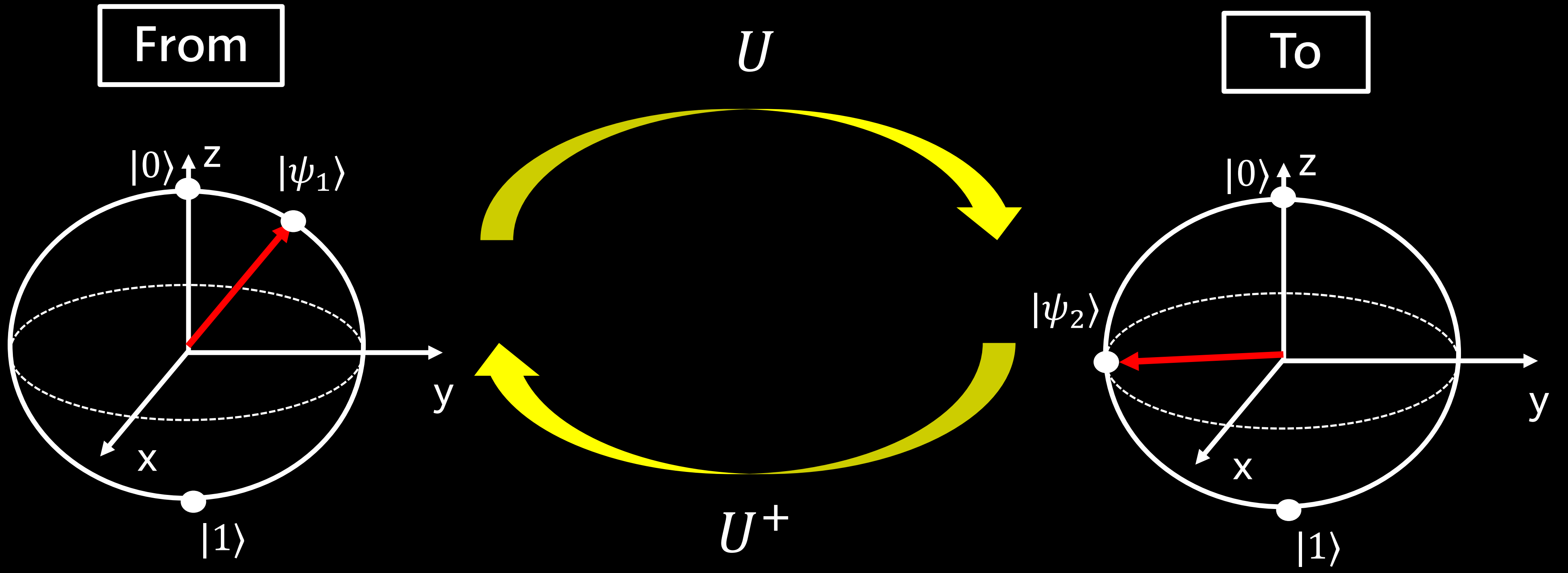
t초 때 양자상태

Unitary operator

0초 때 양자상태 (기존 양자상태)

Unitary Operator

양자상태의 변화를 기술하는 Operator (= 양자상태 $|\psi_1\rangle$에서 양자상태 $|\psi_2\rangle$로 연결)



$$|\psi_1\rangle = U^+|\psi_2\rangle$$

Unitary operator (conjugate)

$$|\psi_2\rangle = U|\psi_1\rangle$$

Unitary operator

Unitary Operator

양자상태가 벡터로 표현되는 것처럼, 양자 상태에 대한 Unitary operator는 행렬로 표현될 수 있음!

성질

U: 사각 행렬 (행과 열수가 같음)

$U^+ = (U^*)^T$ (Hermitian conjugate)

$UU^+ = U^+U = I$ (단위 행렬)

$U^+ = U^{-1}$ (역 행렬)

예시

U

$|0\rangle \quad \frac{|0\rangle + |1\rangle}{\sqrt{2}} := |+\rangle$

$|1\rangle \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}} := |-\rangle$

양자상태를 변환할 때,
Basis를 유지하도록
변환해야함

$$U = |+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Exercise

1. $U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

2. $U = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$

Unitary operator (matrix)인가?

Exercise

1. $U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ Unitary

➡ $U^+ = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, UU^+ = U^+U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

2. $U = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$ Unitary X

➡ $U^+ = \frac{1}{2}\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, UU^+ = U^+U = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq I$

Exercise

3. $U = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

4. $U = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

Unitary operator (matrix)인가?

Exercise

3. $U = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ Unitary X

$$U^{+} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, UU^{+} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq U^{+}U = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

4. $U = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$ Unitary

$$U^{+} = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}, UU^{+} = U^{+}U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Exercise

5. $U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$

6. $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Unitary operator (matrix)인가?

Exercise

5. $U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$

Unitary

$U^+ = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, UU^+ = U^+U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$

6. $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Unitary (Controlled-NOT)

Qubit Unitary Operator

- Qubit에 대한 Unitary operator는 2차원 행렬로 표현될 수 있음!
- 아래 예시들은 양자 컴퓨터에 기본이 되는 operator들이 직접 손/컴퓨터로 적어 보시면서 이해하는 것을 추천합니다.

Identity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad I|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Spin-X (Pauli-X) or NOT

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \qquad \sigma_x|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Spin-Y (Pauli-Y)

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle, \qquad \sigma_y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle$$

Spin-Z (Pauli-Z) or Bit Flip

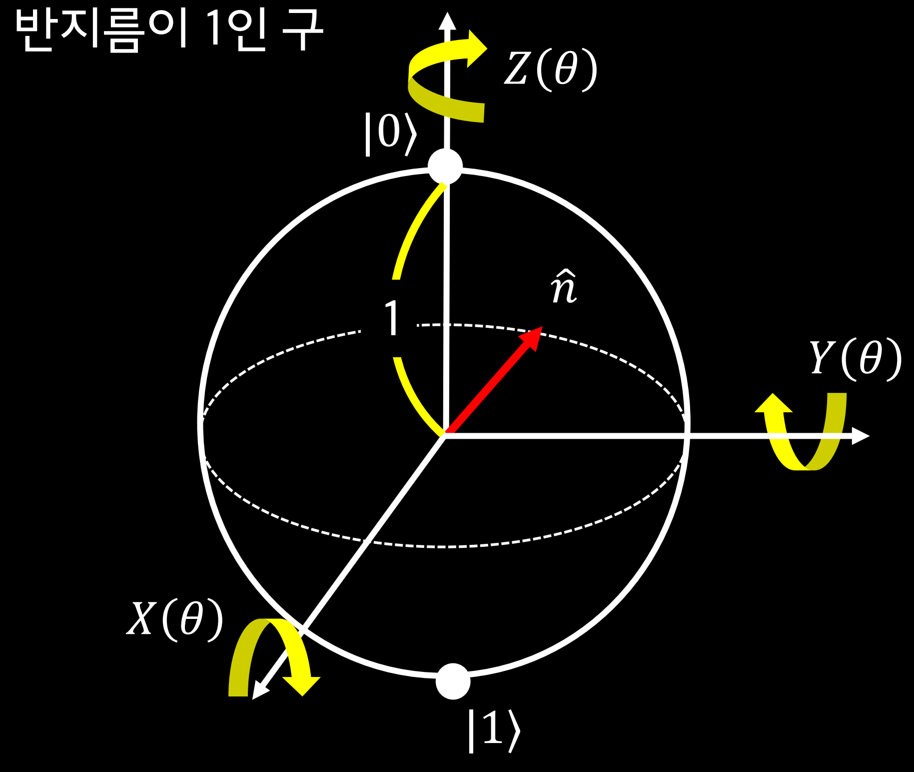
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad \sigma_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

Unitary Operator

앞에 살펴본 $\sigma_x, \sigma_y, \sigma_z$는 Qubit을 해당 축에서 회전시키는 Hamiltonian이 됨!

Qubit Rotation



$\hat{n}$: 단위벡터 $(n_x^2 + n_y^2 + n_z^2 = 1)$

- X rotation (x축을 중심으로 θ도 만큼 돌림)

$$X(\theta) \coloneqq e^{-\frac{i\sigma_x\theta}{2}} = \begin{pmatrix} \cos{}^{\theta}/_{2} & -i\sin{}^{\theta}/_{2} \\ -i\sin{}^{\theta}/_{2} & \cos{}^{\theta}/_{2} \end{pmatrix}$$

- Y rotation (y축을 중심으로 θ도 만큼 돌림)

$$Y(\theta) \coloneqq e^{-\frac{i\sigma_y\theta}{2}} = \begin{pmatrix} \cos{}^{\theta}/_{2} & -\sin{}^{\theta}/_{2} \\ \sin{}^{\theta}/_{2} & \cos{}^{\theta}/_{2} \end{pmatrix}$$

- Z rotation (z축을 중심으로 θ도 만큼 돌림)

$$Z(\theta) \coloneqq e^{-\frac{i\sigma_z\theta}{2}} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

- Rotation (임의의 축을 중심으로 θ도 만큼 돌림)

$$R(\theta) \coloneqq e^{-\frac{i(\hat{n}\cdot\vec{\sigma})\theta}{2}} = e^{-\frac{i(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)\theta}{2}}$$

Unitary Operator

- 아래 예시들도 양자 컴퓨터에 기본이 되는 operator들이 직접 손/컴퓨터로 적어 보시면서 이해하는 것을 추천합니다.

Hadamard operator (중첩을 생성하는 operator)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Ex. $H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = |+\rangle,$ $\quad H|+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\0\end{pmatrix} = |0\rangle$

Ex. $H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle,$ $\quad H|-\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix} = |1\rangle$

Phase operator ($|0\rangle$과 $|1\rangle$의 계수(위상)를 다르게 만드는 operator)

$$P_\theta = \begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}$$

Ex. $P_\theta(|0\rangle + |1\rangle) = |0\rangle + e^{i\theta}|1\rangle$

Ex.

$\pi/2$ - operator

$$S = P_{\theta=\pi/2} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$$

$\pi/4$ - operator

$$T = P_{\theta=\pi/4} = \begin{pmatrix}1 & 0\\ 0 & \frac{1+i}{\sqrt{2}}\end{pmatrix}$$

Spin-Z operator

$$\sigma_z = P_{\theta=\pi} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

5. 코드 구현