Team Note of PS akgwi

Jeounghui Nah, Joowon Oh, Seongseo Lee

Compiled on February 27, 2024

Contents

1 DataStructure

1.1 Erasable Priority Queue

```cpp
template<class T=int, class O=less<T>>
struct pq_set {
  priority_queue<T, vector<T>, O> q, del;
  const T& top() const { return q.top(); }
  int size() const { return int(q.size()-del.size()); }
  bool empty() const { return !size(); }
  void insert(const T x) { q.push(x); flush(); }
  void pop() { q.pop(); flush(); }
  void erase(const T x) { del.push(x); flush(); }
  void flush() { while(del.size() && q.top()==del.top())
  q.pop(), del.pop(); }
};
```

1.2 Convex Hull Trick (Stack, LineContainer)

```cpp
struct Line{ // call init() before use
  ll a, b, c; // y = ax + b, c = line index
  Line(ll a, ll b, ll c) : a(a), b(b), c(c) {}
  ll f(ll x){ return a * x + b; }
};
vector<Line> v; int pv;
void init(){ v.clear(); pv = 0; }
int chk(const Line &a, const Line &b, const Line &c) const {
  return (__int128_t)(a.b - b.b) * (b.a - c.a) <=
  (__int128_t)(c.b - b.b) * (b.a - a.a);
}
void insert(Line l){
  if(v.size() > pv && v.back().a == l.a){ // fix if min query
    if(l.b < v.back().b) l = v.back();    v.pop_back();
  }
  while(v.size() >= pv+2 && chk(v[v.size()-2], v.back(), l))
  v.pop_back();
  v.push_back(l);
}
p query(ll x){ // if min query, then v[pv].f(x) >= v[pv+1].f(x)
  while(pv+1 < v.size() && v[pv].f(x) <= v[pv+1].f(x)) pv++;
  return {v[pv].f(x), v[pv].c};
}
///// line container start (max query) /////
struct Line {
  mutable ll k, m, p;
  bool operator<(const Line& o) const { return k < o.k; }
  bool operator<(ll x) const { return p < x; }
}; // (for doubles, use inf = 1/.0, div(a,b) = a/b)
struct LineContainer : multiset<Line, less<>> {
  static const ll inf = LLONG_MAX; // div: floor
```

```
  ll div(ll a, ll b) { return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) return x->p = inf, 0;
    if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
    else x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(ll k, ll m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p) isect(x,
    erase(y));
  } ll query(ll x) { assert(!empty());
    auto l = *lower_bound(x); return l.k * x + l.m; }
};
```

1.3 Color Processor

```
template<class CT, class T> struct color_processor {
  map<array<CT, 2>, T> v; // CT: coord type
  color_processor(T col={}): v({{{MIN,MAX},col}}){}
  auto get_range(CT p){ return *prev(v.upper_bound({p, MAX})); }
  // Cover [l, r) with color c, amortized O(1) process call
  // process(l, r, pc, c): color of [l, r) change pc -> c
  auto cover(CT l, CT r, T c, auto process){
    array<CT, 2> I{l, l};
    auto it = v.lower_bound(I);
    if(it != v.begin() && l < prev(it)->fi[1]){
      auto x = *--it; v.erase(it);
      v.insert({{x.fi[0],l}, x.se});
      it = v.insert({{l,x.fi[1]}, x.se}).fi;
    }
    while(it != v.end() && it->fi[0] < r){
      if(r < it->fi[1]){
        auto x = *it; v.erase(it);
        it = v.insert({{x.fi[0],r}, x.se}).fi;
        v.insert({{r,x.fi[1]}, x.se});
      }
      process(max(l,it->fi[0]), min(r,it->fi[1]), it->se, c);
      I = {min(I[0],it->fi[0]), max(I[1],it->fi[1])};
      it = v.erase(it);
    } return v.insert({I, c});
  }
  // new_color(l, r, pc): return new color for
  // [l, r) previous color pc O(Number of color ranges affected)
  void recolor(CT l, CT r, auto new_color){
    auto left = v.lower_bound({l, l});
    if(l < left->fi[0]){
      auto [range, c] = *--left; left = v.erase(left);
      left = v.insert(left, {{range[0],l},c});
      left = v.insert(left, {{l,range[1]},c});
    }
    auto right = v.lower_bound({r, r});
    if(r < right->fi[0]){
      auto [range, c] = *--right; right = v.erase(right);
      right = v.insert(right,{{range[0],r},c});
      right = v.insert(right,{{r,range[1]},c});
    }
    for(auto it=left; it!=right; ++it)
      it->se = new_color(it->fi[0], it->fi[1], it->se);
  }
};
```

1.4 Kinetic Segment Tree

```
// 일반적으로 heaten 함수는 교점 s개일 때 O(lambda_{s+2}(n)log^2n)
// update가 insert/delete만 주어지면 O(lambda_{s+1}(n)log^2n)
// update가 없으면 O(lambda_{s}(n)log^2n)
// s = 0: 1 | s = 1: n | s = 2: 2n-1 | s = 3: 2n alpha(n) + O(n)
// s = 4: O(n * 2^alpha(n)) | s = 5: O(n alpha(n) * 2^alpha(n))
// apply_heat(heat): x좌표가 heat 증가했을 때의 증가량을 v에 더함
// heaten(l, r, t): 구간의 온도를 t 만큼 증가
struct line_t{
  ll a, b, v, idx; line_t() : line_t(0, nINF) {}
  line_t(ll a, ll b) : line_t(a, b, -1) {}
  line_t(ll a, ll b, ll idx) : a(a), b(b), v(b), idx(idx) {}
  void apply_heat(ll heat){ v += a * heat; }
  void apply_add(ll lz_add){ v += lz_add; }
  ll cross(const line_t &l) const {
    if(a == l.a) return pINF; ll p = v - l.v, q = l.a - a;
    if(q < 0) p = -p, q = -q;
    return p >= 0 ? (p + q - 1) / q : -p / q * -1;
  } ll cross_after(const line_t &l, ll temp) const {
    ll res = cross(l); return res > temp ? res : pINF; }
};
struct range_kinetic_segment_tree{
  struct node_t{
    line_t v; ll melt, heat, lz_add; node_t():node_t(line_t()){}
    node_t(ll a, ll b, ll idx) : node_t(line_t(a, b, idx)) {}
    node_t(const line_t &v):v(v),melt(pINF),heat(0),lz_add(0){}
    bool operator < (const node_t &o) const { return
    tie(v.v,v.a) < tie(o.v.v,o.v.a); }
    ll cross_after(const node_t &o, ll temp) const { return
    v.cross_after(o.v, temp); }
    void apply_lazy(){ v.apply_heat(heat); v.apply_add(lz_add);
    melt -= heat; }
    void clear_lazy(){ heat = lz_add = 0; }
    void prop_lazy(const node_t &p){ heat += p.heat; lz_add +=
    p.lz_add; }
    bool have_lazy() const { return heat != 0 || lz_add != 0; }
  };
  node_t T[SZ<<1]; range_kinetic_segment_tree(){ clear(); }
  void clear(){ fill(T, T+SZ*2, node_t()); }
  void pull(int node, int s, int e){
    if(s == e) return;
    const node_t &l = T[node<<1], &r = T[node<<1|1];
assert(!l.have_lazy() && !r.have_lazy() &&
!T[node].have_lazy());
    T[node] = max(l, r);
    T[node].melt = min({ l.melt, r.melt, l.cross_after(r, 0) });
  }
  void push(int node, int s, int e){
    if(!T[node].have_lazy()) return; T[node].apply_lazy();
    if(s != e) for(auto c : {node<<1, node<<1|1})
    T[c].prop_lazy(T[node]);
    T[node].clear_lazy();
  }
  void build(const vector<line_t> &lines, int node=1, int s=0,
  int e=SZ-1){
    if(s == e){ T[node] = s < lines.size() ? node_t(lines[s]) :
    node_t(); return; }
    int m = (s + e) / 2;
    build(lines,node*2,s,m); build(lines,node*2+1,m+1,e);
    pull(node, s, e);
  }
  void update(int x, const line_t &v, int node=1, int s=0, int
  e=SZ-1){
    push(node, s, e); int m = (s + e) / 2;
    if(s == e){ T[node] = v; return; }
    if(x <= m)update(x,v, node<<1, s, m), push(node<<1|1, m+1,
    e);
    else update(x, v, node<<1|1, m+1, e), push(node<<1, s, m);
    pull(node, s, e);
  }
  void add(int l, int r, ll v, int node=1, int s=0, int e=SZ-1){
    push(node, s, e); int m = (s + e) / 2;
    if(r < s || e < l) return;
    if(l <= s && e <= r){ T[node].lz_add += v; push(node, s, e);
    return; }
    add(l,r,v,node*2,s,m); add(l,r,v,node*2+1,m+1,e);
    pull(node, s, e);
  }
  void heaten(int l,int r,ll t,int node=1,int s=0,int e=SZ-1){
    push(node, s, e); int m = (s + e) / 2;
    if(r < s || e < l) return;
    if(l <= s && e <= r){ _heat(t, node, s, e); return; }
    heaten(l,r,t,node*2,s,m); heaten(l,r,t,node*2+1,m+1,e);
    pull(node, s, e);
  }
  void _heat(ll t, int node=1, int s=0, int e=SZ-1){
    push(node, s, e); int m = (s + e) / 2;
    if(T[node].melt > t){ T[node].heat += t; push(node, s, e);
    return; }
    _heat(t,node*2,s,m);_heat(t,node*2+1,m+1,e);pull(node,s,e);
  }
  ll query(int l, int r, int node=1, int s=0, int e=SZ-1){
    push(node, s, e); if(r < s || e < l) return nINF;
    if(l <= s && e <= r) return T[node].v.v; int m = (s + e)/2;
return max(query(l,r,node<<1,s,m), query(l,r,node<<1|1,m+1,e));
  } // query end
};
```

1.5 Lazy LiChao Tree

```
/* get_point(x) : get min(f(x)), O(log X)
range_min(l,r) get min(f(x)), l<=x<=r, O(log X)
insert(l,r,a,b) : insert f(x)=ax+b, l<=x<=r, O(log^2 X)
add(l,r,a,b) : add f(x)=ax+b, l<=x<=r, O(log^2 X)
WARNING: a != 0인 add가 없을 때만 range_min 가능 */
template<typename T, T LE, T RI, T INF=(long long)(4e18)>
struct LiChao{
  struct Node{
    int l, r; T a, b, mn, aa, bb;
    Node(){ l = r = 0; a = 0; b = mn = INF; aa = bb = 0; }
    void apply(){ mn += bb; a += aa; b += bb; aa = bb = 0; }
    void add_lazy(T _aa, T _bb){ aa += _aa; bb += _bb; }
    T f(T x) const { return a * x + b; }
  }; vector<Node> seg; LiChao() : seg(2) {}
  void make_child(int n){
    if(!seg[n].l) seg[n].l = seg.size(), seg.emplace_back();
    if(!seg[n].r) seg[n].r = seg.size(), seg.emplace_back();
  }
  void push(int node, T s, T e){
    if(seg[node].aa || seg[node].bb){
      if(s != e){
```

```
      make_child(node);
      seg[seg[node].l].add_lazy(seg[node].aa, seg[node].bb);
      seg[seg[node].r].add_lazy(seg[node].aa, seg[node].bb);
    } seg[node].apply();
  }
}
void insert(T l, T r, T a, T b, int node=1, T s=LE, T e=RI){
  if(r < s || e < l || l > r) return;
  make_child(node); push(node, s, e); T m = (s + e) >> 1;
  seg[node].mn=min({seg[node].mn, a*max(s,l)+b,a*min(e,r)+b});
  if(s < l || r < e){
    if(l <= m) insert(l, r, a, b, seg[node].l, s, m);
    if(m+1 <= r) insert(l, r, a, b, seg[node].r, m+1, e);
    return;
  }
  T &sa = seg[node].a, &sb = seg[node].b;
  if(a*s+b < sa*s+sb) swap(a, sa), swap(b, sb);
  if(a*e+b >= sa*e+sb) return;
  if(a*m+b < sa*m+sb){
    swap(a,sa); swap(b,sb); insert(l,r,a,b,seg[node].l,s,m);
  } else insert(l, r, a, b, seg[node].r, m+1, e);
}
void add(T l, T r, T a, T b, int node=1, T s=LE, T e=RI){
  if(r < s || e < l || l > r) return;
  make_child(node); push(node, s, e); T m = (s + e) >> 1;
  if(s < l || r < e){
    insert(s, m, seg[node].a, seg[node].b, seg[node].l, s, m);
    insert(m+1,e,seg[node].a,seg[node].b,seg[node].r,m+1,e);
    seg[node].a = 0; seg[node].b = seg[node].mn = INF;
    if(l <= m) add(l, r, a, b, seg[node].l, s, m);
    if(m+1 <= r) add(l, r, a, b, seg[node].r, m+1, e);
    seg[node].mn=min(seg[seg[node].l].mn,
    seg[seg[node].r].mn);
    return;
  } seg[node].add_lazy(a, b); push(node, s, e);
}
T get_point(T x, int node=1, T s=LE, T e=RI){
  if(node == 0) return INF; push(node, s, e);
  T m = (s + e) >> 1, res = seg[node].f(x);
  if(x <= m) return min(res, get_point(x, seg[node].l, s, m));
  else return min(res, get_point(x, seg[node].r, m+1, e));
}
T range_min(T l, T r, int node=1, T s=LE, T e=RI){
  if(node == 0 || r < s || e < l || l > r) return INF;
  push(node, s, e); T m = (s + e) >> 1;
  if(l <= s && e <= r) return seg[node].mn;
  return min({ seg[node].f(max(s,l)), seg[node].f(min(e,r)),
     range_min(l, r, seg[node].l, s, m),
     range_min(l, r, seg[node].r, m+1, e) });
}
};
```

1.6 Splay Tree, Link-Cut Tree

```
struct Node{
  Node *l, *r, *p; bool flip; int sz; T now, sum, lz;
  Node(){ l = r = p = nullptr; sz = 1; flip = false; now = sum =
  lz = 0; }
  bool IsLeft() const { return p && this == p->l; }
  bool IsRoot() const { return !p || (this != p->l && this !=
  p->r); }
  friend int GetSize(const Node *x){ return x ? x->sz : 0; }
  friend T GetSum(const Node *x){ return x ? x->sum : 0; }
  void Rotate(){
    p->Push(); Push();
    if(IsLeft()) r && (r->p = p), p->l = r, r = p;
    else l && (l->p = p), p->r = l, l = p;
    if(!p->IsRoot()) (p->IsLeft() ? p->p->l : p->p->r) = this;
    auto t = p; p = t->p; t->p = this; t->Update(); Update();
  }
  void Update(){
    sz = 1 + GetSize(l) + GetSize(r); sum = now + GetSum(l) +
    GetSum(r);
  }
  void Update(const T &val){ now = val; Update(); }
  void Push(){
    Update(now + lz); if(flip) swap(l, r);
    for(auto c : {l, r}) if(c) c->flip ^= flip, c->lz += lz;
    lz = 0; flip = false;
  }
};
Node* rt;
Node* Splay(Node *x, Node *g=nullptr){
  for(g || (rt=x); x->p!=g; x->Rotate()){
    if(!x->p->IsRoot()) x->p->p->Push();
    x->p->Push(); x->Push();
    if(x->p->p != g) (x->IsLeft() ^ x->p->IsLeft() ? x :
    x->p)->Rotate();
  }
  x->Push(); return x;
}
Node* Kth(int k){
  for(auto x=rt; ; x=x->r){
    for(; x->Push(), x->l && x->l->sz > k; x=x->l);
    if(x->l) k -= x->l->sz;
    if(!k--) return Splay(x);
  }
}
Node* Gather(int s, int e){ auto t = Kth(e+1); return Splay(t,
Kth(s-1))->l; }
Node* Flip(int s, int e){ auto x = Gather(s, e); x->flip ^= 1;
return x; }
Node* Shift(int s, int e, int k){
  if(k >= 0){ // shift to right
    k %= e-s+1; if(k) Flip(s, e), Flip(s, s+k-1), Flip(s+k, e);
  }
  else{ // shift to left
    k = -k; k %= e-s+1; if(k) Flip(s, e), Flip(s, e-k),
    Flip(e-k+1, e);
  }
  return Gather(s, e);
}
int Idx(Node *x){ return x->l->sz; }
/////////// Link Cut Tree Start ///////////
Node* Splay(Node *x){
  for(; !x->IsRoot(); x->Rotate()){
    if(!x->p->IsRoot()) x->p->p->Push();
    x->p->Push(); x->Push();
    if(!x->p->IsRoot()) (x->IsLeft() ^ x->p->IsLeft() ? x :
    x->p)->Rotate();
  }
  x->Push(); return x;
}
void Access(Node *x){
  Splay(x); x->r = nullptr; x->Update();
  for(auto y=x; x->p; Splay(x)) y = x->p, Splay(y), y->r = x,
  y->Update();
}
int GetDepth(Node *x){Access(x);x->Push();return GetSize(x->l);}
Node* GetRoot(Node *x){
  Access(x);for(x->Push();x->l;x->Push())x=x->l;return Splay(x);
}
Node* GetPar(Node *x){
  Access(x); x->Push(); if(!x->l) return nullptr;
  x = x->l; for(x->Push(); x->r; x->Push()) x = x->r;
  return Splay(x);
}
void Link(Node *p, Node *c){ Access(c); Access(p); c->l = p;
p->p = c; c->Update(); }
void Cut(Node *c){ Access(c); c->l->p = nullptr; c->l = nullptr;
c->Update(); }
Node* GetLCA(Node *x, Node *y){
  Access(x); Access(y); Splay(x); return x->p ? x->p : x;
}
Node* Ancestor(Node *x, int k){
  k = GetDepth(x) - k; assert(k >= 0);
  for(;;x->Push()){
    int s = GetSize(x->l); if(s == k) return Access(x), x;
    if(s < k) k -= s + 1, x = x->r; else x = x->l;
  }
}
void MakeRoot(Node *x){ Access(x); Splay(x); x->flip ^= 1;
x->Push(); }
bool IsConnect(Node *x, Node *y){return GetRoot(x)==GetRoot(y);}
void PathUpdate(Node *x, Node *y, T val){
  Node *root = GetRoot(x); // original root
  MakeRoot(x); Access(y);  // make x to root, tie with y
  Splay(x); x->lz += val; x->Push();
  MakeRoot(root);    // Revert
  // edge update without edge vertex...
  Node *lca = GetLCA(x, y);
  Access(lca); Splay(lca); lca->Push();
  lca->Update(lca->now - val);
}
T VertexQuery(Node *x, Node *y){
  Node *l = GetLCA(x, y); T ret = l->now;
  Access(x); Splay(l); if(l->r) ret = ret + l->r->sum;
  Access(y); Splay(l); if(l->r) ret = ret + l->r->sum;
  return ret;
}
Node* GetQueryResultNode(Node *u, Node *v){
  if(!IsConnect(u, v)) return 0;
  MakeRoot(u); Access(v); auto ret = v->l;
  while(ret->mx != ret->now){
    if (ret->l && ret->mx == ret->l->mx) ret = ret->l;
    else ret = ret->r;
  }
  Access(ret); return ret;
}
```

2 Geometry

2.1 $O(\log N)$ Point in Convex Polygon

```
bool Check(const vector<Point> &v, const Point &pt){
  if(CCW(v[0], v[1], pt) < 0) return false;
```

```
  int l = 1, r = v.size() - 1;
  while(l < r){
    int m = l + r + 1 >> 1;
    if(CCW(v[0], v[m], pt) >= 0) l = m; else r = m - 1;
  }
  if(l == v.size() - 1) return CCW(v[0], v.back(), pt) == 0 &&
  v[0] <= pt && pt <= v.back();
  return CCW(v[0], v[l], pt) >= 0 && CCW(v[l], v[l+1], pt) >= 0
  && CCW(v[l+1], v[0], pt) >= 0;
}
```

2.2 Segment Distance, Segment Reflect

```
double Proj(Point a, Point b, Point c){
  ll t1 = (b - a) * (c - a), t2 = (a - b) * (c - b);
  if(t1 * t2 >= 0 && CCW(a, b, c) != 0)
    return abs(CCW(a, b, c)) / sqrt(Dist(a, b));
  else return 1e18; // INF
}
double SegmentDist(Point a[2], Point b[2]){
  double res = 1e18; // NOTE: need to check intersect
  for(int i=0; i<4; i++) res=min(res,sqrt(Dist(a[i/2],b[i%2])));
  for(int i=0; i<2; i++) res = min(res, Proj(a[0], a[1], b[i]));
  for(int i=0; i<2; i++) res = min(res, Proj(b[0], b[1], a[i]));
  return res;
}
P Reflect(P p1, P p2, P p3){ // line p1-p2, point p3
  auto [x1,y1] = p1; auto [x2,y2] = p2; auto [x3,y3] = p3;
  auto a = y1-y2, b = x2-x1, c = x1 * (y2-y1) + y1 * (x1-x2);
  auto d = a * y3 - b * x3;
  T x = -(a*c+b*d) / (a*a+b*b), y = (a*d-b*c) / (a*a+b*b);
  return 2 * P(x,y) - p3;
}
```

2.3 Tangent Series

```
template<bool UPPER=true> // O(log N)
Point GetPoint(const vector<Point> &hull, real_t slope){
    auto chk = [slope](real_t dx, real_t dy){ return UPPER ? dy
    >= slope * dx : dy <= slope * dx; };
    int l = -1, r = hull.size() - 1;
    while(l + 1 < r){
        int m = (l + r) / 2;
        if(chk(hull[m+1].x - hull[m].x, hull[m+1].y -
        hull[m].y)) l = m; else r = m;
    }
    return hull[r];
}
int ConvexTangent(const vector<Point> &v, const Point &pt, int
up=1){ //given outer point, O(log N)
  auto sign = [&](ll c){ return c>0 ? up : c==0 ? 0 : -up; };
  auto local = [&](Point p, Point a, Point b, Point c){
    return sign(CCW(p, a, b)) <= 0 && sign(CCW(p, b, c)) >= 0;
  }; // assert(v.size() >= 2);
  int n = v.size() - 1, s = 0, e = n, m;
  if(local(pt, v[1], v[0], v[n-1])) return 0;
  while(s + 1 < e){
    m = (s + e) / 2;
    if(local(pt, v[m-1], v[m], v[m+1])) return m;
    if(sign(CCW(pt, v[s], v[s+1])) < 0){ // up
      if(sign(CCW(pt, v[m], v[m+1])) > 0) e = m;
      else if(sign(CCW(pt, v[m], v[s])) > 0) s = m; else e = m;
    }
    else{ // down
      if(sign(CCW(pt, v[m], v[m+1])) < 0) s = m;
      else if(sign(CCW(pt, v[m], v[s])) < 0) s = m; else e = m;
    }
  }
  if(s && local(pt, v[s-1], v[s], v[s+1])) return s;
  if(e != n && local(pt, v[e-1], v[e], v[e+1])) return e;
  return -1;
}
int Closest(const vector<Point> &v, const Point &out, int now){
  int prv = now > 0 ? now-1 : v.size()-1, nxt = now+1 < v.size()
  ? now+1 : 0, res = now;
  if(CCW(out, v[now], v[prv]) == 0 && Dist(out, v[res]) >
  Dist(out, v[prv])) res = prv;
  if(CCW(out, v[now], v[nxt]) == 0 && Dist(out, v[res]) >
  Dist(out, v[nxt])) res = nxt;
  return res; // if parallel, return closest point to out
} // int point_idx =  Closest(convex_hull, pt,
ConvexTangent(hull + hull[0], pt, +-1) % N);
/////////
double polar(pdd x){ return atan2(x.second, x.first); }
int tangent(circle &A, circle &B, pdd des[4]){ // return angle
  int top = 0; // outer
  double d = size(A.O - B.O), a = polar(B.O - A.O), b = PI + a;
  double t = sq(d) - sq(A.r - B.r);
  if (t >= 0){
    t = sqrt(t); double p = atan2(B.r - A.r, t);
    des[top++] = pdd(a + p + PI / 2, b + p - PI / 2);
    des[top++] = pdd(a - p - PI / 2, b - p + PI / 2);
  }
  t = sq(d) - sq(A.r + B.r); // inner
  if (t >= 0){ t = sqrt(t);
    double p = atan2(B.r + A.r, t);
    des[top++] = pdd(a + p - PI / 2, b + p - PI / 2);
    des[top++] = pdd(a - p + PI / 2, b - p + PI / 2);
  }
  return top;
}
pair<T, T> CirclePointTangent(P o, double r, P p){
  T op=D1(p,o), u=atan2l(p.y-o.y, p.x-o.x), v=acosl(r/op);
  return {u + v, u - v};
} // COORD 1e4 EPS 1e-7 / COORD 1e3 EPS 1e-9 with circleLine
```

2.4 Intersect Series

```
// 0: not intersect, -1: infinity, 4: intersect
// 1/2/3: intersect first/second/both segment corner
// flag, xp, xq, yp, yq : (xp / xq, yp / yq)
using T = __int128_t; // T <= O(COORD^3)
tuple<int,T,T,T,T> SegmentIntersect(P s1, P e1, P s2, P e2){
  if(!Intersect(s1, e1, s2, e2)) return {0, 0, 0, 0, 0};
  auto det = (e1 - s1) / (e2 - s2);
  if(!det){
    if(s1 > e1) swap(s1, e1);
    if(s2 > e2) swap(s2, e2);
    if(e1 == s2) return {3, e1.x, 1, e1.y, 1};
    if(e2 == s1) return {3, e2.x, 1, e2.y, 1};
    return {-1, 0, 0, 0, 0};
  }
  T p = (s2 - s1) / (e2 - s2), q = det, flag = 0;
  T xp = s1.x * q + (e1.x - s1.x) * p, xq = q;
  T yp = s1.y * q + (e1.y - s1.y) * p, yq = q;
  if(xp%xq || yp%yq) return {4,xp,xq,yp,yq};//gcd?
  //if(xq < 0) xp=-xp, xq=-xq; if(yq < 0) yp=-yp, yq=-yq //gcd?
  xp /= xq; yp /= yq;
  if(s1.x == xp && s1.y == yp) flag |= 1;
  if(e1.x == xp && e1.y == yp) flag |= 1;
  if(s2.x == xp && s2.y == yp) flag |= 2;
  if(e2.x == xp && e2.y == yp) flag |= 2;
  return {flag ? flag : 4, xp, 1, yp, 1};
}
P perp() const { return P(-y, x); }
#define arg(p, q) atan2(p.cross(q), p.dot(q))
bool circleIntersect(P a,P b,double r1,double r2,pair<P, P>*
out){
  if (a == b) { assert(r1 != r2); return false; }
  P vec = b-a; double d2 = vec.dist2(), sum = r1+r2, dif =
  r1-r2;
  double p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
  if (sum*sum < d2 || dif*dif > d2) return false; // use EPS
  plz...
  P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
  *out = {mid + per, mid - per}; return true;
}
vector<P> circleLine(P c, double r, P a, P b) {
    P ab = b - a, p = a + ab * (c-a) * ab / D2(ab);
    T s = (b - a) / (c - a), h2 = r*r - s*s / D2(ab);
    if (abs(h2) < EPS) return {p}; if (h2 < 0) return {};
    P h = ab / D1(ab) * sqrtl(h2); return {p - h, p + h};
} // use circleLine if you use double...
int CircleLineIntersect(P o, T r, P p1, P p2, bool segment){
  P s = p1, d = p2 - p1; // line : s + kd, int support
  T a = d * d, b = (s - o) * d * 2, c = D2(s, o) - r * r;
  T det = b * b - 4 * a * c; // solve ak^2 + bk + c = 0, a > 0
  if(!segment) return Sign(det) + 1;
  if(det <= 0) return det ? 0 : 0 <= -b && -b <= a + a;
  bool f11 = b <= 0 || b * b <= det;
  bool f21 = b <= 0 && b * b >= det;
  bool f12 = a+a+b >= 0 && det <= (a+a+b) * (a+a+b);
  bool f22 = a+a+b >= 0 || det >= (a+a+b) * (a+a+b);
  return (f11 && f12) + (f21 && f22);
} // do not use this if you want to use double...
double circlePoly(P c, double r, vector<P> ps){ // return area
  auto tri = [&](P p, P q) { // ps must be ccw polygon
    auto r2 = r * r / 2; P d = q - p;
    auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
    auto det = a * a - b;
    if (det <= 0) return arg(p, q) * r2;
    auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
    if (t < 0 || 1 <= s) return arg(p, q) * r2;
    P u = p + d * s, v = p + d * t;
    return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
  };
  auto sum = 0.0;
  rep(i,0,sz(ps)) sum += tri(ps[i] - c, ps[(i+1)%sz(ps)] - c);
  return sum;
}
// extrVertex: point of hull, max projection onto line
#define cmp(i,j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
int extrVertex(vector<P>& poly, P dir) {
  int n = sz(poly), lo = 0, hi = n;
  if (extr(0)) return 0;
```

```
  while (lo + 1 < hi) {
    int m = (lo + hi) / 2; if (extr(m)) return m;
    int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
    (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
  }
  return lo;
}
//(-1,-1): no collision
//(i,-1): touch corner
//(i,i): along side (i,i+1)
//(i,j): cross (i,i+1)and(j,j+1)
//(i,i+1): cross corner i
// O(log n), ccw no colinear point convex polygon
// P perp() const { return P(-y, x); }
#define cmpL(i) sgn(a.cross(poly[i], b))
array<int, 2> lineHull(P a, P b, vector<P>& poly) { // O(log N)
  int endA = extrVertex(poly, (a - b).perp());
  int endB = extrVertex(poly, (b - a).perp());
  if (cmpL(endA) < 0 || cmpL(endB) > 0) return {-1, -1};
  array<int, 2> res;
  rep(i,0,2) {
    int lo = endB, hi = endA, n = sz(poly);
    while ((lo + 1) % n != hi) {
      int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
      (cmpL(m) == cmpL(endB) ? lo : hi) = m;
    }
    res[i] = (lo + !cmpL(hi)) % n;
    swap(endA, endB);
  }
  if (res[0] == res[1]) return {res[0], -1};
  if (!cmpL(res[0]) && !cmpL(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
      case 0: return {res[0], res[0]};
      case 2: return {res[1], res[1]};
    }
  return res;
}
```

2.5 $O(N^2 \log N)$ Union of Circles Area

```
ld TwoCircleUnion(const Circle &p, const Circle &q) {
  ld d = D1(p.o - q.o); if(d >= p.r+q.r-EPS) return 0;
  else if(d <= abs(p.r-q.r)+EPS) return pow(min(p.r,q.r),2) *
  PI;
  ld pc = (p.r*p.r + d*d - q.r*q.r) / (p.r*d*2), pa = acosl(pc);
  ld qc = (q.r*q.r + d*d - p.r*p.r) / (q.r*d*2), qa = acosl(qc);
  ld ps = p.r*p.r*pa - p.r*p.r*sin(pa*2)/2;
  ld qs = q.r*q.r*qa - q.r*q.r*sin(qa*2)/2;
  return ps + qs;
}
vector<pair<double, double>> CoverSegment(Cir a, Cir b) {
  double d = abs(a.o - b.o); vector<pair<double, double>> res;
  if(sign(a.r + b.r - d) == 0); /* skip */
  else if(d <= abs(a.r - b.r) + eps) {
    if (a.r < b.r) res.emplace_back(0, 2 * pi);
  } else if(d < abs(a.r + b.r) - eps) {
    double o = acos((a.r*a.r + d*d - b.r*b.r) / (2 * a.r * d));
    double z = norm(atan2((b.o - a.o).y, (b.o - a.o).x));
    double l = norm(z - o), r = norm(z + o);
    if(l > r) res.emplace_back(l, 2*pi), res.emplace_back(0,r);
    else res.emplace_back(l, r);
  } return res;
}// circle should be identical
double CircleUnionArea(vector<Cir> c) {
  int n = c.size(); double a = 0, w;
  for (int i = 0; w = 0, i < n; ++i) {
    vector<pair<double, double>> s = {{2 * pi, 9}}, z;
    for (int j = 0; j < n; ++j) if (i != j) {
      z = CoverSegment(c[i], c[j]);
      for (auto &e : z) s.push_back(e); } /* for j */
    sort(s.begin(), s.end());
    auto F = [&] (double t) { return c[i].r * (c[i].r * t +
    c[i].o.x * sin(t) - c[i].o.y * cos(t)); };
    for (auto &e : s) {
      if (e.first > w) a += F(e.first) - F(w);
      w = max(w, e.second); } /* for e */
  } return a * 0.5; }
```

2.6 Segment In Polygon

```
bool segment_in_polygon_non_strict(const vector<Point> &poly,
Point s, Point e){
  if(!PointInPolygon(poly, s, false) || !PointInPolygon(poly, e,
  false)) return false;
  if(s == e) return true; int cnt[4] = {0}; // no, at least one
  corner, mid, inf
  for(int j=(int)poly.size()-1, i=0; i<poly.size(); j=i++){
    int flag = get<0>(SegmentIntersect(poly[i], poly[j], s, e));
    if(flag<=0) flag = flag?3:0; else flag = max(1, flag-2);
    cnt[flag] += 1;
  }
  if(cnt[2] != 0 || cnt[3] > 1) return false;
  if((cnt[3] == 1 || cnt[1] != 0) && !PointInPolygon(poly, (s +
  e) / 2, false)) return false;
  return true;
}
```

2.7 Polygon Cut, Center, Union

```
// Returns the polygon on the left of line l
// *: dot product, ^: cross product
// l = p + d*t, l.q() = l + d
// doubled_signed_area(p,q,r) = (q-p) ^ (r-p)
template<class T> vector<point<T>> polygon_cut(const
vector<point<T>> &a, const line<T> &l){
  vector<point<T>> res;
  for(auto i = 0; i < (int)a.size(); ++ i){
    auto cur = a[i], prev = i ? a[i - 1] : a.back();
    bool side = doubled_signed_area(l.p, l.q(), cur) > 0;
    if(side != (doubled_signed_area(l.p, l.q(), prev) > 0))
      res.push_back(l.p + (cur - l.p ^ prev - cur) / (l.d ^ prev
      - cur) * l.d);
    if(side) res.push_back(cur);
  }
  return res;
}
P polygonCenter(const vector<P>& v){ // center of mass
  P res(0, 0); double A = 0;
  for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
    res = res + (v[i] + v[j]) * v[j].cross(v[i]);
    A += v[j].cross(v[i]);
  } return res / A / 3;
}
// O(points^2), area of union of n polygon, ccw polygon
int sideOf(P s, P e, P p) { return sgn((e-s)/(p-s)); }
int sideOf(const P& s, const P& e, const P& p, double eps) {
  auto a = (e-s)/(p-s); auto l=D1(e-s) * eps;
  return (a > l) - (a < -l);
}
double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y; }
double polyUnion(vector<vector<P>>& poly) { // (points)^2
  double ret = 0;
  rep(i,0,sz(poly)) rep(v,0,sz(poly[i])) {
    P A = poly[i][v], B = poly[i][(v + 1) % sz(poly[i])];
    vector<pair<double, int>> segs = {{0, 0}, {1, 0}};
    rep(j,0,sz(poly)) if (i != j) { // START
      rep(u,0,sz(poly[j])) {
        P C = poly[j][u], D = poly[j][(u + 1) % sz(poly[j])];
        int sc = sideOf(A, B, C), sd = sideOf(A, B, D);
        if (sc != sd) {
          double sa = C.cross(D, A), sb = C.cross(D, B);
          if (min(sc, sd) < 0)
            segs.emplace_back(sa / (sa - sb), sgn(sc - sd));
        }
        else if (!sc && !sd && j<i && sgn((B-A).dot(D-C))>0){
          segs.emplace_back(rat(C - A, B - A), 1);
          segs.emplace_back(rat(D - A, B - A), -1);
    } /*else if*/ } /*rep u*/ } /*rep j*/ // END
    sort(all(segs));
    for (auto& s : segs) s.first = min(max(s.first, 0.0), 1.0);
    double sum = 0; int cnt = segs[0].second;
    rep(j,1,sz(segs)) {
      if (!cnt) sum += segs[j].first - segs[j - 1].first;
      cnt += segs[j].second;
    }
    ret += A.cross(B) * sum;
  } return abs(ret) / 2;
}
```

2.8 Polygon Raycast

```
// ray A + kd and CCW polygon C, return events {k, event_id}
// 0: out->line / 1: in->line / 2: line->out / 3: line->in
// 4: pass corner outside / 5: pass corner inside / 6: out -> in
/ 7: in -> out
// WARNING: C.push_back(C[0]) before working
struct frac{
  ll first, second; frac(){}
  frac(ll a, ll b) : first(a), second(b) {
    if( b < 0 ) first = -a, second = -b; // operator cast int128
  } double v(){ return 1.*first/second; } // operator <,<=,==
};
frac raypoints(vector<pii> &C, pii A, pii d, vector<pair<frac,
int>> &R){
  assert(d != pii(0, 0));
  int g = gcd(abs(d.first), abs(d.second));
  d.first /= g, d.second /= g;
  vector<pair<frac, int>> L;
  for(int i = 0; i+1 < C.size(); i++){
    pii v = C[i+1] - C[i];
    int a = sign(d/(C[i]-A)), b = sign(d/(C[i+1]-A));
    if(a == 0)L.emplace_back(frac(d*(C[i]-A)/size2(d), 1), b);
    if(b == 0)L.emplace_back(frac(d*(C[i+1]-A)/size2(d), 1),a);
    if(a*b == -1) L.emplace_back(frac((A-C[i])/v, v/d), 6);
  }
  sort(L.begin(), L.end());
  int sz = 0;
```

```
  for(int i = 0; i < L.size(); i++){
    // assert(i+2 >= L.size() || !(L[i].first == L[i+2].first));
    if(i+1<L.size()&&L[i].first==L[i+1].first&&L[i].second!=6){
      int a = L[i].second, b = L[i+1].second;
      R.emplace_back(L[i++].first, a*b? a*b > 0? 4:6:(1-a-b)/2);
    }
    else R.push_back(L[i]);
  }
  int state = 0; // 0: out, 1: in, 2: line+ccw, 3: line+cw
  for(auto &e : R){
    int &n = e.second;
    if( n == 6 ) n ^= state, state ^= 1;
    else if( n == 4 ) n ^= state;
    else if( n == 0 ) n = state, state ^= 2;
    else if( n == 1 ) n = state^(state>>1), state ^= 3;
  } return frac(g, 1);
}
bool visible(vector<pii> &C, pii A, pii B){
  if( A == B ) return true;
  char I[4] = "356", O[4] = "157";
  vector<pair<frac, int>> R; vector<frac> E;
  frac s = frac(0, 1), e = raypoints(C, A, B-A, R);
  for(auto e : R){
    int &n = e.second, m;
    if(*find(O, O+3, n+'0')) E.emplace_back(e.first);
    if(*find(I, I+3, n+'0')) E.emplace_back(e.first);
  }
  for(int j = 0; j < E.size(); j += 2) if( !(e <= E[j] || E[j+1]
  <= s) ) return false;
  return true;
}
```

2.9 $O(N \log N)$ Shamos-Hoey

```
struct Line{
  static ll CUR_X; ll x1, y1, x2, y2, id;
  Line(Point p1, Point p2, int id) : id(id) {
    if(p1 > p2) swap(p1, p2);
    tie(x1,y1) = p1; tie(x2,y2) = p2;
  } Line() = default;
  int get_k() const { return y1 != y2 ? (x2-x1)/(y1-y2) : -1; }
  void convert_k(int k){ // x1,y1,x2,y2 = O(COORD^2), use i128
  in ccw
    Line res;
    res.x1 = x1 + y1 * k; res.y1 = -x1 * k + y1;
    res.x2 = x2 + y2 * k; res.y2 = -x2 * k + y2;
    x1 = res.x1; y1 = res.y1; x2 = res.x2; y2 = res.y2;
    if(x1 > x2) swap(x1, x2), swap(y1, y2);
  }
  ld get_y(ll offset=0) const { // OVERFLOW
    ld t = ld(CUR_X-x1+offset) / (x2-x1);
    return t * (y2 - y1) + y1; }
  bool operator < (const Line &l) const {
    return get_y() < l.get_y(); }
  // strict
  /* bool operator < (const Line &l) const {
    auto le = get_y(), ri = l.get_y();
    if(abs(le-ri) > 1e-7) return le < ri;
    if(CUR_X==x1 || CUR_X==l.x1) return get_y(1)<l.get_y(1);
    else return get_y(-1) < l.get_y(-1);
  } */
}; ll Line::CUR_X = 0;
struct Event{ // f=0 st, f=1 ed
  ll x, y, i, f; Event() = default;
  Event(Line l, ll i, ll f) : i(i), f(f) {
    if(f==0) tie(x,y) = tie(l.x1,l.y1);
    else tie(x,y) = tie(l.x2,l.y2);
  }
  bool operator < (const Event &e) const {
    return tie(x,f,y) < tie(e.x,e.f,e.y);
    // strict
    // return make_tuple(x,-f,y) < make_tuple(e.x,-e.f,e.y);
  }
};
tuple<bool,int,int> ShamosHoey(vector<array<Point,2>> v){
  int n = v.size(); vector<int> use(n+1);
  vector<Line> lines; vector<Event> E; multiset<Line> T;
  for(int i=0; i<n; i++){
    lines.emplace_back(v[i][0], v[i][1], i);
    if(int t=lines[i].get_k(); 0<=t && t<=n) use[t] = 1;
  }
  int k = find(use.begin(), use.end(), 0) - use.begin();
  for(int i=0; i<n; i++){ lines[i].convert_k(k);
    E.emplace_back(lines[i], i, 0); E.emplace_back(lines[i], i,
    1);
  } sort(E.begin(), E.end());
  for(auto &e : E){ Line::CUR_X = e.x;
    if(e.f == 0){
      auto it = T.insert(lines[e.i]);
      if(next(it) != T.end() && Intersect(lines[e.i],
      *next(it))) return {true, e.i, next(it)->id};
      if(it != T.begin() && Intersect(lines[e.i], *prev(it)))
        return {true, e.i, prev(it)->id};
    }
    else{
      auto it = T.lower_bound(lines[e.i]);
      if(it != T.begin() && next(it) != T.end() &&
      Intersect(*prev(it), *next(it)))
        return {true, prev(it)->id, next(it)->id};
      T.erase(it);
    }
  }
  return {false, -1, -1};
}
```

2.10 $O(N \log N)$ Half Plane Intersection

```
double CCW(p1, p2, p3); bool same(double a, double b); const
Point o = Point(0, 0);
struct Line{ // ax+by leq c
  double a, b, c; Line() : Line(0, 0, 0) {}
  Line(double a, double b, double c) : a(a), b(b), c(c) {}
  bool operator < (const Line &l) const {
    bool f1 = Point(a, b) > o, f2 = Point(l.a, l.b) > o;
    if(f1 != f2) return f1 > f2;
    double cw = CCW(o, Point(a, b), Point(l.a, l.b));
    return same(cw, 0) ? c * hypot(l.a, l.b) < l.c * hypot(a, b)
    : cw > 0;
  } Point slope() const { return Point(a, b); }
};
Point LineIntersect(Line a, Line b){
  double det = a.a*b.b - b.a*a.b, x = (a.c*b.b - a.b*b.c) / det,
  y = (a.a*b.c - a.c*b.a) / det; return Point(x, y);
}
bool CheckHPI(Line a, Line b, Line c){
  if(CCW(o, a.slope(), b.slope()) <= 0) return 0;
  Point v=LineIntersect(a,b); return v.x*c.a+v.y*c.b>=c.c;
}
vector<Point> HPI(vector<Line> v){
  sort(v.begin(), v.end()); deque<Line> dq; vector<Point> ret;
  for(auto &i : v){
    if(dq.size() && same(CCW(o, dq.back().slope(), i.slope()),
    0)) continue;
    while(dq.size() >= 2 && CheckHPI(dq[dq.size()-2], dq.back(),
    i)) dq.pop_back();
    while(dq.size()>=2&&CheckHPI(i,dq[0],dq[1]))dq.pop_front();
    dq.push_back(i);
  }
  while(dq.size() > 2 && CheckHPI(dq[dq.size()-2], dq.back(),
  dq[0])) dq.pop_back();
  while(dq.size() > 2 && CheckHPI(dq.back(), dq[0], dq[1]))
  dq.pop_front();
  for(int i=0; i<dq.size(); i++){
    Line now = dq[i], nxt = dq[(i+1)%dq.size()];
    if(CCW(o, now.slope(), nxt.slope()) <= eps) return
    vector<Point>();
    ret.push_back(LineIntersect(now, nxt));
  } //for(auto &[x,y] : ret) x = -x, y = -y;
  return ret;
} // MakeLine: left side of ray (x1,y1) -> (x2,y2)
Line MakeLine(T x1, T y1, T x2, T y2){
  T a = y2-y1, b = x1-x2, c = x1*a + y1*b; return {a,b,c}; }
```

2.11 $O(M \log M)$ Dual Graph

```
constexpr int quadrant_id(const Point p){
  constexpr int arr[9] = { 5, 4, 3, 6, -1, 2, 7, 0, 1 };
  return arr[sign(p.x)*3+sign(p.y)+4];
}
pair<vector<int>, int> dual_graph(const vector<Point> &points,
const vector<pair<int,int>> &edges){
  int n = points.size(), m = edges.size();
  vector<int> uf(2*m); iota(uf.begin(), uf.end(), 0);
  function<int(int)> find = [&](int v){ return v == uf[v] ? v :
  uf[v] = find(uf[v]); };
  function<bool(int,int)> merge = [&](int u, int v){ return
  find(u) != find(v) && (uf[uf[u]]=uf[v], true); };
  vector<vector<pair<int,int>>> g(n);
  for(int i=0; i<m; i++){
    g[edges[i].first].emplace_back(edges[i].second, i);
    g[edges[i].second].emplace_back(edges[i].first, i);
  }
  for(int i=0; i<n; i++){
    const auto base = points[i];
    sort(g[i].begin(), g[i].end(), [&](auto a, auto b){
      auto p1=points[a.first]-base, p2=points[b.first]-base;
      return quadrant_id(p1) != quadrant_id(p2) ?
      quadrant_id(p1) < quadrant_id(p2) : p1.cross(p2) > 0;
    });
    for(int j=0; j<g[i].size(); j++){
      int k = j ? j - 1 : g[i].size() - 1;
      int u = g[i][k].second << 1, v = g[i][j].second << 1 | 1;
      auto p1=points[g[i][k].first], p2=points[g[i][j].first];
      if(p1 < base) u ^= 1; if(p2 < base) v ^= 1;
      merge(u, v);
    }
```

```
  }
  vector<int> res(2*m);
  for(int i=0; i<2*m; i++) res[i] = find(i);
  auto comp=res;compress(comp);for(auto &i:res)i=IDX(comp,i);
  int mx_idx = max_element(all(points)) - points.begin();
  return {res, res[g[mx_idx].back().second << 1 | 1]};
}
```

2.12 $O(N^2 \log N)$ Bulldozer Trick

```
struct Line{
  ll i, j, dx, dy; // dx >= 0
  Line(int i, int j, const Point &pi, const Point &pj)
    : i(i), j(j), dx(pj.x-pi.x), dy(pj.y-pi.y) {}
  bool operator < (const Line &l) const {
    return make_tuple(dy*l.dx, i, j) < make_tuple(l.dy*dx, l.i,
    l.j); }
  bool operator == (const Line &l) const {
    return dy * l.dx == l.dy * dx; }
};
void Solve(){ // V.reserve(N*(N-1)/2)
  sort(A+1, A+N+1); iota(P+1, P+N+1, 1); vector<Line> V;
  for(int i=1; i<=N; i++) for(int j=i+1; j<=N; j++)
    V.emplace_back(i, j, A[i], A[j]);
  sort(V.begin(), V.end());
  for(int i=0, j=0; i<V.size(); i=j){
    while(j < V.size() && V[i] == V[j]) j++;
    for(int k=i; k<j; k++){
      int u = V[k].i, v = V[k].j; // point id, index -> Pos[id]
      swap(Pos[u], Pos[v]); swap(A[Pos[u]], A[Pos[v]]);
      if(Pos[u] > Pos[v]) swap(u, v);
      // @TODO
    }
  }
}
```

2.13 $O(N)$ Smallest Enclosing Circle

```
pt getCenter(pt a, pt b){ return pt((a.x+b.x)/2, (a.y+b.y)/2); }
pt getCenter(pt a, pt b, pt c){
  pt aa = b - a, bb = c - a;
  auto c1 = aa*aa * 0.5, c2 = bb*bb * 0.5, d = aa / bb;
  auto x = a.x + (c1 * bb.y - c2 * aa.y) / d;
  auto y = a.y + (c2 * aa.x - c1 * bb.x) / d;
  return pt(x, y); }
Circle solve(vector<pt> v){
  pt p = {0, 0};
  double r = 0; int n = v.size();
  for(int i=0; i<n; i++) if(dst(p, v[i]) > r + EPS){
    p = v[i]; r = 0;
    for(int j=0; j<i; j++) if(dst(p, v[j]) > r + EPS){
      p = getCenter(v[i], v[j]); r = dst(p, v[i]);
      for(int k=0; k<j; k++) if(dst(p, v[k]) > r + EPS){
        p = getCenter(v[i], v[j], v[k]); r = dst(v[k], p);
  }}}
  return {p, r}; }
```

2.14 $O(N + Q \log N)$ K-D Tree

```
T GetDist(const P &a, const P &b){ return (a.x-b.x) * (a.x-b.x)
+ (a.y-b.y) * (a.y-b.y); }
struct Node{
  P p; int idx;
  T x1, y1, x2, y2;
  Node(const P &p, const int idx) : p(p), idx(idx), x1(1e9),
  y1(1e9), x2(-1e9), y2(-1e9) {}
  bool contain(const P &pt)const{ return x1 <= pt.x && pt.x <=
  x2 && y1 <= pt.y && pt.y <= y2; }
  T dist(const P &pt) const { return idx == -1 ? INF :
  GetDist(p, pt); }
  T dist_to_border(const P &pt) const {
    const auto [x,y] = pt;
    if(x1 <= x && x <= x2) return min((y-y1)*(y-y1),
    (y2-y)*(y2-y));
    if(y1 <= y && y <= y2) return min((x-x1)*(x-x1),
    (x2-x)*(x2-x));
    T t11 = GetDist(pt, {x1,y1}), t12 = GetDist(pt, {x1,y2});
    T t21 = GetDist(pt, {x2,y1}), t22 = GetDist(pt, {x2,y2});
    return min({t11, t12, t21, t22});
  }
};
template<bool IsFirst = 1> struct Cmp {
  bool operator() (const Node &a, const Node &b) const {
    return IsFirst ? a.p.x < b.p.x : a.p.y < b.p.y; }
};
struct KDTree { // Warning : no duplicate
  constexpr static size_t NAIVE_THRESHOLD = 16;
  vector<Node> tree;
  KDTree() = default;
  explicit KDTree(const vector<P> &v) {
    for(int i=0; i<v.size(); i++) tree.emplace_back(v[i], i);
    Build(0, v.size());
  }
  template<bool IsFirst = 1>
  void Build(int l, int r) {
    if(r - l <= NAIVE_THRESHOLD) return;
    const int m = (l + r) >> 1;
    nth_element(tree.begin()+l, tree.begin()+m, tree.begin()+r,
    Cmp<IsFirst>{});
    for(int i=l; i<r; i++){
      tree[m].x1 = min(tree[m].x1, tree[i].p.x); tree[m].y1 =
      min(tree[m].y1, tree[i].p.y);
      tree[m].x2 = max(tree[m].x2, tree[i].p.x); tree[m].y2 =
      max(tree[m].y2, tree[i].p.y);
    }
    Build<!IsFirst>(l, m); Build<!IsFirst>(m + 1, r);
  }
  template<bool IsFirst = 1>
  void Query(const P &p, int l, int r, Node &res) const {
    if(r - l <= NAIVE_THRESHOLD){
      for(int i=l; i<r; i++) if(p != tree[i].p && res.dist(p) >
      tree[i].dist(p)) res = tree[i];
    }
    else{ // else 1
      const int m = (l + r) >> 1;
      const T t = IsFirst ? p.x - tree[m].p.x : p.y -
      tree[m].p.y;
      if(p != tree[m].p && res.dist(p) > tree[m].dist(p)) res =
      tree[m];
      if(!tree[m].contain(p) && tree[m].dist_to_border(p) >=
      res.dist(p)) return;
      if(t < 0){
        Query<!IsFirst>(p, l, m, res);
        if(t*t < res.dist(p)) Query<!IsFirst>(p, m+1, r, res);
      }
      else{ // else 2
        Query<!IsFirst>(p, m+1, r, res);
        if(t*t < res.dist(p)) Query<!IsFirst>(p, l, m, res);
  } /*else 1*/ } /*else 2*/ } /*void Query*/
  int Query(const P& p) const {
    Node ret(make_pair<T>(1e9, 1e9), -1); Query(p, 0,
    tree.size(), ret); return ret.idx; }
};
```

2.15 $O(N \log N)$ Voronoi Diagram

```
/*
input: order will be changed, sorted by (y,x) order
vertex: voronoi intersection points, degree 3, may duplicated
edge: may contain inf line (-1)
area
  - (a,b) = i-th element of area
  - (u,v) = i-th element of edge
  - input[a] is located CCW of u->v line
  - input[b] is located CW of u->v line
  - u->v line is a subset of perpendicular bisector of input[a]
to input[b] segment
  - Straight line {a, b}, {-1, -1} through midpoint of input[a]
and input[b]
*/
const double EPS = 1e-9;
int dcmp(double x){ return x < -EPS? -1 : x > EPS ? 1 : 0; }
// sq(x) = x*x, size(p) = hypot(p.x, p.y)
// sz2(p) = sq(p.x)+sq(p.y), r90(p) = (-p.y, p.x)
double sq(double x){ return x*x; }
double size(pdd p){ return hypot(p.x, p.y); }
double sz2(pdd p){ return sq(p.x) + sq(p.y); }
pdd r90(pdd p){ return pdd(-p.y, p.x); }
pdd line_intersect(pdd a, pdd b, pdd u, pdd v){ return u +
(((a-u)/b) / (v/b))*v; }
pdd get_circumcenter(pdd p0, pdd p1, pdd p2){
  return line_intersect(0.5 * (p0+p1), r90(p0-p1), 0.5 *
  (p1+p2), r90(p1-p2)); }
double pb_int(pdd left, pdd right, double sweepline){
  if(dcmp(left.y - right.y) == 0) return (left.x + right.x) /
  2.0;
  int sign = left.y < right.y ? -1 : 1;
  pdd v = line_intersect(left, right-left, pdd(0, sweepline),
  pdd(1, 0));
  double d1 = sz2(0.5 * (left+right) - v), d2 = sz2(0.5 *
  (left-right));
  return v.x + sign * sqrt(std::max(0.0, d1 - d2)); }
struct Beachline{
  struct node{ node(){}
    node(pdd point, int idx):point(point), idx(idx), end(0),
    link{0, 0}, par(0), prv(0), nxt(0) {}
    pdd point; int idx; int end;
    node *link[2], *par, *prv, *nxt; };
  node *root;
  double sweepline;
  Beachline() : sweepline(-1e20), root(NULL){ }
  inline int dir(node *x){ return x->par->link[0] != x; }
  void rotate(node *n){
    node *p = n->par; int d = dir(n);
    p->link[d] = n->link[!d];
    if(n->link[!d]) n->link[!d]->par = p;
    n->par = p->par; if(p->par) p->par->link[dir(p)] = n;
```

```
    n->link[!d] = p; p->par = n;
  } void splay(node *x, node *f = NULL){
    while(x->par != f){
      if(x->par->par == f);
      else if(dir(x) == dir(x->par)) rotate(x->par);
      else rotate(x);
      rotate(x); }
    if(f == NULL) root = x;
  } void insert(node *n, node *p, int d){
    splay(p); node* c = p->link[d];
    n->link[d] = c; if(c) c->par = n;
    p->link[d] = n; n->par = p;
    node *prv = !d?p->prv:p, *nxt = !d?p:p->nxt;
    n->prv = prv;   if(prv) prv->nxt = n;
    n->nxt = nxt;   if(nxt) nxt->prv = n;
  } void erase(node* n){
    node *prv = n->prv, *nxt = n->nxt;
    if(!prv && !nxt){ if(n == root) root = NULL; return; }
    n->prv = NULL; if(prv) prv->nxt = nxt;
    n->nxt = NULL; if(nxt) nxt->prv = prv;
    splay(n);
    if(!nxt){
      root->par = NULL; n->link[0] = NULL;
      root = prv; }
    else{
      splay(nxt, n);   node* c = n->link[0];
      nxt->link[0] = c;  c->par = nxt;     n->link[0] = NULL;
      n->link[1] = NULL; nxt->par = NULL;
      root = nxt; }
  } bool get_event(node* cur, double &next_sweep){
    if(!cur->prv || !cur->nxt) return false;
    pdd u = r90(cur->point - cur->prv->point);
    pdd v = r90(cur->nxt->point - cur->point);
    if(dcmp(u/v) != 1) return false;
    pdd p = get_circumcenter(cur->point, cur->prv->point,
    cur->nxt->point);
    next_sweep = p.y + size(p - cur->point); return true;
  } node* find_bl(double x){
    node* cur = root;
    while(cur){
      double left = cur->prv ? pb_int(cur->prv->point,
      cur->point, sweepline) : -1e30;
      double right = cur->nxt ? pb_int(cur->point,
      cur->nxt->point, sweepline) : 1e30;
      if(left <= x && x <= right){ splay(cur); return cur; }
      cur = cur->link[x > right]; }
  }
}; using BNode = Beachline::node;
static BNode* arr;
static int sz;
static BNode* new_node(pdd point, int idx){
  arr[sz] = BNode(point, idx); return arr + (sz++); }
struct event{
  event(double sweep, int idx):type(0), sweep(sweep), idx(idx){}
  event(double sweep, BNode* cur):type(1), sweep(sweep),
  prv(cur->prv->idx), cur(cur), nxt(cur->nxt->idx){}
  int type, idx, prv, nxt; BNode* cur; double sweep;
  bool operator>(const event &l)const{ return sweep > l.sweep; }
};
void VoronoiDiagram(vector<pdd> &input, vector<pdd> &vertex,
vector<pii> &edge, vector<pii> &area){
  Beachline bl = Beachline();
  priority_queue<event, vector<event>, greater<event>> events;
  auto add_edge = [&](int u, int v, int a, int b, BNode* c1,
  BNode* c2){
    if(c1) c1->end = edge.size()*2;
    if(c2) c2->end = edge.size()*2 + 1;
    edge.emplace_back(u, v); area.emplace_back(a, b);
  };
  auto write_edge = [&](int idx, int v){ idx%2 == 0 ?
  edge[idx/2].x = v : edge[idx/2].y = v; };
  auto add_event = [&](BNode* cur){ double nxt;
  if(bl.get_event(cur, nxt)) events.emplace(nxt, cur); };
  int n = input.size(), cnt = 0;
  arr = new BNode[n*4]; sz = 0;
  sort(input.begin(), input.end(), [](const pdd &l, const pdd
  &r){
    return l.y != r.y ? l.y < r.y : l.x < r.x; });
  BNode* tmp = bl.root = new_node(input[0], 0), *t2;
  for(int i = 1; i < n; i++){
    if(dcmp(input[i].y - input[0].y) == 0){
      add_edge(-1, -1, i-1, i, 0, tmp);
      bl.insert(t2 = new_node(input[i], i), tmp, 1);
      tmp = t2;
    }
    else events.emplace(input[i].y, i);
  }
  while(events.size()){
    event q = events.top(); events.pop();
    BNode *prv, *cur, *nxt, *site;
    int v = vertex.size(), idx = q.idx;
    bl.sweepline = q.sweep;
    if(q.type == 0){
      pdd point = input[idx];
      cur = bl.find_bl(point.x);
      bl.insert(site = new_node(point, idx), cur, 0);
      bl.insert(prv = new_node(cur->point, cur->idx), site, 0);
      add_edge(-1, -1, cur->idx, idx, site, prv);
      add_event(prv); add_event(cur);
    }
    else{
      cur = q.cur, prv = cur->prv, nxt = cur->nxt;
      if(!prv || !nxt || prv->idx != q.prv || nxt->idx != q.nxt)
      continue;
      vertex.push_back(get_circumcenter(prv->point, nxt->point,
      cur->point));
      write_edge(prv->end, v); write_edge(cur->end, v);
      add_edge(v, -1, prv->idx, nxt->idx, 0, prv);
      bl.erase(cur);
      add_event(prv); add_event(nxt);
    }
  }
  delete arr;
}
```

3 Graph

3.1 Euler Tour

```
// Not Directed / Cycle
constexpr int SZ = 1010;
int N, G[SZ][SZ], Deg[SZ], Work[SZ];
void DFS(int v){
  for(int &i=Work[v]; i<=N; i++) while(G[v][i]) G[v][i]--,
  G[i][v]--, DFS(i);
  cout << v << " ";
}
// Directed / Path
void DFS(int v){
  for(int i=1; i<=pv; i++) while(G[v][i]) G[v][i]--, DFS(i);
  Path.push_back(v);
}
void Get(){
  for(int i=1; i<=pv; i++) if(In[i] < Out[i]){ DFS(i); return; }
  for(int i=1; i<=pv; i++) if(Out[i]){ DFS(i); return; }
}// WARNING: path.size() == M + 1 && not trail
```

3.2 2-SAT

```
int SZ; vector<vector<int>> G1, G2;
void Init(int n){ SZ = n; G1 = G2 = vector<vector<int>>(SZ*2); }
int New(){
  for(int i=0;i<2;i++) G1.emplace_back(), G2.emplace_back();
  return SZ++;
}
inline void AddEdge(int s, int e){ G1[s].push_back(e);
G2[e].push_back(s); }
// T(x) = x << 1, F(x) = x << 1 | 1, I(x) = x ^ 1
inline void AddCNF(int a, int b){ AddEdge(I(a), b);
AddEdge(I(b), a); }
void MostOne(vector<int> vec){
  compress(vec);
  for(int i=0; i<vec.size(); i++){
    int now = New();
    AddEdge(vec[i], T(now)); AddEdge(F(now), I(vec[i]));
    if(i == 0) continue;
    AddEdge(T(now-1), T(now)); AddEdge(F(now), F(now-1));
    AddEdge(T(now-1), I(vec[i])); AddEdge(vec[i], F(now-1));
  }
}
```

3.3 Horn SAT

```
/* n : numer of variance
{}, 0 : x1 | {0, 1}, 2 : (x1 and x2) => x3, (-x1 or -x2 or x3)
fail -> empty vector */
vector<int> HornSAT(int n, const vector<vector<int>> &cond,
const vector<int> &val){
  int m = cond.size(); vector<int> res(n), margin(m), stk;
  vector<vector<int>> gph(n);
  for(int i=0; i<m; i++){
    margin[i] = cond[i].size();
    if(cond[i].empty()) stk.push_back(i);
    for(auto j : cond[i]) gph[j].push_back(i);
  }
  while(!stk.empty()){
    int v = stk.back(), h = val[v]; stk.pop_back();
    if(h < 0) return vector<int>();
    if(res[h]) continue; res[h] = 1;
    for(auto i : gph[h]) if(!--margin[i]) stk.push_back(i);
  } return res;
}
```

3.4 2-QBF

```
// con[i] \in {A(∀), E(∃)}, 0-based string
// variable: 1-based(parameter), 0-based(computing)
```

```
// (a or not b) -> {a, -b} in 1-based index
// return empty vector if satisfiable, else any solution
// T(x) = x << 1, F(x) = x << 1 | 1, I(x) = x ^ 1
vector<int> TwoQBF(int n, string con, vector<pair<int,int>>
cnf){
  auto f = [](int v){ return v > 0 ? T(v-1) : F(-v-1); };
  for(auto &[a,b] : cnf) AddCNF(a=f(a), b=f(b));
  if(!TwoSAT(n)) return {}; int k = SCC.size();
  vector<int> has(k,-1), from(k), to(k), res(n, -1);
  for(int i=n-1; i>=0; i--){ // WARNING: index is scc id
    if(has[C[T(i)]] != -1 || has[C[F(i)]] != -1) return {};
    if(con[i] == 'A') has[C[T(i)]] = T(i), has[C[F(i)]] = F(i);
  }
  for(int i=0; i<k; i++) if(has[i] != -1) from[i] = to[i] = 1;
  for(int i=0; i<n+n; i++){
    for(auto j : Gph[i]) if(C[i] != C[j])
    DAG[C[i]].push_back(C[j]);
  }
  for(int i=k-1; i>=0; i--){
    bool flag = false; // i -> A?
    for(auto j : DAG[i]) flag |= to[j];
    if(flag && to[i]) return {}; to[i] |= flag;
  }
  for(int i=0; i<k; i++) for(auto j : DAG[i]) from[j] |=
  from[i]; // A->i?
  for(int i=0; i<k; i++){
    if(has[i] != -1) for(auto v : SCC[i]) res[v/2] = v % 2 ==
    has[i] % 2;
    else if(from[i]) for(auto v : SCC[i]) res[v/2] = v % 2 == 0;
    else if(to[i]) for(auto v : SCC[i]) res[v/2] = v % 2 == 1;
  }
  for(int i=0; i<n; i++) if(res[i]==-1) res[i]=C[F(i)]<C[T(i)];
  return res;
}
```

3.5 BCC

```
// Call tarjan(N) before use!!!
vector<int> G[MAX_V]; int In[MAX_V], Low[MAX_V], P[MAX_V];
void addEdge(int s,int e){G[s].push_back(e);G[e].push_back(s);}
void tarjan(int n){ /// Pre-Process
  int pv = 0;
  function<void(int,int)> dfs = [&pv,&dfs](int v, int b){
    In[v] = Low[v] = ++pv; P[v] = b;
    for(auto i : G[v]){
      if(i == b) continue;
      if(!In[i]) dfs(i, v), Low[v] = min(Low[v], Low[i]); else
      Low[v] = min(Low[v], In[i]);
  }};
  for(int i=1; i<=n; i++) if(!In[i]) dfs(i, -1);
}
vector<int> cutVertex(int n){
  vector<int> res; array<char,MAX_V> isCut; isCut.fill(0);
  function<void(int)> dfs = [&dfs,&isCut](int v){
    int ch = 0;
    for(auto i : G[v]){
      if(P[i] != v) continue; dfs(i); ch++;
      if(P[v] == -1 && ch > 1) isCut[v] = 1;
      else if(P[v] != -1 && Low[i] >= In[v]) isCut[v]=1;
  }};
  for(int i=1; i<=n; i++) if(P[i] == -1) dfs(i);
  for(int i=1; i<=n; i++) if(isCut[i]) res.push_back(i);
  return move(res);
}
vector<PII> cutEdge(int n){
  vector<PII> res;
  function<void(int)> dfs = [&dfs,&res](int v){
    for(int t=0; t<G[v].size(); t++){
      int i = G[v][t]; if(t != 0 && G[v][t-1] == G[v][t])
      continue;
      if(P[i] != v) continue; dfs(i);
      if((t+1 == G[v].size() || i != G[v][t+1]) && Low[i] >
      In[v]) res.emplace_back(min(v,i), max(v,i));
  }}; // sort edges if multi edge exist
  for(int i=1; i<=n; i++) sort(G[i].begin(), G[i].end());
  for(int i=1; i<=n; i++) if(P[i] == -1) dfs(i);
  return move(res); // sort(all(res));
}
vector<int> BCC[MAX_V]; // BCC[v] = components which contains v
void vertexDisjointBCC(int n){ // allow multi edge, no self loop
  int cnt = 0; array<char,MAX_V> vis; vis.fill(0);
  function<void(int,int)> dfs = [&dfs,&vis,&cnt](int v, int c){
    vis[v] = 1; if(c > 0) BCC[v].push_back(c);
    for(auto i : G[v]){
      if(vis[i]) continue;
      if(In[v] <= Low[i]) BCC[v].push_back(++cnt), dfs(i, cnt);
      else dfs(i, c);
  }};
  for(int i=1; i<=n; i++) if(!vis[i]) dfs(i, 0);
  for(int i=1; i<=n; i++) if(BCC[i].empty())
  BCC[i].push_back(++cnt);
}void edgeDisjointBCC(int n){} // remove cut edge, do flood fill
```

3.6 Prufer Sequence

```
vector<pair<int,int>> Gen(int n, vector<int> a){ // a :
[1,n]^(n-2)
  if(n == 1) return {}; if(n == 2) return { make_pair(1, 2) };
  vector<int> deg(n+1); for(auto i : a) deg[i]++;
  vector<pair<int,int>> res; priority_queue<int> pq;
  for(int i=n; i; i--) if(!deg[i]) pq.emplace(i);
  for(auto i : a){
    res.emplace_back(i, pq.top()); pq.pop();
    if(!--deg[i]) pq.push(i);
  }int u = pq.top(); pq.pop(); int v = pq.top(); pq.pop();
  res.emplace_back(u, v); return res;
}
```

3.7 $O(3^{V/3})$ Maximal Clique

```
using B = bitset<128>; template<typename F> //0-based
void maximal_cliques(vector<B>&g,F f,B P=~B(),B X={},B R={}){
  if(!P.any()){ if(!X.any()) f(R); return; }
  auto q = (P|X)._Find_first(); auto c = P & ~g[q];
  for(int i=0; i<g.size(); i++) if(c[i]) {
    R[i] = 1; cliques(g, f, P&g[i], X&g[i], R);
    R[i]=P[i]=0; X[i] = 1; } // faster for sparse gph
} // undirected, self loop not allowed, O(3^{n/3})
B max_independent_set(vector<vector<int>> g){ //g=adj matrix
  int n = g.size(), i, j; vector<B> G(n); B res{};
  auto chk_mx = [&](B a){ if(a.count()>res.count()) res=a; };
  for(i=0; i<n; i++) for(int j=0; j<n; j++)
    if(i!=j && !g[i][j])G[i][j]=1;
  cliques(G, chk_mx); return res; }
```

3.8 $O(V \log V)$ Tree Isomorphism

```
struct Tree{ // (M1,M2)=(1e9+7, 1e9+9), P1,P2 = random int
array(sz >= N+2)
  int N; vector<vector<int>> G; vector<pair<int,int>> H;
  vector<int> S, C; // size,centroid
  Tree(int N) : N(N), G(N+2), H(N+2), S(N+2) {}
  void addEdge(int s, int e){ G[s].push_back(e);
  G[e].push_back(s); }
  int getCentroid(int v, int b=-1){
    S[v] = 1; // do not merge if-statements
    for(auto i : G[v]) if(i!=b) if(int now=getCentroid(i,v);
    now<=N/2) S[v]+=now; else break;
    if(N - S[v] <= N/2) C.push_back(v); return S[v] = S[v];
  }
  int init(){
    getCentroid(1); if(C.size() == 1) return C[0];
    int u = C[0], v = C[1], add = ++N;
    G[u].erase(find(G[u].begin(), G[u].end(), v));
    G[v].erase(find(G[v].begin(), G[v].end(), u));
    G[add].push_back(u); G[u].push_back(add);
    G[add].push_back(v); G[v].push_back(add);
    return add;
  }
  pair<int,int> build(const vector<ll> &P1, const vector<ll>
  &P2, int v, int b=-1){
    vector<pair<int,int>> ch; for(auto i : G[v]) if(i != b)
    ch.push_back(build(P1, P2, i, v));
    ll h1 = 0, h2 = 0; stable_sort(ch.begin(), ch.end());
    if(ch.empty()){ return {1, 1}; }
    for(int i=0; i<ch.size(); i++)
    h1=(h1+(ch[i].first^P1[P1.size()-1-i])*P1[i])%M1,
    h2=(h2+(ch[i].second^P2[P2.size()-1-i])*P2[i])%M2;
    return H[v] = {h1, h2};
  }
  int build(const vector<ll> &P1, const vector<ll> &P2){
    int rt = init(); build(P1, P2, rt); return rt;
  }
};
```

3.9 $O(E \log E)$ Complement Spanning Forest

```
vector<pair<int,int>> ComplementSpanningForest(int n, const
vector<pair<int,int>> &edges){ // V+ElgV
  vector<vector<int>> g(n);
  for(const auto &[u,v] : edges) g[u].push_back(v),
  g[v].push_back(u);
  for(int i=0; i<n; i++) sort(g[i].begin(), g[i].end());
  set<int> alive;
  for(int i=0; i<n; i++) alive.insert(i);
  vector<pair<int,int>> res;
  while(!alive.empty()){
    int u = *alive.begin(); alive.erase(alive.begin());
    queue<int> que; que.push(u);
    while(!que.empty()){
      int v = que.front(); que.pop();
      for(auto it=alive.begin(); it!=alive.end(); ){
        if(auto t=lower_bound(g[v].begin(), g[v].end(), *it); t
        != g[v].end() && *it == *t) ++it;
        else que.push(*it), res.emplace_back(u, *it), it =
        alive.erase(it);
  }}}return res;
}
```

3.10 $O(E\sqrt{V})$ Bipartite Matching, Konig, Dilworth

```
struct HopcroftKarp{
  int n, m; vector<vector<int>> g;
  vector<int> dst, le, ri; vector<char> visit, track;
  HopcroftKarp(int n, int m) : n(n), m(m), g(n), dst(n), le(n,
  -1), ri(m, -1), visit(n), track(n+m) {}
  void add_edge(int s, int e){ g[s].push_back(e); }
  bool bfs(){ bool res = false; queue<int> que;
    fill(dst.begin(), dst.end(), 0);
    for(int i=0; i<n; i++)if(le[i] == -1)que.push(i),dst[i]=1;
    while(!que.empty()){ int v = que.front(); que.pop();
      for(auto i : g[v]){
        if(ri[i] == -1) res = true;
        else if(!dst[ri[i]])dst[ri[i]]=dst[v]+1,que.push(ri[i]);
      }
    }
    return res;
  }
  bool dfs(int v){
    if(visit[v]) return false; visit[v] = 1;
    for(auto i : g[v]){
      if(ri[i] == -1 || !visit[ri[i]] && dst[ri[i]] == dst[v] +
      1 && dfs(ri[i])){ le[v] = i; ri[i] = v; return true; }
    } return false;
  }
  int maximum_matching(){
    int res = 0; fill(all(le), -1); fill(all(ri), -1);
    while(bfs()){
      fill(visit.begin(), visit.end(), 0);
      for(int i=0; i<n; i++) if(le[i] == -1) res += dfs(i);
    } return res;
  }
  vector<pair<int,int>> maximum_matching_edges(){
    int matching = maximum_matching();
    vector<pair<int,int>> edges; edges.reserve(matching);
    for(int i=0; i<n; i++) if(le[i] != -1) edges.emplace_back(i,
    le[i]);
    return edges;
  }
  void dfs_track(int v){
    if(track[v]) return; track[v] = 1;
    for(auto i : g[v]) track[n+i] = 1, dfs_track(ri[i]);
  }
  tuple<vector<int>, vector<int>, int> minimum_vertex_cover(){
    int matching = maximum_matching(); vector<int> lv, rv;
    fill(track.begin(), track.end(), 0);
    for(int i=0; i<n; i++) if(le[i] == -1) dfs_track(i);
    for(int i=0; i<n; i++) if(!track[i]) lv.push_back(i);
    for(int i=0; i<m; i++) if(track[n+i]) rv.push_back(i);
    return {lv, rv, lv.size() + rv.size()}; // s(lv)+s(rv)=mat
  }
  tuple<vector<int>, vector<int>, int>
  maximum_independent_set(){
    auto [a,b,matching] = minimum_vertex_cover();
    vector<int> lv, rv; lv.reserve(n-a.size());
    rv.reserve(m-b.size());
    for(int i=0, j=0; i<n; i++){
      while(j < a.size() && a[j] < i) j++;
      if(j == a.size() || a[j] != i) lv.push_back(i);
    }
    for(int i=0, j=0; i<m; i++){
      while(j < b.size() && b[j] < i) j++;
      if(j == b.size() || b[j] != i) rv.push_back(i);
    } // s(lv)+s(rv)=n+m-mat
    return {lv, rv, lv.size() + rv.size()};
  }
  vector<vector<int>> minimum_path_cover(){ // n == m
    int matching = maximum_matching();
    vector<vector<int>> res; res.reserve(n - matching);
    fill(track.begin(), track.end(), 0);
    auto get_path = [&](int v) -> vector<int> {
      vector<int> path{v}; // ri[v] == -1
      while(le[v] != -1) path.push_back(v=le[v]);
      return path;
    };
    for(int i=0; i<n; i++) if(!track[n+i] && ri[i] == -1)
      res.push_back(get_path(i));
    return res; // sz(res) = n-mat
  }
  vector<int> maximum_anti_chain(){ // n = m
    auto [a,b,matching] = minimum_vertex_cover();
    vector<int> res; res.reserve(n - a.size() - b.size());
    for(int i=0, j=0, k=0; i<n; i++){
      while(j < a.size() && a[j] < i) j++;
      while(k < b.size() && b[k] < i) k++;
      if((j == a.size() || a[j] != i) && (k == b.size() || b[k]
      != i)) res.push_back(i);
    } return res; // sz(res) = n-mat
  }
};
```

3.11 $O(V^2\sqrt{E})$ Push Relabel

```
template<typename flow_t> struct Edge {
  int u, v, r; flow_t c, f; Edge() = default;
  Edge(int u, int v, flow_t c, int r) : u(u), v(v), r(r), c(c),
  f(0) {}
};
template<typename flow_t, size_t _Sz> struct PushRelabel {
  using edge_t = Edge<flow_t>;
  int n, b, dist[_Sz], count[_Sz+1];
  flow_t excess[_Sz]; bool active[_Sz];
  vector<edge_t> g[_Sz]; vector<int> bucket[_Sz];
  void clear(){ for(int i=0; i<_Sz; i++) g[i].clear(); }
  void addEdge(int s, int e, flow_t x){
    g[s].emplace_back(s, e, x, (int)g[e].size());
    if(s == e) g[s].back().r++;
    g[e].emplace_back(e, s, 0, (int)g[s].size()-1);
  }
  void enqueue(int v){
    if(!active[v] && excess[v] > 0 && dist[v] < n){
      active[v] = true; bucket[dist[v]].push_back(v); b = max(b,
      dist[v]); }
  }
  void push(edge_t &e){
    flow_t fl = min(excess[e.u], e.c - e.f);
    if(dist[e.u] == dist[e.v] + 1 && fl > flow_t(0)){
      e.f += fl; g[e.v][e.r].f -= fl; excess[e.u] -= fl;
      excess[e.v] += fl; enqueue(e.v); }
  }
  void gap(int k){
    for(int i=0; i<n; i++){
      if(dist[i] >= k) count[dist[i]]--, dist[i] = max(dist[i],
      n), count[dist[i]]++;
      enqueue(i); }
  }
  void relabel(int v){
    count[dist[v]]--; dist[v] = n;
    for(const auto &e : g[v]) if(e.c - e.f > 0) dist[v] =
    min(dist[v], dist[e.v] + 1);
    count[dist[v]]++; enqueue(v);
  }
  void discharge(int v){
    for(auto &e : g[v]) if(excess[v] > 0) push(e); else break;
    if(excess[v] > 0) if(count[dist[v]] == 1) gap(dist[v]);
    else relabel(v);
  }
  flow_t maximumFlow(int _n, int s, int t){
    // memset dist, excess, count, active 0
    n = _n; b = 0; for(auto &e : g[s]) excess[s] += e.c;
    count[s] = n; enqueue(s); active[t] = true;
    while(b >= 0){
      if(bucket[b].empty()) b--;
      else{
        int v = bucket[b].back(); bucket[b].pop_back();
        active[v] = false; discharge(v);
      } /*else*/ } /*while*/ return excess[t];
  }
};
```

3.12 LR Flow, Circulation

```
struct LR_Flow{ LR_Flow(int n) : F(n+2), S(0) {}
  void add_edge(int s, int e, flow_t l, flow_t r){
    S += abs(l); F.add_edge(s+2, e+2, r-l);
    if(l > 0) F.add_edge(s+2, 1, l), F.add_edge(0, e+2, l);
    else F.add_edge(0, s+2, -l), F.add_edge(e+2, 1, -l);
  } Dinic<flow_t, MAX_U> F; flow_t S;
  bool solve(int s, int t){//maxflow: run F.maximum_flow(s, t)
    if(s != -1) F.add_edge(t+2, s+2, MAX_U); //min cost circ
    return F.maximum_flow(0,1) == S; }
  flow_t get_flow(int s, int e) const { s += 2; e += 2;
    for(auto i : F.g[s]) if(i.c > 0 && i.v == e) return i.f; }
}; struct Circulation{ // demand[i] = in[i] - out[i]
  Circulation(int n, const vector<flow_t> &demand) : F(n+2) {
    // demand[i] > 0: add_edge(0, i+2, demand[i], demand[i])
    // demand[i] <= 0: add_edge(i+2, 1, -demand[i], demand[i])
  } LR_Flow<flow_t, MAX_U> F;
  void add_edge(int s, int e, flow_t l, flow_t r){
    F.add_edge(s+2, e+2, l, r); }
  bool feasible(){ return F.feasible(0, 1); } };
```

3.13 Min Cost Circulation

```
template <class T> struct MinCostCirculation {
  const int SCALE = 3; // scale by 1/(1 << SCALE)
  const T INF = numeric_limits<T>::max() / 2;
  struct EdgeStack { int s, e; T l, r, cost; };
  struct Edge { int pos, rev; T rem, cap, cost; };
  int n; vector<vector<Edge>> g; vector<EdgeStack> estk;
  LR_Flow<T, 1LL<<60> circ; vector<T> p;
  MinCostCirculation(int k) : n(k), g(k), circ(k), p(k) {}
  void add_edge(int s, int e, T l, T r, T cost){
    estk.push_back({s, e, l, r, cost}); }
  pair<bool, T> solve(){
```

```
    for(auto &i:estk)if(i.s!=i.e)circ.add_edge(i.s,i.e,i.l,i.r);
    if(!circ.solve(-1, -1)) return make_pair(false, T(0));
    vector<int> cnt(n); T eps = 0;
    for(auto &i : estk){ T curFlow;
      auto &edge = circ.F.g[i.s+2][cnt[i.s]];
      if(i.s != i.e) curFlow = i.r - (edge.c - edge.f);
      else curFlow = i.r;
      int srev = sz(g[i.e]), erev = sz(g[i.s]);
      if(i.s == i.e) srev++;
    g[i.s].push_back(i.e,srev,+i.r-curFlow,+i.r,+i.cost*(n+1));
    g[i.e].push_back(i.s,erev,-i.l+curFlow,-i.l,-i.cost*(n+1));
      eps = max(eps, abs(i.cost) * (n+1));
      if(i.s != i.e) cnt[i.s] += 2, cnt[i.e] += 2;
    }
    while(true){ eps=0; auto cost=[&](Edge &e, int s, int t){
        return e.cost + p[s] - p[t]; };
      for(int i = 0; i < n; i++) for(auto &e : g[i])
        if(e.rem > 0) eps = max(eps, -cost(e, i, e.pos));
      if(eps <= T(1)) break;
      eps = max(T(1), eps >> SCALE);
      vector<T> excess(n); queue<int> que;
      auto push = [&](Edge &e, int src, T flow){
        e.rem -= flow; g[e.pos][e.rev].rem += flow;
        excess[src] -= flow; excess[e.pos] += flow;
        if(excess[e.pos] <= flow && excess[e.pos] > 0)
        que.push(e.pos);
      }; vector<int> ptr(n);
      auto relabel = [&](int v){
        ptr[v] = 0; p[v] = -INF;
        for(auto &e : g[v])
          if(e.rem>0) p[v] = max(p[v], p[e.pos]-e.cost-eps);
      };
      for(int i = 0; i < n; i++) for(auto &j : g[i])
        if(j.rem>0 && cost(j, i, j.pos)<0) push(j, i, j.rem);
      while(sz(que)){
        int x = que.front(); que.pop();
        while(excess[x] > 0){
          for(; ptr[x] < sz(g[x]); ptr[x]++){
            Edge &e = g[x][ptr[x]];
            if(e.rem > 0 && cost(e, x, e.pos) < 0){
              push(e, x, min(e.rem, excess[x]));
              if(excess[x] == 0) break;
          } /* if end */ } /* for end*/
          if(excess[x] == 0) break; relabel(x);
      } /* excess end */  } /* que end */
    } /* while true end */ T ans = 0;
    for(int i=0; i<n; i++) for(auto &j : g[i])
      j.cost /= n + 1, ans += j.cost * (j.cap - j.rem);
    return make_pair(true, ans / 2);
  }
  void bellmanFord(){
    fill(p.begin(), p.end(), T(0)); bool upd = 1;
    while(upd){ upd = 0;
      for(int i = 0; i < n; i++) for(auto &j : g[i])
        if(j.rem > 0 && p[j.pos] > p[i] + j.cost) p[j.pos] =
        p[i] + j.cost, upd = 1;
    }
  }
};
```

3.14 $O(V^3)$ Hungarian Method

```
// 1-based, only for min matching, max matching may get TLE
template<typename cost_t=int, cost_t _INF=0x3f3f3f3f>
struct Hungarian{
  int n; vector<vector<cost_t>> mat;
  Hungarian(int n) : n(n), mat(n+1, vector<cost_t>(n+1, _INF))
  {}
  void addEdge(int s, int e, cost_t x){ mat[s][e] =
  min(mat[s][e], x); }
  pair<cost_t, vector<int>> run(){
    vector<cost_t> u(n+1), v(n+1), m(n+1);
    vector<int> p(n+1), w(n+1), c(n+1);
    for(int i=1,a,b; i<=n; i++){
      p[0] = i; b = 0; fill(m.begin(), m.end(), _INF);
      fill(c.begin(), c.end(), 0);
      do{
        int nxt; cost_t delta = _INF; c[b] = 1; a = p[b];
        for(int j=1; j<=n; j++){
          if(c[j]) continue;
          cost_t t = mat[a][j] - u[a] - v[j];
          if(t < m[j]) m[j] = t, w[j] = b;
          if(m[j] < delta) delta = m[j], nxt = j;
        }
        for(int j=0; j<=n; j++){
          if(c[j]) u[p[j]] += delta, v[j] -= delta; else m[j] -=
          delta;
        }
        b = nxt;
      }while(p[b] != 0);
      do{int nxt = w[b]; p[b] = p[nxt]; b = nxt;}while(b!=0);
    }
    vector<int> assign(n+1);for(int i=1;i<=n;i++)assign[p[i]]=i;
    return {-v[0], assign};
  }
};
```

3.15 $O(V^3)$ Global Min Cut

```
template<typename T, T INF>// 0-based, adj matrix
pair<T, vector<int>> GetMinCut(vector<vector<T>> g){
  int n=g.size(); vector<int> use(n), cut, mn_cut; T mn=INF;
  for(int phase=n-1; phase>=0; phase--){
    vector<int> w=g[0], add=use; int k=0, prv;
    for(int i=0; i<phase; i++){ prv = k; k = -1;
      for(int j=1; j<n; j++) if(!add[j] && (k==-1 || w[j] >
      w[k])) k=j;
      if(i + 1 < phase){
        for(int j=0; j<n; j++) w[j] += g[k][j];
        add[k] = 1; continue; }
      for(int j=0; j<n; j++) g[prv][j] += g[k][j];
      for(int j=0; j<n; j++) g[j][prv] = g[prv][j];
      use[k] = 1; cut.push_back(k);
      if(w[k] < mn) mn_cut = cut, mn = w[k];
    }
  } return {mn, mn_cut};
}
```

3.16 $O(V^2 + V \times \text{Flow})$ Gomory-Hu Tree

```
//0-based, S-T cut in graph=S-T cut in gomory-hu tree (path min)
vector<Edge> GomoryHuTree(int n, const vector<Edge> &e){
  Dinic<int,100> Flow; vector<Edge> res(n-1); vector<int> pr(n);
  for(int i=1; i<n; i++, Flow.clear()){ // // bi-directed edge
    for(const auto &[s,e,x] : e) Flow.AddEdge(s, e, x);
    int fl = Flow.MaxFlow(pr[i], i);
    for(int j=i+1; j<n; j++){
      if(!Flow.Level[i] == !Flow.Level[j] && pr[i] == pr[j])
      pr[j] = i;
    } /*for-j end*/ res[i-1] = Edge(pr[i], i, fl);
  } /*for-i end*/ return res; }
```

3.17 $O(V + E\sqrt{V})$ Count/Find 3/4 Cycle

```
vector<tuple<int,int,int>> Find3Cycle(int n, const
vector<pair<int,int>> &edges){ // N+MsqrtN
  int m = edges.size();
  vector<int> deg(n), pos(n), ord; ord.reserve(n);
  vector<vector<int>> gph(n), que(m+1), vec(n);
  vector<vector<tuple<int,int,int>>> tri(n);
  vector<tuple<int,int,int>> res;
  for(auto [u,v] : edges) deg[u]++, deg[v]++;
  for(int i=0; i<n; i++) que[deg[i]].push_back(i);
  for(int i=m; i>=0; i--) ord.insert(ord.end(), que[i].begin(),
  que[i].end());
  for(int i=0; i<n; i++) pos[ord[i]] = i;
  for(auto [u,v] : edges) gph[pos[u]].push_back(pos[v]),
  gph[pos[v]].push_back(pos[u]);
  for(int i=0; i<n; i++){
    for(auto j : gph[i]){
      if(i > j) continue;
      for(int x=0, y=0; x<vec[i].size() && y<vec[j].size(); ){
        if(vec[i][x] == vec[j][y]) res.emplace_back(ord[i],
        ord[j], ord[vec[i][x]]), x++, y++;
        else if(vec[i][x] < vec[j][y]) x++; else y++;
      }
      vec[j].push_back(i);
    }
  }
  for(auto &[u,v,w] : res){
    if(pos[u] < pos[v]) swap(u, v);
    if(pos[u] < pos[w]) swap(u, w);
    if(pos[v] < pos[w]) swap(v, w);
    tri[u].emplace_back(u, v, w);
  }
  res.clear();
  for(int i=n-1; i>=0; i--) res.insert(res.end(),
  tri[ord[i]].begin(), tri[ord[i]].end());
  return res;
}
bitset<500> B[500]; // N3/w
long long Count3Cycle(int n, const vector<pair<int,int>>
&edges){
  long long res = 0;
  for(int i=0; i<n; i++) B[i].reset();
  for(auto [u,v] : edges) B[u].set(v), B[v].set(u);
  for(int i=0; i<n; i++) for(int j=i+1; j<n; j++)
  if(B[i].test(j)) res += (B[i] & B[j]).count();
  return res / 3;
}
// O(n + m * sqrt(m) + th) for graphs without loops or
multiedges
void Find4Cycle(int n, const vector<array<int, 2>> &edge, auto
process, int th = 1){
  int m = (int)edge.size();
  vector<int> deg(n), order, pos(n);
```

```
  vector<vector<int>> appear(m+1), adj(n), found(n);
  for(auto [u, v]: edge) ++deg[u], ++deg[v];
  for(auto u=0; u<n; u++) appear[deg[u]].push_back(u);
  for(auto d=m; d>=0; d--) order.insert(order.end(),
  appear[d].begin(), appear[d].end());
  for(auto i=0; i<n; i++) pos[order[i]] = i;
  for(auto i=0; i<m; i++){
    int u = pos[edge[i][0]], v = pos[edge[i][1]];
    adj[u].push_back(v), adj[v].push_back(u);
  }
  T res = 0; vector<int> cnt(n);
  for(auto u=0; u<n; u++){
    for(auto v: adj[u]) if(u < v) for(auto w: adj[v]) if(u < w)
    cnt[w] = 0;
    for(auto v: adj[u]) if(u < v) for(auto w: adj[v]) if(u < w)
    res += cnt[w] ++;
  }
  for(auto u=0; u<n; u++){
    for(auto v: adj[u]) if(u < v) for(auto w: adj[v]) if(u < w)
    found[w].clear();
    for(auto v: adj[u]) if(u < v) for(auto w: adj[v]) if(u < w)
    {
      for(auto x: found[w]){
        if(!th--) return;
        process(order[u], order[v], order[w], order[x]);
      }
      found[w].push_back(v);
    }
  }
}
```

3.18 $O(V \log V)$ Rectlinear MST

```
template<class T> vector<tuple<T, int, int>>
rectilinear_minimum_spanning_tree(vector<point<T>> a){
  int n = a.size(); vector<int> ind(n);
  iota(ind.begin(),ind.end(),0); vector<tuple<T,int,int>> edge;
  for(int k=0; k<4; k++){ map<T, int> mp;
    sort(ind.begin(), ind.end(), [&](int i,int j){
      return a[i].x-a[j].x < a[j].y-a[i].y;});
    for(auto i: ind){
      for(auto it=mp.lower_bound(-a[i].y); it!=mp.end();
      it=mp.erase(it)){
        int j = it->second; point<T> d = a[i] - a[j];
        if(d.y > d.x) break; edge.push_back({d.x+d.y,i,j});
      }
      mp.insert({-a[i].y, i});
    }
    for(auto &p: a) if(k & 1) p.x = -p.x; else swap(p.x, p.y);
  } /*for-k end*/ sort(edge.begin(), edge.end());
  disjoint_set dsu(n); vector<tuple<T, int, int>> res;
  for(auto [x, i, j]: edge) if(dsu.merge(i, j))
    res.push_back({x, i, j});
  return res; }
```

3.19 $O(VE)$ Shortest Mean Cycle

```
template<typename T, T INF> vector<int> // T = V*E*max(C)
min_mean_cycle(int n, const vector<tuple<int,int,T>> &edges){
  vector<vector<T>>dp(n+1,vector<T>(n,INF)); // int support!
  vector<vector<int>>pe(n+1,vector<int>(n,-1));
  fill(dp[0].begin(),dp[0].end(),0); //0-based,directed
  for(int x=1; x<=n; x++){ int id=0; // bellman
    for(auto [u,v,w] : edges){
      if(dp[x-1][u] != INF && dp[x-1][u] + w < dp[x][v])
        dp[x][v] = dp[x-1][u] + w, pe[x][v] = id;
      id++; } // range based for end!
  } T p=1; int q=0, src=-1; //fraction
  for(auto u=0; u<n; u++){ if(dp[n][u] == INF) continue;
    T cp=-1, cq=0; // ↓ overflow!!!
    for(int x=0;x<=n;x++)if(cp*(n-x) < (dp[n][u]-dp[x][u])*cq)
      cp = dp[n][u] - dp[x][u], cq = n - x;
    if(p * cq > cp * q) src = u, p = cp, q = cq;
  } if(src == -1) return {};
  vector<int> res, po(n, -1);
  for(int u=src, x = n; ; u=get<0>(edges[pe[x--][u]])){
    if(po[u] != -1)return
    vector<int>{res.rbegin(),res.rend()-po[u]};
    po[u] = res.size(); res.push_back(pe[x][u]);
  } assert(false);
} // return edge index
```

3.20 $O(V^2)$ Stable Marriage Problem

```
// man : 1~n, woman : n+1~2n
struct StableMarriage{
  int n; vector<vector<int>> g;
  StableMarriage(int n) : n(n), g(2*n+1) { for(int i=1; i<=n+n;
  i++) g[i].reserve(n); }
  void addEdge(int u, int v){  g[u].push_back(v); } // insert in
  decreasing order of preference.
  vector<int> run(){
    queue<int> q; vector<int> match(2*n+1), ptr(2*n+1);
    for(int i=1; i<=n; i++) q.push(i);
    while(q.size()){
      int i = q.front(); q.pop();
      for(int &p=ptr[i]; p<g[i].size(); p++){
        int j = g[i][p];
        if(!match[j]){ match[i] = j; match[j] = i; break; }
        int m = match[j], u = -1, v = -1;
        for(int k=0; k<g[j].size(); k++){
          if(g[j][k] == i) u = k; if(g[j][k] == m) v = k;
        }
        if(u < v){
          match[m] = 0; q.push(m); match[i] = j; match[j] = i;
          break;
    } /*if u < v*/ } /*for-p*/ } /*while*/
    return match; } /*vector<int> run*/
};
```

3.21 $O((V+E) \log V)$ Dominator Tree

```
vector<int> DominatorTree(const vector<vector<int>> &g, int
src){ // // 0-based
  int n = g.size();
  vector<vector<int>> rg(n), buf(n);
  vector<int> r(n), val(n), idom(n, -1), sdom(n, -1), o, p(n),
  u(n);
  iota(all(r), 0); iota(all(val), 0);
  for(int i=0; i<n; i++) for(auto j : g[i]) rg[j].push_back(i);
  function<int(int)> find = [&](int v){
    if(v == r[v]) return v;
    int ret = find(r[v]);
    if(sdom[val[v]] > sdom[val[r[v]]]) val[v] = val[r[v]];
    return r[v] = ret;
  };
  function<void(int)> dfs = [&](int v){
    sdom[v] = o.size(); o.push_back(v);
    for(auto i : g[v]) if(sdom[i] == -1) p[i] = v, dfs(i);
  };
  dfs(src); reverse(all(o));
  for(auto &i : o){
    if(sdom[i] == -1) continue;
    for(auto j : rg[i]){
      if(sdom[j] == -1) continue;
      int x = val[find(j), j];
      if(sdom[i] > sdom[x]) sdom[i] = sdom[x];
    }
    buf[o[o.size() - sdom[i] - 1]].push_back(i);
    for(auto j : buf[p[i]]) u[j] = val[find(j), j];
    buf[p[i]].clear();
    r[i] = p[i];
  }
  reverse(all(o)); idom[src] = src;
  for(auto i : o){ // WARNING : if different, takes idom
    if(i != src) idom[i] = sdom[i] == sdom[u[i]] ? sdom[i] :
    idom[u[i]];
  }
  for(auto i : o) if(i != src) idom[i] = o[idom[i]];
  return idom; // unreachable -> ret[i] = -1
}
```

3.22 $O(VE)$ Vizing Theorem

```
// Graph coloring with (max-degree)+1 colors, O(N^2)
int C[MX][MX] = {}, G[MX][MX] = {}; // MX ~= 2500
void solve(vector<pii> &E, int N, int M){
  int X[MX] = {}, a, b;
  auto update = [&](int u){ for(X[u] = 1; C[u][X[u]]; X[u]++);
  };
  auto color = [&](int u, int v, int c){
    int p = G[u][v]; G[u][v] = G[v][u] = c;
    C[u][c] = v; C[v][c] = u; C[u][p] = C[v][p] = 0;
    if( p ) X[u] = X[v] = p; else update(u), update(v);
    return p; }; // end of function : color
  auto flip = [&](int u, int c1, int c2){
    int p = C[u][c1], q = C[u][c2];
    swap(C[u][c1], C[u][c2]);
    if( p ) G[u][p] = G[p][u] = c2;
    if( !C[u][c1] ) X[u] = c1; if( !C[u][c2] ) X[u] = c2;
    return p; }; // end of function : flip
  for(int i = 1; i <= N; i++) X[i] = 1;
  for(int t = 0; t < E.size(); t++){
    int u=E[t].first, v0=E[t].second, v=v0, c0=X[u], c=c0, d;
    vector<pii> L; int vst[MX] = {};
    while(!G[u][v0]){
      L.emplace_back(v, d = X[v]);
      if(!C[v][c]) for(a = (int)L.size()-1; a >= 0; a--) c =
      color(u, L[a].first, c);
      else if(!C[u][d])for(a=(int)L.size()-1;a>=0;a--)
      color(u,L[a].first,L[a].second);
      else if( vst[d] ) break;
      else vst[d] = 1, v = C[u][d];
    }
    if( !G[u][v0] ){
      for(;v; v = flip(v, c, d), swap(c, d));
      if(C[u][c0]){
```

```
        for(a=(int)L.size()-2; a>=0 && L[a].second!=c; a--);
        for(; a >= 0; a--) color(u, L[a].first, L[a].second);
      } else t--;
    }
  }
}
```

3.23 $O(E + V^3 + V3^T + V^2 2^T)$ Minimum Steiner Tree

```
struct SteinerTree{ // O(E + V^3 + V 3^T + V^2 2^T)
  constexpr static int V = 33, T = 8;
  int n, G[V][V], D[1<<T][V], tmp[V];
  void init(int _n){ n = _n;
    memset(G, 0x3f, sizeof G); for(int i=0; i<n; i++) G[i][i]=0;
  } void shortest_path(){ /*floyd 0..n-1*/ }
  void add_edge(int u, int v, int w){
    G[u][v] = G[v][u] = min(G[v][u], w); }
  int solve(const vector<int>& ter){
    int t = (int)ter.size(); memset(D, 0x3f, sizeof D);
    for(int i=0; i<n; i++) D[0][i] = 0;
    for(int msk=1; msk<(1 << t); msk++){
      if(msk == (msk & (-msk))){ int who = __lg(msk);
        for(int i=0; i<n; i++) D[msk][i] = G[ter[who]][i];
        continue;
      }
      for(int i=0; i<n; i++)
        for(int sub=(msk-1)&msk; sub; sub=(sub-1)&msk)
          D[msk][i] = min(D[msk][i], D[sub][i] + D[msk^sub][i]);
      memset(tmp, 0x3f, sizeof tmp);
      for(int i=0; i<n; i++) for(int j=0; j<n; j++)
        tmp[i] = min(tmp[i], D[msk][j] + G[j][i]);
      for(int i=0; i<n; i++) D[msk][i] = tmp[i];
    }
    return *min_element(D[(1<<t)-1], D[(1<<t)-1]+n);
  }
};
```

3.24 $O(E \log V)$ Directed MST

```
using D = int; struct edge { int u, v; D w; };
vector<int> DirectedMST(vector<edge> &e, int n, int root){
  using T = pair<D, int>;  // 0-based, return index of edges
  using PQ = pair<priority_queue <T,vector<T>,greater<T>>, D>;
  auto push = [](PQ &pq, T v){
  pq.first.emplace(v.first-pq.second, v.second); };
  auto top = [](const PQ &pq) -> T {
    auto r = pq.first.top(); return {r.first + pq.second,
    r.second}; };
  auto join = [&push, &top](PQ &a, PQ &b) {
    if(a.first.size() < b.first.size()) swap(a, b);
    while(!b.first.empty()) push(a, top(b)), b.first.pop(); };
  vector<PQ> h(n * 2);
  for(int i=0; i<e.size(); i++) push(h[e[i].v], {e[i].w, i});
  vector<int> a(n*2), v(n*2, -1), pa(n*2, -1), r(n*2);
  iota(a.begin(), a.end(), 0);
  auto o = [&](int x) { int y; for(y=x; a[y]!=y; y=a[y]);;;
    for(int ox=x; x!=y; ox=x) x = a[x], a[ox] = y;
    return y; };
  v[root] = n + 1; int pc = n;
  for(int i=0; i<n; i++) if(v[i] == -1) {
    for(int p=i; v[p]==-1 || v[p]==i; p=o(e[r[p]].u)){
      if(v[p] == i){ int q = p; p = pc++;
        do{ h[q].second = -h[q].first.top().first;
          join(h[pa[q]=a[q]=p], h[q]);
        }while((q=o(e[r[q]].u)) != p);
      } v[p] = i;
      while(!h[p].first.empty() && o(e[top(h[p]).second].u) ==
      p) h[p].first.pop();
      r[p] = top(h[p]).second;
    }
  }
  vector<int> ans;
  for(int i=pc-1; i>=0; i--) if(i != root && v[i] != n) {
    for(int f=e[r[i]].v; f!=-1 && v[f]!=n; f=pa[f]) v[f] = n;;;
    ans.push_back(r[i]);
  }
  return ans;
}
```

3.25 $O(E \log V + K \log K)$ K Shortest Walk

```
int rnd(int l, int r){ /* return random int [l,r] */ }
struct node{ // weight>=0, allow multi edge, self loop
  array<node*, 2> son; pair<ll, ll> val;
  node() : node(make_pair(-1e18, -1e18)) {}
  node(pair<ll, ll> val) : node(nullptr, nullptr, val) {}
  node(node *l, node *r, pair<ll,ll> val):son({l,r}),val(val){}
};
node* copy(node *x){ return x ? new node(x->son[0], x->son[1],
x->val) : nullptr; }
node* merge(node *x, node *y){ // precondition: x, y both points
to new entity
  if(!x || !y) return x ? x : y;
  if(x->val > y->val) swap(x, y);
  int rd = rnd(0, 1); if(x->son[rd])
  x->son[rd]=copy(x->son[rd]);
  x->son[rd] = merge(x->son[rd], y); return x;
}
struct edge{
  ll v, c, i; edge() = default;
  edge(ll v, ll c, ll i) : v(v), c(c), i(i) {}
};
vector<vector<edge>> gph, rev; int idx;
void init(int n){ gph = rev = vector<vector<edge>>(n); idx=0; }
void add_edge(int s, int e, ll x){
  gph[s].emplace_back(e, x, idx);
  rev[e].emplace_back(s, x, idx);
  assert(x >= 0); idx++;
}
vector<int> par, pae; vector<ll> dist; vector<node*> heap;
void dijkstra(int snk){ // replace this to SPFA if edge weight
is negative
  int n = gph.size();
  par = pae = vector<int>(n, -1);
  dist = vector<ll>(n, 0x3f3f3f3f3f3f3f3f);
  heap = vector<node*>(n, nullptr);
  priority_queue<pair<ll,ll>,vector<pair<ll,ll>>,greater<>>pq;
  auto enqueue = [&](int v, ll c, int pa, int pe){
    if(dist[v] > c) dist[v] = c, par[v] = pa, pae[v] = pe,
    pq.emplace(c, v);
  }; enqueue(snk, 0, -1, -1); vector<int> ord;
  while(!pq.empty()){
    auto [c,v] = pq.top(); pq.pop(); if(dist[v] != c) continue;
    ord.push_back(v); for(auto e : rev[v]) enqueue(e.v, c+e.c,
    v, e.i);
  }
  for(auto &v : ord){
    if(par[v] != -1) heap[v] = copy(heap[par[v]]);
    for(auto &e : gph[v]){
      if(e.i == pae[v]) continue;
      ll delay = dist[e.v] + e.c - dist[v];
      if(delay < 1e18) heap[v] = merge(heap[v], new
      node(make_pair(delay, e.v)));
    }
  }
}
vector<ll> run(int s, int e, int k){
  using state = pair<ll, node*>; dijkstra(e); vector<ll> ans;
  priority_queue<state, vector<state>, greater<state>> pq;
  if(dist[s] > 1e18) return vector<ll>(k, -1);
  ans.push_back(dist[s]);
  if(heap[s]) pq.emplace(dist[s] + heap[s]->val.first, heap[s]);
  while(!pq.empty() && ans.size() < k){
    auto [cst, ptr] = pq.top(); pq.pop(); ans.push_back(cst);
    for(int j=0; j<2; j++) if(ptr->son[j])
      pq.emplace(cst-ptr->val.first + ptr->son[j]->val.first,
      ptr->son[j]);
    int v = ptr->val.second;
    if(heap[v]) pq.emplace(cst + heap[v]->val.first, heap[v]);
  }
  while(ans.size() < k) ans.push_back(-1);
  return ans;
}
```

3.26 $O(V + E)$ Chordal Graph, Tree Decomposition

```
struct Set { list<int> L; int last; Set() { last = 0; } };
struct PEO {
  int N; list<Set> L;
  vector<vector<int>> g; vector<int> vis, res;
  vector<list<Set>::iterator> ptr;
  vector<list<int>::iterator> ptr2;
  PEO(int n, vector<vector<int> > _g) {
    N = n; g = _g;
    for (int i = 1; i <= N; i++) sort(g[i].begin(), g[i].end());
    vis.resize(N + 1); ptr.resize(N + 1); ptr2.resize(N + 1);
    L.push_back(Set());
    for (int i = 1; i <= N; i++) {
      L.back().L.push_back(i);
      ptr[i] = L.begin(); ptr2[i] = prev(L.back().L.end());
    }
  }
  pair<bool, vector<int>> Run() {
    // lexicographic BFS
    int time = 0;
    while (!L.empty()) {
      if (L.front().L.empty()) { L.pop_front(); continue; }
      auto it = L.begin();
      int n = it->L.front(); it->L.pop_front();
      vis[n] = ++time;
      res.push_back(n);
      for (int next : g[n]) {
        if (vis[next]) continue;
        if (ptr[next]->last != time) {
          L.insert(ptr[next], Set()); ptr[next]->last = time;
        }
        ptr[next]->L.erase(ptr2[next]); ptr[next]--;
```

```cpp
      ptr[next]->L.push_back(next);
      ptr2[next] = prev(ptr[next]->L.end());
    }
  }
  // PEO existence check
  for (int n = 1; n <= N; n++) {
    int mx = 0;
    for (int next : g[n]) if (vis[n] > vis[next]) mx = max(mx,
    vis[next]);
    if (mx == 0) continue;
    int w = res[mx - 1];
    for (int next : g[n]) {
      if (vis[w] > vis[next] && !binary_search(g[w].begin(),
      g[w].end(), next)){
        vector<int> chk(N+1), par(N+1, -1); // w와 next가
        이어져 있지 않다면 not chordal
        deque<int> dq{next}; chk[next] = 1;
        while (!dq.empty()) {
          int x = dq.front(); dq.pop_front();
          for (auto y : g[x]) {
            if (chk[y] || y == n || y != w &&
            binary_search(g[n].begin(), g[n].end(), y))
            continue;
            dq.push_back(y); chk[y] = 1; par[y] = x;
          }
        }
        vector<int> cycle{next, n};
        for (int x=w; x!=next; x=par[x]) cycle.push_back(x);
        return {false, cycle};
      }
    }
  }
  reverse(res.begin(), res.end());
  return {true, res};
 }
};
bool vis[200201]; // 배열 크기 알아서 수정하자.
int p[200201], ord[200201], P = 0; // P=정점 개수
vector<int> V[200201], G[200201]; // V=bags, G=edges
void tree_decomposition(int N, vector<vector<int> > g) {
  for(int i=1; i<=N; i++) sort(g[i].begin(), g[i].end());
  vector<int> peo = PEO(N, g).Run(), rpeo = peo;
  reverse(rpeo.begin(), rpeo.end());
  for(int i=0; i<peo.size(); i++) ord[peo[i]] = i;
  for(int n : rpeo) { // tree decomposition
    vis[n] = true;
    if (n == rpeo[0]) { // 처음
      P++; V[P].push_back(n); p[n] = P; continue;
    }
    int mn = INF, idx = -1;
    for(int next : g[n]) if (vis[next] && mn > ord[next]) mn =
    ord[next], idx = next;
    assert(idx != -1); idx = p[idx];
    // 두 set인 V[idx]와 g[n](visited ver)가 같나?
    // V[idx]의 모든 원소가 g[n]에서 나타나는지 판별로 충분하다.
    int die = 0;
    for(int x : V[idx]) {
      if (!binary_search(g[n].begin(), g[n].end(), x)) { die =
      1; break; }
    }
    if (!die) { V[idx].push_back(n), p[n] = idx; } // 기존 집합에
    추가
    else { // 새로운 집합을 자식으로 추가
      P++;
      G[idx].push_back(P); // 자식으로만 단방향으로 잇자.
      V[P].push_back(n);
      for(int next : g[n]) if (vis[next]) V[P].push_back(next);
      p[n] = P;
    }
  }
  for(int i=1; i<=P; i++) sort(V[i].begin(), V[i].end());
}
```

3.27 $O(V^3)$ General Matching

```cpp
int N, M, R, Match[555], Par[555], Chk[555], Prv[555], Vis[555];
vector<int> G[555];
int Find(int x){return x == Par[x] ? x : Par[x] = Find(Par[x]);}
int LCA(int u, int v){ static int cnt = 0;
  for(cnt++; Vis[u]!=cnt; swap(u, v)) if(u) Vis[u] = cnt, u =
  Find(Prv[Match[u]]);
  return u; }
void Blossom(int u, int v, int rt, queue<int> &q){
  for(; Find(u)!=rt; u=Prv[v]){
    Prv[u] = v; Par[u] = Par[v=Match[u]] = rt;
    if(Chk[v] & 1) q.push(v), Chk[v] = 2;
} }
bool Augment(int u){ // iota Par 0, fill Chk 0
  queue<int> Q; Q.push(u); Chk[u] = 2;
  while(!Q.empty()){ u = Q.front(); Q.pop();
    for(auto v : G[u]){
      if(Chk[v] == 0){
        Prv[v]=u; Chk[v]=1; Q.push(Match[v]); Chk[Match[v]]=2;
        if(!Match[v]){ for(; u; v=u) u = Match[Prv[v]],
        Match[Match[v]=Prv[v]] = v;;; return true; }
      }
      else if(Chk[v] == 2){ int l = LCA(u, v); Blossom(u, v, l,
      Q), Blossom(v, u, l, Q); }
  } /* for v */ } /* while */
  return 0; }
void Run(){ for(int i=1; i<=N; i++) if(!Match[i]) R +=
Augment(i); }
```

3.28 $O(V^3)$ Weighted General Matching

```cpp
namespace weighted_blossom_tree{
  #define d(x) (lab[x.u]+lab[x.v]-e[x.u][x.v].w*2)
  const int N=403*2; using ll = long long; using T = int; // sum
  of weight, single weight
  const T inf=numeric_limits<T>::max()>>1;
  struct Q{ int u, v; T w; } e[N][N]; vector<int> p[N];
  int n, m=0, id, h, t, lk[N], sl[N], st[N], f[N], b[N][N],
  s[N], ed[N], q[N]; T lab[N];
  void upd(int u, int v){ if (!sl[v] || d(e[u][v]) <
  d(e[sl[v]][v])) sl[v] = u; }
  void ss(int v){
    sl[v]=0; for(int u=1; u<=n; u++) if(e[u][v].w > 0 && st[u]
    != v && !s[st[u]]) upd(u, v);
  }
  void ins(int u){ if(u <= n) q[++t] = u; else for(int v : p[u])
  ins(v); }
  void mdf(int u, int w){ st[u]=w; if(u > n) for(int v : p[u])
  mdf(v, w); }
  int gr(int u,int v){
    if ((v=find(p[u].begin(), p[u].end(), v) - p[u].begin()) &
    1){
      reverse(p[u].begin()+1, p[u].end()); return
      (int)p[u].size() - v;
    }
    return v; }
  void stm(int u, int v){
    lk[u] = e[u][v].v;
    if(u <= n) return; Q w = e[u][v];
    int x = b[u][w.u], y = gr(u,x);
    for(int i=0; i<y; i++) stm(p[u][i], p[u][i^1]);
    stm(x,v);rotate(p[u].begin(), p[u].begin()+y, p[u].end()); }
  void aug(int u, int v){
    int w = st[lk[u]]; stm(u, v); if (!w) return;
    stm(w, st[f[w]]); aug(st[f[w]], w); }
  int lca(int u, int v){
    for(++id; u|v; swap(u, v)){
      if(!u) continue; if(ed[u] == id) return u;
      ed[u] = id; if(u = st[lk[u]]) u = st[f[u]]; // not ==
    }
    return 0; }
  void add(int u, int a, int v){
    int x = n+1; while(x <= m && st[x]) x++;
    if(x > m) m++;
    lab[x] = s[x] = st[x] = 0; lk[x] = lk[a];
    p[x].clear(); p[x].push_back(a);
    for(int i=u, j; i!=a; i=st[f[j]]) p[x].push_back(i),
    p[x].push_back(j=st[lk[i]]), ins(j);
    reverse(p[x].begin()+1, p[x].end());
    for(int i=v, j; i!=a; i=st[f[j]]) p[x].push_back(i),
    p[x].push_back(j=st[lk[i]]), ins(j);
    mdf(x,x); for(int i=1; i<=m; i++) e[x][i].w=e[i][x].w=0;
    memset(b[x]+1, 0, n*sizeof b[0][0]);
    for (int u : p[x]){
      for(v=1; v<=m; v++) if(!e[x][v].w || d(e[u][v]) <
      d(e[x][v])) e[x][v] = e[u][v],e[v][x] = e[v][u];
      for(v=1; v<=n; v++) if(b[u][v]) b[x][v] = u;
    }
    ss(x); }
  void ex(int u){  // s[u] == 1
    for(int x : p[u]) mdf(x, x);
    int a = b[u][e[u][f[u]].u],r = gr(u, a);
    for(int i=0; i<r; i+=2){
      int x = p[u][i], y = p[u][i+1];
      f[x] = e[y][x].u; s[x] = 1; s[y] = 0; sl[x] = 0; ss(y);;;
      ins(y); }
    s[a] = 1; f[a] = f[u];
    for(int i=r+1;i<p[u].size();i++)s[p[u][i]]=-1, ss(p[u][i]);
    st[u] = 0; }
  bool on(const Q &e){
    int u=st[e.u], v=st[e.v], a;
    if(s[v] == -1) f[v] = e.u, s[v] = 1, a = st[lk[v]], sl[v] =
    sl[a] = s[a] = 0, ins(a);
    else if(!s[v]){
      a = lca(u, v); if(!a) return aug(u,v), aug(v,u), true;
      else add(u,a,v);
    }
    return false; }
  bool bfs(){
```

```
    memset(s+1, -1, m*sizeof s[0]); memset(sl+1, 0, m*sizeof sl[0]);
    h = 1; t = 0; for(int i=1; i<=m; i++) if(st[i] == i && !lk[i]) f[i] = s[i] = 0, ins(i);
    if(h > t) return 0;
    while (true){
      while (h <= t){
        int u = q[h++];
        if (s[st[u]] != 1) for (int v=1; v<=n; v++) if (e[u][v].w > 0 && st[u] != st[v])
          if(d(e[u][v])) upd(u, st[v]); else if(on(e[u][v]))
          return true;
      }
      T x = inf;
      for(int i=n+1; i<=m; i++) if(st[i] == i && s[i] == 1) x = min(x, lab[i]>>1);
      for(int i=1; i<=m; i++) if(st[i] == i && sl[i] && s[i] != 1) x = min(x, d(e[sl[i]][i])>>s[i]+1);
      for(int i=1; i<=n; i++) if(~s[st[i]]) if((lab[i] += (s[st[i]]*2-1)*x) <= 0) return false;
      for(int i=n+1 ;i<=m; i++) if(st[i] == i && ~s[st[i]]) lab[i] += (2-s[st[i]]*4)*x;
      h = 1; t = 0;
      for(int i=1; i<=m; i++) if(st[i] == i && sl[i] && st[sl[i]] != i && !d(e[sl[i]][i]) && on(e[sl[i]][i]))
      return true;
      for(int i=n+1; i<=m; i++) if(st[i] == i && s[i] == 1 && !lab[i]) ex(i);
    }
    return 0; }
  template<typename TT> pair<int,ll> run(int N, const vector<tuple<int,int,TT>> &edges){ // 1-based
    memset(ed+1, 0, m*sizeof ed[0]); memset(lk+1, 0, m*sizeof lk[0]);
    n = m = N; id = 0; iota(st+1, st+n+1, 1); T wm = 0; ll r = 0;
    for(int i=1; i<=n; i++) for(int j=1; j<=n; j++) e[i][j] = {i,j,0};
    for(auto [u,v,w] : edges) wm = max(wm, e[v][u].w=e[u][v].w=max(e[u][v].w,(T)w));
    for(int i=1; i<=n; i++) p[i].clear();
    for(int i=1; i<=n; i++) for (int j=1; j<=n; j++) b[i][j] = i*(i==j);
    fill_n(lab+1, n, wm); int match = 0; while(bfs()) match++;
    for(int i=1; i<=n; i++) if(lk[i]) r += e[i][lk[i]].w;
    return {match, r/2};
  }
  #undef d
} using weighted_blossom_tree::run, weighted_blossom_tree::lk;
```

4 Math

4.1 Extend GCD, CRT, Combination

```
// ll gcd(ll a, ll b), ll lcm(ll a, ll b), ll mod(ll a, ll b)
tuple<ll,ll,ll> ext_gcd(ll a, ll b){ // return [g,x,y] s.t. ax+by=gcd(a,b)=g
  if(b == 0) return {a, 1, 0}; auto [g,x,y] = ext_gcd(b, a % b);
  return {g, y, x - a/b * y};
}
ll inv(ll a, ll m){ //return x when ax mod m = 1, fail -> -1
  auto [g,x,y] = ext_gcd(a, m); return g == 1 ? mod(x, m) : -1;
}
void DivList(ll n){ // {n/1, n/2, ... , n/n}, size <= 2 sqrt n
  for(ll i=1, j=1; i<=n; i=j+1) cout << i << " " << (j=n/(n/i)) << " " << n/i << "\n"; }
void Div2List(ll n){// n/(i^2), n^{3/4}
  for(ll i=1, j=1; i*i<=n; i=j+1){
    j = (ll)floorl(sqrtl(n/(n/(i*i))));
    cout << i << " " << j << " " << n/(i*i) << " ";
} }//square free: sum_{i=1..sqrt n} mu(i)floor(n/(i^2))
pair<ll,ll> crt(ll a1, ll m1, ll a2, ll m2){
  ll g = gcd(m1, m2), m = m1 / g * m2;
  if((a2 - a1) % g) return {-1, -1};
  ll md = m2/g, s = mod((a2-a1)/g, m2/g);
  ll t = mod(get<1>(ext_gcd(m1/g%md, m2/g)), md);
  return { a1 + s * t % md * m1, m };
}
pair<ll,ll> crt(const vector<ll> &a, const vector<ll> &m){
  ll ra = a[0], rm = m[0];
  for(int i=1; i<m.size(); i++){
    auto [aa,mm] = crt(ra, rm, a[i], m[i]);
    if(mm == -1) return {-1, -1}; else tie(ra,rm) = tie(aa,mm);
  } return {ra, rm};
}
struct Lucas{ // init : O(P), query : O(log P)
  const size_t P;
  vector<ll> fac, inv;
  ll Pow(ll a, ll b){ /* return a^b mod P */ }
  Lucas(size_t P):P(P),fac(P),inv(P){ /* init fac, facinv */ }
  ll small(ll n, ll r) const { /* n! / r! / (n-r)! */ }
  ll calc(ll n, ll r) const { if(n<r || n<0 || r<0) return 0;
    if(!n || !r || n == r) return 1;
    else return small(n%P, r%P) * calc(n/P, r/P) % P; }
};
template<ll p, ll e> struct CombinationPrimePower{
  vector<ll> val; ll m; // init : O(p^e), query : O(log p)
  CombinationPrimePower(){
    m=1; for(int i=0; i<e; i++) m *= p; val.resize(m); val[0]=1;
    for(int i=1; i<m; i++)val[i] = val[i-1] * (i%p ? i : 1) % m;
  }
  pair<ll,ll> factorial(int n){ if(n < p) return {0, val[n]};
    int k = n / p; auto v = factorial(k);
    int cnt = v.first + k, kp = n / m, rp = n % m;
    ll ret = v.second * Pow(val[m-1], kp%2, m) % m * val[rp] % m;
    return {cnt, ret}; }
  ll calc(int n, int r){ if(n < 0 || r < 0 || n < r) return 0;
    auto v1=factorial(n), v2=factorial(r), v3=factorial(n-r);
    ll cnt = v1.first - v2.first - v3.first;
    ll ret = v1.second * inv(v2.second, m) % m * inv(v3.second, m) % m;
    if(cnt >= e) return 0;
    for(int i=1; i<=cnt; i++) ret = ret * p % m;
    return ret; }
};
```

4.2 Partition Number

```
for(int j=1; j*(3*j-1)/2<=i; j++) P[i] += (j%2?1:-1)*P[i-j*(3*j-1)/2], P[i] %= MOD;
for(int j=1; j*(3*j+1)/2<=i; j++) P[i] += (j%2?1:-1)*P[i-j*(3*j+1)/2], P[i] %= MOD;
vector<ModInt> res(sz+1); res[0] = 1; int sq=sqrt(sz);
vector<vector<ModInt>> p(2, vector<ModInt>(sz+1)), d=p;
for(int k=1; k<sq; k++){ p[0][0] = k == 1; // calc p[k][n]
  for(int n=1; n<=sz; n++){
    p[k&1][n] = p[k-1&1][n-1] + (n-k>=0 ? p[k&1][n-k] : 0);
    res[n] += p[k&1][n]; }}
for(int a=sq; a>=0; a--) for(int b=sq; b<=sz; b++){
  d[a&1][b] = d[a+1&1][b-sq] + p[sq-1&1][b-1] + (b-a-1>=0 ? d[a&1][b-a-1] : 0);
  if(a == 0) res[b] += d[a&1][b];
}
```

4.3 Diophantine

```
// solutions to ax + by = c where x in [xlow, xhigh] and y in [ylow, yhigh]
// cnt, leftsol, rightsol, gcd of a and b
template<class T> array<T, 6> solve_linear_diophantine(T a, T b, T c, T xlow, T xhigh, T ylow, T yhigh){
    T g, x, y = euclid(a >= 0 ? a : -a, b >= 0 ? b : -b, x, y);
    array<T, 6> no_sol{0, 0, 0, 0, 0, g};
    if(c % g) return no_sol; x *= c / g, y *= c / g;
    if(a < 0) x = -x; if(b < 0) y = -y;
    a /= g, b /= g, c /= g;
    auto shift = [&](T &x, T &y, T a, T b, T cnt){ x += cnt * b, y -= cnt * a; };
    int sign_a = a > 0 ? 1 : -1, sign_b = b > 0 ? 1 : -1;
    shift(x, y, a, b, (xlow - x) / b);
    if(x < xlow) shift(x, y, a, b, sign_b);
    if(x > xhigh) return no_sol;
    T lx1 = x; shift(x, y, a, b, (xhigh - x) / b);
    if(x > xhigh) shift(x, y, a, b, -sign_b);
    T rx1 = x; shift(x, y, a, b, -(ylow - y) / a);
    if(y < ylow) shift(x, y, a, b, -sign_a);
    if(y > yhigh) return no_sol;
    T lx2 = x; shift(x, y, a, b, -(yhigh - y) / a);
    if(y > yhigh) shift(x, y, a, b, sign_a);
    T rx2 = x; if(lx2 > rx2) swap(lx2, rx2);
    T lx = max(lx1, lx2), rx = min(rx1, rx2);
    if(lx > rx) return no_sol;
    return {(rx - lx) / (b >= 0 ? b : -b) + 1, lx, (c - lx * a) / b, rx, (c - rx * a) / b, g};
}
```

4.4 FloorSum

```
// sum of floor((A*i+B)/M) over 0 <= i < N in O(log(N+M+A+B))
// Also, sum of i * floor((A*i+B)/M) and floor((A*i+B)/M)^2
template<class T, class U> // T must be able to hold arg^2
array<U, 3> weighted_floor_sum(T n, T m, T a, T b){
  array<U, 3> res{}; auto[qa,ra]=div(a,m); auto[qb,rb]=div(b,m);
  if(T n2 = (ra * n + rb) / m){
    auto prv=weighted_floor_sum<T,U>(n2, ra, m, m-rb-1);
    res[0] += U(n-1)*n2 - prv[0];
    res[1] += (U(n-1)*n*n2 - prv[0] - prv[2]) / 2;
    res[2] += U(n-1)*(n2-1)*n2 - 2*prv[1] + res[0];
  }
  res[2] += U(n-1)*n*(2*n-1)/6 * qa*qa + U(n)*qb*qb;
  res[2] += U(n-1)*n * qa*qb + 2*res[0]*qb + 2*res[1]*qa;
  res[0] += U(n-1)*n/2 * qa + U(n)*qb;
  res[1] += U(n-1)*n*(2*n-1)/6 * qa + U(n-1)*n/2 * qb;
  return res;
}
ll modsum(ull to, ll c, ll k, ll m){
```

```
  c = (c % m + m) % m; k = (k % m + m) % m;
  return to*c + k*sumsq(to) - m*divsum(to, c, k, m);
} // sum (ki+c)%m 0<=i<to, O(log m) large constant
```

4.5 XOR Basis (XOR Maximization)

```
vector<ll> basis; // ascending
for(int i=0; i<n; i++){ ll x; cin >> x;
for(int j=(int)basis.size()-1; j>=0; j--) x=min(x,basis[j]^x);
if(x)basis.insert(lower_bound(basis.begin(),basis.end(), x),x);
}//xor maximization, reverse -> for(auto i:basis)r=max(r,r^i);
```

4.6 Stern Brocot Tree

```
pair<ll,ll> Solve(ld l, ld r){//find l<p/q<r -> min q -> min p
  auto g=[](ll v,pair<ll,ll>a,pair<ll,ll>b)->pair<ll,ll>{
    return { v * a.first + b.first, v * a.second + b.second };
    };
  auto f = [g](ll v, pair<ll,ll> a, pair<ll,ll> b) -> ld {
    auto [p,q] = g(v, a, b); return ld(p) / q; };
  pair<ll,ll> s(0, 1), e(1, 0);
  while(true){
    pair<ll,ll> m(s.first+e.first, s.second+e.second);
    ld v = 1.L * m.first / m.second;
    if(v >= r){
      ll ks = 1, ke = 1; while(f(ke, s, e) >= r) ke *= 2;
      while(ks <= ke){
        ll km = (ks + ke) / 2;
        if(f(km, s, e) >= r) ks = km + 1; else ke = km - 1;
      } e = g(ke, s, e);
    }
    else if(v <= l){
      ll ks = 1, ke = 1; while(f(ke, e, s) <= l) ke *= 2;
      while (ks <= ke){
        ll km = (ks + ke) / 2;
        if(f(km, e, s) <= l) ks = km + 1; else ke = km - 1;
      } s = g(ke, e, s);
    }
    else return m;
  }
}
struct Frac { ll p, q; };//find smallest 0 <= p/q <= 1 (p,q<=N)
template<class F> Frac fracBS(F f, ll N) { // s.t. f(p/q) true
  bool dir = 1, A = 1, B = 1; // O(log N)
  Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
  if(f(lo)) return lo; assert(f(hi));
  while(A != 0 || B != 0){
    ll adv = 0, step = 1; // move hi if dir, else lo
    for(int si=0; step; (step*=2)>>=si){ adv += step;
      Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
      if(abs(mid.p)>N || mid.q>N || dir != f(mid))
        adv -= step, si = 2;
    }
    hi.p += lo.p * adv; hi.q += lo.q * adv;
    dir = !dir; swap(lo, hi); A = B; B = adv != 0;
  }
  return dir ? hi : lo;
}
```

4.7 $O(N^3 \log 1/\epsilon)$ Polynomial Equation

```
vector<double> poly_root(vector<double> p, double xmin, double
xmax){
  if(p.size() == 2){ return {-p[0] / p[1]}; }
  vector<double> ret, der(p.size()-1);
  for(int i=0; i<der.size(); i++) der[i] = p[i+1] * (i + 1);
  auto dr = poly_root(der, xmin, xmax);
  dr.push_back(xmin-1); dr.push_back(xmax+1);
  sort(dr.begin(), dr.end());
  for(int i=0; i+1<dr.size(); i++){
    double l = dr[i], h = dr[i+1]; bool sign = calc(p, l) > 0;
    if (sign ^ (calc(p, h) > 0)){
      for(int it=0; it<60; it++){ // while(h-l > 1e-8)
        double m = (l + h) / 2, f = calc(p, m);
        if ((f <= 0) ^ sign) l = m; else h = m;
      }
      ret.push_back((l + h) / 2);
    }
  }
  return ret;
}
```

4.8 Gauss Jordan Elimination

```
template<typename T> // return {rref, rank, det, inv}
tuple<vector<vector<T>>, int, T, vector<vector<T>>>
Gauss(vector<vector<T>> a, bool square=true){ // n500 -400ms
  int n = a.size(), m = a[0].size(), rank = 0;//bitset 4096-700
  vector<vector<T>> out(n, vector<T>(m, 0)); T det = T(1);
  for(int i=0; i<n; i++) if(square) out[i][i] = T(1);
  for(int i=0; i<m; i++){
    if(rank == n) break;
    if(IsZero(a[rank][i])){
      T mx = T(0); int idx = -1; // fucking precision error
      for(int j=rank+1; j<n; j++) if(mx < abs(a[j][i])) mx =
      abs(a[j][i]), idx = j;
      if(idx == -1 || IsZero(a[idx][i])){ det = 0; continue; }
      for(int k=0; k<m; k++){
        a[rank][k] = Add(a[rank][k], a[idx][k]);
        if(square)out[rank][k]=Add(out[rank][k],out[idx][k]);
      }
    }
    det = Mul(det, a[rank][i]);
    T coeff = Div(T(1), a[rank][i]);
    for(int j=0; j<m; j++) a[rank][j] = Mul(a[rank][j], coeff);
    for(int j=0; j<m; j++) if(square) out[rank][j] =
    Mul(out[rank][j], coeff);
    for(int j=0; j<n; j++){
      if(rank == j) continue;
      T t = a[j][i]; // Warning: [j][k], [rank][k]
      for(int k=0; k<m; k++) a[j][k] = Sub(a[j][k],
      Mul(a[rank][k], t));
      for(int k=0; k<m; k++) if(square) out[j][k] =
      Sub(out[j][k], Mul(out[rank][k], t));
    }
    rank++; // linear system: warning len(A) != len(A[0])
  } return {a, rank, det, out}; // linear system: get RREF(A|b)
} // 0 0 ... 0 b[i]: inconsistent, rank < len(A[0]): multiple
// get det(A) mod M, M can be composite number
// remove mod M -> get pure det(A) in integer
ll Det(vector<vector<ll>> a){//destroy matrix, n500 -400ms
  int n = a.size(); ll ans = 1;
  for(int i=0; i<n; i++){
    for(int j=i+1; j<n; j++){
      while(a[j][i] != 0){ // gcd step
        ll t = a[i][i] / a[j][i];
        if(t)for(int k=i;k<n;k++) a[i][k]=(a[i][k]-a[j][k]*t)%M;
        swap(a[i], a[j]); ans *= -1;
      }
    }
    ans = ans * a[i][i] % M; if(!ans) return 0;
  } return (ans + M) % M;
}
```

4.9 Berlekamp + Kitamasa

```
const int mod = 1e9+7; ll pw(ll a, ll b){/*a^b mod M*/}
vector<int> berlekamp_massey(vector<int> x){
  int n = x.size(),L=0,m=0; ll b=1; if(!n) return {};
  vector<int> C(n), B(n), T; C[0]=B[0]=1;
  for(int i=0; ++m && i<n; i++){ ll d = x[i] % mod;
    for(int j=1; j<=L; j++) d = (d + 1LL * C[j] * x[i-j]) % mod;
    if(!d) continue; T=C; ll c = d * pw(b, mod-2) % mod;
    for(int j=m; j<n; j++) C[j] = (C[j] - c * B[j-m]) % mod;
    if(2 * L <= i) L = i-L+1, B = T, b = d, m = 0;
  }
  C.resize(L+1); C.erase(C.begin());
  for(auto &i : C) i = (mod - i) % mod; return C;
} // O(NK + N log mod)
int get_nth(vector<int> rec, vector<int> dp, ll n){
  int m = rec.size(); vector<int> s(m), t(m); ll ret=0;
  s[0] = 1; if(m != 1) t[1] = 1; else t[0] = rec[0];
  auto mul = [&rec](vector<int> v, vector<int> w){
    int m = v.size(); vector<int> t(2*m);
    for(int j=0; j<m; j++) for(int k=0; k<m; k++){
      t[j+k] = (t[j+k] + 1LL * v[j] * w[k]) % mod;
    }
    for(int j=2*m-1; j>=m; j--) for(int k=1; k<=m; k++){
      t[j-k] = (t[j-k] + 1LL * t[j] * rec[k-1]) % mod;
    }
    t.resize(m); return t;
  };
  for(; n; n>>=1, t=mul(t,t)) if(n & 1) s=mul(s,t);
  for(int i=0; i<m; i++) ret += 1LL * s[i] * dp[i] % mod;
  return ret % mod;
} // O(N2 log X)
int guess_nth_term(vector<int> x, ll n){
  if(n < x.size()) return x[n];
  vector<int> v = berlekamp_massey(x);
  return v.empty() ? 0 : get_nth(v, x, n);
}
struct elem{int x, y, v;}; // A_(x, y) <- v, 0-based. no
duplicate please..
vector<int> get_min_poly(int n, vector<elem> M){
  // smallest poly P such that A^i = sum_{j < i} {A^j \times
  P_j}
  vector<int> rnd1, rnd2, gobs; mt19937 rng(0x14004);
  auto gen = [&rng](int lb, int ub){ return
  uniform_int_distribution<int>(lb, ub)(rng); };
  for(int i=0; i<n; i++) rnd1.push_back(gen(1, mod-1)),
  rnd2.push_back(gen(1, mod-1));
  for(int i=0; i<2*n+2; i++){ int tmp = 0;
    for(int j=0; j<n; j++) tmp = (tmp + 1LL * rnd2[j] * rnd1[j])
    % mod;
    gobs.push_back(tmp); vector<int> nxt(n);
    for(auto &j : M) nxt[j.x] = (nxt[j.x] + 1LL * j.v *
    rnd1[j.y]) % mod;
```

```
    rnd1 = nxt;
  } auto v = berlekamp_massey(gobs);
  return vector<int>(v.rbegin(), v.rend());
}
ll det(int n, vector<elem> M){
  vector<int> rnd; mt19937 rng(0x14004);
  auto gen = [&rng](int lb, int ub){ return
  uniform_int_distribution<int>(lb, ub)(rng); };
  for(int i=0; i<n; i++) rnd.push_back(gen(1, mod-1));
  for(auto &i : M) i.v = 1LL * i.v * rnd[i.y] % mod;
  auto sol = get_min_poly(n, M)[0]; if(n % 2 == 0) sol = mod -
  sol;
  for(auto &i : rnd) sol = 1LL * sol * pw(i, mod-2) % mod;
  return sol;
}
```

4.10 Linear Sieve

```
// sp : 최소 소인수, 소수라면 0
// tau : 약수 개수, sigma : 약수 합
// phi : n 이하 자연수 중 n과 서로소인 개수
// mu : non square free이면 0, 그렇지 않다면 (-1)^(소인수 종류)
// e[i] : 소인수분해에서 i의 지수
vector<int> prime;
int sp[sz], e[sz], phi[sz], mu[sz], tau[sz], sigma[sz];
phi[1] = mu[1] = tau[1] = sigma[1] = 1;
for(int i=2; i<=n; i++){
  if(!sp[i]){
    prime.push_back(i);
    e[i] = 1; phi[i] = i-1; mu[i] = -1; tau[i] = 2; sigma[i] =
    i+1;
  }
  for(auto j : prime){
    if(i*j >= sz) break;
    sp[i*j] = j;
    if(i % j == 0){
      e[i*j] = e[i]+1; phi[i*j] = phi[i]*j; mu[i*j] = 0;
      tau[i*j] = tau[i]/e[i*j]*(e[i*j]+1);
      sigma[i*j] = sigma[i]*(j-1)/(pw(j, e[i*j])-1)*(pw(j,
      e[i*j]+1)-1)/(j-1);//overflow
      break;
    }
    e[i*j] = 1; phi[i*j] = phi[i] * phi[j]; mu[i*j] = mu[i] *
    mu[j];
    tau[i*j] = tau[i] * tau[j]; sigma[i*j] = sigma[i] *
    sigma[j];
  }
}
```

4.11 Xudyh Sieve

```
/* e(x) = [x==1], 1(x) = 1, id_k(x) = x^k
mu: mobius function, id(x) = x
phi: euler totient function
sigma_k: sum of k-th power of divisors
sigma = sigma_1, d = tau = sigma_0
sigma_k = id_k * 1  | sigma = id * 1
id_k = sigma_k * mu | id = sigma * mu
e = 1 * mu | d = 1 * 1 | 1 = d * mu
phi * 1 = id | phi = id * mu | sigma = phi * d
g = f * 1 iff f = g * mu */
template<class T, class F1, class F2, class F3>
struct xudyh_sieve{
  T th; // threshold, 2(single query) ~ 5 * MAXN^2/3
  F1 pf; F2 pg; F3 pfg;
  // prefix sum of f(up to th), g(easy to calc), f*g(easy to
  calc)
  unordered_map<T, T> mp; // f * g means dirichlet conv.
  xudyh_sieve(T th,F1 pf,F2 pg,F3
  pfg):th(th),pf(pf),pg(pg),pfg(pfg){}
  // Calculate the preix sum of a multiplicative f up to n
  T query(T n){ // O(n^2/3)
    if(n <= th) return pf(n); if(mp.count(n)) return mp[n];
    T res = pfg(n);
    for(T low = 2, high = 2; low <= n; low = high + 1){
      high = n / (n / low);
      res -= (pg(high) - pg(low - 1)) * query(n / low); // MOD
    }
    return mp[n] = res / pg(1);//Pow(pg(1),MOD-2)?
  }
};
```

4.12 Miller Rabin + Pollard Rho

```
constexpr int SZ = 10'000'000; bool PrimeCheck[SZ+1];
vector<int> Primes;
void Sieve(){ memset(PrimeCheck, true, sizeof PrimeCheck); /*
Sieve */ }
ull MulMod(ull a, ull b, ull c){ return (__uint128_t)a * b % c;
}
// 32bit : 2, 7, 61
// 64bit : 2, 325, 9375, 28178, 450775, 9780504, 1795265022
bool MillerRabin(ull n, ull a){
  if(a % n == 0) return true;
  int cnt = __builtin_ctzll(n - 1);
  ull p = PowMod(a, n >> cnt, n);
  if(p == 1 || p == n - 1) return true;
  while(cnt--) if((p=MulMod(p,p,n)) == n - 1) return true;
  return false;
}
bool IsPrime(ll n){
  if(n <= SZ) return PrimeCheck[n];
  if(n <= 2) return n == 2;
  if(n % 2 == 0 || n % 3 == 0 || n % 5 == 0 || n % 7 == 0 || n %
  11 == 0) return false;
  for(int p : {2, 325, 9375, 28178, 450775, 9780504,
  1795265022}) if(!MillerRabin(n, p)) return false;
  return true;
}
ll Rho(ll n){
  while(true){
    ll x = rand() % (n - 2) + 2, y = x, c = rand() % (n - 1) +
    1;
    while(true){
      x = (MulMod(x,x,n)+c) % n; y = (MulMod(y,y,n)+c) % n; y =
      (MulMod(y,y,n)+c) % n;
      ll d = __gcd(abs(x - y), n); if(d == 1) continue;
      if(IsPrime(d)) return d; else{ n = d; break; }
    }
  }
}
vector<pair<ll,ll>> Factorize(ll n){
  vector<pair<ll,ll>> v;
  int two = __builtin_ctzll(n);
  if(two > 0) v.emplace_back(2, two), n >>= two;
  if(n == 1) return v;
  while(!IsPrime(n)){
    ll d = Rho(n), cnt = 0; while(n % d == 0) cnt++, n /= d;
    v.emplace_back(d, cnt); if(n == 1) break;
  }
  if(n != 1) v.emplace_back(n, 1); return v;
}
```

4.13 Primitive Root, Discrete Log/Sqrt

```
ll PrimitiveRoot(ll p){ // order p-1
  vector<pair<ll,ll>> v = Factorize(p-1);
  for(ll r=1; ; r++){
    bool flag = true; // Warning: 64bit Pow
    for(auto [d,e] : v) if(PowMod(r, (p-1)/d, p) == 1){ flag =
    false; break; }
    if(flag) return r;
  }
}
// Given A, B, P, solve A^x === B mod P, return smallest value
ll DiscreteLog(ll A, ll B, ll P){ // O(sqrt P) with hash set
  __gnu_pbds::gp_hash_table<ll,__gnu_pbds::null_type> st;
  ll t = ceil(sqrt(P)), k = 1; // use binary search?
  for(int i=0; i<t; i++) st.insert(k), k = k * A % P;
  ll inv = Pow(k, P-2, P);
  for(int i=0, s=1; i<t; i++, s=s*inv%P){
    ll x = B * s % P;
    if(st.find(x) == st.end()) continue;
    for(int j=0, f=1; j<t; j++, f=f*A%P){
      if(f == x) return i * t + j;
    }
  }
  return -1;
}
// Given A, P, solve X^2 === A mod P, return arbitrary
ll DiscreteSqrt(ll A, ll P){//O(log^2P), O(logP) in random data
  if(A == 0) return 0;
  if(Pow(A, (P-1)/2, P) != 1) return -1;
  if(P % 4 == 3) return Pow(A, (P+1)/4, P);
  ll s = P - 1, n = 2, r = 0, m;
  while(~s & 1) r++, s >>= 1;
  while(Pow(n, (P-1)/2, P) != P-1) n++;
  ll x = Pow(A, (s+1)/2, P), b = Pow(A, s, P), g = Pow(n, s, P);
  for(;; r=m){
    ll t = b; for(m=0; m<r && t!=1; m++) t = t * t % P;
    if(!m) return x;
    ll gs = Pow(g, 1LL << (r-m-1), P);
    g = gs * gs % P; x = x * gs % P; b = b * g % P;
  }
}
```

4.14 Power Tower

```
bool PowOverflow(ll a, ll b, ll c){
  __int128_t res = 1;
  bool flag = false;
  for(; b; b >>= 1, a = a * a){
    if(a >= c) flag = true, a %= c;
    if(b & 1){
      res *= a; if(flag || res >= c) return true;
    }
  }
  return false;
```

```
}
ll Recursion(int idx, ll mod, const vector<ll> &vec){
  if(mod == 1) return 1;
  if(idx + 1 == vec.size()) return vec[idx];
  ll nxt = Recursion(idx+1, phi[mod], vec);
  if(PowOverflow(vec[idx], nxt, mod)) return Pow(vec[idx], nxt,
  mod) + mod; else return Pow(vec[idx], nxt, mod);
}
ll PowerTower(const vector<ll> &vec, ll mod){ //
vec[0]^(vec[1]^(vec[2]^(...)))
  if(vec.size() == 1) return vec[0] % mod;
  else return Pow(vec[0], Recursion(1, phi[mod], vec), mod);
}
```

4.15 De Bruijn Sequence

```
// Create cyclic string of length k^n that contains every length
n string as substring. alphabet = [0, k - 1]
int res[10000000], aux[10000000]; // >= k^n
int de_bruijn(int k, int n) { // Returns size (k^n)
  if(k == 1) { res[0] = 0; return 1; }
  for(int i = 0; i < k * n; i++) aux[i] = 0;
  int sz = 0;
  function<void(int, int)> db = [&](int t, int p) {
    if(t > n) {
      if(n % p == 0) for(int i=1; i<=p; i++) res[sz++]=aux[i];
    }
    else {
      aux[t] = aux[t - p]; db(t + 1, p);
      for(int i=aux[t-p]+1; i<k; i++) aux[t]=i, db(t+1, t);
    }
  }; db(1, 1); return sz;
}
```

4.16 Simplex / LP Duality

```
// Solves the canonical form: maximize c^T x, subject to ax <= b
and x >= 0.
template<class T> // T must be of floating type
struct linear_programming_solver_simplex{
  int m, n; vector<int> nn, bb; vector<vector<T>> mat;
  static constexpr T eps = 1e-8, inf = 1/.0;
  linear_programming_solver_simplex(const vector<vector<T>> &a,
  const vector<T> &b, const vector<T> &c) : m(b.size()),
  n(c.size()), nn(n+1), bb(m), mat(m+2, vector<T>(n+2)){
    for(int i=0; i<m; i++) for(int j=0; j<n; j++) mat[i][j] =
    a[i][j];
    for(int i=0; i<m; i++) bb[i] = n + i, mat[i][n] = -1,
    mat[i][n + 1] = b[i];
    for(int j=0; j<n; j++) nn[j] = j, mat[m][j] = -c[j];
    nn[n] = -1; mat[m + 1][n] = 1;
  }
  void pivot(int r, int s){
    T *a = mat[r].data(), inv = 1 / a[s];
    for(int i=0; i<m+2; i++) if(i != r && abs(mat[i][s]) > eps)
    {
      T *b = mat[i].data(), inv2 = b[s] * inv;
      for(int j=0; j<n+2; j++) b[j] -= a[j] * inv2;
      b[s] = a[s] * inv2;
    }
    for(int j=0; j<n+2; j++) if(j != s) mat[r][j] *= inv;
    for(int i=0; i<m+2; i++) if(i != r) mat[i][s] *= -inv;
    mat[r][s] = inv; swap(bb[r], nn[s]);
  }
  bool simplex(int phase){
    for(auto x=m+phase-1; ; ){
      int s = -1, r = -1;
      for(auto j=0; j<n+1; j++) if(nn[j] != -phase) if(s == -1
      || pair(mat[x][j], nn[j]) < pair(mat[x][s], nn[s])) s = j;
      if(mat[x][s] >= -eps) return true;
      for(auto i=0; i<m; i++){
        if(mat[i][s] <= eps) continue;
        if(r == -1 || pair(mat[i][n + 1] / mat[i][s], bb[i]) <
        pair(mat[r][n + 1] / mat[r][s], bb[r])) r = i;
      }
      if(r == -1) return false;
      pivot(r, s);
    }
  }
  // Returns -inf if no solution, {inf, a vector satisfying the
  constraints}
  //   if there are abritrarily good solutions, or {maximum c^T
  x, x} otherwise.
  // O(n m (# of pivots)), O(2 ^ n) in general.
  pair<T, vector<T>> solve(){
    int r = 0;
    for(int i=1; i<m; i++) if(mat[i][n+1] < mat[r][n+1]) r = i;
    if(mat[r][n+1] < -eps){
      pivot(r, n);
      if(!simplex(2) || mat[m+1][n+1] < -eps) return {-inf, {}};
      for(int i=0; i<m; i++) if(bb[i] == -1){
          int s = 0;
          for(int j=1; j<n+1; j++) if(s == -1 || pair(mat[i][j],
          nn[j]) < pair(mat[i][s], nn[s])) s = j;
          pivot(i, s);
        }
    }
    bool ok = simplex(1);
    vector<T> x(n);
    for(int i=0; i<m; i++) if(bb[i] < n) x[bb[i]] = mat[i][n +
    1];
    return {ok ? mat[m][n + 1] : inf, x};
  }
};
```

Simplex Example

Maximize $p = 6x + 14y + 13z$

Constraints

- $0.5x + 2y + z \le 24$
- $x + 2y + 4z \le 60$

Coding

- $n = 2, m = 3, a = \begin{pmatrix} 0.5 & 2 & 1 \\ 1 & 2 & 4 \end{pmatrix}, b = \begin{pmatrix} 24 \\ 60 \end{pmatrix}, c = [6, 14, 13]$

LP Duality & Example

tableu를 대각선으로 뒤집고 음수 부호를 붙인 답 = -(원 문제의 답)

- Primal : $n = 2, m = 3, a = \begin{pmatrix} 0.5 & 2 & 1 \\ 1 & 2 & 4 \end{pmatrix}, b = \begin{pmatrix} 24 \\ 60 \end{pmatrix}, c = [6, 14, 13]$
- Dual : $n = 3, m = 2, a = \begin{pmatrix} -0.5 & -1 \\ -2 & -2 \\ -1 & -4 \end{pmatrix}, b = \begin{pmatrix} -6 \\ -14 \\ -13 \end{pmatrix}, c = [-24, -60]$

공식

- Primal : $\max_x c^T x$, Constraints $Ax \le b, x \ge 0$
- Dual : $\min_y b^T y$,Constraints $A^T y \ge c, y \ge 0$

4.17 Polynomial & Convolution

```
// 998,244,353 = 119 23, w 3 | 2,281,701,377 = 17 27, w 3
// 167,772,161 =  10 25, w 3 | 2,483,027,969 = 37 26, w 3
// 469,762,049 =  26 26, w 3 | 2,013,265,921 = 15 27, w 31
using real_t = double; using cpx = complex<real_t>;
void FFT(vector<cpx> &a, bool inv_fft=false){
  int N = a.size(); vector<cpx> root(N/2);//root[0]=1
  for(int i=1, j=0; i<N; i++){ int bit = N / 2;
    while(j >= bit) j -= bit, bit >>= 1;
    if(i < (j += bit)) swap(a[i], a[j]);
  } long double ang = 2 * acosl(-1) / N * (inv_fft ? -1 : 1);
  for(int i=0; i<N/2; i++)root[i]=cpx(cosl(ang*i),sinl(ang*i));
/* NTT : ang = pow(w, (mod-1)/n) % mod, inv_fft -> ang^{-1},
root[i] = root[i-1] * ang
XOR Convolution: roots[*]=1, a[j+k] = u+v, a[j+k+i/2] = u-v
 OR Convolution: roots[*]=1, a[j+k+i/2] += inv_fft ? -u : u;
AND Convolution: roots[*]=1, a[j+k    ] += inv_fft ? -v : v; */
  for(int i=2; i<=N; i<<=1){ int step = N / i;
    for(int j=0; j<N; j+=i) for(int k=0; k<i/2; k++){
        cpx u = a[j+k], v = a[j+k+i/2] * root[step * k];
        a[j+k] = u+v; a[j+k+i/2] = u-v;
  } } // inv_fft: skip for AND/OR convolution.
  if(inv_fft) for(int i=0; i<N; i++) a[i] /= N;
}
vector<ll> Mul(const vector<ll> &_a, const vector<ll> &_b){
  vector<cpx> a(all(_a)), b(all(_b)); // (NTT) 2^19 700ms
  int N = 2; while(N < a.size() + b.size()) N <<= 1;
  a.resize(N); b.resize(N); FFT(a); FFT(b);
  for(int i=0; i<N; i++) a[i] *= b[i]; // mod?
  vector<ll> ret(N); FFT(a, 1); // NTT : just return a
  for(int i=0; i<N; i++) ret[i] = llround(a[i].real());
  while(ret.size() > 1 && ret.back() == 0) ret.pop_back();
  return ret; }
vector<ll> MulMod(const vector<ll> &a, const vector<ll> &b,
const unsigned long long mod){ // (FFT) 2^19 1000ms
  int N = 2; while(N < a.size() + b.size()) N <<= 1;
  vector<cpx> v1(N), v2(N), r1(N), r2(N);
  for(int i=0;i<a.size();i++)v1[i] = cpx(a[i]>>15, a[i]&32767);
  for(int i=0;i<b.size();i++)v2[i] = cpx(b[i]>>15, b[i]&32767);
  FFT(v1); FFT(v2);
  for(int i=0; i<N; i++){ int j = i ? N-i : i;
    cpx ans1 = (v1[i] + conj(v1[j])) * cpx(0.5, 0);
    cpx ans2 = (v1[i] - conj(v1[j])) * cpx(0, -0.5);
    cpx ans3 = (v2[i] + conj(v2[j])) * cpx(0.5, 0);
    cpx ans4 = (v2[i] - conj(v2[j])) * cpx(0, -0.5);
    r1[i] = (ans1 * ans3) + (ans1 * ans4) * cpx(0, 1);
    r2[i] = (ans2 * ans3) + (ans2 * ans4) * cpx(0, 1);
  } vector<ll> ret(N); FFT(r1, true); FFT(r2, true);
  for(int i=0; i<N; i++){
    ll av = llround(r1[i].real()) % mod;
    ll bv = (llround(r1[i].imag()) + llround(r2[i].real()))%mod;
    ll cv = llround(r2[i].imag()) % mod;
    ret[i] = (av << 30) + (bv << 15) + cv;
    ret[i] %= mod; ret[i] += mod; ret[i] %= mod;
  } while(ret.size() > 1 && ret.back() == 0) ret.pop_back();
  return ret; }
template<char op>vector<ll>FWHT_Conv(vector<ll> a,vector<ll> b){
  int n = max({(int)a.size(), (int)b.size()-1, 1});//2^20 700ms
  if(__builtin_popcount(n) != 1) n = 1 << (__lg(n) + 1);
  a.resize(n); b.resize(n); FWHT<op>(a); FWHT<op>(b);
  for(int i=0; i<n; i++) a[i] = a[i] * b[i] % M;
```

```
  FWHT<op>(a, true); return a;
} // subset: C[k] = sum_{i and j = 0, i or j = k} A[i] * B[j]
vector<ll> SubsetConvolution(vector<ll> p,vector<ll> q){//Nlog2N
  int n = max({(int)p.size(), (int)q.size()-1, 1}), w=__lg(n);
  if(__builtin_popcount(n) != 1) n = 1 << (w + 1); // 2^20 4s
  p.resize(n); q.resize(n); vector<ll> res(n); // SOS DP: 2.5s
  vector<vector<ll>> a(w+1, vector<ll>(n)), b(a);
  for(int i=0; i<n; i++) a[__builtin_popcount(i)][i] = p[i];
  for(int i=0; i<n; i++) b[__builtin_popcount(i)][i] = q[i];
  for(int bit=0; bit<=w; bit++) FWHT<'|'>(a[bit]),
  FWHT<'|'>(b[bit]);
  for(int bit=0; bit<=w; bit++){
    vector<ll> c(n); // Warning : MOD
    for(int i=0; i<=bit; i++) for(int j=0; j<n; j++) c[j] +=
    a[i][j] * b[bit-i][j] % M;
    for(auto &i : c) i %= M;
    FWHT<'|'>(c, true);
    for(int i=0; i<n; i++) if(__builtin_popcount(i) == bit)
    res[i] = c[i];
  } return res; }
vector<ll> Trim(vector<ll> a, size_t sz){ a.resize(min(a.size(),
sz)); return a; }
vector<ll> Inv(const vector<ll> &a, size_t sz){ // 5e5 2s
  vector<ll> q(1, Pow(a[0], M-2)); // 1/a[0], a[0] != 0
  for(int i=1; i<sz; i<<=1){ // - : polynomial minus
    auto p = vector<ll>{2} - Mul(q, Trim(a, i*2));
    q = Trim(Mul(p, q), i*2);
  } return Trim(q, sz); }
vector<ll> Div(const vector<ll> &a, const vector<ll> &b){
  if(a.size() < b.size()) return {}; // 5e5 4s
  size_t sz = a.size() - b.size() + 1; auto ra = a, rb = b;
  reverse(ra.begin(), ra.end()); ra = Trim(ra, sz);
  reverse(rb.begin(), rb.end()); rb = Inv(Trim(rb,sz), sz);
  auto res = Trim(Mul(ra, rb), sz); res.resize(sz);
  reverse(res.begin(), res.end());
  while(!res.empty() && !res.back()) res.pop_back();
  return res; }
vector<ll> Mod(const vector<ll> &a, const vector<ll> &b){ return
a - Mul(b, Div(a, b)); }
ll Evaluate(const vector<ll> &a, ll x){ ll res = 0;
  for(int i=(int)a.size()-1; i>=0; i--) res = (res*x+a[i]) % M;
  return res >= 0 ? res : res + M; }
vector<ll> Derivative(const vector<ll> &a){
  if(a.size() <= 1) return {}; vector<ll> res(a.size()-1);
  for(int i=0; i+1<a.size(); i++) res[i] = (i+1) * a[i+1] % M;
  return res; }
vector<ll> Integrate(const vector<ll> &a){
  int n = a.size(); vector<ll> res(n+1);
  for(int i=0; i<n; i++) res[i+1] = a[i] * Pow(i+1, M-2) % M;
  return res; }
vector<ll> MultipointEvaluation(vector<ll> a, vector<ll> x){
  if(x.empty()) return {}; int n = x.size(); // 2^17 7s
  vector<vector<ll>> up(n*2), dw(n*2);
  for(int i=0; i<n; i++) up[i+n] = {x[i]?M-x[i]:0, 1};
  for(int i=n-1; i; i--) up[i] = Mul(up[i*2], up[i*2+1]);
  dw[1] = Mod(a, up[1]);
  for(int i=2; i<n*2; i++) dw[i] = Mod(dw[i/2], up[i]);
  vector<ll> y(n); for(int i=0; i<n; i++) y[i] = dw[i+n][0];
  return y; }
vector<ll> Interpolation(vector<ll> x, vector<ll> y){//2^17 10s
  int n = x.size(); vector<vector<ll>> up(n*2), dw(n*2);
  for(int i=0; i<n; i++) up[i+n] = {x[i]?M-x[i]:0, 1};
  for(int i=n-1; i; i--) up[i] = Mul(up[i*2], up[i*2+1]);
  vector<ll> a = MultipointEvaluation(Derivative(up[1]), x);
  for(int i=0; i<n; i++) a[i] = y[i] * Pow(a[i], M-2) % M;
  for(int i=0; i<n; i++) dw[i+n] = {a[i]};
  for(int i=n-1; i; i--){
    auto l = Mul(dw[i*2],up[i*2+1]), r = Mul(dw[i*2+1],up[i*2]);
    dw[i].resize(l.size());
    for(int j=0; j<l.size(); j++) dw[i][j] = (l[j] + r[j]) % M;
  } return dw[1]; }
vector<ll> Log(const vector<ll> &a, size_t sz){ // 5e5 3.5s
  assert(a.size() > 0 && a[0] == 1); // int f'(x)/f(x), resize!
  return Trim(Integrate(Mul(Derivative(a), Inv(a,sz))), sz); }
vector<ll> Exp(const vector<ll> &a, size_t sz){ // 5e5 5s
  vector<ll> res = {1}; if(a.empty()) return {1};
  assert(a.size() > 0 && a[0] == 0);
  for(int i=1; i<sz; i<<=1){
    auto t = Trim(a, i*2) - Log(res, i*2);
    if(++t[0] == M) t[0] = 0; // t[0] += 1, mod
    res = Trim(Mul(res, t), i*2);
  } return Trim(res, sz); } // need resize
vector<ll> Pow(const vector<ll> f, ll e, int sz){ // 5e5 8s
  if(e == 0){ vector<ll> res(sz); res[0] = 1; return res; }
  ll p = 0; while(p < f.size() && f[p] == 0 && p*e < sz) p++;
  if(p == f.size() || p*e >= sz) return vector<ll>(sz, 0);
  vector<ll> a(f.begin()+p, f.end()); ll k = a[0]; // not f[0]
  for(auto &i : a) i = mul(i, Pow(k, M-2));
  a = Log(a, sz); for(auto &i : a) i = mul(i, e%M);
  a = Exp(a, sz); for(auto &i : a) i = mul(i, Pow(k, e));
  vector<ll> res(p*e); res.insert(res.end(), a.begin(),
  a.end());
  res.resize(sz); return res; }
vector<ll> SqrtImpl(vector<ll> a){
  if (a.empty()) return {0}; int inv2=(M+1)/2;
  int z = DiscreteSqrt(a[0], M), n = a.size();
  if (z == -1) return {-1}; vector<ll> q(1, z);
  for(int m=1; m<n; m<<=1){
    if(n < m*2) a.resize(m*2);;; q.resize(m*2);
    auto f2 = Mul(q, q); f2.resize(m*2);
    for(int i=0; i<m*2; i++) f2[i] = sub(f2[i], a[i]);
    f2 = Mul(f2, Inv(q, q.size())); f2.resize(m*2);
    for(int i=0; i<m*2; i++) q[i] = sub(q[i], mul(f2[i], inv2));
  } q.resize(n); return q; }
vector <ll> Sqrt(vector <ll> a){ // nlgn, fail -> -1, 5e5 5.5s
  int n = a.size(), m = 0; while(m < n && a[m] == 0) m++;
  if(m == n) return vector<ll>(n); if(m & 1) return {-1};
  auto s = SqrtImpl(vector<ll>(a.begin()+m, a.end()));
  if(s[0] == -1) return {-1}; vector<ll> res(n);
  for(int i=0; i<s.size(); i++) res[i+m/2] = s[i];
  return res; }
vector<ll> TaylorShift(vector<ll> a, ll c){//f(x+c), 2^19 700ms
  int n = a.size(); // fac[i] = i!, ifc[i] = inv(i!)
  for(int i=0; i<n; i++) a[i] = mul(a[i], fac[i]);
  reverse(all(a)); vector<ll> b(n); ll w = 1;
  for(int i=0; i<n; i++) b[i] = mul(ifc[i], w), w = mul(w, c);
  a = Mul(a, b); a.resize(n); reverse(all(a));
  for(int i=0; i<n; i++) a[i] = mul(a[i], ifc[i]);
  return a; }
vector<ll> SamplingShift(vector<ll> a, ll c, int m){ // 2^19 ~2s
  // given f(0), f(1), ..., f(n - 1), warning: fac size
  // return f(c), f(c + 1), ..., f(c + m - 1)
  int n = a.size(); vector<ll> b(ifc.begin(), ifc.begin()+n);
  for(int i=0; i<n; i++) a[i] = mul(a[i],ifc[i]);
  for(int i=1; i<n; i+=2) b[i] = sub(0, b[i]);
  a = Mul(a, b); a.resize(n); ll w = 1;
  for(int i=0; i<n; i++) a[i] = mul(a[i], fac[i]);;
  reverse(all(a));
  for(int i=0; i<n; w=mul(w, sub(c,i++))) b[i] = mul(ifc[i],w);
  a = Mul(a, b); a.resize(n); reverse(all(a)); //warning: N->M
  for(int i=0; i<n; i++) a[i] = mul(a[i], ifc[i]);; a.resize(m);
  b = vector<ll>(ifc.begin(), ifc.begin()+m);
  a = Mul(a, b); a.resize(m);
  for(int i=0; i<m; i++) a[i] = mul(a[i], fac[i]);
  return a; }
vector<double> interpolate(vector<double> x, vector<double> y,
int n){ // n^2
  vector<double> res(n), temp(n);
  for(int k=0; k<n-1; k++) for(int i=k+1; i<n; i++) y[i] = (y[i]
  - y[k]) / (x[i] - x[k]);
  double last = 0; temp[0] = 1;
  for(int k=0; k<n; k++){
  for(int i=0; i<n; i++) res[i] += y[k] * temp[i], swap(last,
  temp[i]), temp[i] -= last * x[k];
  }
  return res; }//for numerical precision, x[k]=c*cos(k*pi/(n-1))
vector<ll> Interpolation_0_to_n(vector<ll> y){ // n^2
  int n = y.size();
  vector<ll> res(n), tmp(n), x; // x[i] = i / (i+1)
  for(int i=0; i<n; i++) x.push_back(Pow(i+1, M-2));
  for(int k=0; k+1<n; k++) for(int i=k+1; i<n; i++)
    y[i] = (y[i] - y[k] + M) * x[i-k-1] % M;
  ll lst = 0; tmp[0] = 1;
  for(int k=0; k<n; k++) for(int i=0; i<n; i++) {
    res[i] = (res[i] + y[k] * tmp[i]) % M;
    swap(lst, tmp[i]);
    tmp[i] = (tmp[i] - lst * k) % M;
    if(tmp[i] < 0) tmp[i] += M;
  } return res; }
```

4.18 Matroid Intersection

```
struct Matroid{
  virtual bool check(int i) = 0; // O(R^2N), O(R^2N)
  virtual void insert(int i) = 0; // O(R^3), O(R^2N)
  virtual void clear() = 0; // O(R^2), O(RN)
};
template<typename cost_t>
vector<cost_t> MI(const vector<cost_t> &cost, Matroid *m1,
Matroid *m2){
  int n = cost.size();
  vector<pair<cost_t, int>> dist(n+1);
  vector<vector<pair<int, cost_t>>> adj(n+1);
  vector<int> pv(n+1), inq(n+1), flag(n); deque<int> dq;
  auto augment = [&]() -> bool {
    fill(dist.begin(), dist.end(),
    pair(numeric_limits<cost_t>::max()/2, 0));
    fill(adj.begin(), adj.end(), vector<pair<int, cost_t>>());
    fill(pv.begin(),pv.end(),-1); fill(inq.begin(),inq.end(),0);
    dq.clear(); m1->clear(); m2->clear();
    for(int i=0;i<n;i++)if(flag[i])m1->insert(i),m2->insert(i);
    for(int i=0; i<n; i++){
      if(flag[i]) continue;
      if(m1->check(i))
```

```
      dist[pv[i]=i] = {cost[i], 0}, dq.push_back(i), inq[i]=1;
    if(m2->check(i)) adj[i].emplace_back(n, 0);
  }
  for(int i=0; i<n; i++){
    if(!flag[i]) continue; m1->clear(); m2->clear();
    for(int j=0; j<n; j++) if(i != j && flag[j])
    m1->insert(j), m2->insert(j);
    for(int j=0; j<n; j++){
      if(flag[j]) continue;
      if(m1->check(j)) adj[i].emplace_back(j, cost[j]);
      if(m2->check(j)) adj[j].emplace_back(i, -cost[i]);
    }
  }
  while(dq.size()){
    int v = dq.front(); dq.pop_front(); inq[v] = 0;
    for(const auto &[i,w] : adj[v]){
      pair<cost_t, int> nxt{dist[v].ff+w, dist[v].ss+1};
      if(nxt < dist[i]){
        dist[i] = nxt; pv[i] = v;
        if(!inq[i]) dq.push_back(i), inq[i] = 1;
      } /* if */ } /* for [i,w] */ } /* while */
  if(pv[n] == -1) return false;
  for(int i=pv[n]; ; i=pv[i]){
    flag[i] ^= 1; if(i == pv[i]) break;
  } return true;
}; vector<cost_t> res;
while(augment()){
  cost_t now = cost_t(0);
  for(int i=0; i<n; i++) if(flag[i]) now += cost[i];
  res.push_back(now);
} return res;
}
```

5 String

5.1 KMP, Hash, Manacher, Z

```
vector<int> getFail(const container &pat){
  vector<int> fail(pat.size());
//match: pat[0..j] and pat[j-i..i] is equivalent
//ins/del: manipulate corresponding range to pattern starts at 0
//         (insert/delete pat[i], manage pat[j-i..i])
  function<bool(int, int)> match = [&](int i, int j){ };
  function<void(int)> ins = [&](int i){ };
  function<void(int)> del = [&](int i){ };
  for(int i=1, j=0; i<pat.size(); i++){
    while(j && !match(i, j)){
      for(int s=i-j; s<i-fail[j-1]; s++) del(s);
      j = fail[j-1]; }
    if(match(i, j)) ins(i), fail[i] = ++j;
  } return fail;
}
vector<int> doKMP(const container &str, const container &pat){
  vector<int> ret, fail = getFail(pat);
//match: pat[0..j] and str[j-i..i] is equivalent
//ins/del: manipulate corresponding range to pattern starts at 0
//          (insert/delete str[i], manage str[j-i..i])
  function<bool(int, int)> match = [&](int i, int j){ };
  function<void(int)> ins = [&](int i){ };
  function<void(int)> del = [&](int i){ };
  for(int i=0, j=0, s; i<str.size(); i++){
    while(j && !match(i, j)){
      for(int s=i-j; s<i-fail[j-1]; s++) del(s);
      j = fail[j-1]; }
    if(match(i, j)){
      if(j+1 == pat.size()){
        ret.push_back(i-j); for(s=i-j;s<i-fail[j]+1; s++)del(s);
        j = fail[j];
      } else ++j;      ins(i);
    } } return ret;
}
// # a # b # a # a # b # a #
// 0 1 0 3 0 1 6 1 0 3 0 1 0
vector<int> Manacher(const string &inp){
  int n = inp.size() * 2 + 1; vector<int> ret(n);
  string s = "#"; for(auto i : inp) s += i, s += "#";
  for(int i=0, p=-1, r=-1; i<n; i++){
    ret[i] = i <= r ? min(r-i, ret[2*p-i]) : 0;
    while(i-ret[i]-1 >= 0 && i+ret[i]+1 < n && s[i-ret[i]-1] ==
    s[i+ret[i]+1]) ret[i]++;
    if(i+ret[i] > r) r = i+ret[i], p = i;
  } return ret;
}
// input: manacher array, 1-based hashing structure
// output: set of pair(hash_val, length)
set<pair<hash_t,int>> UniquePalindrome(const vector<int> &dp,
const Hashing &hashing){
  set<pair<hash_t,int>> st;
  for(int i=0,s,e; i<dp.size(); i++){
    if(!dp[i]) continue;
    if(i & 1) s = i/2 - dp[i]/2 + 1, e = i/2 + dp[i]/2 + 1;
    else s = (i-1)/2 - dp[i]/2 + 2, e = (i+1)/2 + dp[i]/2;
    for(int l=s, r=e; l<=r; l++, r--){
      auto now = hashing.get(l, r);
      auto [iter,flag] = st.emplace(now, r-l+1);
      if(!flag) break;
    }
  } return st;
}
//z[i]=match length of s[0,n-1] and s[i,n-1]
vector<int> Z(const string &s){
  int n = s.size(); vector<int> z(n); z[0] = n;
  for(int i=1, l=0, r=0; i<n; i++){
    if(i < r) z[i] = min(r-i-1, z[i-l]);
    while(i+z[i] < n && s[i+z[i]] == s[z[i]]) z[i]++;
    if(i+z[i] > r) r = i+z[i], l = i;
  } return z;
}
```

5.2 Aho-Corasick

```
struct Node{
  int g[26], fail, out;
  Node() { memset(g, 0, sizeof g); fail = out = 0; }
};
vector<Node> T(2); int aut[100101][26];
void Insert(int n, int i, const string &s){
  if(i == s.size()){ T[n].out++; return; }
  int c = s[i] - 'a';
  if(T[n].g[c] == 0) T[n].g[c] = T.size(), T.emplace_back();
  Insert(T[n].g[c], i+1, s);
}
int go(int n, int i){ // DO NOT USE `aut` DIRECTLY
  int &res = aut[n][i]; if(res) return res;
  if(n != 1 && T[n].g[i] == 0) res = go(T[n].fail, i);
  else if(T[n].g[i] != 0) res = T[n].g[i]; else res = 1;
  return res;
}
void Build(){
  queue<int> q; q.push(1); T[1].fail = 1;
  while(!q.empty()){
    int n = q.front(); q.pop();
    for(int i=0; i<26; i++){
      int next = T[n].g[i]; if(next == 0) continue;
     if(n == 1)T[next].fail=1;else T[next].fail=go(T[n].fail,i);
      q.push(next); T[next].out += T[T[next].fail].out;
    } /* for i */ } /* while q */ } /* build */
bool Find(const string &s){
  int n = 1, ok = 0;
  for(int i=0; i<s.size(); i++){
    n = go(n, s[i] - 'a'); if(T[n].out != 0) ok = 1;
  } return ok;
}
```

5.3 $O(N \log N)$ SA + LCP

```
pair<vector<int>, vector<int>> SuffixArray(const string &s){
  int n = s.size(), m = max(n, 256);
  vector<int> sa(n), lcp(n), pos(n), tmp(n), cnt(m);
  auto counting_sort = [&](){
    fill(cnt.begin(), cnt.end(), 0);
    for(int i=0; i<n; i++) cnt[pos[i]]++;
    partial_sum(cnt.begin(), cnt.end(), cnt.begin());
    for(int i=n-1; i>=0; i--) sa[--cnt[pos[tmp[i]]]] = tmp[i];
  };
  for(int i=0; i<n; i++) sa[i] = i, pos[i] = s[i], tmp[i] = i;
  counting_sort();
  for(int k=1; ; k<<=1){ int p = 0;
    for(int i=n-k; i<n; i++) tmp[p++] = i;
    for(int i=0; i<n; i++) if(sa[i] >= k) tmp[p++] = sa[i] - k;
    counting_sort(); tmp[sa[0]] = 0;
    for(int i=1; i<n; i++){
      tmp[sa[i]] = tmp[sa[i-1]];
      if(sa[i-1]+k < n && sa[i]+k < n && pos[sa[i-1]] ==
      pos[sa[i]] && pos[sa[i-1]+k] == pos[sa[i]+k]) continue;
      tmp[sa[i]] += 1;
    }
    swap(pos, tmp); if(pos[sa.back()] + 1 == n) break;
  }
  for(int i=0, j=0; i<n; i++, j=max(j-1,0)){
    if(pos[i] == 0) continue;
    while(sa[pos[i]-1]+j < n && sa[pos[i]]+j < n &&
    s[sa[pos[i]-1]+j] == s[sa[pos[i]]+j]) j++;
    lcp[pos[i]] = j;
  } return {sa, lcp};
}
auto [SA,LCP] = SuffixArray(S); RMQ<int> rmq(LCP);
vector<int> Pos(N); for(int i=0; i<N; i++) Pos[SA[i]] = i;
auto get_lcp = [&](int a, int b){
    if(Pos[a] > Pos[b]) swap(a, b);
    return a == b ? (int)S.size() - a : rmq.query(Pos[a]+1,
    Pos[b]);
};
vector<pair<int,int>> can; // common substring {start, lcp}
vector<tuple<int,int,int>> valid; // valid substring [string,
end_l~end_r]
```

```
for(int i=1; i<N; i++){
  if(SA[i] < X && SA[i-1] > X) can.emplace_back(SA[i], LCP[i]);
  if(i+1 < N && SA[i] < X && SA[i+1] > X)
  can.emplace_back(SA[i], LCP[i+1]);
}
for(int i=0; i<can.size(); i++){
  int skip = i > 0 ? min({can[i-1].second, can[i].second,
  get_lcp(can[i-1].first, can[i].first)}) : 0;
  valid.emplace_back(can[i].first, can[i].first + skip,
  can[i].first + can[i].second - 1);
}
```

5.4 $O(N \log N)$ Tandem Repeats

```
// return O(n log n) tuple {l, r, p} that
// [i, i+p) = [i+p, i+2p) for all l <= i < r
vector<tuple<int,int,int>> TandemRepeat(const string &s){
  int n = s.size(); vector<tuple<int,int,int>> res;
  string t = s; reverse(t.begin(), t.end());
  // WARNING: add empty suffix!!
  // sa.insert(sa.begin(), n) before calculate lcp/pos
  auto [sa_s,lcp_s,pos_s] = SuffixArray(s);
  auto [sa_t,lcp_t,pos_t] = SuffixArray(t);
  RMQ<int> rmq_s(lcp_s), rmq_t(lcp_t);
  auto get = [n](const vector<int> &pos, const RMQ<int> &rmq,
  int a, int b){
    if(pos[a] > pos[b]) swap(a, b);
    return a == b ? n - a : rmq.query(pos[a] + 1, pos[b]);
  };
  for(int p=1; p*2<=n; p++){
    for(int i=0, j=-1; i+p<=n; i+=p){
      int l = i - get(pos_t, rmq_t, n-i-p, n-i);
      int r = i - p + get(pos_s, rmq_s, i, i+p);
      if(l <= r && l != j) res.emplace_back(j=l, r+1, p);
  }} return res;
} // Check p = 0, time complexity O(n log n)
```

5.5 Suffix Automaton

```
template<typename T, size_t S, T init_val>
struct initialized_array : public array<T, S> {
  initialized_array(){ this->fill(init_val); }
};
template<class Char_Type, class Adjacency_Type>
struct suffix_automaton{
  // Begin States
  // len: length of the longest substring in the class
  // link: suffix link
  // firstpos: minimum value in the set endpos
  vector<int> len{0}, link{-1}, firstpos{-1}, is_clone{false};
  vector<Adjacency_Type> next{{}};
  ll ans{0LL}; // 서로 다른 부분 문자열 개수
  // End States
  void set_link(int v, int lnk){
    if(link[v] != -1) ans -= len[v] - len[link[v]];
    link[v] = lnk;
    if(link[v] != -1) ans += len[v] - len[link[v]];
  }
  int new_state(int l, int sl, int fp, bool c, const
  Adjacency_Type &adj){
    int now = len.size(); len.push_back(l); link.push_back(-1);
    set_link(now, sl); firstpos.push_back(fp);
    is_clone.push_back(c); next.push_back(adj); return now;
  } int last = 0;
  void extend(const vector<Char_Type> &s){
    last = 0; for(auto c: s) extend(c); }
  void extend(Char_Type c){
    int cur = new_state(len[last] + 1, -1, len[last], false,
    {}), p = last;
    while(~p && !next[p][c]) next[p][c] = cur, p = link[p];
    if(!~p) set_link(cur, 0);
    else{
      int q = next[p][c];
      if(len[p] + 1 == len[q]) set_link(cur, q);
      else{
        int clone = new_state(len[p] + 1, link[q], firstpos[q],
        true, next[q]);
        while(~p && next[p][c] == q) next[p][c] = clone, p =
        link[p];
        set_link(cur, clone); set_link(q, clone);
      }
    }
    last = cur;
  }
  int size() const {  return (int)len.size(); } // # of states
}; suffix_automaton<int, initialized_array<int,26,0>> T;
// for(auto c : s) if((x=T.next[x][c]) == 0) return false;
```

5.6 Bitset LCS

```
#include <x86intrin.h>
template<size_t _Nw> void _M_do_sub(_Base_bitset<_Nw> &A, const
_Base_bitset<_Nw> &B){
  for(int i=0, c=0; i<_Nw; i++) c = _subborrow_u64(c, A._M_w[i],
  B._M_w[i], (ull*)&A._M_w[i]);
}
void _M_do_sub(_Base_bitset<1> &A, const _Base_bitset<1> &B){
A._M_w -= B._M_w; }
template<size_t _Nb> bitset<_Nb>& operator-=(bitset<_Nb> &A,
const bitset<_Nb> &B){
  _M_do_sub(A, B); return A;
}
template<size_t _Nb> inline bitset<_Nb> operator-(const
bitset<_Nb> &A, const bitset<_Nb> &B){
  bitset<_Nb> C(A); return C -= B;
}
char s[50050], t[50050];
int lcs(){ // O(NM/64)
  bitset<50050> dp, ch[26];
  int n = strlen(s), m = strlen(t);
  for(int i=0; i<m; i++) ch[t[i]-'A'].set(i);
  for(int i=0; i<n; i++){ auto x = dp | ch[s[i]-'A']; dp = dp -
  (dp ^ x) & x; }
  return dp.count();
}
```

5.7 Lyndon Factorization, Minimum Rotation

```
// link[i]: length of smallest suffix of s[0..i-1]
// factorization result: s[res[i]..res[i+1]-1]
vector<int> Lyndon(const string &s){
  int n = s.size(); vector<int> link(n);
  for(int i=0; i<n; ){
    int j=i+1, k=i; link[i] = 1;
    for(; j<n && s[k]<=s[j]; j++){
      if(s[j] == s[k]) link[j] = link[k], k++;
      else link[j] = j - i + 1, k = i;
    } for(; i<=k; i+=j-k);
  } vector<int> res;
  for(int i=n-1; i>=0; i-=link[i]) res.push_back(i-link[i]+1);
  reverse(res.begin(), res.end()); return res;
}
// rotate(v.begin(), v.begin()+min_rotation(v), v.end());
template<typename T> int min_rotation(T s){ // O(N)
  int a = 0, N = s.size();
  for(int i=0; i<N; i++) s.push_back(s[i]);
  for(int b=0; b<N; b++) for(int k=0; k<N; k++){
    if(a+k == b || s[a+k] < s[b+k]){ b += max(0, k-1); break; }
    if(s[a+k] > s[b+k]){ a = b; break; }
  }
  return a;
}
```

5.8 All LCS

```
void AllLCS(const string &s, const string &t){
  vector<int> h(t.size()); iota(h.begin(), h.end(), 0);
  for(int i=0, v=-1; i<s.size(); i++, v=-1){
    for(int r=0; r<t.size(); r++){
      if(s[i] == t[r] || h[r] < v) swap(h[r], v);
      //LCS(s[0..i],t[l..r]) = r-l+1 - sum([h[x] >= l] | x <= r)
} /*for r*/ } /* for i */ } /* end*/
```

6 Misc

6.1 CMakeLists.txt

```
set(CMAKE_CXX_STANDARD 17)
set(CMAKE_CXX_FLAGS "-DLOCAL -lm -g -Wl,--stack,268435456")
add_compile_options(-Wall -Wextra -Winvalid-pch -Wfloat-equal
-Wno-sign-compare -Wno-misleading-indentation -Wno-parentheses)
# add_compile_options(-O3 -mavx -mavx2 -mfma)
```

6.2 Stack Hack

```
int main2(){ return 0; }
int main(){
  size_t  sz = 1<<29;  // 512MB
  void* newstack = malloc(sz);
  void* sp_dest = newstack + sz - sizeof(void*);
  asm  __volatile__("movq %0, %%rax\n\t"
          "movq %%rsp , (%%rax)\n\t"
          "movq %0, %%rsp\n\t": : "r"(sp_dest): );
  main2();
  asm  __volatile__("pop %rsp\n\t");
  return 0; }
```

6.3 Python Decimal

```
from fractions import Fraction
from decimal import Decimal, getcontext
getcontext().prec = 250 # set precision
N, two, itwo = 200, Decimal(2), Decimal(0.5)
# sin(x) = sum (-1)^n x^(2n+1) / (2n+1)!
# cos(x) = sum (-1)^n x^(2n) / (2n)!
def angle(cosT):
  #given cos(theta) in decimal return theta
  for i in range(N): cosT=((cosT+1)/two)**itwo
  sinT = (1-cosT*cosT)**itwo
  return sinT*(2**N)
pi = angle(Decimal(-1))
```

6.4 Java I/O

```
// java.util.*, java.math.*, java.io.*
public class Main{ // BufferedReader, BufferedWriter
public static void main(String[] args) throws IOException {
br=new BufferedReader(new InputStreamReader(System.in));
bw=new BufferedWriter(new OutputStreamWriter(System.out));
String[] ar = br.readLine().split(" ");
int a=Integer.parseInt(ar[0]), b=Integer.parseInt(ar[1]);
bw.write(String.valueOf(a+b)+"\n");br.close();bw.close();
ArrayList<Integer> a = new ArrayList<>();
a.add(1234); a.get(0); a.remove(a.size()-1); a.clear();
}}
```

6.5 Calendar

```
int f(int y,int m,int d){// 0: Sat, 1: Sun, ...
  if (m<=2) y--, m+=12; int c=y/100; y%=100;
  int w=((c>>2)-(c<<1)+y+(y>>2)+(13*(m+1)/5)+d-1)%7;
  if (w<0) w+=7; return w; }
```

6.6 Ternary Search

```
while(s + 3 <= e){
  T l = (s + s + e) / 3, r = (s + e + e) / 3;
  if(Check(l) > Check(r)) s = l; else e = r;
}// get minimum / when multiple answer, find minimum `s`
T mn = INF, idx = s;
for(T i=s; i<=e; i++) if(T now = Check(i); now < mn) mn = now,
idx = i;
```

6.7 Add/Mul Update, Range Sum Query

```
struct Lz{
  ll a, b; // constructor, clear(a = 1, b = 0)
  Lz& operator+=(const Lz &t); // a *= t.a, b = t.a * b + t.b
};
struct Ty{
  ll cnt, sum; // constructor cnt=1, sum=0
  Ty& operator += (const Ty &t); // cnt += t.cnt, sum += t.sum
  Ty* operator += (const Lz &t); // sum= t .a * sum + cnt * t.b}
};
```

6.8 $O(N \times \max W)$ Subset Sum (Fast Knapsack)

```
// O(N*maxW), maximize sumW <= t
int Knapsack(vector<int> w, int t){
  int a = 0, b = 0, x;
  while(b < w.size() && a + w[b] <= t) a += w[b++];
  if(b == w.size()) return a;
  int m = *max_element(w.begin(), w.end());
  vector<int> u, v(2*m, -1); v[a+m-t] = b;
  for(int i=b; (u=v,i<w.size()); i++){
    for(x=0; x<m; x++) v[x+w[i]] = max(v[x+w[i]], u[x]);
    for(x=2*m; --x>m; ) for(int j=max(0,u[x]); j<v[x]; j++)
    v[x-w[j]] = max(v[x-w[j]], j);
  } for(a=t; v[a+m-t]<0; a--);;; return a;
}
```

6.9 Monotone Queue Optimization

```
template<class T, bool GET_MAX = false> // D[i] = func_{0 <= j <
i} D[j] + cost(j, i)
pair<vector<T>, vector<int>> monotone_queue_dp(int n, const
vector<T> &init, auto cost){
  assert((int)init.size() == n + 1); // cost function -> auto,
  do not use std::function
  vector<T> dp = init; vector<int> prv(n+1);
  auto compare = [](T a, T b){ return GET_MAX ? a < b : a > b;
  };
  auto cross = [&](int i, int j){
    int l = j, r = n + 1;
    while(l < r){
      int m = (l + r + 1) / 2;
      if(compare(dp[i] + cost(i, m), dp[j] + cost(j, m))) r = m
      - 1; else l = m;
  } return l; };
  deque<int> q{0};
  for(int i=1; i<=n; i++){
    while(q.size() > 1 && compare(dp[q[0]] + cost(q[0], i),
    dp[q[1]] + cost(q[1], i))) q.pop_front();
    dp[i] = dp[q[0]] + cost(q[0], i); prv[i] = q[0];
    while(q.size() > 1 && cross(q[q.size()-2], q.back()) >=
    cross(q.back(), i)) q.pop_back();
    q.push_back(i);
  } /*for end*/ return {dp, prv}; }
```

6.10 Slope Trick

```
//NOTE: f(x)=min{f(x+i),i<a}+|x-k|+m -> pf(k)sf(k)ab(-a,m)
//NOTE: sf_inc에 답구하는게 들어있어서, 반드시 한 연산에 대해
pf_dec->sf_inc순서로 호출
struct LeftHull{
  void pf_dec(int x){ pq.empl(x-bias); }//x이하의 기울기들 -1
  int sf_inc(int x){//x이상의 기울기들 +1, pop된 원소 반환(Right
  Hull관리에 사용됨)
    if(pq.empty() or argmin()<=x) return x; ans += argmin()-x;//
    이 경우 최솟값이 증가함
    pq.empl(x-bias);/*x 이하 -1*/int r=argmin(); pq.pop();/*전체
    +1*/
    return r;
  }
  void add_bias(int x,int y){ bias+=x; ans+=y; } int minval(){
  return ans; } //x축 평행이동, 최소값
  int argmin(){return pq.empty()?-inf<int>():pq.top()+bias;}//
  최소값 x좌표
  void operator+=(LeftHull& a){ ans+=a.ans; while(sz(a.pq))
  pf_dec(a.argmin()), a.pq.pop(); }
  int size()const{return sz(pq);} PQMax<int> pq; int ans=0,
  bias=0;
};
//NOTE: f(x)=min{f(x+i),a<i<b}+|x-k|+m -> pf(k)sf(k)ab(-a,b,m)
struct SlopeTrick{
  void pf_dec(int x){l.pf_dec(-r.sf_inc(-x));}
  void sf_inc(int x){r.pf_dec(-l.sf_inc(x));}
  void add_bias(int lx,int rx,int
  y){l.add_bias(lx,0),r.add_bias(-rx,0),ans+=y;}
  int minval(){return ans+l.minval()+r.minval();}
  pint argmin(){return {l.argmin(),-r.argmin()};}
  void operator+=(SlopeTrick& a){
    while(sz(a.l.pq)) pf_dec(a.l.argmin()),a.l.pq.pop();
    l.ans+=a.l.ans;
    while(sz(a.r.pq)) sf_inc(-a.r.argmin()),a.r.pq.pop();
    r.ans+=a.r.ans; ans+=a.ans;
  } LeftHull l,r; int ans=0;
  int size()const{return l.size()+r.size();}
};
//LeftHull 역추적 방법: 스텝i의 argmin값을 am(i)라고 하자. 스텝n부터
스텝1까지 ans[i]=min(ans[i+1],am(i))하면 된다. 아래는 증명..은
아니고 간략한 이유
//am(i)<=ans[i+1]일때: ans[i]=am(i)
//x[i]>ans[i+1]일때: ans[i]=ans[i+1] 왜냐하면 f(i,a)는 a<x[i]에서
감소함수이므로 가능한 최대로 오른쪽으로 붙은 ans[i+1]이 최적.
//스텝i에서 add_bias(k,0)한다면 간격제한k가 있는것이므로
ans[i]=min(ans[i+1]-k,x[i])으로 수정.
//LR Hull 역추적은 케이스나눠서 위 방법을 확장하면 될듯
```

6.11 Aliens Trick

```
// pair<T, vector<int>> f(T c): return opt_val, prv
// cost function must be multiplied by 2
template<class T, bool GET_MAX = false>
pair<T,vector<int>> AliensTrick(int n,int k,auto f,T lo,T hi){
    T l = lo, r = hi; while(l < r) {
        T m = (l + r + (GET_MAX?1:0)) >> 1;
        vector<int> prv = f(m*2+(GET_MAX?-1:+1)).second;
        int cnt = 0; for(int i=n; i; i=prv[i]) cnt++;
        if(cnt <= k) (GET_MAX?l:r) = m;
        else (GET_MAX?r:l) = m + (GET_MAX?-1:+1);
    }
    T opt_value = f(l*2).first / 2 - k*l;
    vector<int> prv1 = f(l*2+(GET_MAX?1:-1)).second, p1{n};
    vector<int> prv2 = f(l*2-(GET_MAX?1:-1)).second, p2{n};
    for(int i=n; i; i=prv1[i]) p1.push_back(prv1[i]);
    for(int i=n; i; i=prv2[i]) p2.push_back(prv2[i]);
    reverse(p1.begin(),p1.end());reverse(p2.begin(),p2.end());
    assert(p2.size() <= k+1 && k+1 <=p1.size());
    if(p1.size() == k+1) return {opt_value, p1};
    if(p2.size() == k+1) return {opt_value, p2};
    for(int i=1, j=1; i<p1.size(); i++){
        while(j < p2.size() && p2[j] < p1[i-1]) j++;
        if(p1[i] <= p2[j] && i - j == k+1 - (int)p2.size()){
            vector<int> res;
            res.insert(res.end(), p1.begin(), p1.begin()+i);
            res.insert(res.end(), p2.begin()+j, p2.end());
            return {opt_value, res};
    } /* if */ } /* for */ assert(false);
}
```

6.12 SWAMK, Min Plus Convolution

```
// find the indices of row maxima, smallest index when tie
template <class F, class T=long long>
vector<int> SMAWK(F f, int n, int m){
  vector<int> ans(n, -1);
  auto rec = [&](auto self, int*const rs, int x, int*const cs,
  int y){
    const int t = 8;
    if(x <= t || y <= t){
      for(int i=0; i<x; i++){ int r = rs[i]; T mx;
        for(int j=0; j<y; j++){
          int c = cs[j]; T w = f(r, c);
          if(ans[r] == -1 || w > mx) ans[r] = c, mx = w;
    }} /* for j i */ return; } /* base case */
    if(x < y){ int s = 0;
      for(int i=0; i<y; i++){ int c = cs[i];
        while(s && f(rs[s-1], cs[s-1]) < f(rs[s-1], c)) s--;
        if(s < x) cs[s++] = c;
      } y = s;
    }
    int z=0, k=0, *a=rs+x, *b=cs+y;
```

```
  for(int i=1; i<x; i+=2) a[z++] = rs[i];
  for(int i=0; i<y; i++) b[i] = cs[i];
  self(self, a, z, b, y);
  for(int i=0; i<x; i+=2){
    int to = i+1 < x ? ans[rs[i+1]] : cs[y-1]; T mx;
    while(true){
      T w = f(rs[i], cs[k]);
      if(ans[rs[i]] == -1 || w > mx) ans[rs[i]]=cs[k], mx=w;
      if(cs[k] == to) break; k++;
  }}
};
int *rs = new int[n*2]; iota(rs,rs+n,0);
int *cs = new int[max(m, n*2)]; iota(cs,cs+m,0);
rec(rec,rs,n,cs,m);delete[]rs;delete[]cs;return ans;
}
// A: convex, B: arbitrary, O((N+M) log M)
int N, M, A[1<<19], B[1<<19], C[1<<20];
void DnC(int s, int e, int l, int r){
  if(s > e) return;
  int m = (s+e)/2, opt = -1, &mn = C[m];
  for(int i=l; i<=min(m,r); i++){
    if(m - i >= N) continue;
    if(opt == -1 || A[m-i] + B[i] < mn) mn=A[m-i]+B[i], opt=i;
  } DnC(s, m-1, l, opt); DnC(m+1, e, opt, r);
} // or...
int f(int r, int c){//O(N+M) but not fast
  if(0 <= r-c && r-c < N) return -(A[r-c] + B[c]);
  else return -21e8 - (r - c); // min
} SMAWK(f, N+M-1, M); // DnC opt 163ms SMAWK 179ms N,M=2^19
```

6.13 Money for Nothing (WF17D)

```
ll MoneyForNothing(vector<Point> lo, vector<Point> hi){
  sort(lo.begin(), lo.end()); sort(hi.rbegin(), hi.rend());//rev
  vector<Point> a, b; ll res = 0;
  for(auto p:lo)if(a.empty() || a.back().y > p.y)a.push_back(p);
  for(auto p:hi)if(b.empty() || b.back().y < p.y)b.push_back(p);
  reverse(b.begin(),b.end()); if(a.empty()||b.empty()) return 0;
  queue<tuple<int,int,int,int>> que;
  que.emplace(0, (int)a.size()-1, 0, (int)b.size()-1);
  while(!que.empty()){
    auto [s,e,l,r] = que.front(); que.pop();
    int m = (s + e) / 2, pos = l; ll mx = -4e18;
    for(int i=l; i<=r; i++){
      ll dx = b[i].x - a[m].x, dy = b[i].y - a[m].y;
      ll now = (dx < 0 && dy < 0) ? 0 : dx * dy;
      if(now > mx) mx = now, pos = i;
    } res = max(res, mx);
    if(s < m) que.emplace(s, m-1, l, pos);
    if(m < e) que.emplace(m+1, e, pos, r);
  } return res;
}
```

6.14 Exchange Argument on Tree (WF16L,CERC13E)

```
struct Info{ // down a -> up b, a b >= 0
  ll a, b, idx; Info() : Info(0, 0, 0) {}
  Info(ll a, ll b, ll idx) : a(a), b(b), idx(idx) {}
  bool operator < (const Info &t) const {
    ll le = b - a, ri = t.b - t.a;
    if(le >= 0 && ri < 0) return false;
    if(le < 0 && ri >= 0) return true;
    if(le < 0 && b != t.b) return b < t.b;
    if(le >= 0 && a != t.a) return a > t.a;
    return idx < t.idx;
  }
  Info& operator += (const Info &v){
    ll aa = min(-a, -a+b-v.a), bb = -a+b-v.a+v.b;
    a = -aa; b = bb - aa; return *this;
  }
};
void MonsterHunter(int root=1){
  set<Info> T(A+1, A+N+1); T.erase(A[root]);
  while(!T.empty()){
    auto v = *T.rbegin(); T.erase(prev(T.end()));
    int now = v.idx, nxt = Find(Par[v.idx]); // @TODO
    UF[now] = nxt; T.erase(A[nxt]); A[nxt] += A[now];
    if(nxt != root) T.insert(A[nxt]);
  } // @TODO
}
```

6.15 Hook Length Formula

```
int HookLength(const vector<int> &young){
  if(young.empty()) return 1;
  vector<int> len(young[0]);
  ll num = 1, div = 1, cnt = 0;
  for(int i=(int)young.size()-1; i>=0; i--){
    for(int j=0; j<young[i]; j++){
      num = num * ++cnt % MOD;
      div = div * (++len[j] + young[i] - j - 1) % MOD;
    }
  }
  return num * Pow(div, MOD-2) % MOD;
}
```

6.16 Floating Point Add (Kahan)

```
template<typename T> T float_sum(vector<T> v){
  T sum=0, c=0, y, t; // sort abs(v[i]) increase?
  for(T i:v) y=i-c, t=sum+y, c=(t-sum)-y, sum=t;
  return sum; //worst O(eps*N), avg O(eps*sqrtN)
}//dnc: worst O(eps*logN), avg O(eps*sqrtlogN)
```

6.17 Random, PBDS, Bit Trick, Bitset

```
mt19937
rd((unsigned)chrono::steady_clock::now().time_since_epoch().count()
uniform_int_distribution<int> rnd_int(l, r); // rnd_int(rd)
uniform_real_distribution<double> rnd_real(0, 1);// rnd_real(rd)
// ext/pb_ds/assoc_container.hpp, tree_policy.hpp, rope
// namespace __gnu_pbds (find_by_order, order_of_key)
// namespace __gnu_cxx (append(str), substr(l, r), at(idx))
template <typename T> using ordered_set = tree<T, null_type,
less<T>, rb_tree_tag, tree_order_statistics_node_update>;
bool next_combination(T &bit, int N){
  T x = bit & -bit, y = bit + x;
  bit = (((bit & ~y) / x) >> 1) | y;
  return (bit < (1LL << N)); }
long long next_perm(long long v){
  long long t = v | (v-1);
  return (t + 1) | (((~t & -~t) - 1) >> (__builtin_ctz(v) + 1));
} // __builtin_clz/ctz/popcount
for(submask=mask; submask; submask=(submask-1)&mask);
for(supermask=mask; supermask<(1<<n);
supermask=(supermask+1)|mask);
int frq(int n, int i) { int j, r = 0; // # of digit i in [1, n]
  for (j = 1; j <= n; j *= 10) if (n / j / 10 >= !i) r += (n /
  10 / j - !i) * j + (n / j % 10 > i ? j : n / j % 10 == i ? n %
  j + 1 : 0);
  return r; }
bitset<17> bs; bs[1] = bs[7] = 1; assert(bs._Find_first() == 1);
assert(bs._Find_next(0) == 1 && bs._Find_next(1) == 7);
assert(bs._Find_next(3) == 7 && bs._Find_next(7) == 17);
cout << bs._Find_next(7) << "\n";
template <int len = 1> // Arbitrary sized bitset
void solve(int n){ // solution using bitset<len>
  if(len < n){ solve<std::min(len*2, MAXLEN)>(n); return; } }
```

6.18 Fast I/O, Fast Div, Fast Mod

```
namespace io { // thanks to cgiosy
  const signed IS=1<<20; char I[IS+1],*J=I;
  inline void daer(){if(J>=I+IS-64){
    char*p=I;do*p++=*J++;
    while(J!=I+IS);p[read(0,p,I+IS-p)]=0;J=I;}}
  template<int N=10,typename T=int>inline T getu(){
    daer();T x=0;int k=0;do x=x*10+*J-'0';
    while(*++J>='0'&&++k<N);++J;return x;}
  template<int N=10,typename T=int>inline T geti(){
    daer();bool e=*J=='-';J+=e;return(e?-1:1)*getu<N,T>();}
  struct f{f(){I[read(0,I,IS)]=0;}}flu; };
struct FastMod{ // typedef __uint128_t L;
  ull b, m; FastMod(ull b) : b(b), m(ull((L(1) << 64) / b)) {}
  ull reduce(ull a){ // can be proven that 0 <= r < 2*b
    ull q = (ull)((L(m) * a) >> 64), r = a - q * b;
    return r >= b ? r - b : r;
} };
inline pair<uint32_t, uint32_t> Div(uint64_t a, uint32_t b){
  if(__builtin_constant_p(b)) return {a/b, a%b};
  uint32_t lo=a, hi=a>>32;
  __asm__("div %2" : "+a,a" (lo), "+d,d" (hi) : "r,m" (b));
  return {lo, hi}; // BOJ 27505, q r < 2^32
} // divide 10M times in ~400ms
ull mulmod(ull a, ull b, ull M){ // ~2x faster than int128
  ll ret = a * b - M * ull(1.L / M * a * b);
  return ret + M * (ret < 0) - M * (ret >= (ll)M);
} // safe for 0 ≤ a,b < M < (1<<63) when long double is 80bit
```

6.19 DP Optimization

```
// Quadrangle Inequality : C(a, c)+C(b, d) ≤ C(a, d)+C(b, c)
// Monotonicity : C(b, c) ≤ C(a, d)
// CHT, DnC Opt(Quadrangle), Knuth(Quadrangle and Monotonicity)
// Knuth: K[i][j-1] <= K[i][j] <= K[i+1][j]
// 1. Calculate D[i][i], K[i][i]
// 2. Calculate D[i][j], K[i][j] (i < j)
// Another: D[i][j] = min(D[i-1][k] + C[k+1][j]), C quadrangle
// i=1..k  j=n..1  k=K[i-1,j]..K[i,j+1] update, vnoi/icpc22_mn_c
```

6.20 Highly Composite Numbers, Large Prime

< 10^k	number	divisors	2 3 5 71113171923293137
1	6	4	1 1
2	60	12	2 1 1
3	840	32	3 1 1 1
4	7560	64	3 3 1 1
5	83160	128	3 3 1 1 1
6	720720	240	4 2 1 1 1 1

```
7               8648640           448    6 3 1 1 1 1
8              73513440           768    5 3 1 1 1 1 1
9             735134400          1344    6 3 2 1 1 1 1
10           6983776800          2304    5 3 2 1 1 1 1 1
11          97772875200          4032    6 3 2 2 1 1 1 1
12         963761198400          6720    6 4 2 1 1 1 1 1 1
13        9316358251200         10752    6 3 2 1 1 1 1 1 1 1
14       97821761637600         17280    5 4 2 2 1 1 1 1 1 1
15      866421317361600         26880    6 4 2 1 1 1 1 1 1 1 1
16     8086598962041600         41472    8 3 2 2 1 1 1 1 1 1 1
17    74801040398884800         64512    6 3 2 2 1 1 1 1 1 1 1 1
18   897612484786617600        103680    8 4 2 2 1 1 1 1 1 1 1 1

< 10^k    prime   # of prime          < 10^k               prime
------------------------------------------------------------------
1             7            4          10              9999999967
2            97           25          11             99999999977
3           997          168          12            999999999989
4          9973         1229          13           9999999999971
5         99991         9592          14          99999999999973
6        999983        78498          15         999999999999989
7       9999991       664579          16        9999999999999937
8      99999989      5761455          17       99999999999999997
9     999999937     50847534          18      999999999999999989
```

6.21 DLAS(Diversified Late Acceptance Search)

```cpp
template<class T, class U>
T incMod(T x, U mod) { x += 1; return x == mod ? 0 : x; }
template<class Domain, class CoDomain, size_t LEN = 5>
pair<Domain, CoDomain> dlas( function<CoDomain(Domain&)> f,
  function<void(Domain&)> mutate,
  Domain const& initial, u64 maxIdleIters = -1ULL) {
  array<Domain, 3> S{initial, initial, initial};
  CoDomain curF = f(S[0]), minF = curF;
  size_t curPos = 0, minPos = 0, k = 0;
  array<CoDomain, LEN> fitness; fitness.fill(curF);
  for(u64 idleIters=0; idleIters<maxIdleIters; idleIters++){
    CoDomain prvF = curF;
    size_t newPos = incMod(curPos, 3);
    if (newPos == minPos) newPos = incMod(newPos, 3);
    Domain &curS = S[curPos], &newS = S[newPos];
    newS = curS; mutate(newS); CoDomain newF = f(newS);
    if(newF < minF) idleIters=0, minPos=newPos, minF=newF;
    if(newF == curF || newF < *max_element(all(fitness))){
      curPos = newPos; curF = newF;
    } CoDomain& fit = fitness[k]; k = incMod(k, LEN);
    if(curF > fit || curF < fit && curF < prvF) fit = curF;
  } return { S[minPos], minF };
} // 점수 최소화하는 함수, f: 상태의 점수를 반환
//dlas<state_type, score_type>(f, mutate, initial, maxIdleIter)
//initial:초기 상태, mutate:상태를 참조로 받아서 임의로 수정(반환X)
//maxIdleIters:지역 최적해에서 얼마나 오래 기다릴지
```

7 Notes

7.1 Triangles/Trigonometry

- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$, $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, $\cos 2\theta = 1 - 2\sin^2\theta$
- $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$
- 반각 공식: $\sin^2\theta/2 = \frac{1-\cos\theta}{2}$, $\cos^2\theta/2 = \frac{1+\cos\theta}{2}$, $\tan^2\theta/2 = \frac{1-\cos\theta}{1+\cos\theta}$
- 변 길이 $a, b, c; p = (a+b+c)/2$
- 넓이 $A = \sqrt{p(p-a)(p-b)(p-c)}$
- 외접원 반지름 $R = abc/4A$, 내접원 반지름 $r = A/p$
- 중선 길이 $m_a = 0.5\sqrt{2b^2+2c^2-a^2}$, 각 이등분선 $s_a = \sqrt{bc(1-\frac{a}{b+c}^2)}$
- 사인 법칙 $\frac{sin A}{a} = 1/2R$, 코사인 법칙 $a^2 = b^2 + c^2 - 2bc\cos A$
- 탄젠트 법칙 $\frac{a+b}{a-b} = \frac{\tan(A+B)/2}{\tan(A-B)/2}$
- 중심 좌표 $(\frac{\alpha x_a+\beta x_b+\gamma x_c}{\alpha+\beta+\gamma}, \frac{\alpha y_a+\beta y_b+\gamma y_c}{\alpha+\beta+\gamma})$

이름	α	β	γ	
외심	$a^2\mathcal{A}$	$b^2\mathcal{B}$	$c^2\mathcal{C}$	$\mathcal{A} = b^2 + c^2 - a^2$
내심	a	b	c	$\mathcal{B} = a^2 + c^2 - b^2$
무게중심	1	1	1	$\mathcal{C} = a^2 + b^2 - c^2$
수심	$\mathcal{BC}$	$\mathcal{CA}$	$\mathcal{AB}$	
방심(A)	$-a$	b	c	

7.2 Calculus, Newton's Method

- $(\arcsin x)' = 1/\sqrt{1-x^2}$
- $(\tan x)' = 1 + \tan^2 x$
- $\int tanax = -\ln|\cos ax|/a$
- $(\arccos x)' = -1/\sqrt{1-x^2}$
- $(\arctan x)' = 1/(1+x^2)$
- $\int x\sin ax = (\sin ax - ax\cos ax)/a^2$
- Newton: $x_{n+1} = x_n - f(x_n)/f'(x_n)$
- $\oint_C (Ldx + Mdy) = \int\int_D (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})dxdy$
- where C is positively oriented, piecewise smooth, simple, closed; D is the region inside C; L and M have continuous partial derivatives in D.

7.3 Counting

	조건 없음	단사 함수	전사 함수
함수 일치	k^n	$k!/(k-n)!$	$k! \times S_2(n,k)$
공 구별 X	$C(n+k-1, n)$	$C(k,n)$	$C(n-1, n-k)$ 조심
상자 구별 X	$B(n,k)$	$[n \le k]$	$S_2(n,k)$
모두 구별 X	$P_k(n+k)$	$[n \le k]$	$P_k(n)$

- 단사 함수: 상자에 공 최대 1개, 전사 함수: 상자에 공 최소 1개
- 중복 조합: $C(n+k-1, n) = H(n,k)$
- 공 구별 X, 전사 함수에서 $n = k = 0$일 때 정답 1인 거 조심
- 교란 순열: 모든 i에 대해 $\pi(i) \ne i$가 성립하는 길이 n 순열 개수
  초항(0-based): 1, 0, 1, 2, 9, 44, 265, 1854
  일반항: $D(n) = \sum_{k=0}^{n}(-1)^k n!/k!$, $D(n) \approx n!/e$
  점화식: $D(0) = 1; D(1) = 0; D(n) = (n-1)(D(n-1) + D(n-2))$
  생성함수(EGF): $e^{-x}/(1-x)$
- 카탈란 수
  초항(0-based): 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786
  일반항: $Cat(n) = Comb(2n,n)/(n+1) = C(2n,n) - C(2n,n+1)$
  점화식: $Cat(0) = 1; Cat(n) = \sum_{k=0}^{n-1} Cat(k) \times Cat(n-k-1)$
  생성함수(OGF): $(1-\sqrt{1-4x})/(2x)$
  여는 괄호 n개, 닫는 괄호 $k(\le n)$개 $= C(n+k,k) \times (n-k+1)/(n+1)$
- 제 1종 스털링 수: k개의 사이클로 구성된 길이 n 순열 개수
  초항(0-based, 삼각형): 1, 0, 1, 0, 1, 1, 0, 2, 3, 1, 0, 6, 11, 6, 1
  점화식: $S_1(n,0) = [n=0]; S_1(n,k) = S_1(n-1,k-1) + S_1(n-1,k) \times (n-1)$, 생성함수(EGF): $(-\ln(1-x))^k/k!$
  성질: $\sum_{k=0}^{n} S_1(n,k) = n!$
- 제 2종 스털링 수: n개의 원소를 k개의 공집합이 아닌 부분집합으로 분할
  초항(0-based, 삼각형): 1, 0, 1, 0, 1, 1, 0, 1, 3, 1, 0, 1, 7, 6, 1
  일반항: $S_2(n,k) = 1/k! \times \sum_{i=1}^{k}(-1)^{k-i} \times C(k,i) \times i^n$, 단 $S(0,0) = 1$
  점화식: $S_2(n,0) = [n=0]; S_2(n,k) = S_2(n-1,k-1) + S_2(n-1,k) \times k$
  생성함수(EGF): $(\exp(x)-1)^k/k!$
  성질: A, B를 $n-k$, $\lfloor(k-1)/2\rfloor$의 켜진 비트 위치라고 하면, $S_2(n,k)$ mod $2 = [A \cap B = \emptyset]$
  성질: $S_2(2n,n)$은 n이 fibbinary number(연속한 1 없음) 일 때만 홀수
- 벨 수 $B(n)$: n개의 원소를 분할하는 방법의 수
  초항(0-based): 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975
  일반항: $B(n) = \sum_{k=0}^{n} S_2(n,k)$, 몇 개의 상자를 버릴지 다 돌아보기
  점화식: $B(0) = 1; B(n) = \sum_{k=0}^{n-1} C(n-1,k) \times B(k)$
  생성함수(EGF): $\exp(\exp(x)-1)$
- 벨 수 $B(n,k)$: n개를 집합 k개로 분할하는 방법의 수(공집합 허용)
  초항(0-based, 삼각형): 1 0 1 0 1 2 0 1 4 5 0 1 8 14 15
  일반항: $B(n,k) = \sum_{i=0}^{k} S_2(n,i) = \sum_{i=0}^{k} \frac{i^n}{i!} \sum_{j=0}^{k-i} \frac{(-1)^j}{j!}$
- 분할 수 $P(n)$: n을 자연수 몇 개의 합으로 나타내는 방법의 수, 순서 X
  초항(0-based): 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42
  점화식: $P(0) = 1$, 팀노트 어딘가에 있는 코드로 $O(n\sqrt{n})$ 가능
- 분할 수 $P(n,k)$: n을 k개의 자연수의 합으로 나타내는 방법의 수, 순서 X
  초항(0-based, 삼각형): 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 2, 1, 1, 0, 1, 2, 2, 1, 1
  점화식: $P(0,0) = 1; P(n,0) = 0; P(n,k) = P(n-1,k-1) + P(n-k,k)$
  성질: $\sum_{k=0}^{n} P(n,k) = P(n)$
- 벨 수 $B(n,k)$ 식 전개
  $B(n,k) = \sum_{j=0}^{k} S(n,j) = \sum_{j=0}^{k} 1/j! \sum_{i=0}^{j}(-1)^{j-i} jCi \times i^n$
  $= \sum_{j=0}^{k}\sum_{i=0}^{j} \frac{(-1)^{j-i}}{i!(j-i)!} i^n = \sum_{i=0}^{k}\sum_{j=i}^{k} \frac{(-1)^{j-i}}{i!(j-i)!} i^n$
  $= \sum_{i=0}^{k}\sum_{j=0}^{k-i} \frac{(-1)^j}{i!j!} i^n = \sum_{i=0}^{k} \frac{i^n}{i!} \sum_{j=0}^{k-i} \frac{(-1)^j}{j!}$

7.4 Generating Function

- 등차수열: $(pn+q)x^n = p/(1-x) + q/(1-x)^2$
- 등비수열: $(rx)^n = (1-rx)^-1$
- 조합: $C(m,n)x^n = (1+x)^m$
- 중복조합: $C(m+n-1,n)x^n = (1-x)^{-m}$
- $f(n) = \sum_{k=0}^{n} k! \times S_2(n,k)$의 EGF: $1/(2-e^x)$
- Ordinary Generating Function $A(x) = \sum_{i \ge 0} a_i x^i$
  $A(rx) \Rightarrow r^n a_n$, $xA(x)' \Rightarrow na_n$
  $A(x) + B(x) \Rightarrow a_n + b_n$, $A(x)B(x) \Rightarrow \sum_{i=0}^{n} a_i b_{n-i}$
  $A(x)^k \Rightarrow \sum_{i_1+i_2+\cdots+i_k=n} a_{i_1} a_{i_2} \ldots a_{i_k}$
  $\frac{A(x)}{1-x} \Rightarrow \sum_{i=0}^{n} a_i$
- Exponential Generating Function $A(x) = \sum_{i \ge 0} \frac{a_i}{i!} x_i$
  $A(x) + B(x) \Rightarrow a_n + b_n$, $A(x)B(x) \Rightarrow \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$
  $A^{(k)}(x) \Rightarrow a_{n+k}$, $xA(x) \Rightarrow na_n$
  $A(x)^k \Rightarrow \sum_{i_1+i_2+\cdots+i_k=n} \binom{n}{i_1, i_2, \ldots, i_k} a_{i_1} a_{i_2} \ldots a_{i_k}$

7.5 Faulhaber's Formula ($\sum_{k=1}^{n} k^c$)

- B_n: 베르누이 수
- 생성함수(egf): $\frac{x}{e^x-1} = \frac{1}{(e^x-1)/x} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$
- 일반항: $B_n = \sum_{k=0}^{n} \frac{k!(-1)^k}{k+1} S_2(n,k)$
- 점화식: $B_0 = 1$; $B_n = -\frac{1}{n+1} \sum_{r=0}^{n-1} \binom{n+1}{r} B_r$
- $\sum_{k=1}^{n} k^c = \sum_{k=0}^{c} \frac{(-1)^k}{c+1} \binom{c+1}{k} B_k n^{c+1-k}$

7.6 Zeta/Mobius Transform

- Subset Zeta/Mobius Transform
  - for i=0..n-1 for j=0..2^n-1 if(i and j) v[j] ±= v[i xor j]
- Superset Zeta/Mobius Transform
  - for i=0..n-1 for j=0..2^n-1 if(i and j) v[i xor j] ±= v[j]
- Divisor Zeta/Mobius Transform
  - for p:Prime for i=1..n/p v[i*p] += v[i]
  - for p:Prime for i=n/p..1 v[i*p] -= v[i]

- Multiple Zeta/Mobius Transform
  - for p:Prime for i=n/p..1 v[i] += v[i*p]
  - for p:Prime for i=1..n/p v[i] -= v[i*p]
- AND Convolution: SupZeta(A), SupZeta(B), SupMobius(mul)
- OR Conv.: Subset, GCD Conv.: Multiple, LCM Conv.: Divisor
- AND/OR: 2^20 0.3s, Subset: 2^{20} 2.5s, GCD/LCM: 1e6 0.3s

7.7 About Graph Degree Sequence

- 단순 무향 그래프(Erdos-Gallai): 차수열 $d_1 \geq \cdots \geq d_n$의 합이 짝수 and 모든 $1 \leq k \leq n$에 대해 $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$
- 단순 이분 그래프(Gale-Ryser): 차수열 $a_1 \geq \cdots \geq a_n$, b_i에서 $sum(a) = sum(b)$ and 모든 $1 \leq k \leq n$에 대해 $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} \min(b_i, k)$
- 단순 방향 그래프(Fulkerson-Chen-Anstee): $a_1 \geq \cdots \geq a_n$를 만족하는 진입/진출 차수열 $(a_1, b_1), \cdots, (a_n, b_n)$에서 $sum(a) = sum(b)$ and 모든 $1 \leq k \leq n$에 대해 $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \min(b_i, k-1) + \sum_{i=k+1}^{n} \min(b_i, k)$

7.8 Burnside, Grundy, Pick, Hall, Simpson, Area of Quadrangle, Fermat Point, Euler

- Burnside's Lemma
  - 수식
  G=(X,A): 집합X와 액션A로 정의되는 군G에 대해, $|A||X/A| = sum(|\text{Fixed points of a}|, \text{for all a in A})$
  X/A 는 Action으로 서로 변형가능한 X의 원소들을 동치로 묶었을때 동치류(파티션) 집합
  - 풀어쓰기
  orbit: 그룹에 대해 두 원소 a,b와 액션f에 대해 f(a)=b인거에 간선연결한 컴포넌트(연결집합)
  orbit개수 = sum(각 액션 g에 대해 f(x)=x인 x(고정점)개수)/액션개수
  - 자유도 치트시트
  회전 n개: 회전i의 고정점 자유도=gcd(n,i)
  임의뒤집기 n=홀수: n개 원소중심축(자유도 (n+1)/2)
  임의뒤집기 n=짝수: n/2개 원소중심축(자유도 n/2+1) + n/2개 원소안지나는축(자유도 n/2)
- 알고리즘 게임
  - Nim Game의 해법 : 각 더미의 돌의 개수를 모두 XOR했을 때 0 이 아니면 첫번째, 0 이면 두번째 플레이어가 승리.
  - Grundy Number : 어떤 상황의 Grundy Number는, 가능한 다음 상황들의 Grundy Number를 모두 모은 다음, 그 집합에 포함 되지 않는 가장 작은 수가 현재 state의 Grundy Number가 된다. 만약 다음 state가 독립된 여러개의 state들로 나뉠 경우, 각각의 state의 Grundy Number의 XOR 합을 생각한다.
  - Subtraction Game : 한 번에 k 개까지의 돌만 가져갈 수 있는 경우, 각 더미의 돌의 개수를 k + 1로 나눈 나머지를 XOR 합하여 판단한다.
  - Index-k Nim : 한 번에 최대 k개의 더미를 골라 각각의 더미에서 아무렇게나 돌을 제거할 수 있을 때, 각 binary digit에 대하여 합을 k + 1로 나눈 나머지를 계산한다. 만약 이 나머지가 모든 digit에 대하여 0이라면 두번째, 하나라도 0이 아니라면 첫번째 플레이어가 승리.
  - Misere Nim : 모든 돌 무더기가 1이면 N이 홀수일 때 후공 승, 그렇지 않은 경우 XOR 합 0이면 후공 승
- Pick's Theorem
  격자점으로 구성된 simple polygon이 주어짐. I 는 polygon 내부의 격자점 수, B 는 polygon 선분 위 격자점 수, A는 polygon의 넓이라고 할 때, 다음과 같은 식이 성립한다. $A = I + B/2 - 1$
- 홀의 결혼 정리 : 이분그래프(L-R)에서, 모든 L을 매칭하는 필요충분 조건 = L에서 임의의 부분집합 S를 골랐을 때, 반드시 (S의 크기) <= (S와 연결되어있는 모든 R의 크기)이다.
- Simpson 공식 (적분) : Simpson 공식, $S_n(f) = \frac{h}{3}[f(x_0) + f(x_n) + 4\sum f(x_{2i+1}) + 2\sum f(x_{2i})]$
  - $M = \max|f^4(x)|$이라고 하면 오차 범위는 최대 $E_n \leq \frac{M(b-a)}{180}h^4$
- 브라마굽타 : 원에 내접하는 사각형의 각 선분의 길이가 a, b, c, d일 때 사각형의 넓이 $S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, $s = (a+b+c+d)/2$
- 브레치나이더 : 임의의 사각형의 각 변의 길이를 a, b, c, d라고 하고, 마주보는 두 각의 합을 2로 나눈 값을 θ라 하면, $S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \times cos^2\theta}$
- 페르마 포인트 : 삼각형의 세 꼭짓점으로부터 거리의 합이 최소가 되는 점
  $2\pi/3$ 보다 큰 각이 있으면 그 점이 페르마 포인트, 그렇지 않으면 각 변마다 정삼각형 그린 다음, 정삼각형의 끝점에서 반대쪽 삼각형의 꼭짓점으로 연결한 선분의 교점
  $2\pi/3$ 보다 큰 각이 없으면 거리의 합은 $\sqrt{(a^2 + b^2 + c^2 + 4\sqrt{3}S)/2}$, S는 넓이
- 오일러 정리: 서로소인 두 정수 a, n에 대해 $a^{\phi(n)} \equiv 1 \pmod n$
  모든 정수에 대해 $a^n \equiv a^{n-\phi(n)} \pmod n$
  $m \geq log_2 n$이면 $a^m \equiv a^{m\%\phi(n)+\phi(n)} \pmod n$
- $g^0 + g^1 + g^2 + \cdots g^{p-2} \equiv -1 \pmod p$ iff $g = 1$, otherwise 0.
- if $n \equiv 0 \pmod 2$, then $1^n + 2^n + \cdots + (n-1)^n \equiv 0 \pmod n$
- Eulerian numbers
  Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.
  $E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$
  $E(n, 0) = E(n, n-1) = 1$
  $E(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$

7.9 About Graph Minimum Cut

- N개의 boolean 변수 $v_1, \cdots, v_n$을 정해서 비용을 최소화하는 문제 =true인 점은 T, false인 점은 F와 연결되게 분할하는 민컷 문제
  1. v_i가 T일 때 비용 발생: i에서 F로 가는 비용 간선
  2. v_i가 F일 때 비용 발생: i에서 T로 가는 비용 간선
  3. v_i가 T이고 v_j가 F일 때 비용 발생: i에서 j로 가는 비용 간선
  4. $v_i \neq v_j$일 때 비용 발생: i에서 j로, j에서 i로 가는 비용 간선
  5. v_i가 T면 v_j도 T여야 함: i에서 j로 가는 무한 간선
  6. v_i가 F면 v_j도 F여야 함: j에서 i로 가는 무한 간선
- 5/6번 + v_i와 v_j가 달라야 한다는 조건이 있으면 MAX-2SAT
- Maximum Density Subgraph (NEERC'06H, BOJ 3611 팀의 난이도)
  - density $\geq x$인 subgraph가 있는지 이분 탐색
  - 정점 N개, 간선 M개, 차수 D_i개
  - 그래프의 간선마다 용량 1인 양방향 간선 추가
  - 소스에서 정점으로 용량 M, 정점에서 싱크로 용량 $M - D_i + 2x$
  - min cut에서 S와 붙어 있는 애들이 1개 이상이면 x 이상이고, 그게 subgraph의 정점들
  - while(r-l $\geq$ 1.0/(n*n)) 으로 해야 함. 너무 많이 돌리면 실수 오차

7.10 Matrix with Graph(Kirchhoff, Tutte, LGV)

- Kirchhoff's Theorem : 그래프의 스패닝 트리 개수
  - m[i][j] := -(i-j 간선 개수) (i $\neq$ j)
  - m[i][i] := 정점 i의 degree
  - res = (m의 첫 번째 행과 첫 번째 열을 없앤 (n-1) by (n-1) matrix의 행렬식)
- Tutte Matrix : 그래프의 최대 매칭
  - m[i][j] := 간선 (i, j)가 없으면 0, 있으면 $i < j ? r : -r$, r은 $[0, P)$ 구간의 임의의 정수
  - $rank(m)/2$가 높은 확률로 최대 매칭
- LGV Theorem: 간선에 가중치 있는 DAG에서 어떤 경로 P의 간선 가중치 곱을 $w(P)$, 모든 $a \to b$ 경로들의 $w(P)$의 합을 $e(a, b)$라고 하자. n개의 시작점 a_i와 도착점 b_j가 주어졌을 때, 서로 정점이 겹치지 않는 n개의 경로로 시작점과 도착점을 일대일 대응시키는 모든 경우에서 $w(P)$의 곱의 합은 det $M(i, j) = e(a_i, b_j)$와 같음. 따라서 모든 가중치를 1로 두면 서로소 경로 경우의 수를 구함

7.11 About Graph Matching(Graph with $|V| \leq 500$)

- **Game on a Graph** : s에 토큰이 있음. 플레이어는 각자의 턴마다 토큰을 인접한 정점으로 옮기고 못 옮기면 짐.
  s를 포함하지 않는 최대 매칭이 존재함 $\leftrightarrow$ 후공이 이김
- **Chinese Postman Problem** : 모든 간선을 방문하는 최소 가중치 Walk를 구하는 문제.
  Floyd를 돌린 다음, 홀수 정점들을 모아서 최소 가중치 매칭 (홀수 정점은 짝수 개 존재)
- **Unweighted Edge Cover** : 모든 정점을 덮는 가장 작은(minimum cardinality/weight) 간선 집합을 구하는 문제
  $|V| - |M|$, 길이 3짜리 경로 없음, star graph 여러 개로 구성
- **Weighted Edge Cover** : $sum_{v \in V}(w(v)) - sum_{(u,v) \in M}(w(u) + w(v) - d(u, v))$, $w(x)$는 x와 인접한 간선의 최소 가중치
- **NEERC'18 B** : 각 기계마다 2명의 노동자가 다뤄야 하는 문제.
  기계마다 두 개의 정점을 만들고 간선으로 연결하면 정답은 $|M| - |$기계$|$임. 정답에 1/2씩 기여한다는 점을 생각해보면 좋음.
- **Min Disjoint Cycle Cover** : 정점이 중복되지 않으면서 모든 정점을 덮는 길이 3 이상의 사이클 집합을 찾는 문제.
  모든 정점은 2개의 서로 다른 간선, 일부 간선은 양쪽 끝점과 매칭되어야 하므로 플로우를 생각할 수 있지만 용량 2짜리 간선에 유량을 1만큼 흘릴 수 있으므로 플로우는 불가능.
  각 정점과 간선을 2개씩((v, v'), $(e_{i,u}, e_{i,v})$)로 복사하자. 모든 간선 $e = (u, v)$에 대해 e_u와 e_v를 잇는 가중치 w짜리 간선을 만들고(like NEERC18), $(u, e_{i,u}), (u', e_{i,u}), (v, e_{i,v}), (v', e_{i,v})$를 연결하는 가중치 0짜리 간선을 만들자. Perfect 매칭이 존재함 $\Leftrightarrow$ Disjoint Cycle Cover 존재. 최대 가중치 매칭 찾은 뒤 모든 간선 가중치 합에서 매칭 빼면 됨.
- **Two Matching** : 각 정점이 최대 2개의 간선과 인접할 수 있는 최대 가중치 매칭 문제.
  각 컴포넌트는 정점 하나/경로/사이클이 되어야 함. 모든 서로 다른 정점 쌍에 대해 가중치 0짜리 간선 만들고, 가중치 0짜리 (v, v') 간선 만들면 Disjoing Cycle Cover 문제가 됨. 정점 하나만 있는 컴포넌트는 self-loop, 경로 형태의 컴포넌트는 양쪽 끝점을 연결한다고 생각하면 편함.

7.12 Checklist

- (예비소집) bits/stdc++.h, int128, long double 80bit, avx2 확인
- (예비소집) 스택 메모리(지역 변수, 재귀, 람다 재귀), 제출 파일 크기 확인
- (예비소집) MLE(힙,스택), stderr 출력 RTE?, 줄 앞뒤 공백 채점 결과
- 비슷한 문제를 풀어본 적이 있는가?
- 단순한 방법에서 시작할 수 있을까? (Brute Force)
- 내가 문제를 푸는 과정을 수식화할 수 있을까? (예제를 직접 해결하면서)
- 문제를 단순화할 수 없을까? / 그림으로 그려볼 수 있을까?
- 수식으로 표현할 수 있을까? / 문제를 분해할 수 있을까?
- 뒤에서부터 생각해서 풀 수 있을까? / 순서를 강제할 수 있을까?
- 특정 형태의 답만을 고려할 수 있을까? (정규화)
- 구간을 통째로 가져간다 : 플로우 + 적당한 자료구조
  $(i, i+1, k, 0), (s, e, 1, w), (N, T, k, 0)$
- a = b : a만 이동, b만 이동, 두 개 동시에 이동, 반대로 이동
- 말도 안 되는 것 / 당연하다고 생각한 것 다시 생각해 보기
- Directed MST / Dominator Tree
- 일정 비율 충족 or 2 3개로 모두 커버 : 랜덤
- 확률 : DP, 이분 탐색(NYPC 2019 Finals C)
- 최대/최소 : 이분 탐색, 그리디(Prefix 고정, Exchange Argument), DP(순서 고정)
- 냅색: 파라미터 순서 변경, min plus convolution, FFT
- signal(SIGSEGV,[](int){_Exit(0);}); converts segfaults into WA. SIGABRT(assertion fail), SIGFPE(0div)
- feenableexcept(29) kills problem on NaNs(1), 0div(4), inf(8), denormals(16)