Statistical Modelling III
Assignment 3

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Q1

(a)

We have:

(i) Treatment A and controls ($x_i = 0, x_j = 0$):

$$\begin{aligned}\log(\lambda_{11}) &= \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j \\ &= \gamma_0 + \gamma_1(0) + \gamma_2(0) + \gamma_3(0)(0) \\ &= \gamma_0\end{aligned}$$

(ii) Treatment A and cases ($x_i = 0, x_j = 1$):

$$\begin{aligned}\log(\lambda_{12}) &= \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j \\ &= \gamma_0 + \gamma_1(0) + \gamma_2(1) + \gamma_3(0)(1) \\ &= \gamma_0 + \gamma_2\end{aligned}$$

(iii) Treatment B and controls ($x_i = 1, x_j = 0$):

$$\begin{aligned}\log(\lambda_{21}) &= \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j \\ &= \gamma_0 + \gamma_1(1) + \gamma_2(0) + \gamma_3(1)(0) \\ &= \gamma_0 + \gamma_1\end{aligned}$$

(iv) Treatment B and cases ($x_i = 1, x_j = 1$):

$$\begin{aligned}\log(\lambda_{22}) &= \gamma_0 + \gamma_1 x_i + \gamma_2 x_j + \gamma_3 x_i x_j \\ &= \gamma_0 + \gamma_1(1) + \gamma_2(1) + \gamma_3(1)(1) \\ &= \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3\end{aligned}$$

(b)

We have $Y_{i1} \sim Po(\lambda_{i1}),\ Y_{i2} \sim Po(\lambda_{i2})$ and they are independent:

$$\begin{aligned}
P(Y_{i2}|Y_{i1}+Y_{i2}=n_i) &= \frac{P(Y_{i2}=y\ \cap Y_{i1}+Y_{i2}=n_i)}{P(Y_{i1}+Y_{i2}=n_i)} \\
&= \frac{P(Y_{i2}=y\ \cap Y_{i1}=n_i-y)}{P(Y_{i1}+Y_{i2}=n_i)} \\
&= \frac{\frac{e^{-\lambda_{i2}}\lambda_{i2}^{y}}{y!}\cdot\frac{e^{-\lambda_{i1}}\lambda_{i1}^{n_i-y}}{(n_i-y)!}}{\frac{e^{-(\lambda_{i1}+\lambda_{i2})}(\lambda_{i1}+\lambda_{i2})^{n_i}}{n_i!}} \\
&= \frac{\frac{\lambda_{i2}^{y}}{y!}\cdot\frac{\lambda_{i1}^{n_i-y}}{(n_i-y)!}}{\frac{(\lambda_{i1}+\lambda_{i2})^{n_i}}{n_i!}} \\
&= \frac{n_i!}{y!(n_i-y)!}\cdot\frac{\lambda_{i2}^{y}\lambda_{i1}^{n_i-y}}{(\lambda_{i1}+\lambda_{i2})^{n_i}} \\
\frac{n_i!}{y!(n_i-y)!} &= \binom{n_i}{y} \\
\frac{\lambda_{i2}^{y}\lambda_{i1}^{n_i-y}}{(\lambda_{i1}+\lambda_{i2})^{n_i}} &= \frac{\lambda_{i2}^{y}\lambda_{i1}^{n_i-y}}{(\lambda_{i1}+\lambda_{i2})^{n_i}}\cdot\frac{(\lambda_{i1}+\lambda_{i2})^{-y}}{(\lambda_{i1}+\lambda_{i2})^{-y}} \\
&= \left(\frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}\right)^{n_i-y}\cdot\left(\frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}\right)^{y}
\end{aligned}$$

Let $\pi_i = \frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}$, then $1-\pi_i = \frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}$ (Y_{i1} and Y_{i2} are from the same treatment)

Therefore, we have:

$$P(Y_{i2}|Y_{i1}+Y_{i2}=n_i) = \binom{n_i}{y}\pi_i^{y}(1-\pi_i)^{(n_i-y)}$$

$$\therefore Y_{i2}|(Y_{i1}+Y_{i2}=n_i) \sim Bin(n_i,\pi_i)$$

(c)

For treatment A, $x_i = 0$:

$$\begin{aligned}
log\left(\frac{\pi_1}{1-\pi_1}\right) &= \beta_0+\beta_1 x_i \\
&= \beta_0+\beta_1(0) \\
&= \beta_0
\end{aligned}$$

For treatment B, $x_i = 1$:

$$\begin{aligned}
log\left(\frac{\pi_2}{1-\pi_2}\right) &= \beta_0+\beta_1 x_i \\
&= \beta_0+\beta_1(1) \\
&= \beta_0+\beta_1
\end{aligned}$$

Subtracting the equation for treatment A from the equation for treatment B, we have:

$$\begin{aligned}
log\left(\frac{\pi_2}{1-\pi_2}\right) - log\left(\frac{\pi_1}{1-\pi_1}\right) &= (\beta_0+\beta_1)-\beta_0 \\
&= \beta_1
\end{aligned}$$

In addition, $\pi_i = \frac{\lambda_{i2}}{\lambda_{i1}+\lambda_{i2}}$ and $1 - \pi_i = \frac{\lambda_{i1}}{\lambda_{i1}+\lambda_{i2}}$:

$$\begin{aligned}
\beta_1 &= log\left(\frac{\pi_2}{1-\pi_2}\right) - log\left(\frac{\pi_1}{1-\pi_1}\right) \\
&= log\left(\frac{\frac{\lambda_{22}}{\lambda_{21}+\lambda_{22}}}{\frac{\lambda_{21}}{\lambda_{21}+\lambda_{22}}}\right) - log\left(\frac{\frac{\lambda_{12}}{\lambda_{i1}+\lambda_{12}}}{\frac{\lambda_{11}}{\lambda_{i1}+\lambda_{12}}}\right) \\
&= log\left(\frac{\lambda_{22}}{\lambda_{21}}\right) - log\left(\frac{\lambda_{12}}{\lambda_{11}}\right) \\
&= log(\lambda_{22}) - log(\lambda_{21}) - log(\lambda_{12}) + log(\lambda_{11}) \\
&= (\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) - (\gamma_0 + \gamma_1) - (\gamma_0 + \gamma_2) + (\gamma_0) \\
&= \gamma_3 \\
\therefore \beta_i &= \gamma_3
\end{aligned}$$

(d)

Testing that treatment has no effect on the probability of being a case is equivalent to testing if $\beta_1 = 0$ in the logistic regression model. From part (c), we know that $\beta_1 = \gamma_3$. Therefore, testing for $\beta_1 = 0$ is equivalent to testing for no interaction in the Poisson model ($\gamma_3 = 0$).