The Substitution Model

Concepts of Programming Languages
Lecture 15

CAS CS 320

Outline

» Look formally at the **lambda calculus** and its semantics

» Discuss **substitution** and the pitfalls to avoid

Recap

Recall: Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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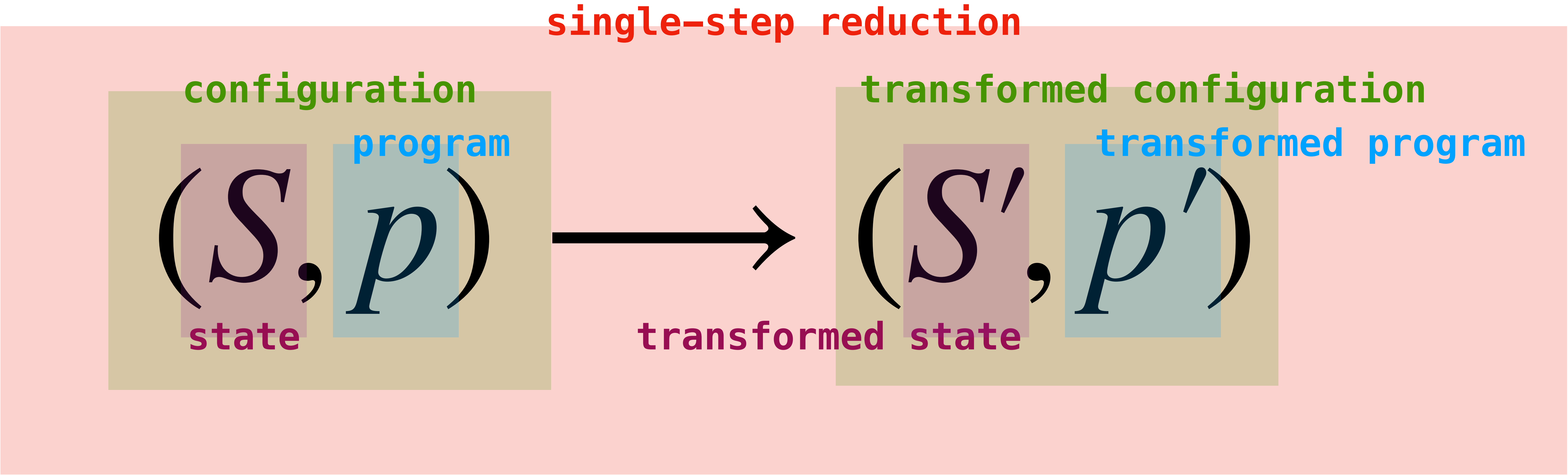

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When we define the small-step semantics of PL, we need to define three things:

» What kind of **state** are we manipulating?

» What **rules** describe how to transform configurations?

» What are the **values** of our PL (i.e., when are we done reducing)?

Recall: Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
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Rules:

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{n\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{add}\ n\ e_2) \longrightarrow (\texttt{add}\ n\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1'\ e_2)}\ \texttt{sub-left}$$

$$\frac{n\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{sub}\ n\ e_2) \longrightarrow (\texttt{sub}\ n\ e_2')}\ \texttt{sub-right}$$

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Values: `<int>` (i.e., numbers)

Recall: Example

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<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
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<int>  ::= ...
```

$$\frac{n \texttt{ is a number}}{n \Downarrow n}\ \texttt{numEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{add}\ e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{sub}\ e_1\ e_2) \Downarrow v_1 - v_2}\ \texttt{subEval}$$

We can also give a big-step semantics to this system

```
<prog> ::= { <stmt> ; }
<stmt> ::= rot90 | refX | refY
```

Practice Problem

$$\frac{}{(s,\ \texttt{rot90;}\ P) \longrightarrow (s\ \texttt{rotated 90 deg. clockwise},\ P)}$$

$$\frac{}{(s,\ \texttt{refX;}\ P) \longrightarrow (s\ \texttt{reflected across x-axis},\ P)}$$

$$\frac{}{(s,\ \texttt{refY;}\ P) \longrightarrow (s\ \texttt{reflected across y-axis},\ P)}$$

What does ($\triangle$, `rot90; refY; rot90; refX;`) *evaluate to? Give a sequence of single step reductions (you do not need to give the full multi-step derivation)*

Answer

```
<prog> ::= { <stmt> ; }
<stmt> ::= rot90 | refX | refY
```

(△, rot90; refY; rot90; refX;)

The Lambda Calculus

High-Level View

```
(fun x -> x x)(fun x -> x x)
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lambda term called Ω

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The **lambda calculus** is the *simplest functional programming language*. It only has:

» variables
» anonymous functions
» function application

High-Level View

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(fun x -> x x)(fun x -> x x)
```

lambda term called Ω

The **lambda calculus** is the *simplest functional programming language*. It only has:

» variables
» anonymous functions
» function application

It's also **untyped**, so *anything* can be applied to *anything*

demo

(OCaml and Python)

History

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The lambda calculus **as powerful** as every model of computation (Turing Machines, Register Machines, etc.)

Syntax

```
<expr> ::= fun <var> -> <expr>
         | <var>
         | <expr> <expr>
         | ( <expr> )
<var>  ::= a | b | ... | y | z
```

Syntax

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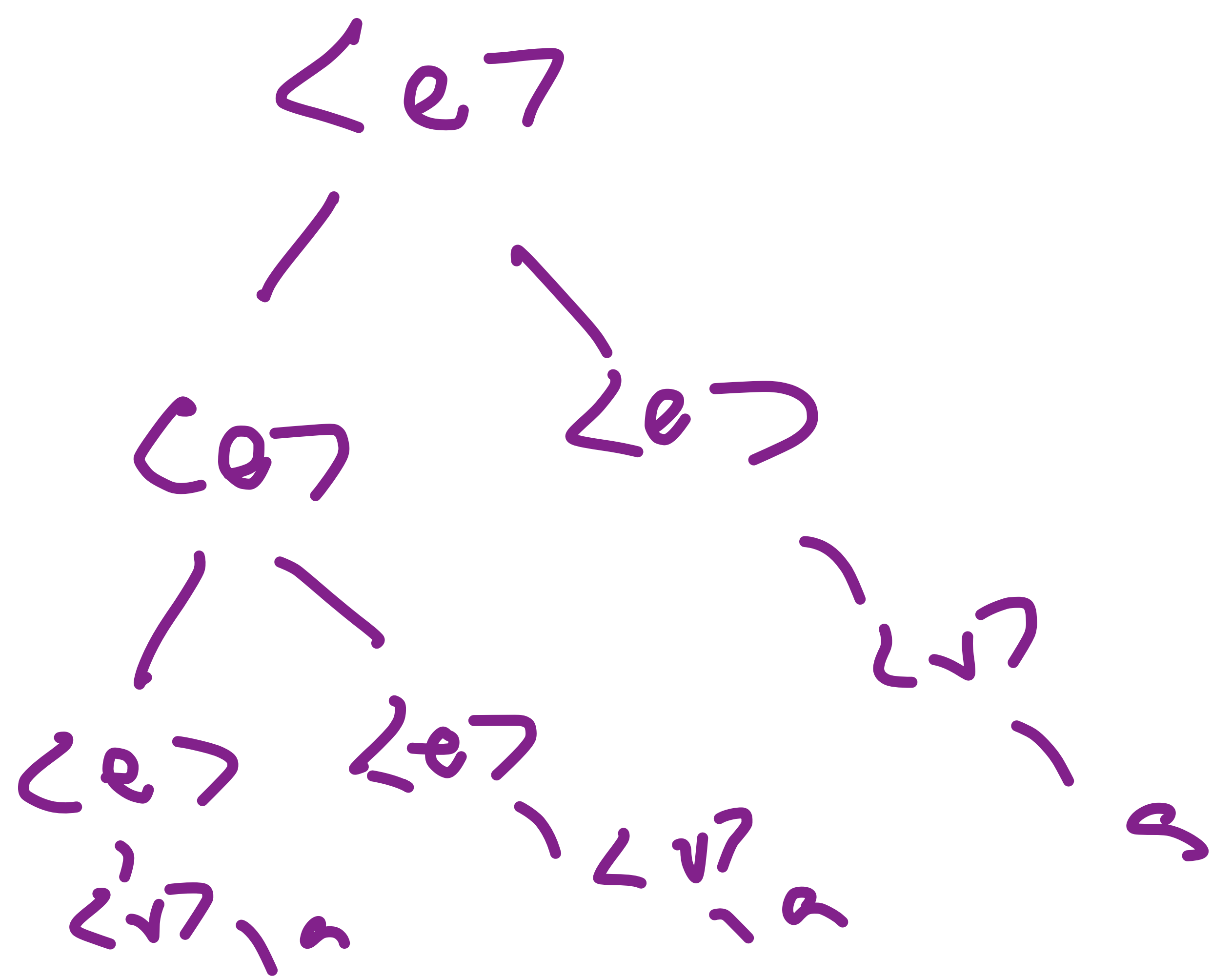
This presentation is technically ambiguous (why?)

aaa



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```

This presentation is technically ambiguous (why?)

We assume application is **left-associative** and has higher precedence than the anonymous function syntax

Syntax (Unambiguous)

```
<expr>  ::= fun <var> -> <expr>
          | <expr2> { <expr2> }
<expr2> ::= <var> | ( <expr> )
<var>   ::= a | b | ... | y | z
```

In this grammar we can only use variables or functions in parentheses in applications

Syntax (Mathematical)

```
<expr> ::= λ<var>.<expr>
         | <var>
         | <expr><expr>
```

In mathematical settings, we use more compact syntax

Parentheses, precedence, and variables are often left implicit

Warning: Get used to λ

$$\lambda x . e \qquad \equiv \qquad \texttt{fun } x \texttt{ -> } e$$

`We will use these syntaxes interchangeably starting now`

`These are the same thing, get used to it`

Values

```
<val> ::= λ<var>.<expr>
```

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`In arithmetic, values are` **`numbers`**

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<val> ::= λ<var>.<expr>
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`In arithmetic, values are` **`numbers`**

`In the lambda calculus, values are` **`functions`**

`(We often use BNF syntax to specify our values)`

Small-Step Semantics

(1) $$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}$$

(2) $$\frac{e_2 \longrightarrow e_2'}{(\lambda x\,.\,e_1)e_2 \longrightarrow (\lambda x\,.\,e_1)e_2'}$$

(3) $$\frac{}{(\lambda x\,.\,e)(\lambda y\,.\,e') \longrightarrow [(\lambda y\,.\,e')/x]e}$$

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1. We can reduce the LHS of an application
2. We can reduce the RHS of an application *if the LHS is already a function*

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(1) $$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}$$

(2) $$\frac{e_2 \longrightarrow e_2'}{\boxed{(\lambda x \,.\, e_1)} e_2 \longrightarrow (\lambda x \,.\, e_1) e_2'}$$

(3) $$\frac{}{\boxed{(\lambda x \,.\, e)}\boxed{(\lambda y \,.\, e')} \longrightarrow [(\lambda y \,.\, e')/x]e}$$

values

1. We can reduce the LHS of an application
2. We can reduce the RHS of an application *if the LHS is already a function*
3. We can apply a function to another function by **substitution.** This is also called β**–reduction**

Example

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$$(3)\ \frac{}{(\lambda x\,.\,e)(\lambda y\,.\,e') \longrightarrow [(\lambda y\,.\,e')/x]e}$$

$$(\lambda f\,.(\lambda x\,.\,fx))(\lambda y\,.\,y) \xrightarrow{(3)}$$

$$(\lambda f.\ \boxed{}\ )(\lambda y.y)$$

$$[\boxed{\lambda y.y}/f](\lambda x.\boxed{f}\,x) = \lambda x.((\lambda y.y)x)$$

Small-Step Semantics (Another Form)

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1. We can reduce the LHS of an application
2. We can apply a function to *any* expression via substitution

Example

$$(1)\ \frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2} \qquad (2)\ \frac{}{(\lambda x\,.\,e)e' \longrightarrow [e'/x]e}$$

$$(\lambda x\,.\,y)((\lambda z\,.\,z)(\lambda w\,.\,w)) \xrightarrow{(2)}$$

$$[(\lambda z.z)(\lambda w.w)/x]\,y \;=\; y \not\longrightarrow$$

stuck

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» **value:** we reach the end of our computation and the value of our program

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» **stuck:** we reach an expression that cannot be reduced, but that is not a value

`y (fun x -> x)` $\not\rightarrow$

» **diverge:** the computation never reaches a point where the expression is not reducible

`(fun x -> x x) (fun x -> x x)` $\longrightarrow^\star$ `(fun x -> x x) (fun x -> x x)`

Stuck Terms

<val> ::= λ<var>.<expr>

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2} \qquad \frac{}{(\lambda x\,.\,e)e' \longrightarrow [e'/x]e}$$

$$(\lambda x\,.\,yx)(\lambda x\,.\,x) \longrightarrow [\lambda x.x \,/\, x]\, y x$$

$$= y\,(\lambda x.x) \not\longrightarrow$$

Based on our operational semantics, it's possible for the above expression to reduce to a value

Non-Termination

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2} \qquad \frac{}{(\lambda x\,.\,e)e' \longrightarrow [e'/x]e}$$

$$(\lambda x\,.\,xx)(\lambda x\,.\,xx) \longrightarrow [\boxed{\lambda x.xx}/x](xx)$$

$$(\lambda x.xx)(\lambda x.xx) \longrightarrow$$

$$\longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow \ldots$$

And unlike with arithmetic, it's now possible to define expressions which **do not terminate.** These expression do not have values, but also don't get stuck

Big-Step Semantics

$$\text{(1)}\ \frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e}$$

$$\text{(2)}\ \frac{e_1 \Downarrow \lambda x\,.\,e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

Big-Step Semantics

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1. A function evaluates to a function value (itself)

Big-Step Semantics

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e}\;(1)$$

$$\frac{e_1 \Downarrow \lambda x\,.\,e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}\;(2)$$

1. A function evaluates to a function value (itself)
2. If e_1 evaluates to the function $\lambda x\,.\,e$ and e_2 evaluates to the value v_2 and e with v_2 substituted for x evaluates to v, then the application $e_1 e_2$ evaluates to v

Big-Step Semantics (Another Form)

$$\frac{}{\lambda x \,.\, e \Downarrow \lambda x \,.\, e}$$

$$\frac{e_1 \Downarrow \lambda x \,.\, e \qquad [e_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

These are the same rules as before except we're not required to evaluate e_2 first

Practice Problem

$$(\lambda x\,.\,\lambda y\,.\,y)((\lambda z\,.\,z)(\lambda q\,.\,q)) \Downarrow \lambda y\,.\,y$$

Give a derivation of the above judgment in both versions of the big-step semantics

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e}$$

$$\frac{e_1 \Downarrow \lambda x\,.\,e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e}$$

$$\frac{e_1 \Downarrow \lambda x\,.\,e \qquad [e_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$(\lambda x \,.\, \lambda y \,.\, y)((\lambda z \,.\, z)(\lambda q \,.\, q)) \Downarrow \lambda y \,.\, y$$

Answer

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}\;\texttt{(CBV)}$$

$$\frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad [e_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}\;\texttt{(CBN)}$$

Call-by-Value vs. Call-by-Name

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The two versions of semantics we've given correspond to **call-by-value** (CBV) and **call-by-name** (CBN). These are **evaluation strategies** for functional languages

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x \,.\, e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}\ \texttt{(CBV)}$$

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<u>CBV:</u> evaluate the argument of a function *before* substituting it in the function

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}\ (\texttt{CBV})$$

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The two versions of semantics we've given correspond to **call-by-value** (CBV) and **call-by-name** (CBN). These are **evaluation strategies** for functional languages

<u>CBV:</u> evaluate the argument of a function *before* substituting it in the function

<u>CBN:</u> substitute the expression *directly* into the function

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e} \qquad \frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}$$

Benefits of CBV

$$(\lambda x\,.\,x + x + x + x)e$$

$$\frac{}{\lambda x \,.\, e \Downarrow \lambda x \,.\, e} \qquad \frac{e_1 \Downarrow \lambda x \,.\, e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}$$

Benefits of CBV

$$(\lambda x \,.\, x + x + x + x)e$$

If we compute the value of an argument before substituting it into the expression, we only have to compute the expression *once*

Benefits of CBV

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$$(\lambda x\,.\,x + x + x + x)e \xrightarrow{\text{CBN}} e + e + e + e$$

If we compute the value of an argument before substituting it into the expression, we only have to compute the expression *once*

This is good if the variable appears several times in the body of our function

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e} \qquad \frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}$$

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This is also called **eager**, or **applicative**, or **strict** evaluation (and is what OCaml does)

Benefits of CBN

$$\frac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e} \qquad \frac{e_1 \Downarrow \lambda x\,.\,e_1' \qquad [e_2/x]e_1' \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$(\lambda x\,.\,\lambda y\,.\,x)e_1 e_2$$

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Aside. It's possible to simulate CBN in CBV. Think about it for a bit, ask me after if you're interested

Side Effects

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let f x = x + x in
let y =
  let _ = print_int 2 in
  2
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CBV

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CBN

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Different evaluation strategies yield different side-effectful behavior!

Aside: Evaluation Strategies

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There are *a lot* more evaluation strategies, all of which optimize *something*

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There are *a lot* more evaluation strategies, all of which optimize *something*

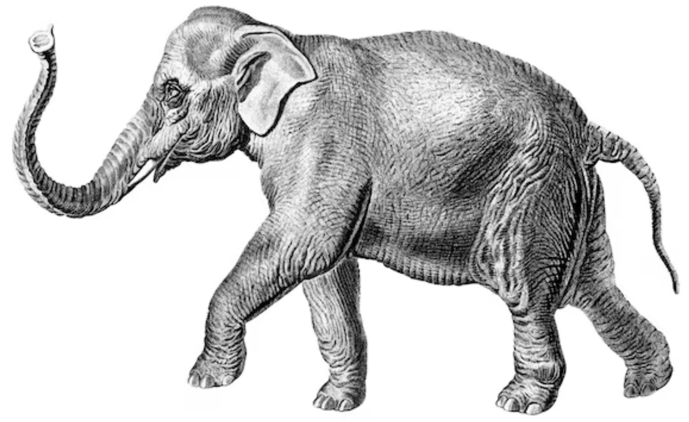
Haskell uses **lazy** evaluation also called **call-by-need**

In languages with pointers we also often have the option to use **call-by-reference** evaluation or **call-by-sharing**

*We will exclusively implement **call-by-value** (because, again, this is what OCaml does)*

Substitution

Substitution



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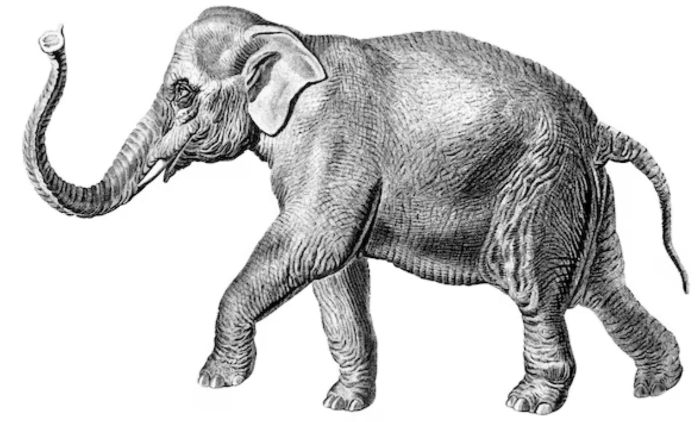

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`It's time to get more formal about` **`substitution`**

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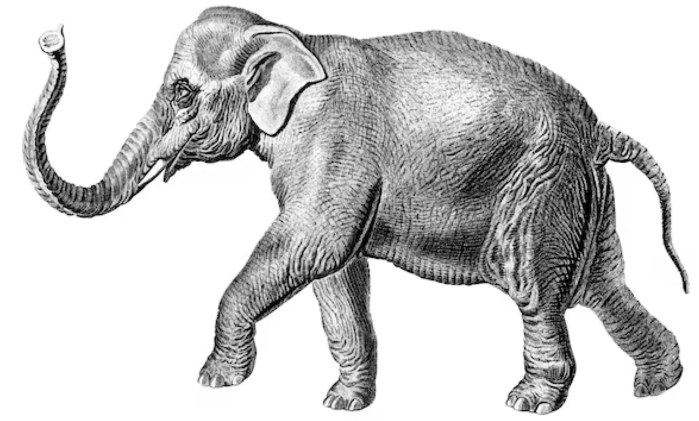

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We've been able to get by on our intuitions for a while, but our intuitions won't help us *implement* substitution (which is *difficult*)

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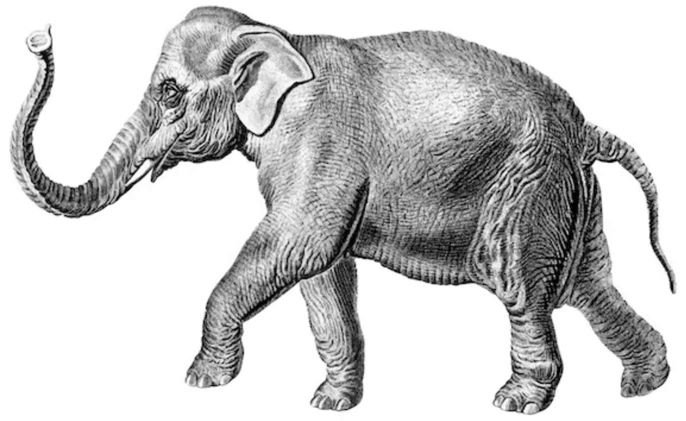

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We've been able to get by on our intuitions for a while, but our intuitions won't help us *implement* substitution (which is *difficult*)

We need to understand why...

Notation

$$[y/x](\lambda x \,.\, y)$$

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Informally. *Replace every instance of x with v*

Already things start to break down with this informal definition, e.g., consider the above substitution...

The Idea

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However we define substitution shouldn't *change the underlying behavior of a function*

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<u>The Key Point:</u> A function does not depend on our choice of variable names

α-Equivalence

```
let x = 2 in x + 1
```

$=_\alpha$

```
let z = 2 in z + 1
```

OCaml

$$\lambda x\,.\,\lambda y\,.\,x =_\alpha \lambda v\,.\,\lambda w\,.\,v$$

λ-calculus

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The **principle of name irrelevance** says that any two programs that are the same up to "renaming of variables" should behave exactly the same way (they are α**-equivalent**)

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We say that a variable x is **bound** is an expression if it appears in the expression as $(\ldots \lambda x . e \ldots)$

Substitution should preserve this

Definition (First Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases} \tag{1}$$

$$[v/y](\lambda x\,.\,e) = \lambda x\,.\,[v/y]e \tag{2}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2) \tag{3}$$

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1. Replace every y with v, leave other variables
2. Replace y with v in the body of a function
3. Replace y with v in both subexpressions of an application

(This is an example of an *inductive definition*)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x \,.\, e) = \lambda x \,.\, [v/y]e$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Problem Case I

$$[y/x](\lambda x \,.\, x)$$

We shouldn't be allowed to substitute x if it's the argument of a function

This may *change the behavior* of a function

Definition (Second Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \boxed{\begin{cases} \lambda x\,.\,e & x = y \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

We can handle the problem case directly in our definition. *Check the bound variable before we substitute in the body of a function*

Is there still a problem?

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x \, . \, e) = \begin{cases} \lambda x \, . \, e & x = y \\ \lambda x \, . \, [v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Problem Case II

$$[y/x](\lambda y \, . \, x)$$

We're not replacing a bound variable, but we *are* substituting an expression that has variables which *became* bound

The variable y is said to be **captured** in this (incorrect) substitution

Free and Bound Variables

$$FV(x) = \{x\} \tag{1}$$

$$FV(\lambda x \,.\, e) = FV(e) \backslash \{x\} \tag{2}$$

$$FV(e_1 e_2) = FV(e_1) \cup FV(e_2) \tag{3}$$

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Definition. A variable x is **free** in e if it does not appear **bound** by a λ. Formally:

1. x is free in x
2. x is free in $\lambda y\,.\,e$ if it is free in e and $x \neq y$
3. x is free in $e_1 e_2$ if x is free in e_1 or e_2

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<u>Definition.</u> A variable x is **free** in e if it does not appear **bound** by a λ. Formally:

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2. x is free in $\lambda y\,.\,e$ if it is free in e and $x \neq y$

3. x is free in $e_1 e_2$ if x is free in e_1 or e_2

<u>Definition.</u> A variable x is **free** in e if $x \in FV(e)$ as above

Definition (Third Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[v/y][z/x]e & x \in FV(v) \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Since we're interested in α-equivalence, we can first *replace* the bound variable and *substitute* it in the body of the function. This is called **α-renaming**

Is there still a problem?

Problem Case III

$$FV(x) = \{x\}$$
$$FV(\lambda x\,.\,e) = FV(e) \backslash \{x\}$$
$$FV(e_1 e_2) = FV(e_1) \cup FV(e_2)$$

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$
$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[w/z][z/x]e & x \in FV(v) \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$
$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

$$[x/y](\lambda x\,.\,xyz)$$

This isn't exactly a problem, but *we have to be careful about which variable to replace the bound variable* x *with*

If we choose z, then we capture a *different* variable!

"Correct" Definition

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[v/y][z/x]e & x \in FV(v), z \texttt{ is fresh} \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Finally a definition, that works. Sort of...

The only problem with this definition is that it now poses an implementation issue. **How do we come up with z?**

In mathematics, we can say it's **always possible** to come up with a variable z, but when we're implementing a programming language, we need an *actual* procedure

Well-Scopedness and Closedness

open

$$\lambda x\,.\,y$$

closed

$$\lambda x\,.\,\lambda y\,.\,y$$

Definition. *(informal)* An expression e is **well-scoped** if every free variable in e is "in scope" (more on that on Thursday)

Definition. An expression e is **closed** if it has no free variables

Every closed term is well-scoped

One Solution: Well-Scopedness Check

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x \,.\, e) = \begin{cases} \lambda x \,.\, e & x = y \\ \lambda z \,.\, [v/y][z/x]e & x \in FV(v), z \texttt{ is fresh} \\ \lambda x \,.\, [v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

If we only work with closed (well-scoped) expressions, then we don't need to worry about captured variables

The condition requiring α-renaming never holds!

The Takeaway: In mini-project 1, you should check if the expression has a free variable *before* you evaluate it

Summary

The **lambda calculus** is a simple function programming language useful for learning basic functional concepts

Different **evaluation strategies** have different benefits (and may yield different programming behaviors)

Substitution is a bit tricky to define correctly (!) but any definition must preserve α−equivalence