Berner Fachhochschule - Technik und Informatik

Object-Oriented Programming 2

Recursion

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Outline

Recursive Algorithms

Examples

MiniMax Algorithm for Two-Player Games

ExpectedMax Algorithm for One-Player Games with Chance

Outline

Divide and Conquer

- Divide and conquer is a powerful algorithm design paradigm
  - → Break down a problem recursively into two or more sub-problems of the same (or related) type
  - → Continue until the problem becomes simple enough to be solved directly
  - → Merge the solutions of the sub-problems to get a solution for the original problem
- Benefits of divide and conquer
  - → Powerful tool for solving conceptually difficult problems
  - → Often provides a natural way to design efficient algorithms
  - → Sub-problems can be executed in parallel
- Decrease and conquer is a variation of divide and conquer, where the problem is simplified into a single sub-problem

Recursive Algorithm

Recursive Algorithm
: Algorithm that uses itself as part of the solution

Recursive Call
: A method call in which the method being called is the same as the one making the call

Direct Recursion
: Recursion in which a method directly calls itself

Indirect Recursion
: Recursion in which a chain of two or more method calls returns to the method that originated the chain, e.g. A calls B, B calls C, and C calls A

Example: Factorial (Recursion)

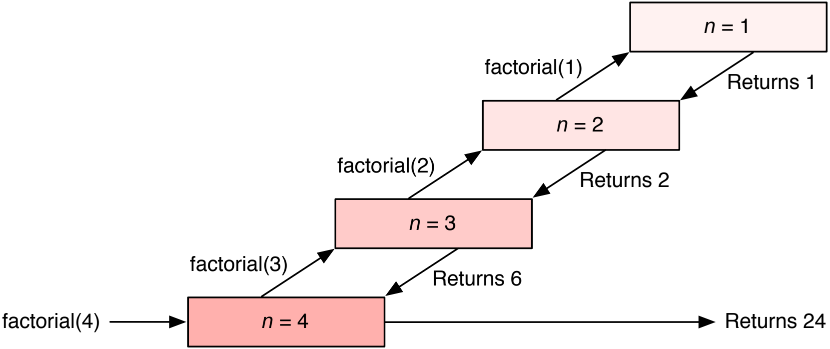
Compute the factorial of integer n:

$$n! = \underbrace{1 * 2 * \cdots * n-1}_{(n-1)!} * n = \begin{cases} 1 & n = 1 \\ n * (n-1)! & n > 1 \end{cases}$$

```
algorithm factorial(n)
if n = 1 then
return 1 // base case of the recursion
else
return n * factorial(n − 1) // recursive call
```

Example: Factorial (Recursion)

The execution of a recursive algorithm uses a call stack to keep track of previous method call



Remarks

- Each recursion must have a base case to stop
- Each recursive call must reduce the problem size, i.e. leading inescapably to the base case
- A recursion is called tail-recursive, if the recursive call is always the last executed instruction
- Each recursive algorithm can be transformed into an iterative algorithm (and vice versa)
  - → trivial for tail-recursions
  - → not so easy in general (requires a stack to replace the call stack)
- In a recursive algorithm, the stack is hidden from the user

Iterative Algorithm

- In general, an iterative algorithm is a non-recursive one
- A typical iterative algorithm contains one or several, possibly nested loops (repetitions of instructions):
  - → `For ... Do`
  - → `While ... Do`
  - → `Repeat ... Until`
- Iteration also stands for the style of programming used in imperative programming languages (e.g. Assembler, Basic, Pascal, C++, Java, . . .)
- This contrasts with recursion, which is more declarative
  - → Functional programming languages (e.g. Scheme, Haskell, . . .)
  - → Logical programming languages (e.g. Prolog)

Example: Factorial (Iteration)

The tail-recursive factorial algorithm can be replaced by a simple iterative loop

```
algorithm factorial(n)
result ← 1
while n > 0 do
result ← result ∗ n
n ← n − 1
return result
```

Recursive vs. Iterative Algorithms

Recursive algorithms are sometimes less efficient than their iterative counterparts

- The overhead of managing the call stack is non-negligible
- The space complexity is often $O(n)$ or $O(\log n)$ instead of $O(1)$
- Risk to get `"stack_overflow"` errors
- Some (functional) programming languages such as Scheme, Lisp, or Haskell detect tail-recursions automatically

Nevertheless, recursion has been used to solve some of the biggest and hardest algorithmic problems in computer science !!!

Outline

Example 1: Fibonacci

Compute the Fibonacci number

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

Input Integer $n \geq 0$

Ouput $F(n)$

Examples

`fibonacci`$(10) \rightarrow 55$
`fibonacci`$(21) \rightarrow 10946$

Example 1: Fibonacci (cont.)

Pseudo-code solution:

```
algorithm fibonacci(n)
if n ≤ 1 then
  return n
else
  return fibonacci(n − 1) + fibonacci(n − 2)
```

Example 2: Palindrome

Detect whether a string is a palindrome

Input

String s

Ouput

true or *false*

Examples

`palindrome("amanaplanacanalpanama")` $\rightarrow$ *true*

`palindrome("recursioniscomplicated")` $\rightarrow$ *false*

Auxiliary Functions

`size`(s), `first`(s), `last`(s), `removeFirst`(s), `removeLast`(s)

Example 2: Palindrome (cont.)

Pseudo-code solution:

```
algorithm palindrome(s)
if size(s) ≤ 1 then
  return true
else if first(s) = last(s) then
  return palindrome(removeFirst(removeLast(s)))
else
  return false
```

Example 3: Integer Printing

Print integers relative to various bases

Input

Integer $n \geq 0$, base $b \geq 2$

Ouput

Screen output of $[d_k \cdots d_1 d_0]_b$ for $n = d_k b^k + \cdots + d_1 b + d_0$

Examples

$\texttt{printInteger}(61, 2) \to 111101$
$\texttt{printInteger}(61, 10) \to 61$
$\texttt{printInteger}(61, 16) \to 3D$

Auxiliary Function

$\texttt{printDigit}(d)$

Example 3: Integer Printing (cont.)

Pseudo-code solution:

```
algorithm printInteger(n, b)
if n < b then
  printDigit(n)
else
  printInteger(⌊n/b⌋, b)
  printDigit(n mod b)
```

Example 4: Line Plotter

Plot a line between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$

Input

Points x_1, y_1, x_2, y_2

Ouput

Plot the connecting line on the screen

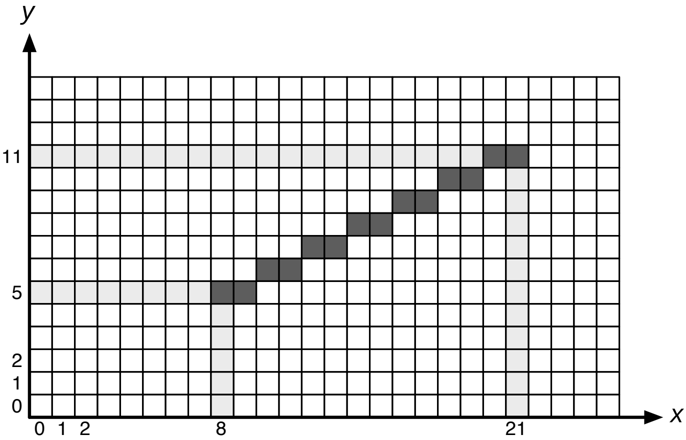
Example

$\texttt{plotLine}(8, 5, 21, 11) \rightarrow$ see next slide

Auxiliary Functions

$\texttt{plotPixel}(x, y)$

Example 4: Line Plotter (cont.)



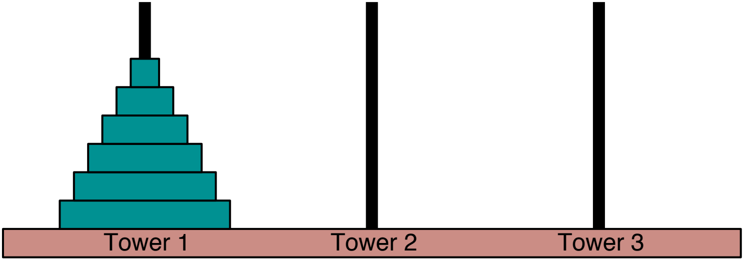
Example 4: Line Plotter (cont.)

Pseudo-code solution:

```
algorithm plotLine(x1, y1, x2, y2)
if x1 = x2 and y1 = y2 then
  plotPixel(x1, y1)
else if |x1 − x2| ≤ 1 and |y1 − y2| ≤ 1 then
  plotPixel(x1, y1)
  plotPixel(x2, y2)
else
  x ← ⌊(x1 + x2)/2⌋
  y ← ⌊(y1 + y2)/2⌋
  plotLine(x1, y1, x, y)
  plotLine(x, y, x2, y2)
```

Example 5: Towers of Hanoi

Move all the blocks from the first to the third tower, one after another, and without ever putting a block on a smaller one



Example 5: Towers of Hanoi (cont.)

Input
: Integer n (number of blocks)

Ouput
: Print sequence of instructions on screen

Example
: $\texttt{hanoi}(3) \rightarrow 1{\Rightarrow}3, 1{\Rightarrow}2, 3{\Rightarrow}2, 1{\Rightarrow}3, 2{\Rightarrow}1, 2{\Rightarrow}3, 1{\Rightarrow}3$

Auxiliary Functions
: $\texttt{print}(t_1, t_2)$

Example 5: Towers of Hanoi (cont.)

Pseudo-code solution:

```
algorithm hanoi(n)
  recHanoi(n, 1, 3)

algorithm recHanoi(n, from, to)
if n = 1 then
  print(from, to)
else
  other ← 6 − (from + to)
  recHanoi(n − 1, from, other)
  recHanoi(1, from, to)
  recHanoi(n − 1, other, to)
```

Outline

The MiniMax-Algorithm

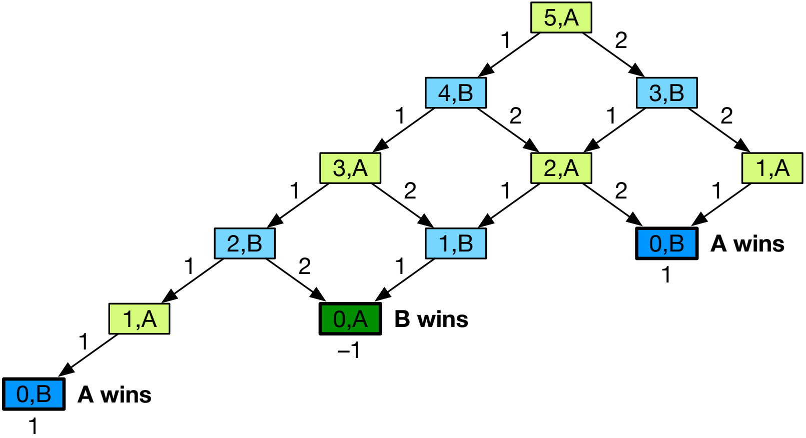
- The MiniMax algorithm is an optimal game playing algorithm for two player games such as Tic-Tac-Toe, Chess, Go, Uril, ...
- Let A and B be the two players and S the finite set of possible game states, where ...
  - → $s_0 \in S$ is the initial state
  - → $S^* \subseteq S$ are final states, in which the game ends
- For all final states $s^* \in S^*$, the winner of the game is defined by a function $E : S^* \to [-1, 1]$, where

$$E(s^*) = \begin{cases} 1, & \text{if } A \text{ wins} \\ 0, & \text{if the game ends as draw} \\ -1, & \text{if } B \text{ wins} \end{cases}$$

Example: Subtraction Game

- In a (k, n)-subtraction game, A and B take turns in removing up to k objects from a pile with initially n objects
  - → A begins
  - → Whoever removes the last object from the pile wins
- Example: $(2,5)$-subtraction game
  - → States: $S = \{(5,A),(3,A),\ldots,(0,A),(4,B),\ldots,(0,B)\}$
  - → Initial state: $s_0 = (5,A)$
  - → Final states: $S^* = \{(0,A),(0,B)\}$
  - → $E(s^*) = \begin{cases} 1, & \text{for } s^* = (0,B) \\ -1, & \text{for } s^* = (0,A) \end{cases}$

Example: Subtraction Game



MiniMax Algorithm: General Idea

- Let $next(s) \subseteq S$ be the reachable states from $s \in S$
- Example: $(2, 5)$-subtraction game

  → $next((5, A)) = \{(4, B), (3, B)\}$
  → $next((4, B)) = \{(3, A), (2, A)\}$
  $\vdots$
  → $next((0, A)) = next(0, B) = \{\}$

- The MiniMax algorithm extends E from $E : S^* \rightarrow [-1, 1]$ to $E' : S \rightarrow [-1, 1]$ by computing $E'(s)$ recursively for all $s \in S$

$$E'(s) = \begin{cases} E(s), & \text{if } s \in S^* \\ \max\{E'(s') : s' \in next(s)\}, & \text{if it is } A\text{'s turn} \\ \min\{E'(s') : s' \in next(s)\}, & \text{if it is } B\text{'s turn} \end{cases}$$

MiniMax Algorithm: Pseudocode

Algorithm: MiniMax(s, d)

if $s \in S^*$ **then** // game ends
  return $E(s)$

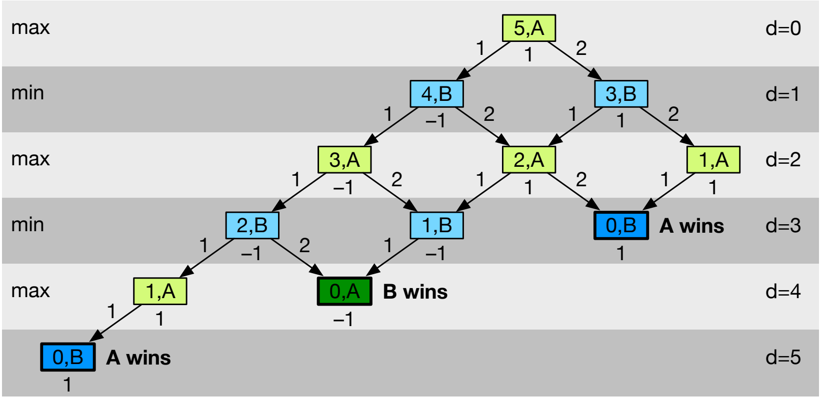
if $d \bmod 2 = 0$ **then** // A's turn
  $m \leftarrow -1$
  for $s' \in next(s)$ **do**
    $m \leftarrow \max(m, \text{MiniMax}(s', d+1))$

else // B's turn
  $m \leftarrow 1$
  for $s' \in next(s)$ **do**
    $m \leftarrow \min(m, \text{MiniMax}(s', d+1))$

return m

Example: Subtraction Game

Initial call: $\mathrm{MiniMax}((5, A), 0)$
Return value: $E'((5, A)) = 1$, i.e. A wins by removing 2 objects



MiniMax Algorithm: Performance

- The branching factor b of a game is the average number legal moves (children in the game tree)
- Examples:
  - → (k, n)-subtraction game: $b \leq k$, $h \leq n$
  - → Chess: $b \approx 35$, $h \approx 80$
  - → Go: $b \approx 250$
- The MiniMax algorithms runs in $O(b^h)$ time, where h denotes the height (maximal depth) of the game tree
- In other words, exploring the full game tree is impossible for most non-trivial games

Pruned MiniMax Algorithm

- To apply the MiniMax algorithm to non-trivial games, the game tree exploration must be pruned
- The simplest pruning method is to stop the recursion when a maximal depth $d_{\max}$ is reached
- When the recursion stops at state $s \in S$, then a evaluation function $\tilde{E} : S \to [-1, 1]$ is applied to s
  - → For $s \in S^*$, let $\tilde{E}(s) = E(s)$
  - → Otherwise, let $\tilde{E}(s)$ be an estimate of the advantage of state s relative to A and B, such that $\tilde{E}(s) = 1$ means maximal advantage for A and $\tilde{E}(s) = -1$ maximal advantage for B
- The quality of the estimate $\tilde{E}(s)$ and $d_{\max}$ determine the quality and accuracy of the final MiniMax return value

Pruned MiniMax Algorithm: Pseudocode

Algorithm: MiniMax(s, d, d_{max})

```
if s ∈ S* or d = d_max then                          // game ends
  return Ẽ(s)
if d mod 2 = 0 then                                  // A's turn
  m ← −1
  for s′ ∈ next(s) do
    m ← max(m, MiniMax(s′, d + 1, d_max))
else                                                 // B's turn
  m ← 1
  for s′ ∈ next(s) do
    m ← min(m, MiniMax(s′, d + 1, d_max))
return m
```

Defining the Evaluation Function

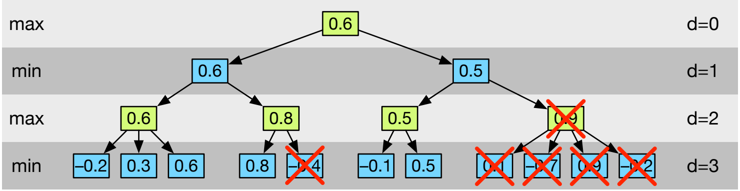
- One popular strategy for constructing an evaluation function $\tilde{E} : S \to [-1, 1]$ is as a weighted sum

$$\tilde{E}(s) = \frac{1}{W} \sum_{i=1}^{k} w_i \cdot \tilde{E}_i(s)$$

  of k individual evaluation criteria
- The value $w_i \in [0, 1]$ denotes the weight of criterion i and $W = \sum_{i=1}^{k} w_i$ denotes the total weight of all criteria
- $\tilde{E}_i : S \to [-1, 1]$ defines the evaluation function of criterion i
- Example from chess:
  $\tilde{E}(s) = 9 \cdot (Q - Q') + 5 \cdot \frac{R-R'}{2} + 3 \cdot \frac{B-B'}{2} + 3 \cdot \frac{N-N'}{2} + \frac{P-P'}{8} + \ldots$

Alpha-Beta Pruning

- There are many ways of optimizing the MiniMax algorithm
- The general idea is to prune branches of the game tree that will not influence the final MiniMax return value
- The simplest optimization is known as alpha-beta pruning



- The following version of the MiniMax algorithm is initially called with $MiniMax(s_0, 0, d_{\max}, -1, 1)$

Alpha-Beta Pruning: Pseudocode I

```
Algorithm: MiniMax(s, d, d_max, α, β)
if s ∈ S* or d = d_max then                          // game ends
    return Ẽ(s)
if d mod 2 = 0 then                                  // A's turn
    m ← −1
    for s' ∈ next(s) do
        m ← max(m, MiniMax(s', d + 1, d_max, α, β))
        α ← max(α, m)
        if α ≥ β then
            return m                                 // β cutoff
```

Alpha-Beta Pruning: Pseudocode II

```
else                                              // B's turn
    m ← 1
    for s' ∈ next(s) do
        m ← min(m, MiniMax(s', d + 1, d_max, α, β))
        β ← min(β, m)
        if α ≥ β then
            return m                              // α cutoff
return m
```

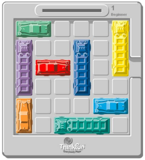
Outline

One-Player Games

- In one-player games, the goal is often to maximize the value $E(s^*)$ when reaching a final state $s^* \in S^*$, for

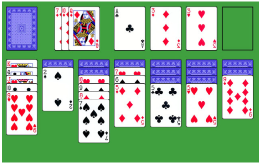
$$E : S^* \to \mathbb{R}^+$$

- The highest such value reached in all previously played games is called high score
- A puzzle is a one-player game with complete information and no chance
- Examples: Peg Solitaire, Rubik's Cube, Rush Hour, ...



One-Player Games with Chance

- Adding chance (or hidden information) makes one-player games more interesting
- Examples: Minesweepeer, Solitaire, 2048, ...


- Probabilistic events can be seen as an auxiliary player playing chance moves, each leading to a new state $s' \in next(s)$
- For $n = |next(s)|$ different chance moves, let $0 < p(s'_i) \leq 1$ be the probability of $s'_i \in next(s)$ to occur, i.e. $\sum_{i=1}^{n} p(s'_i) = 1$

ExpectedMax Algorithm

- Let's assume that a chance move takes place after every move by the player (if not, think of a chance move with $p(s') = 1$)
- The ExpectedMax algorithm extends E from $E : S^* \to \mathbb{R}^+$ to $E' : S \to \mathbb{R}^+$ by computing $E'(s)$ recursively for all $s \in S$

$$E'(s) = \begin{cases} E(s), & \text{if } s \in S^* \\ \max\{E'(s') : s' \in \mathit{next}(s)\}, & \text{if it is the player's turn} \\ \sum\{p(s')\cdot E'(s') : s' \in \mathit{next}(s)\}, & \text{if it is a chance move,} \end{cases}$$

  where $\sum p(s')\cdot E'(s')$ is the expected value of the game state before a chance move
- Note that chance moves can also be added to two-player or multi-player games

ExpectedMax Algorithm: Pseudocode

Algorithm: ExpectedMax(s, d)

```
if s ∈ S* then                                   // game ends
    return E(s)
if d mod 2 = 0 then                              // Player's turn
    m ← 0
    for s′ ∈ next(s) do
        m ← max(m, ExpectedMax(s′, d + 1))
else                                             // Chance move
    e ← 0
    for s′ ∈ next(s) do
        e ← e + p(s′) · ExpectedMax(s′, d + 1)
return m
```

Pruned ExpectedMax Algorithm: Pseudocode

```
Algorithm: ExpectedMax(s, d, d_max)
if s ∈ S* or d = d_max then                      // game ends
    return Ẽ(s)
if d mod 2 = 0 then                              // Player's turn
    m ← 0
    for s′ ∈ next(s) do
        m ← max(m, ExpectedMax(s′, d + 1, d_max))
else                                             // Chance move
    e ← 0
    for s′ ∈ next(s) do
        e ← e + p(s′) · ExpectedMax(s′, d + 1, d_max)
return m
```