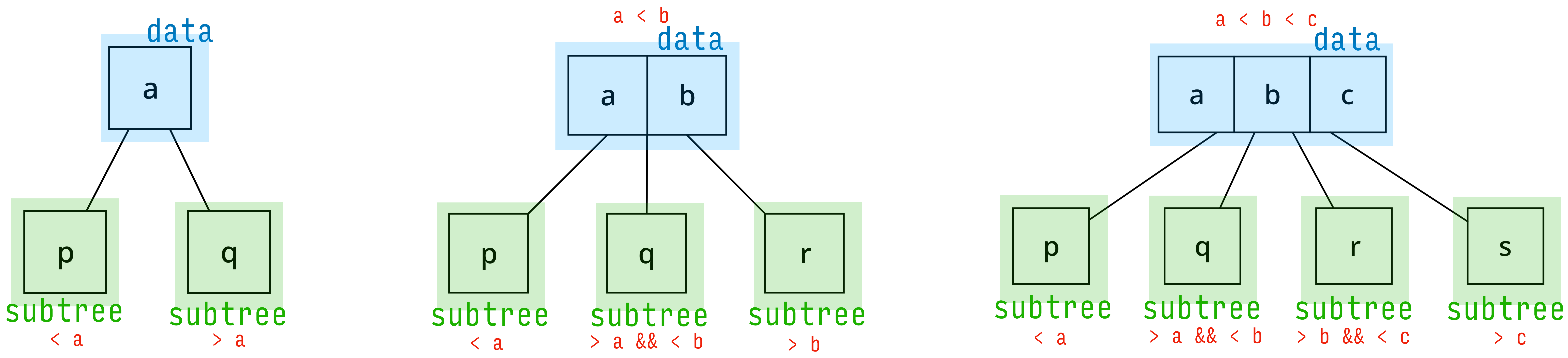
Derivations

Concepts of Programming Languages
Lecture 6

CAS CS 320

Practice Problem



A **2-3-4 tree** is a self-balancing tree structure with three possible kinds of nodes, shown above. Write an ADT to represent 2-3-4 trees

(If you have extra time, try implementing search for 2-3-4 trees)

Outline

- Discuss derivations in general
- See how to read and write derivations
- Go through a couple examples

Recap

Recall: Inference Rules

$$\dfrac{\overset{\texttt{premise}}{P_1} \qquad \overset{\texttt{premise}}{P_2} \qquad \cdots \qquad \overset{\texttt{premise}}{P_k}}{\underset{\texttt{conclusion}}{C}} \; \overset{\texttt{name}}{\texttt{my-rule}}$$

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The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **<u>judgments</u>**

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The general form of an inference rule has a collection of **premises** and a **conclusion** all of which are **judgments**

There may be no premises, this is called an **axiom**

Recall: Inference Rules

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<u>We can read this as:</u>

*If the judgments P_1 through P_k hold, then the judgment C holds (by **my-rule**)*

Typing Judgments

$$\underbrace{\Gamma}_{\text{context}} \vdash \underbrace{e}_{\text{expression}} : \underbrace{\tau}_{\text{type}}$$

A **typing judgment** a compact way of representing the statement:

e is of type τ in the context Γ

A **typing rule** is an inference rule whose premises and conclusion are typing judgments

Recall: Contexts

$$\Gamma = \{\ \texttt{x : int, y : string, z : int -> string}\ \}$$

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A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ"

Recall: Contexts

$$\Gamma = \{\ \texttt{x : int, y : string, z : int -> string}\ \}$$

A **context** is a set of **variable declarations**

A variable declaration $(x : \tau)$ says: "I declare that the variable x is of type τ"

A context keeps track of all the types of variables in the "environment"

Recall: Reading Typing Judgements

$$\{\texttt{b} : \texttt{bool}\} \vdash \texttt{if b then 2 else 3} : \texttt{int}$$

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In English: *Given that b is a bool, the expression if b then 2 else 3 is an int*

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In English: *Given that b is a bool, the expression if b then 2 else 3 is an int*

The context allows us to determine the type of an expression *relative to the types of variables*

Recall: Integer Addition Typing Rule

$$\frac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 \texttt{ + } e_2 : \texttt{int}} \text{ (addInt)}$$

If e_1 *is an* `int` *(in any context* Γ*) and* e_2 *is an* `int` *then (in any context* Γ*)* $e_1 + e_2$ *is an* `int` *(in any context* Γ*)*

Recall: Judgements are Statements

$$\{\texttt{b} : \texttt{bool}\} \vdash \texttt{if b then 2 else 3} : \texttt{string}$$

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We haven't *proved* anything by writing down a typing judgment

Today: We will talk about **typing derivations**, which are used to demonstrate that expressions *actually* have their expected types in our PL

Derivations

High Level

$$\dfrac{\dfrac{}{\{\} \vdash \mathtt{2} : \mathtt{int}}\,(\mathsf{intLit}) \qquad \dfrac{\dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\mathsf{var}) \qquad \dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\mathsf{var})}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} + \mathtt{y} : \mathtt{int}}\,(\mathsf{intAdd})}{\{\} \vdash \mathtt{let\ y\ =\ 2\ in\ y} + \mathtt{y} : \mathtt{int}}\,(\mathsf{let})$$

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Derivations allow us to *prove* that a typing judgment holds with respect to a collection of inference rules

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Formally, a **derivation** is a tree in which:

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» each node is labeled with a typing judgment

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Formally, a **derivation** is a tree in which:

» each node is labeled with a typing judgment

» and typing judgment *follows* from the typing judgments at it's children by an inference rule

Applying Rules

$$\frac{}{\Gamma \vdash \texttt{[]} : \tau\ \texttt{list}}\ \text{(nil)} \qquad \frac{\Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau\ \texttt{list}}{\Gamma \vdash e_1\ \texttt{::}\ e_2 : \tau\ \texttt{list}}\ \text{(cons)}$$

$$\frac{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int} \qquad \{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\ \text{(cons)}$$

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So far, we've used rules as ways of describing the behavior of a PL

Applying Rules

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So far, we've used rules as ways of describing the behavior of a PL

When we build typing derivations, we *instantiate* the meta-variables in the rule at *particular* expressions, contexts, etc.

Building from the Ground Up

$$\dfrac{\dfrac{\dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x} : \texttt{int}}\,(\text{var}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{1} : \texttt{int}}\,(\text{intLit})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int}}\,(\text{intAdd}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}\,(\text{nil})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\,(\text{cons})$$

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

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But we can't *just* apply rules, because it's possible that the premises of a rule **also need to be demonstrated**

This is how we get our tree structure: we apply rules from the ground up

Axioms (When are we done?)

$$
\dfrac{
  \dfrac{
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We know that we can stop building a derivation once we need to derive a premise with an **axiom**, i.e., a rule with no premises

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In our case, this will almost always be "literal" or "variable" rules

Integer Literals

(1)

$$\frac{\texttt{n is an int lit}}{\Gamma \vdash \texttt{n} : \texttt{int}}\ (\text{intLit})$$

(2)

$$\frac{\texttt{n is an int lit}}{\texttt{n} \Downarrow n}\ (\text{intLitEval})$$

1. If n is an integer literal, then it is of type int in any context

2. If n is an integer literal, then it evaluates to the number it represents

A Note about Side Conditions

we don't write "1 is an integer literal"

$$\dfrac{\dfrac{\dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x} : \texttt{int}}\,(\text{var}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{1} : \texttt{int}}\,(\text{intLit})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{x + 1} : \texttt{int}}\,(\text{intAdd}) \qquad \dfrac{}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{[]} : \texttt{int list}}\,(\text{nil})}{\{\texttt{x} : \texttt{int}\} \vdash \texttt{(x + 1) :: []} : \texttt{int list}}\,(\text{cons})$$

If a premise is a side-condition this *it is not included in the derivation*

Side conditions need to hold in order to apply the rule, but they don't appear in the derivation itself

We will try to always write side conditions in green

Float Literals

(1)
$$\frac{\texttt{n is an float lit}}{\Gamma \vdash \texttt{n} : \texttt{float}} \text{(floatLit)}$$

(2)
$$\frac{\texttt{n is an float lit}}{\texttt{n} \Downarrow n} \text{(floatLitEval)}$$

1. If n is an float literal, then it is of type float in any context

2. If n is an float literal, then it evaluates to the number it represents

Boolean Literals

(1)

$$\frac{}{\Gamma \vdash \texttt{true} : \texttt{bool}} \ (\text{trueLit})$$

(2)

$$\frac{}{\Gamma \vdash \texttt{false} : \texttt{bool}} \ (\text{falseLit})$$

(3)

$$\frac{}{\texttt{true} \Downarrow \top} \ (\text{trueLitEval})$$

(4)

$$\frac{}{\texttt{false} \Downarrow \bot} \ (\text{falseLitEval})$$

1. `true` is of type `bool` in any context
2. `false` if of type `bool` in any context
3. `true` evaluates to the value $\top$
4. `false` evaluates to the value $\bot$

Variables

$$\frac{(v : \tau) \in \Gamma}{\Gamma \vdash v : \tau} \text{(intLit)}$$

If v is declared to be of type τ in the context Γ, then v is of type τ in Γ

Variables cannot be evaluated (more on this when we talk about substitution and well-scopedness)

Back to the Example

$$\dfrac{\dfrac{}{\{\} \vdash 2 : \texttt{int}}\,(\text{intLit}) \qquad \dfrac{\dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\,(\text{var}) \qquad \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\,(\text{var})}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} + \texttt{y} : \texttt{int}}\,(\text{intAdd})}{\{\} \vdash \texttt{let y = 2 in y} + \texttt{y} : \texttt{int}}\,(\text{let})$$

We need $\{\} \vdash 2 : \texttt{int}$ in order to proof that the bottom typing judgment holds

Now we know that this follows from the **intLit** rule, which says that 2 is always an int, *by fiat*

Okay, I know that was a lot, let's take a step back

Derivations Encode Natural Language Arguments

$$\dfrac{\dfrac{}{\{\} \vdash \mathtt{2} : \mathtt{int}}\,(\mathsf{intLit}) \qquad \dfrac{\dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\mathsf{var}) \qquad \dfrac{}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} : \mathtt{int}}\,(\mathsf{var})}{\{\mathtt{y} : \mathtt{int}\} \vdash \mathtt{y} + \mathtt{y} : \mathtt{int}}\,(\mathsf{intAdd})}{\{\} \vdash \mathtt{let\ y\ =\ 2\ in\ y} + \mathtt{y} : \mathtt{int}}\,(\mathsf{let})$$

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A derivation is just a math-y way of writing a natural language prove that a typing derivation holds

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*(In fact, most mathematical arguments can be represented formally as derivation trees, this is the called **proof theory**)*

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The expression **let y = 2 in y + y** is an **int** *because*

Derivations Encode Natural Language Arguments

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» **2** is an **int** by fiat (and so **y** is being assigned to a well-typed expression)

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$$\dfrac{\texttt{n} \text{ is an integer literal}}{\Gamma \vdash \texttt{n} : \texttt{int}}\,(\text{intLit})$$

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The expression **let y = 2 in y + y** is an **int** *because*

» **2** is an **int** by fiat (and so **y** is being assigned to a well-typed expression)

» and, assuming **y** is an int, **y + y** is an **int** *because*

$$\dfrac{\Gamma \vdash e_1 : \texttt{int} \qquad \Gamma \vdash e_2 : \texttt{int}}{\Gamma \vdash e_1 + e_2 : \texttt{int}}\,(\text{addInt})$$

Derivations Encode Natural Language Arguments

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Derivations Encode Natural Language Arguments

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Derivations Encode Natural Language Arguments

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and so integer-adding these two expressions (**y** and **y**) yields an **int**

Derivations Encode Natural Language Arguments

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And all this works for
semantics judgements as well

Recall: Semantic Judgements

$$\overset{\texttt{expression}}{\boxed{e}} \Downarrow \overset{\texttt{value}}{\boxed{v}}$$

A **semantic judgment** is a compact way of representing the statement:

The expression e evaluates to the value v

A **semantic rule** is an inference rule with semantic judgments

Recall: Integer Addition Semantic Rule

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 + v_2 = v}{e_1 + e_2 \Downarrow v} \text{(evalInt)}$$

If e_1 evaluates to the (integer) v_1 and e_2 evaluates to the (integer) v_2, and $v_1 + v_2 = v$, then $e_1 + e_2$ evaluates to the (integer) v

Semantic Derivations

$$\dfrac{\dfrac{}{\texttt{true} \Downarrow \top}\,(\mathsf{trueEval}) \qquad \dfrac{}{\texttt{2} \Downarrow 2}\,(\mathsf{intEval})}{\texttt{if true then 2 else 3} \Downarrow 2}\,(\mathsf{ifEval})$$

We can also write derivations to prove semantic judgments

The principle is the same, except that the judgments are semantic judgments instead of typing judgments

Examples

Example (Typing)

$\{\} \vdash$ `if true then 2 else 5 : int`

Example (Evaluation)

`if true then 2 else 5` $\Downarrow$ `2`

Example (Typing)

$$\{\} \vdash \texttt{2 + 3 <> 4 : bool}$$

Example (Evaluation)

$\texttt{2 + 3 <> 4} \Downarrow \texttt{true}$

Example (Evaluation)

$$\texttt{let x = 2 in x + x} \Downarrow \texttt{4}$$

Summary

» Derivations are tree-like proofs that judgments hold with respect to a collection of inference rules

» Derivations are compact mathematical representations of English language arguments

» Learning to write derivations takes *practice*