Lab 11: Closures Practice

Consider the λ-calculus^{+}:

$$e ::= x \mid \lambda x.e \mid ee \mid \text{let } x = e \text{ in } e$$
$$x ::= a \mid b \mid c \mid \cdots$$

$$\frac{}{\langle \mathcal{E}, x\rangle \Downarrow \mathcal{E}(x)}\ \text{var}$$

$$\frac{}{\langle \mathcal{E}, \lambda x.e\rangle \Downarrow \langle \mathcal{E}, \lambda x.e\rangle}\ \text{fun}$$

$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow \langle \mathcal{E}', \lambda x.e\rangle \qquad \langle \mathcal{E}, e_2\rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], e\rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2\rangle \Downarrow v}\ \text{app}$$

1 Write a rule for *let* based on the typical behavior of let-expressions.

$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow v_1 \qquad \langle \mathcal{E}[x \mapsto v_1], e_2\rangle \Downarrow v_2}{\langle \mathcal{E}, \text{ let } x = e_1 \text{ in } e_2\rangle \Downarrow v_2}\ \text{let}$$

2 Show how let-expressions can be treated as sugar for the proper λ-calculus, so that the semantics of a *let* term are equivalent to that of the desugared term.

$$\text{let } x = e_1 \text{ in } e_2 \quad \longrightarrow \quad (\lambda x.e_2)\, e_1$$

$$\frac{\langle \mathcal{E}, \lambda x.e_2\rangle \Downarrow \langle \mathcal{E}, \lambda x.e_2\rangle \qquad \langle \mathcal{E}, e_1\rangle \Downarrow v_1 \qquad \langle \mathcal{E}[x \mapsto v_1], e_2\rangle \Downarrow v_2}{\langle \mathcal{E}, (\lambda x.e_2)e_1\rangle \Downarrow v_2}\ \text{app (let)}$$

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$$\frac{}{\langle \mathcal{E}, x\rangle \Downarrow \mathcal{E}(x)}\ \text{var}$$

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$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow \langle \mathcal{E}', \lambda x.\, e\rangle \qquad \langle \mathcal{E}, e_2\rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], e\rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2\rangle \Downarrow v}\ \text{app}$$

Consider an alternate *app* rule:

$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow \langle \mathcal{E}', \lambda x.\, e\rangle \qquad \langle \mathcal{E}, e_2\rangle \Downarrow v_2 \qquad \boxed{\langle \mathcal{E}[x \mapsto v_2], e\rangle \Downarrow v}}{\langle \mathcal{E}, e_1 e_2\rangle \Downarrow \langle \mathcal{E}, v\rangle}\ \text{app*}$$

1 Give a **closed** λ-calculus^{+} expression that:

- does **not** feature shadowing
- evaluates **correctly** if we use the *app* rule
- **fails** to evaluate if instead we use the *app** rule

2 Give a λ-calculus^{+} expression that:

- does **not** feature shadowing
- **fails** to evaluate if we use the *app* rule
- evaluates (**incorrectly**) if we use the *app** rule

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$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x.\, e \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}[x \mapsto v_2], e \rangle \Downarrow \text{✗}}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow \text{✗}}\ \text{app*}$$

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$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x.\, y \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], y \rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow v} \text{ app}$$

$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x.\, y \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}[x \mapsto v_2], y \rangle \Downarrow \text{X}}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow \text{X}} \text{ app*}$$

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$$\frac{\langle \emptyset, e_1\rangle \Downarrow \langle \mathcal{E}', \lambda x.\, y\rangle \qquad \langle \emptyset, e_2\rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y\rangle \Downarrow \text{X}}{\langle \emptyset, e_1 e_2\rangle \Downarrow \text{X}}\ \text{app*}$$

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$$\frac{\langle \emptyset, e_1\rangle \Downarrow \langle y \mapsto v', \lambda x.\, y\rangle \qquad \langle \emptyset, e_2\rangle \Downarrow v_2 \qquad \langle \{y \mapsto v'\}[x \mapsto v_2], y\rangle \Downarrow v}{\langle \emptyset, e_1 e_2\rangle \Downarrow v}\ \text{app}$$

$$\frac{\langle \emptyset, e_1\rangle \Downarrow \langle y \mapsto v', \lambda x.\, y\rangle \qquad \langle \emptyset, e_2\rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y\rangle \Downarrow \text{X}}{\langle \emptyset, e_1 e_2\rangle \Downarrow \text{X}}\ \text{app*}$$

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$$\frac{\langle \emptyset, e_1 \rangle \Downarrow \langle y \mapsto v', \lambda x.\, y \rangle \qquad \langle \emptyset, e_2 \rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y \rangle \Downarrow \text{✗}}{\langle \emptyset, e_1 e_2 \rangle \Downarrow \text{✗}}\ \text{app*}$$

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- does **not** feature shadowing
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$$\frac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \qquad \langle \emptyset, e_2 \rangle \Downarrow v_2 \qquad \langle \{y \mapsto v'\}[x \mapsto v_2], y \rangle \Downarrow v'}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)e_2 \rangle \Downarrow v'} \text{ app}$$

$$\frac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \qquad \langle \emptyset, e_2 \rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y \rangle \Downarrow \times}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)e_2 \rangle \Downarrow \times} \text{ app*}$$

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$$\frac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \qquad \langle \emptyset, \lambda z.z \rangle \Downarrow v_2 \qquad \langle \{y \mapsto v'\}[x \mapsto v_2], y \rangle \Downarrow v'}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)(\lambda z.z) \rangle \Downarrow v'}\ \text{app}$$

$$\frac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \qquad \langle \emptyset, \lambda z.z \rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y \rangle \Downarrow \text{X}}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)(\lambda z.z) \rangle \Downarrow \text{X}}\ \text{app*}$$

1 Give a **closed** λ-calculus⁺ expression that:

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$$\dfrac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \quad \langle \emptyset, \lambda z.z \rangle \Downarrow \langle \emptyset, \lambda z.z \rangle \quad \langle \{y \mapsto v'\}[x \mapsto \langle \emptyset, \lambda z.z \rangle], y \rangle \Downarrow v'}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)(\lambda z.z) \rangle \Downarrow v'}\ \text{app}$$

$$\dfrac{\langle \emptyset, (\lambda y.\lambda x.y)v' \rangle \Downarrow \langle y \mapsto v', \lambda x.y \rangle \quad \langle \emptyset, \lambda z.z \rangle \Downarrow \langle \emptyset, \lambda z.z \rangle \quad \langle \emptyset[x \mapsto \langle \emptyset, \lambda z.z \rangle], y \rangle \Downarrow \times}{\langle \emptyset, \big((\lambda y.\lambda x.y)v'\big)(\lambda z.z) \rangle \Downarrow \times}\ \text{app*}$$

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$$\dfrac{\begin{array}{c}\langle \emptyset, \lambda z.z\rangle \Downarrow \langle \emptyset, \lambda z.z\rangle \\ \langle \emptyset, (\lambda y.\lambda x.y)(\lambda u.u)\rangle \Downarrow \langle y \mapsto \langle \emptyset, \lambda u.u\rangle, \lambda x.y\rangle \qquad \langle \{y \mapsto \langle \emptyset, \lambda u.u\rangle\}[x \mapsto \langle \emptyset, \lambda z.z\rangle], y\rangle \Downarrow \langle \emptyset, \lambda u.u\rangle\end{array}}{\langle \emptyset, \big((\lambda y.\lambda x.y)(\lambda u.u)\big)(\lambda z.z)\rangle \Downarrow \langle \emptyset, \lambda u.u\rangle}\ \text{app}$$

$$\dfrac{\langle \emptyset, (\lambda y.\lambda x.y)(\lambda u.u)\rangle \Downarrow \langle y \mapsto \langle \emptyset, \lambda u.u\rangle, \lambda x.y\rangle \quad \langle \emptyset, \lambda z.z\rangle \Downarrow \langle \emptyset, \lambda z.z\rangle \quad \langle \emptyset[x \mapsto \langle \emptyset, \lambda z.z\rangle], y\rangle \Downarrow \text{✗}}{\langle \emptyset, \big((\lambda y.\lambda x.y)(\lambda u.u)\big)(\lambda z.z)\rangle \Downarrow \text{✗}}\ \text{app*}$$

$$\big((\lambda y.\,\lambda x.\,y)(\lambda u.\,u)\big)(\lambda z.\,z)$$
$$\langle y \mapsto \langle \emptyset, \lambda u.\,u\rangle, \lambda x.\,y\rangle(\lambda z.\,z)$$
$$\langle y \mapsto \langle \emptyset, \lambda u.\,u\rangle; x \mapsto \langle \emptyset, \lambda z.\,z\rangle, y\rangle$$
$$\langle \emptyset, \lambda u.\,u\rangle$$

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$$\dfrac{\langle \emptyset, (\lambda y.\,\lambda x.\,y)(\lambda u.\,u)\rangle \Downarrow \langle y \mapsto \langle \emptyset, \lambda u.\,u\rangle, \lambda x.\,y\rangle \quad \langle \emptyset, \lambda z.\,z\rangle \Downarrow \langle \emptyset, \lambda z.\,z\rangle \quad \langle \emptyset[x \mapsto \langle \emptyset, \lambda z.\,z\rangle], y\rangle \Downarrow \times}{\langle \emptyset, \big((\lambda y.\,\lambda x.\,y)(\lambda u.\,u)\big)(\lambda z.\,z)\rangle \Downarrow \times}\ \text{app*}$$

$$\big((\lambda y.\,\lambda x.\,y)(\lambda u.\,u)\big)(\lambda z.\,z) \longleftrightarrow$$

```
let f =
    let y = fun u → u
    in fun x → y
in f (fun z → z)
```

2 Give a λ-calculus$^+$ expression that:

- does **not** feature shadowing
- **fails** to evaluate if we use the *app* rule
- evaluates (**incorrectly**) if we use the *app** rule

$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x. e \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], e \rangle \Downarrow ✗}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow ✗}\ \text{app}$$

$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x. e \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}[x \mapsto v_2], e \rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow v}\ \text{app*}$$

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$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x. \color{red}{y} \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], \color{red}{y} \rangle \Downarrow \text{✗}}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow \text{✗}} \text{ app}$$

$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \mathcal{E}', \lambda x. \color{red}{y} \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}[x \mapsto v_2], \color{red}{y} \rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow v} \text{ app*}$$

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$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow \langle \emptyset, \lambda x.\, y \rangle \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}[x \mapsto v_2], y \rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow v}\ \text{app*}$$

2 Give a λ-calculus⁺ expression that:

- does **not** feature shadowing
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$$\frac{\langle y \mapsto v, e_1\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \qquad \langle y \mapsto v, e_2\rangle \Downarrow v_2 \qquad \langle \emptyset[x \mapsto v_2], y\rangle \Downarrow \text{✗}}{\langle y \mapsto v, e_1 e_2\rangle \Downarrow \text{✗}}\ \text{app}$$

$$\frac{\langle y \mapsto v, e_1\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \qquad \langle y \mapsto v, e_2\rangle \Downarrow v_2 \qquad \langle \{y \mapsto v\}[x \mapsto v_2], y\rangle \Downarrow v}{\langle y \mapsto v, e_1 e_2\rangle \Downarrow v}\ \text{app*}$$

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$$\frac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \quad \langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, e_2\rangle \Downarrow v_2 \quad \langle \emptyset[x \mapsto v_2], y\rangle \Downarrow \text{✗}}{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y e_2\rangle \Downarrow \text{✗}}\ \text{app}$$

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$$\dfrac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \qquad \dfrac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle \Downarrow {\color{red}\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle}}{\langle \emptyset[x \mapsto {\color{red}\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle}], y\rangle \Downarrow \text{✗}}}{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y \lambda z.\, z\rangle \Downarrow \text{✗}} \text{ app}$$

$$\dfrac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \qquad \dfrac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle \Downarrow {\color{red}\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle}}{\langle \{y \mapsto \langle \emptyset, \lambda x.\, y\rangle\}[x \mapsto {\color{red}\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle}], y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle}}{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y \lambda z.\, z\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle} \text{ app*}$$

$(\lambda y.\, y\lambda z.\, z)(\lambda x.\, y)$

$\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\lambda z.\, z\rangle$

$\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \langle \emptyset, \lambda x.\, y\rangle\lambda z.\, z\rangle$

$\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle; x \mapsto \langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle, y\rangle$

$\langle \emptyset, \lambda x.\, y\rangle$

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$$\dfrac{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle \qquad \langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle \Downarrow \langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle \qquad \langle \{y \mapsto \langle \emptyset, \lambda x.\, y\rangle\}[x \mapsto \langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, \lambda z.\, z\rangle], y\rangle \Downarrow \langle \emptyset, \lambda x.\, y\rangle}{\boxed{\langle y \mapsto \langle \emptyset, \lambda x.\, y\rangle, y\lambda z.\, z\rangle} \Downarrow \langle \emptyset, \lambda x.\, y\rangle}\ \text{app*}$$

$\boxed{(\lambda y.\, y\lambda z.\, z)(\lambda x.\, y)}$ $\leftrightarrow$

```
let f = fun y →
    y (fun z → z)
in f (fun x → y)
```