

Formal Semantics

Concepts of Programming Languages
Lecture 14

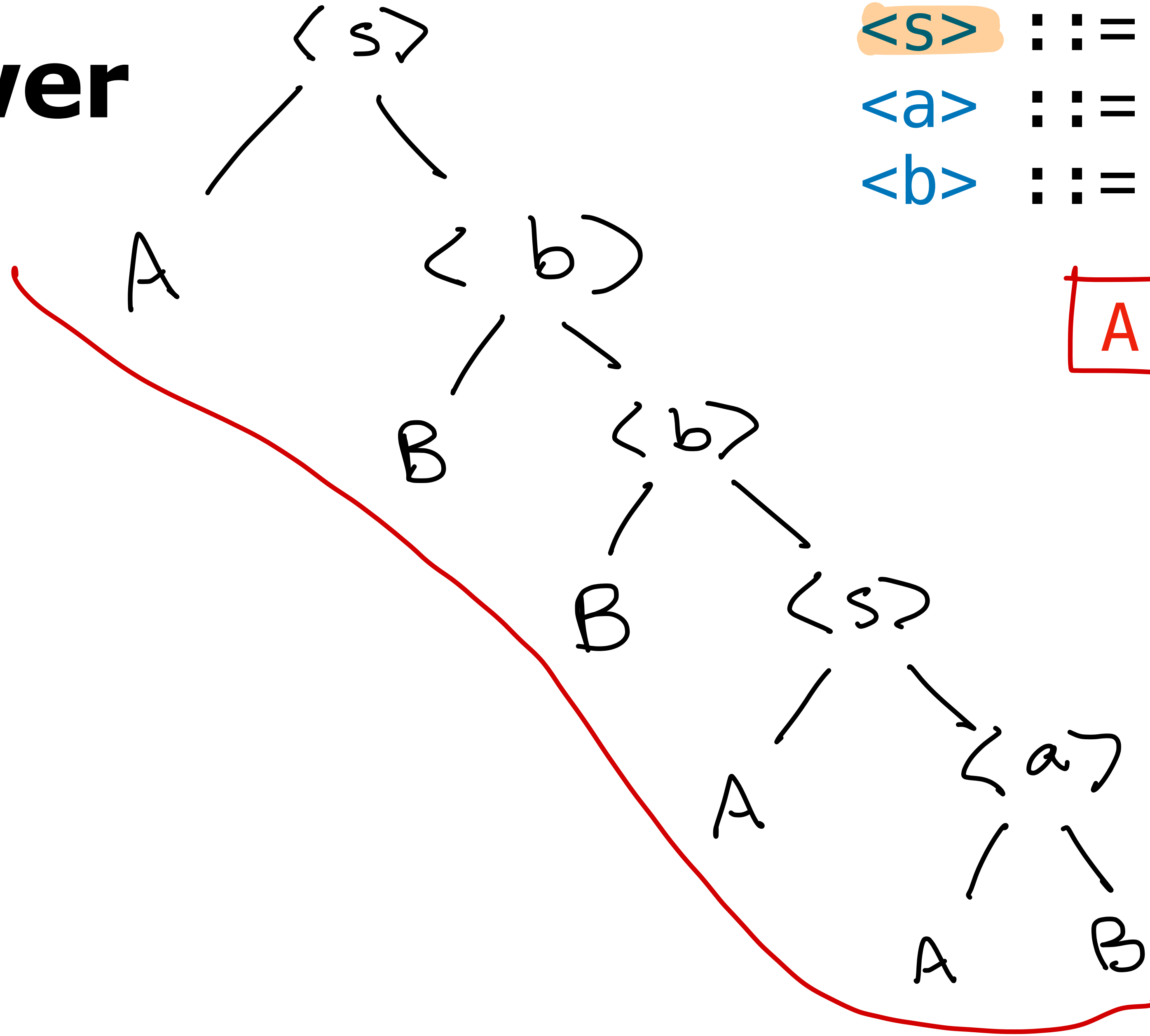
Practice Problem

$$\begin{aligned} \langle s \rangle &::= A \langle a \rangle \mid A \langle b \rangle \\ \langle a \rangle &::= A B \\ \langle b \rangle &::= B \langle b \rangle \mid B \langle s \rangle \end{aligned}$$

Is the following sentence recognized by the above grammar?

A B B A A B

Answer



$\langle s \rangle$	$::=$	$A \langle a \rangle \mid A \langle b \rangle$
$\langle a \rangle$	$::=$	$A B$
$\langle b \rangle$	$::=$	$B \langle b \rangle \mid B \langle s \rangle$

ABBAAB

Outline

- » Discuss **formal semantics** in general
- » Look at **small-step** and **big-step** semantics with some examples

Introduction

A Thought Experiment

```
x=3
function f () {
    x=2
}
f
echo $x
```

Bash

```
x = 3
def f():
    x = 2
f()
print(x)
```

Python

```
let x = 3
let f () =
    let x = 2 in
    ()
let _ = f ()
let _ = print_int x
```

OCaml

A Thought Experiment

```
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OCaml

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Question. *How do we know what will happen when a program executes?*

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Usually we build intuitions by writing programs and reading manuals

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OCaml

Question. *How do we know what will happen when a program executes?*

Usually we build intuitions by writing programs and reading manuals

But many decisions about what it means to execute a program are arbitrary (or based on concerns like efficiency)

Meaning

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Syntax is interested in the *form*
of a program

```
let x = 2 in  
let y = 3 in  
x + y
```

Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

```
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let y = 3 in  
x + y
```



```
let y = 3 in  
2 + y
```



```
2 + 3
```



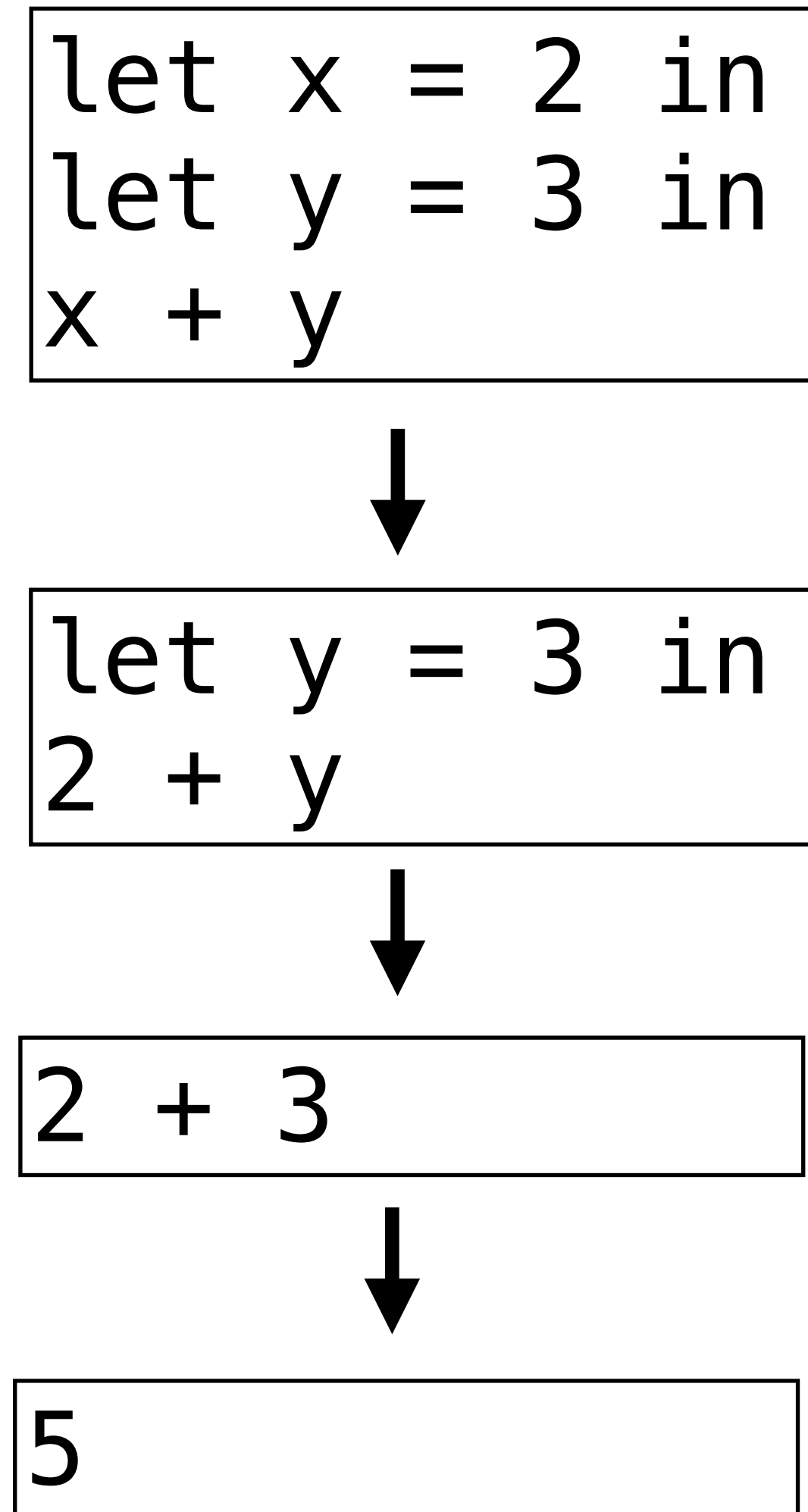
```
5
```

Meaning

Syntax is interested in the *form* of a program

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What is the meaning of meaning?



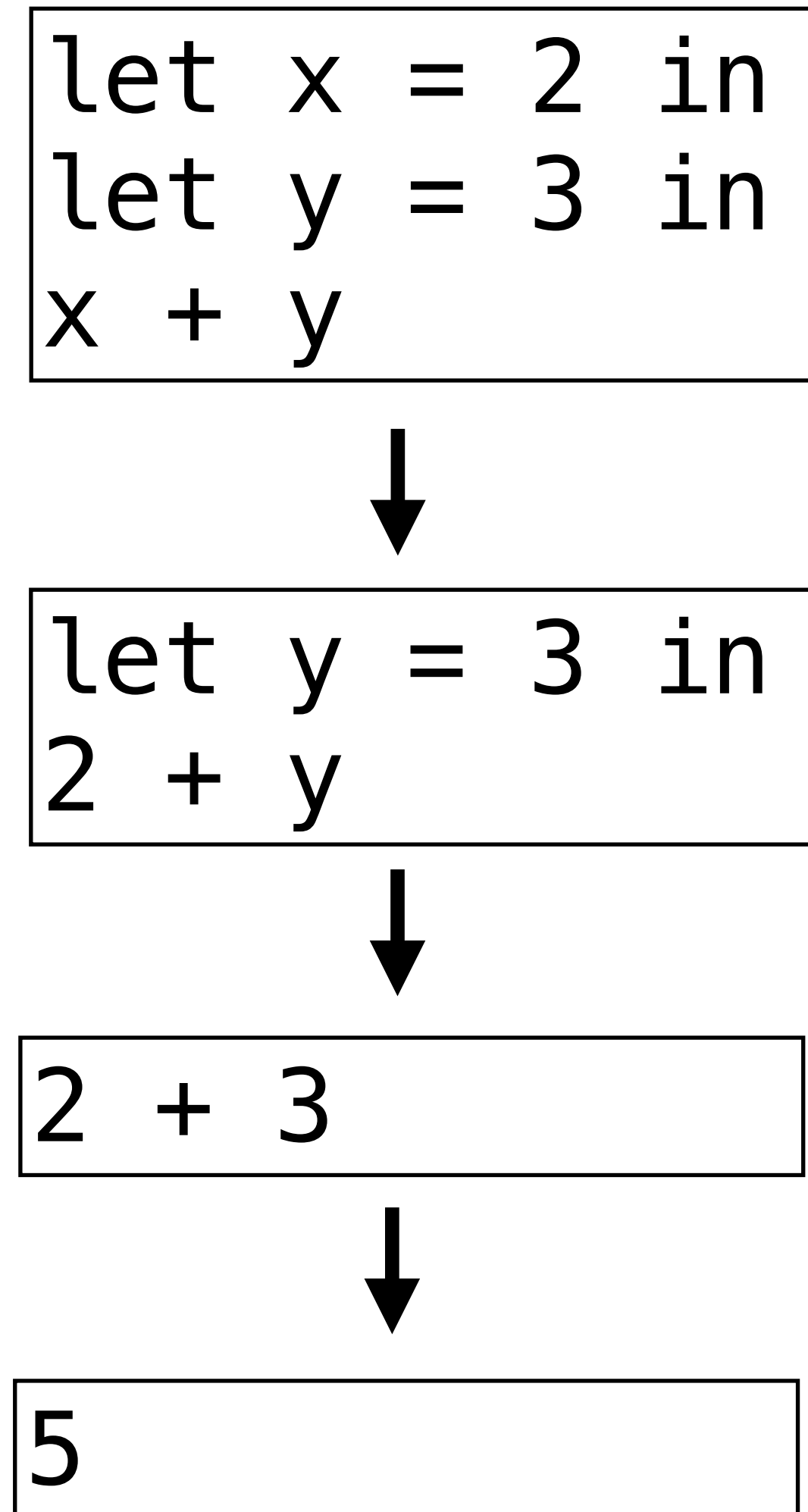
Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

What is the meaning of meaning?

Formal semantics is the mathematical study of meaning



Aside: Denotational vs. Operational Semantics

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Denotational semantics is interested in what a syntactic object "denotes" i.e. in interpreting programs as *objects in a mathematical space*

$$1 + 2 * 3 + 4 = 11$$
$$1 + 12 - 2 = 11$$

Aside: Denotational vs. Operational Semantics

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Operational semantics is interested in how a programming language "operates" i.e. how a program *behaves* during execution

$$\begin{aligned}1 + 2 * 3 + 4 &\longrightarrow 1 + 6 + 4 \\ &\longrightarrow 7 + 4 \\ &\longrightarrow 11\end{aligned}$$

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This course

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Small-Step vs. Big-Step Semantics

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Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

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let x = 2 in  
let y = 3 in  
x + y
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2 + y
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```
2 + 3
```



```
5
```

Small-Step vs. Big-Step Semantics

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Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

$$\frac{2 \Downarrow 2}{\text{let } x = 2 \text{ in } \frac{\frac{3 \Downarrow 3}{\text{let } y = 3 \text{ in } 2 + y \Downarrow 5}}{2 + 3 \Downarrow 5}} \Downarrow 5$$

Small-Step vs. Big-Step Semantics

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Mini-projects

Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

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Static vs. Dynamic Semantics

Static vs. Dynamic Semantics

Static semantics

refers to the meaning
given to a program
before it is evaluated

```
% ocaml silly.ml
```

```
File "./silly.ml", line 1, characters 8-9:
```

```
1 | let x = 2 +. 3.
```

^

Error: This expression has type int but an expression was
expected of type
float

Hint: Did you mean '2.'?

Static vs. Dynamic Semantics

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Dynamic semantics

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utop # let x = 2 + 3;;
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```
val x : int = 5
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Type checking

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Evaluation

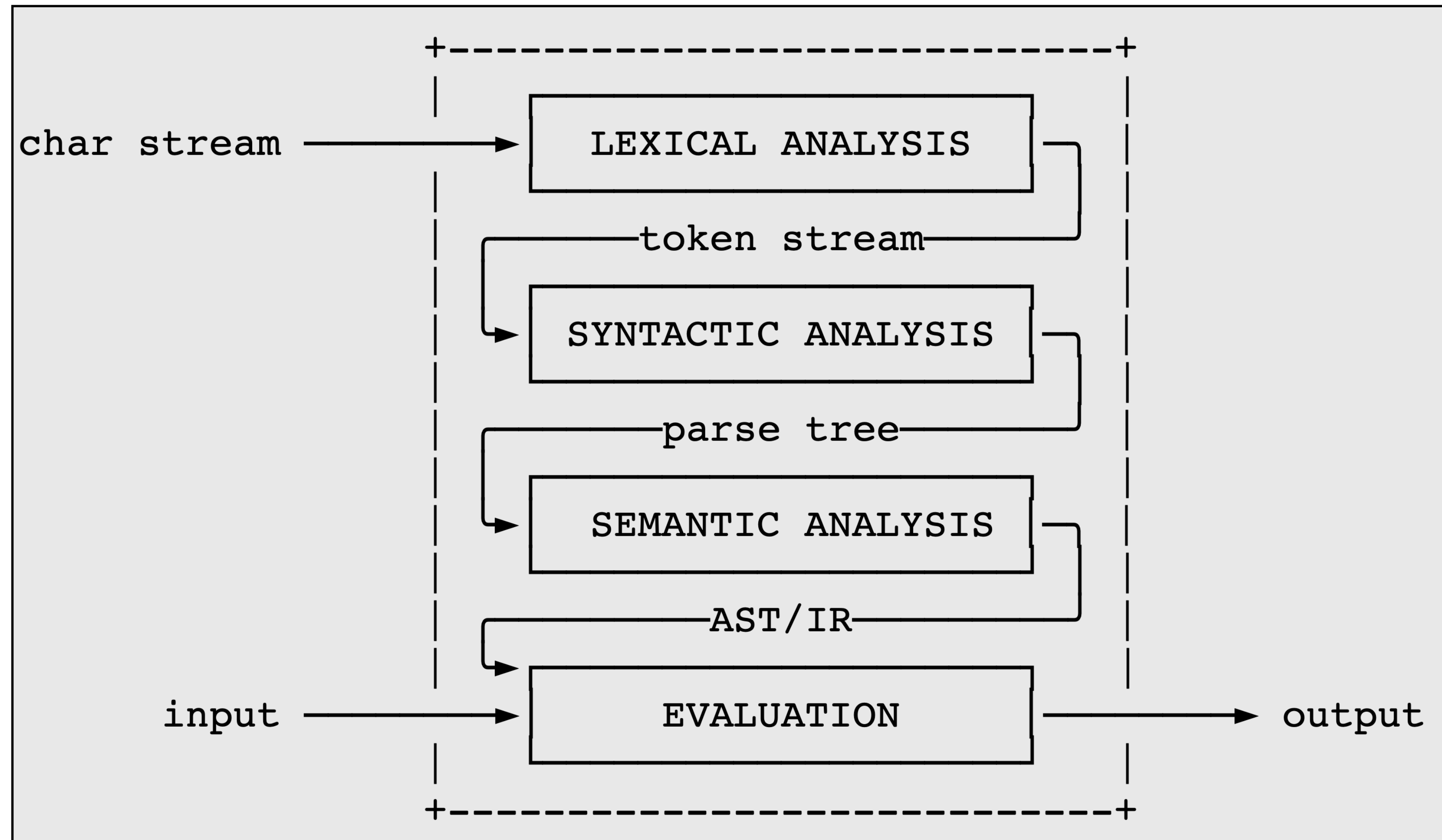
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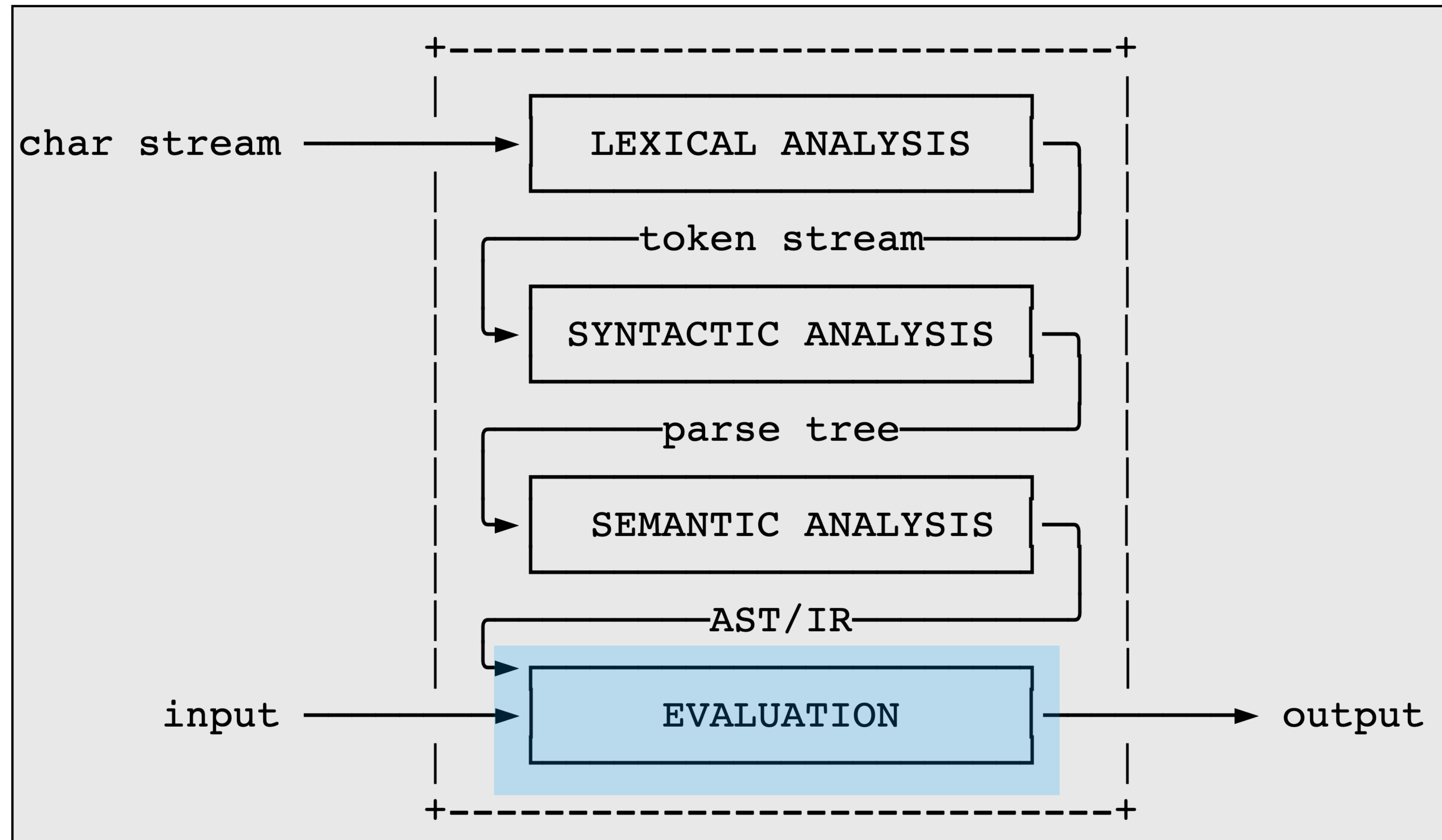
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Recall: The Picture

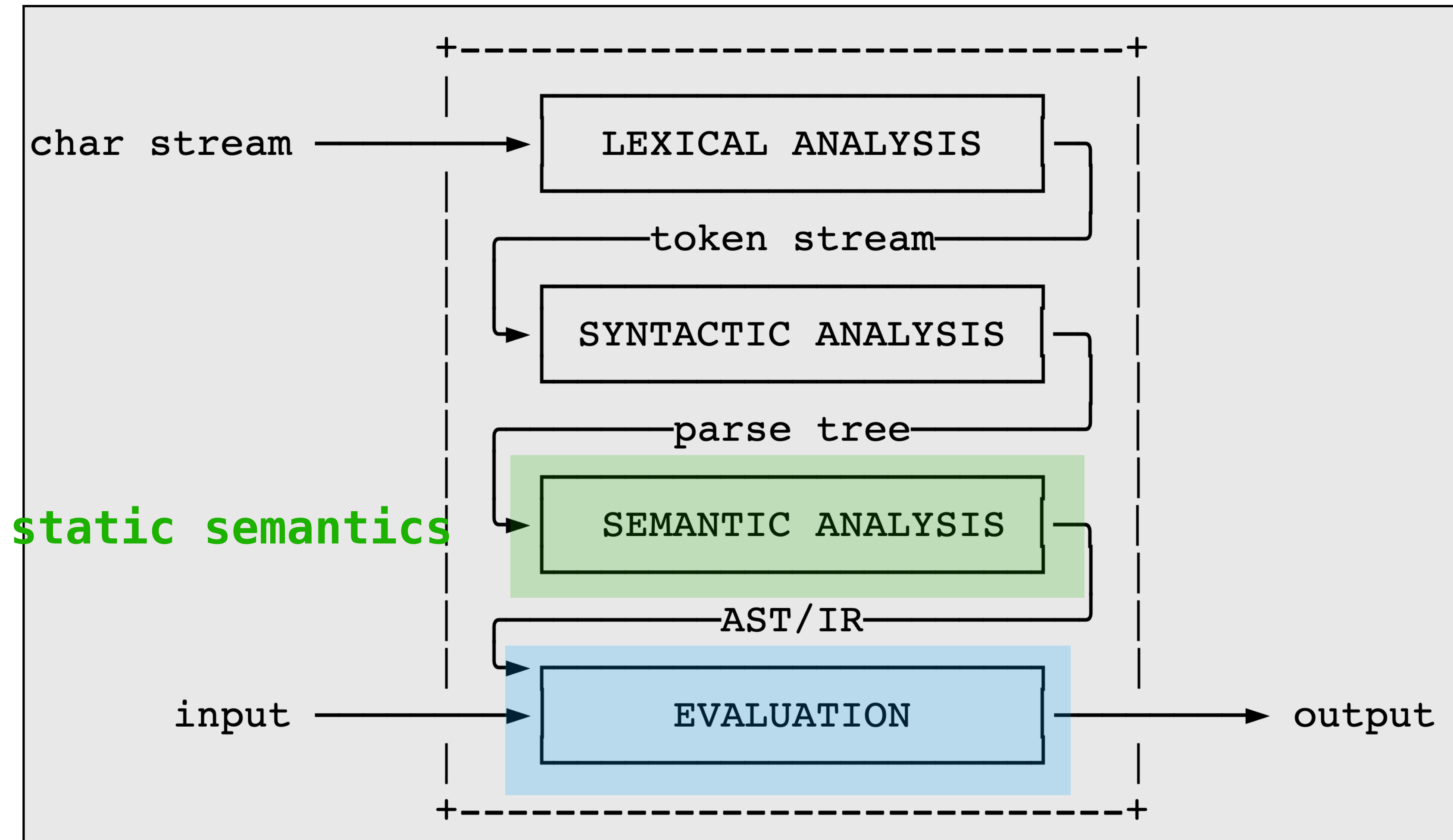


Recall: The Picture



dynamic semantics (this week + next week)

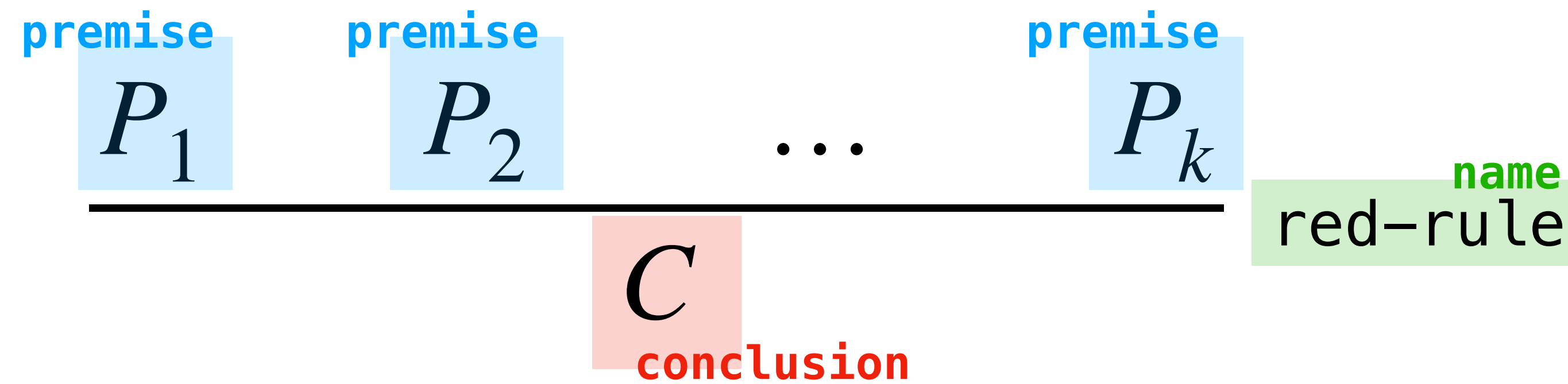
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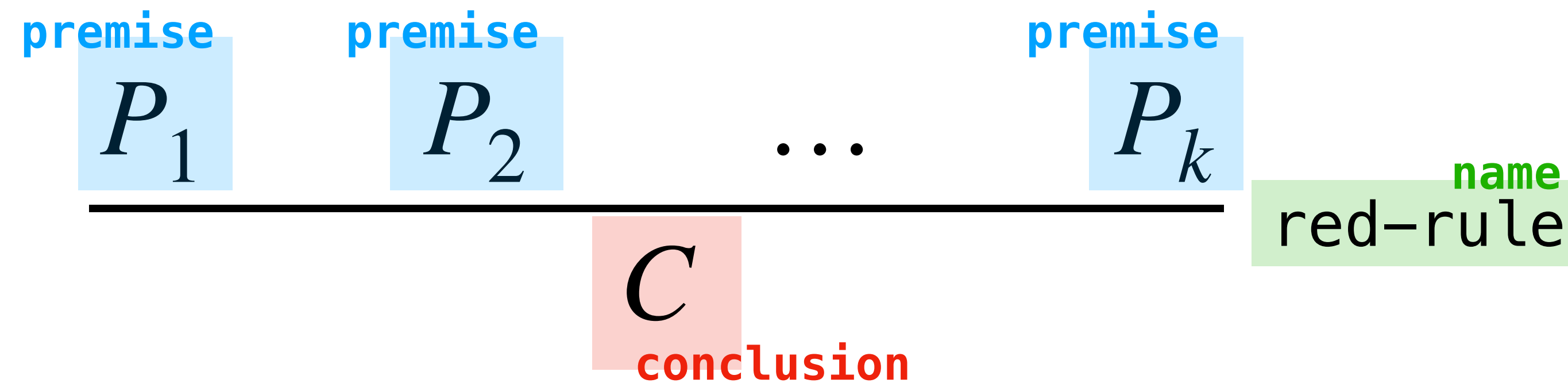
dynamic semantics (this week + next week)

Operational Semantics

Recall: Inference Rules

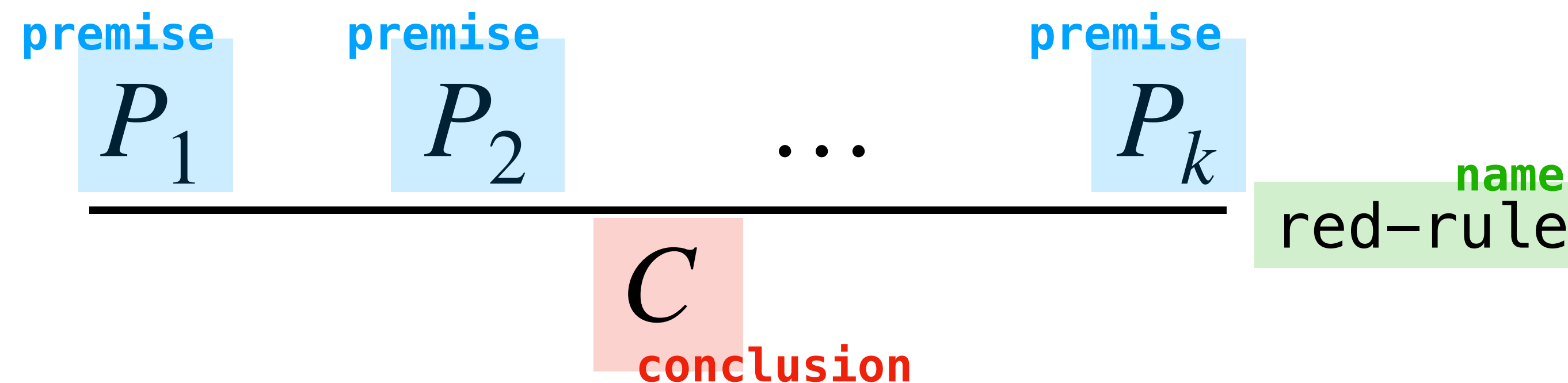


Recall: Inference Rules



Then general form of a reduction rule has a collection of **premises** and a **conclusion**

Recall: Inference Rules



Then general form of a reduction rule has a collection of **premises** and a **conclusion**

There may be no premises, this is called an **axiom**

Example

$$\frac{e_1 \xrightarrow{\text{premise}} e'_1}{(\text{add } e_1 \ e_2) \xrightarrow{\text{conclusion}} (\text{add } e'_1 \ e_2)} \text{add-left}$$

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<expr> ::= ( <op> <expr> <expr> )  
          | <bool> | <int>  
<op>    ::= add | sub | eq  
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```

Example Programs:

```
(add 2 3)  
(add (add 2 3) 5)  
(eq (add 2 3) (sub 7 2))  
(add true 2)
```

Example

$$\frac{e_1 \xrightarrow{\text{premise}} e'_1}{(\text{add } e_1 \ e_2) \xrightarrow{\text{add-left}} (\text{add } e'_1 \ e_2)} \text{conclusion}$$

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Example Programs:

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(add 2 3)  
(add (add 2 3) 5)  
(eq (add 2 3) (sub 7 2))  
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```

If e_1 reduces to e'_1 in one step, then $\text{add } e_1 \ e_2$ reduces to $\text{add } e'_1 \ e_2$ in one step

Another Example

add 2 3 \rightarrow 5

n_1 is a number

n_2 is a number

add-ok

$(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2$

If n_1 and n_2 are numbers then $(\text{add } n_1 \ n_2)$ reduces in one step to the number $n_1 + n_2$

In this case, the premises are side-conditions

(We'll come back to these examples)

Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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In general, we define small-step semantics on a **configuration**, which is a program together with some stateful information

Small-Step Semantics

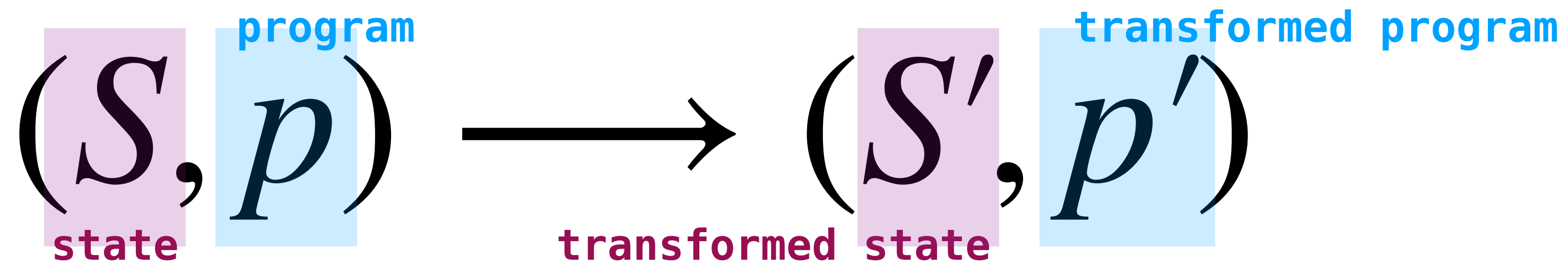
$$(S, \overset{\text{program}}{p}) \longrightarrow (S', \overset{\text{transformed program}}{p'})$$

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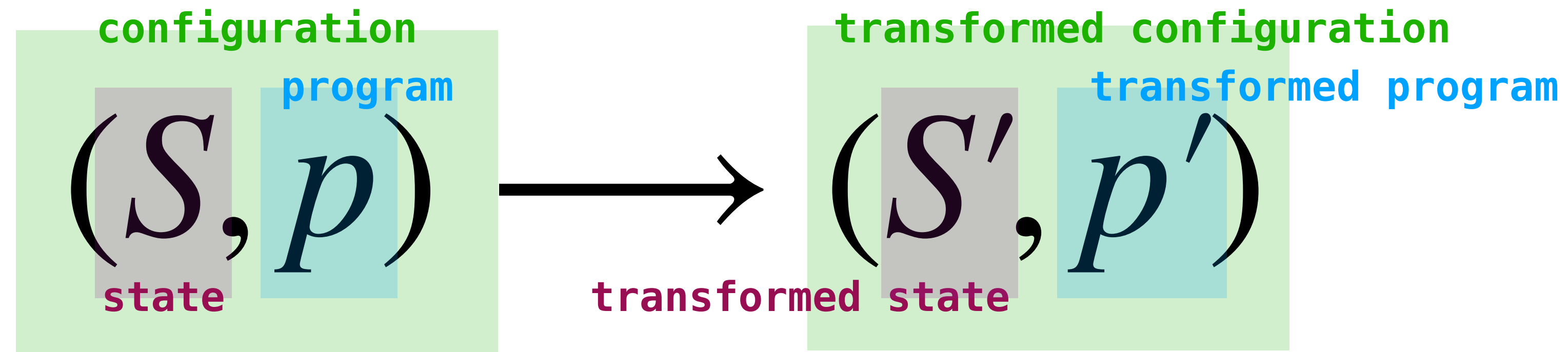


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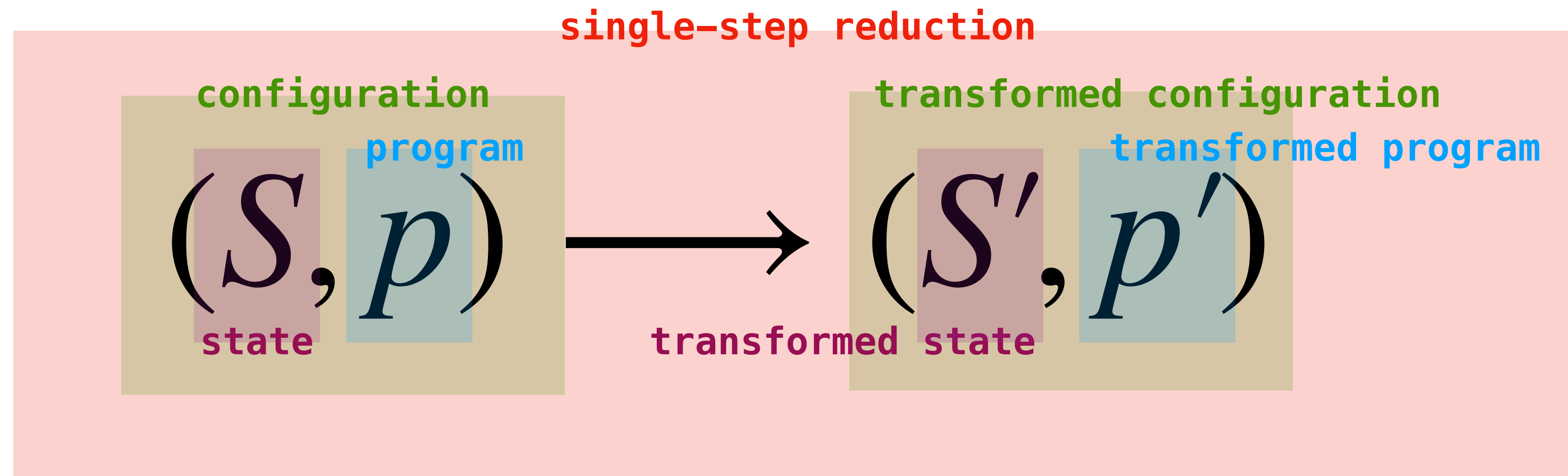


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Example: Arithmetic Expressions

$$\left(\underbrace{\emptyset}_{\text{state}}, \underbrace{10 \times (2 + 3)}_{\text{program}} \right) \longrightarrow (\emptyset, 10 \times 5) \longrightarrow (\emptyset, 50)$$

State: none

Program: arithmetic expression

Example: (Fragment of) OCaml

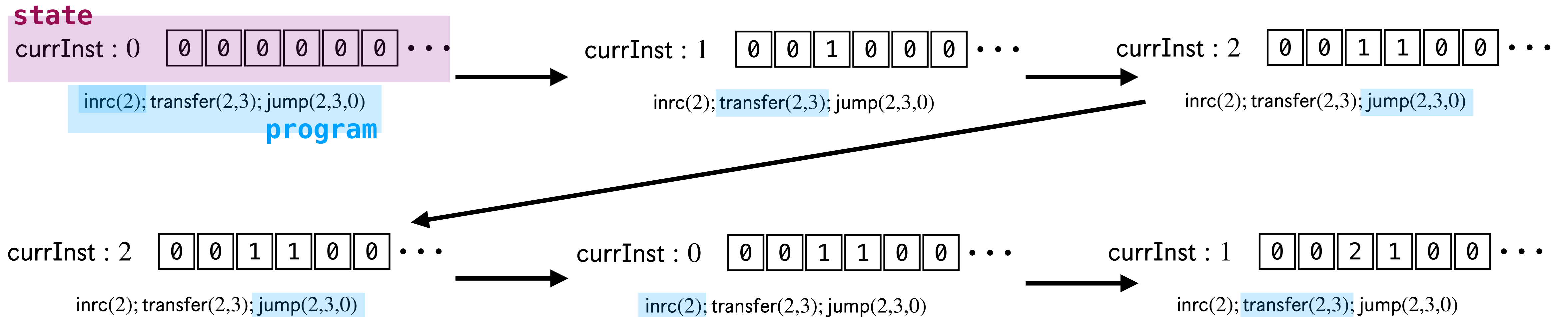
$(\emptyset, \text{let } x = 3 \text{ in if } x > 10 \text{ then } 4 \text{ else } 5)$ $\longrightarrow (\emptyset, \text{if } 3 > 10 \text{ then } 4 \text{ else } 5)$
 $\longrightarrow (\emptyset, \text{if false then } 4 \text{ else } 5)$
 $\longrightarrow (\emptyset, 5)$

State: none

Program: OCaml expression

For purely functional languages
there is no state

Example: Unlimited Register Machines



State: (current instruction pointer) +
 (collection of number registers)

Program: sequence of commands for updating registers
values and current instruction

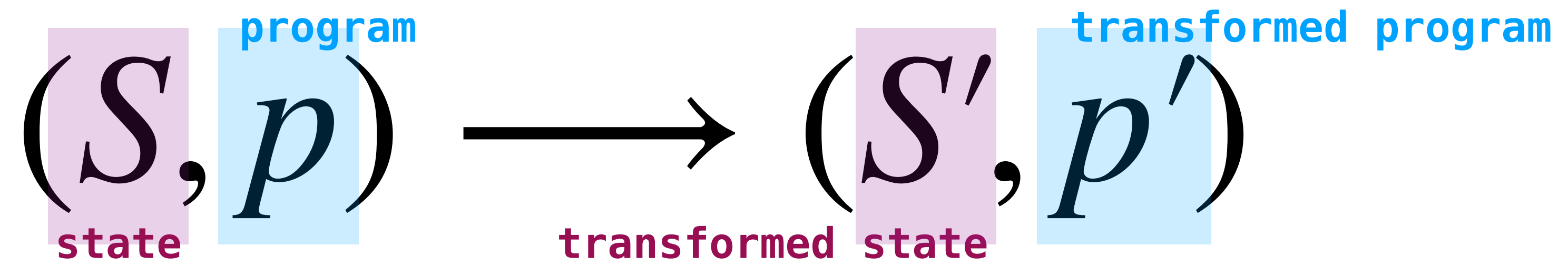
Example: Stack-Oriented Language

^{state}
(\emptyset , ^{program} push 2; push 3; add) \longrightarrow
(2 :: \emptyset , push 3; add) \longrightarrow
(3 :: 2 :: \emptyset , add) \longrightarrow
(5 :: \emptyset , ϵ)

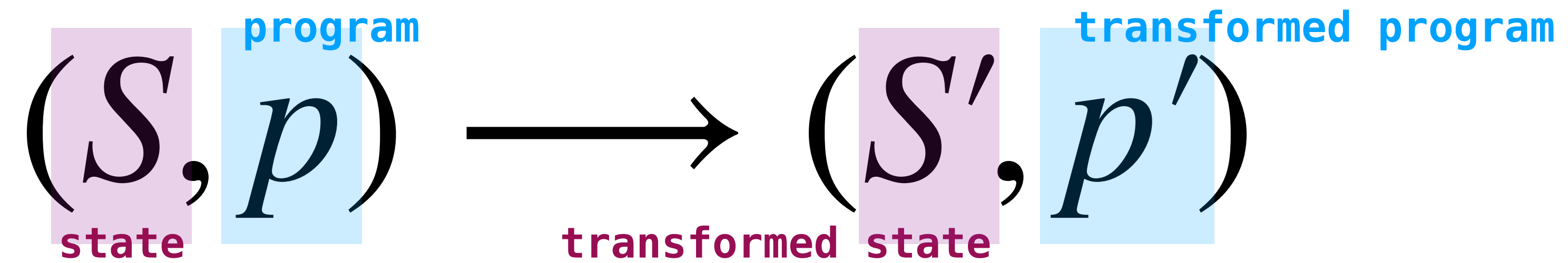
State: stack (i.e., list) of values

Program: sequence of commands for manipulating the stack

High-Level

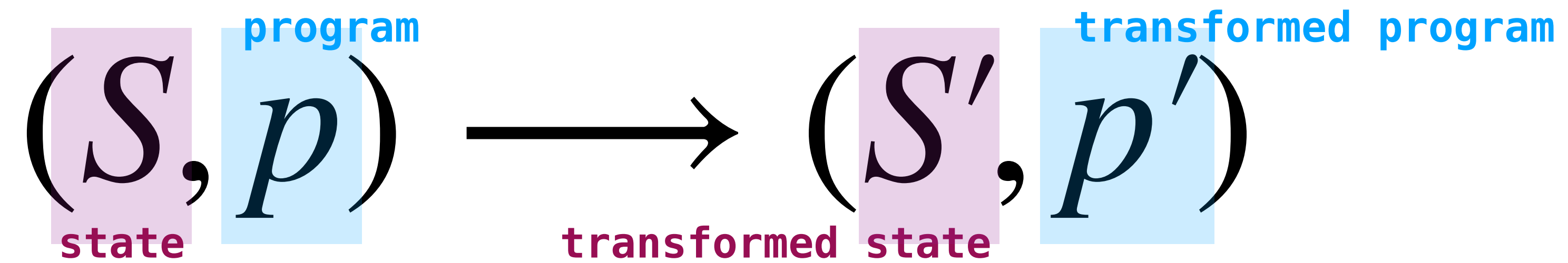


High-Level



When we define the small-step semantics of PL, we need to define two things:

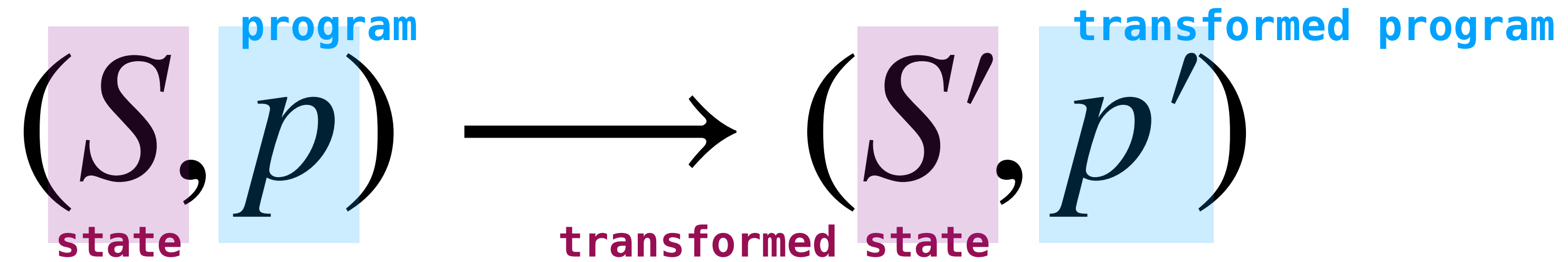
High-Level



When we define the small-step semantics of PL, we need to define two things:

» What kind of **state** are we manipulating?

High-Level



When we define the small-step semantics of PL, we need to define two things:

- » What kind of **state** are we manipulating?
- » What **rules** describe how to transform configurations?

Example

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

Example

<code><expr></code>	<code>::=</code>	<code>(<op> <expr> <expr>)</code>
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<code><op></code>	<code>::=</code>	<code>add sub eq</code>
<code><bool></code>	<code>::=</code>	<code>true false</code>
<code><int></code>	<code>::=</code>	<code>...</code>

$$\frac{e_1 \longrightarrow e'_1}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e'_1 \ e_2)} \text{ add-left}$$

$$\frac{e_2 \longrightarrow e'_2}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e_1 \ e'_2)} \text{ add-right}$$

$$\frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok}$$

$(\text{add } \overset{e_1}{\boxed{(\text{add } 1 \ 2)}} \ (\text{add } 2 \ 3)) \longrightarrow \}$ add left
 $(\text{add } \overset{e_1}{\boxed{3}} \ (\text{add } 2 \ 3)) \longrightarrow$
 $(\text{add } 3 \ 5) \longrightarrow 8$

add-over

(add 1 2) \rightarrow 3

add-left

(add (add 1 2) 3) \rightarrow (add 3 3)

Example

<code><expr></code>	<code>::=</code>	<code>(<op> <expr> <expr>)</code>
		<code> <bool> <int></code>
<code><op></code>	<code>::=</code>	<code>add sub eq</code>
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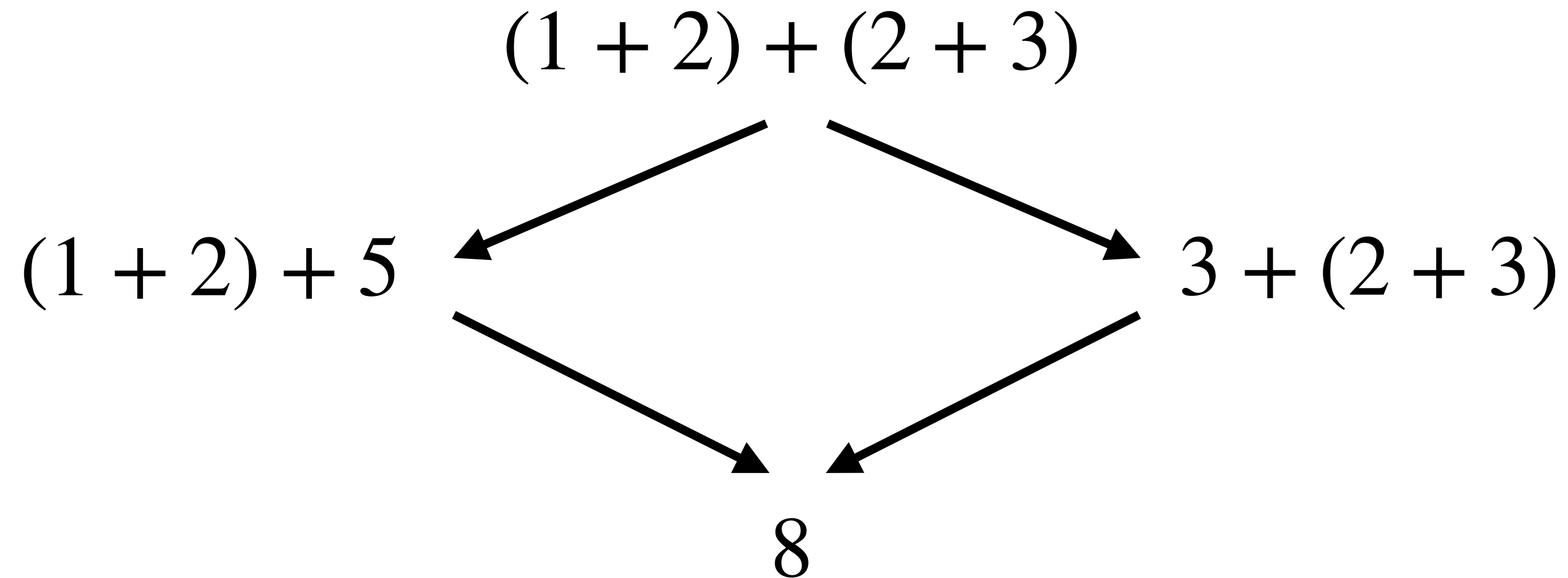
$$\frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok}$$

$$\frac{e_1 \longrightarrow e'_1}{(\text{sub } e_1 \ e_2) \longrightarrow (\text{sub } e'_1 \ e_2)} \text{ sub-left}$$

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$$\frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{sub } n_1 \ n_2) \longrightarrow n_1 - n_2} \text{ sub-ok}$$

Reduction is a Relation



It's important to recognize that **reduction is a *relation***

This means there may be **multiple choices of reductions**

When possible, we try to design our rules to avoid this

Reduction is a Relation

$$\frac{\text{add } 1 \ 2 \longrightarrow 3}{(\text{add } (\text{add } 1 \ 2) \ (\text{add } 2 \ 3)) \longrightarrow (\text{add } 3 \ (\text{add } 2 \ 3))} \text{ add-left}$$

$$\frac{\text{add } 2 \ 3 \longrightarrow 5}{(\text{add } (\text{add } 1 \ 2) \ (\text{add } 2 \ 3)) \longrightarrow (\text{add } (\text{add } 1 \ 2) \ 5)} \text{ add-right}$$

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There are two reductions from `(add (add 1 2) (add 2 3))` in our current rule set

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There are two reductions from `(add (add 1 2) (add 2 3))` in our current rule set

We can avoid this by *breaking symmetry*. We will enforce that the right argument can be reduced only when the `left argument is completely reduced`

Example: Addition

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

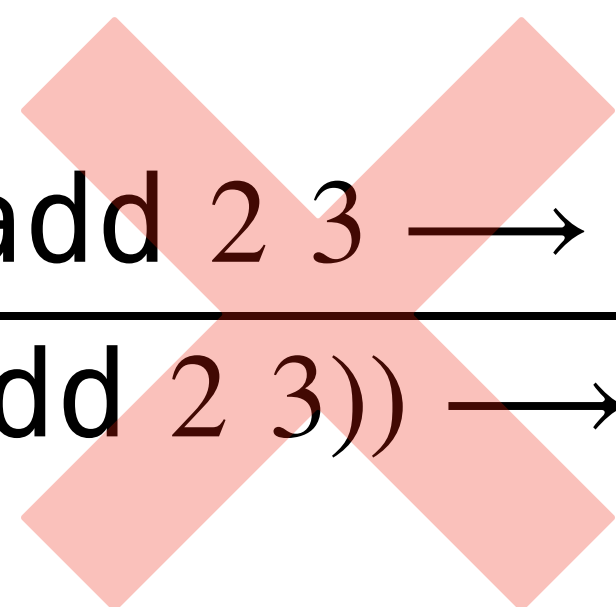
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$$\frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok}$$

Enforcing an Evaluation Order

not a number →

$$\frac{\text{add } 1 \ 2 \longrightarrow 3}{(\text{add } (\text{add } 1 \ 2) \ (\text{add } 2 \ 3)) \longrightarrow (\text{add } 3 \ (\text{add } 2 \ 3))} \text{ add-left}$$
$$\frac{\text{add } 2 \ 3 \longrightarrow 5}{(\text{add } (\text{add } 1 \ 2) \ (\text{add } 2 \ 3)) \longrightarrow (\text{add } (\text{add } 1 \ 2) \ 5)} \text{ add-right}$$


The new rule enforces that arguments of **add** are evaluated from left to right

Example

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<op>    ::= add | sub | eq  
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$$\frac{n \text{ is a number} \quad e_2 \longrightarrow e'_2}{(\text{add } n \ e_2) \longrightarrow (\text{add } n \ e'_2)} \text{ add-right}$$

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Example

<code><expr></code>	<code>::=</code>	<code>(<op> <expr> <expr>)</code>
		<code> <bool> <int></code>
<code><op></code>	<code>::=</code>	<code>add sub eq</code>
<code><bool></code>	<code>::=</code>	<code>true false</code>
<code><int></code>	<code>::=</code>	<code>...</code>

$$\frac{e_1 \longrightarrow e'_1}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e'_1 \ e_2)} \text{ add-left}$$

$$\frac{n \text{ is a number} \quad e_2 \longrightarrow e'_2}{(\text{add } n \ e_2) \longrightarrow (\text{add } n \ e'_2)} \text{ add-right}$$

$$\frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok}$$

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Practice Problem

```
<expr> ::= ( <op> <expr> <expr> )  
        | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

*Write down the reduction rules for **eq** (to the best of your ability) so that the left argument is evaluated before the right argument*

Answer

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
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Two Questions

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Two Questions

Once we have an operational semantics, there are **two questions** we can ask (as PL designers and on the final exam):

- » Show that $C \longrightarrow C'$ (add (add 2 3) 5) $\stackrel{?}{\longrightarrow}$ (add 6 8)
- » Given C , determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds)

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Once we have an operational semantics, there are **two questions** we can ask (as PL designers and on the final exam):

» Show that $C \longrightarrow C'$

» Given C , determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds)

Recall: Derivations

$$\frac{\frac{\frac{}{(add\ 1\ 2) \longrightarrow 3} \text{ add-ok}}{(add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow (add\ 3\ (add\ 2\ 3))} \text{ add-left}}{sub\ 10\ (add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow sub\ 10\ (add\ 3\ (add\ 2\ 3))} \text{ sub-right}$$

Recall: Derivations

$$\frac{\frac{\frac{}{(add\ 1\ 2) \longrightarrow 3} \text{ add-ok}}{(add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow (add\ 3\ (add\ 2\ 3))} \text{ add-left}}{sub\ 10\ (add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow sub\ 10\ (add\ 3\ (add\ 2\ 3))} \text{ sub-right}$$

A **derivation** is a tree of reductions, gotten by applying reduction rules. The leaves are trivial premises

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A derivation is a **proof** that the reduction step is valid in the operational semantics

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A derivation is a **proof** that the reduction step is valid in the operational semantics

We've done this!

Recall: Building Derivations

sub 10 (add (add 1 2) (add 2 3)) \longrightarrow sub 10 (add 3 (add 2 3))

Recall: Building Derivations

`sub 10 (add (add 1 2) (add 2 3)) → sub 10 (add 3 (add 2 3))`

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Recall: Building Derivations

$$\frac{\frac{\text{sub } 10 \text{ (add (add 1 2) (add 2 3))} \longrightarrow \text{sub } 10 \text{ (add 3 (add 2 3))}}{\text{(add (add 1 2) (add 2 3))} \longrightarrow \text{(add 3 (add 2 3))}} \text{ sub-right} \quad \frac{\text{(add 1 2)} \longrightarrow 3}{\text{(add (add 1 2) (add 2 3))} \longrightarrow \text{(add 3 (add 2 3))}} \text{ add-left}$$

We can build derivations from the ground up, applying rules in reverse

If the reduction is valid, then at each step we should be able to find a rule to apply

Recall: Building Derivations

$$\frac{\frac{\frac{}{(add\ 1\ 2) \longrightarrow 3} \text{ add-ok}}{(add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow (add\ 3\ (add\ 2\ 3))} \text{ add-left}}{sub\ 10\ (add\ (add\ 1\ 2)\ (add\ 2\ 3)) \longrightarrow sub\ 10\ (add\ 3\ (add\ 2\ 3))} \text{ sub-right}$$

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Two Questions

Once we have a small-step semantics, there are **two questions** we can ask (as PL designers and on the final exam):

» Show that $C \longrightarrow C'$

» Given C , determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds)

Single-Step Evaluation

$(\text{sub } 10 (\text{add } (\text{add } 1 \ 2) (\text{add } 2 \ 3))) \longrightarrow ???$

$(\text{sub } 10 (\text{add } 3 (\text{add } 2 \ 3)))$

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The more "realistic" situation is to be given a program and then try to `figure out what it evaluates to` in a single step

Single-Step Evaluation

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This is why we want to be careful about how we design our rules: *we don't want to get too caught up on which rule to apply*

Example

$$\frac{e_1 \longrightarrow e'_1}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e'_1 \ e_2)} \text{ add-left}$$

$$\frac{n \text{ is a number} \quad e_2 \longrightarrow e'_2}{(\text{add } n \ e_2) \longrightarrow (\text{add } n \ e'_2)} \text{ add-right}$$

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$(\text{sub } 10 \ (\text{add} \ (\text{add } 1 \ 2) \ (\text{add } 2 \ 3))) \longrightarrow ???$

$(\text{add } 1 \ 2) \longrightarrow 3$

add-ok

$(\text{add } (\text{add } 1 \ 2) \ (\text{add } 2 \ 3)) \longrightarrow (\text{add } 3 \ (\text{add } 2 \ 3))$

add-left

$(\text{sub } 10 \ (\text{add} \ (\text{add } 1 \ 2) \ (\text{add } 2 \ 3))) \longrightarrow (\text{sub } 10 \ (\text{add } 3 \ (\text{add } 2 \ 3)))$

add-right

Practice Problem

$$\begin{array}{c} \frac{e_1 \longrightarrow e'_1}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e'_1 \ e_2)} \text{ add-left} \qquad \frac{e_2 \longrightarrow e'_2}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e_1 \ e'_2)} \text{ add-right} \\[10pt] \frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok} \\[10pt] \frac{e_1 \longrightarrow e'_1}{(\text{sub } e_1 \ e_2) \longrightarrow (\text{sub } e'_1 \ e_2)} \text{ sub-left} \qquad \frac{e_2 \longrightarrow e'_2}{(\text{sub } e_1 \ e_2) \longrightarrow (\text{sub } e_1 \ e'_2)} \text{ sub-right} \\[10pt] \frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{sub } n_1 \ n_2) \longrightarrow n_1 - n_2} \text{ sub-ok} \end{array}$$

$(\text{sub } 10 \ (\text{add } 3 \ (\text{add } 2 \ 3))) \longrightarrow (\text{sub } 10 \ (\text{add } 3 \ 5))$

Give a derivation of the above reduction

Answer

$$\overbrace{(\text{add } 2 \ 3) \rightarrow 5}^{\text{add-ok}}$$

add-right

$$(\text{add } 3 \ \boxed{(\text{add } 2 \ 3)}) \rightarrow (\text{add } 3 \ \boxed{5})$$

sub-right

$$(\text{sub } 10 (\text{add } 3 (\text{add } 2 \ 3))) \rightarrow (\text{sub } 10 (\text{add } 3 \ 5))$$

Multi-Step Reduction Relation

$$\frac{}{C \longrightarrow^{\star} C} \text{ refl} \qquad \frac{C \longrightarrow C' \quad C' \longrightarrow^{\star} D}{C \longrightarrow^{\star} D} \text{ trans}$$

Handwritten red annotations:
- Under the first rule, a red arrow points from the star to a red '0'.
- Under the second rule, a red 'n' is above the star and a red 'n+1' is below the star.

Given any single-step reduction relation, we can derive the **multi-step reduction relation**:

- » Every configuration reduces to itself **(reflexivity)**
- » Every \longrightarrow^{\star} reduction can be extended by a single step **(transitivity)**

Two Questions (Again)

Once we have an operational semantics, there are **two questions** we can ask (as PL designers and on the final exam):

- » Show that $C \longrightarrow^* C'$
- » Given C , determine a configuration C' such that $C \longrightarrow^* C'$ and C' cannot be reduced

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How To: Derivations of Multi-Step Reductions

`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow^* 2`
want to show

» Derive all necessary single-step evaluations

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`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow^* 2`
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`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow sub 10 (add 3 (add 2 3))` (we did this)

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`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow^* 2`
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`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow sub 10 (add 3 (add 2 3))` (we did this)

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`sub 10 (add 3 5) \longrightarrow sub 10 8` (exercise)

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`sub 10 (add 3 5) \longrightarrow sub 10 8` (exercise)

`sub 10 8 \longrightarrow 2` (easy)

» Derive all necessary single-step evaluations

How To: Derivations of Multi-Step Reductions

`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow^* 2`

- » Derive all necessary single-step evaluations
- » Combine them with the **transitivity rule**

How To: Derivations of Multi-Step Reductions

$$\frac{\begin{array}{c} \text{(we did this)} \\ \vdots \\ s\ 10\ (a\ (a\ 1\ 2)\ (a\ 2\ 3)) \longrightarrow s\ 10\ (a\ 3\ (a\ 2\ 3)) \end{array} \quad s\ 10\ (a\ 3\ (a\ 2\ 3)) \longrightarrow^* 2}{\text{sub}\ 10\ (\text{add}\ (\text{add}\ 1\ 2)\ (\text{add}\ 2\ 3)) \longrightarrow^* 2} \text{trans}$$

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 \hline
 \text{sub 10 (add (add 1 2) (add 2 3))} \longrightarrow^* 2
 \end{array}
 \quad
 \begin{array}{c}
 \text{(you did this)} \\
 \vdots \\
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 \begin{array}{c}
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 \text{(easy)} \\
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 \hline
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 \begin{array}{c}
 \text{refl} \\
 \hline
 \text{trans}
 \end{array}$$

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Two Questions (Again)

Once we have an operational semantics, there are **two questions** we can ask (as PL designers and on the final exam):

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want to show

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If our rules are well defined, then should be easy:

Solve this single-step evaluation problem until you reach a configuration that cannot be further reduced

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How To: Evaluation

`sub 10 (add (add 1 2) (add 2 3)) \longrightarrow^* 2`
want to show

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`sub 10 (add 3 5) \longrightarrow sub 10 8`

`sub 10 8 \longrightarrow 2`

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`y (fun x -> x) ↯`

When are we done?

When evaluating, there are **three** "end" cases:

» **value:** we reach the end of our computation and the value of our program

$(\text{fun } x \rightarrow x) (2 + 3) \rightarrow^* 5$

» **stuck:** we reach an expression that cannot be reduced, but that is not a value

$y (\text{fun } x \rightarrow x) \nrightarrow$

» **diverge:** the computation never reaches a point where the expression is not reducible

$(\text{fun } x \rightarrow x x) (\text{fun } x \rightarrow x x) \rightarrow^*$
 $(\text{fun } x \rightarrow x x) (\text{fun } x \rightarrow x x)$

moving onto big-step...

Big-Step Semantics

$(\text{sub } 10 (\text{add } (\text{add } 1 \ 2) (\text{add } 2 \ 3))) \Downarrow 2$

Big-Step Semantics

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Big-step semantics deals only with a program and its value

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Notation: We write $e \Downarrow v$ to mean that e evaluates to the value v

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This is what we've been doing in this course so far

Example

<code><expr></code>	<code>::=</code>	<code>(</code>	<code><op></code>	<code><expr></code>	<code><expr></code>	<code>)</code>
				<code><bool></code>	<code> </code>	<code><int></code>
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<code><bool></code>	<code>::=</code>	<code>true</code>	<code> </code>	<code>false</code>		
<code><int></code>	<code>::=</code>	<code>...</code>				

$$\frac{n \text{ is a number}}{n \Downarrow n} \text{ numEval}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{add } e_1 \ e_2) \Downarrow v_1 + v_2} \text{ addEval}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{sub } e_1 \ e_2) \Downarrow v_1 - v_2} \text{ subEval}$$

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```
<expr> ::= ( <op> <expr> <expr> )  
          | <bool> | <int>  
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$$\frac{n \text{ is a number}}{n \Downarrow n} \text{ numEval}$$
$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{add } e_1 \ e_2) \Downarrow v_1 + v_2} \text{ addEval}$$
$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{sub } e_1 \ e_2) \Downarrow v_1 - v_2} \text{ subEval}$$

we'll remove these side conditions once we have type-checking

Practice Problem

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

Write the rule for eq

Answer

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

Relation to Small-Step

$$e \longrightarrow^{\star} v \qquad \approx \qquad e \Downarrow v$$

The big-step relation "**cuts out the middle steps**" of a small-step relation

This means fewer and clearer rules, but less fine-grain control of the evaluation sequence

Note: We can't always have both small-step and big-step!

Order of Evaluation

.....order of evaluation.....▶

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{add } e_1 \ e_2) \Downarrow v_1 + v_2} \text{addEval}$$

With small-step semantics, we can choose the order of evaluations based on the rules

With big-step semantics, we can't because our relation only deals with the *final* value, nothing intermediate

We will take the order of operations to be from left to right

Summary

big-step

$$e \Downarrow v$$

*e evaluates
to v*

single-step

$$e \longrightarrow e'$$

*e reduces to e'
in a single step*

multi-step

$$e \longrightarrow^{\star} e'$$

*e reduces to e'
in many steps*