STATS 3001 / STATS 4101 / STATS 7054
Statistical Modelling III
Tutorial 1
2022

QUESTIONS:

1. Suppose A and B are matrices of dimension $n \times m$ and $m \times n$ respectively. Prove that the trace satisfies

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

2. If A is a constant $n \times n$ matrix and $\mathcal{Y}$ is an $n \times n$ matrix whose elements are random variables, prove that

$$E[\mathrm{tr}(A\mathcal{Y})] = \mathrm{tr}(AE[\mathcal{Y}]).$$

3. If X is a matrix of dimension $n \times p$ with linearly independent columns (so that X^TX is invertible) then prove that the matrix

$$P = X(X^TX)^{-1}X^T$$

satisfies

$$P^T = P = P^2$$

and hence show that

$$(I - P)^T = (I - P) = (I - P)^2.$$

4. Suppose $\boldsymbol{y} \in \mathbb{R}^n$ and X is a $n \times p$ matrix with linearly independent columns. Prove that

$$Q(\boldsymbol{\beta}) = \|\boldsymbol{y} - X\boldsymbol{\beta}\|^2$$

is uniquely minimized by

$$\hat{\boldsymbol{\beta}} = (X^TX)^{-1}X^T\boldsymbol{y} :$$

(a) Algebraically by considering

$$Q(\boldsymbol{\beta}) = \|\boldsymbol{y} - X\hat{\boldsymbol{\beta}} + X\hat{\boldsymbol{\beta}} - X\boldsymbol{\beta}\|^2.$$

(a) Using calculus to set up system of equations

$$\frac{\partial Q}{\partial \beta_j} = 0, \quad j = 1, 2, \ldots, p$$

and solving for $\boldsymbol{\beta}$.

Hint: This can be done easily by observing (or proving) first that

$$\frac{\partial}{\partial \boldsymbol{u}} \boldsymbol{u}^T \boldsymbol{u} = 2\boldsymbol{u}^T$$

and then applying the chain rule of multivariable calculus to $\boldsymbol{u} = (\boldsymbol{y} - X\boldsymbol{\beta})$ to obtain

$$\frac{\partial Q}{\partial \boldsymbol{\beta}} = -2(\boldsymbol{y} - X\boldsymbol{\beta})^T X.$$