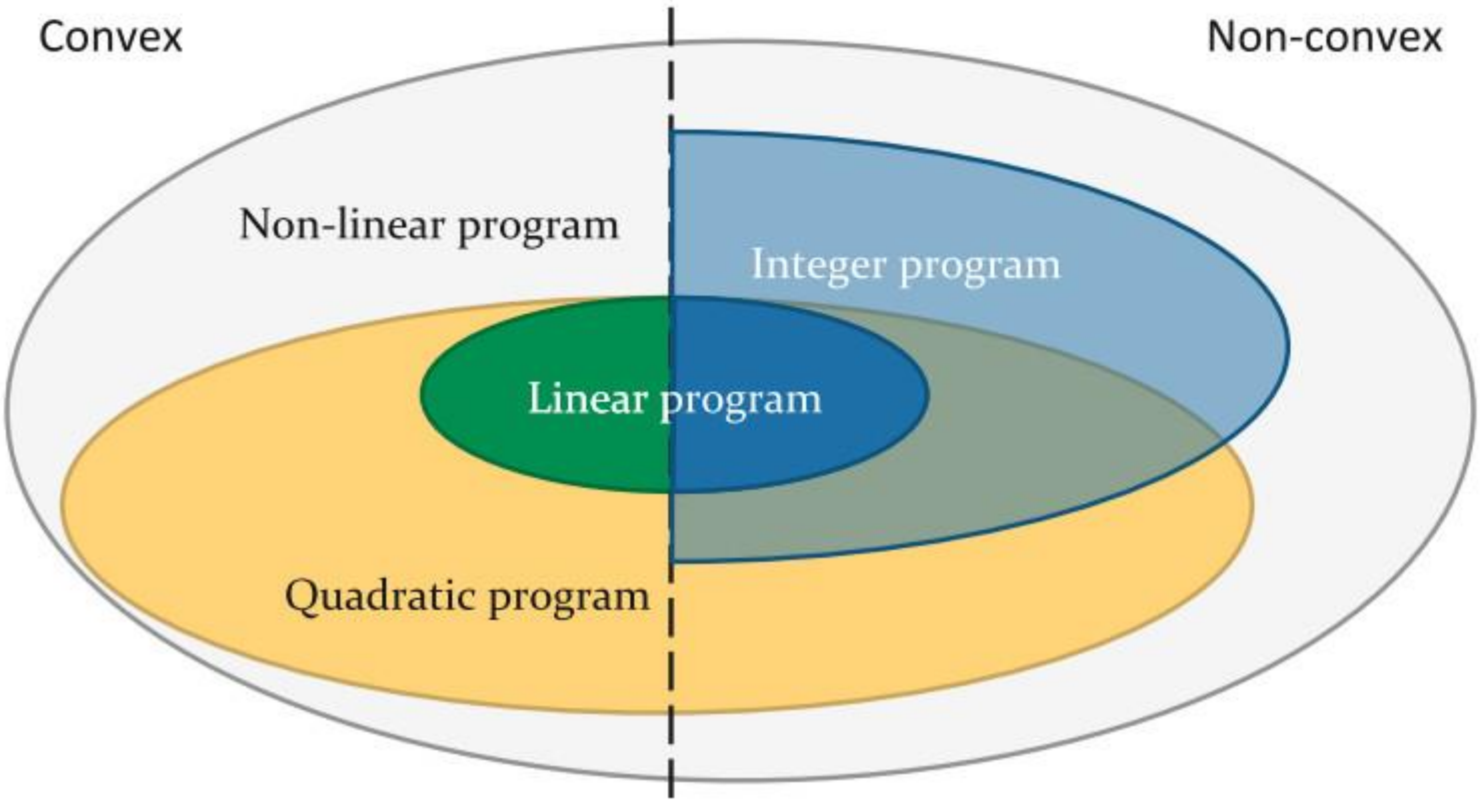
Brandenburgische
Technische Universität
Cottbus - Senftenberg

Energy systems modelling

Tutorial 11

Iegor Riepin

Introduction to complementarity problems



Source: [1]

- MCP (mixed complementarity programming) is a common modelling approach to describe various energy markets around the world.

- Complementarity models generalize linear programs (LP), quadratic programs (QP) and (convex) nonlinear programs (NLPs)

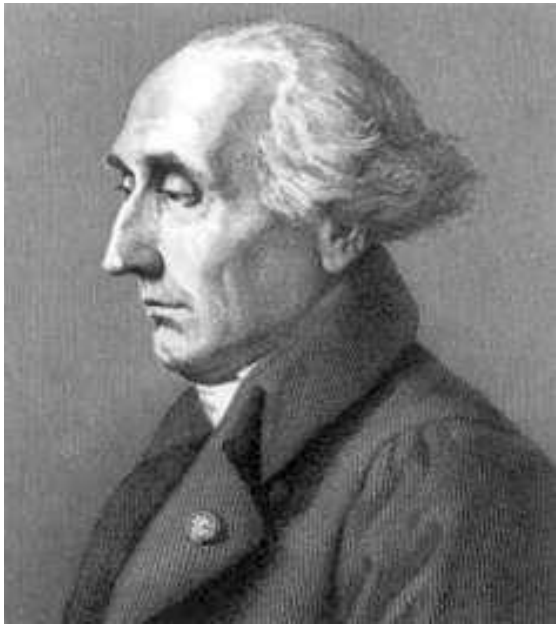
- Complementarity problems are appropriate for modelling the regulated/deregulated, perfect/imperfect competition that characterizes today’s energy markets

Method of Lagrange multipliers: problem definition

In mathematical optimization, the method of Lagrange multipliers *is a strategy for finding the local maxima and minima of a function subject to equality constraints:*

$$max\, f(x,y)$$

$$s.t.\, g(x,y) = c$$



Joseph-Louis Lagrange

Method of Lagrange multipliers: problem definition

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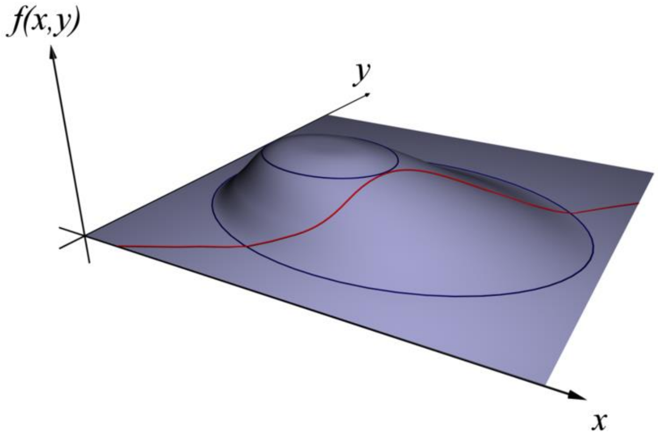
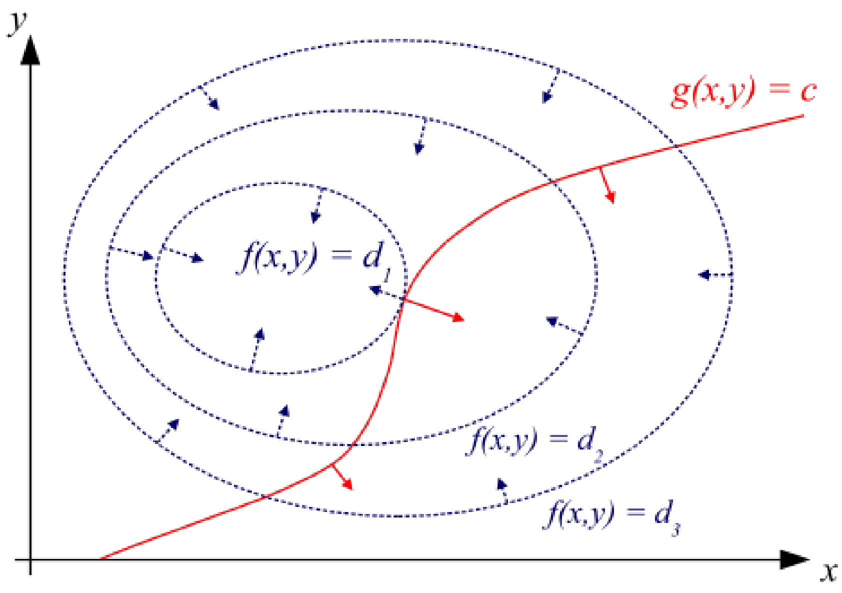




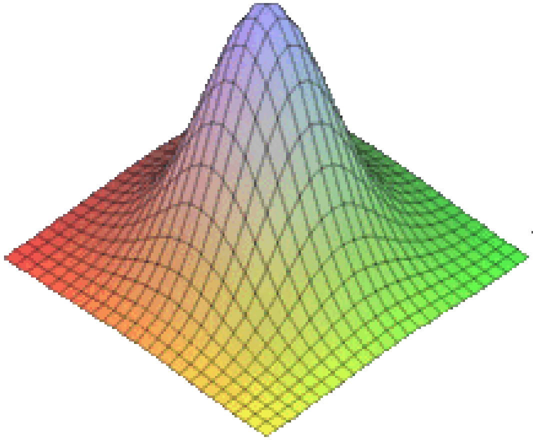
Method of Lagrange multipliers: gradient

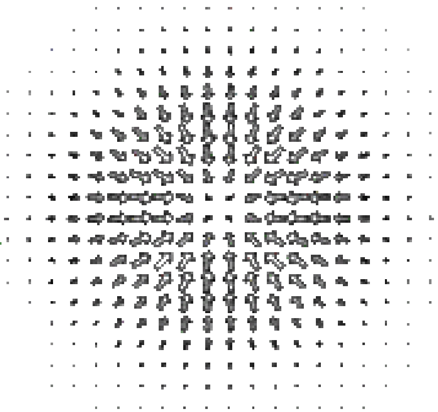
The gradient is a generalization of the usual concept of derivative of a function in one dimension to a function in several dimensions.

- Gradient points in the <u>direction of the greatest rate of increase</u> of the function and its <u>magnitude is the slope of the graph</u> in that direction

$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n$$

where the e_i are the orthogonal unit vectors pointing in the coordinate directions.





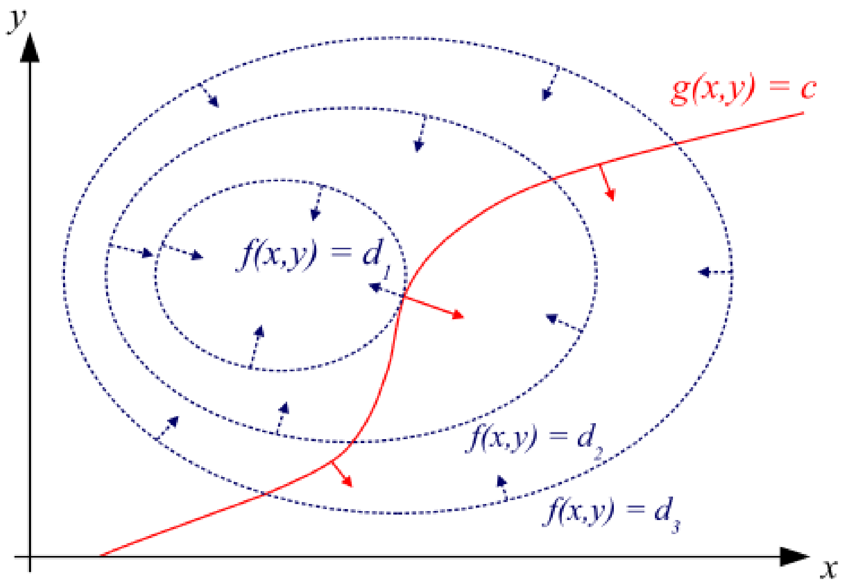
Method of Lagrange multipliers

The method of Lagrange multipliers *<u>is a strategy for finding the local maxima and minima of a function subject to equality constraints:</u>*

$$max\ f(x, y)$$

$$s.t.\ g(x, y) = c$$

Important observation: if 2 curves are tangent at the same point -> they have the same slope



Method of Lagrange multipliers

The method of Lagrange multipliers *is a strategy for finding the local maxima and minima of a function subject to equality constraints:*

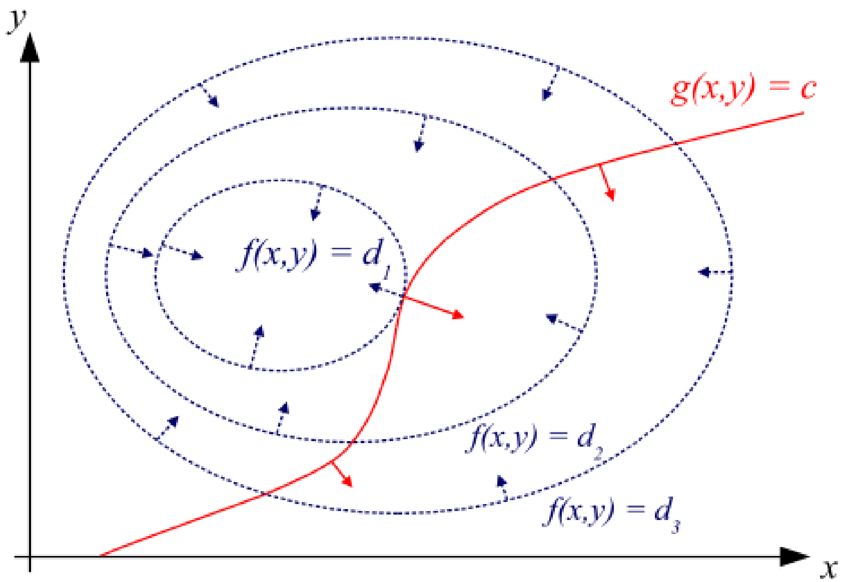
$$max\, f(x, y)$$

$$s.t.\, g(x, y) = c$$

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\nabla g(x, y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix}$$

Important observation: if 2 curves are tangent at the same point -> they have the same slope



Method of Lagrange multipliers

In mathematical optimization, the method of Lagrange multipliers *is a strategy for finding the local maxima and minima of a function subject to equality constraints:*

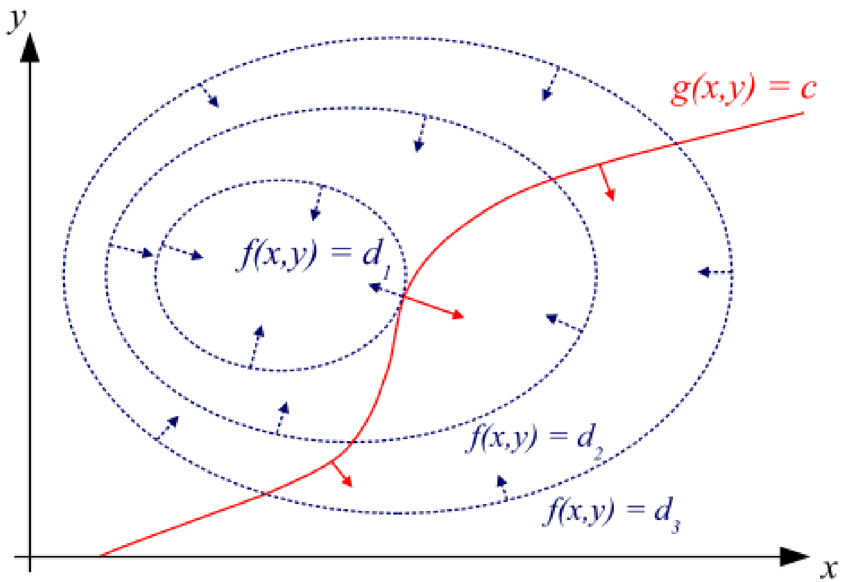
$$max\, f(x, y)$$

$$s.t.\, g(x, y) = c$$

$$\nabla f(x, y) = \boldsymbol{\lambda} \cdot \nabla g(x, y)$$

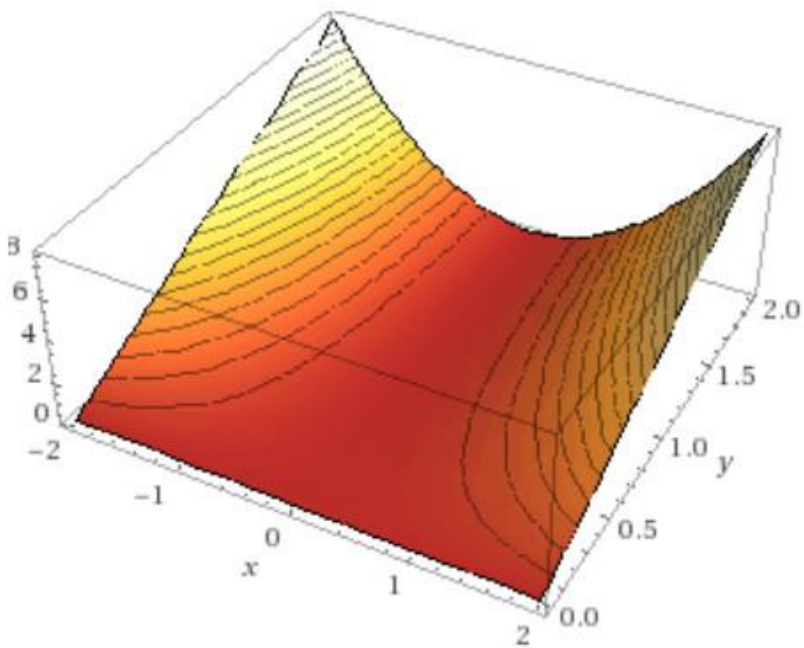
This condition ensures that isolines (contour curves) are tangent

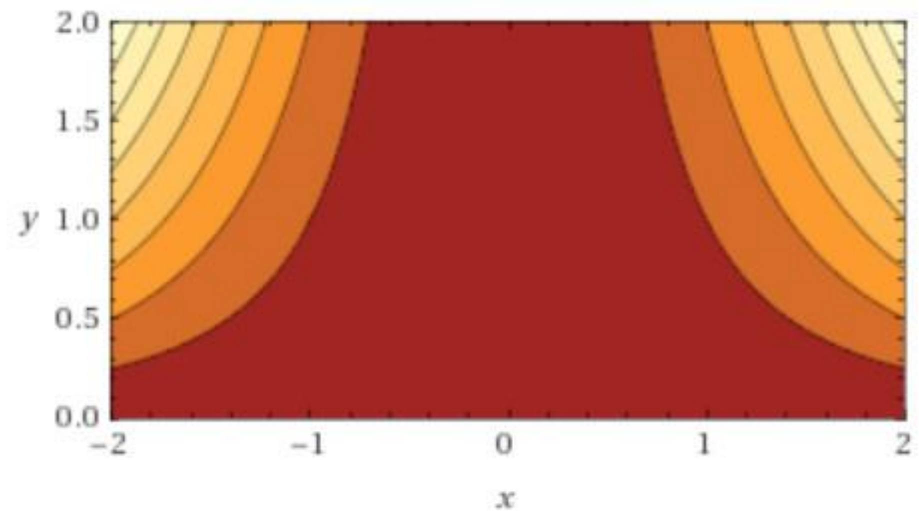
Important observation: if 2 curves are tangent at the same point -> they have the same slope -> their gradient vectors are parallel



Method of Lagrange multipliers: an example

$$\boxed{max\, x^2 y}$$

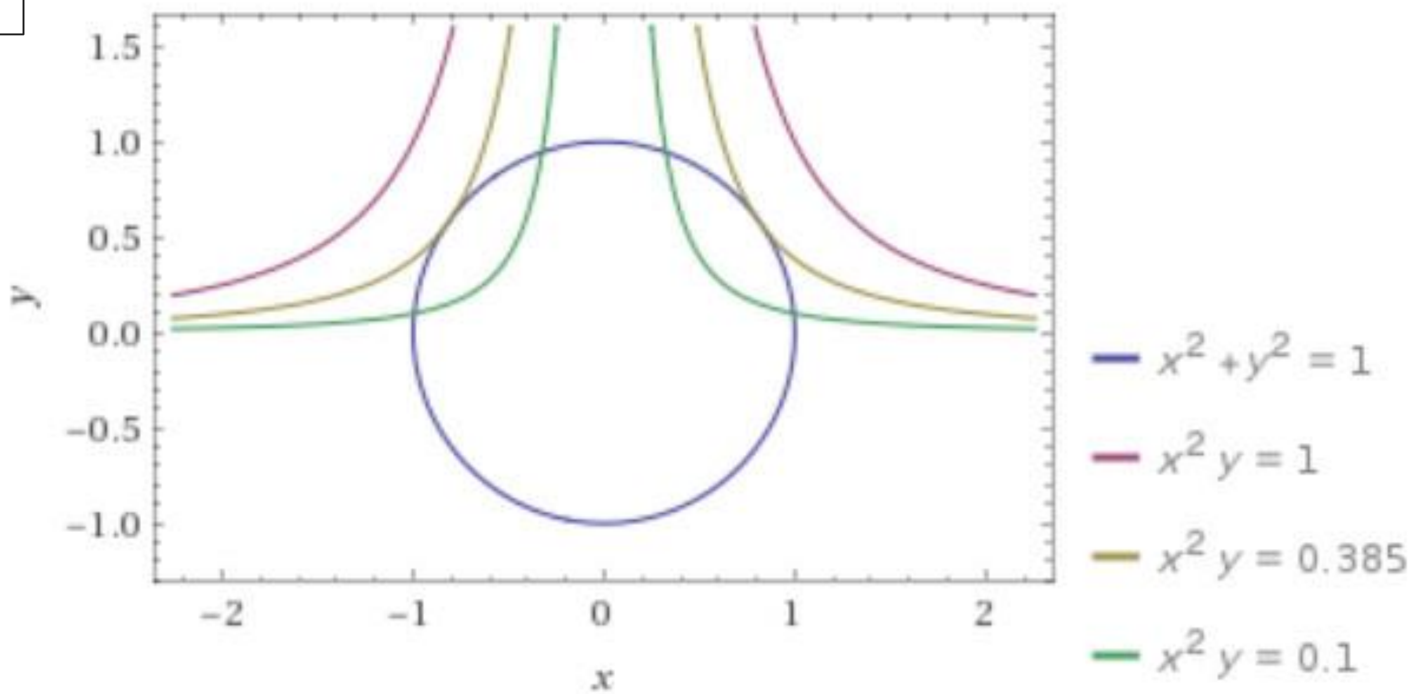




Own visualisations

Method of Lagrange multipliers: an example

$$\max x^2 y$$
$$s.t.$$
$$x^2 + y^2 = 1$$



Own visualisations

Method of Lagrange multipliers: economics

- ✓ In economics the optimal profit to a player is calculated subject to a constrained space of actions, where a Lagrange multiplier *is the change in the optimal value of the objective function (profit) due to the relaxation of a given constraint*

$$L(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$$

$$\Downarrow$$

in such a context λ is the marginal cost of the constraint, and is referred as the shadow price

Karush–Kuhn–Tucker conditions

- ✓ The Karush–Kuhn–Tucker (KKT) conditions *are first order necessary conditions for a solution in nonlinear programming to be optimal*, provided that some regularity conditions are satisfied.

- ✓ Allowing inequality constraints, the KKT approach applied to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints.

Karush–Kuhn–Tucker conditions

- Let us consider the problem:

$$\min_{x} F(x) \qquad [1.1]$$

$$s.t. \quad g_i(x) \leq 0 \quad (\lambda_i \geq 0) \quad \forall i = 1, \dots n \qquad [1.2]$$

$$h_j(x) = 0 \quad (\mu_j \; free) \;\; \forall j = 1, \dots m \qquad [1.3]$$

- For this problem, the KKT conditions are:

$$\nabla f(x) + \sum_{i=1}^{n} \lambda_i \nabla \mathrm{g}_i(\mathrm{x}) + \sum_{j=1}^{n} \mu_i \nabla \mathrm{h}_j(\mathrm{x}) \leq 0 \perp x \geq 0 \qquad [1.4]$$

$$0 \geq g_i(x) \perp \lambda_i \geq 0 \qquad \forall i = 1, \dots n \qquad [1.5]$$

$$0 = h_j(x) \qquad \mu_j \text{ free} \qquad \forall j = 1, \dots m \qquad [1.6]$$

The solution stationarity is ensured by the equation [1.4].
Equations [1.5] and [1.6] ensure complementarity and feasibility of a solution

Example: perfectly competitive market two producers & single demand

Producer's objective function is to maximize profits P by selling quantity of product q at price p bearing costs of production $C(q)$:

$$\max_{q_i \geq 0} P_i = q_i p - C_i(q_i) \ \forall i \quad [1.7*]$$

$$s.t.\, q_i \leq Q_i \ \ \forall i \quad [1.8]$$

where:

q	- quantity of product sold by producer i at price p
$C_i(q_i)$	- production costs

**if price is an exogenous variable from producers perspective;*

Example: perfectly competitive market two producers & single demand

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Optimization problem:

$$\max_{q_i \geq 0} P_i = q_i p - C_i(q_i) \; \forall i$$
$$s.t. \, q_i \leq Q_i \;\; \forall i$$

The KKTs for producer i are as follows:

$$0 \leq q_i \perp p - C_i'(q_i) + \lambda_i \leq 0 \qquad [1.9]$$

$$0 \leq \lambda_i \perp (q_i - Q_i) \leq 0 \qquad [1.10]$$

Symbol $\perp$ states orthogonality

$$0 \leq a \perp b \geq 0$$

Implies:

$$a \geq 0, b \geq 0 \; and \; ab = 0$$

Eq. [1.9] is a short way to express the following:

$$0 \leq q_i$$
$$p - C_i{}'(q_i) + \lambda_i \leq 0$$
$$q_i(p - C_i{}'(q_i) + \lambda_i) = 0$$

Hometask - simple LP / MCP problem in GAMS
/I know it might be challenging…/

Solve the following problem using LP and MCP formulations:

$$\max_{\boldsymbol{x},\boldsymbol{y}} \mathbf{4}\boldsymbol{x} + \mathbf{5}\boldsymbol{y}$$

$$\boldsymbol{s.t.}$$

$$\boldsymbol{x} + \boldsymbol{y} \le \mathbf{24}$$

$$\boldsymbol{x} \in R_+, \boldsymbol{y} \in R_+$$

Compare the marginal value of the constraint to the values of Lagrange multipliers in MCP model

$$\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$$

in such context λ is the <u>marginal cost</u> of the constraint, and is referred as the <u>shadow price</u>

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See you next class!

Appendix A: Duality concept

Some (classes of) optimization problems have a "twin" problem,

⇒ This is called the "dual problem"

Illustration using a simple linear program:

Mathematical formulation	Example of interpretation
$\min_{x \in \mathbb{R}} \; a^T x$ $\text{s.t.} \; Ax \leq b \quad (y)$	Minimize cost of supplying electricity subject to engineering and power flow constraints
$\max_{y \in \mathbb{R}_+} \; b^T y$ $\text{s.t.} \; A^T y = a \quad (x)$	Maximize pay-off such that the dual constraints are satisfied

⇒ If the optimal objective values are identical, we call it **strong duality**

Source: [1]

Appendix B: KKT formulations

- The "classical" formulation of Karush-Kuhn-Tucker conditions:

$$0 = \nabla_u f(x^*) + \sum_i \lambda_i^* \nabla_u g_i(x^*) + \sum_j \mu_j^* \nabla_u h_j(x^*) \quad , \quad x_u^* \ \text{(free)}$$

$$0 \geq g_i(x^*) \ \perp \ \lambda_i^* \geq 0$$

$$0 = h_j(x^*) \quad , \quad \mu_j^* \ \text{(free)}$$

- One alternative formulation (of many) :

$$0 \leq \nabla_u f(x^*) + \sum_i \lambda_i^* \nabla_u g_i(x^*) + \sum_j \mu_j^* \nabla_u h_j(x^*) \ \perp \ x_u^* \geq 0$$

$$0 \leq -g_i(x^*) \ \perp \ \lambda_i^* \geq 0$$

$$0 = h_j(x^*) \quad , \quad \mu_j^* \ \text{(free)}$$

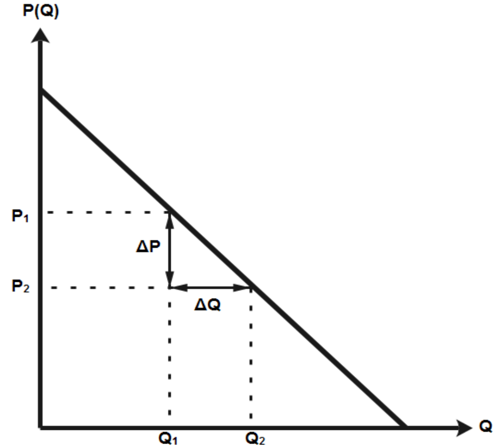
Appendix C: inverse demand function

The affine inverse demand function is commonly expressed in the following way:

$$P(Q) = a + b \cdot Q$$

where *P(Q)* represents the price of a good as a function of quantity demanded *(Q).* The constant *b* represents a slope of the function and the constant *a* is an intersection point with the vertical axis.

Inverse demand function is plotted on a coordinate system with the price on the vertical axis and quantity on the horizontal axis:



Appendix D: literature used

1. Huppmann, D., 2014. One and Two-level Energy Market Equilibrium Modelling

2. Bertsekas, D.P., 1999. Nonlinear programming.