Deep Learning
Recurrent Networks

2/26/2018

Which open source project?

```
/*
 * Increment the size file of the new incorrect UI_FILTER group information
 * of the size generatively.
 */
static int indicate_policy(void)
{
  int error;
  if (fd == MARN_EPT) {
    /*
     * The kernel blank will coeld it to userspace.
     */
    if (ss->segment < mem_total)
      unblock_graph_and_set_blocked();
    else
      ret = 1;
    goto bail;
  }
  segaddr = in_SB(in.addr);
  selector = seg / 16;
  setup_works = true;
  for (i = 0; i < blocks; i++) {
    seq = buf[i++];
    bpf = bd->bd.next + i * search;
    if (fd) {
      current = blocked;
    }
  }
  rw->name = "Getjbbregs";
  bprm_self_clearl(&iv->version);
  regs->new = blocks[(BPF_STATS << info->historidac)] | PFMR_CLOBATHINC_SECON
  return segtable;
}
```

Related math. What is it talking about?

Proof. Omitted. □

Lemma 0.1. *Let* $\mathcal{C}$ *be a set of the construction.*

Let $\mathcal{C}$ *be a gerber covering. Let* $\mathcal{F}$ *be a quasi-coherent sheaves of* $\mathcal{O}$-modules. We have to show that

$$\mathcal{O}_{\mathcal{O}_X} = \mathcal{O}_X(\mathcal{L})$$

.

Proof. This is an algebraic space with the composition of sheaves $\mathcal{F}$ on $X_{\acute{e}tale}$ we have

$$\mathcal{O}_X(\mathcal{F}) = \{morph_1 \times_{\mathcal{O}_X} (\mathcal{G}, \mathcal{F})\}$$

where $\mathcal{G}$ defines an isomorphism $\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}$-modules. □

Lemma 0.2. *This is an integer* $\mathcal{Z}$ *is injective.*

Proof. See Spaces, Lemma ??. □

Lemma 0.3. *Let* S *be a scheme. Let* X *be a scheme and* X *is an affine open covering. Let* $\mathcal{U} \subset \mathcal{X}$ *be a canonical and locally of finite type. Let* X *be a scheme. Let* X *be a scheme which is equal to the formal complex.*

The following to the construction of the lemma follows.

Let X *be a scheme. Let* X *be a scheme covering. Let*

$$b : X \to Y' \to Y \to Y \to Y' \times_X Y \to X.$$

be a morphism of algebraic spaces over S *and* Y.

Proof. Let X be a nonzero scheme of X. Let X be an algebraic space. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent

(1) $\mathcal{F}$ is an algebraic space over S.
(2) If X is an affine open covering.

Consider a common structure on X and X the functor $\mathcal{O}_X(U)$ which is locally of finite type. □

This since $\mathcal{F} \in \mathcal{F}$ and $x \in \mathcal{G}$ the diagram

is a limit. Then $\mathcal{G}$ is a finite type and assume S is a flat and $\mathcal{F}$ and $\mathcal{G}$ is a finite type f_*. This is of finite type diagrams, and
- the composition of $\mathcal{G}$ is a regular sequence,
- $\mathcal{O}_{X'}$ is a sheaf of rings. □

Proof. We have see that $X = \mathrm{Spec}(R)$ and $\mathcal{F}$ is a finite type representable by algebraic space. The property $\mathcal{F}$ is a finite morphism of algebraic stacks. Then the cohomology of X is an open neighbourhood of U. □

Proof. This is clear that $\mathcal{G}$ is a finite presentation, see Lemmas ??.
A *reduced above* we conclude that U is an open covering of $\mathcal{C}$. The functor $\mathcal{F}$ is a "field

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{F}_{\overline{x}} \quad -1(\mathcal{O}_{X_{\acute{e}tale}}) \longrightarrow \mathcal{O}_{X_\ell}^{-1}\mathcal{O}_{X_\lambda}(\mathcal{O}_{X_n}^{\overline{v}})$$

is an isomorphism of covering of $\mathcal{O}_{X_i}$. If $\mathcal{F}$ is the unique element of $\mathcal{F}$ such that X is an isomorphism.
The property $\mathcal{F}$ is a disjoint union of Proposition ?? and we can filtered set of presentations of a scheme $\mathcal{O}_X$-algebra with $\mathcal{F}$ are opens of finite type over S.
If $\mathcal{F}$ is a scheme theoretic image points. □

If $\mathcal{F}$ is a finite direct sum $\mathcal{O}_{X_\lambda}$ is a closed immersion, see Lemma ??. This is a sequence of $\mathcal{F}$ is a similar morphism.

And a Wikipedia page explaining it all

Naturalism and decision for the majority of Arab countries' capitalide was grounded by the Irish language by [[John Clair]], [[An Imperial Japanese Revolt]], associated with Guangzham's sovereignty. His generals were the powerful ruler of the Portugal in the [[Protestant Immineners]], which could be said to be directly in Cantonese Communication, which followed a ceremony and set inspired prison, training. The emperor travelled back to [[Antioch, Perth, October 25|21]] to note, the Kingdom of Costa Rica, unsuccessful fashioned the [[Thrales]], [[Cynth's Dajoard]], known in western [[Scotland]], near Italy to the conquest of India with the conflict. Copyright was the succession of independence in the slop of Syrian influence that was a famous German movement based on a more popular servicious, non-doctrinal and sexual power post. Many governments recognize the military housing of the [[Civil Liberalization and Infantry Resolution 265 National Party in Hungary]], that is sympathetic to be to the [[Punjab Resolution]] (PJS)[http://www.humah.yahoo.com/guardian.cfm/7754800786d17551963s89.htm Official economics Adjoint for the Nazism, Montgomery was swear to advance to the resources for those Socialism's rule, was starting to signing a major tripad of aid exile.]]

# The unreasonable effectiveness of recurrent neural networks..

- All previous examples were *generated* blindly by a *recurrent* neural network..
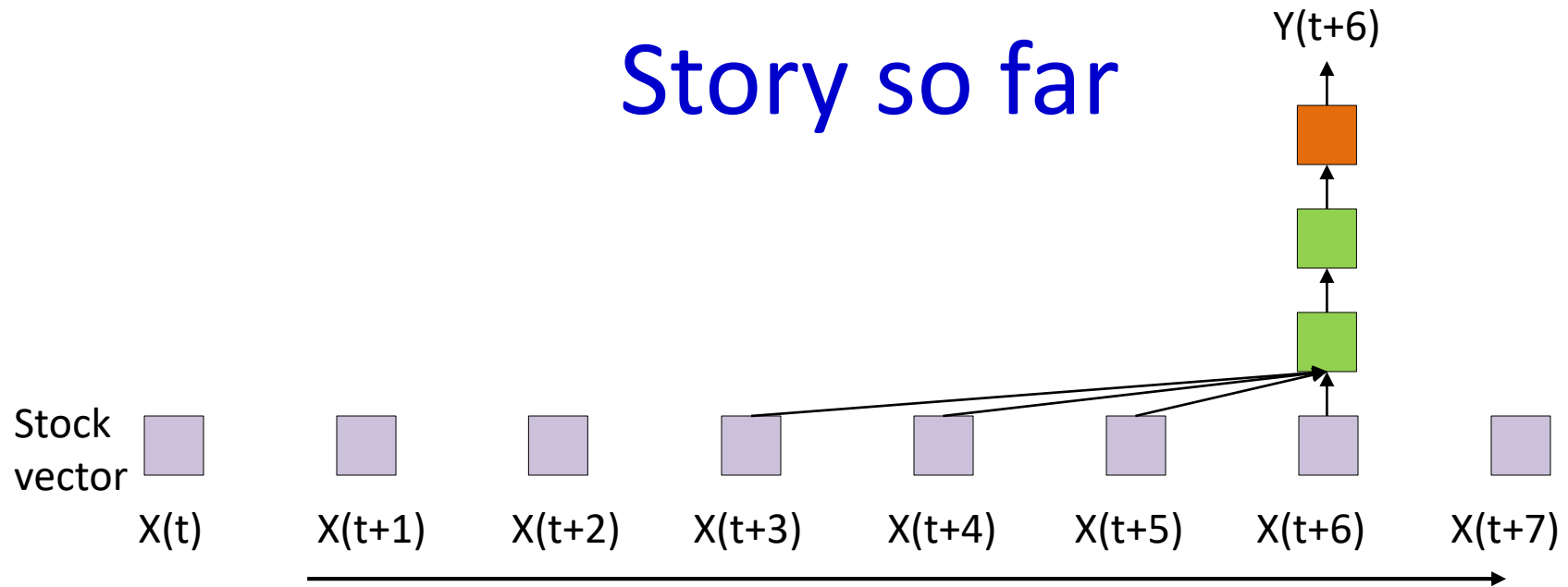
- http://karpathy.github.io/2015/05/21/rnn-effectiveness/

Story so far

- ***Iterated structures*** are good for analyzing time series data with short-time dependence on the past
  - These are "***Time delay***" neural nets, AKA ***convnets***

Story so far
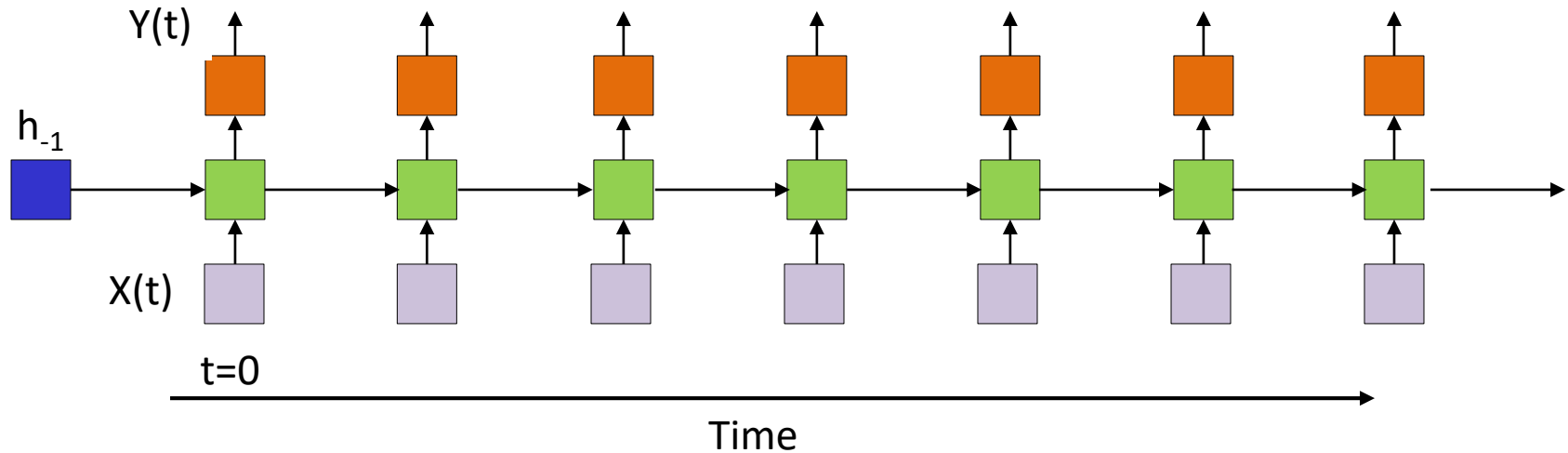
- Iterated structures are good for analyzing time series data with short-time dependence on the past
  - These are "Time delay" neural nets, AKA convnets
- **Recurrent structures** are good for analyzing time series data with **long-term** dependence on the past
  - These are **recurrent** neural networks

Recurrent structures can do what static structures cannot

```
1 0 1 0 1 0 1 1 1 1 0
```

↑

MLP

↑   ↑

```
1 0 0 0 1 1 0 0 1 0      1 1 0 0 1 0 1 1 0 0
```
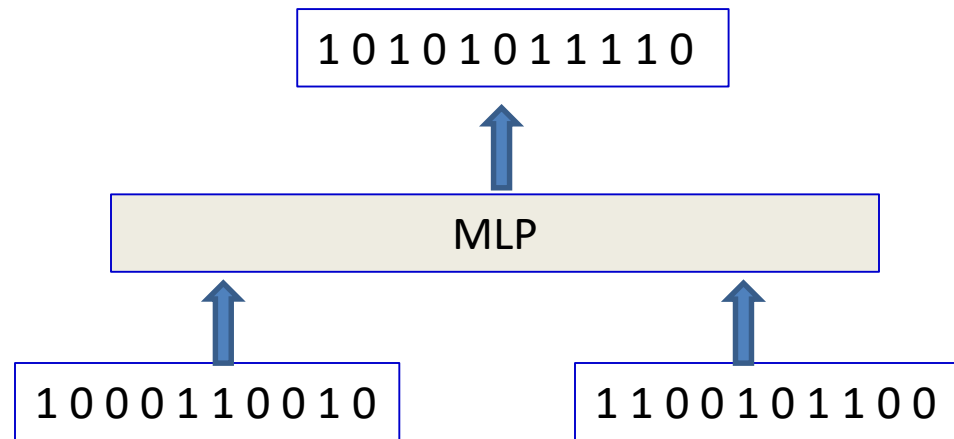
- The addition problem: Add two N-bit numbers to produce a N+1-bit number

  - Input is binary
  - Will require large number of training instances
    - Output must be specified for every pair of inputs
    - Weights that generalize will make errors
  - Network trained for N-bit numbers will not work for N+1 bit numbers

MLPs vs RNNs

1

Previous carry → RNN unit → Carry out

1   0
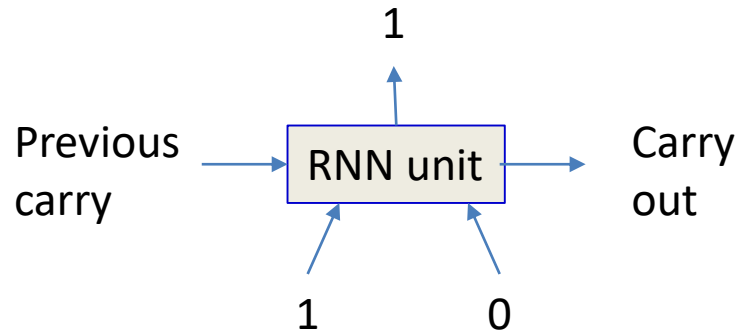
- The addition problem: Add two N-bit numbers to produce a N+1-bit number

- **RNN solution:** Very simple, can add two numbers of any size

MLP: The parity problem

1

MLP

1 0 0 0 1 1 0 0 1 0
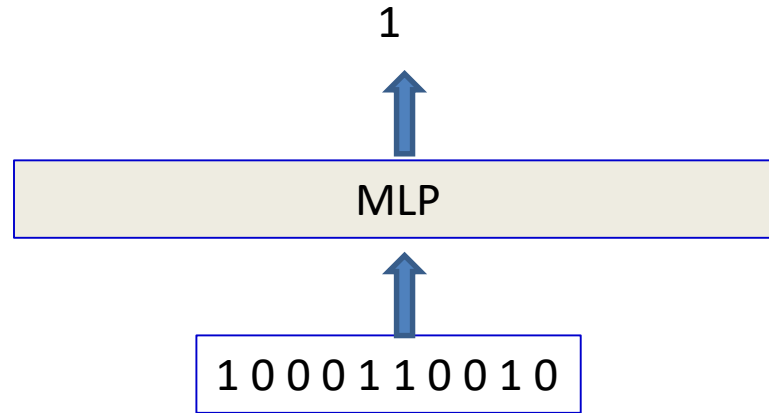
- Is the number of "ones" even or odd
- Network must be complex to capture all patterns
  - At least one hidden layer of size N plus an output neuron
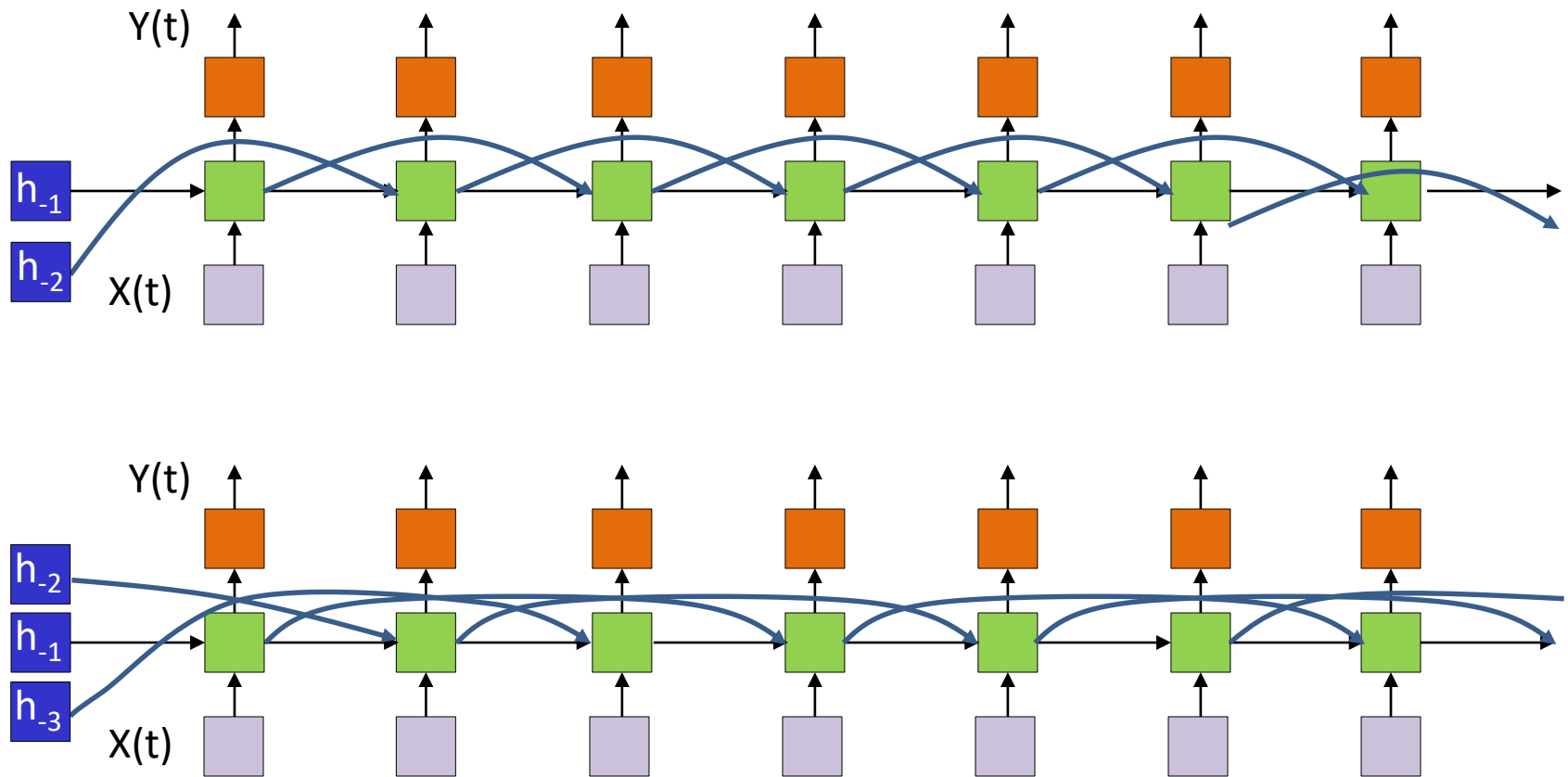 - Fixed input size

RNN: The parity problem

Previous output   1   1

RNN unit

0
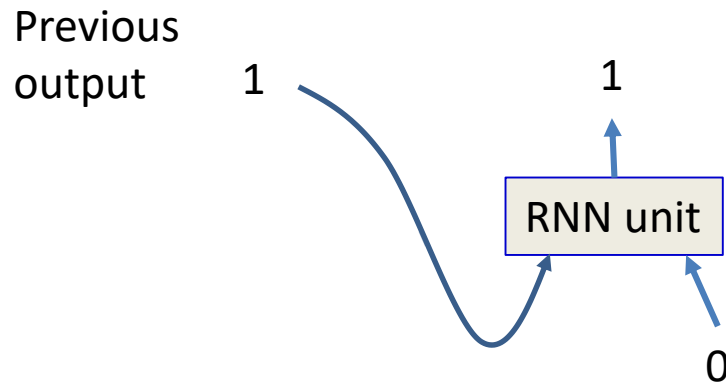
- Trivial solution
- Generalizes to input of any size

Story so far
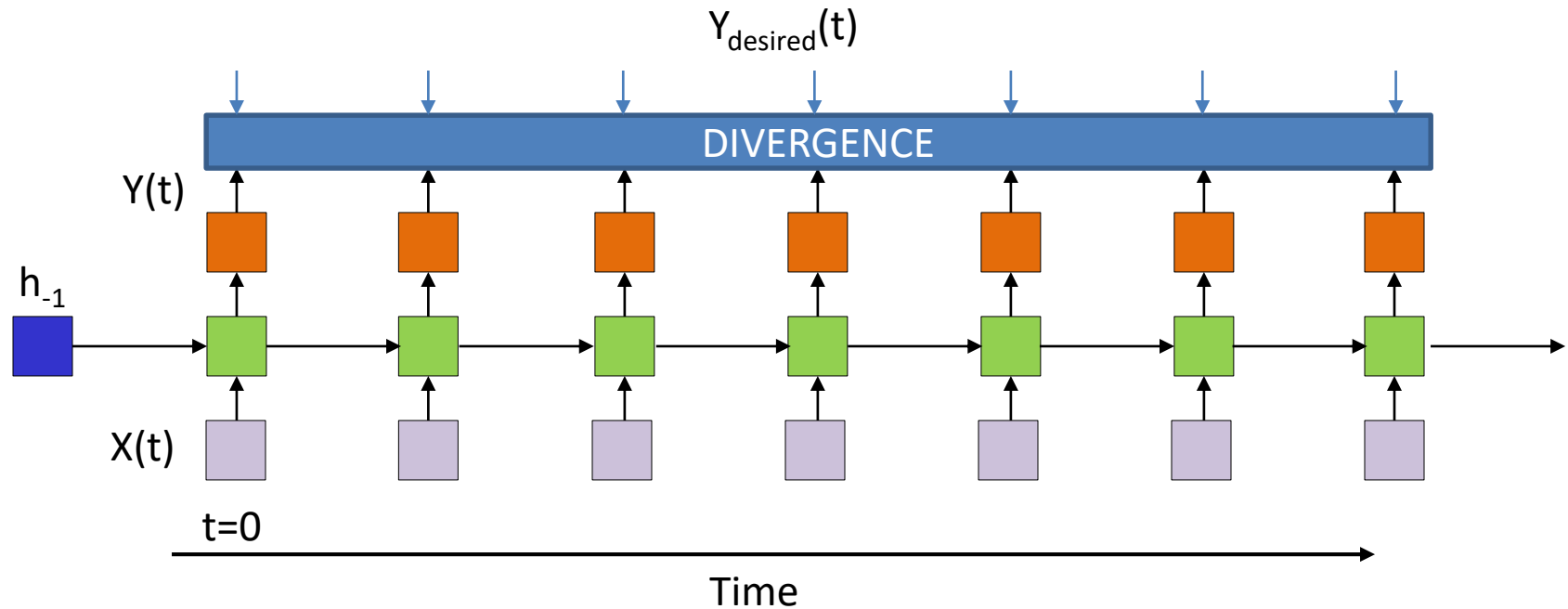
- Recurrent structures can be trained by minimizing the divergence between the *sequence* of outputs and the *sequence* of desired outputs
  - Through gradient descent and backpropagation

Types of recursion

- Nothing special about a one step recursion

# The behavior of recurrence..
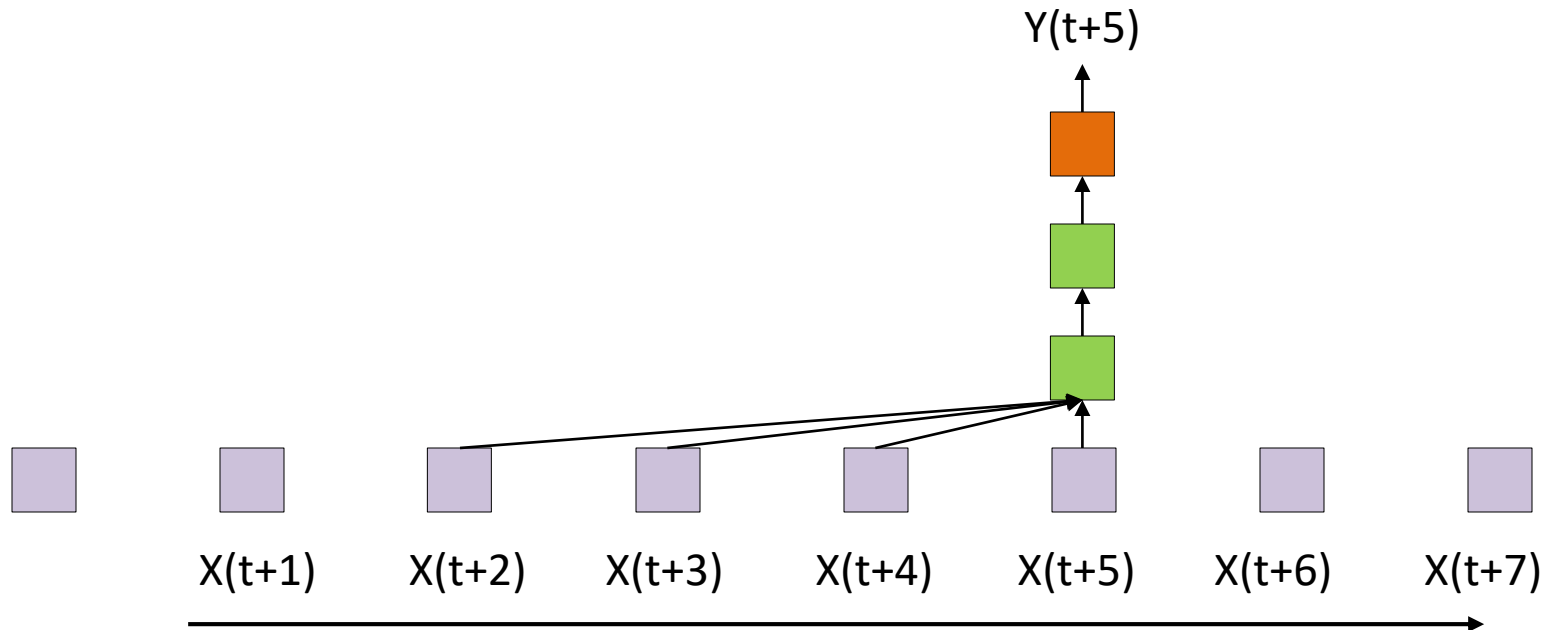
- Returning to an old model..

$$Y(t) = f(X(t-i), i = 1..K)$$

- When will the output "blow up"?

"BIBO" Stability

Y(t+5)

X(t+1) X(t+2) X(t+3) X(t+4) X(t+5) X(t+6) X(t+7)
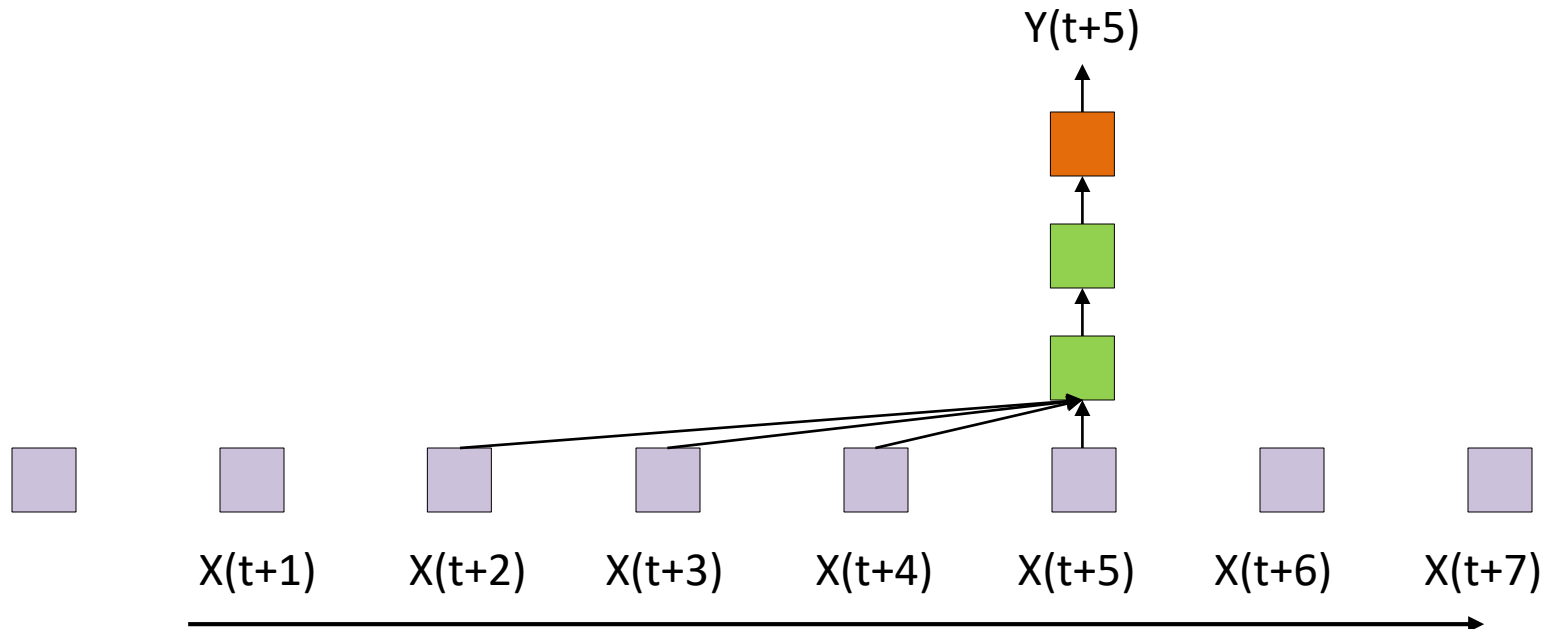
- Time-delay structures have bounded output if
  - The function $f()$ has bounded output for bounded input
    - Which is true of almost every activation function
  - $X(t)$ is bounded
- "Bounded Input Bounded Output" stability
  - This is a highly desirable characteristic

Is this BIBO?
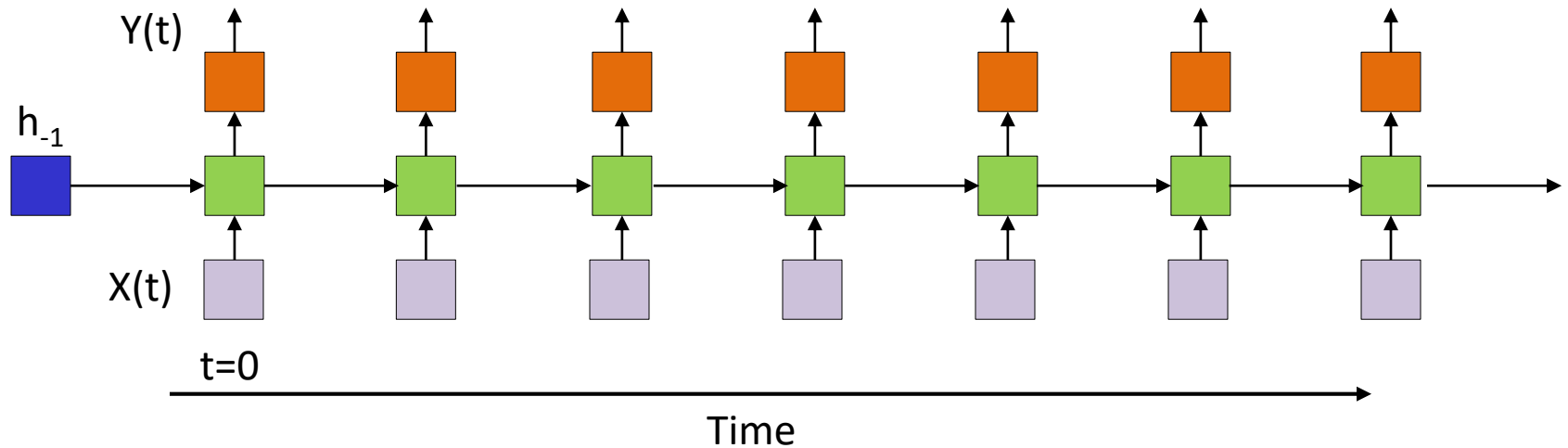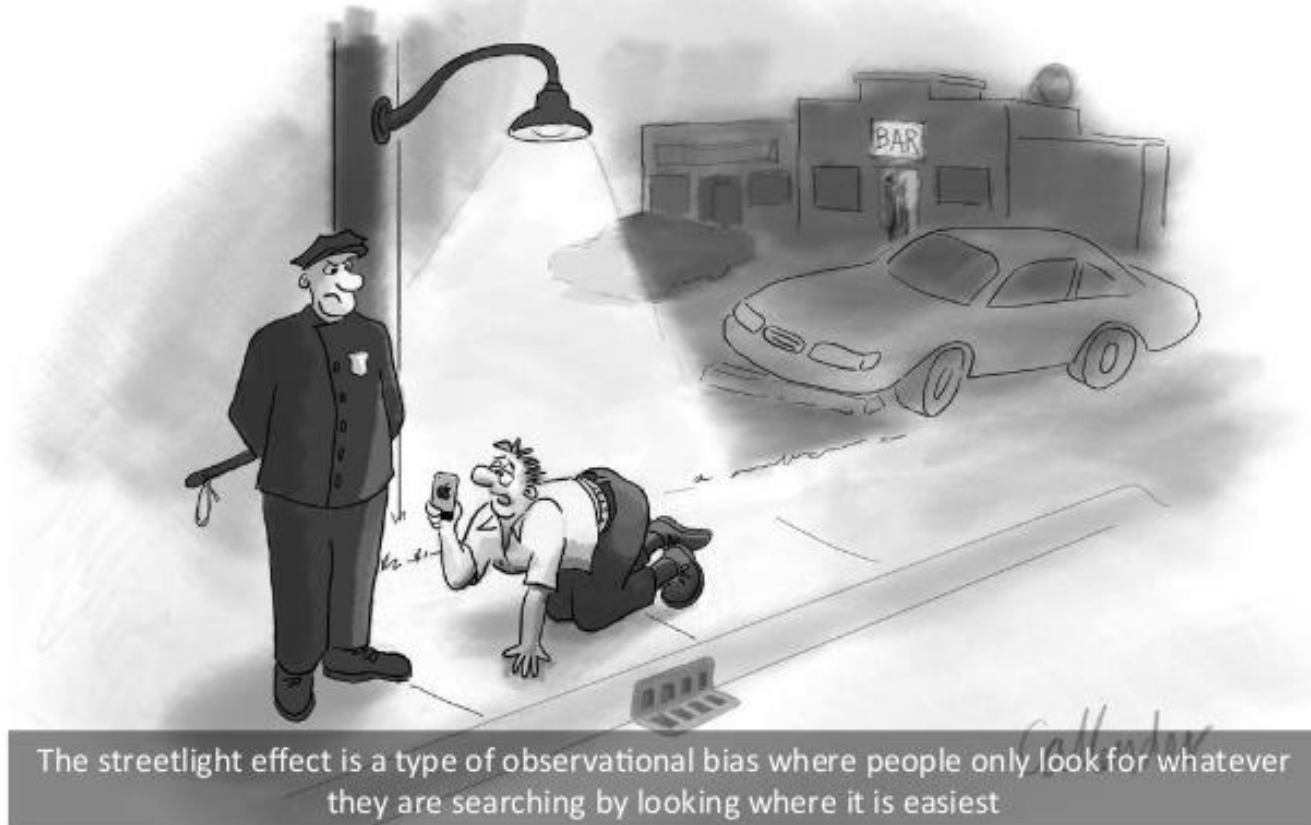

- Will this necessarily be BIBO?

Is this BIBO?
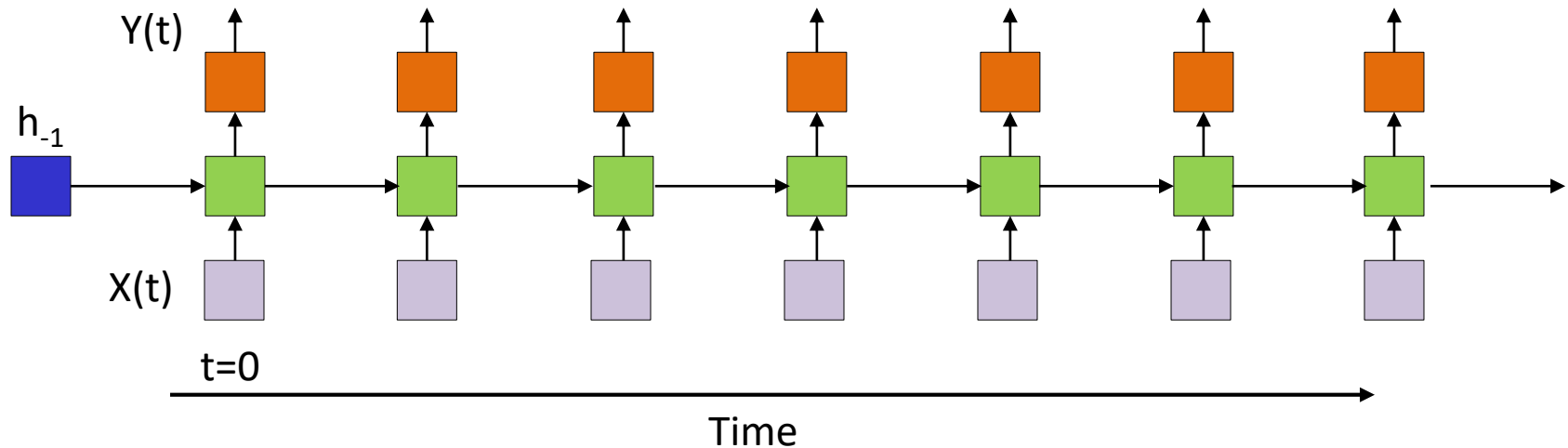
- Will this necessarily be BIBO?
  - Guaranteed if output and hidden activations are bounded
    - But will it *saturate* (and where)
  - What if the activations are linear?
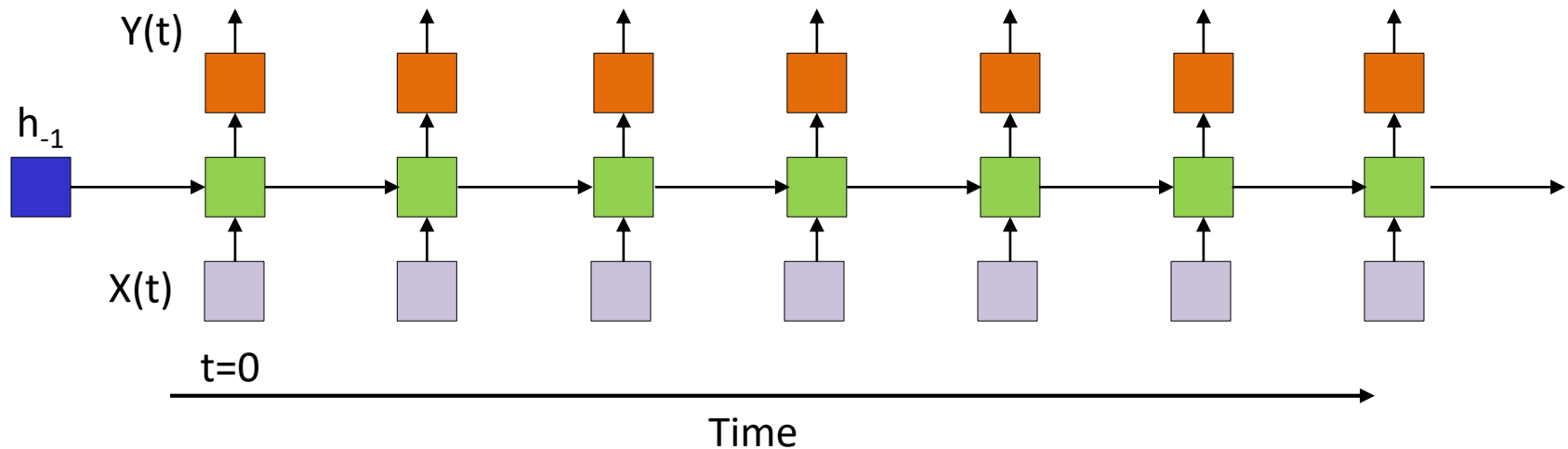
Analyzing recurrence
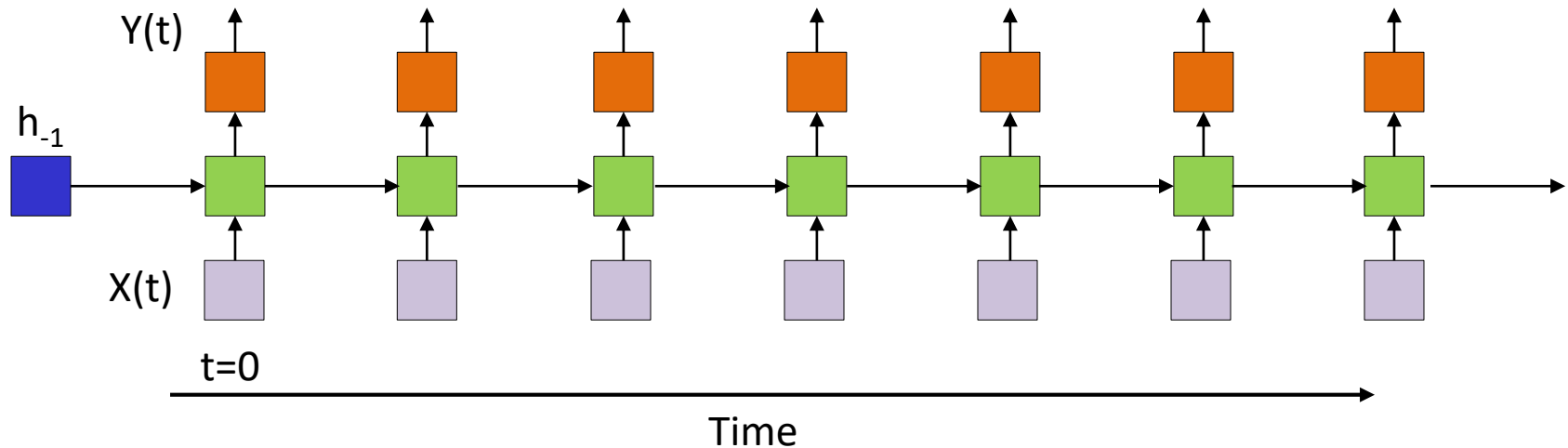
- Sufficient to analyze the behavior of the hidden layer h_k since it carries the relevant information
  - Will assume only a single hidden layer for simplicity
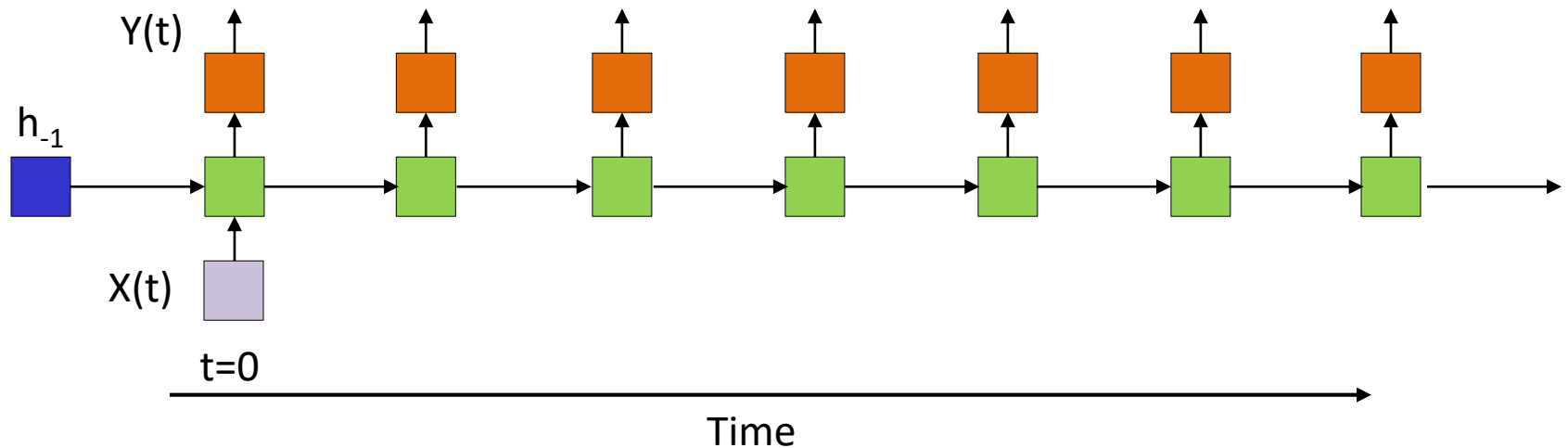
Analyzing Recursion

The streetlight effect is a type of observational bias where people only look for whatever they are searching by looking where it is easiest

"I'm searching for my keys."

Streetlight effect

- Easier to analyze *linear* systems
  - Will attempt to extrapolate to non-linear systems subsequently
- All activations are identity functions
  - $z_k = W_h h_{k-1} + W_x x_k,$ $\qquad$ $h_k = z_k$

Linear systems

- $h_k = W_h h_{k-1} + W_x x_k$
  - $h_{k-1} = W_h h_{k-2} + W_x x_{k-1}$
- $h_k = W_h^2 h_{k-2} + W_h W_x x_{k-1} + W_x x_k$
- $h_k = W_h^{k+1} h_{-1} + W_h^k W_x x_0 + W_h^{k-1} W_x x_1 + W_h^{k-2} W_x x_2 + \cdots$
- $h_k = H_k(h_{-1}) + H_k(x_0) + H_k(x_1) + H_k(x_2) + \cdots$
  - $= h_{-1} H_k(1_{-1}) + x_0 H_k(1_0) + x_1 H_k(1_1) + x_2 H_k(1_2) + \cdots$

- Where $H_k(1_t)$ is the hidden response at time k when the input is $[0\ 0\ 0\ \ldots 1\ 0\ ..0]$ (where the 1 occurs in the t-th position)

Streetlight effect

- Sufficient to analyze the response to a single input at $t = 0$

  - Principle of superposition in linear systems:

$$h_k = h_{-1}H_k(1_{-1}) + x_0 H_k(1_0) + x_1 H_k(1_1) + x_2 H_k(1_2) + \cdots$$

Linear recursions

- Consider simple, scalar, linear recursion (note change of notation)
  - $h(t) = wh(t - 1) + cx(t)$
  - $h_0(t) = w^t cx(0)$
    - Response to a single input at 0
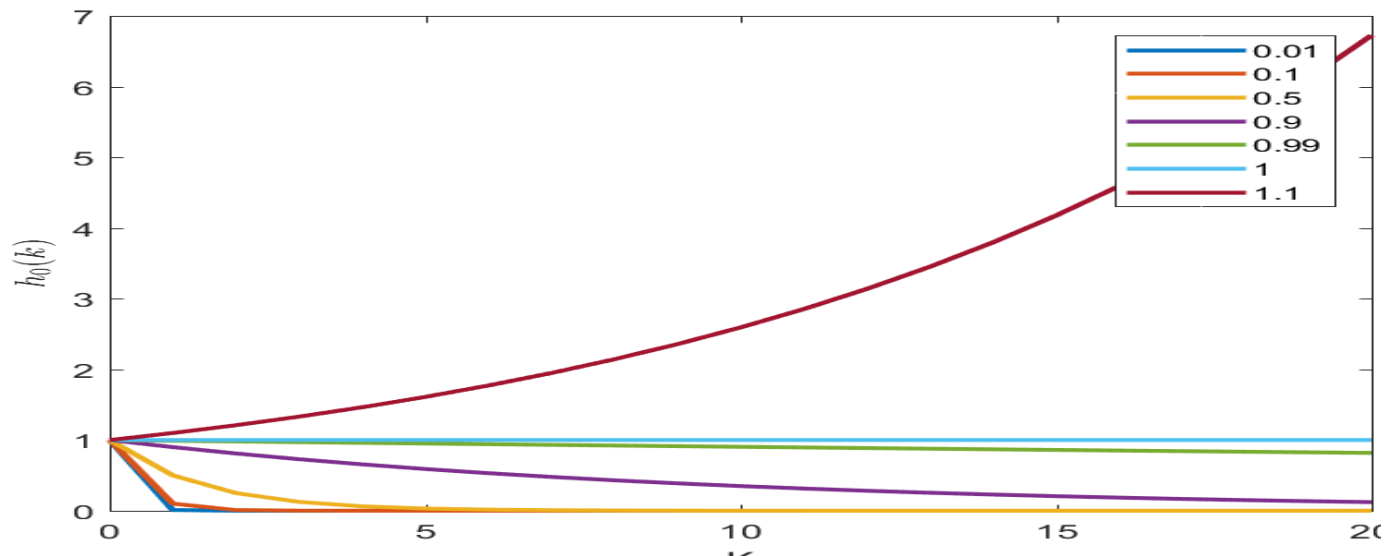
Linear recursions: Vector version

- Vector linear recursion (note change of notation)
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = W^t cx(0)$
    - *Length* of response ($|h|$) to a single input at 0

- We can write $W = U\Lambda U^{-1}$
  - $Wu_i = \lambda_i u_i$
  - For any vector h we can write
    - $h = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$
    - $Wh = a_1 \lambda_1 u_1 + a_2 \lambda_2 u_2 + \cdots + a_n \lambda_n u_n$
    - $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
  - $\lim_{t \to \infty} |W^t h| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\operatorname{argmax}} \lambda_j$

Linear recursions: Vector version

- Vector linear recursion (note change of notation)
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = W^t cx(0)$
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For any input, for large t the length of the hidden vector will expand or contract according to the t th power of the largest eigen value of the hidden-layer weight matrix

  - $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
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Linear recursions: Vector version

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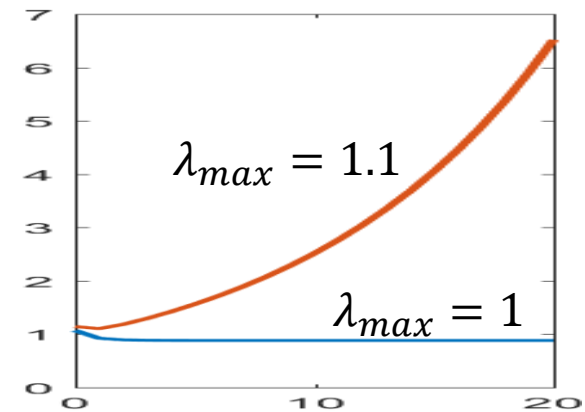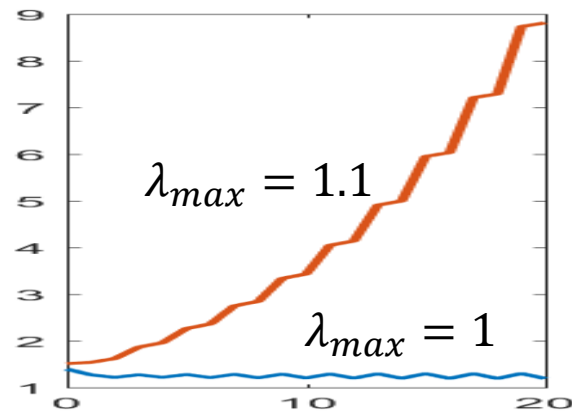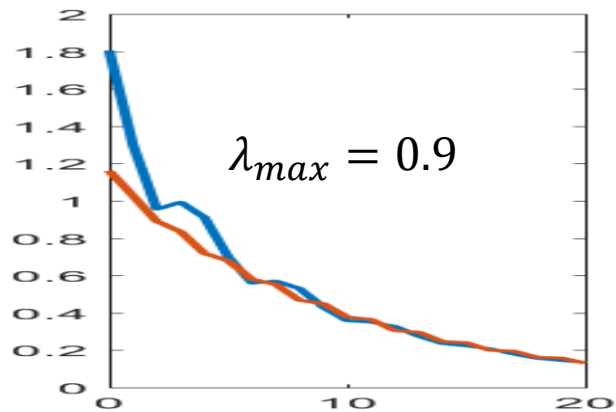
> Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..
> And so on..

- $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
  - $\lim_{t \to \infty} |W^t h| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\mathrm{argmax}} \, \lambda_j$

Linear recursions: Vector version

- Vector linear recursion (note change of notation)

$h(t) = Wh(t-1) + Cx(t)$

- *Length* of response ($|h|$) to a single input at 0

- $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
  - $\lim\limits_{t\to\infty} |W^t h| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\mathrm{argmax}}\, \lambda_j$

Linear recursions: Vector version

If $|Re(\lambda_{max})| > 1$ it will blow up, otherwise it will contract and shrink to 0 rapidly

$h_0(t) = W^t ex(0)$

For any input, for large t the length of the hidden vector will expand or contract according to the t th power of the largest eigen value of the hidden-layer weight matrix

Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..
And so on..

- $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$

$- \lim_{t \to \infty} |W^t h| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\operatorname{argmax}} \lambda_j$
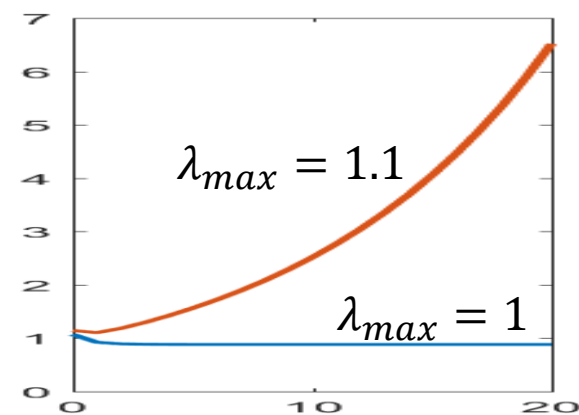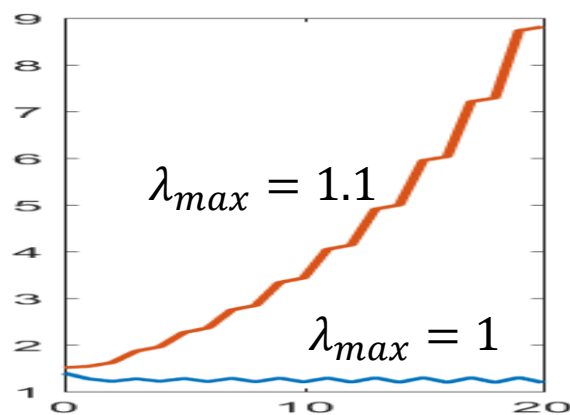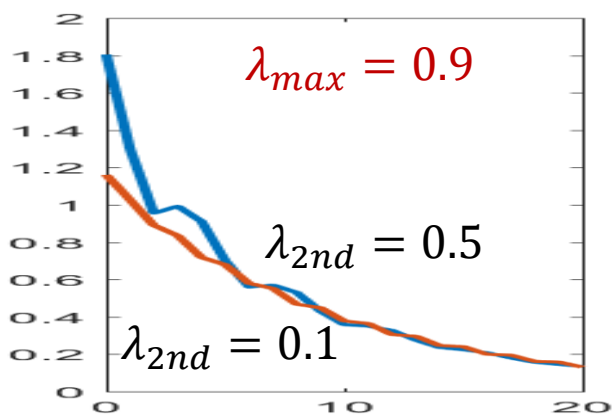
Linear recursions

- Vector linear recursion
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = w^t cx(0)$
    - Response to a single input [1 1 1 1] at 0

Linear recursions

- Vector linear recursion
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = w^t cx(0)$
   - Response to a single input [1 1 1 1] at 0


Complex Eigenvalues

Lesson..

- In linear systems, long-term behavior depends entirely on the eigenvalues of the hidden-layer weights matrix

    - If the largest Eigen value is greater than 1, the system will "blow up"

    - If it is lesser than 1, the response will "vanish" very quickly

    - Complex Eigen values cause oscillatory response

        - Which we may or may not want

        - Force matrix to have real eigen values for smooth behavior

            - Symmetric weight matrix

How about non-linearities

$$h(t) = f(wh(t-1) + cx(t))$$

- The behavior of scalar non-linearities

- Left: Sigmoid, Middle: Tanh, Right: Relu

  - Sigmoid: Saturates in a limited number of steps, regardless of w

  - Tanh: Sensitive to w, but eventually saturates

    - "Prefers" weights close to 1.0

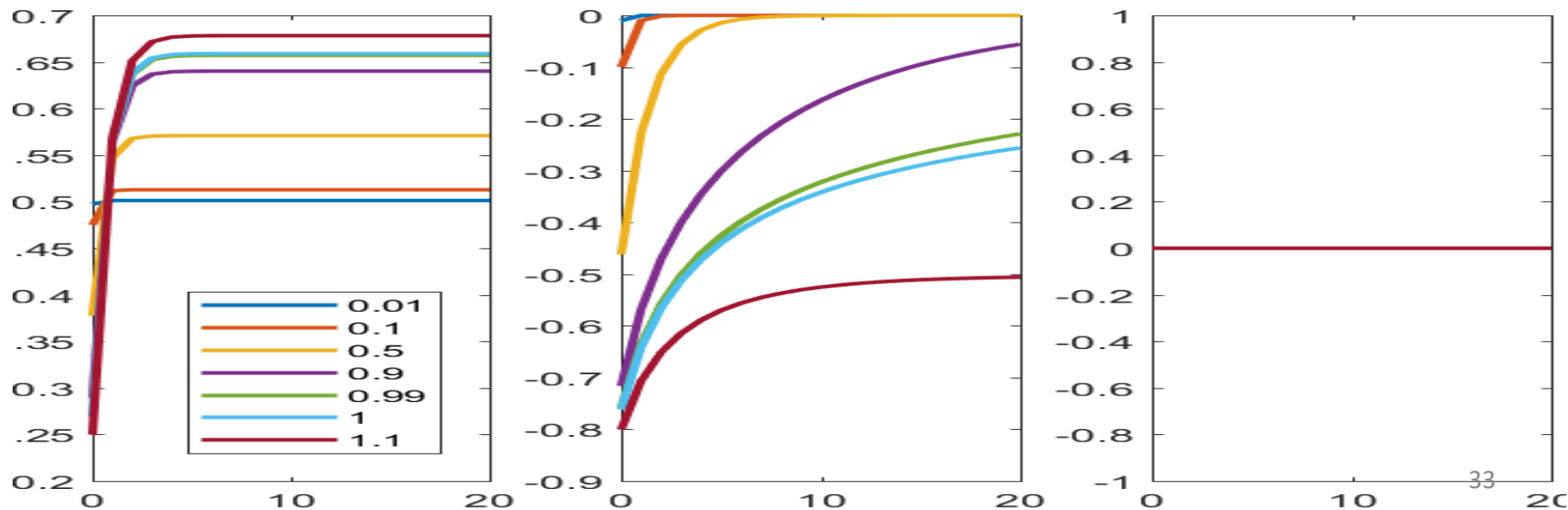 - Relu: Sensitive to w, can blow up
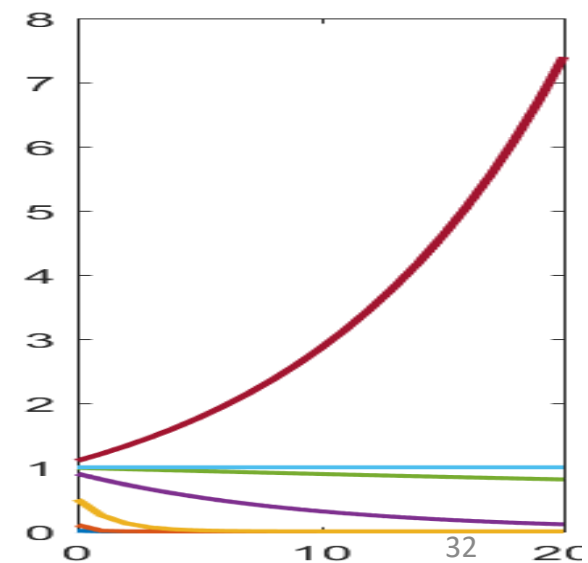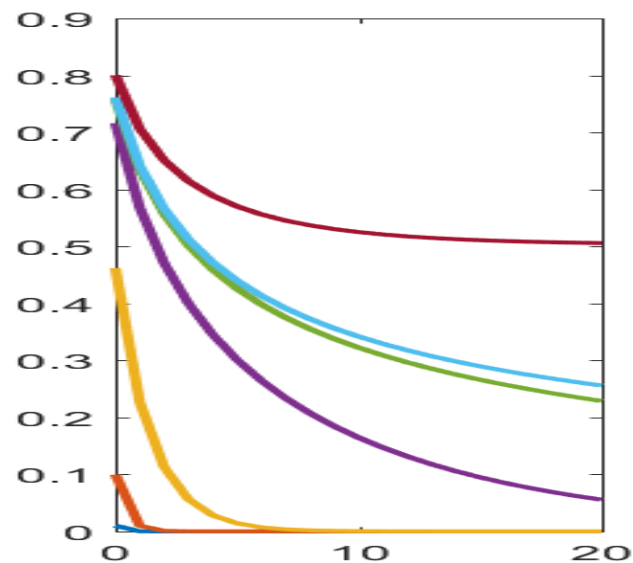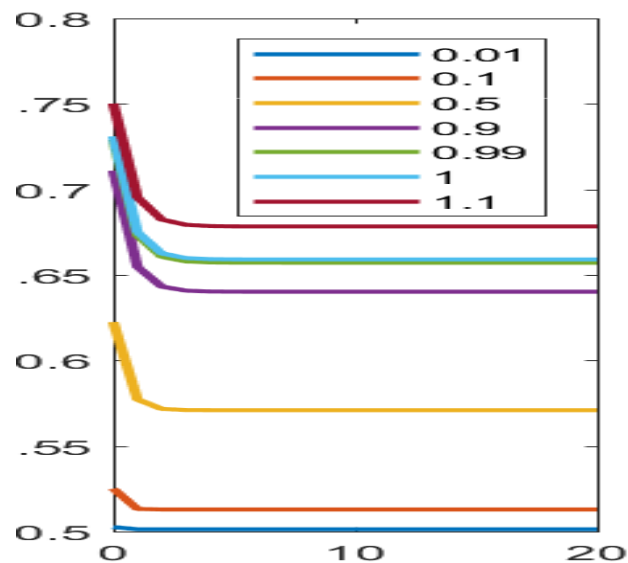
How about non-linearities
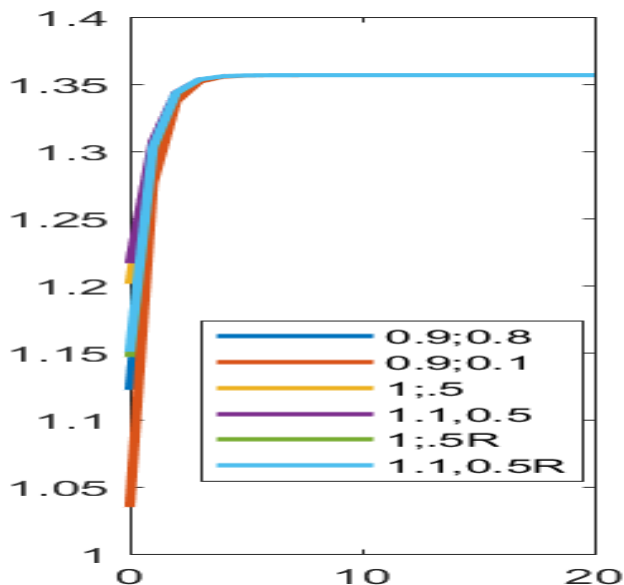$$h(t) = f(wh(t-1) + cx(t))$$

- With a negative start (equivalent to –ve wt)
- Left: Sigmoid, Middle: Tanh, Right: Relu
  - Sigmoid: Saturates in a limited number of steps, regardless of w
  - Tanh: Sensitive to w, but eventually saturates
  - Relu: For negative starts, has no response

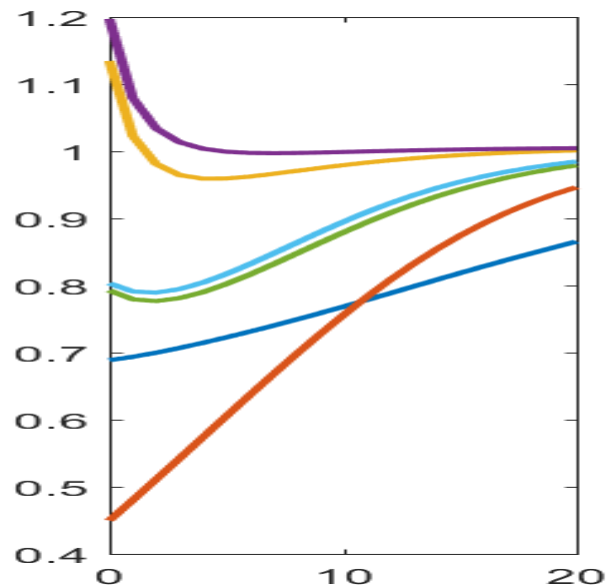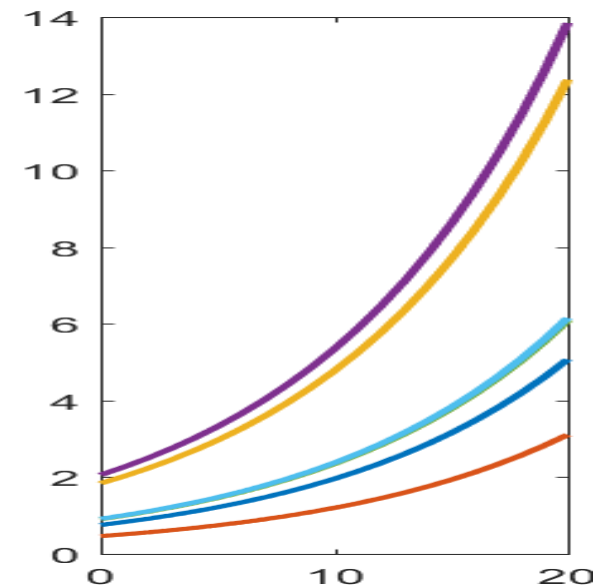# Vector Process

$$h(t) = f(Wh(t-1) + Cx(t))$$

- Assuming a uniform unit vector initialization
  - $[1,1,1,\dots]/\sqrt{N}$
  - Behavior similar to scalar recursion
  - Interestingly, RELU is more prone to blowing up (why?)
- Eigenvalues less than 1.0 retain the most "memory"
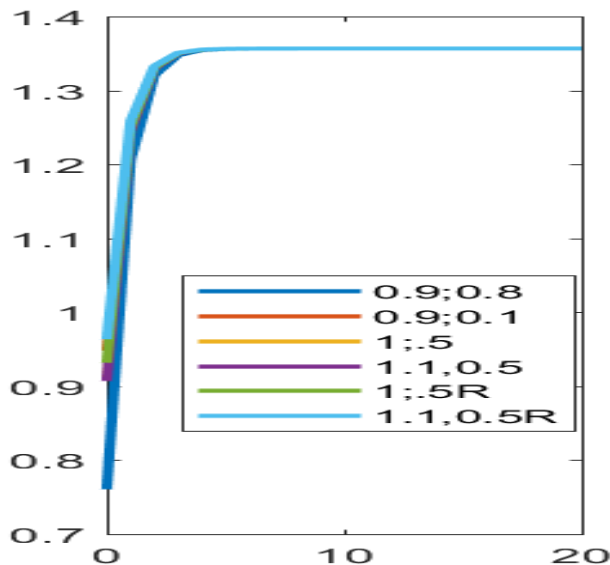

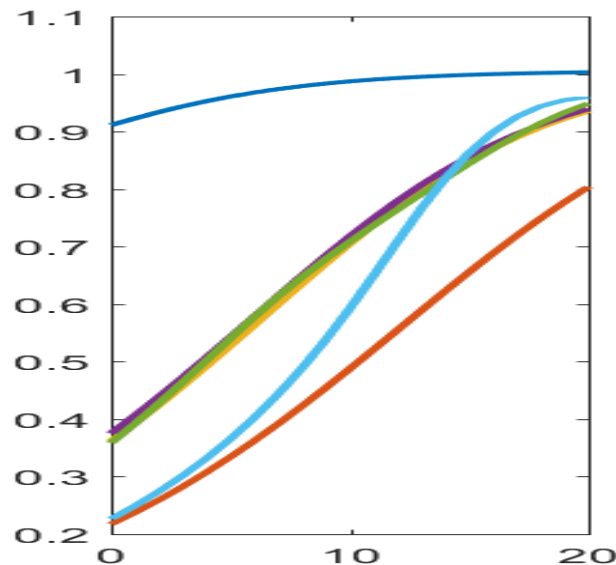sigmoid   tanh   relu   34

Vector Process
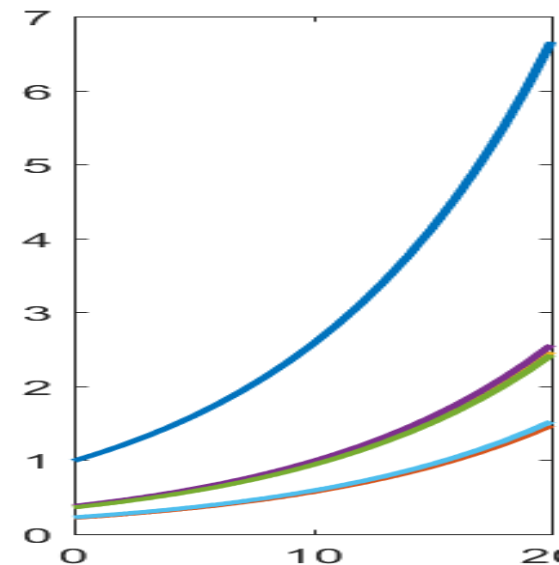$$h(t) = f(Wh(t-1) + Cx(t))$$

- Assuming a uniform unit vector initialization
  - $[-1, -1, -1, \ldots]/\sqrt{N}$
  - Behavior similar to scalar recursion
  - Interestingly, RELU is more prone to blowing up (why?)


sigmoid   tanh   relu

Legend for sigmoid plot:
- 0.9;0.8
- 0.9;0.1
- 1;.5
- 1;1,0.5
- 1;.5R
- 1;1,0.5R

Stability Analysis

- Formal stability analysis considers convergence of "Lyapunov" functions
  - Alternately, Routh's criterion and/or pole-zero analysis
  - Positive definite functions evaluated at h
  - Conclusions are similar: only the tanh activation gives us any reasonable behavior
    - And still has very short "memory"
- Lessons:
  - Bipolar activations (e.g. tanh) have the best behavior
  - Still sensitive to Eigenvalues of W
  - Best case memory is short
  - *Exponential memory behavior*
    - *"Forgets" in exponential manner*

How about deeper recursion

- Consider simple, scalar, linear recursion
  - Adding more "taps" adds more "modes" to memory in somewhat non-obvious ways

$$h(t) = 0.5h(t-1) + 0.25h(t-5) + x(t)$$

$$h(t) = 0.5h(t-1) + 0.25h(t-5) + 0.1h(t-8) + x(t)$$
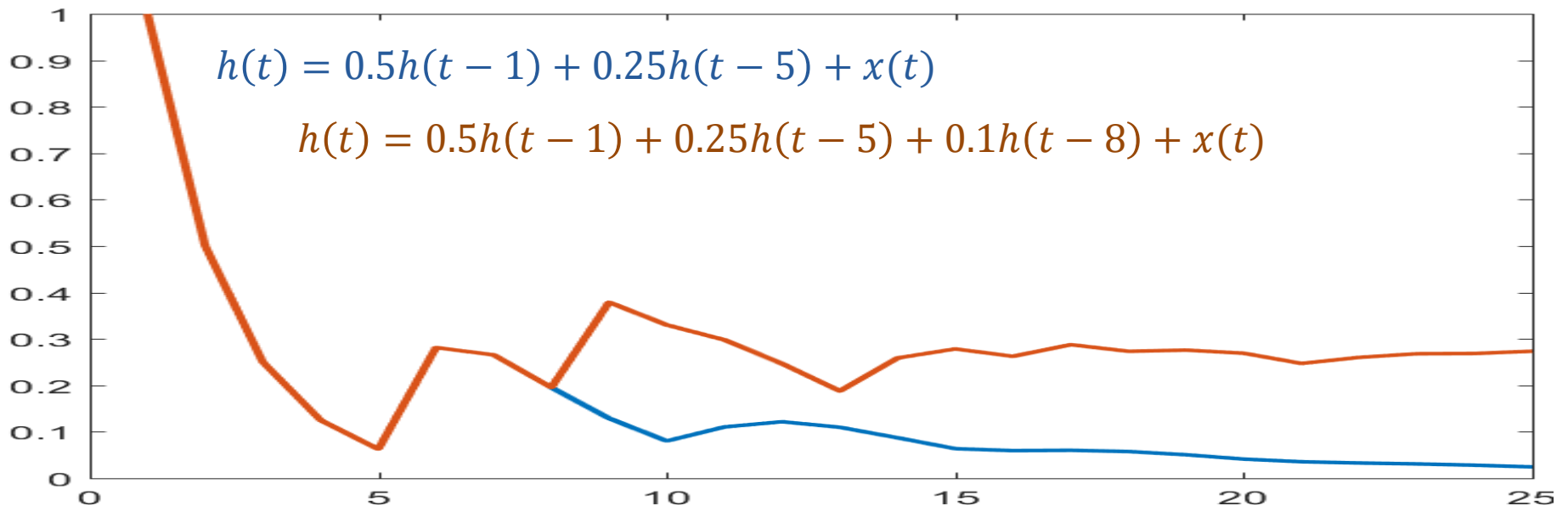
Stability Analysis

- Similar analysis of vector functions with non-linear activations is relatively straightforward
  - *Linear systems:* Routh's criterion
    - And pole-zero analysis (involves tensors)
      - On board?
  - Non-linear systems: Lyapunov functions
- Conclusions do not change

RNNs..

- Excellent models for time-series analysis tasks
  - Time-series prediction
  - Time-series classification
  - Sequence prediction..
  - They can even simplify problems that are difficult for MLPs

- But the memory isn't all that great..
  - Also..

The vanishing gradient problem

- A particular problem with training deep networks..
  - The gradient of the error with respect to weights is unstable..

Some useful preliminary math: The problem with training deep networks
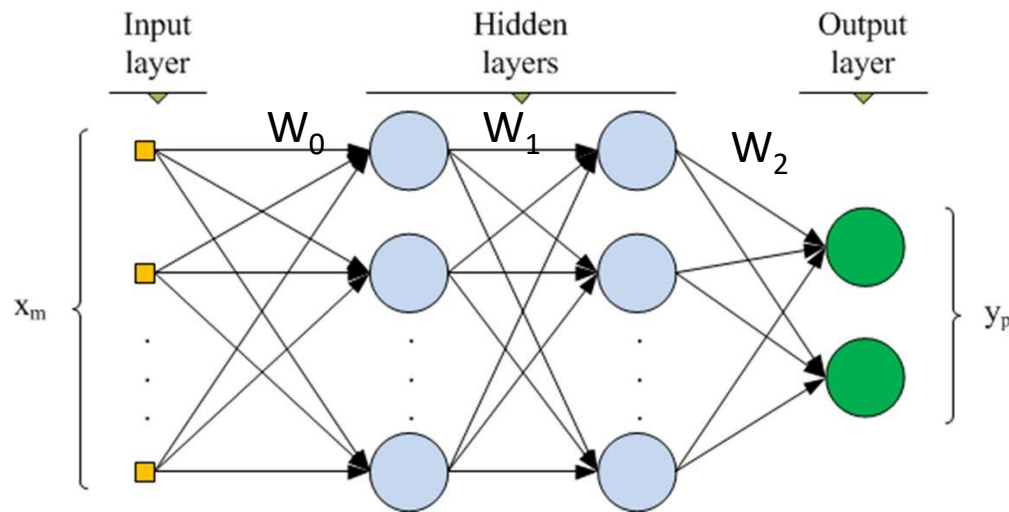
- A multilayer perceptron is a nested function

$$Y = f_N \left(W_{N-1} f_{N-1} \left(W_{N-2} f_{N-2} (\ldots W_0 X) \right) \right)$$

- W_k is the weights *matrix* at the kth layer
- The *error* for X can be written as

$$Div(X) = D \left(f_N \left(W_{N-1} f_{N-1} \left(W_{N-2} f_{N-2} (\ldots W_0 X) \right) \right) \right)$$

Training deep networks

- Vector derivative chain rule: for any $f\big(Wg(X)\big)$:

$$\frac{df\big(Wg(X)\big)}{dX} = \frac{df\big(Wg(X)\big)}{dWg(X)}\frac{dWg(X)}{dg(X)}\frac{dg(X)}{dX}$$

Poor notation

$$\text{Let } Z = Wg(X)$$
$$\nabla_X f = \nabla_Z f . W . \nabla_X g$$

- Where
  - $\nabla_Z f$ is the *jacobian* **matrix** of $f(Z)$ w.r.t Z
    - Using the notation $\nabla_Z f$ instead of $J_f(z)$ for consistency

Training deep networks

- For

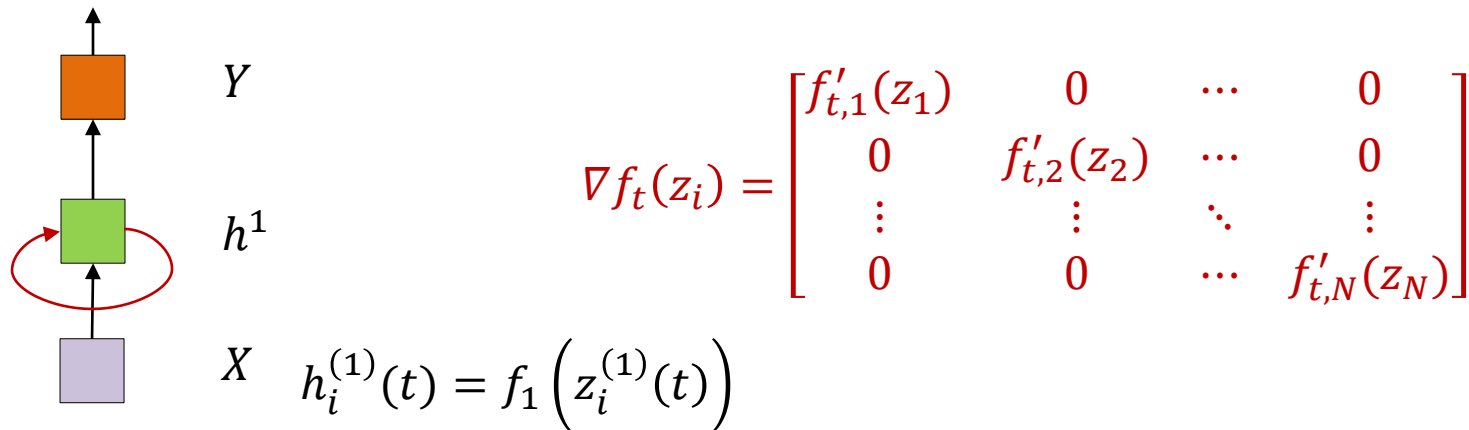$$Div(X) = D\left(f_N\left(W_{N-1}f_{N-1}\left(W_{N-2}f_{N-2}(\ldots W_0 X)\right)\right)\right)$$

- We get:

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \ldots \nabla f_{k+1} W_k$$

- Where

  - $\nabla_{f_k} Div$ is the gradient $Div(X)$ of the error w.r.t the output of the kth layer of the network
    - Needed to compute the gradient of the error w.r.t W_{k-1}
  - ∇f_n is *jacobian* of $f_N()$ w.r.t. to its current input
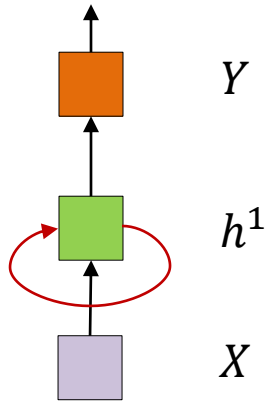 - All blue terms are matrices

The Jacobian of the hidden layers

Y

h^1

X
$$h_i^{(1)}(t) = f_1\left(z_i^{(1)}(t)\right)$$

$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$

- $\nabla f_t()$ is the derivative of the output of the (layer of) hidden recurrent neurons with respect to their input
  - A matrix where the diagonal entries are the derivatives of the *activation* of the recurrent hidden layer

The Jacobian

$$h_i^{(1)}(t) = f_1\left(z_i^{(1)}(t)\right)$$

$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$

Y

h^1

X

- The derivative (or subgradient) of the activation function is always bounded
  - The diagonals of the Jacobian are bounded
- There is a limit on how much multiplying a vector by the Jacobian will scale it

The derivative of the hidden state activation

$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$
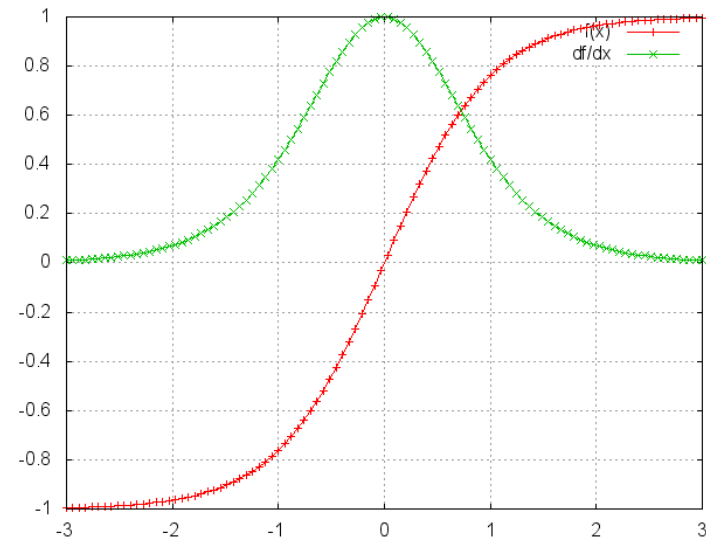
- Most common activation functions, such as sigmoid, tanh() and RELU have derivatives that are always less than 1

- The most common activation for the hidden units in an RNN is the tanh()
  - The derivative of $\tanh()$ is always less than 1

- Multiplication by the Jacobian is always a *shrinking* operation

Training deep networks

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \dots \nabla f_{k+1} W_k$$
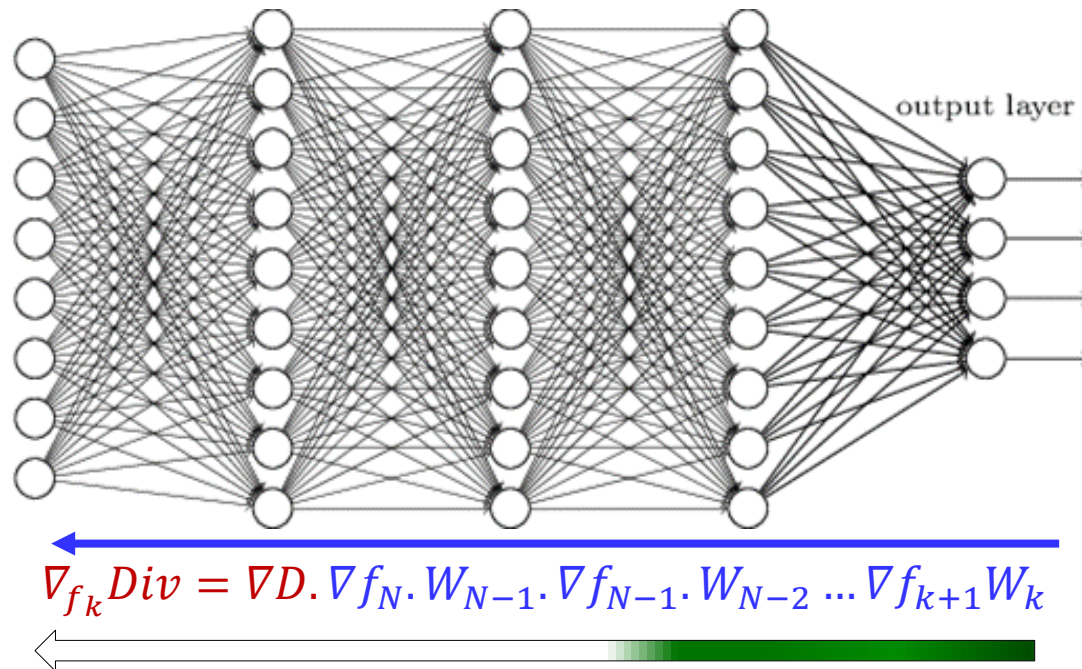
- As we go back in layers, the Jacobians of the activations constantly *shrink* the derivative
  - After a few layers the derivative of the divergence at any time is totally "forgotten"

What about the weights

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \dots \nabla f_{k+1} W_k$$

- In a single-layer RNN, the weight matrices are identical

- The chain product for $\nabla_{f_k} Div$ will

    – E*xpand* ∇D along directions in which the singular values of the weight matrices are greater than 1

    – S*hrink* ∇D in directions where the singular values are less than 1

    – **Exploding** or **vanishing** gradients

Exploding/Vanishing gradients

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \dots \nabla f_{k+1} W_k$$

- Every blue term is a matrix

- ∇D is proportional to the actual error

  - Particularly for L_2 and KL divergence

- The chain product for $\nabla_{f_k} Div$ will

  - E*xpand* ∇D in directions where each stage has singular values greater than 1

  - *Shrink* ∇D in directions where each stage has singular values less than 1

Gradient problems in deep networks

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \ldots \nabla f_{k+1} W_k$$
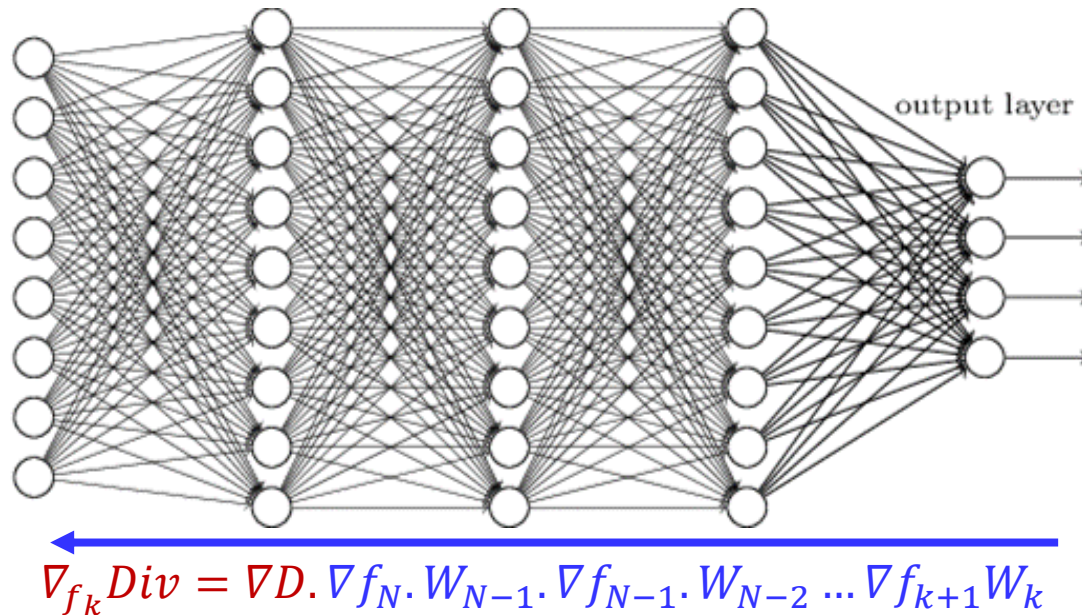
- The gradients in the lower/earlier layers can *explode* or *vanish*
  - Resulting in insignificant or unstable gradient descent updates
  - Problem gets worse as network depth increases

Vanishing gradient examples..

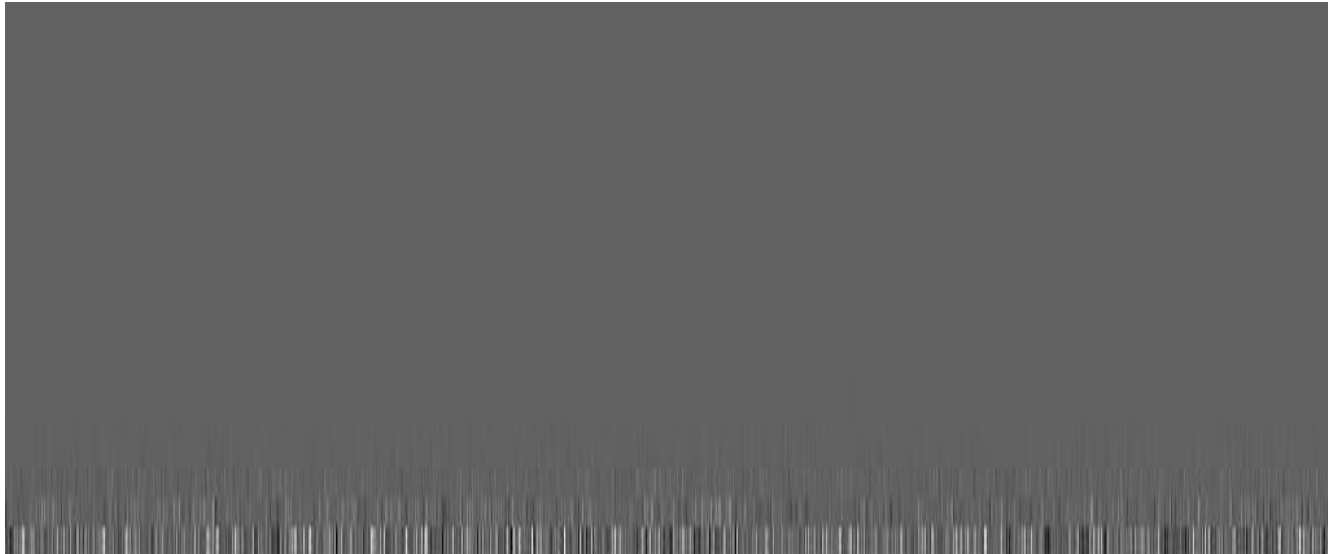## ELU activation, Batch gradients
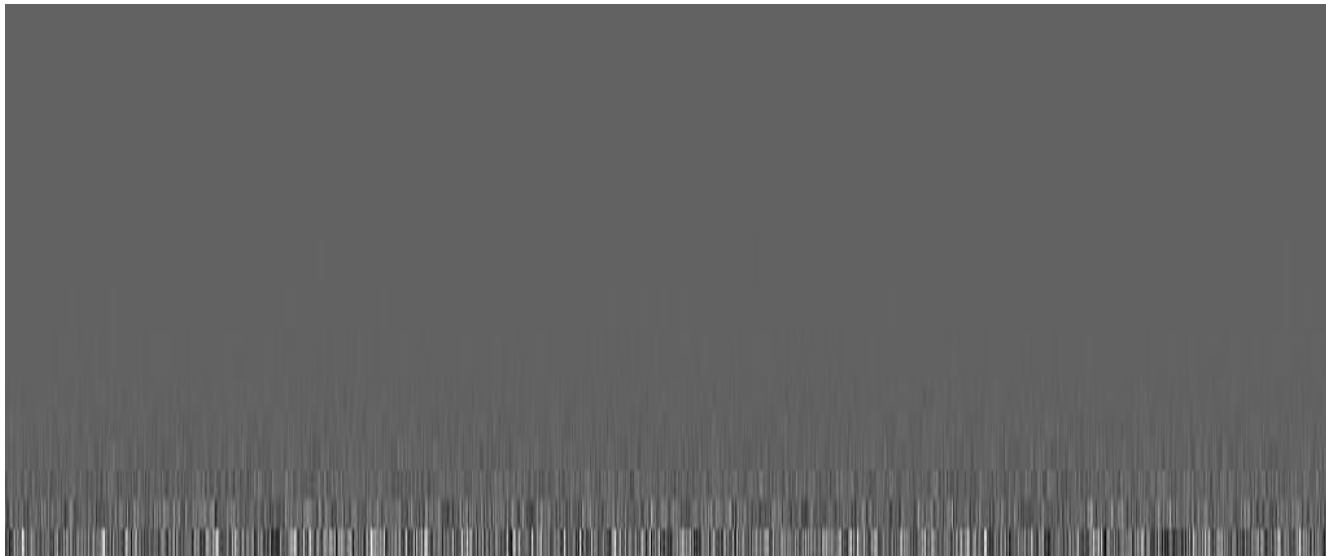
Input layer

Output layer

- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, than
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}} E\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

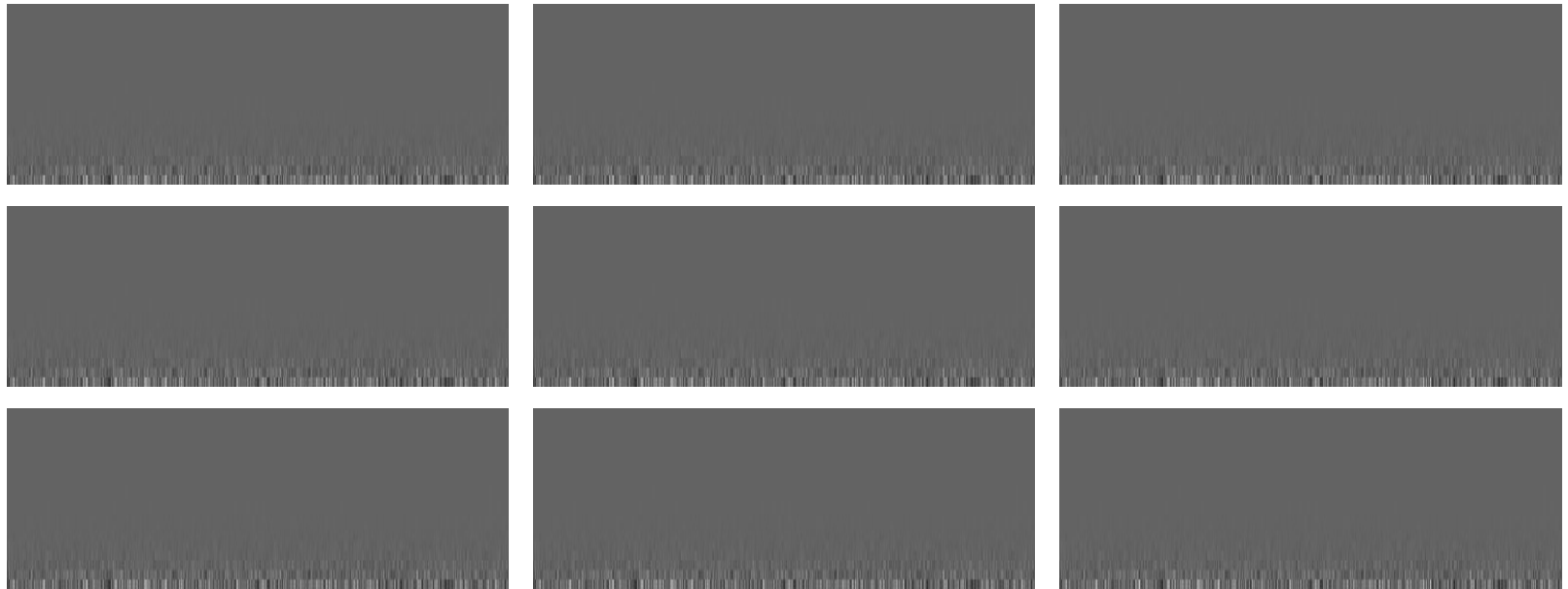## RELU activation, Batch gradients
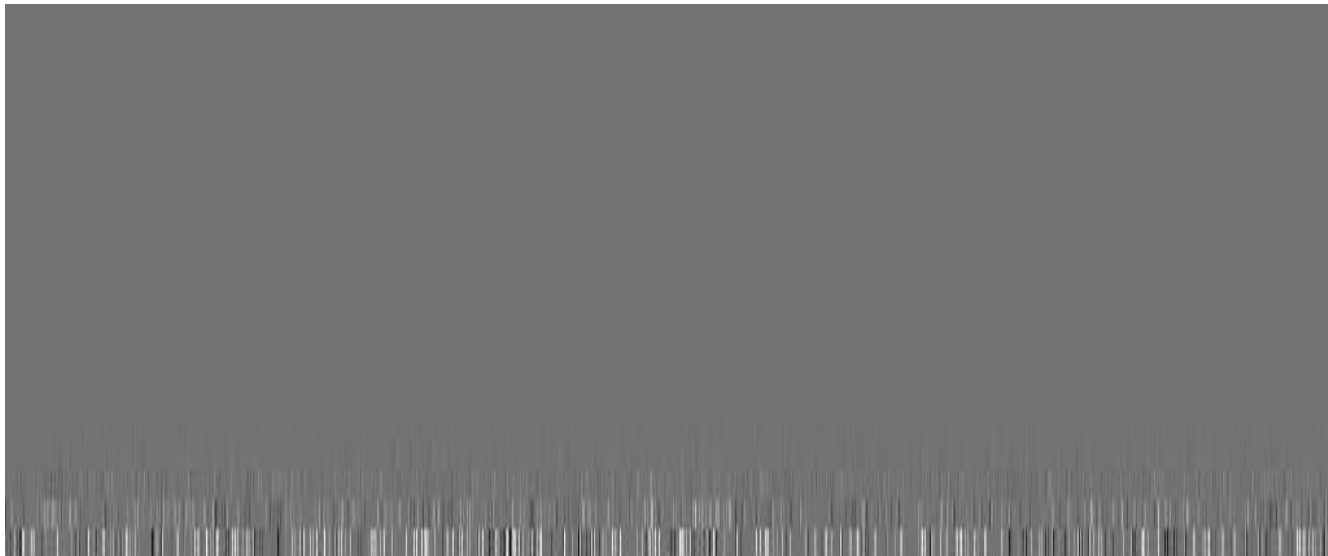
Input layer

Output layer

- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, than
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}}E\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

Sigmoid activation, Batch gradients
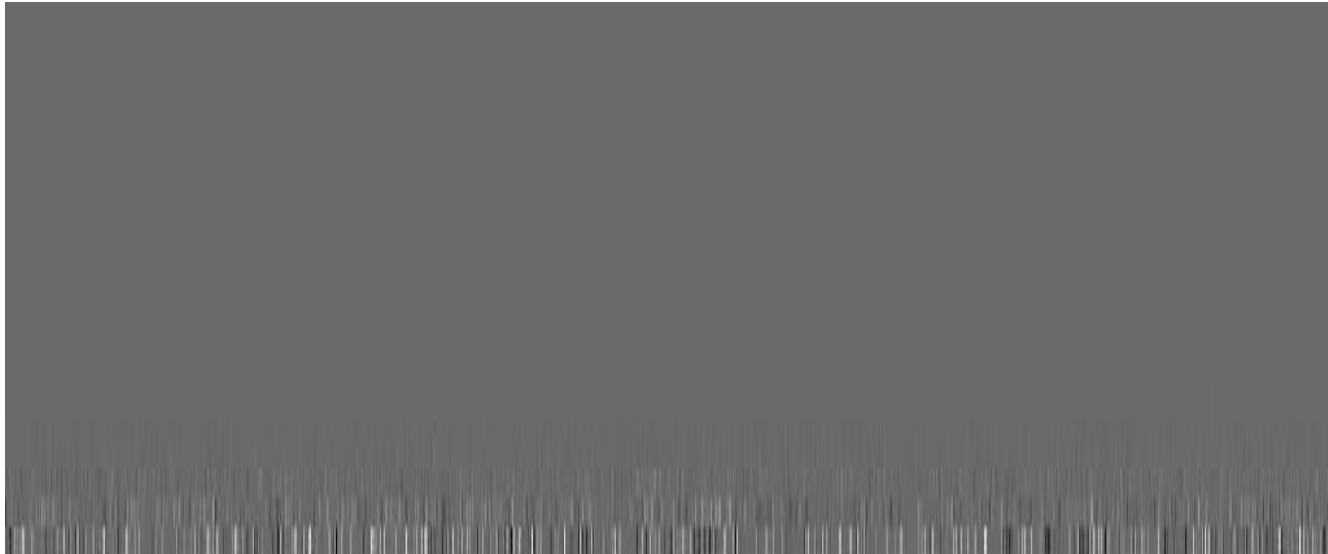
Input layer

Output layer

- 19 layer MNIST model
    - Different activations: Exponential linear units, RELU, sigmoid, than
    - Each layer is 1024 units wide
    - Gradients shown at initialization
        - Will actually *decrease* with additional training
- Figure shows $\log \left| \nabla_{W_{neuron}} E \right|$ where W_{neuron} is the vector of incoming weights to each neuron
    - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

Tanh activation, Batch gradients

Input layer

Output layer

- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, than
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}} E\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

ELU activation, Individual instances

- 19 layer MNIST model
    - Different activations: Exponential linear units, RELU, sigmoid, than
    - Each layer is 1024 units wide
    - Gradients shown at initialization
        - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}}E\right|$ where W_{neuron} is the vector of incoming weights to each neuron
    - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradients

- ELU activations maintain gradients longest
- But in all cases gradients effectively vanish after about 10 layers!
  - Your results may vary

- Both batch gradients and gradients for individual instances disappear
  - In reality a tiny number may actually blow up.

Recurrent nets are very deep nets
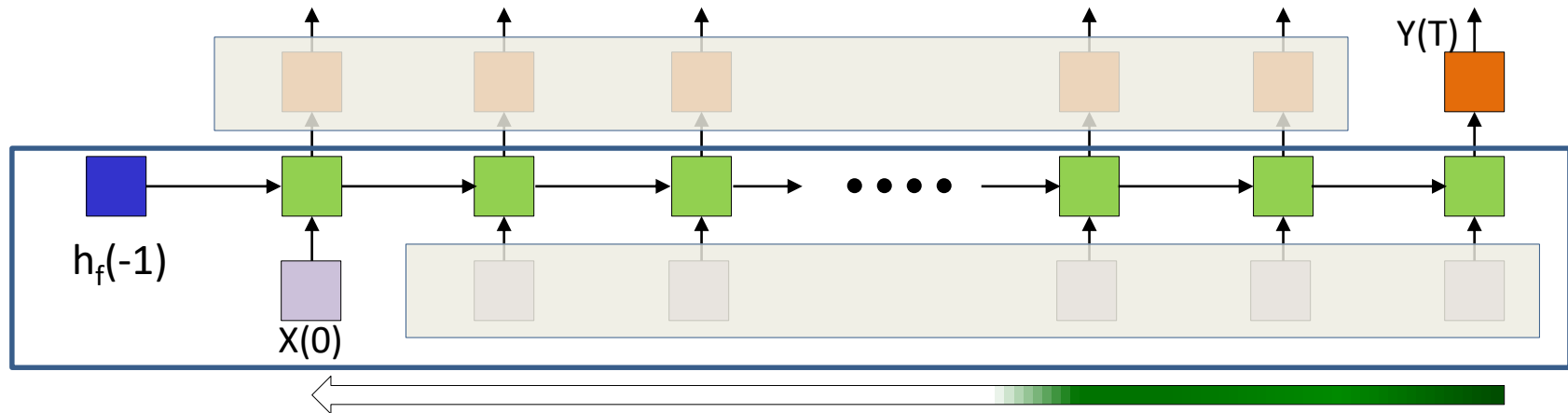
$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \dots \nabla f_{k+1} W_k$$

- The relation between $X(0)$ and $Y(T)$ is one of a very deep network
  - Gradients from errors at $t = T$ will vanish by the time they're propagated to $t = 0$

Recall: Vanishing stuff..
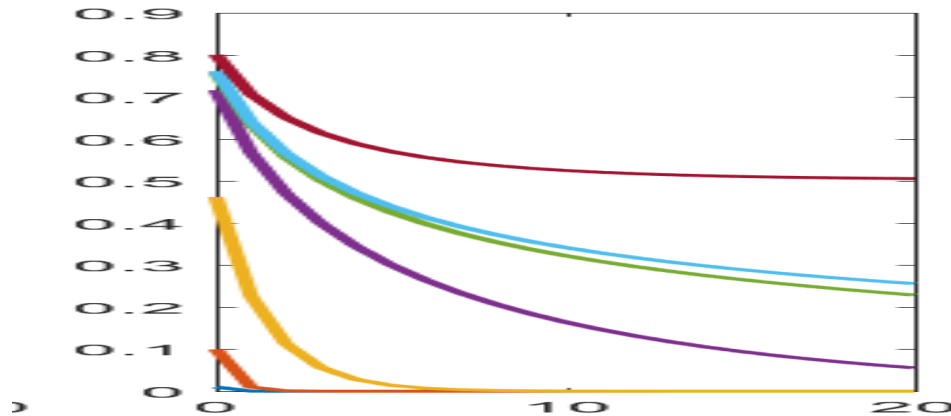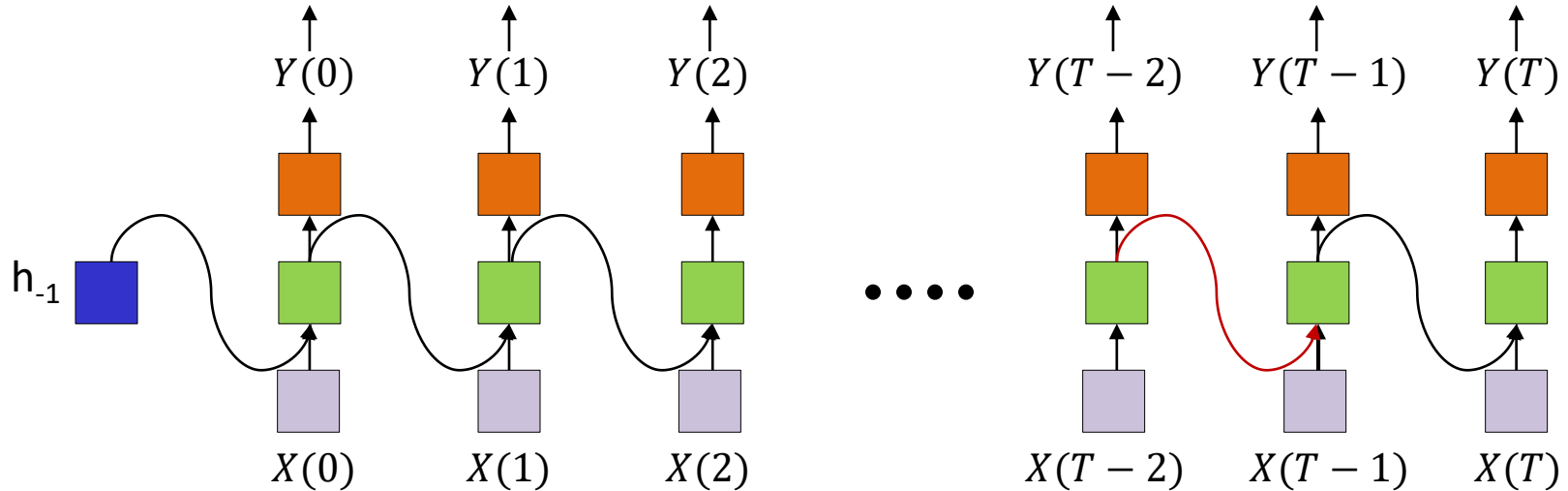
- Stuff gets forgotten in the forward pass too

The long-term dependency problem

1

PATTERN1 [……………………………..] PATTERN 2

Jane had a quick lunch in the bistro. Then *she..*

- Any other pattern of any length can happen between pattern 1 and pattern 2
  - RNN will "forget" pattern 1 if intermediate stuff is too long
  - "Jane" → the next pronoun referring to her will be "she"
- Must know to "remember" for extended periods of time and "recall" when necessary
  - Can be performed with a multi-tap recursion, but how many taps?
  - Need an alternate way to "remember" stuff

And now we enter the domain of..
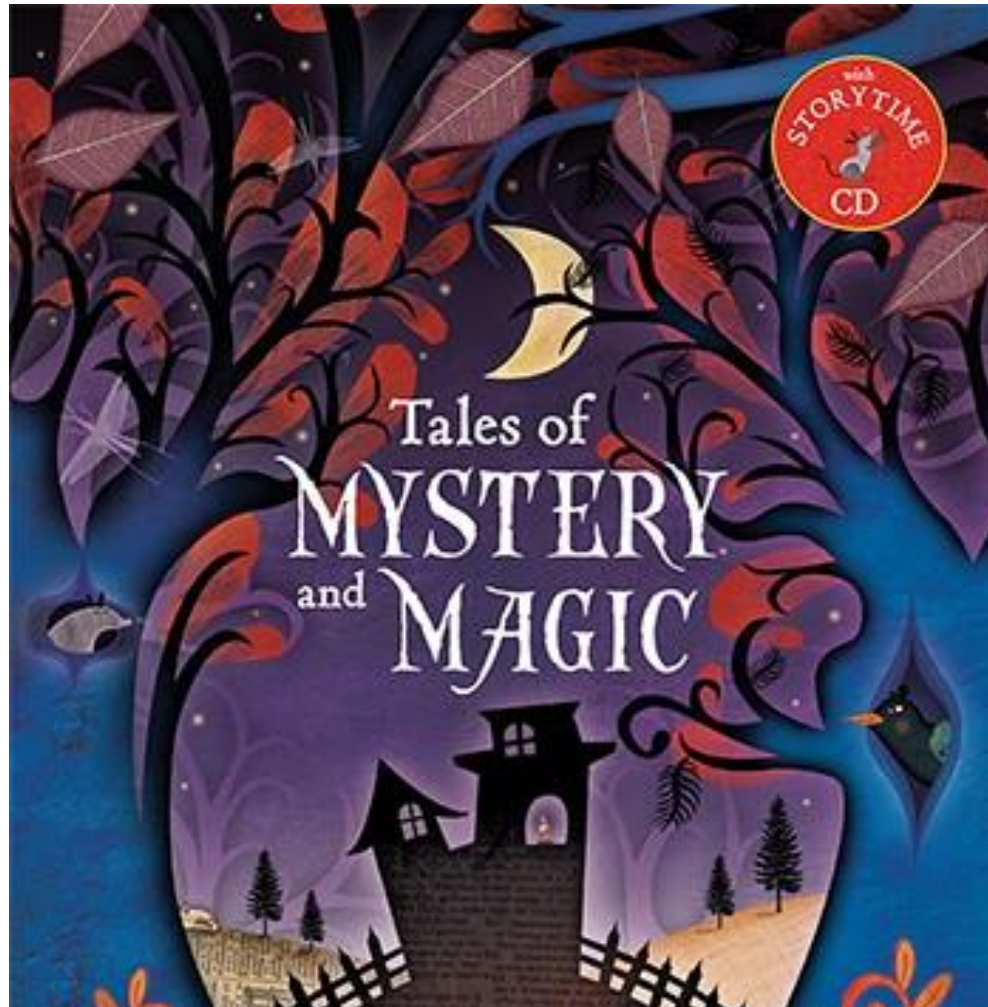
Exploding/Vanishing gradients

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \ldots \nabla f_{k+1} W_k$$

- Can we replace this with something that doesn't fade or blow up?

- $\nabla_{f_k} Div = \nabla D C \sigma_N C \sigma_{N-1} C \ldots \sigma_k$

- Can we have a network that just "remembers" arbitrarily long, to be recalled on demand?

Enter – the constant error carousel
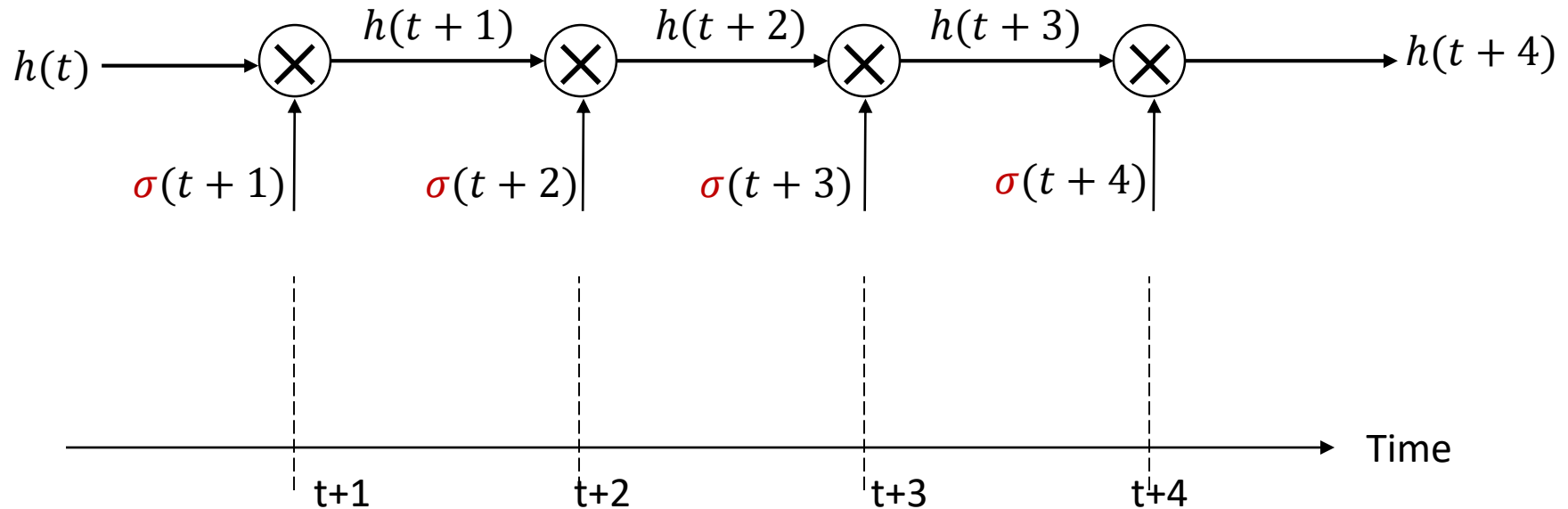
- History is carried through uncompressed
  - No weights, no nonlinearities
  - Only scaling is through the σ "gating" term that captures other triggers
  - E.g. "Have I seen Pattern2"?

Enter – the constant error carousel
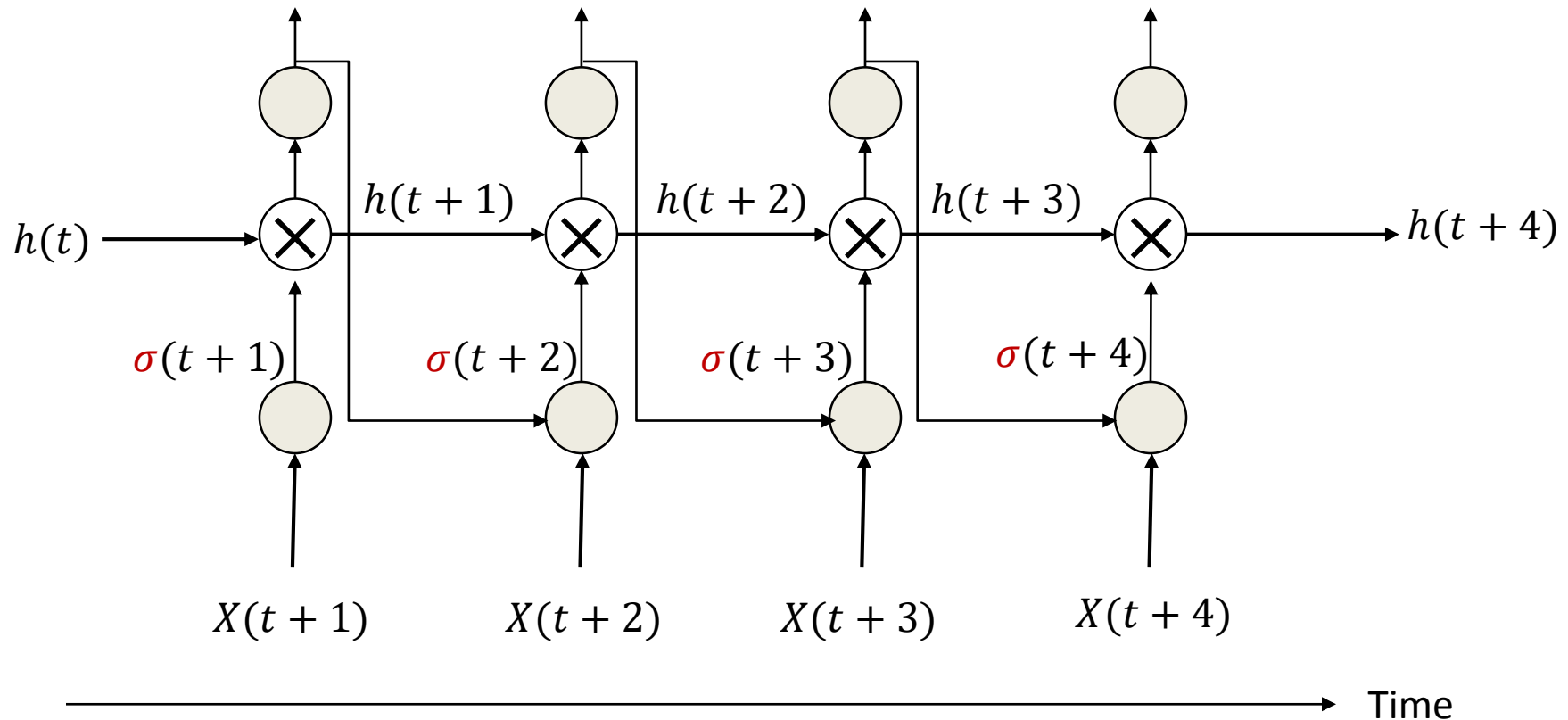
- Actual non-linear work is done by other portions of the network

Enter – the constant error carousel
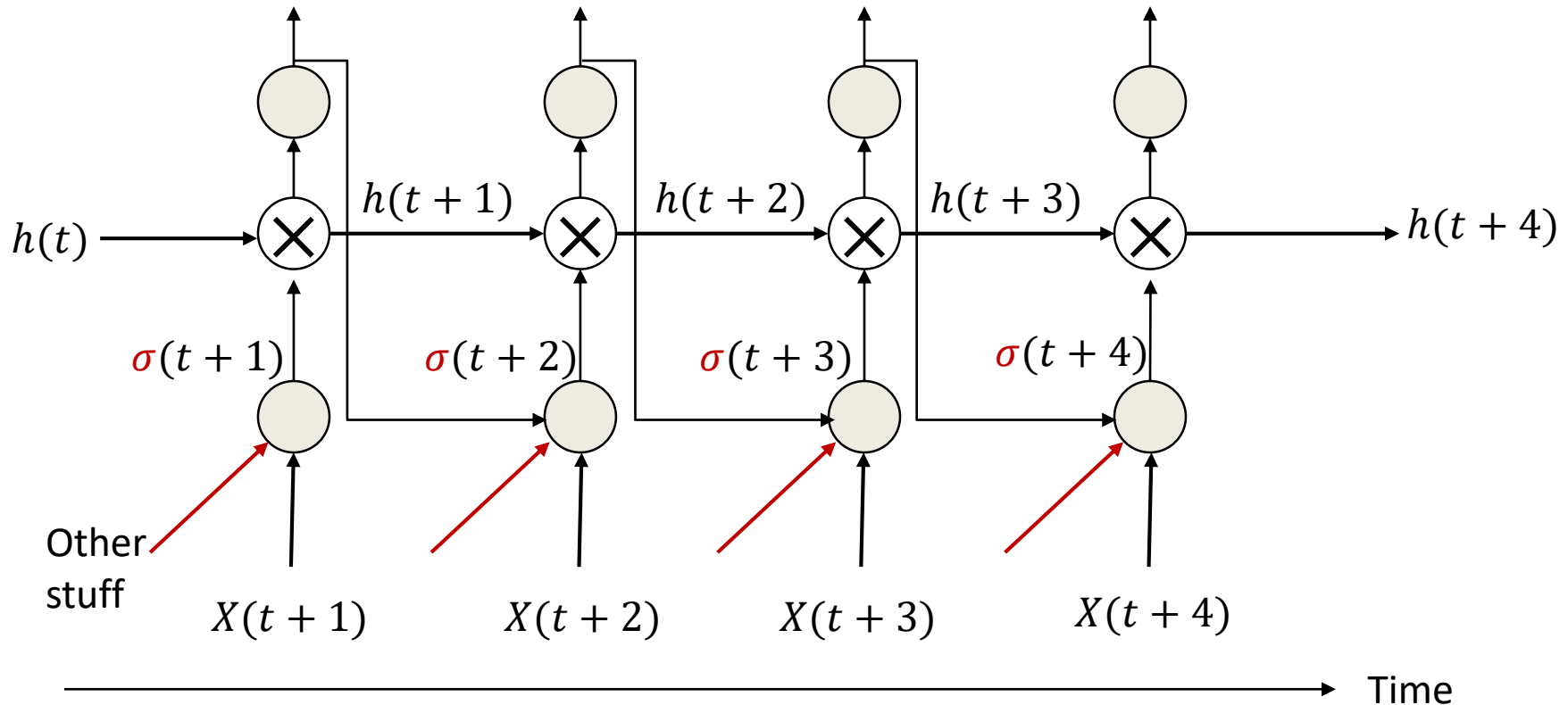
- Actual non-linear work is done by other portions of the network

Enter – the constant error carousel
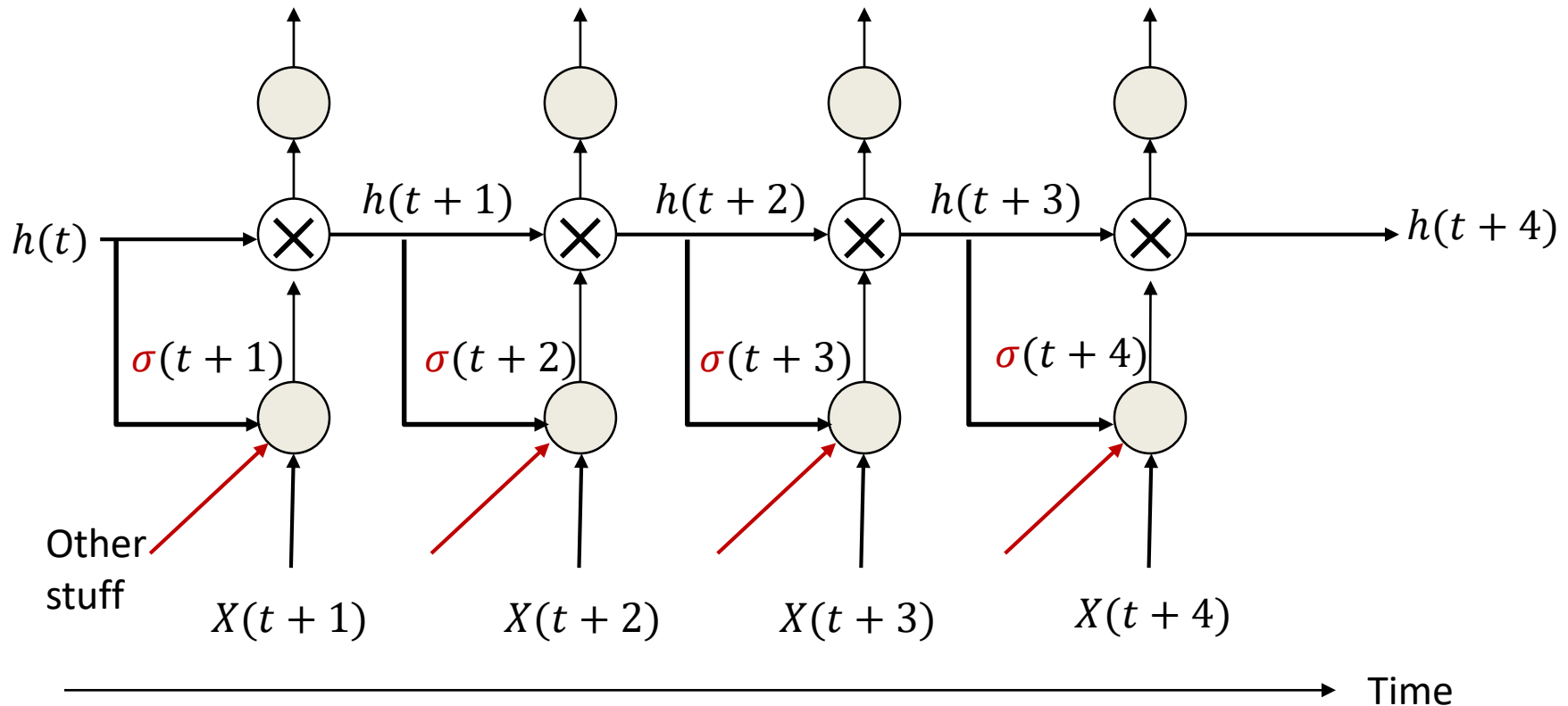
- Actual non-linear work is done by other portions of the network

Enter – the constant error carousel
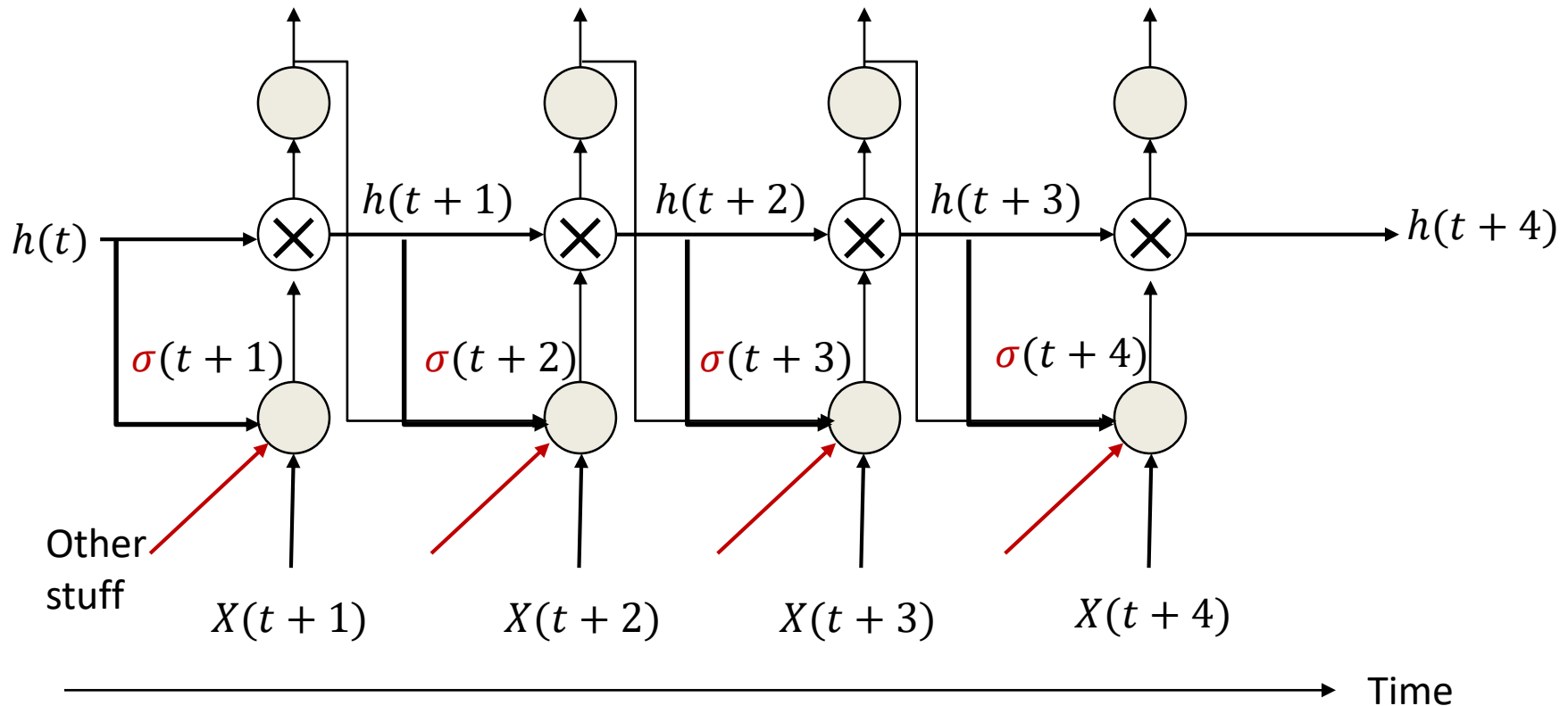
- Actual non-linear work is done by other portions of the network

Enter the *LSTM*

- *Long Short-Term Memory*
- Explicitly latch information to prevent decay / blowup

- Following notes borrow liberally from
- http://colah.github.io/posts/2015-08-Understanding-LSTMs/

Standard RNN
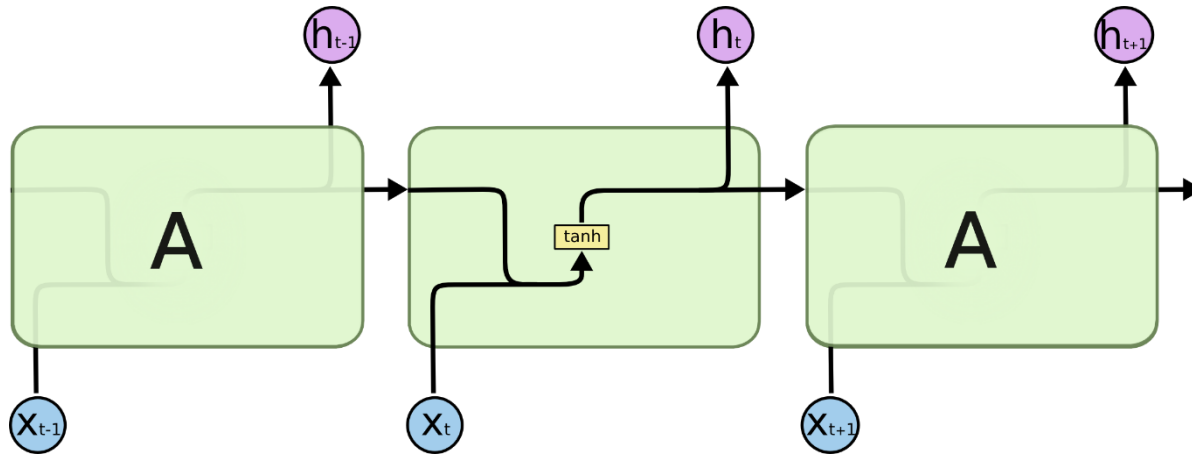
- Recurrent neurons receive past recurrent outputs and current input as inputs

- Processed through a tanh() activation function

  – As mentioned earlier, tanh() is the generally used activation for the hidden layer

- Current recurrent output passed to next higher layer and next time instant

Long Short-Term Memory
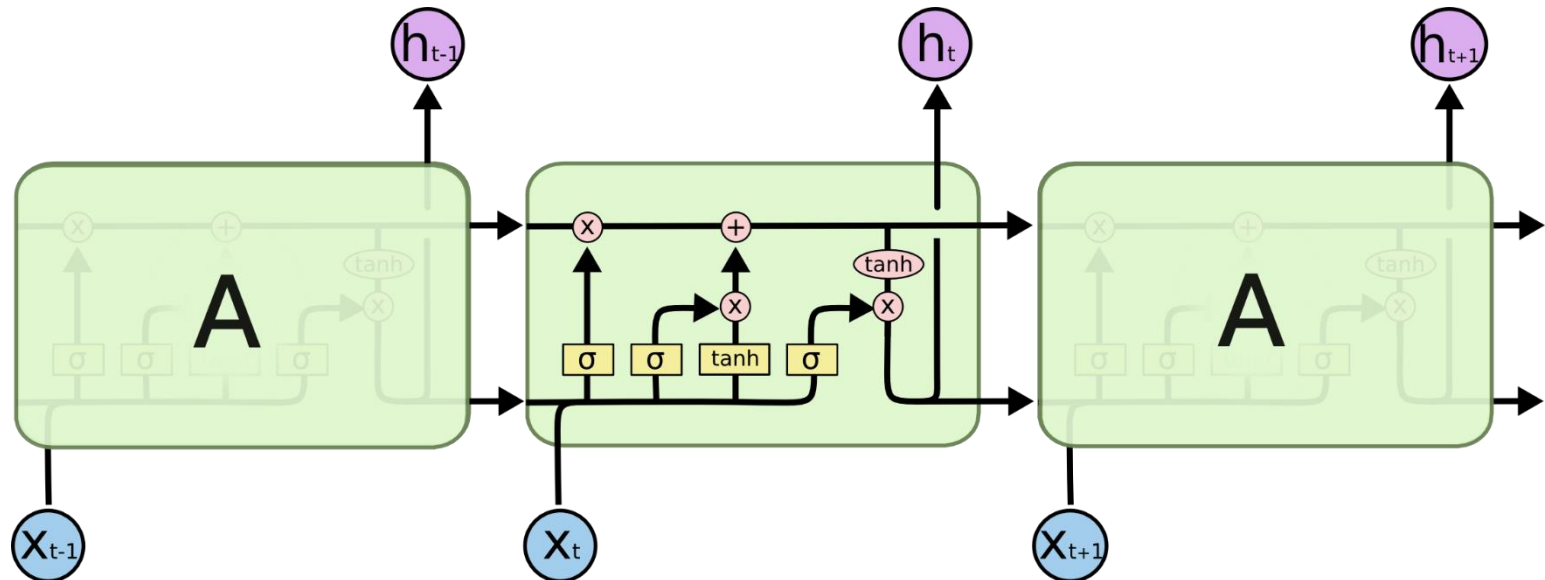
- The $\sigma()$ are *multiplicative gates* that decide if something is important or not

- Remember, every line actually represents a *vector*

LSTM: Constant Error Carousel
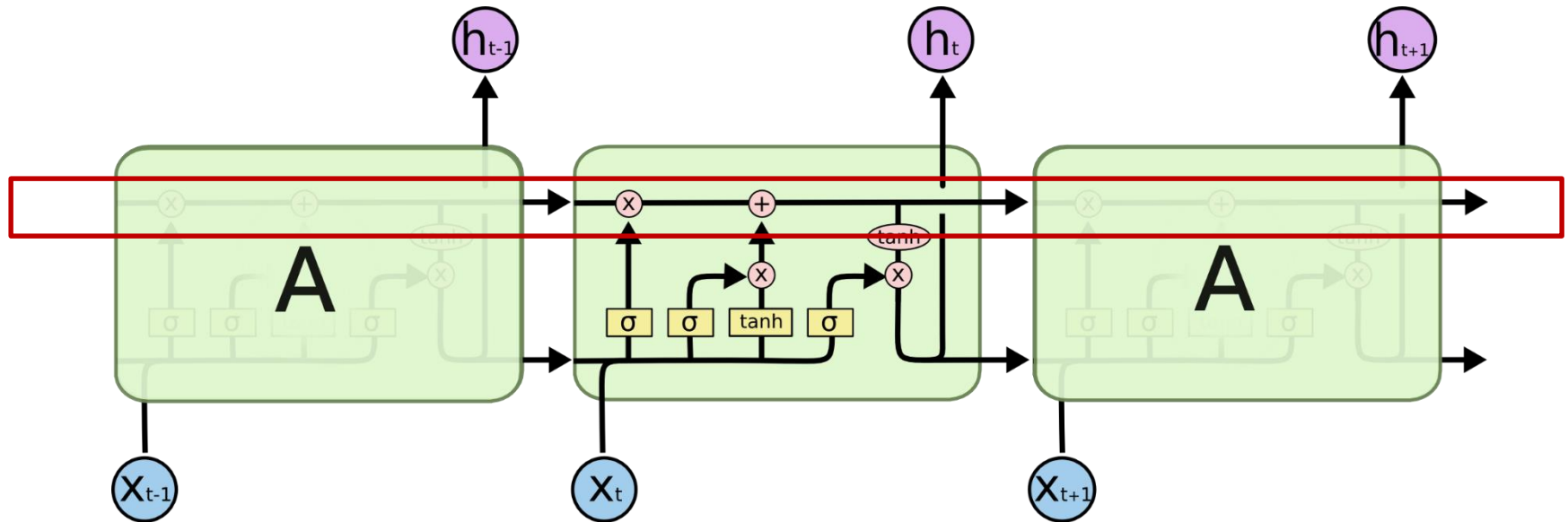
- Key component: a *remembered cell state*

LSTM: CEC
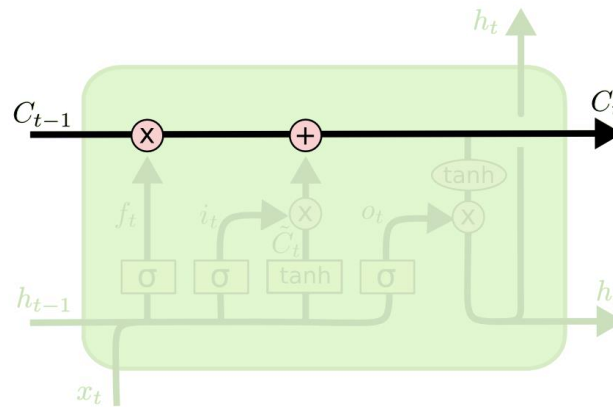
- C_t is the linear history carried by the *constant-error carousel*
- Carries information through, only affected by a gate
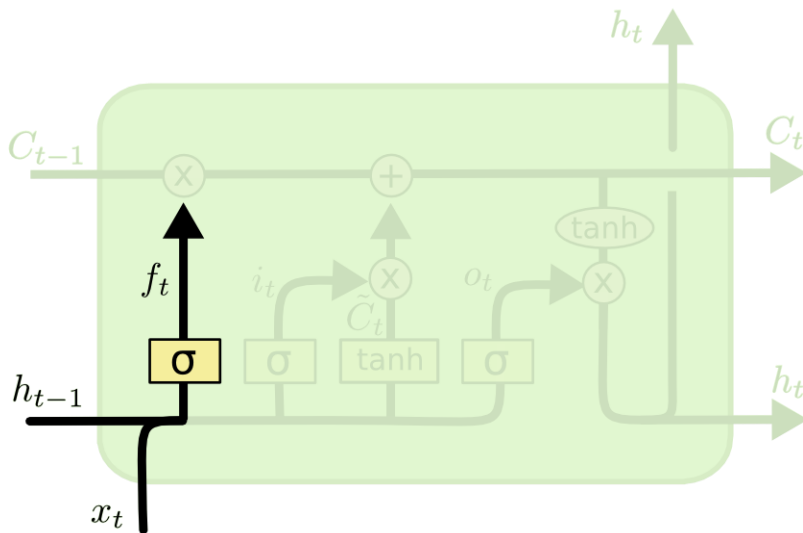 - And *addition of history,* which too is gated..

LSTM: Gates
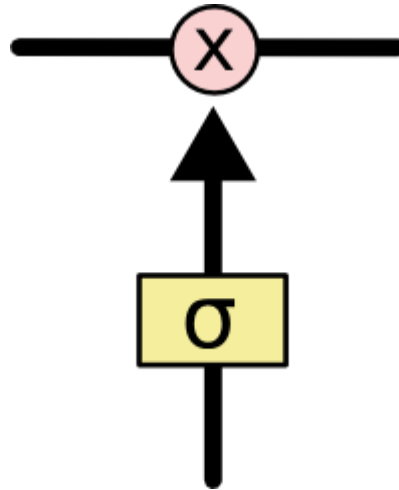
- Gates are simple sigmoidal units with outputs in the range (0,1)
- Controls how much of the information is to be let through

LSTM: Forget gate

$$f_t = \sigma \left(W_f \cdot [h_{t-1}, x_t] \; + \; b_f \right)$$

- The first gate determines whether to carry over the history or to forget it
  - More precisely, how much of the history to carry over
  - Also called the "forget" gate
  - Note, we're actually distinguishing between the cell memory C and the state h that is coming over time! They're related though
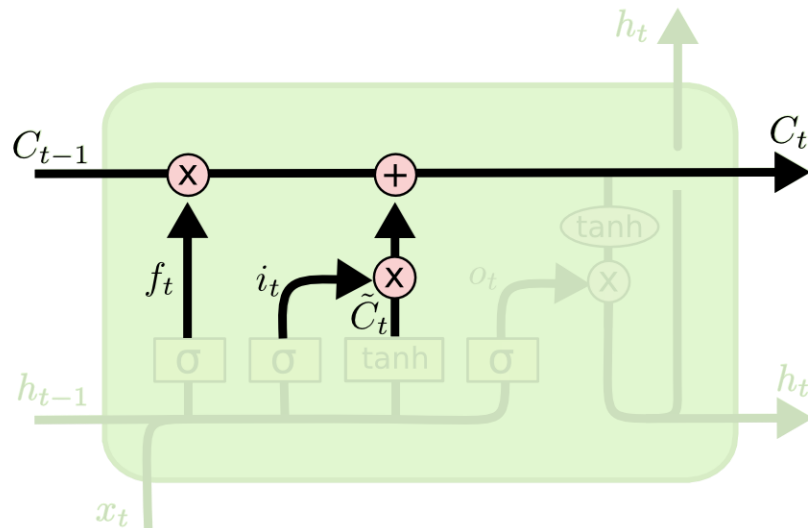
LSTM: Input gate
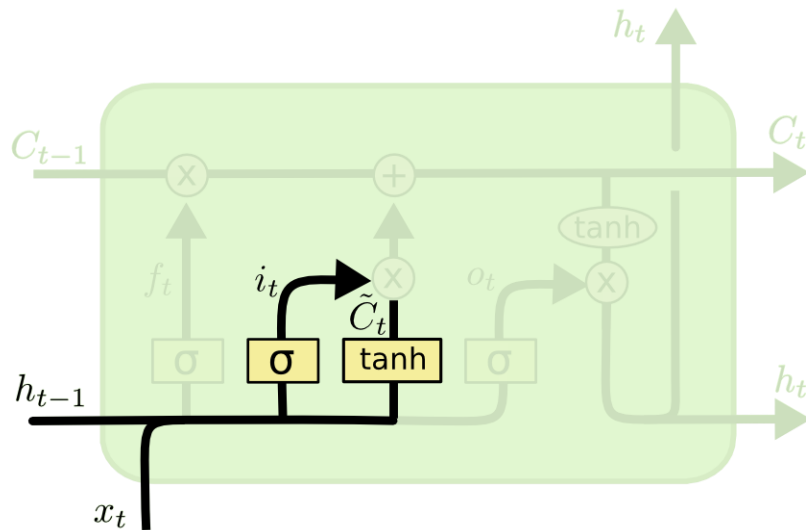
$$i_t = \sigma\left(W_i \cdot [h_{t-1}, x_t] + b_i\right)$$
$$\tilde{C}_t = \tanh(W_C \cdot [h_{t-1}, x_t] + b_C)$$

- The second gate has two parts
  - A perceptron layer that determines if there's something interesting in the input
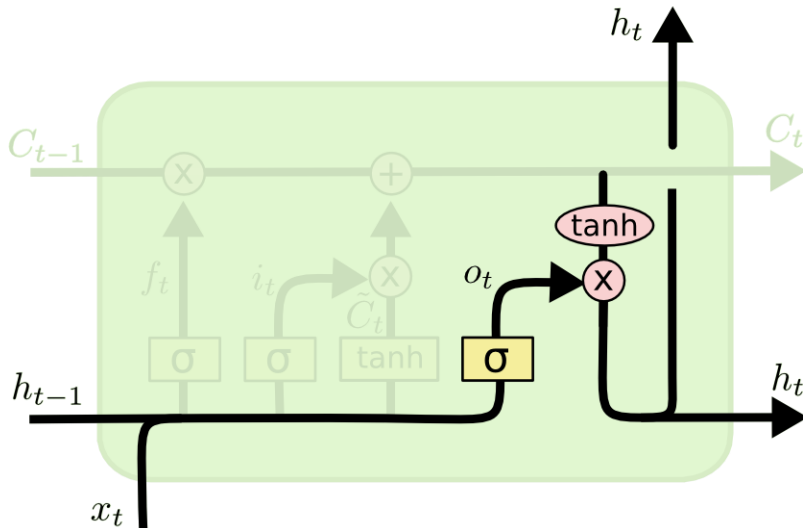 - A gate that decides if its worth remembering

LSTM: Memory cell update

$$C_t = f_t * C_{t-1} + i_t * \tilde{C}_t$$

- The second gate has two parts
  - A perceptron layer that determines if there's something interesting in the input
  - A gate that decides if its worth remembering
  - **If so its added to the current memory cell**
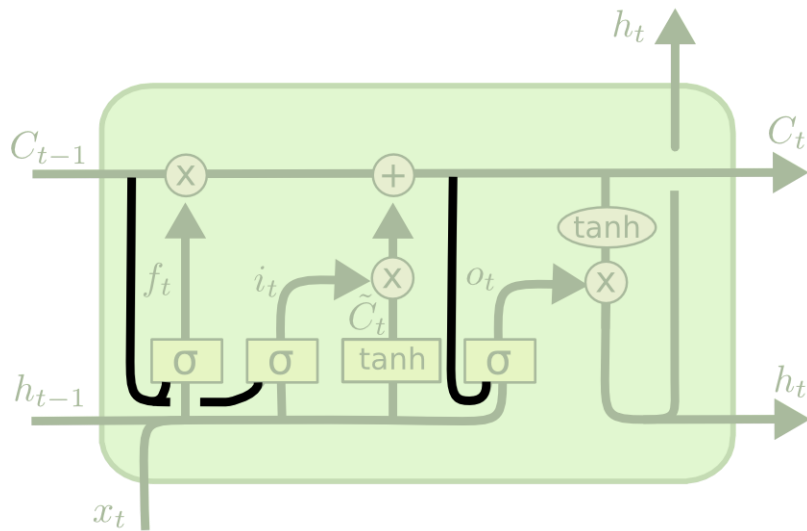
LSTM: Output and Output gate

$$o_t = \sigma\left(W_o\left[h_{t-1}, x_t\right] + b_o\right)$$

$$h_t = o_t * \tanh\left(C_t\right)$$

- The *output* of the cell
  - Simply compress it with tanh to make it lie between 1 and -1
    - Note that this compression no longer affects our ability to *carry* memory forward
  - While we're at it, lets toss in an output gate
    - To decide if the memory contents are worth reporting at *this* time
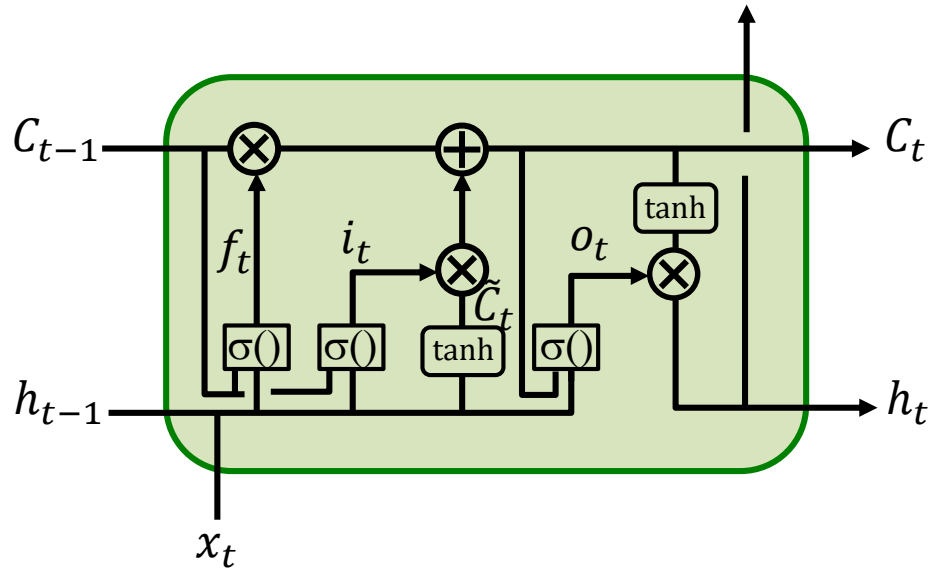
LSTM: The "Peephole" Connection

$$f_t = \sigma\left(W_f \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] + b_f\right)$$

$$i_t = \sigma\left(W_i \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] + b_i\right)$$

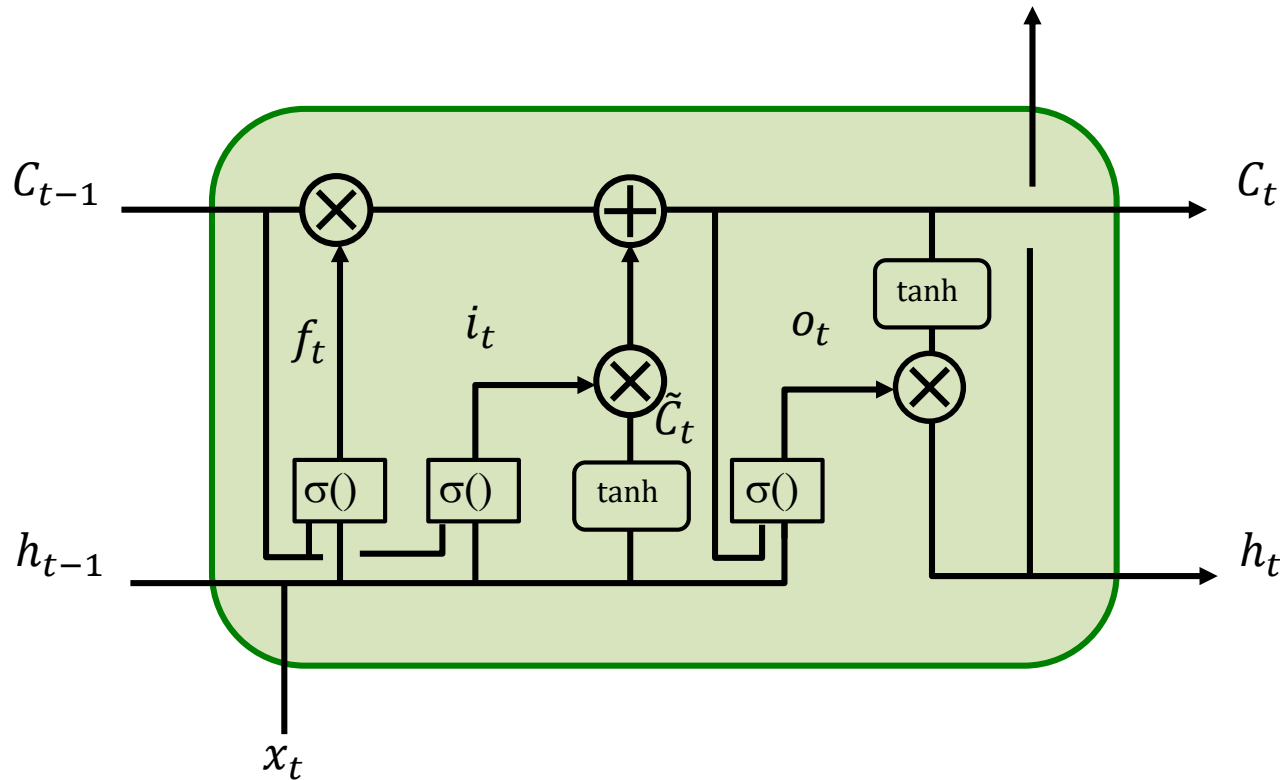$$o_t = \sigma\left(W_o \cdot [\boldsymbol{C_t}, h_{t-1}, x_t] + b_o\right)$$

- Why not just let the cell directly influence the gates while at it
  - Party!!

The complete LSTM unit

- With input, output, and forget gates and the peephole connection..

Backpropagation rules: Forward

• Forward rules:

Gates
$$f_t = \sigma\left(W_f \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \; + \; b_f\right)$$
$$i_t = \sigma\left(W_i \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \; + \; b_i\right)$$
$$o_t = \sigma\left(W_o \cdot [\boldsymbol{C_t}, h_{t-1}, x_t] \; + \; b_o\right)$$

Variables
$$\tilde{C}_t = \tanh(W_C \cdot [h_{t-1}, x_t] \; + \; b_C)$$
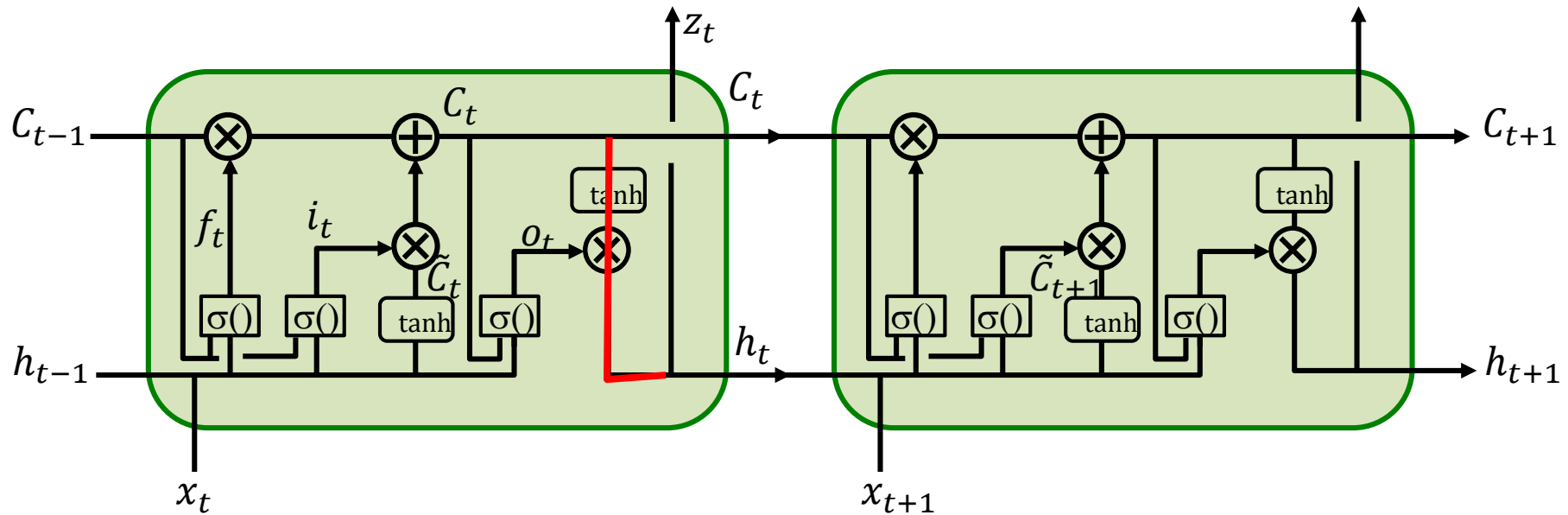$$C_t = f_t * C_{t-1} + i_t * \tilde{C}_t$$
$$h_t = o_t * \tanh(C_t)$$

Backpropagation rules: Backward
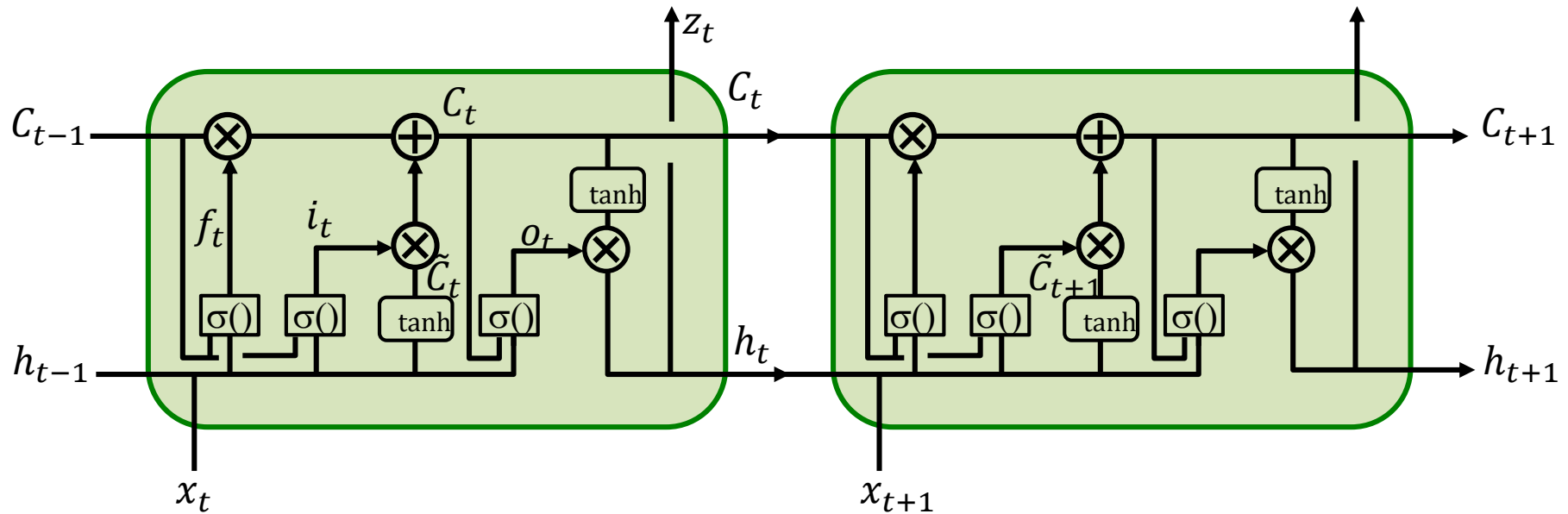
$$\nabla_{C_t} Div =$$

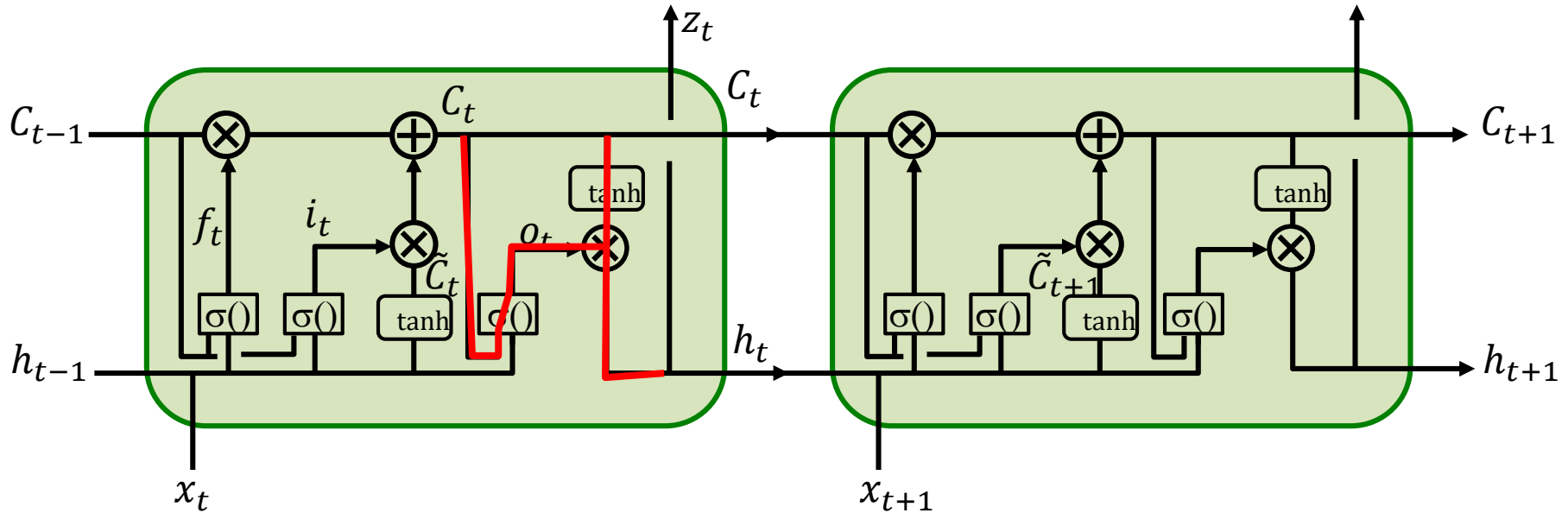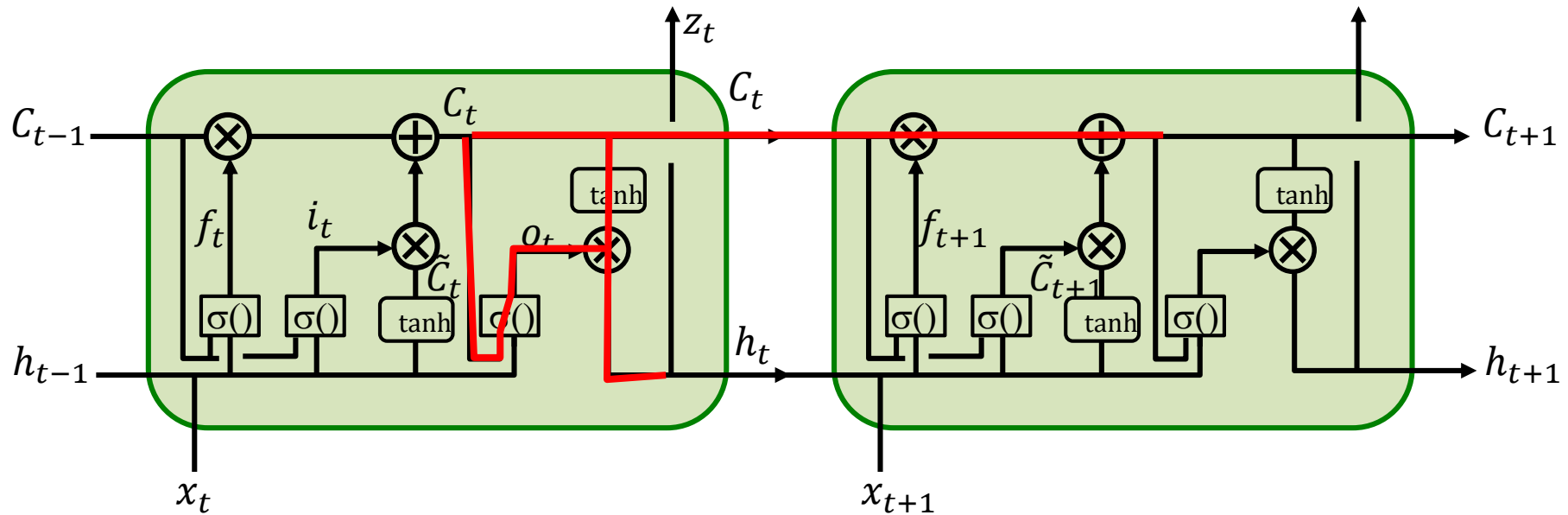# Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ o_t \circ tanh'(.)W_{Ch}$$

Backpropagation rules: Backward

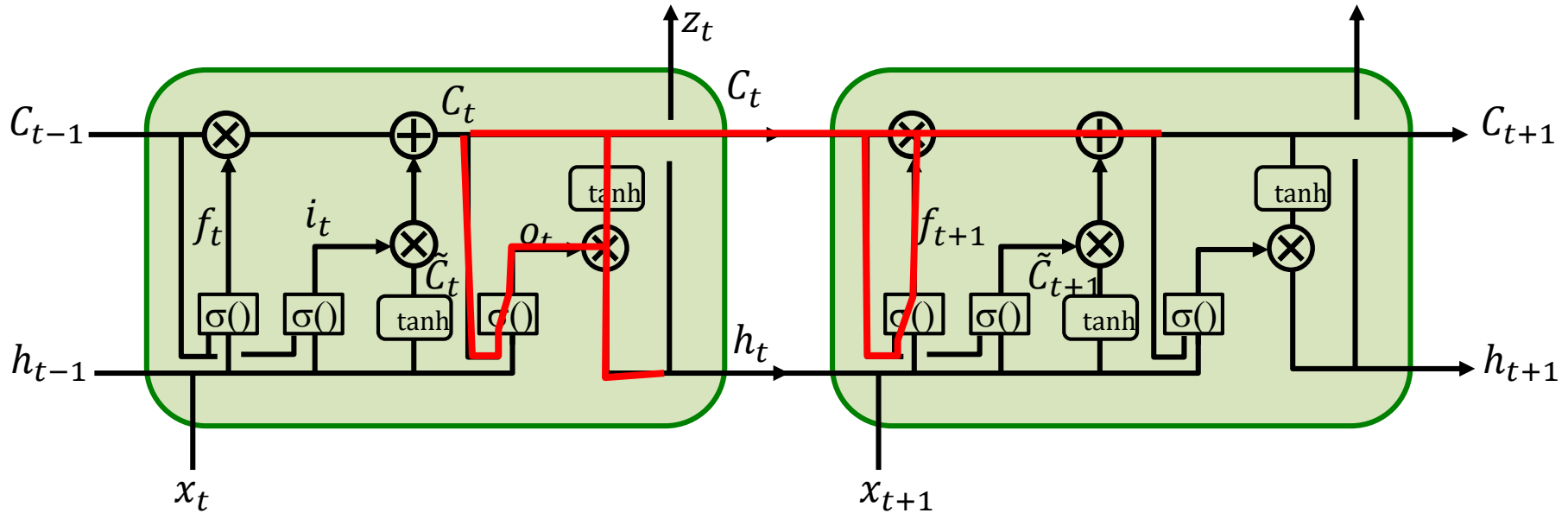

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co})$$

Backpropagation rules: Backward

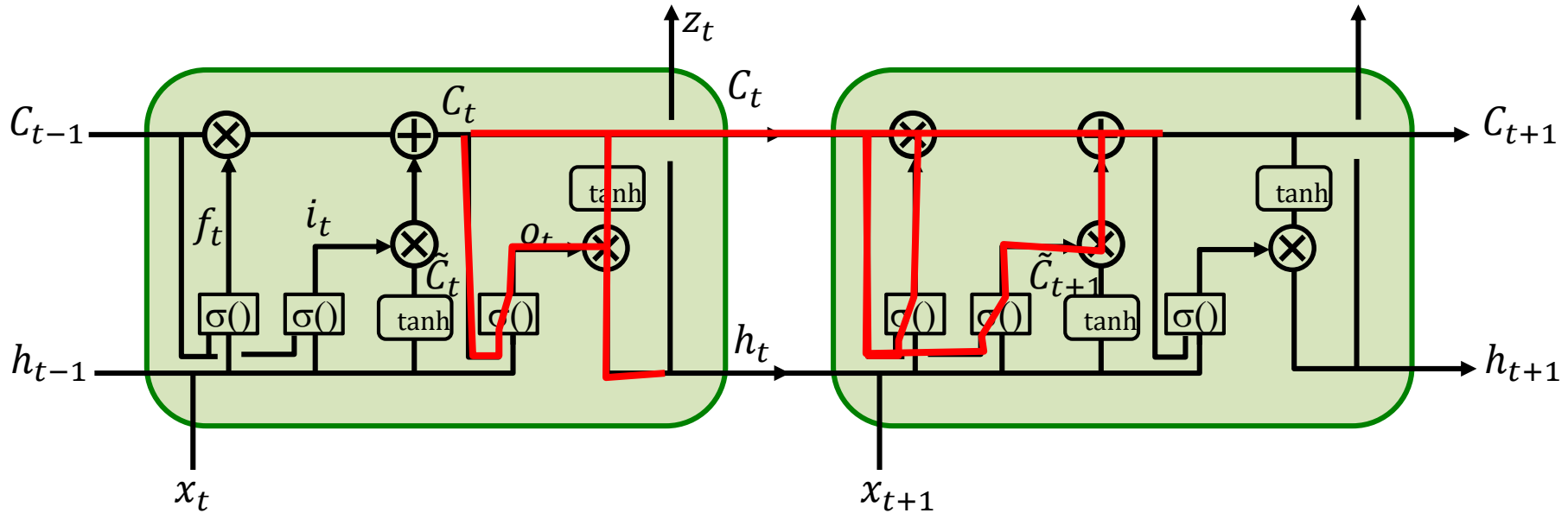$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{h_t} C_{t+1} \circ f_{t+1} +$$

Backpropagation rules: Backward

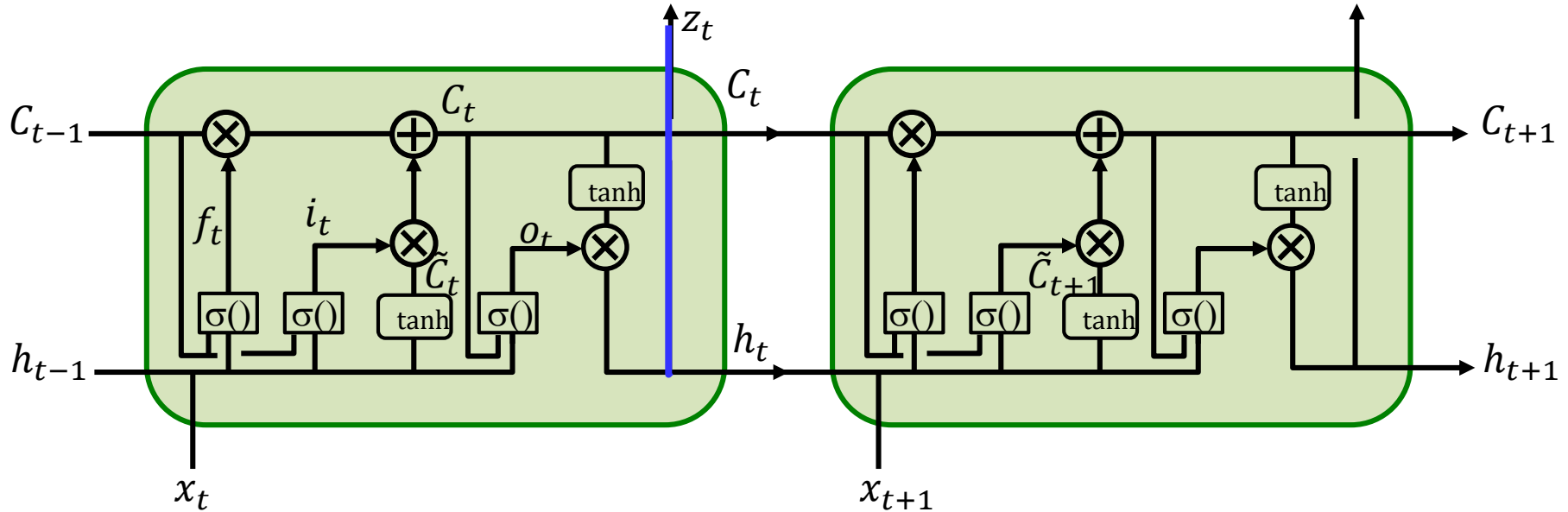$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{h_t} C_{t+1} \circ (f_{t+1} + C_t \circ \sigma'(.)W_{Cf})$$

Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
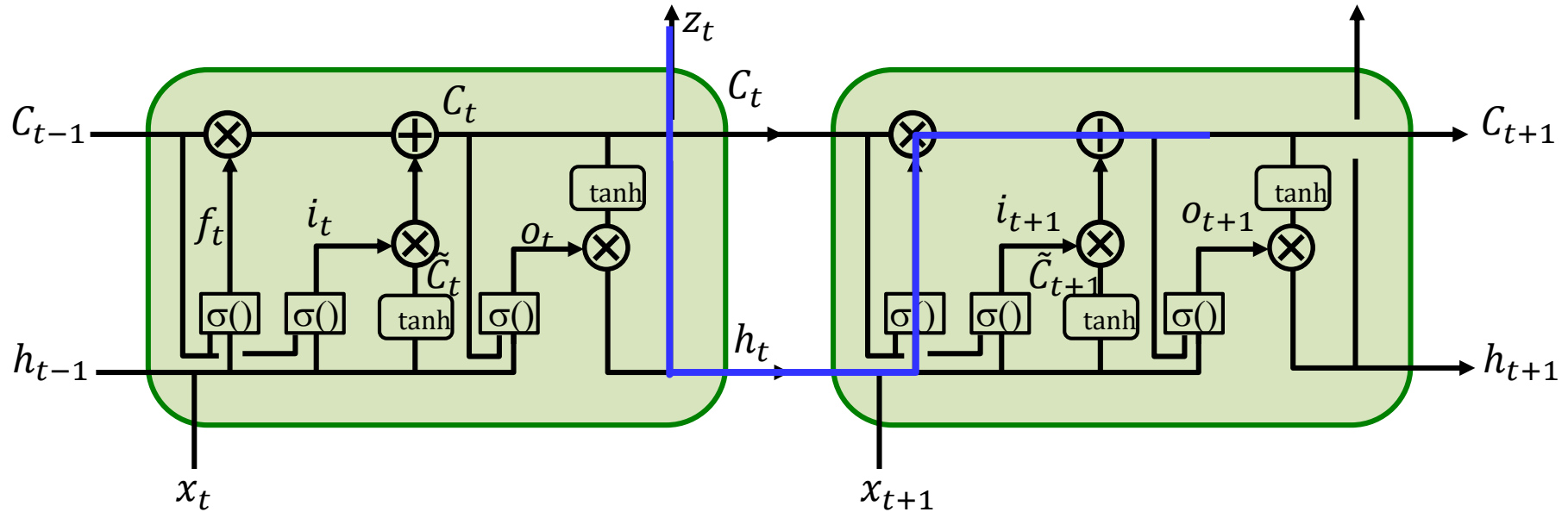$$\nabla_{h_t} C_{t+1} \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{h_t} C_{t+1} \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t$$

Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{h_t} C_{t+1} \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci} \right)$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{h_t} C_{t+1} \circ C_t \circ \sigma'(.)W_{hf}$$
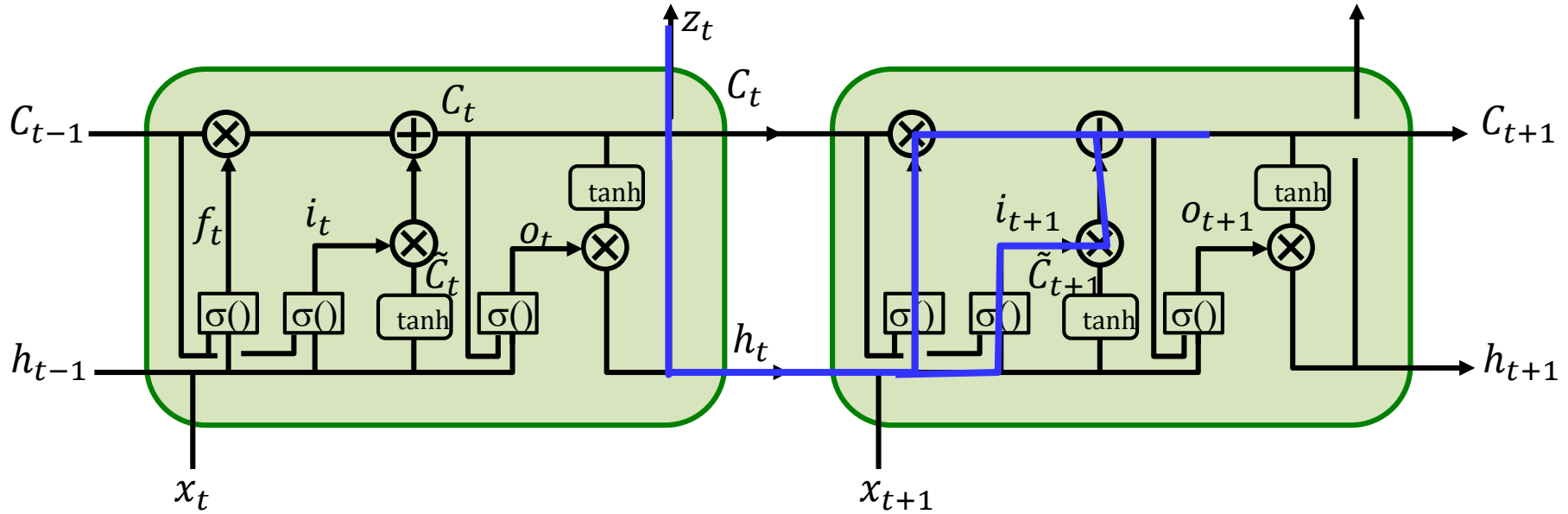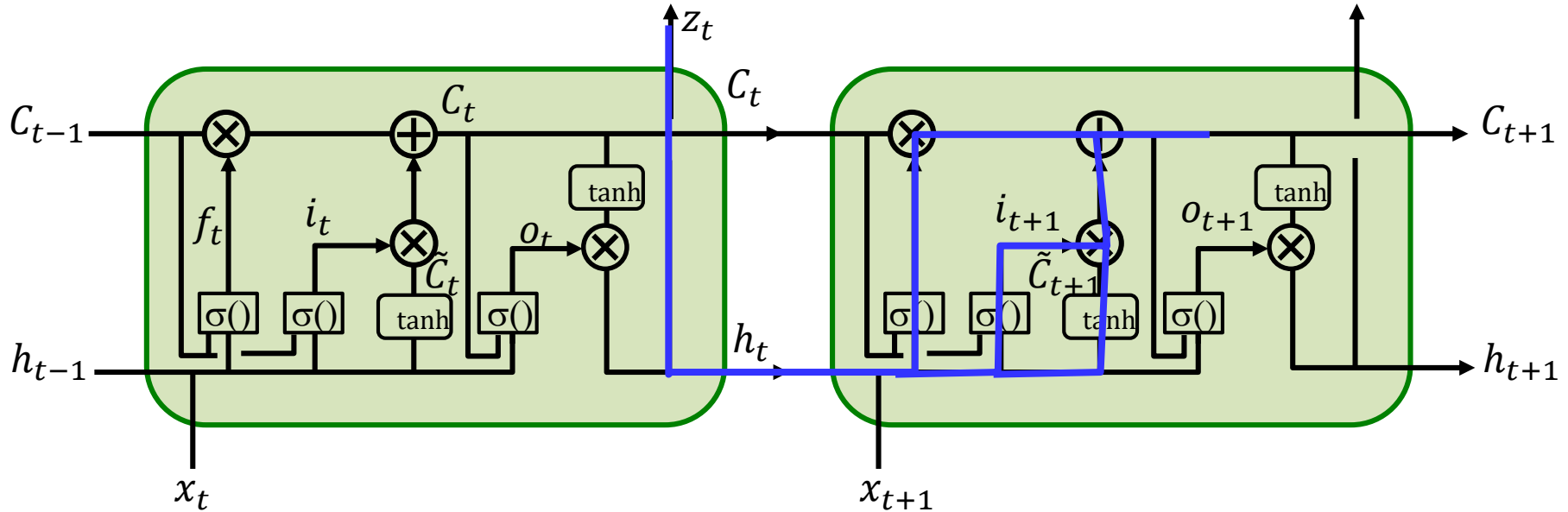
Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
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$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{h_t} C_{t+1} \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi} \right)$$

Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
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$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{h_t} C_{t+1} \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi} \right) +$$
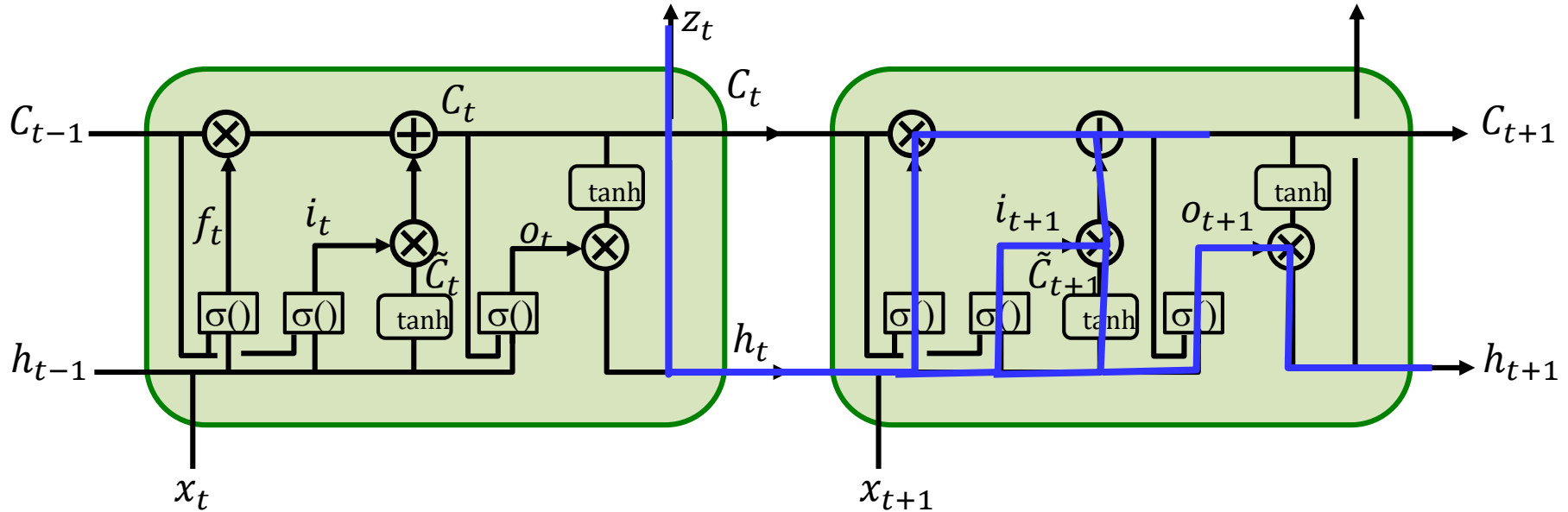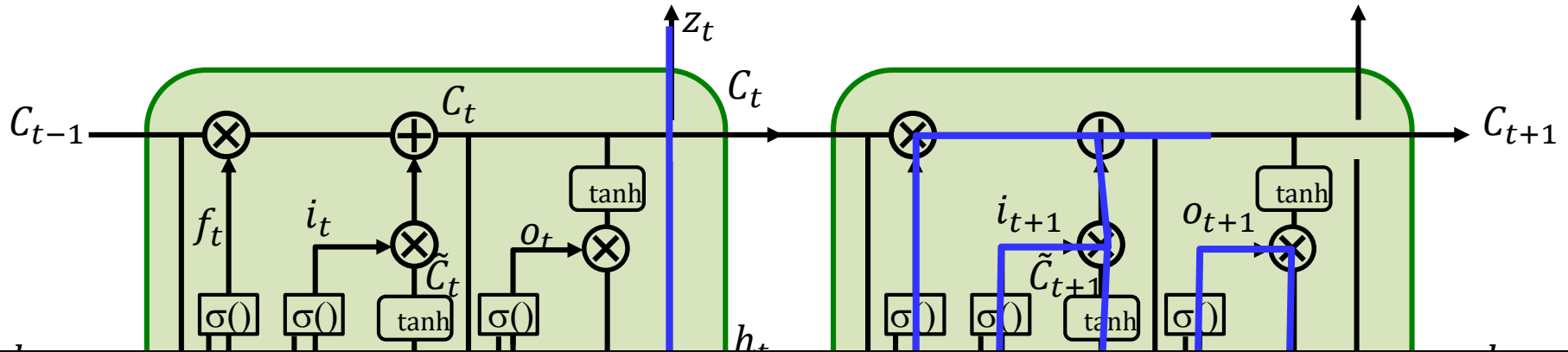$$\nabla_{C_{t+1}} Div \circ i_{t+1} \circ tanh'(.)W_{hi}$$

Backpropagation rules: Backward

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{h_t} C_{t+1} \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{h_t} C_{t+1} \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi}\right) +$$
$$\nabla_{C_{t+1}} Div \circ o_{t+1} \circ tanh'(.)W_{hi} + \nabla_{h_{t+1}} Div \circ tanh(.) \circ \sigma'(.)W_{ho}$$
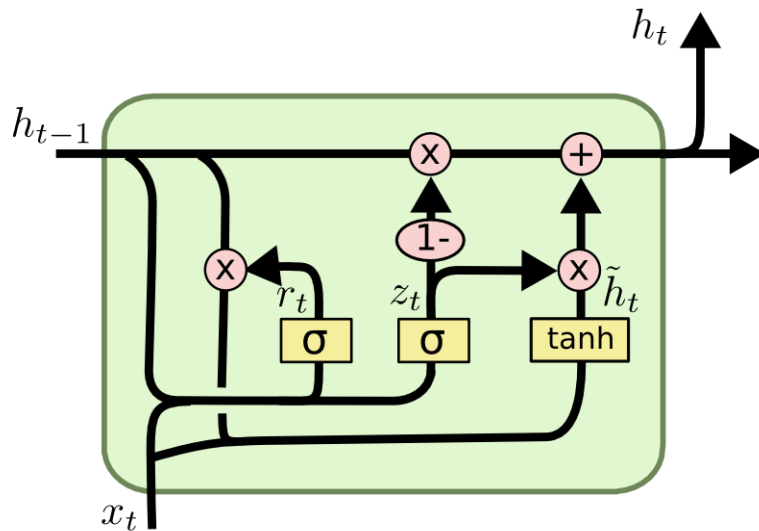
Backpropagation rules: Backward

Not explicitly deriving the derivatives w.r.t weights; Left as an exercise

$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) + \nabla_{h_t} C_{t+1} \circ (f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci})$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{h_t} C_{t+1} \circ (C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi}) + \nabla_{C_{t+1}} Div \circ o_{t+1} \circ tanh'(.)W_{hi} + \nabla_{h_{t+1}} Div \circ tanh(.) \circ \sigma'(.)W_{ho}$$

Gated Recurrent Units: Lets simplify the LSTM
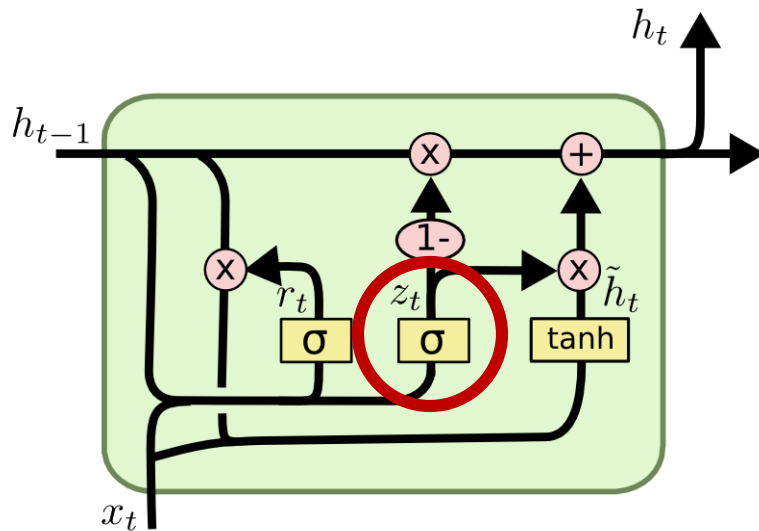

$$z_t = \sigma \left(W_z \cdot [h_{t-1}, x_t] \right)$$

$$r_t = \sigma \left(W_r \cdot [h_{t-1}, x_t] \right)$$

$$\tilde{h}_t = \tanh \left(W \cdot [r_t * h_{t-1}, x_t] \right)$$

$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Simplified LSTM which addresses some of your concerns of *why*

Gated Recurrent Units: Lets simplify the LSTM

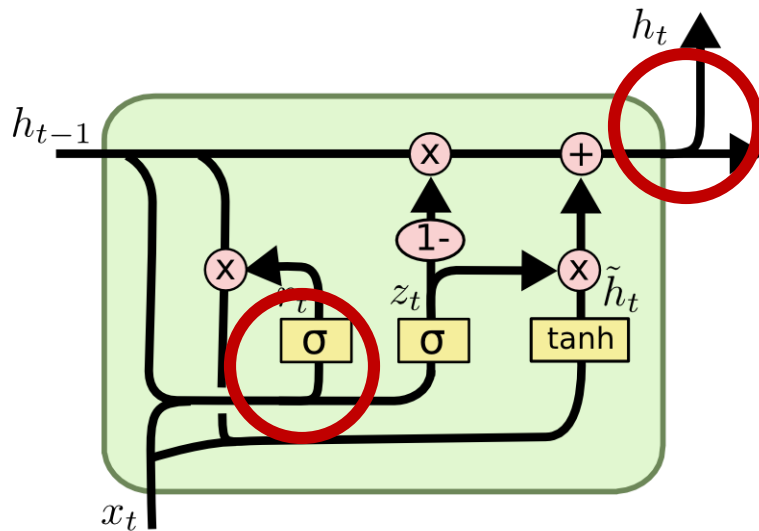$$z_t = \sigma\left(W_z \cdot [h_{t-1}, x_t]\right)$$

$$r_t = \sigma\left(W_r \cdot [h_{t-1}, x_t]\right)$$

$$\tilde{h}_t = \tanh\left(W \cdot [r_t * h_{t-1}, x_t]\right)$$

$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Combine forget and input gates
  - In new input is to be remembered, then this means old memory is to be forgotten
    - Why compute twice?

Gated Recurrent Units: Lets simplify the LSTM

$$z_t = \sigma \left(W_z \cdot [h_{t-1}, x_t] \right)$$

$$r_t = \sigma \left(W_r \cdot [h_{t-1}, x_t] \right)$$

$$\tilde{h}_t = \tanh \left(W \cdot [r_t * h_{t-1}, x_t] \right)$$

$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Don't bother to separately maintain compressed and regular memories
  - Pointless computation!
- But compress it before using it to decide on the usefulness of the current input!

LSTM Equations

- $i = \sigma\left(x_t U^i + s_{t-1} W^i\right)$
- $f = \sigma\left(x_t U^f + s_{t-1} W^f\right)$
- $o = \sigma(x_t U^o + s_{t-1} W^o)$
- $g = \tanh(x_t U^g + s_{t-1} W^g)$
- $c_t = c_{t-1} \circ f + g \circ i$
- $s_t = \tanh(c_t) \circ o$
- $y = softmax(V s_t)$

- i: **input gate, how much of the new information will be let through the memory cell.**

- f: **forget gate, responsible for information should be thrown away from memory cell.**

- o: **output gate, how much of the information will be passed to expose to the next time step.**

- g: **self-recurrent which is equal to standard RNN**

- c_t: **internal memory of the memory cell**

- s_t: **hidden state**

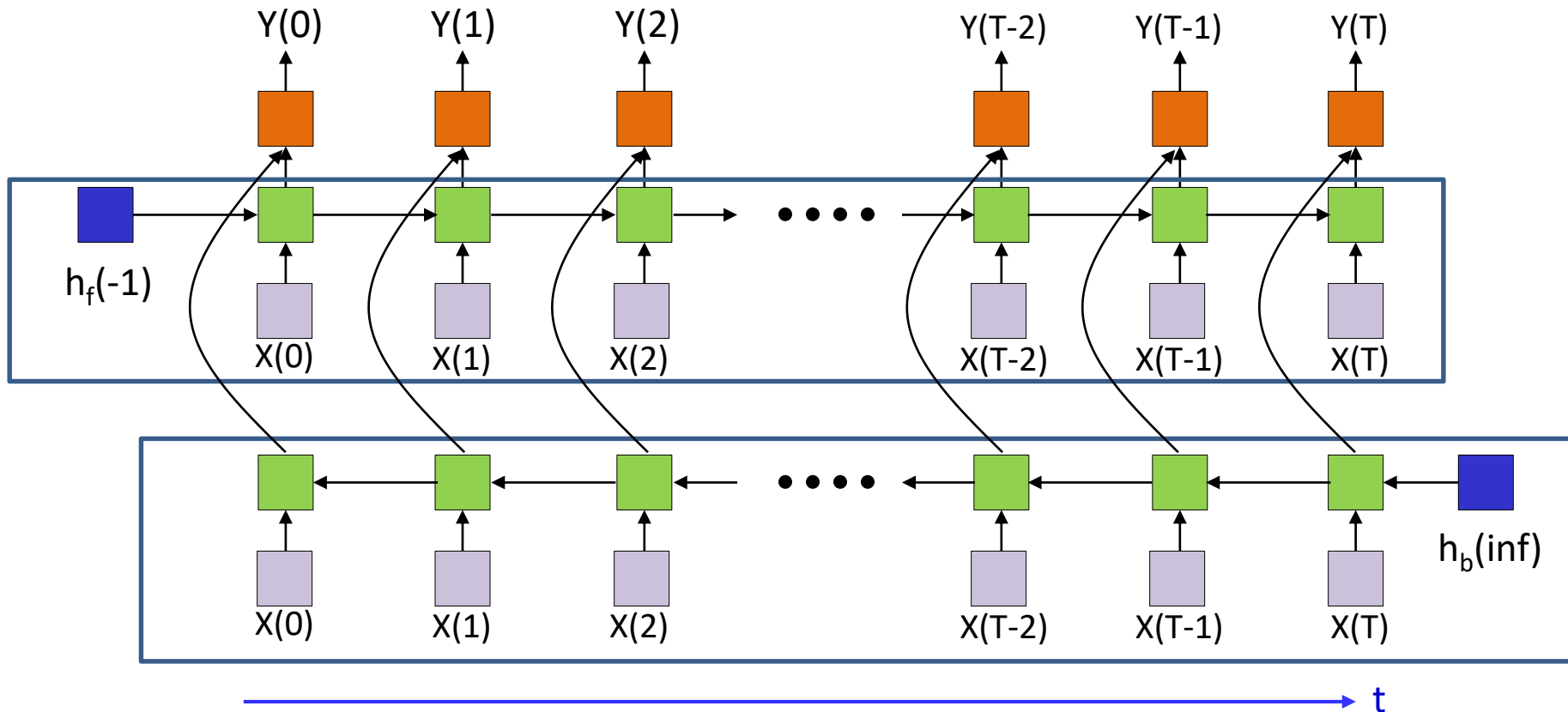- y: **final output**
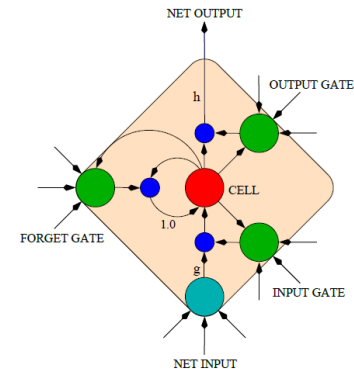
LSTM Memory Cell

LSTM architectures example
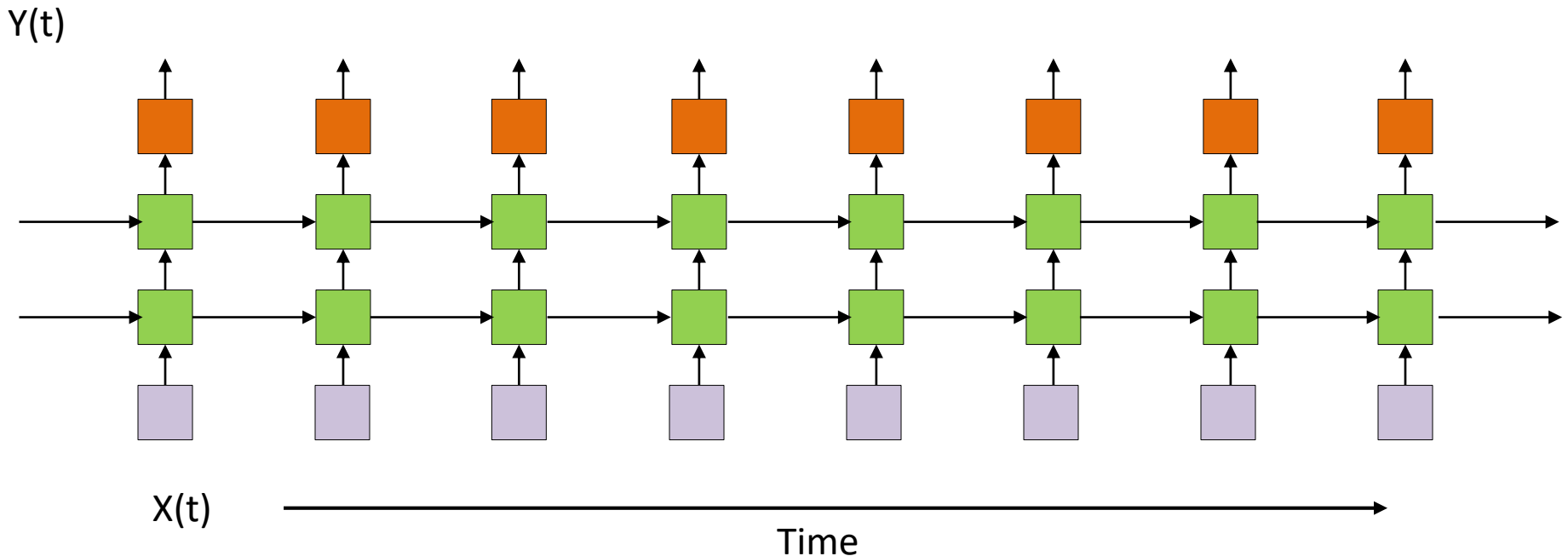
- Each green box is now an entire LSTM or GRU unit
- Also keep in mind each box is an *array* of units

Bidirectional LSTM

- Like the BRNN, but now the hidden nodes are LSTM units.
- Can have multiple layers of LSTM units in either direction
  - Its also possible to have MLP feed-forward layers between the hidden layers..
- The output nodes (orange boxes) may be complete MLPs

Significant issue left out

- The Divergence