

The Substitution Model

Concepts of Programming Languages
Lecture 15

Outline

- » Look formally at the **lambda calculus** and its semantics
- » Discuss **substitution** and the pitfalls to avoid

Recap

Recall: Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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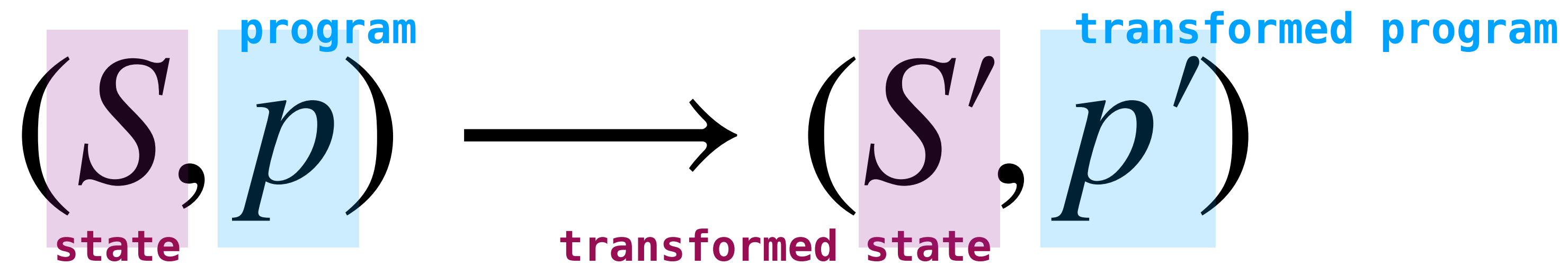
$$(S, \overset{\text{program}}{p}) \longrightarrow (S', \overset{\text{transformed program}}{p'})$$

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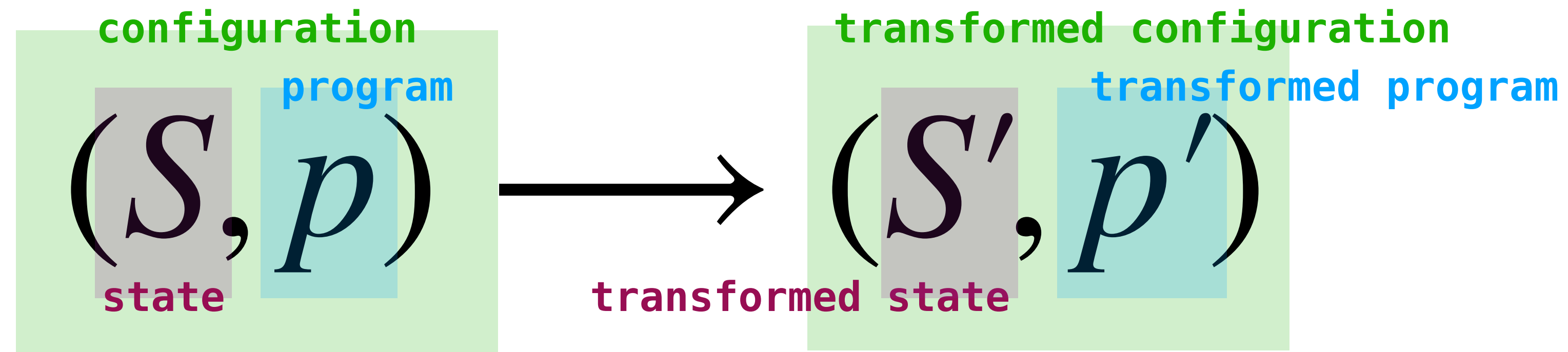


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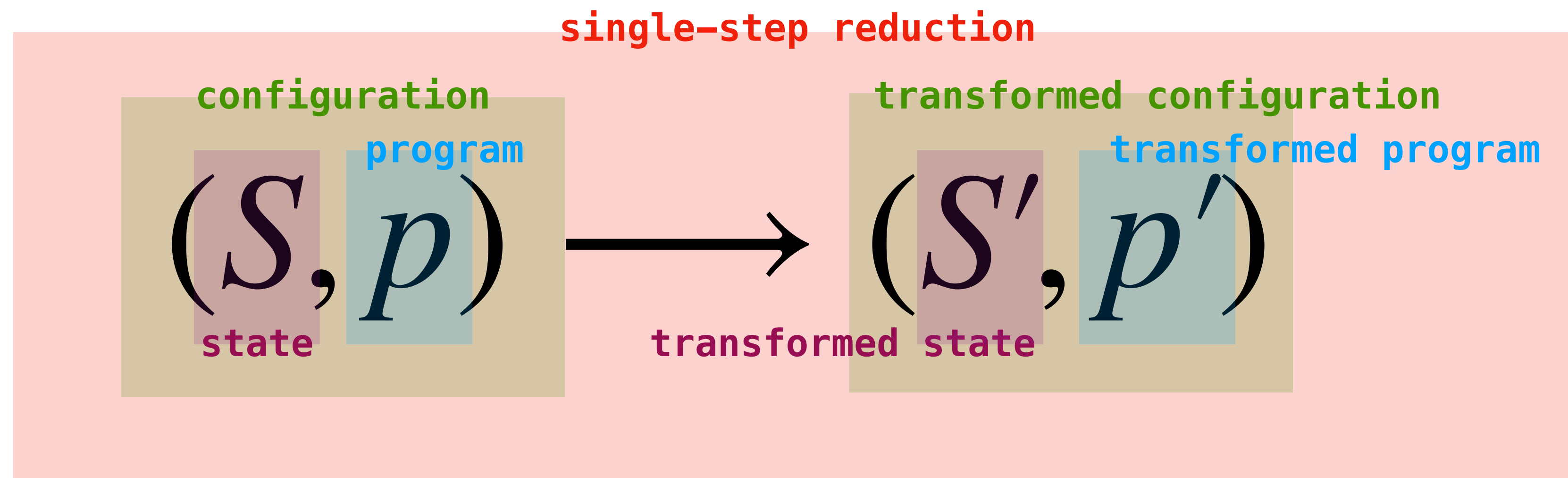


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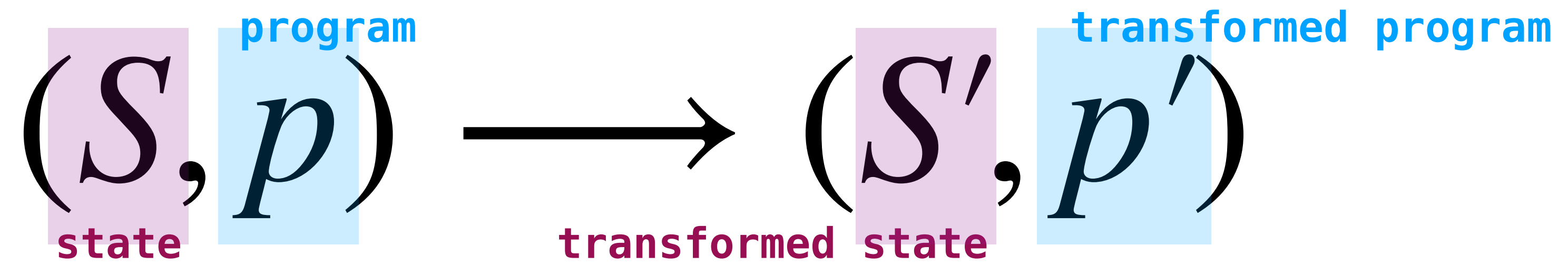


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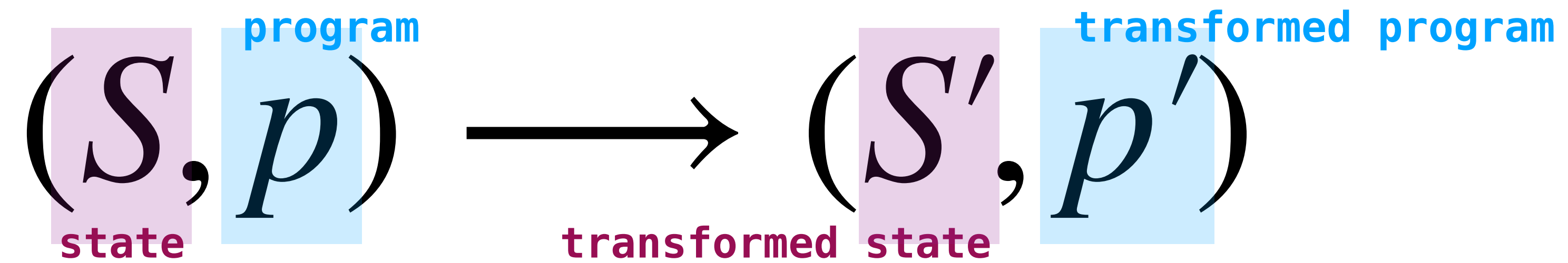
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Recall: High-Level

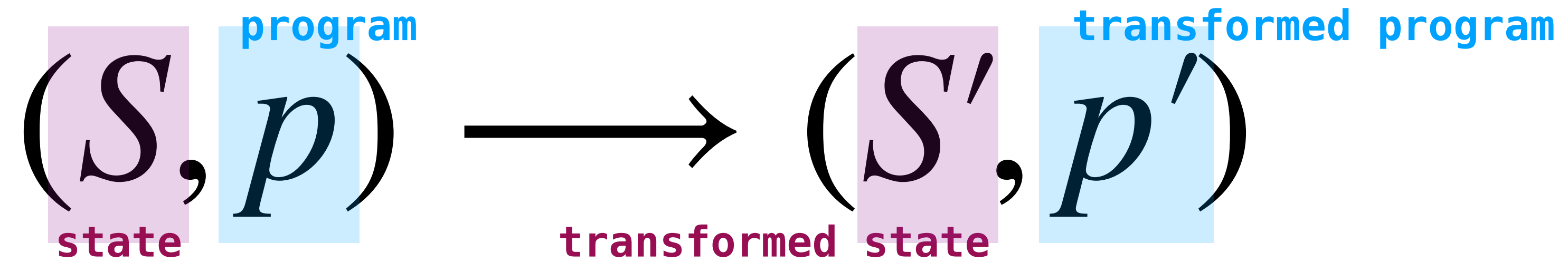


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When we define the small-step semantics of PL, we need to define **three** things:

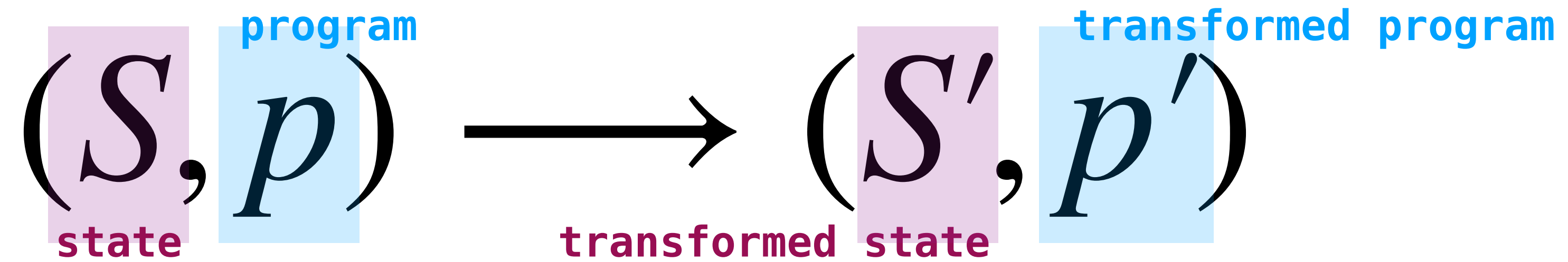
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When we define the small-step semantics of PL, we need to define **three** things:

» What kind of **state** are we manipulating?

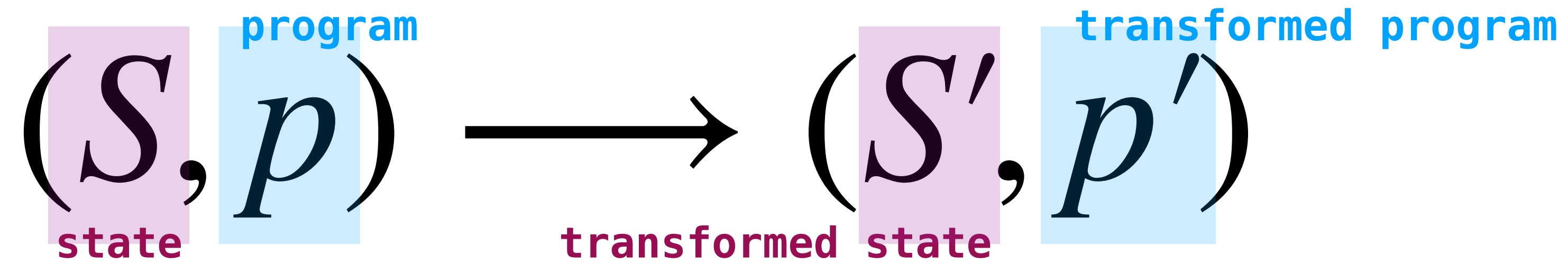
Recall: High-Level



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- » What kind of **state** are we manipulating?
- » What **rules** describe how to transform configurations?

Recall: High-Level



When we define the small-step semantics of PL, we need to define **three** things:

- » What kind of **state** are we manipulating?
- » What **rules** describe how to transform configurations?
- » What are the **values** of our PL (i.e., when are we done reducing)?

Recall: Example

```
<expr> ::= ( <op> <expr> <expr> )  
         | <bool> | <int>  
<op>    ::= add | sub | eq  
<bool>  ::= true | false  
<int>   ::= ...
```

Recall: Example

State: \emptyset

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Recall: Example

State: \emptyset

Rules:

$\langle \text{expr} \rangle$	$::=$	$(\langle \text{op} \rangle \langle \text{expr} \rangle \langle \text{expr} \rangle)$
		$ \langle \text{bool} \rangle \quad \quad \langle \text{int} \rangle$
$\langle \text{op} \rangle$	$::=$	$\text{add} \quad \quad \text{sub} \quad \quad \text{eq}$
$\langle \text{bool} \rangle$	$::=$	$\text{true} \quad \quad \text{false}$
$\langle \text{int} \rangle$	$::=$	\dots

$$\begin{array}{c} \frac{e_1 \longrightarrow e'_1}{(\text{add } e_1 \ e_2) \longrightarrow (\text{add } e'_1 \ e_2)} \text{ add-left} \qquad \frac{n \text{ is a number} \quad e_2 \longrightarrow e'_2}{(\text{add } n \ e_2) \longrightarrow (\text{add } n \ e'_2)} \text{ add-right} \\[10pt] \frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{add } n_1 \ n_2) \longrightarrow n_1 + n_2} \text{ add-ok} \\[10pt] \frac{e_1 \longrightarrow e'_1}{(\text{sub } e_1 \ e_2) \longrightarrow (\text{sub } e'_1 \ e_2)} \text{ sub-left} \qquad \frac{n \text{ is a number} \quad e_2 \longrightarrow e'_2}{(\text{sub } n \ e_2) \longrightarrow (\text{sub } n \ e'_2)} \text{ sub-right} \\[10pt] \frac{n_1 \text{ is a number} \quad n_2 \text{ is a number}}{(\text{sub } n_1 \ n_2) \longrightarrow n_1 - n_2} \text{ sub-ok} \end{array}$$

Recall: Example

<code><expr></code>	<code>::=</code>	<code>(<op> <expr> <expr>)</code>
		<code> <bool> <int></code>
<code><op></code>	<code>::=</code>	<code>add sub eq</code>
<code><bool></code>	<code>::=</code>	<code>true false</code>
<code><int></code>	<code>::=</code>	<code>...</code>

State: \emptyset

Rules:

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Values: `<int>` (i.e., numbers)

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         | <bool> | <int>  
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```

$$\frac{n \text{ is a number}}{n \Downarrow n} \text{ numEval}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{add } e_1 \ e_2) \Downarrow v_1 + v_2} \text{ addEval}$$

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v_1 \text{ is a number} \quad v_2 \text{ is a number}}{(\text{sub } e_1 \ e_2) \Downarrow v_1 - v_2} \text{ subEval}$$

We can also give a big-step semantics to this system

Practice Problem

$\langle \text{prog} \rangle ::= \{ \langle \text{stmt} \rangle ; \}$
$\langle \text{stmt} \rangle ::= \text{rot90} \mid \text{refX} \mid \text{refY}$

$$(s, \text{rot90}; P) \longrightarrow (s \text{ rotated } 90 \text{ deg. clockwise}, P)$$

$$(s, \text{refX}; P) \longrightarrow (s \text{ reflected across x-axis}, P)$$

$$(s, \text{refY}; P) \longrightarrow (s \text{ reflected across y-axis}, P)$$

What does $(\triangle, \text{rot90}; \text{refY}; \text{rot90}; \text{refX};)$ evaluate to?

Give a sequence of single step reductions (you do not need to give the full multi-step derivation)

Answer

```
<prog> ::= { <stmt> ; }  
<stmt> ::= rot90 | refX | refY  
          (Δ, rot90; refY; rot90; refX;)
```

The Lambda Calculus

High-Level View

```
(fun x -> x x) (fun x -> x x)
```

lambda term called Ω

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The **lambda calculus** is the *simplest functional programming language*. It only has:

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The **lambda calculus** is the *simplest functional programming language*. It only has:

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The **lambda calculus** is the *simplest functional programming language*. It only has:

- » variables
- » anonymous functions

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The **lambda calculus** is the *simplest functional programming language*. It only has:

- » variables
- » anonymous functions
- » function application

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```
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The **lambda calculus** is the *simplest functional programming language*. It only has:

- » variables
- » anonymous functions
- » function application

It's also **untyped**, so *anything* can be applied to *anything*

demo

(OCaml and Python)

History



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The lambda calculus was introduced by **Alonzo Church** in the 1930s



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The lambda calculus **as powerful** as every model of computation (Turing Machines, Register Machines, etc.)



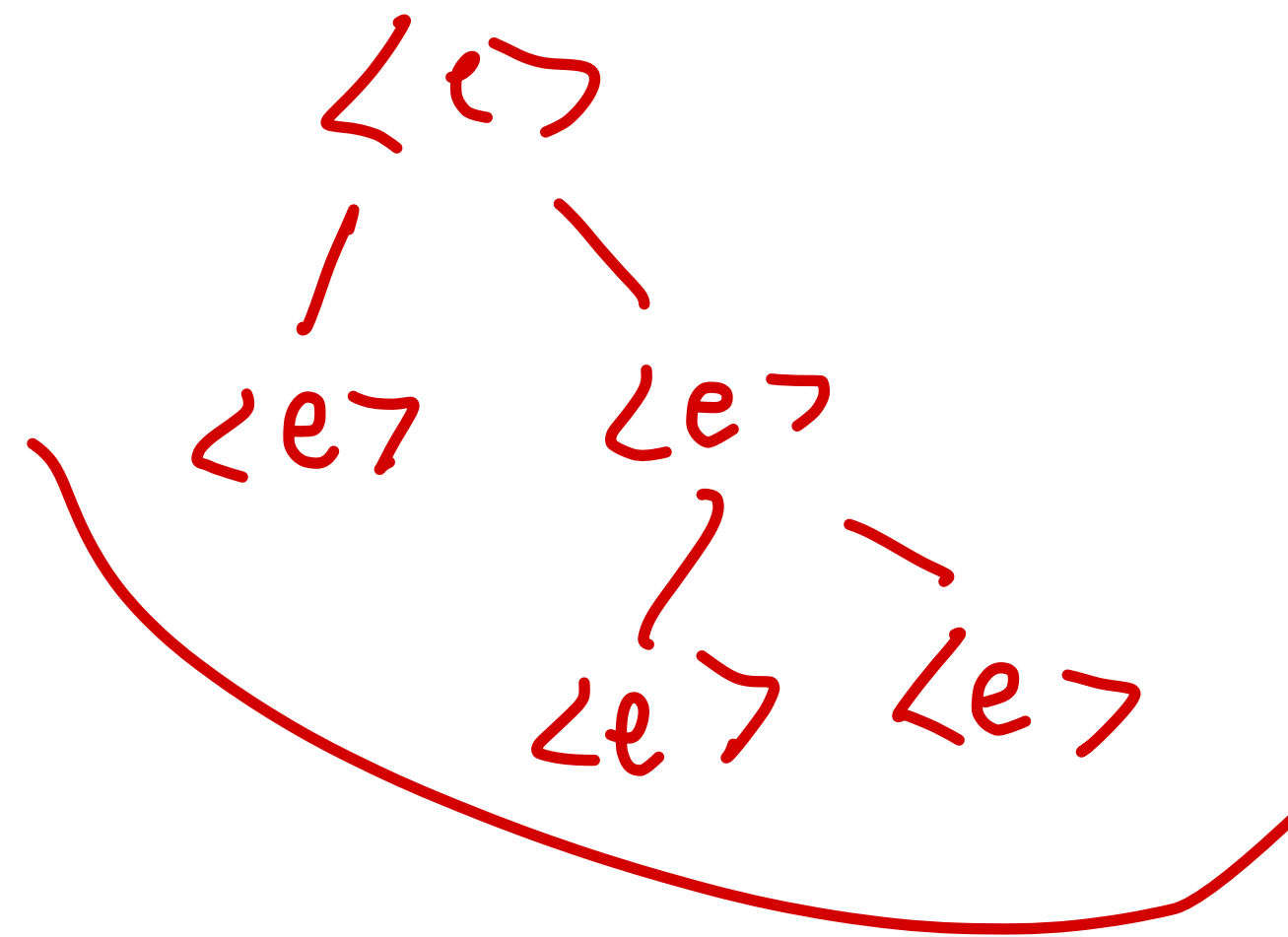
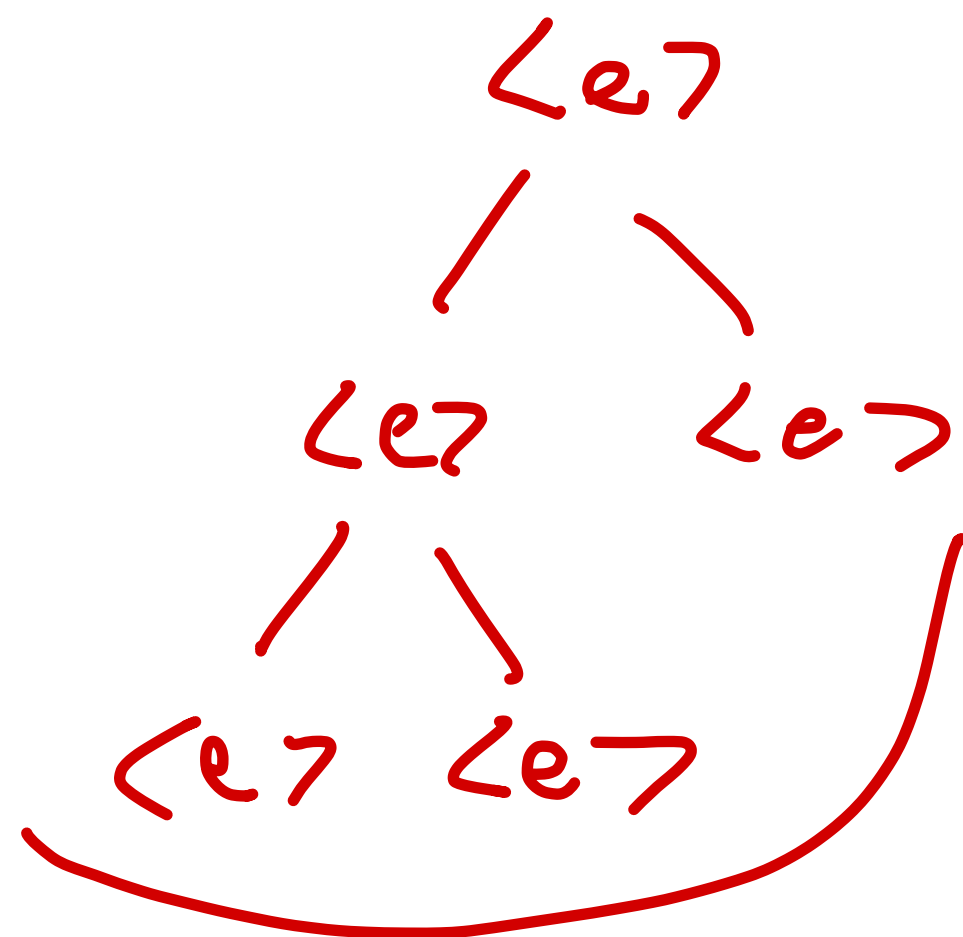
Syntax

```
<expr> ::= fun <var> -> <expr>
          | <var>
          | <expr> <expr>
          | ( <expr> )
<var>   ::= a | b | . . . | y | z
```

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This presentation is technically ambiguous (why?)



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This presentation is technically ambiguous (why?)

We assume application is **left-associative** and has higher precedence than the anonymous function syntax

Syntax (Unambiguous)

<code><expr></code>	<code>::=</code>	<code>fun</code>	<code><var></code>	<code>-></code>	<code><expr></code>
			<code> </code>	<code><expr2></code>	<code>{ <expr2> }</code>
<code><expr2></code>	<code>::=</code>	<code><var></code>	<code> </code>	<code>(</code>	<code><expr></code>
<code><var></code>	<code>::=</code>	<code>a</code>	<code> </code>	<code>b</code>	<code> ... y z</code>

In this grammar we can only use variables or functions in parentheses in applications

Syntax (Mathematical)

$\langle \text{expr} \rangle$	$::=$	$\lambda \langle \text{var} \rangle . \langle \text{expr} \rangle$
	$ $	$\langle \text{var} \rangle$
	$ $	$\langle \text{expr} \rangle \langle \text{expr} \rangle$

In mathematical settings, we use more compact syntax

Parentheses, precedence, and variables are often left implicit

Warning: Get used to λ

$$\lambda x . e \quad \equiv \quad \text{fun } x \text{ -> } e$$

We will use these syntaxes interchangeably
starting now

These are the same thing, get used to it

Values

$\langle val \rangle ::= \lambda \langle var \rangle . \langle expr \rangle$

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(We often use BNF syntax to specify our values)

Small-Step Semantics

$$(1) \frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$(2) \frac{e_2 \longrightarrow e'_2}{(\lambda x . e_1) e_2 \longrightarrow (\lambda x . e_1) e'_2}$$

$$(3) \frac{}{(\lambda x . e)(\lambda y . e') \longrightarrow [(\lambda y . e')/x]e}$$

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2. We can reduce the RHS of an application *if the LHS is already a function*

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Handwritten red annotations: The word "value" is written above the expression $(\lambda y. e')$, which is also enclosed in a red box. A red arrow points from the word "value" to the boxed expression.

1. We can reduce the LHS of an application
2. We can reduce the RHS of an application *if the LHS is already a function*
3. We can apply a function to another function by **substitution**. This is also called **β -reduction**

Example

$$(\lambda f. (\lambda x. fx))(\lambda y. y) \xrightarrow{(3)} \lambda x. (\lambda y. y) x$$

$$(1) \frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2} \quad (2) \frac{e_2 \longrightarrow e'_2}{(\lambda x. e_1) e_2 \longrightarrow (\lambda x. e_1) e'_2}$$

$$(3) \frac{}{(\lambda x. e)(\lambda y. e') \longrightarrow [(\lambda y. e')/x]e}$$

$$[\underline{\lambda y. y} / \boxed{f}] (\lambda x. fx)$$

Small-Step Semantics (Another Form)

$$(1) \frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

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1. We can reduce the LHS of an application
2. We can apply a function to *any* expression via substitution


Example

$$(1) \frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$(2) \frac{}{(\lambda x . e) e' \longrightarrow [e' / x] e}$$

$$(\lambda x . y)((\lambda z . z)(\lambda w . w)) \xrightarrow{(2)} [(\lambda z . z)(\lambda w . w) / \boxed{x}] y$$

y



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`(fun x -> x) (2 + 3) →★ 5`

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» **value:** we reach the end of our computation and the value of our program

» **stuck:** we reach an expression that cannot be reduced, but that is not a value

`(fun x -> x) (2 + 3) →★ 5`

`y (fun x -> x) ↯`

Recall: Values

When evaluating, there are **three** "end" cases:

» **value:** we reach the end of our computation and the value of our program

$(\text{fun } x \rightarrow x) (2 + 3) \rightarrow^{\star} 5$

» **stuck:** we reach an expression that cannot be reduced, but that is not a value

$y (\text{fun } x \rightarrow x) \nrightarrow$

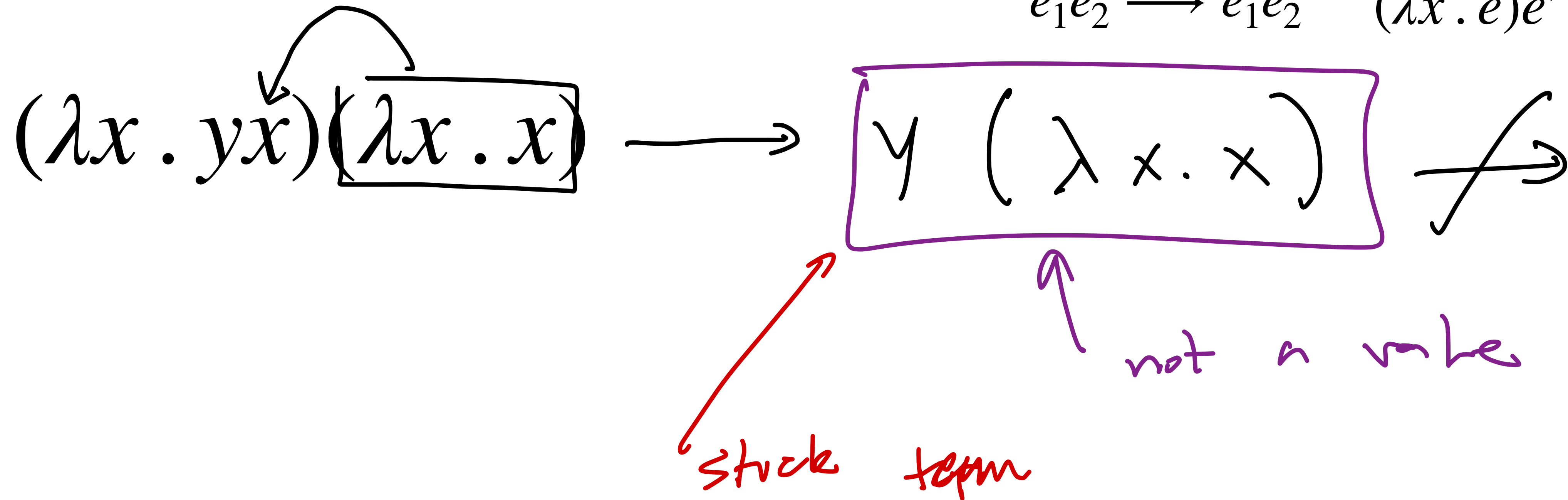
» **diverge:** the computation never reaches a point where the expression is not reducible

$(\text{fun } x \rightarrow x x) (\text{fun } x \rightarrow x x) \rightarrow^{\star}$
 $(\text{fun } x \rightarrow x x) (\text{fun } x \rightarrow x x)$

Stuck Terms

$\langle \text{val} \rangle ::= \lambda \langle \text{var} \rangle . \langle \text{expr} \rangle$

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2} \quad \frac{}{(\lambda x . e) e' \longrightarrow [e' / x] e}$$

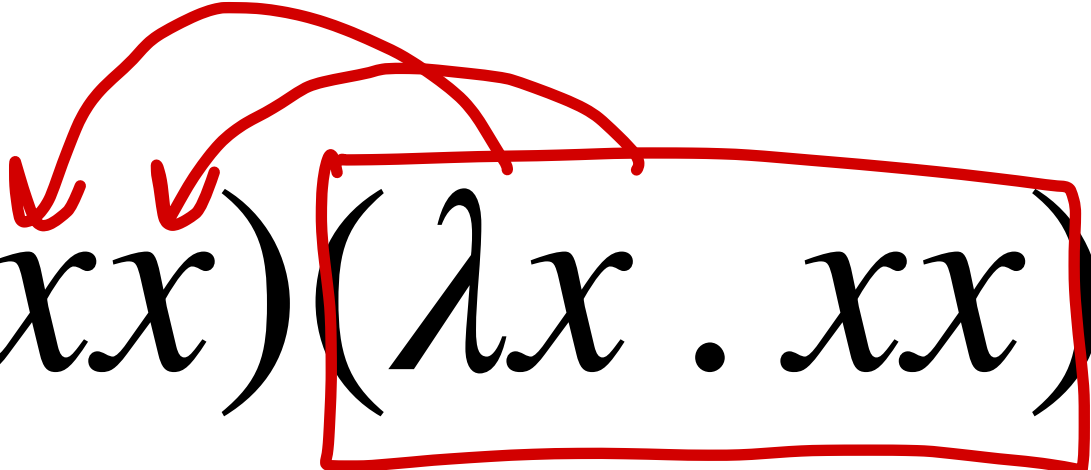


Based on our operational semantics, it's possible for the above expression to reduce to a value

Non-Termination

$$\frac{e_1 \longrightarrow e'_1}{e_1 e_2 \longrightarrow e'_1 e_2}$$

$$\frac{}{(\lambda x. e) e' \longrightarrow [e'/x]e}$$

$$(\lambda x. xx)(\lambda x. xx) \longrightarrow [\lambda x. xx / x] xx$$


$$(\lambda x. xx)(\lambda x. xx) \longrightarrow \dots$$

And unlike with arithmetic, it's now possible to define expressions which **do not terminate**. These expressions do not have values, but also don't get stuck

Big-Step Semantics

$$(1) \frac{}{\lambda x . e \Downarrow \lambda x . e}$$

$$(2) \frac{e_1 \Downarrow \lambda x . e \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

Big-Step Semantics

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$$(2) \frac{e_1 \Downarrow \lambda x . e \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

1. A function evaluates to a function value (itself)

Big-Step Semantics

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$$(2) \frac{e_1 \Downarrow \lambda x . e \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

1. A function evaluates to a function value (itself)
2. If e_1 evaluates to the function $\lambda x . e$ and e_2 evaluates to the value v_2 and e with v_2 substituted for x evaluates to v , then the application $e_1 e_2$ evaluates to v

Big-Step Semantics (Another Form)

$$\frac{}{\lambda x . e \Downarrow \lambda x . e}$$

$$\frac{e_1 \Downarrow \lambda x . e \quad [e_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}$$

These are the same rules as before except we're not required to evaluate e_2 first

Practice Problem

$$(\lambda x . \lambda y . y)((\lambda z . z)(\lambda q . q)) \Downarrow \lambda y . y$$

Give a derivation of the above judgment in both versions of the big-step semantics

$$\frac{\frac{\overline{\lambda x . e \Downarrow \lambda x . e} \quad e_1 \Downarrow \lambda x . e \quad e_2 \Downarrow v_2 \quad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}}{\lambda x . e \Downarrow \lambda x . e}$$

$$\frac{\frac{\overline{\lambda x . e \Downarrow \lambda x . e} \quad [e_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}}{e_1 \Downarrow \lambda x . e}$$

Answer

$$(\lambda x . \lambda y . y)((\lambda z . z)(\lambda q . q)) \Downarrow \lambda y . y$$

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x . e'_1 \quad e_2 \Downarrow v_2 \quad [v_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v} \text{ (CBV)}$$

$$\frac{e_1 \Downarrow \lambda x . e'_1 \quad [e_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v} \text{ (CBN)}$$

Call-by-Value vs. Call-by-Name

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The two versions of semantics we've given correspond to **call-by-value** (CBV) and **call-by-name** (CBN). These are **evaluation strategies** for functional languages

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x. e'_1 \quad e_2 \Downarrow v_2 \quad [v_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v} \text{ (CBV)}$$

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The two versions of semantics we've given correspond to **call-by-value** (CBV) and **call-by-name** (CBN). These are **evaluation strategies** for functional languages

CBV: evaluate the argument of a function *before* substituting it in the function

Call-by-Value vs. Call-by-Name

$$\frac{e_1 \Downarrow \lambda x. e'_1 \quad e_2 \Downarrow v_2 \quad [v_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v} \text{ (CBV)}$$

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The two versions of semantics we've given correspond to **call-by-value** (CBV) and **call-by-name** (CBN). These are **evaluation strategies** for functional languages

CBV: evaluate the argument of a function *before* substituting it in the function

CBN: substitute the expression *directly* into the function

Benefits of CBV

$$\frac{}{\lambda x . e \Downarrow \lambda x . e} \quad \frac{e_1 \Downarrow \lambda x . e'_1 \quad e_2 \Downarrow v_2 \quad [v_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$(\lambda x . x + x + x + x)e$$

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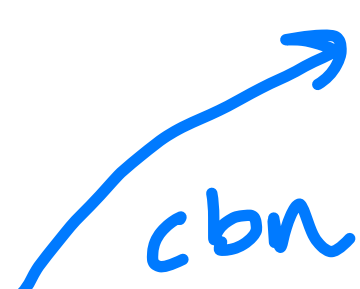
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This is good if the variable appears several times in the body of our function

This is also called **eager**, or **applicative**, or **strict** evaluation (and is what OCaml does)

Benefits of CBN

$$\frac{}{\lambda x . e \Downarrow \lambda x . e}$$

$$\frac{e_1 \Downarrow \lambda x . e'_1 \quad [e_2/x]e'_1 \Downarrow v}{e_1 e_2 \Downarrow v}$$

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Aside. It's possible to simulate CBN in CBV. Think about it for a bit, ask me after if you're interested

Side Effects

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let f x = x + x in  
let y =  
    let _ = print_int 2 in  
    2  
in f y
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Different evaluation strategies yield different side-effectful behavior!

Aside: Evaluation Strategies

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There are *a lot* more evaluation strategies, all of which optimize *something*

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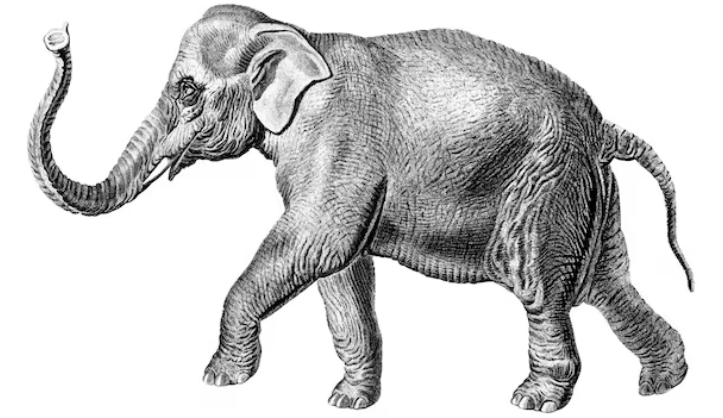
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*We will exclusively implement **call-by-value** (because, again, this is what OCaml does)*

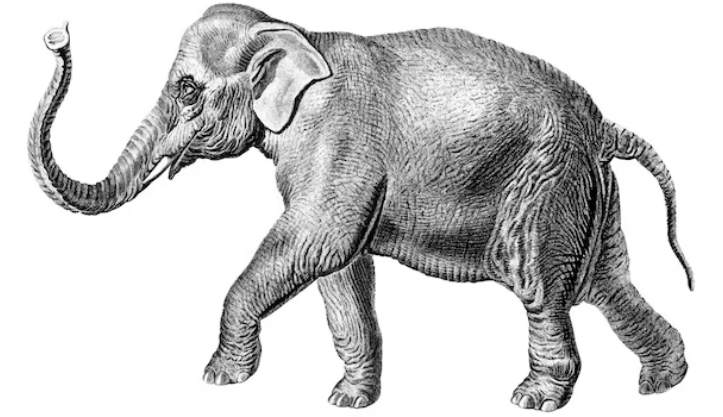
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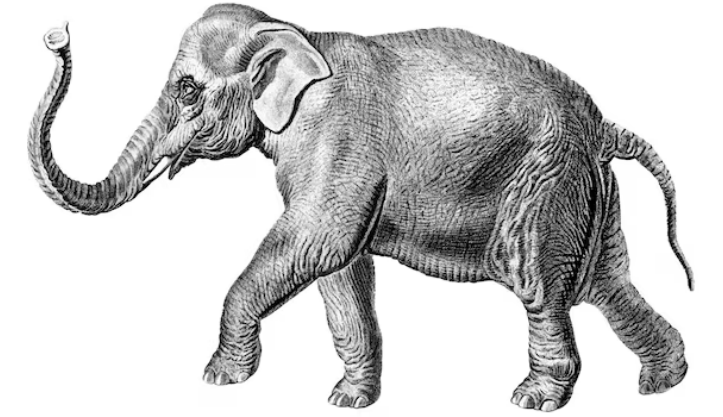
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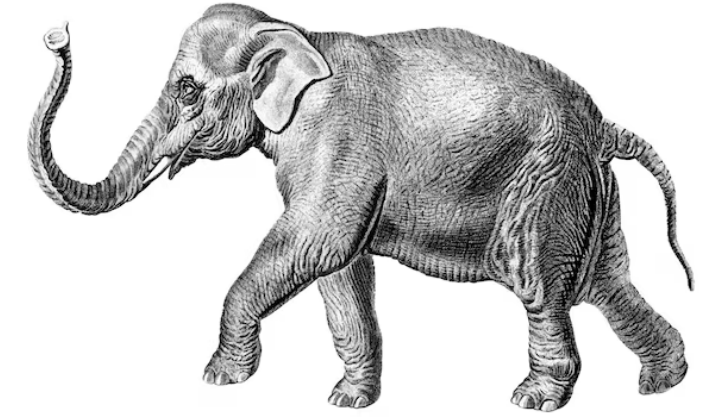


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We need to understand why...

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$$[y/x](\lambda x . y)$$

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Already things start to break down with this informal definition, e.g., consider the above substitution...

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However we define substitution shouldn't *change the underlying behavior of a function*

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However we define substitution shouldn't *change the underlying behavior of a function*

The Key Point: A function does not depend on our choice of variable names

α -Equivalence

let x = 2 in x + 1

$=_{\alpha}$

let z = 2 in z + 1

OCaml

$\lambda x . \lambda y . x =_{\alpha} \lambda v . \lambda w . v$

λ -calculus

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The **principle of name irrelevance** says that any two programs that are the same up to "renaming of variables" should behave exactly the same way (they are α -**equivalent**)

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Substitution should preserve this

Definition (First Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \text{else} \end{cases} \quad (1)$$

$$[v/y](\lambda x . e) = \lambda x . [v/y]e \quad (2)$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2) \quad (3)$$

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1. Replace every y with v , leave other variables
2. Replace y with v in the body of a function
3. Replace y with v in both subexpressions of an application

(This is an example of an *inductive definition*)

Problem Case I

$$[v/y]x = \begin{cases} v & x = y \\ x & \text{else} \end{cases}$$

$$[v/y](\lambda x . e) = \lambda x . [v/y]e$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

$$[y/x](\lambda x . x)$$

We shouldn't be allowed to substitute x if it's the argument of a function

This may *change the behavior* of a function

Definition (Second Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \text{else} \end{cases}$$

$$[v/y](\lambda x . e) = \begin{cases} \lambda x . e & x = y \\ \lambda x . [v/y]e & \text{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

We can handle the problem case directly in our definition. *Check the bound variable before we substitute in the body of a function*

Is there still a problem?

Problem Case II

$$[v/y]x = \begin{cases} v & x = y \\ x & \text{else} \end{cases}$$

$$[v/y](\lambda x . e) = \begin{cases} \lambda x . e & x = y \\ \lambda x . [v/y]e & \text{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

$$[y/x](\lambda y . x)$$

We're not replacing a bound variable, but we *are* substituting an expression that has variables which *became* bound

The variable y is said to be **captured** in this (incorrect) substitution

Free and Bound Variables

$$FV(x) = \{x\} \quad (1)$$

$$FV(\lambda x . e) = FV(e) \setminus \{x\} \quad (2)$$

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3. x is free in $e_1 e_2$ if x is free in e_1 or e_2

Definition. A variable x is **free** in e if $x \in FV(e)$ as above

Definition (Third Attempt)

$$\begin{aligned}[v/y]x &= \begin{cases} v & x = y \\ x & \text{else} \end{cases} \\ [v/y](\lambda x. e) &= \begin{cases} \lambda x. e & x = y \\ \lambda z. [v/y][z/x]e & x \in FV(v) \\ \lambda x. [v/y]e & \text{else} \end{cases} \\ [v/y](e_1 e_2) &= ([v/y]e_1)([v/y]e_2)\end{aligned}$$

Since we're interested in α -equivalence, we can first *replace* the bound variable and *substitute* it in the body of the function. This is called **α -renaming**

Is there still a problem?

Problem Case III

$$\begin{aligned}FV(x) &= \{x\} \\FV(\lambda x . e) &= FV(e) \setminus \{x\} \\FV(e_1 e_2) &= FV(e_1) \cup FV(e_2)\end{aligned}$$

$$\begin{aligned}[v/y]x &= \begin{cases} v & x = y \\ x & \text{else} \end{cases} \\[v/y](\lambda x . e) &= \begin{cases} \lambda x . e & x = y \\ \lambda z . [w/z][z/x]e & x \in FV(v) \\ \lambda x . [v/y]e & \text{else} \end{cases} \\[v/y](e_1 e_2) &= ([v/y]e_1)([v/y]e_2)\end{aligned}$$

$$[x/y](\lambda x . xyz)$$

This isn't exactly a problem, but we *have to be careful about which variable to replace the bound variable x with*

If we choose z , then we capture a *different* variable!

"Correct" Definition

$$\begin{aligned} [v/y]x &= \begin{cases} v & x = y \\ x & \text{else} \end{cases} \\ [v/y](\lambda x. e) &= \begin{cases} \lambda x. e & x = y \\ \lambda z. [v/y][z/x]e & x \in FV(v), z \text{ is fresh} \\ \lambda x. [v/y]e & \text{else} \end{cases} \\ [v/y](e_1 e_2) &= ([v/y]e_1)([v/y]e_2) \end{aligned}$$

Finally a definition, that works. Sort of...

The only problem with this definition is that it now poses an implementation issue. **How do we come up with z ?**

In mathematics, we can say it's **always possible** to come up with a variable z , but when we're implementing a programming language, we need an *actual* procedure

Well-Scopedness and Closedness

open
 $\lambda x . y$

closed
 $\lambda x . \lambda y . y$

Definition. (*informal*) An expression e is **well-scoped** if every free variable in e is "in scope" (more on that on Thursday)

Definition. An expression e is **closed** if it has no free variables

Every closed term is well-scoped

One Solution: Well-Scopedness Check

$$\begin{aligned}[v/y]x &= \begin{cases} v & x = y \\ x & \text{else} \end{cases} \\ [v/y](\lambda x. e) &= \begin{cases} \lambda x. e & x = y \\ \lambda z. [v/y][z/x]e & x \in FV(v), z \text{ is fresh} \\ \lambda x. [v/y]e & \text{else} \end{cases} \\ [v/y](e_1 e_2) &= ([v/y]e_1)([v/y]e_2) \end{aligned}$$

If we only work with closed (well-scoped) expressions, then we don't need to worry about captured variables

The condition requiring α -renaming never holds!

The Takeaway: In mini-project 1, you should check if the expression has a free variable *before* you evaluate it

Summary

The **lambda calculus** is a simple function programming language useful for learning basic functional concepts

Different **evaluation strategies** have different benefits (and may yield different programming behaviors)

Substitution is a bit tricky to define correctly (!) but any definition must preserve α -equivalence