Formal Semantics

Concepts of Programming Languages
Lecture 14

CAS CS 320

Practice Problem

```
<s> ::= A <a> | A <b>
<a> ::= A B
<b> ::= B <b> | B <s>
```

Is the following sentence recognized by the above grammar?

```
A B B A A B
```

Answer

```
<s> ::= A <a> | A <b>
<a> ::= A B
<b> ::= B <b> | B <s>
```

A B B A A B

Outline

- » Discuss **formal semantics** in general
- » Look at **small-step** and **big-step** semantics with some examples

Introduction

A Thought Experiment

```bash
x=3
function f () {
    x=2
}
f
echo $x
```

Bash

```python
x = 3
def f():
    x = 2
f()
print(x)
```

Python

```ocaml
let x = 3
let f () =
  let x = 2 in
  ()
let _ = f ()
let _ = print_int x
```

OCaml

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Question. *How do we know what will happen when a program executes?*

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Usually we build intuitions by writing programs and reading manuals

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OCaml

Question. *How do we know what will happen when a program executes?*

Usually we build intuitions by writing programs and reading manuals

But many decisions about what it means to execute a program are arbitrary (or based on concerns like efficiency)

Meaning

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Syntax is interested in the *form* of a program

```
let x = 2 in
let y = 3 in
x + y
```

Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

```
let x = 2 in
let y = 3 in
x + y
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```
let y = 3 in
2 + y
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```
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Meaning

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What is the meaning of meaning?

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Meaning

Syntax is interested in the *form* of a program

Semantics is interested in the *meaning* of a program

What is the meaning of meaning?

Formal semantics is the mathematical study of meaning

```
let x = 2 in
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Aside: Denotational vs. Operational Semantics

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Denotational semantics is interested in what a syntactic object "denotes" i.e. in interpreting programs as *objects in a mathematical space*

$$1 + 2 * 3 + 4 = 11$$

$$1 + 12 - 2 = 11$$

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Operational semantics is interested in how a programming language "operates" i.e. how a program *behaves* during execution

$$\begin{aligned} 1 + 2 * 3 + 4 &\longrightarrow 1 + 6 + 4 \\ &\longrightarrow 7 + 4 \\ &\longrightarrow 11 \end{aligned}$$

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This course

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Small-Step vs. Big-Step Semantics

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Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

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let x = 2 in
let y = 3 in
x + y
```

↓

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↓

```
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Small-Step vs. Big-Step Semantics

Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

$$\dfrac{2 \Downarrow 2 \qquad \dfrac{3 \Downarrow 3 \qquad \dfrac{2 \Downarrow 2 \qquad 3 \Downarrow 3}{2 + 3 \Downarrow 5}}{\texttt{let y = 3 in 2 + y} \Downarrow 5}}{\texttt{let x = 2 in let y = 3 in x + y} \Downarrow 5}$$

Small-Step vs. Big-Step Semantics

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Mini-projects

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Static vs. Dynamic Semantics

Static vs. Dynamic Semantics

Static semantics refers to the meaning given to a program *before* it is evaluated

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% ocaml silly.ml
File "./silly.ml", line 1, characters 8-9:
1 | let x = 2 +. 3.
            ^
Error: This expression has type int but an expression was
       expected of type
         float
  Hint: Did you mean '2.'?
```

Static vs. Dynamic Semantics

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Dynamic semantics
refers to the behavior of a program *during* evaluation

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utop # let x = 2 + 3;;
val x : int = 5
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Evaluation

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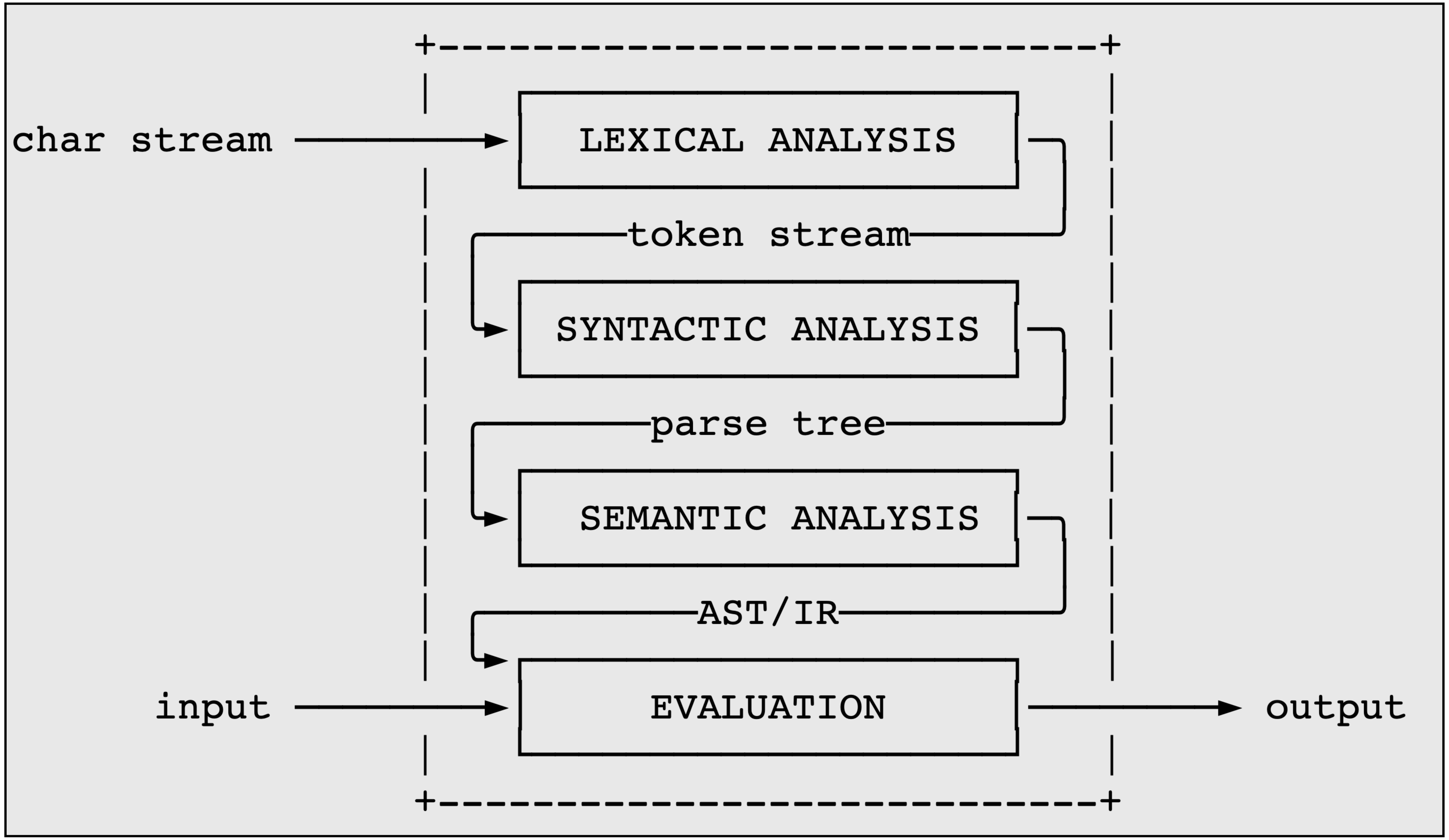
Recall: The Picture



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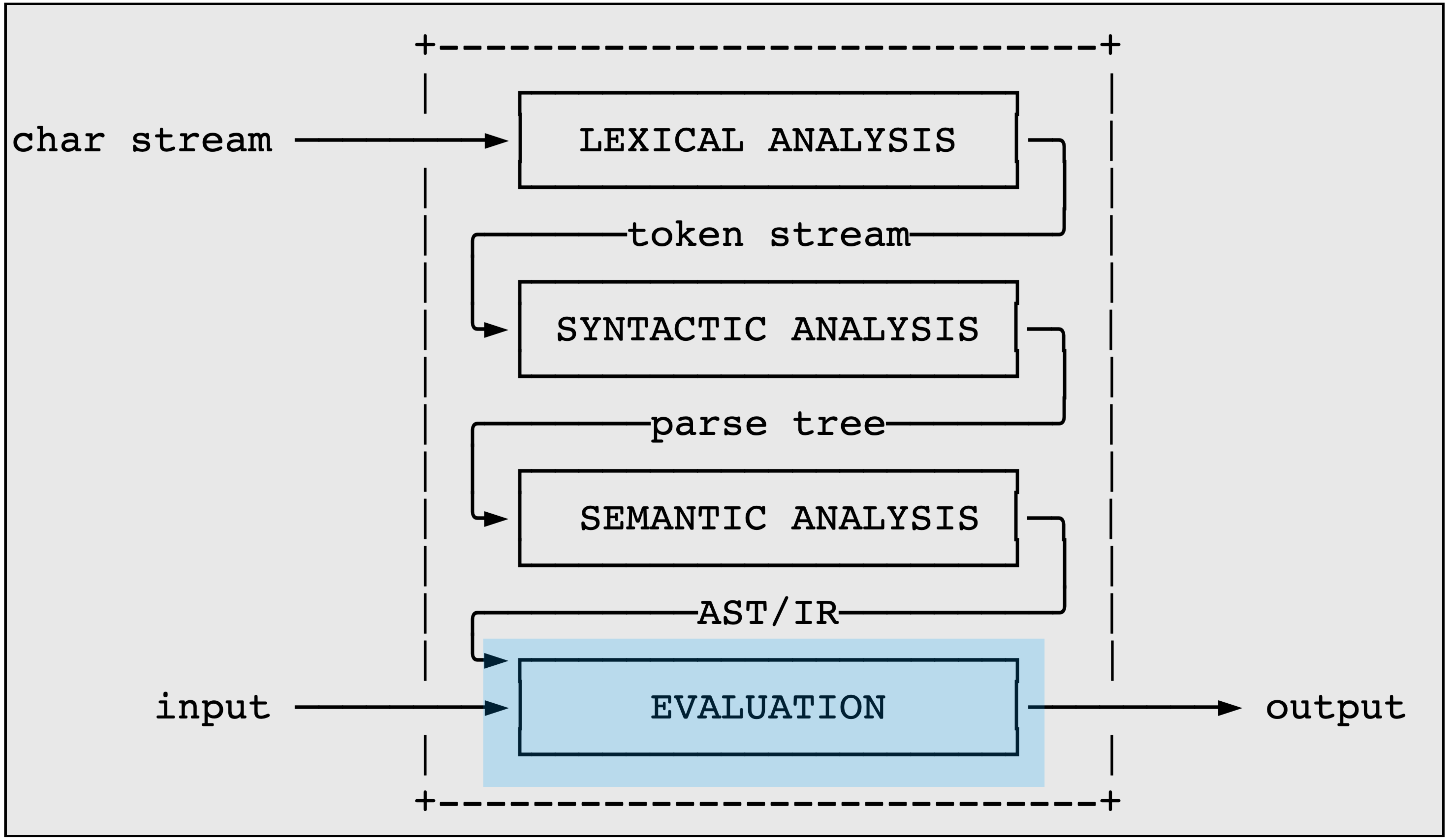

dynamic semantics (this week + next week)

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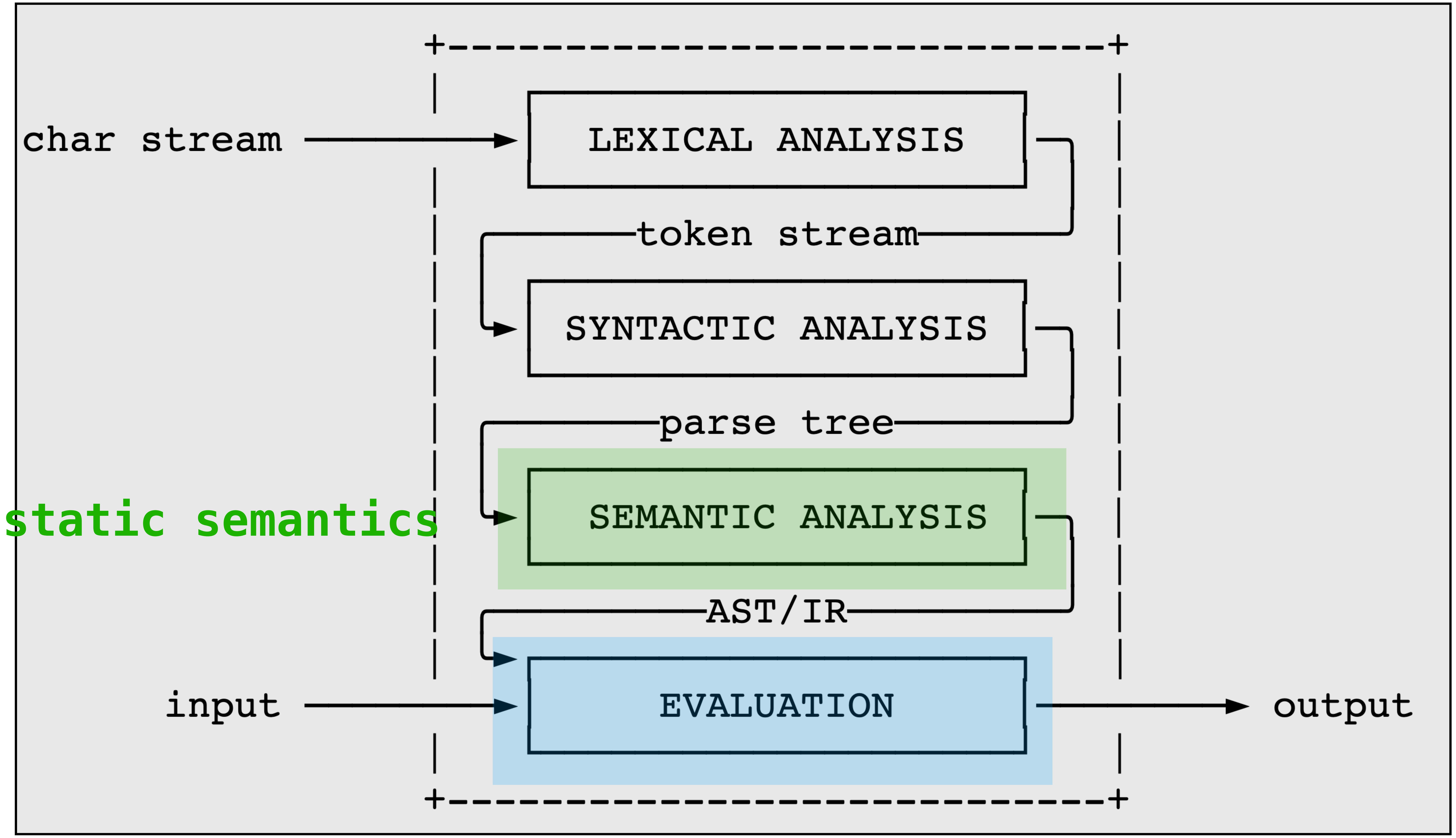

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Operational Semantics

Recall: Inference Rules

$$\frac{\overset{\texttt{premise}}{P_1} \qquad \overset{\texttt{premise}}{P_2} \qquad \ldots \qquad \overset{\texttt{premise}}{P_k}}{\underset{\texttt{conclusion}}{C}}\ \overset{\texttt{name}}{\texttt{red-rule}}$$

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Then general form of a reduction rule has a collection of **premises** and a **conclusion**

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Then general form of a reduction rule has a collection of **premises** and a **conclusion**

There may be no premises, this is called an **axiom**

Example

premise

$$\frac{e_1 \longrightarrow e_1'}{(\text{add } e_1\ e_2) \longrightarrow (\text{add } e_1'\ e_2)}\ \texttt{add-left}$$

conclusion

```
<expr> ::= ( <op> <expr> <expr> )
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Example Programs:

```
(add 2 3)
(add (add 2 3) 5)
(eq (add 2 3) (sub 7 2))
(add true 2)
```

Example

premise

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Example Programs:
(add 2 3)
(add (add 2 3) 5)
(eq (add 2 3) (sub 7 2))
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If e_1 *reduces to* e_1' *in one step, then* add e_1 e_2 *reduces to* add e_1' e_2 *in one step*

Another Example

$$\frac{n_1 \texttt{ is a number} \qquad n_2 \texttt{ is a number}}{(\mathsf{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

If n_1 and n_2 are numbers then $(\mathsf{add}\ n_1\ n_2)$ reduces in one step to the number $n_1 + n_2$

In this case, the premises are side-conditions

(We'll come back to these examples)

Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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In general, we define small-step semantics on a **configuration,** which is a program together with some stateful information

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Small-Step Semantics

$$\left(\underbrace{S}_{\text{state}}, \overbrace{p}^{\text{program}}\right) \longrightarrow \left(\underbrace{S'}_{\text{transformed state}}, \overbrace{p'}^{\text{transformed program}}\right)$$

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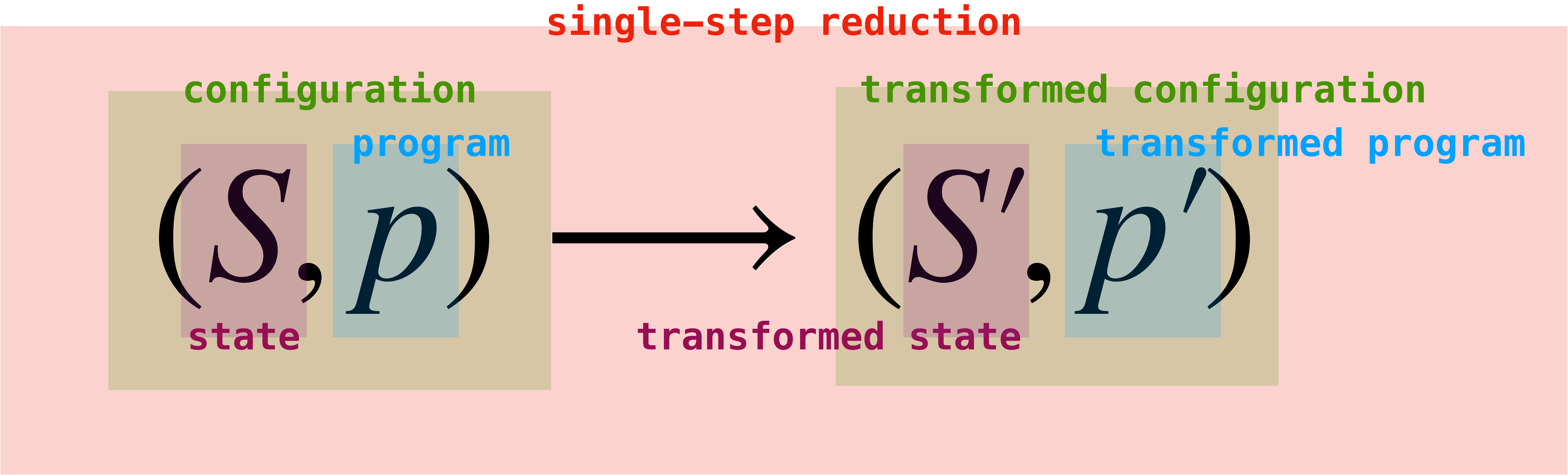

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Example: Arithmetic Expressions

$$\underbrace{(\varnothing}_{\text{state}}, \overbrace{10 \times (2+3)}^{\text{program}}) \longrightarrow (\varnothing, 10 \times 5) \longrightarrow (\varnothing, 50)$$

State: none

Program: arithmetic expression

Example: (Fragment of) OCaml

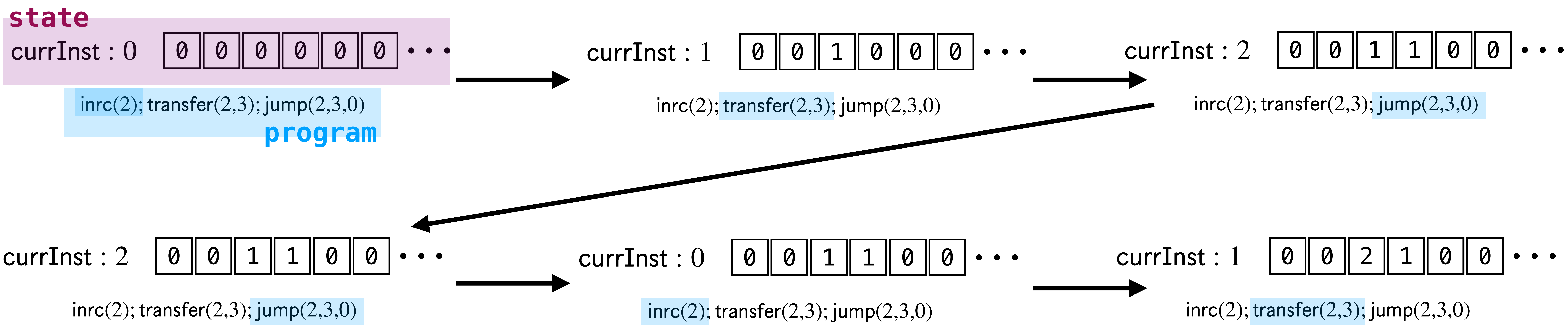
$$(\underbrace{\varnothing}_{\text{state}}, \overbrace{\texttt{let x = 3 in if x > 10 then 4 else 5}}^{\text{program}}) \longrightarrow (\varnothing, \texttt{if 3 > 10 then 4 else 5})$$
$$\longrightarrow (\varnothing, \texttt{if false then 4 else 5})$$
$$\longrightarrow (\varnothing, \texttt{5})$$

State: none

Program: OCaml expression

For purely functional languages
there is no state

Example: Unlimited Register Machines



<u>State:</u> (current instruction pointer) + (collection of number registers)

<u>Program:</u> sequence of commands for updating registers values and current instruction

Example: Stack-Oriented Language

$$\begin{array}{ll}
(\overset{\text{state}}{\varnothing}, \overset{\text{program}}{\texttt{push 2; push 3; add}}) & \longrightarrow \\
(2 :: \varnothing, \texttt{push 3; add}) & \longrightarrow \\
(3 :: 2 :: \varnothing, \texttt{add}) & \longrightarrow \\
(5 :: \varnothing, \epsilon) &
\end{array}$$

<u>State:</u> stack (i.e., list) of values

<u>Program:</u> sequence of commands for manipulating the stack

High-Level

$$(\underset{\texttt{state}}{S}, \overset{\texttt{program}}{p}) \longrightarrow (\underset{\texttt{transformed state}}{S'}, \overset{\texttt{transformed program}}{p'})$$

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$$(\underbrace{S}_{\text{state}}, \overbrace{p}^{\text{program}}) \longrightarrow (\underbrace{S'}_{\text{transformed state}}, \overbrace{p'}^{\text{transformed program}})$$

When we define the small-step semantics of PL, we need to define two things:

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When we define the small-step semantics of PL, we need to define two things:

» What kind of **state** are we manipulating?

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When we define the small-step semantics of PL, we need to define two things:

» What kind of **state** are we manipulating?

» What **rules** describe how to transform configurations?

Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
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$$\frac{e_2 \longrightarrow e_2'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

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$$\frac{e_1 \longrightarrow e_1'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1'\ e_2)}\ \texttt{sub-left} \qquad \frac{e_2 \longrightarrow e_2'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1\ e_2')}\ \texttt{sub-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{sub}\ n_1\ n_2) \longrightarrow n_1 - n_2}\ \texttt{sub-ok}$$

Reduction is a Relation

$$\begin{tikzcd}[column sep=small]
 & (1+2)+(2+3) \arrow[dl] \arrow[dr] & \\
(1+2)+5 \arrow[dr] & & 3+(2+3) \arrow[dl] \\
 & 8 &
\end{tikzcd}$$

It's important to recognize that **reduction is a *relation***

This means there may be multiple choices of reductions

When possible, we try do design our rules to avoid this

Reduction is a Relation

$$\frac{\mathsf{add}\ 1\ 2 \longrightarrow 3}{(\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3)) \longrightarrow (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))}\ \texttt{add-left}$$

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We can avoid this by *breaking symmetry*. We will enforce that the right argument can reduced only when the left argument is completely reduced

Example: Addition

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{v\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{add}\ v\ e_2) \longrightarrow (\texttt{add}\ v\ e_2')}\ \texttt{add-right}$$

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Enforcing an Evaluation Order

$$\frac{\mathsf{add}\ 1\ 2 \longrightarrow 3}{(\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3)) \longrightarrow (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))}\ \texttt{add-left}$$

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`The new rule enforces that arguments of` **`add`** `are evaluated from left to right`

Example

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Practice Problem

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Write down the reduction rules for ***eq*** *(to the best of your ability) so that the left argument is evaluated before the right argument*

Answer

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» Given C, determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds)

Two Questions

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A derivation is a proof that the reduction step is valid in the operational semantics

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We've done this!

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The more "realistic" situation is to be given a program and then try to figure out what it evaluates to in a single step

This is why we want to be careful about how we design our rules: *we don't want to get too caught up on which rule to apply*

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{n\ \texttt{is a number} \quad e_2 \longrightarrow e_2'}{(\texttt{add}\ n\ e_2) \longrightarrow (\texttt{add}\ n\ e_2')}\ \texttt{add-right}$$

Example

$$\frac{n_1\ \texttt{is a number} \quad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

$$(\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))) \longrightarrow ???$$

Practice Problem

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left} \qquad \frac{e_2 \longrightarrow e_2'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1\ e_2')}\ \texttt{add-right}$$

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$$\frac{e_1 \longrightarrow e_1'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1'\ e_2)}\ \texttt{sub-left} \qquad \frac{e_2 \longrightarrow e_2'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1\ e_2')}\ \texttt{sub-right}$$

$$\frac{n_1\ \texttt{is a number} \quad n_2\ \texttt{is a number}}{(\texttt{sub}\ n_1\ n_2) \longrightarrow n_1 - n_2}\ \texttt{sub-ok}$$

$$(\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))) \longrightarrow (\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5))$$

Give a derivation of the above reduction

Answer

$$(\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))) \longrightarrow (\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ 5))$$

Multi-Step Reduction Relation

$$\frac{}{C \longrightarrow^{\star} C}\ \texttt{refl} \qquad \frac{C \longrightarrow C' \qquad C' \longrightarrow^{\star} D}{C \longrightarrow^{\star} D}\ \texttt{trans}$$

Given any single-step reduction relation, we can derive the **multi-step reduction relation:**

» Every configuration reduces to itself **(reflexivity)**

» Every $\longrightarrow^{\star}$ reduction can be extended by a single step **(transitivity)**

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Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

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How To: Derivations of Multi-Step Reductions

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^\star 2$$

want to show

» Derive all necessary single-step evaluations

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$\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5) \longrightarrow \texttt{sub}\ 10\ 8$ (exercise)

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$\texttt{sub}\ 10\ 8 \longrightarrow 2$ (easy)

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$$\dfrac{\dfrac{\texttt{(we did this)}}{\vdots}\quad \texttt{s}\ 10\ (\texttt{a}\ (\texttt{a}\ 1\ 2)\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \qquad \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow^{\star} 2}{\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2}\ \texttt{trans}$$

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» **stuck:** we reach an expression that cannot be reduced, but that is not a value

`y (fun x -> x)` $\not\rightarrow$

» **diverge:** the computation never reaches a point where the expression is not reducible

`(fun x -> x x) (fun x -> x x)` $\longrightarrow^{\star}$ `(fun x -> x x) (fun x -> x x)` moving onto big-step...

Big-Step Semantics

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This is what we've been doing in this course so far

Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{n \texttt{ is a number}}{n \Downarrow n}\ \texttt{numEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{add}\ e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{sub}\ e_1\ e_2) \Downarrow v_1 - v_2}\ \texttt{subEval}$$

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$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1\ \texttt{is a number} \qquad v_2\ \texttt{is a number}}{(\texttt{sub}\ e_1\ e_2) \Downarrow v_1 - v_2}\ \texttt{subEval}$$

we'll remove these side conditions once we have type-checking

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Practice Problem

Write the rule for ***eq***

Answer

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Relation to Small-Step

$$e \longrightarrow^{\star} v \qquad \approx \qquad e \Downarrow v$$

The big-step relation **"cuts out the middle steps"** of a small-step relation

This means fewer and clearer rules, but less fine-grain control of the evaluation sequence

Note: We can't always have both small-step and big-step!

Order of Evaluation

order of evaluation

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{add } e_1 \ e_2) \Downarrow v_1 + v_2} \ \texttt{addEval}$$

With small-step semantics, we can choose the order of evaluations based on the rules

With big-step semantics, we can't because our relation only deals with the *final* value, nothing intermediate

We will take the order of operations to be from left to right

Summary

big-step	single-step	multi-step
$$e \Downarrow v$$	$$e \longrightarrow e'$$	$$e \longrightarrow^{\star} e'$$
e evaluates to v	*e reduces to e' in a single step*	*e reduces to e' in many steps*