SMIII lectures

Week 9

Poisson regression

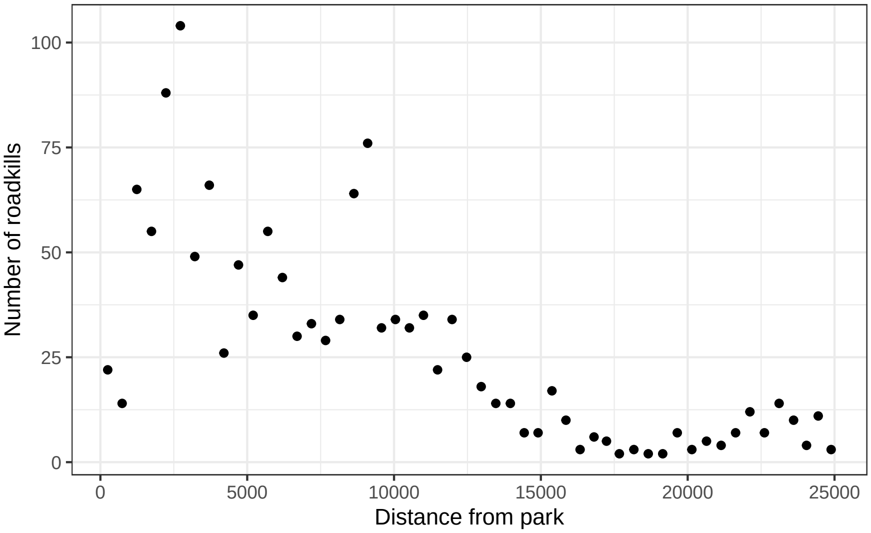
Poisson data

- Independent observations
- Constant rate of success λ (time based intervals)
- The mean rate λ is equal to the variance λ

Example - roadkills

```
## Rows: 52 Columns: 23
## -- Column specification --------------------------------------------------------
## Delimiter: "\t"
## dbl (23): Sector, X, Y, BufoCalamita, TOT.N, S.RICH, OPEN.L, OLIVE, MONT.S, ...
##
## i Use ‘spec()‘ to retrieve the full column specification for this data.
## i Specify the column types or set ‘show_col_types = FALSE‘ to quiet this message.
```

distance_park	roadkills
250.214	22
741.179	14
1240.080	65
1739.885	55
2232.130	88
2724.089	104



Poisson Regression Models

Consider data

$$(y_1, \boldsymbol{x}_1), (y_2, \boldsymbol{x}_2), \ldots, (y_n, \boldsymbol{x}_n)$$

For count data consider a Poisson model

$$Y_i \sim Po(\lambda_i) \text{ independently, for } i = 1, 2, \ldots, n.$$

The regression problem is to relate the Poisson mean, λ_i, to the predictor $\boldsymbol{x}_i$.

That is, we seek a suitable model

$$M : \lambda_i = f(\boldsymbol{x}_i).$$

Log-linear Models

To ensure that the Poisson mean λ_i is positive, we define

$$\eta_i = \log(\lambda_i).$$

Consider log linear regression models

$$M : \eta_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik}$$

In matrix notation,

$$\boldsymbol{\eta} = X\boldsymbol{\beta}.$$

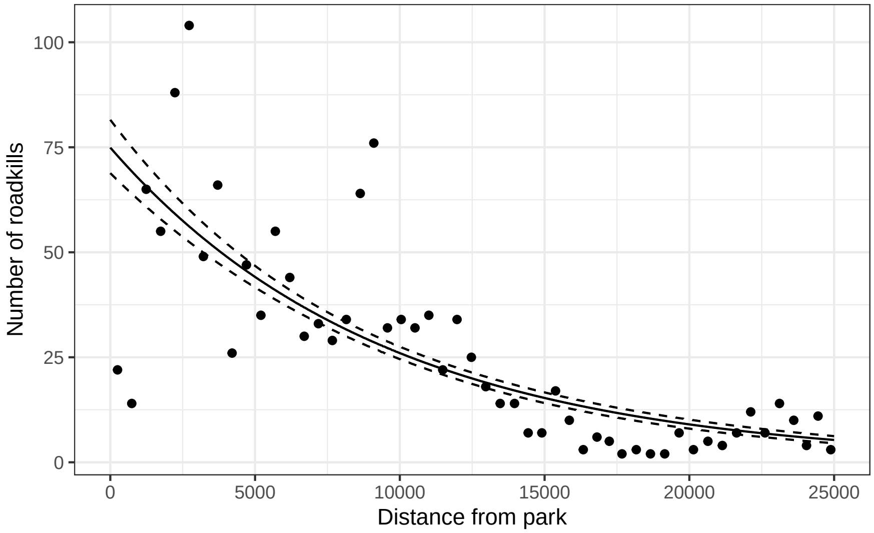
Generalised linear models

Regression Model	Response	Link Function
Linear	$Y_i \sim N(\mu_i, \sigma^2)$	$\eta_i = \mu_i$
Logistic	$Y_i \sim B(n_i, \pi_i)$	$\eta_i = \log\left(\frac{\pi_i}{1-\pi_i}\right)$
Poisson	$Y_i \sim Po(\lambda_i)$	$\eta_i = \log(\lambda_i)$

Fitting in R

```r
roadkill_glm <- glm(roadkills ~ distance_park, data = roadkill, family = poisson())
summary(roadkill_glm)
```

```
##
## Call:
## glm(formula = roadkills ~ distance_park, family = poisson(),
##     data = roadkill)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -8.1100  -1.6950  -0.4708   1.4206   7.3337
##
## Coefficients:
##                 Estimate Std. Error z value Pr(>|z|)
## (Intercept)    4.316e+00  4.322e-02   99.87   <2e-16 ***
## distance_park -1.059e-04  4.387e-06  -24.13   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 1071.4  on 51  degrees of freedom
## Residual deviance:  390.9  on 50  degrees of freedom
## AIC: 634.29
##
## Number of Fisher Scoring iterations: 4
```



Interpretation of coefficients

$$\log(Y_i) = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

Consider increasing x_i by 1

$$
\begin{aligned}
\log(Y_i') &= \hat{\beta}_0 + \hat{\beta}_1(x_i + 1) \\
&= \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\beta}_1 \\
&= \log(Y_i) + \hat{\beta}_1 \\
\Rightarrow Y_i' = e^{\hat{\beta}_1} Y_i
\end{aligned}
$$

Example - roadkill

```r
b <- coef(roadkill_glm)
exp(b[2])
```

```
## distance_park
##     0.9998942
```

So increasing distance by 1 km, decreases by 1%.

Poisson rate regression

Poisson rate regression

$$
\begin{aligned}
\log(r_i) = \log\left(\frac{\lambda_i}{t_i}\right) &= \beta_0 + \beta_1 x_i \\
\Rightarrow \log(\lambda_i) = \log(t_i) + \beta_0 + \beta_1 x_i
\end{aligned}
$$

Example - insurance

An insurance company recorded the number of policies held and the number of accident claims in 64 categories defined by

- `District`: 1-4, where 4 represents major cities;
- `Engine`: Engine capacity of the car,
  - < 1 litre,
  - $1 - 1.5$ litre,
  - $1.5 - 2$ litre,
  - > 2 litre;
- `Age`:
  - < 25,
  - $25 - 29$,
  - $30 - 35$,
  - > 35.

```
## Rows: 64 Columns: 5
## -- Column specification --------------------------------------------------------
## Delimiter: ","
## chr (3): District, Engine, Age
## dbl (2): Policies, Claims
##
## i Use ‘spec()‘ to retrieve the full column specification for this data.
## i Specify the column types or set ‘show_col_types = FALSE‘ to quiet this message.
```

District	Engine	Age	Policies	Claims
D1	<1l	<25	197	38
D1	<1l	25-29	264	35
D1	<1l	30-35	246	20
D1	<1l	35+	1680	156
D1	1-1.5l	<25	284	63
D1	1-1.5l	25-29	536	84

Since it is reasonable to expect the number of claims to be proportional to the number of policies within each class, the number of policies is used as an offset.

Sometimes we wish to include a predictor variable with a fixed coefficient.

This is implemented in the `glm` function using the `offset` argument.

Fitting an offset in R

```
M1 <- glm(Claims ~ District + Engine + Age,
          family=poisson,
          offset=log(Policies),
          data=insurance
)
```

```
summary(M1)
```

```
##
## Call:
## glm(formula = Claims ~ District + Engine + Age, family = poisson,
##     data = insurance, offset = log(Policies))
##
## Deviance Residuals:
##      Min        1Q    Median        3Q       Max
## -2.46558  -0.50802  -0.03198   0.55555   1.94026
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -1.82174    0.07679 -23.724  < 2e-16 ***
## DistrictD2    0.02587    0.04302   0.601 0.547597
## DistrictD3    0.03852    0.05051   0.763 0.445657
## DistrictD4    0.23421    0.06167   3.798 0.000146 ***
## Engine1-1.5l  0.16134    0.05053   3.193 0.001409 **
## Engine1.5-2l  0.39281    0.05500   7.142 9.18e-13 ***
## Engine2+l     0.56341    0.07232   7.791 6.65e-15 ***
## Age25-29     -0.19101    0.08286  -2.305 0.021149 *
## Age30-35     -0.34495    0.08137  -4.239 2.24e-05 ***
## Age35+       -0.53667    0.06996  -7.672 1.70e-14 ***
## ---
## Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 236.26  on 63  degrees of freedom
## Residual deviance:  51.42  on 54  degrees of freedom
## AIC: 388.74
##
## Number of Fisher Scoring iterations: 4
```