Simple Types

Concepts of Programming Languages
Lecture 18

CAS CS 320

Outline

Have a high-level discussion of **type theory** in general

Introduce and analyze the **simply-typed lambda calculus** (STLC)

Demo an **implementation** of the STLC ?

Recap

Recall: The Environment Model

$$\langle \mathscr{E}, e \rangle \Downarrow v$$

Recall: The Environment Model

$$\langle \mathcal{E}, e \rangle \Downarrow v$$

`Idea.` `We keep track of their values in an` *`environment`*

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Idea. We keep track of their values in an *environment*

And evaluate *relative* to the environment, *lazily* filling in variable values along the way

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Idea. We keep track of their values in an *environment*

And evaluate *relative* to the environment, *lazily* filling in variable values along the way

Now the **configurations** in our semantics have nonempty state

Recall: Closures

$$(\mathcal{E}, e)$$

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Definition. A **closure** is an expression together with an environment

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The environment *captures* bindings which a function needs

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<u>Definition.</u> A **closure** is an expression together with an environment

The environment *captures* bindings which a function needs

Functions need to *remember* what the environment looks like in order to behave correctly according to lexical scoping

Recall: Named Closures

$$(\text{name}, \mathscr{E}, \lambda x \,.\, e)$$

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To implement recursion, we need to be able to *name* closures

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To implement recursion, we need to be able to *name* closures

The idea. Named closures will put themselves into their environment *when they're called*

Recall: Lambda Calculus^{++} (Syntax)

```
<expr> ::= λ<var>.<expr>
         | <var>
         | <expr><expr>
         | let <var> = <expr>
           in <expr>
         | let rec <var> <var> = <expr>
           in <expr>
         | <num>
```

Recall: Lambda Calculus^{++} (Semantics)

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values and variables

$$\frac{}{\langle \mathscr{E}, \lambda x \,.\, e \rangle \Downarrow (\mathscr{E}, \lambda x \,.\, e)} \qquad \frac{}{\langle \mathscr{E}, n \rangle \Downarrow n} \qquad \frac{\mathscr{E}(x) \neq \bot}{\langle \mathscr{E}, x \rangle \Downarrow \mathscr{E}(x)}$$

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substi:

$[v_2 / x] e$

application (unnamed closure)

$$\frac{\langle \mathcal{E}, e_1 \rangle \Downarrow (\mathcal{E}', \lambda x . e) \qquad \langle \mathcal{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathcal{E}'[x \mapsto v_2], e \rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2 \rangle \Downarrow v}$$

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application (named closure)

$$\frac{\langle \mathcal{E}, e_1\rangle \Downarrow \boxed{(f, \mathcal{E}', \lambda x\,.\,e)} \qquad \langle \mathcal{E}, e_2\rangle \Downarrow v_2 \qquad \langle \mathcal{E}'\boxed{[f \mapsto (f, \mathcal{E}', \lambda x\,.\,e)]}[x \mapsto v_2], e\rangle \Downarrow v}{\langle \mathcal{E}, e_1 e_2\rangle \Downarrow v}$$

recursive

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let x = 0 in
let f = fun y -> x in
let x = 1 in
f 10

$\langle \{ x \mapsto 1, f \mapsto (\{ x \mapsto 0 \}, \lambda y . x) \} \ldots$

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$$\frac{\langle \mathscr{E}, e_1 \rangle \Downarrow (\mathscr{E}', \lambda x . e) \qquad \langle \mathscr{E}, e_2 \rangle \Downarrow v_2 \qquad \langle \mathscr{E}'[x \mapsto v_2], e \rangle \Downarrow v}{\langle \mathscr{E}, e_1 e_2 \rangle \Downarrow v}$$

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let expressions

$$\frac{\langle \mathscr{E}, e_1 \rangle \Downarrow v_1 \qquad \langle \mathscr{E}[x \mapsto v_1], e_2 \rangle \Downarrow v_2}{\langle \mathscr{E}, \text{let } x = e_1 \text{ in } e_2 \rangle \Downarrow v_2} \qquad \frac{\langle \mathscr{E}[f \mapsto (f, \mathscr{E}, \lambda x . e_1)], e_2 \rangle \Downarrow v_2}{\langle \mathscr{E}, \text{let rec } f\ x = e_1 \text{ in } e_2 \rangle \Downarrow v_2}$$

Practice Problem

```
let x = 0 in
let g = fun y -> x + 1 in
let x = 1 in
let f = fun y -> g x in
let x = 2 in
f
```

What (closure) does the following expression evaluate to? You don't need to give the derivation

Answer

```
let x = 0 in
let g = fun y -> x + 1 in
let x = 1 in
let f = fun y -> g x in
let x = 2 in
f
```

$\emptyset$

$\{x \mapsto 0\}$

$\{x \mapsto 0,\ g \mapsto (\{x \mapsto 0\},\ \lambda y.\ x+1)\}$

$\{x \mapsto 1,\ g \mapsto \overset{C_2}{(\{x \mapsto 0\},\ \lambda y.\ x+1)}\}$

C_1

$\boxed{(\{x \mapsto 1,\ g \mapsto (\{x \mapsto 0\},\ \lambda y.\ x+1)\},\ \lambda y.\ g\ x)}$

$\{x \mapsto 1,\ g \mapsto C_2,\ f \mapsto C_1\}$

$\{x \mapsto 2,\ g \mapsto C_2,\ f \mapsto C_1\}$

demo

Type Theory

What is a Type?

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let f : int -> int = ...
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Types help us delineate "well-behaved" programs

Trade-offs

$$(\lambda x\,.\,xx)(\lambda x\,.\,xx)$$

`lambda term called Ω`

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» Simplicity/Usability

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» Expressivity

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The goal is to balance:

» Simplicity/Usability
» Expressivity
» Safety/Theoretical Guarantees

OCaml

```
# let big_omega =
    let little_omega x = x x in
    little_omega little_omega;;
Error: This expression has type 'a -> 'b
       but an expression was expected of type 'a
       The type variable 'a occurs inside 'a -> 'b
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The more expressive, the more complex the the type system, designing programming languages is finding the balance that works for you

Recall: Typing Judgments

$$\Gamma \vdash e : \tau$$

This judgment reads:

e *has type* τ *in the context* Γ

We say that e is **well-typed** if $\cdot \vdash e : \tau$ for some type τ

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This judgment reads:

e has type τ in the context Γ

We say that e is **well-typed** if $\cdot \vdash e : \tau$ for some type τ

Most of what type theorists do is come up with rules for deriving typing judgments

Recall: Contexts

$$\Gamma ::= \cdot \mid \Gamma, x : \tau$$
$$x ::= \texttt{vars}$$
$$\tau ::= \texttt{types}$$

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<u>In Practice:</u> A context is a set (or ordered list, in some cases) of **variable declarations**

Recall: Contexts

empty $\emptyset$

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$$x ::= \texttt{vars}$$
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<u>In Theory:</u> A context is an inductively-defined syntactic object, just like a type or a expression

<u>In Practice:</u> A context is a set (or ordered list, in some cases) of **variable declarations**

(a variable declaration is a variable together with a type)

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$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \ldots \qquad \Gamma \vdash e_k : \tau_k}{\Gamma \vdash e : \tau}$$

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» How do we know what rules to include?

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<u>The questions we need to answer:</u>

» How do we know what rules to include?

» How do we know if we've chosen *good* rules?

Simply-Typed Lambda Calculus

STLC Syntax

type annotations

```
<e>  ::= () | <v> | <e> <e>
       | fun ( <v> : <ty> ) -> <e>
<ty> ::= unit | <ty> -> <ty>
<v>  ::= a | ... | z
```

type

STLC Syntax

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- we include a unit expression
- we have types, which annotate arguments

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The syntax is the same as that of the lambda calculus except:

» we include a unit expression

» we have types, which annotate arguments

This is the first time that **types are a part of our syntax**

Syntax

unit ty. ann.

$$e ::= \bullet \mid x \mid \lambda x^{\tau} . e \mid ee$$

$$\tau ::= \top \mid \tau \rightarrow \tau$$

unit ty. func.

$$x ::= \textit{variables}$$

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These rules enforce that a function can only be applied if we *know* that it's a function

Type Annotations?

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No, but it does change the way typing works

If we include annotations we're using **Church-style typing.** If we drop annotations, we're using **Curry-style typing**

Aside: Church vs. Curry Typing

```
fun x -> x

fun (x : unit) -> x
```

Aside: Church vs. Curry Typing

```
fun x -> x

fun (x : unit) -> x
```

: bool -> bool
: unit -> unit
: int -> int

: unit -> unit

What is the type of the first expression? How about the second?

Aside: Church vs. Curry Typing

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fun x -> x

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In **Curry-style typing**, the type of an expression is *extrinsic*, the expression is just an expression in the lambda calculus

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In **Church-style typing**, it's *intrinsic*, built into the expression and the semantics

Using Curry-style typing is not the same as having polymorphism

Uniqueness of Types

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> **Lemma.** If $\Gamma \vdash e : \tau_1$ and $\Gamma \vdash e : \tau_2$ then $\tau_1 = \tau_2$

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In the simply typed lambda calculus with Church-style typing, every expression has a *unique type*

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In the simply typed lambda calculus with Church-style typing, every expression has a *unique type*

In particular, the function `type_of` is well-defined

STLC Semantics (Review)

$$\frac{}{\langle \mathcal{E}, \lambda x^{\tau} . e \rangle \Downarrow (\mathcal{E}, \lambda x . e)}\ \texttt{fun} \qquad \frac{}{\langle \mathcal{E}, \bullet \rangle \Downarrow \bullet}\ \texttt{unit} \qquad \frac{}{\langle \mathcal{E}, x \rangle \Downarrow \mathcal{E}(x)}\ \texttt{variable}$$

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This is part of the point. Type-checking only determines *whether* we go on to evaluate the program (whether it makes sense to)

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The semantics are identical

This is part of the point. Type-checking only determines *whether* we go on to evaluate the program (whether it makes sense to)

It doesn't determine **how** we evaluate the program

Example (Church)

$$\lambda x^{\tau}.\, xx$$

What happens if we try to give a type to the above expression? What should τ be?

$\tau = \tau_1 \to \tau_2$ $\qquad$ $\tau = \tau_1$

$$\dfrac{\dfrac{\dfrac{}{x : \tau \vdash x : \tau} \qquad \dfrac{}{x : \tau \vdash x : \tau}}{x : \tau \vdash x\,x : ?}}{\emptyset \vdash \lambda x^{\tau}.\, x\,x : ?}$$

Practice Problem

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}$$

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$$\cdot \vdash \lambda f^{\top \to \top} . \lambda x^{\top} . fx : (\top \to \top) \to \top \to \top$$

Give a derivation for the above judgment

Answer

$$\cdot \vdash \lambda f^{\top \to \top}.\, \lambda x^{\top}.\, f x : (\top \to \top) \to \top \to \top$$

How do we know if we've defined a "good" programming language?

Type Safety

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> **Theorem.** If $\cdot \vdash e : \tau$ then there is a value v such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

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>
> (never get stuck)
>
> » *(progress)* either e is a value or there is an e' such that $e \longrightarrow e'$
>
> » *(preservation)* If $\cdot \vdash e : \tau$ and $e \longrightarrow e'$ then $\cdot \vdash e' : \tau$

if true then 1 else false $\rightarrow$ 1

no preservation.

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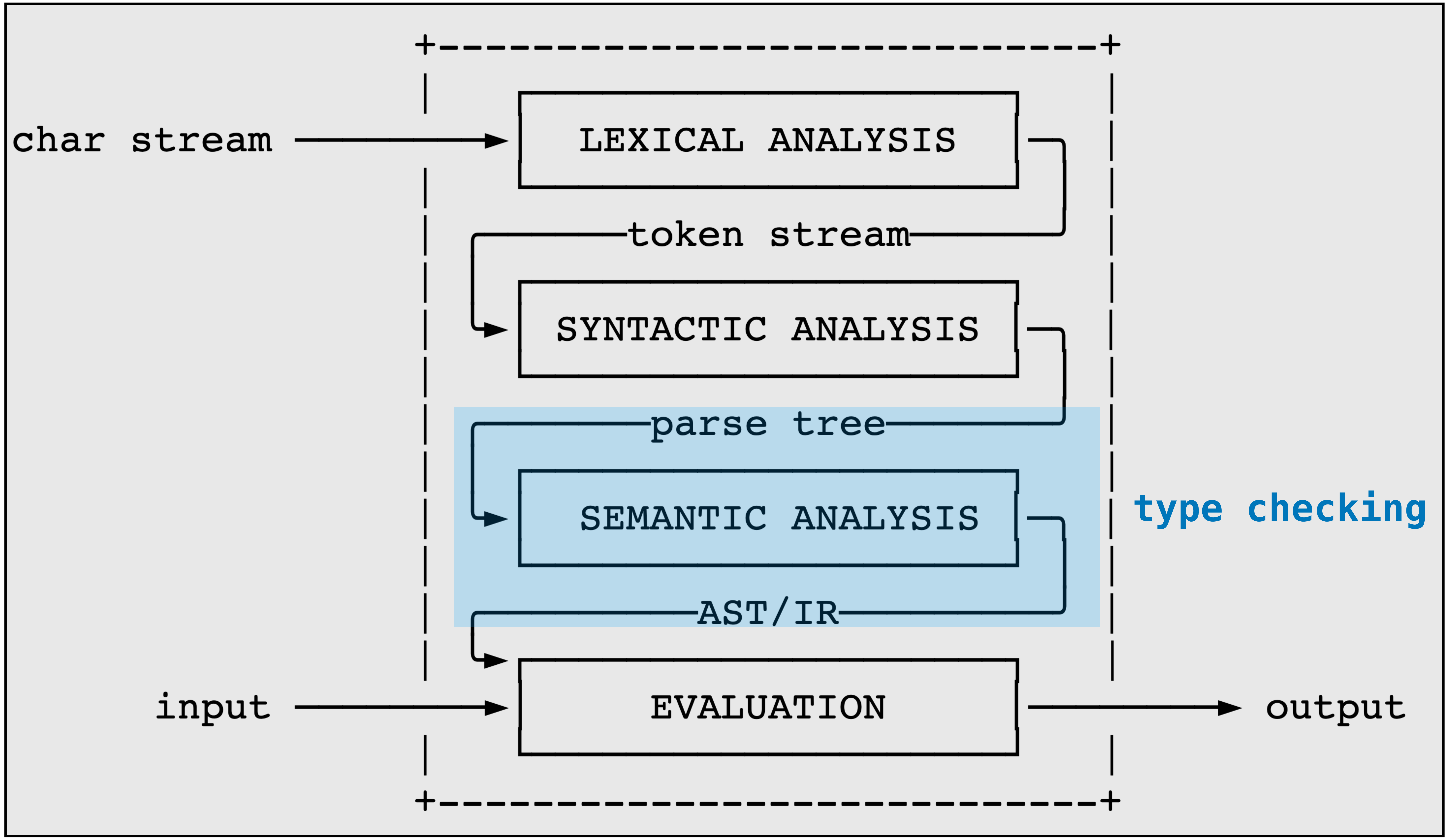
These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

Type Checking

The Picture


char stream
LEXICAL ANALYSIS
token stream
SYNTACTIC ANALYSIS
parse tree
SEMANTIC ANALYSIS
type checking
AST/IR
input
EVALUATION
output

Type Checking vs. Type Inference

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type_check : expr -> ty -> bool

type_of : expr -> ty option
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Theoretically, these two problems can be very different

For STLC, they are both easy

The One Issue

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}$$

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Our solution: We'll just use type inference demo

Summary

Type systems delineate well-behaved expressions

Type inference can sometimes be easier to implement