STATS 3001 / STATS 4101 / STATS 7054
Statistical Modelling III
Tutorial 4
2022
Solutions

QUESTIONS:

1. Show for $\pi_1,\ \pi_2$ both small that

$$\log\left(\frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)}\right) \approx \log\left(\frac{\pi_1}{\pi_2}\right).$$

Does this make the interpretation of the log-odds ratio easier?

SOLUTIONS:

$$\log\left(\frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)}\right) = \log\left(\frac{\pi_1}{\pi_2}\right) - \log(1-\pi_1) + \log(1-\pi_2).$$

Now for $\pi_i \approx 0$

$$1-\pi_i \approx 1 \Rightarrow \log(1-\pi_i) \approx 0$$

and hence

$$\log\left(\frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)}\right) \approx \log\left(\frac{\pi_1}{\pi_2}\right).$$

Such situations make the interpretation of the log-odds ratio much easier as the quantity (approximately) becomes a log ratio of the probabilities of interest. The probability π_1 can be stated in relative terms (relative increase or decrease) to the probability π_2 when the quantity is exponentiated.

---

2. If

$$\pi = \frac{e^{\eta}}{(1+e^{\eta})}$$

then show that

$$\frac{d\pi}{d\eta} = \pi(1-\pi).$$

SOLUTIONS:
Using the quotient rule:

$$\begin{aligned}\frac{d\pi}{d\eta} &= \frac{e^\eta(1+e^\eta) - e^\eta e^\eta}{(1+e^\eta)^2} \\ &= \frac{e^\eta}{(1+e^\eta)^2} \\ &= \frac{e^\eta}{1+e^\eta}.\frac{1}{1+e^\eta} \\ &= \pi(1-\pi)\end{aligned}$$

since

$$\pi = \frac{e^\eta}{1+e^\eta} \text{ and } 1-\pi = \frac{1}{1+e^\eta}.$$

---

3. Consider the log-likelihood function

$$\ell(\boldsymbol{\beta}; \boldsymbol{y}) = \sum_{i=1}^m y_i \log \pi_i + \sum_{i=1}^m (n_i - y_i) \log(1-\pi_i) + \log \prod_{i=1}^m \binom{n_i}{y_i}$$

where

$$\pi_i = \frac{\exp(\boldsymbol{x}_i^T \boldsymbol{\beta})}{1 + \exp(\boldsymbol{x}_i^T \boldsymbol{\beta})}.$$

Evaluate

$$\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k}$$

and hence derive the Fisher information matrix directly.

SOLUTIONS:
Note that

$$\eta_i = \boldsymbol{x}_i^T \boldsymbol{\beta}$$

$$\begin{aligned}
\ell(\boldsymbol{\beta}; \boldsymbol{y}) &= \sum_{i=1}^m y_i \log \frac{\pi_i}{1-\pi_i} + \sum_{i=1}^m (n_i) \log(1-\pi_i) + \log \prod_{i=1}^m \binom{n_i}{y_i} \\
&= \sum_{i=1}^m y_i \eta_i - \sum_{i=1}^m n_i \log(1+\exp(\eta_i)) + \log \prod_{i=1}^m \binom{n_i}{y_i}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta_j} &= \sum_{i=1}^m \frac{\partial}{\partial \eta_i}(y_i \eta_i - n_i \log(1+\exp(\eta_i))) \frac{\partial \eta_i}{\partial \beta_j} \\
&= \sum_{i=1}^m \left(y_i - n_i \frac{\exp(\eta_i)}{1+\exp(\eta_i)} \right) \frac{\partial \eta_i}{\partial \beta_j} \\
&= \sum_{i=1}^m (y_i - n_i \pi_i) x_{ij}. \\
\Rightarrow \frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k} &= \sum_{i=1}^m \frac{\partial}{\partial \eta_i}(y_i - n_i \pi_i) x_{ij} \frac{\partial \eta_i}{\partial \beta_j} \\
&= -\sum_{i=1}^m n_i \frac{\partial}{\partial \eta_i} \pi_i x_{ij} x_{ik} \\
&= -\sum_{i=1}^m n_i \pi_i (1-\pi_i) x_{ij} x_{ik}.
\end{aligned}$$

Since this expression is constant with respect to Y, it follows that

$$E\left(-\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k}\right) = \sum_{i=1}^m n_i \pi_i (1-\pi_i) x_{ij} x_{ik}$$

which is the jk^{th} element of

$$X^T D X,$$

where

$$D = \mathrm{diag}(n_1\pi_1(1-\pi_1), n_2\pi_2(1-\pi_2), \ldots, n_m\pi_m(1-\pi_m)).$$