This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2964558, IEEE Transactions on Automatic Control

Lyapunov Stability for Impulsive Systems via Event-triggered Impulsive Control

Xiaodi Li, Dongxue Peng, and Jinde Cao, *Fellow, IEEE*

***Abstract*—In this paper, we investigate the Lyapunov stability problem for impulsive systems via event-triggered impulsive control, where dynamical systems evolve according to continuous-time equations most of time, but occasionally exhibit instantaneous jumps when impulsive events are triggered. We provide some Lyapunov-based sufficient conditions for uniform stability (*US*) and globally asymptotical stability (*GAS*). Unlike normal time-triggered impulsive control, event-triggered impulsive control is triggered only when an event occurs. Thus our stability conditions rely greatly on the event-triggering mechanism (*ETM*) given in terms of Lyapunov functions. Moreover, the Zeno behaviour can be excluded in our results. Then we apply the theoretical results to nonlinear impulsive control system, where event-triggered impulsive control strategies are designed to achieve stability of the addressed system. Finally, two numerical examples and their simulations are provided to demonstrate the effectiveness of the proposed results.**

***Index Terms*—Lyapunov stability; Impulsive systems; Zeno behaviour; Event-triggered impulsive control.**

I. Introduction

IMPULSIVE system is an important class of hybrid systems and has received considerable attention due to its wide applications in many fields such as communication networks, control technology, engineering sciences, and biology [1], [2], [3], [4]. The stability of such systems have been extensively studied in many literatures [5], [6], [7], [8], [9], [10]. Generally, from the perspective of impulsive effects, the study on the stability of impulsive systems can be divided into two classes: impulsive perturbation and impulsive control. Actually, impulsive perturbation considers the robustness of a system in which the destabilizing impulses are usually involved. While impulsive control considers the stabilization of a system in which the stabilizing impulses are usually involved. As a discontinuous control method, impulsive control has been widely used in satellite orbit transfer [4], currency supply control in the financial market [11], chaos synchronization [15] and communication security [10] due to its distinctive advantages such as lower control cost, higher confidentiality, and stronger robustness. Till now, a large number of scholars from different fields have focused on the investigation of designing impulsive control strategies for various modelings, see [14], [15], [16] and the references therein.

In the control community, two main methods, namely, time-triggered control and event-triggered control, are commonly used to determine the transmission and sampling instants so that the stability or other system performances can be achieved. As a traditional control method, time-triggered control is often adopted in control systems, where information transmissions and samplings are determined by preset time-triggered instants. One of advantages of time-triggered control is that it can make the control strategies easy to be implemented [5], [6], [7], [8]. However, in general, the sampling rate of time-triggered control is determined artificially, which results in the unnecessary resources consumption and control waste in the process of information transmission. While event-triggered control is an effective control strategy that can avoid the unnecessary waste of resources. Its control instants are determined by the occurrence of some well-designed event conditions which are related to the state of system [12], [13], [17], [18], [19], [27], [28], [29]. In recent years, various event-triggered control strategies have been proposed for many practical control systems, such as [12], [20] for networked control systems, [21] for state observation systems, and [22], [27], [29] for multi-agent systems. Event-triggered impulsive control that combines the advantages of event-triggered control and impulsive control has gained increasing attention in recent years. Some related control strategies have been proposed in [23], [24], [25]. In those works, impulsive control happens only when an event is triggered. Therefore, no signal transmission is needed between consecutive triggering instants, which can effectively reduce the communication cost. Nevertheless, those results only apply for some specific systems and moreover, there are some drawbacks. For example, [23] investigated leader-following consensus problem of multi-agent systems via a distributed event-triggered impulsive control method. However, the proposed conditions depending on the triggering intervals are practically not easy to implement. Although [24] investigated the stability problem of control system based on event-triggered impulsive control, those stability criteria require the exact information of the upper bound of the time intervals between two event-triggers. In [25], the stabilization for input-to-state stability of continuous dynamical systems with external inputs/disturbances was considered by event-triggered impulsive control, where a constant period Δ is enforced to restrict the upper bound of the time intervals in the event-triggered impulsive control scheme. Moreover, besides the above mentioned works, to the best of the authors' knowledge, literatures dealing with event-triggered impulsive control

This work was supported in part by National Natural Science Foundation of China (61673247), and the Research Fund for Excellent Youth Scholars of Shandong Province (JQ201719), and in part by the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence (BM2017002).

Xiaodi Li and Dongxue Peng are with the School of Mathematics and Statistics, Shandong Normal University, Ji'nan, 250014, PR China (e-mail: lxd@sdnu.edu.cn).

Jinde Cao is with the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence, and School of Mathematics, Southeast University, Nanjing 210096, PR China. (e-mail: jdcao@seu.edu.cn).

Manuscript received xxxx. 2019; revised xxx xx, xxxx.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2964558, IEEE Transactions on Automatic Control

appears to be scarce due to the difficulties in mathematical analysis for flows involving "beating" phenomenon. Since the well-known Lyapunov method has been successfully utilized in the investigation of various stabilities, one will naturally ask whether we can establish Lyapunov results for stability of general impulsive systems via event-triggered impulsive control.

Motivated by the aforementioned discussion, in this paper, we develop the Lyapunov method for stability analysis of general nonlinear impulsive systems via event-triggered impulsive control. Some Lyapunov-based sufficient conditions for *US* and *GAS* are derived, where a class of forced impulse sequences is involved to ensure the asymptotic behavior. Moreover, Zeno behaviour can be excluded in those results. The designed *ETM* is given in terms of Lyapunov function, which can be applied to different control systems by choosing different Lyapunov functions. As for application, we apply our theoretical results to nonlinear impulsive control systems. A control strategy composed of *ETM* and control input is designed to the stability problem of nonlinear impulsive control systems. The outline of this paper is as follows. In Section II, a definition and some preliminary knowledge are stated. Some sufficient conditions are presented in Section III to ensure the Lyapunov stability (*US/ GAS*). In Section IV, the effectiveness of the theoretical results are shown by an application. In Section V, two numerical examples and their simulations are provided and a conclusion is finally given in Section VI.

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the n-dimensional and $n\times m$-dimensional real spaces equipped with the Euclidean norm $|\cdot|$, respectively, $\mathbb{Z}_+$ the set of positive integer numbers. $A>0$ or $A<0$ denotes that the matrix A is a symmetric and positive or negative definite matrix. I is the identity matrix with appropriate dimensions. $\max\{a,b\}$ and $\min\{a,b\}$ are the maximum and minimum of a and b, respectively. $\mathcal{K}=\{a\in C(\mathbb{R}_+,\mathbb{R}_+)|\ a(0)=0$ and $a(s)>0$ for $s>0$ and a is increasing in $s\}$. In addition, if a is unbounded, it is of class $\mathcal{K}_\infty$.

II. Preliminaries

We consider the general impulsive system

$$\begin{cases} \dot{x}(t)=f(x(t)), & t\neq t_k,\ t\geq t_0, \\ x(t^+)=g_k(x(t)), & t=t_k,\ k\in\mathbb{Z}_+, \end{cases} \tag{1}$$

where $x\in\mathbb{R}^n$ denotes the state, $\dot{x}$ denotes the left-hand derivative of x, f and g_k are functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, with f locally Lipschitz. Assume that $f(0)=0$ and $g_k(0)=0$ such that system (1) admits a trivial solution $x\equiv 0$. The sequence $\{t_k,k\in\mathbb{Z}_+\}$ is the set of impulse instants. It describes that the continuous dynamics from equation $\dot{x}=f$ are activated when $t\neq t_k$ and the discrete dynamics from the impulse condition $x(t^+)=g_k$ are activated when $t=t_k$. Here we assume that the state variables of system (1) are left continuous at each t_k, i.e., $x(t_k)=x(t_k^-)$; in other words, $x(t)$ is continuous at each interval $(t_{k-1},t_k],k\in\mathbb{Z}_+$.

Given a locally Lipschitz function $V:\mathbb{R}^n\to\mathbb{R}_+$, the left-upper Dini derivative of V along system (1) is defined by

$$D^-V[x(t)]f(x(t))=\lim_{h\to 0^-}\sup\frac{1}{h}\big[V(x+hf(x(t)))-V(x(t))\big].$$

Next, let us recall the Lyapunov stability definitions.

Definition 1: Suppose that a sequence $\{t_k,k\in\mathbb{Z}_+\}$ is given. Let $x(t)=x(t,t_0,x_0)$ be the solution of system (1) through (t_0,x_0). Then system (1) is said to be

(i) *stable* if for any $\varepsilon>0,t_0\geq 0$, there exists a $\delta=\delta(t_0,\varepsilon)>0$ such that $\forall x_0:|x_0|<\delta$ implies that $|x(t)|<\varepsilon$ for all $t\geq t_0$;

(ii) *uniformly stable (US)* if the δ in (i) is independent of t_0;

(iii) *globally asymptotically stable (GAS)* if it is stable and globally attractive, namely, $x(t)\to 0$ as $t\to+\infty$, for any $x_0\in\mathbb{R}^n$ and $t_0\geq 0$.

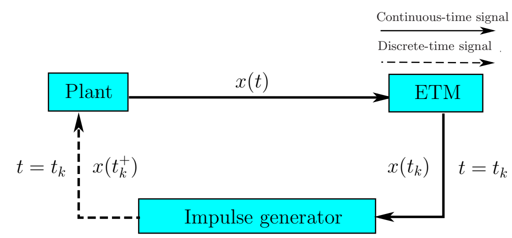

Fig. 1. Event-triggered impulsive control loop.

The aim of this paper is to design the event-based impulsive control strategies such that system (1) is *US* and *GAS*. The block diagram of event-triggered impulsive control is shown in Fig. 1. It can be observed that under a given *ETM*, the plant state is sampled only when the designed event-triggering condition is activated and meanwhile the triggering instant t_k (i.e., sampling time) is determined. Then the impulse generator is activated at the triggering instant t_k and generates an updated state $x(t_k^+)$ for the plant. Then the plant will start the next loop based on the updated state and event. Note that different from the general event-triggered control such as those in [12], [13], [17], [18], [19] in which the state information of sampling instants is needed to be transmitted all the time until it is updated by the next event-trigger, information transmission in event-triggered impulsive control is only needed to be activated at triggering instants, that is, there is no need for any information transmission during two consecutive triggering instants, which can effectively reduce the communication cost.

III. Event-based Lyapunov theorems

In this section, some sufficient conditions for Lyapunov stability of system (1) are presented based on event-based impulsive control approach. First, consider the following *ETM*:

$$t_k=\inf\big\{t\geq t_{k-1}:V(x(t))\geq e^{a_k}V(x(t_{k-1}^+))\big\}, \tag{2}$$

where $a_k\in\mathbb{R}_+$ are triggering parameters satisfying

$$\sum_{k=1}^{m}a_k\to+\infty \text{ as } m\to+\infty. \tag{3}$$

$V(x(t))$ and $V(x(t_{k-1}^+))$ denote the Lyapunov functions which depend on the state trajectory $x(t)$ of system (1) at time t and the last time event-triggering instant t_{k-1}, respectively. $t_0\geq 0$ is a given initial instant and $x(t_0^+)=x_0$ is the initial state of system (1) at $t=t_0$. Note that *ETM* (2) indicates that event-triggering instants $\{t_k, k \in \mathbb{Z}_+\}$ may be different with the difference of Lyapunov functions V. Based on such *ETM*, the following result can be derived.

Theorem 1: System (1) is *US* under *ETM* (2) if there exist functions $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$, and positive constants $c,\ a_k,\ d_k,\ M$, such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|),\ \forall x \in \mathbb{R}^n, \tag{4}$$

$$V(g_k(x)) \le e^{-d_k} V(x),\ \forall x \in \mathbb{R}^n, \forall k \in \mathbb{Z}_+, \tag{5}$$

and the derivative of V along the solution $x(t) = x(t, t_0, x_0)$ of system (1) with $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ satisfies

$$D^- V[x(t)] f(x(t)) \le cV(x(t)),\ t \ne t_k, \tag{6}$$

where $\{t_k,\ k \in \mathbb{Z}_+\}$ are event-triggering instants generated by (2) and constants $a_k, d_k, k \in \mathbb{Z}_+$ satisfy

$$\sum_{k=1}^{m}(a_k - d_k) + a_{m+1} \le M,\ \forall m \in \mathbb{Z}_+. \tag{7}$$

Proof: Assume that $x(t) = x(t, t_0, x_0)$ is the solution of system (1) through initial value (t_0, x_0). For any $\varepsilon > 0$, we choose $\delta = \alpha_2^{-1}\big(\exp(-\Upsilon)\alpha_1(\varepsilon)\big)$, where $\Upsilon = \max\{a_1, M\}$. We show that $|x_0| < \delta$ implies that $|x(t)| < \varepsilon,\ t \ge t_0$. Note that it is possible that the solution $x(t)$ of system (1) is subject to finite or infinite impulse jumps. Thus our proof considers two cases: event-trigger in finite number of times and event-trigger in infinitely many times.

First, if the event occurs finite number of times, assume that $t_1 < t_2 < \cdots < t_N$ are the trigger instants. It then follows from the definition of *ETM* (2) that

$$V(x(t_1)) = e^{a_1} V(x_0) \text{ and } V(x(t)) \le e^{a_1} V(x_0), \forall t \in [t_0, t_1].$$

Moreover, we obtain that at event-trigger instant t_1,

$$V(x(t_1^+)) = V(g_1(x(t_1))) \le e^{-d_1} V(x(t_1)) \le e^{a_1 - d_1} V(x_0).$$

Similarly, we have $V(x(t_2)) = e^{a_2} V(x(t_1^+))$ and

$$V(x(t)) \le e^{a_2} V(x(t_1^+)) \le e^{a_1 + a_2 - d_1} V(x_0), \forall t \in (t_1, t_2].$$

Recursively repeating this procedure for $m = 1, 2, \cdots, N$, it follows that

$$V(x(t)) \le \exp\Big(\sum_{k=0}^{m-1}(a_k - d_k) + a_m\Big) V(x_0), \tag{8}$$

for $\forall t \in (t_{m-1}, t_m]$, where $a_0 = d_0 = 0$. Since t_N is the last event-trigger instant, the event generated by (2) will not occur on $t \in (t_N, +\infty)$, which indicates that

$$\begin{aligned} V(x(t)) &< e^{a_{N+1}} V(x(t_N^+)) \le e^{a_{N+1} - d_N} V(x(t_N)) \\ &\le \exp\Big(\sum_{k=0}^{N}(a_k - d_k) + a_{N+1}\Big) V(x_0),\ \forall t > t_N. \end{aligned} \tag{9}$$

It then follows from (7), (8), and (9) that

$$V(x(t)) \le \exp(\Upsilon) V(x_0),\ \forall t \ge t_0.$$

Note that $\delta = \alpha_2^{-1}\big(\exp(-\Upsilon)\alpha_1(\varepsilon)\big)$, it is easy to derive that $|x_0| < \delta$ implies that $|x(t)| < \varepsilon,\ \forall t \ge t_0$. This shows that the impulsive system (1) is *US* for the case of event-trigger in finite number of times

Second, if the event occurs infinitely many times, we assume that $t_1 < t_2 < \cdots < t_k < \ldots$ are the trigger instants. We firstly claim that there is no Zeno behavior. In fact, note that at the first event-triggering instant t_1, it holds that

$$V(x(t_1)) = e^{a_1} V(x_0) \le e^{c(t_1 - t_0)} V(x_0),$$

which implies that $t_1 - t_0 \ge a_1/c$. Similarly, it follows from the above discussion that at the second event-trigger instant t_2, $t_2 - t_1 \ge a_2/c$. In this way, at the mth event-trigger instant t_m, we conclude that $t_m - t_{m-1} \ge a_m/c$. It then follows that

$$t_m \ge \frac{1}{c}\sum_{k=1}^{m} a_k + t_0.$$

Together with (3), we obtain that $t_m \to +\infty$ as $m \to +\infty$, which implies that the Zeno behavior is excluded. Next, we show that system (1) is *US*. For every $m \in \mathbb{Z}_+$, since an event is always triggered, it follows from (8) and (9) that

$$V(x(t)) \le \exp\Big(\sum_{k=0}^{m-1}(a_k - d_k) + a_m\Big) V(x_0), \forall t \in (t_{m-1}, t_m],$$

which implies that

$$V(x(t)) \le \exp(\Upsilon) V(x_0),\ \forall t \in (t_{m-1}, t_m].$$

Note that $\delta = \alpha_2^{-1}\big(\exp(-\Upsilon)\alpha_1(\varepsilon)\big)$, which indicates if $|x_0| < \delta$, then $|x(t)| < \varepsilon,\ \forall t \ge t_0$. Hence, no matter the event generated by (2) occurs finite number of times or infinitely many times, we have shown that system (1) is *US*. The proof is completed. ■

Remark 1: Traditional event-triggered control is based on a fact that the flow is continuous. However, when the system is subject to impulses, i.e., beating phenomenon, the traditional approach is infeasible. To overcome this difficulty, *Theorem 1* presents some Lyapunov-based sufficient conditions for *US* of impulsive system (1) according to *ETM* (2), where condition (6) governs the continuous dynamics of system between impulse events, condition (5) governs discrete dynamics of system when an impulse event occurs, which can be regarded as impulse generator, and condition (7) establishes the relationship between triggering parameters a_k and impulse parameters d_k, which is crucial to our proposed *ETM* (2). Compared with the traditional time-triggered impulsive schemes, such as those in [5], [6], [7], [8], [14], [15], *Theorem 1* does not impose any direct restriction on time intervals, and the triggering time is completely determined by the designed *ETM*. In addition, triggering parameters a_k in *ETM* (2) can be appropriately changed to adjust the possible triggering time. Observe that choosing a small value of a_k can potentially reduce the value of the inter-event bound and thus increase the frequency of the impulse triggers, which will speed up the convergence rate. Conversely, choosing a large one will potentially increase the value of the inter-event bound and thus decrease the frequency of the impulse triggers, which will slow down the convergence rate. Based on such condition, the *US* of system (1) can be guaranteed even the events are only activated in finite number of times or do not happen. In particular, if we strengthen some restrictions on condition (7), then the following result can be derived.

Corollary 1: Under conditions in *Theorem 1*, system (1) is *US* if condition (7) is replaced by the following condition: the sequence $\{a_k,\ k \in \mathbb{Z}_+\}$ is bounded and $d_k \geq a_k, \forall k \in \mathbb{Z}_+$.

Remark 2: Note that there is no restriction on positive constant c in condition (7) of *Theorem 1*. That is to say, under our designed *ETM* (2), *Theorem 1* allows the continuous flow to be increasing sharply or slowly between two consecutive event triggers, which is different from those in traditional time triggered impulsive schemes.

Remark 3: As a special case, when the stabilizing effect of the impulsive actions gradually vanish with time, i.e., $d_k \to 0$ with increasing k, we must require continuous flows to be persistently interrupted by impulses in order to guarantee the stability. In this case, condition (7) enforces a restriction between triggering parameters a_k and impulse parameters d_k such that impulsive actions appear more and more frequently, but still the Zeno behavior is avoided under the help of condition (3).

Next we study the *GAS* of the impulsive system (1) via *ETM*. In order to ensure the asymptotic behavior of the solution, the forced impulse time sequence $\{\tau_k\}_{k=1}^{\infty}$ is introduced into *ETM* (2). Then the updated *ETM* is given by

$$\begin{aligned} t_k &= \min\{t_k^*, \tau_k\}, \\ t_k^* &= \inf\left\{t \geq t_{k-1} : V(x(t)) \geq e^{a_k} V(x(t_{k-1}^+))\right\}, \end{aligned} \tag{10}$$

where $a_k \in \mathbb{R}_+$ are known triggering parameters satisfying (3) and the forced impulse time sequence $\{\tau_k\}_{k=1}^{\infty}$ satisfies $\inf\{\tau_k - \tau_{k-1}|\ k \in \mathbb{Z}_+\} > 0$. We denote such kind of impulse time sequences by set $\mathcal{S}_0$ for later use.

Theorem 2: System (1) is *GAS* under *ETM* (10) if there exist functions $\alpha_1,\ \alpha_2 \in \mathcal{K}_\infty$, a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ and positive constants $c,\ a_k,\ d_k$, such that (4), (5) and (6) hold, where event-triggering instants $\{t_k,\ k \in \mathbb{Z}_+\}$ are generated by (10) and constants $a_k, d_k, k \in \mathbb{Z}_+$ satisfy

$$\sum_{k=1}^{m-1}(a_k - d_k) + a_m \to -\infty \text{ as } m \to +\infty. \tag{11}$$

Proof: It follows from the definition of *ETM* (10) that the event-based impulses occur infinitely many times. Assume that $t_1 < t_2 < \cdots < t_k < \ldots$ are the impulse instants.

To study Lyapunov stability, first of all, we need to prove that there is no Zeno behavior for system (1) under *ETM* (10). To do this, we consider the following three cases: Case (a). Impulse sequence $\{t_k\}_{k=1}^{\infty}$ is fully generated by the forced time sequence $\{\tau_k\}_{k=1}^{\infty}$. In this case, it follows from the assumption $\inf\{\tau_k - \tau_{k-1}|\ k \in \mathbb{Z}_+\} > 0$ that Zeno behavior is excluded. Case (b). There exist some event-trigger instants t_j^* in impulse sequence $\{t_k\}_{k=1}^{\infty}$, i.e., $\{t_j^*, j \in \mathbb{Z}_+\} \cap \{t_k\}_{k=1}^{\infty} \neq \emptyset$. In this case, if there is Zeno behavior for system (1), then there exist infinite impulse jumps within a finite time interval. Suppose that $[t_0, T]$ denotes the finite time interval which exhibits Zeno behavior and T denotes the Zeno time (accumulation time). Then there are infinitely many impulsive instants in the interval $[T - \frac{\eta}{2}, T]$, where $\eta := \inf\{\tau_k - \tau_{k-1}|\ k \in \mathbb{Z}_+\} > 0$. Let $\{t_{N_0+j}\}_{j=1}^{\infty} \in [T - \frac{\eta}{2}, T]$ be the subsequence of the impulsive sequence $\{t_k\}_{k=1}^{\infty}$ satisfying $t_{N_0+j} \to T$ as $j \to \infty$, where N_0 is a nonnegative integer. If there exists $\tau_l \in \{t_{N_0+j}\}_{j=1}^{\infty}$ for some $l \in \mathbb{Z}_+$, then it follows from the definition of η that there must be one and only one $\tau_l \in \{t_{N_0+j}\}_{j=1}^{\infty}$. It indicates that the impulsive instant $t_{N_0+j} = t_{N_0+j}^*$ when $N_0 + j \geq l + 1$. Then by applying similar arguments as in *Theorem 1*, together with (6) and (10), for $t \in (t_s, t_{s+1}]$, $s \geq l + 1$, we conclude that

$$V(x(t_{s+1})) = e^{a_{s+1}} V(x(t_s^+)) \leq e^{c(t_{s+1} - t_s)} V(x(t_s^+)),$$

which implies that $t_{s+1} - t_s \geq a_{s+1}/c$. It then can be deduced that

$$t_{s+m} \geq \frac{1}{c}\sum_{j=1}^{m} a_{s+j} + t_s.$$

Together with (3), we obtain that $t_{s+m} \to +\infty$ as $m \to +\infty$, which is a contradiction with the definition of Zeno time T. If there is no $\tau_j \in \{t_{N_0+j}\}_{j=1}^{\infty}$, namely, $\{\tau_j, j \in \mathbb{Z}_+\} \cap \{t_{N_0+j}\}_{j=1}^{\infty} = \emptyset$. Then applying the similar discussion as above case, we conclude that $t_{N_0+j} \to +\infty$ as $j \to +\infty$, which is also a contradiction. Thus there is no Zeno behavior for Case (b). Case (c). Impulse sequence $\{t_k\}_{k=1}^{\infty}$ is fully generated by the event-triggering instants $\{t_k^*\}_{k=1}^{\infty}$, which reduces to the same case as in *Theorem 1*. It then follows from the proof of *Theorem 1* that the Zeno behavior is excluded. Hence, we have proven that there is no Zeno behavior for system (1) under *ETM* (10).

Next we show that system (1) is *GAS*. It follows from the condition $t_k = \min\{t_k^*,\ \tau_k\}$ in (10) that

$$V(x(t)) \leq e^{a_k} V(x(t_{k-1}^+)),\ \forall t \in (t_{k-1}, t_k],\ k \in \mathbb{Z}_+.$$

Then applying the similar discussion as the proof of *Theorem 1*, for $m \geq 1$, we have

$$V(x(t)) \leq \exp\Big(\sum_{k=0}^{m-1}(a_k - d_k) + a_m\Big)V(x_0),\ \forall t \in (t_{m-1}, t_m].$$

The *GAS* of system (1) follows from this and (4), (11) by standard arguments. The proof is completed. ■

Remark 4: It is worth noting that the *ETM* (10) is different from the one in (2) since the forced impulse time sequence $\{\tau_k\}_{k=1}^{\infty}$ is involved. As mentioned in *Theorem 1*, based on *ETM* (2), the events may occur finite number of times or infinitely many times. While when the forced impulse time sequence $\{\tau_k\}_{k=1}^{\infty}$ is involved, the events in (10) composed of forced impulses and triggered events must occur infinitely many times, which is a crucial condition for *GAS*. Note that the condition (11) in *Theorem 2* establishes the relationship between trigger parameters a_k and impulse parameters d_k for *GAS*, which is a little stronger restriction than condition (7) for *US*. Based on condition (11), the following corollary can be derived.

Corollary 2: Assume that $a_{2k-1} - d_{2k-1} = \eta$ and $a_{2k} - d_{2k} = -\gamma$ for all $k \in \mathbb{Z}_+$, where η and γ are two positive constants with $\gamma > \eta$. Under conditions in *Theorem 2*, system (1) is *GAS* if the sequence $\{a_k,\ k \in \mathbb{Z}_+\}$ is bounded.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2964558, IEEE Transactions on Automatic Control

IV. Application

In this section, we apply the theoretical results to ***nonlinear impulsive control system*** to verify the feasibility and effectiveness of our analytical results.

Consider the following nonlinear control system:

$$\dot{x}(t) = Ax(t) + Bf(x(t)) + u(t), \tag{12}$$

where $x(t) \in \mathbb{R}^n$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a global Lipschitz function with Lipschitz matrix L and $f(0) = 0$, A and B are two $n \times n$ real matrices, $u(t) \in \mathbb{R}^n$ is the Dirac control ([30]) input given by

$$u(t) = \sum_{k \in \mathbb{Z}_+} Dx(t)\delta(t - t_k), \tag{13}$$

where $\{t_k,\ k \in \mathbb{Z}_+\}$ is the event-based impulse sequence and D is an $n \times n$ control gain matrix. Our purpose is to design the event-based impulse sequence $\{t_k,\ k \in \mathbb{Z}_+\}$ and control gain D such that system (12) is *GAS*. Note that system (12) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)),\ t \neq t_k, \\ x(t^+) = (I + D)x(t),\ t = t_k. \end{cases}$$

Based on *Theorem 2*, let Lyapunov function $V(x) = x^T P x$, then the following result can be derived.

Theorem 3: Assume that there exist constants a, c, $d \in \mathbb{R}_+$ with $d > a$, $n \times n$ matrix $P > 0$, $n \times n$ diagonal matrix $Q > 0$, and $n \times n$ real matrix R such that the following LMIs hold:

$$\begin{bmatrix} PA + A^T P + L^T Q L - cP & PB \\ \star & -Q \end{bmatrix} < 0. \tag{14}$$

$$\begin{bmatrix} -e^{-d} P & P + R \\ \star & -P \end{bmatrix} < 0. \tag{15}$$

Then system (12) is *GAS* under the control gain $D = P^{-1}R^T$ and *ETM*:

$$\begin{aligned} t_k &= \min\{t_k^*, \tau_k\}, \\ t_k^* &= \inf\{t \geq t_{k-1} : x^T(t)Px(t) \geq e^a x^T(t_{k-1}^+)Px(t_{k-1}^+)\}, \end{aligned} \tag{16}$$

where $\{\tau_k\}_{k=1}^{\infty} \in \mathcal{S}_0$.

Corollary 3: Assume that there exist positive constants a and c, $n \times n$ matrix $P > 0$, $n \times n$ diagonal matrix $Q > 0$, and $n \times n$ real matrix R such that (14) and LMI (15) with $d = a$ hold. Then system (12) is *US* under the control gain $D = P^{-1}R^T$ and *ETM* in (16) with $t_k = t_k^*$.

Remark 5: Theorem 3 presents an LMI-based condition to design the Dirac control input u and *ETM* such that system (12) is *GAS*. When applying *Theorem 3*, it is necessary to give proper constants c and d a priori such that the LMIs in (14) and (15) hold. Then the matrix P for *ETM* (16) and gain matrix D for control input (13) can be determined, respectively. In above analysis, intuitively, a large c is preferred for the feasibility of LMI in (14). In fact, condition (6) in *Theorem 2* gives the basic guideline for choosing the parameter c. In addition, when the matrix P is derived by LMI toolbox, the constant a in *ETM* can be appropriately changed to adjust the possible triggering time, which will influence the convergence rate.

Remark 6: The stability problem of nonlinear impulsive control systems has been extensively studied, for example, [31] dealt with the impulsive pinning control, [32] the impulsive distributed control, [33] the dual-stage impulsive control, and [34], [35] the *ADT* method. Compared with those existing results in which the impulsive instants are prescribed and independent of the states of systems, which may result in unnecessary information transmission, *Theorem 3* in this paper presents an impulsive control strategy for *GAS*, where the impulsive instants are determined by appropriately designed *ETM*. Such kind of controllers is state-dependent and can reduce the probability of redundant information transmission.

V. Example

In this section, two numerical examples are presented to demonstrate the validity of our proposed results.

Example 1: Consider the following scalar time-varying system

$$\dot{x}(t) = \cos(t)x(t),\ t \geq 0 \tag{17}$$

subjects to impulse

$$x(t^+) = e^{-0.5}x(t),\ t = t_k, \tag{18}$$

where $\{t_k,\ k \in \mathbb{Z}_+\}$ is the event-based impulse sequence to be designed. When there is no impulsive control, it is obvious that continuous system (17) is not asymptotically stable. Let $a_k = 0.8$ and $d_k = 1$ for all $k \in \mathbb{Z}_+$. Choose a forced impulse time sequence $\{\tau_k\}_{k=1}^{\infty} = \{4k\}_{k=1}^{\infty} \in \mathcal{S}_0$. Then the *ETM* (10) is given by

$$\begin{aligned} t_k &= \min\{t_k^*, 4k\}, \\ t_k^* &= \inf\left\{t \geq t_{k-1} : |x(t)| \geq e^{0.4}|x(t_{k-1}^+)|\right\}. \end{aligned} \tag{19}$$

By *Theorem 2*, system (17)-(18) is *GAS* under *ETM* (19), see Fig. 2 (blue curve). By simulation, one may find that triggering instants $t_3 = \tau_3 = 12, t_7 = \tau_7 = 28$, and others $t_n = t_n^*$ during the time interval $[0, 30]$.

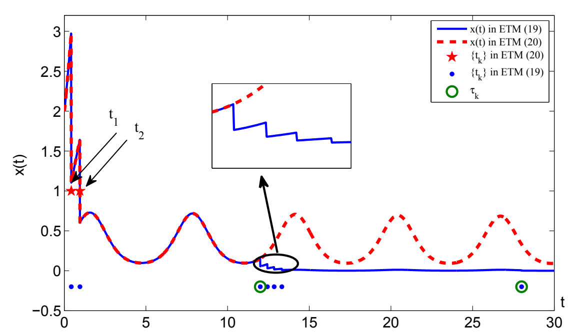

Fig. 2. Simulations in Example 1 with $x(0) = 2$.

Under same conditions, if we ignore the forced impulse sequence $\{\tau_k\}_{k=1}^{\infty}$, then *ETM* (19) reduces to

$$t_k = \inf\left\{t \geq t_{k-1} : |x(t)| \geq e^{0.4}|x(t_{k-1}^+)|\right\}. \tag{20}$$

Under such *ETM*, system (17)-(18) is not *GAS*, see Fig. 2 (red curve). In this case, it follows from simulation that system (17)-(18) encounters only two event-triggers, i.e., t_1 and t_2, which is shown in Fig. 2 (red star). The above analysis shows that the forced impulse time sequence introduced in *ETM* (10) plays an important role to guarantee the feasibility of *Theorem 2*.

Example 2: Consider system (12) with $f_1(x) = f_2(x) = \tanh(x)$, $L = I$, and

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 0.5 & 0.2 \\ 0.5 & -1 \end{pmatrix}.$$

When there is no control input (i.e., $u = 0$), by simulation, system (12) is not *GAS*, see Fig. 3. (a), where $x_0 = (1, -2)^T$.

Let the forced impulsive sequence $\tau_k = 5k$, $k \in \mathbb{Z}_+$ and choose $a = 0.48$, $c = 3$, and $d = 0.5$. It then follows from *Theorem 3* that the *ETM* can be designed by

$$\begin{aligned} t_k &= \min\{t_k^*, 5k\}, \ k \in \mathbb{Z}_+, \\ t_k^* &= \inf \Bigg\{ t \geq t_{k-1} : x^T(t) \begin{pmatrix} 110.1028 & -25.8098 \\ -25.8098 & 190.4531 \end{pmatrix} x(t) \\ &\quad \geq e^{0.48} x^T(t_{k-1}^+) \begin{pmatrix} 110.1028 & -25.8098 \\ -25.8098 & 190.4531 \end{pmatrix} x(t_{k-1}^+) \Bigg\}, \end{aligned} \tag{21}$$

and the control gain D is given by

$$D = \begin{pmatrix} -1.3816 & -0.0873 \\ -0.3314 & -1.3600 \end{pmatrix}. \tag{22}$$

Hence, system (12) is *GAS* under the *ETM* (21) and control gain D in (22), which is shown in Fig. 3. (b). Note that in this case impulse instants $\{t_k\} = \{t_1^*, \tau_2, t_3^*, t_4^*, ...\}$. That is to say, the forced impulses $\{\tau_k\}$ is only active at time $t = 10$, i.e., $t_2 = \tau_2$. In particular, if there is a slight change on the control gain D in (22) and assume that it is replaced by

$$D^* = \begin{pmatrix} -1.3816 & -0.0873 \\ -0.3314 & -1.600 \end{pmatrix}.$$

Then it is easy to check that the LMIs in *Theorem 3* do not hold. In such case, by simulation, one may observe from the Fig. 3. (c) that the *GAS* cannot be achieved, which shows the efficiency and sharpness of our proposed criteria. Fig. 3. (d) shows the triggering impulse instants for different a. Observe that the numbers of event-trigger become fewer with the increasing of the parameter a. In real applications, one can choose appropriate parameter a to change the convergence rate or control the dynamical performance of system.

VI. CONCLUSION

We derived some Lyapunov stability theorems for impulsive systems via event-triggered impulsive control. Then we applied our theorems to stabilization of nonlinear impulsive control system. The designed event-triggered impulsive control strategy can be derived by solving LMIs, which is easier to implement. Possible future work includes extensions for hybrid systems with Zeno solution in the framework of [26] or impulsive systems subject to external perturbations. Another interesting topic is the development of distributed event-triggered impulsive control strategies for consensus problem of multi-agent systems.

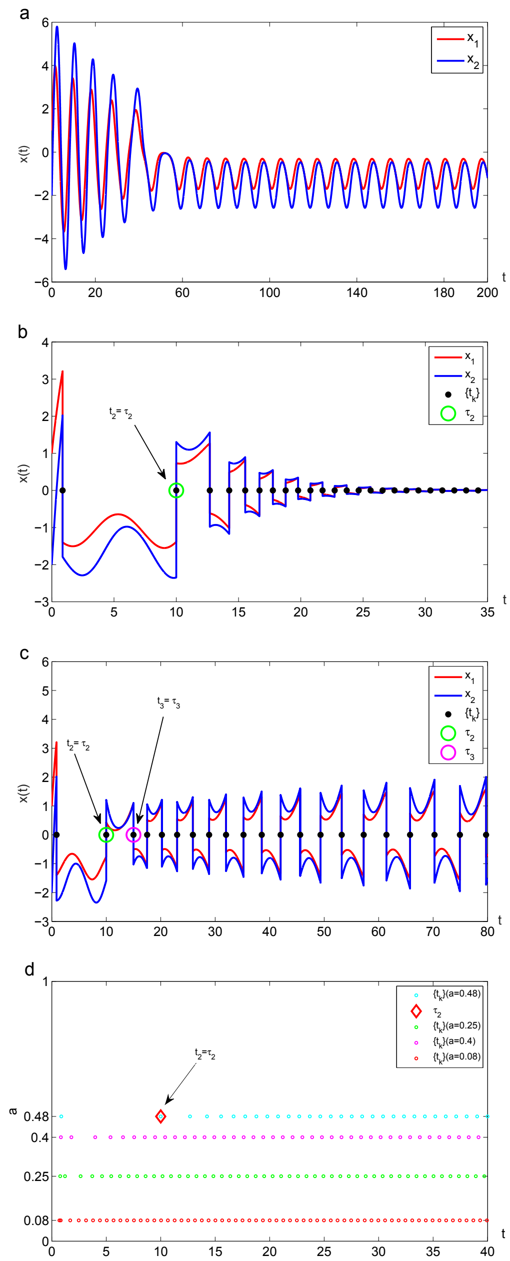

Fig. 3. Simulations in Example 2 with $x_0 = (1, -2)^T$.

REFERENCES

[1] W. Haddad, V. Chellaboina, and S. Nersesov, *Impulsive and hybrid dynamcial systems: stability, dissipativity and control*. New Jersey: Princeton University Press, 2006.

[2] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[3] E. Verriest, "Regularization method for optimally switched and impulsive systems with biomedical applications," *42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475)*, vol. 3, pp. 2156–2161, 2003.

[4] E. Gilbert and G. Harasty, "A class of fixed-time fuel-optimal impulsive control problems and an efficient algorithm for their solution," *IEEE Transactions on Automatic Control*, vol. 16, pp. 1–11, 1971.

[5] P. Naghshtab, J. P. Hespanha, and A. R. Teel, "Exponential stability of impulsive systems with application to uncertain sampled-data systems," *Systems Control & Letters*, vol. 57, pp. 378–385, 2008.

[6] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, "Stability

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.2964558, IEEE Transactions on Automatic Control

of interconnected impulsive systems with and without time delays, using lyapunov methods," *Nonlinear Analysis: Hybrid Systems*, vol. 6, pp. 899–915, 2012.

[7] S. Dashkovskiy and A. Mironchenko, "Input-to-state stability of nonlinear impulsive systems," *SIAM Journal on Control and Optimization*, vol. 51, pp. 1962–1987, 2013.

[8] J. Wu and X. Li, "Stability of nonlinear differential systems with state-dependent delayed impulses," *Automatica*, vol. 64, pp. 63–69, 2016.

[9] V. Lakshmikantham, D. Bainov, and P. Simeonov, *Theory of Impulsive Differential Equations*. World Scientific, 1989.

[10] W. He, X. Gao, W. Zhong, and F. Qian, "Secure impulsive synchronization control of multi-agent systems under deception attacks," *Information Sciences*, vol. 459, pp. 354–368, 2018.

[11] T. Yang, *Impulsive Control Theory*. Springer Science and Business Media, 2001.

[12] V. Dolk, D. Borgers, and W. Heemels, "Output-based and decentralized dynamic event-triggered control with guaranteed $\mathcal{L}_p$-gain performance and zeno-freeness," *IEEE Transactions on Automatic Control*, vol. 62, pp. 34–49, 2017.

[13] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1680–1685, 2007.

[14] X. Li and S. Song, "Stabilization of delay systems: Delay-dependent impulsive control," *IEEE Transactions on Automatic Control*, vol. 62, pp. 406–411, 2017.

[15] X. Liu and K. Zhang, "Input-to-state stability of time-delay systems with delay-dependent impulses ," *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2019.2930239.

[16] J. Lu, D. W. C. Ho, and J. Cao, "A unified synchronization criterion for impulsive dynamical networks," *Automatica*, vol. 46, pp. 1215–1221, 2010.

[17] L. Wu, Y. Gao, J. Liu, and H. Li, "Event-triggered sliding mode control of stochastic systems via output feedback," *Automatica*, vol. 82, pp. 79–92, 2017.

[18] M. Abdelrahim, R. Postoyan, J. Daafouz, and D. Nesic, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96–108, 2017.

[19] W. Xu and D. W. C. Ho, "Clustered event-triggered consensus analysis: An impulsive framework," *IEEE Transactions on Industrial Electronics*, vol. 63, pp. 7133–7143, 2016.

[20] Y. Guan, Q. L. Han, and X. Ge, "On asynchronous event-triggered control of decentralized networked systems," *Information Sciences*, vol. 425, pp. 127–139, 2018.

[21] Z. Zhang and G. Yang, "Event-triggered fault detection for a class of discrete-time linear systems using interval observers," *ISA transactions*, vol. 68, pp. 160–169, 2017.

[22] X. Meng, L. Xie, and Y. C. Soh, "Asynchronous periodic event-triggered consensus for multi-agent systems," *Automatica*, vol. 84, pp. 214–220, 2017.

[23] X. Tan, J. Cao, and X. Li, "Consensus of leader-following multiagent systems: a distributed event-triggered impulsive control strategy," *IEEE Transactions on Cybernetics*, vol. 49, pp. 792-801, 2019.

[24] W. Zhu, D. Wang, and L. Liu, "Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 28, pp. 3599–3609, 2018.

[25] B. Liu, D. Hill, and Z. Sun, "Stabilisation to input-to-state stability for continuous-time dynamical systems via event-triggered impulsive control with three levels of events," *IET Control Theory & Applications*, vol. 12, pp. 1167–1179, 2018.

[26] S. Dashkovskiy and P. Feketa, "Asymptotic properties of zeno solutions," *Nonlinear Analysis: Hybrid Systems*, vol. 30, pp. 256–265, 2018.

[27] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, pp. 245–252, 2013.

[28] Y. Wu, X. Meng, L. Xie, R. Lu, H. Su, and Z. G. Wu, "An input-based triggering approach to leader-following problems," *Automatica*, vol. 75, pp. 221–228, 2017.

[29] D. Yang, W. Ren, and X. Liu, "Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs," *Automatica*, vol. 69, pp. 242–249, 2016.

[30] Z. Guan, G. Chen, and T. Ueta, "On impulsive control of a periodically forced chaotic pendulum system," *IEEE Transactions on Automatic Control*, vol. 45, pp. 1724–1727, 2000.

[31] J. Lu, C. Ding, J. Lou, and J. Cao, "Outer synchronization of partially coupled dynamical networks via pinning impulsive controllers," *Journal of the Franklin Institute*, vol. 352, pp. 5024–5041, 2015.

[32] Z. Guan, Z. Liu, and G. Feng, "Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, pp. 2182–2195, 2010.

[33] H. Zhang, T. Ma, Z. Wang and G. Huang, "Robust global exponential synchronization of uncertain chaotic delayed neural networks via dual-stage impulsive control ," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* , vol. 40, pp. 831–844, 2010.

[34] X. Li, P. Li, and Q. Wang, "Input/output-to-state stability of impulsive switched systems," *Systems & Control Letters*, vol. 116, pp. 1–7, 2018.

[35] S. Dashkovskiy and P. Feketa, "Input-to-state stability of impulsive systems and their networks," *Nonlinear Analysis: Hybrid Systems*, vol. 26, pp. 190–200, 2017.