Formal Semantics

Concepts of Programming Languages
Lecture 14

CAS CS 320

Outline

Discuss formal semantics in general

Look at small-step and big-step semantics with some examples

demo

(finish up the Menhir demo)

Introduction

A Thought Experiment

```bash
x=3
function f () {
    x=2
}
f
echo $x
```

Bash

```python
x = 3
def f():
    x = 2
f()
print(x)
```

Python

```ocaml
let x = 3
let f () =
  let x = 2 in
  ()
let _ = f ()
let _ = print_int x
```

OCaml

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OCaml

Question. *How do we know what will happen when a program executes?*

Usually we build intuitions by writing programs and reading manuals

But many decisions about what it means to execute a program are arbitrary (or based on concerns like efficiency)

Meaning

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Syntax is interested in the *form* of a program

```
let x = 2 in
let y = 3 in
x + y
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Semantics is interested in the *meaning* of a program

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let y = 3 in
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```
let y = 3 in
2 + y
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```
2 + 3
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```
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Meaning

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What is the meaning of meaning?

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Semantics is interested in the *meaning* of a program

What is the meaning of meaning?

Formal semantics is the mathematical study of meaning

```
let x = 2 in
let y = 3 in
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↓

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Aside: Denotational vs. Operational Semantics

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$$1 + 2 * 3 + 4 = 11$$
$$1 + 12 - 2 = 11$$

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Operational semantics is interested in how a programming language "operates" i.e. how a program *behaves* during execution

$$\begin{aligned} 1 + 2 * 3 + 4 &\longrightarrow 1 + 6 + 4 \\ &\longrightarrow 7 + 4 \\ &\longrightarrow 11 \end{aligned}$$

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This course

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Small-Step vs. Big-Step Semantics

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Small-step operational semantics is interested in *program transformation*, i.e., how a program transforms "one step at a time"

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Small-Step vs. Big-Step Semantics

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Big-step operational semantics is interested in *evaluation*, i.e., what is the value of the program once a program has finished evaluating

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Small-Step vs. Big-Step Semantics

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Mini-projects

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Static vs. Dynamic Semantics

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Static semantics refers to the meaning given to a program *before* it is evaluated

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% ocaml silly.ml
File "./silly.ml", line 1, characters 8-9:
1 | let x = 2 +. 3.
            ^
Error: This expression has type int but an expression was
       expected of type
         float
  Hint: Did you mean `2.'?
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Dynamic semantics refers to the behavior of a program *during* evaluation

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utop # let x = 2 + 3;;
val x : int = 5
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Evaluation

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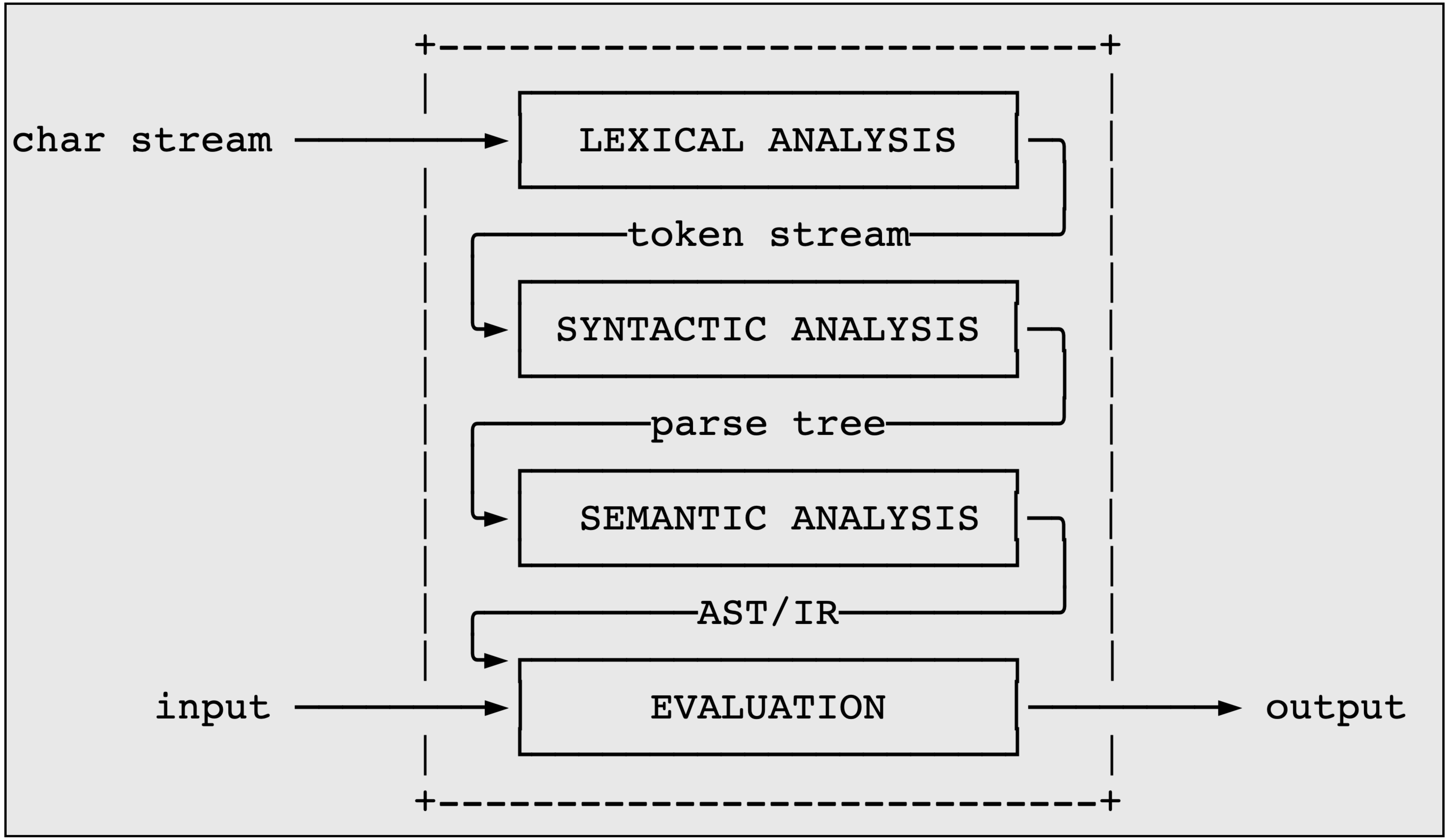
Recall: The Picture



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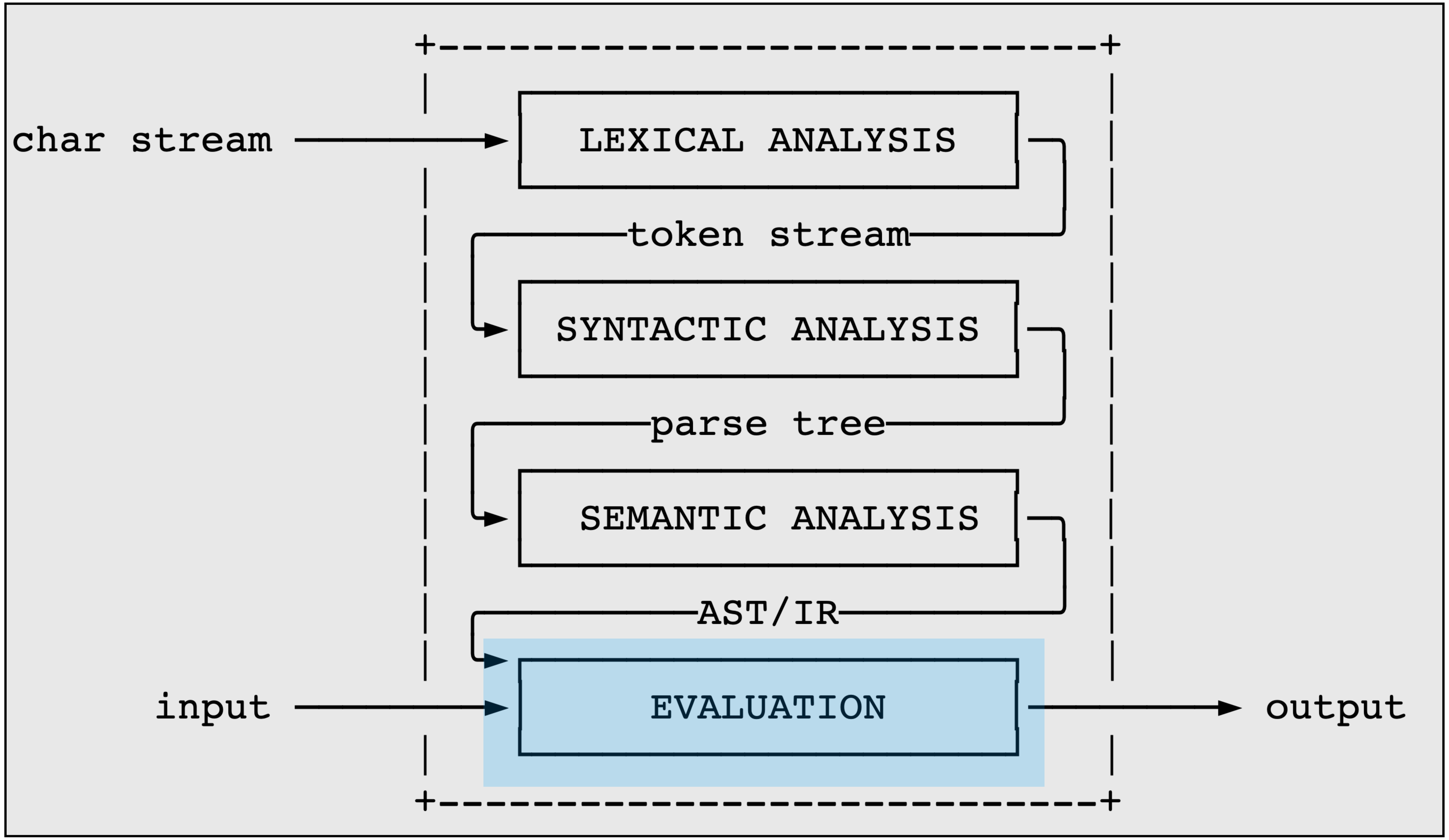

dynamic semantics (this week + next week)

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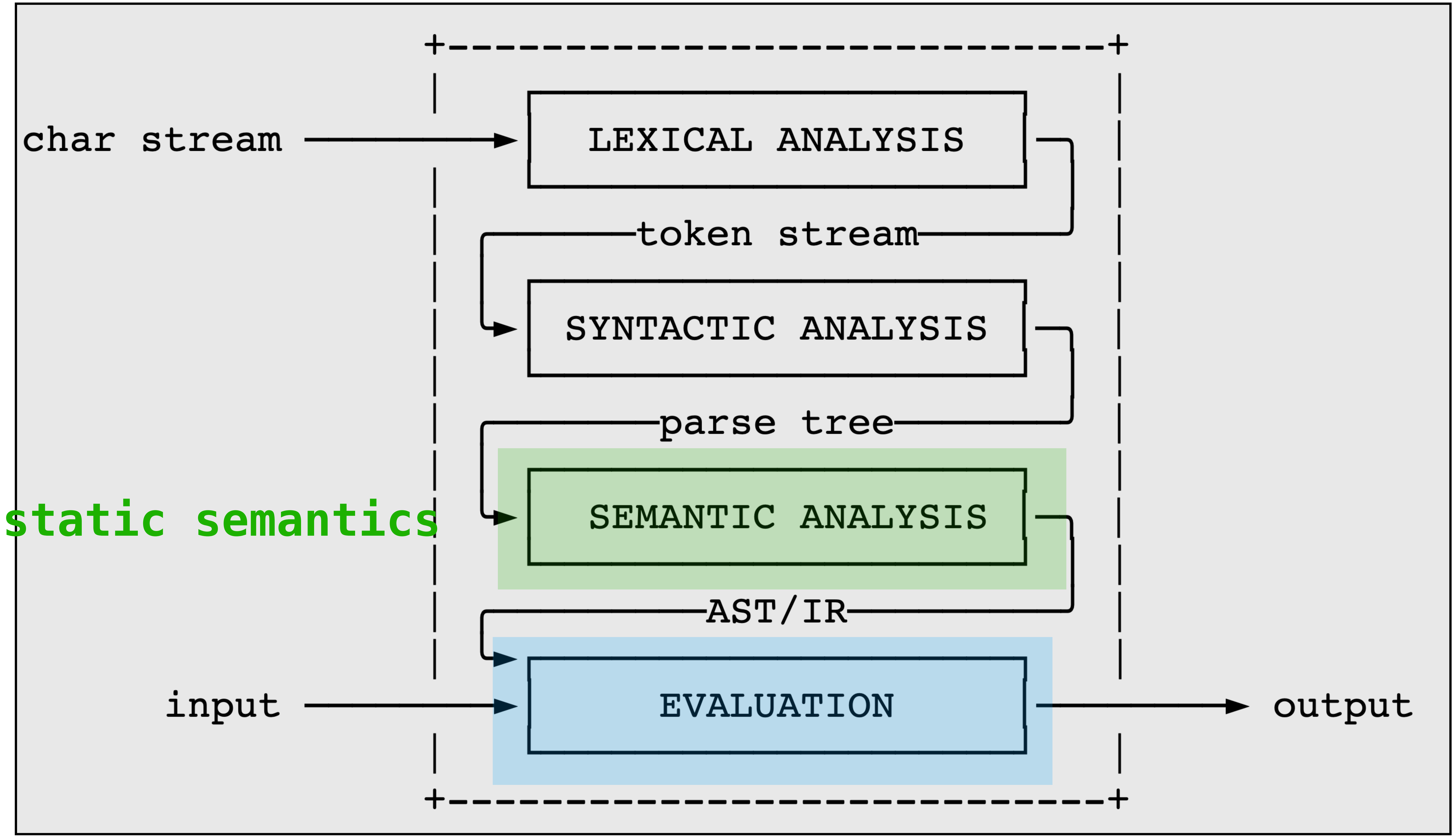

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Operational Semantics

Recall: Inference Rules

$$\dfrac{\overset{\texttt{premise}}{P_1} \qquad \overset{\texttt{premise}}{P_2} \qquad \cdots \qquad \overset{\texttt{premise}}{P_k}}{\underset{\texttt{conclusion}}{C}}\ \overset{\texttt{name}}{\texttt{red-rule}}$$

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Then general form of a reduction rule has a collection of **premises** and a **conclusion**

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Then general form of a reduction rule has a collection of **premises** and a **conclusion**

There may be no premises, this is called an **axiom**

Example

premise

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

conclusion

```
<expr> ::= ( <op> <expr> <expr> )
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```
Example Programs:
(add 2 3)
(add (add 2 3) 5)
(eq (add 2 3) (sub 7 2))
(add true 2)
```

Example

premise

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<u>Example Programs:</u>

```
(add 2 3)
(add (add 2 3) 5)
(eq (add 2 3) (sub 7 2))
(add true 2)
```

If e_1 *reduces to* e_1' *in one step, then* add e_1 e_2 *reduces to* add e_1' e_2 *in one step*

Another Example

$$\frac{n_1 \texttt{ is a number} \qquad n_2 \texttt{ is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

If n_1 *and* n_2 *are numbers then* $(\texttt{add}\ n_1\ n_2)$ *reduces in one step to the number* $n_1 + n_2$

(In this case, the premises are side-conditions)

We'll come back to these examples...

Small-Step Semantics

$$(S, p) \longrightarrow (S', p')$$

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In general, we define small-step semantics on a **configuration,** which is a program together with some stateful information

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$$(\underbrace{S}_{\text{state}}, \overbrace{p}^{\text{program}}) \longrightarrow (\underbrace{S'}_{\text{transformed state}}, \overbrace{p'}^{\text{transformed program}})$$

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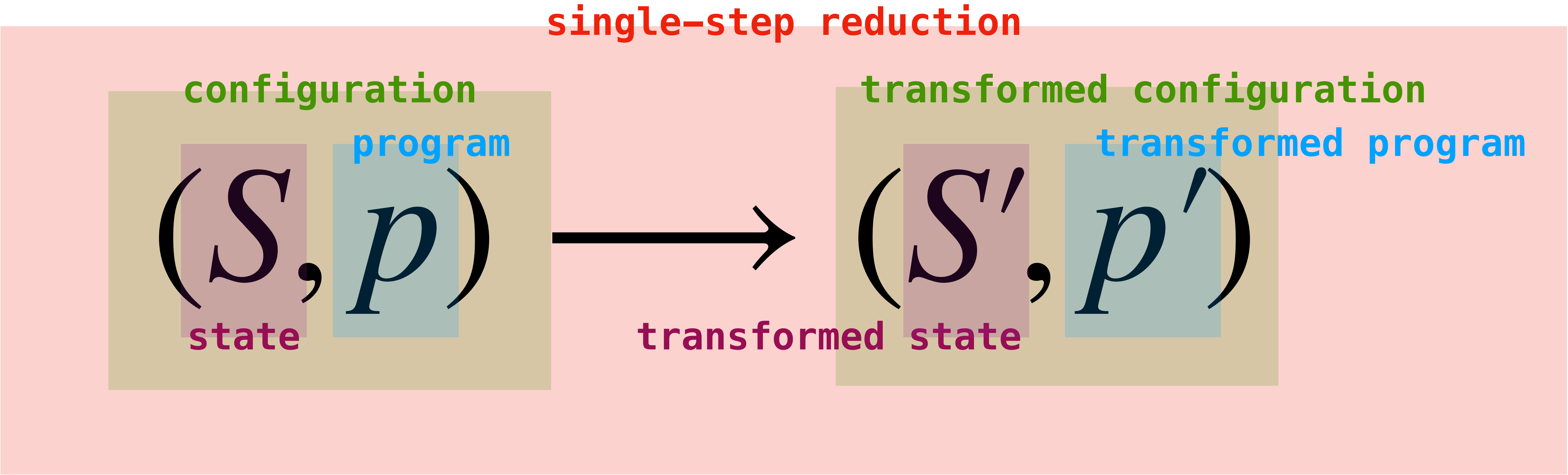

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Example: Arithmetic Expressions

$$\underbrace{(\varnothing}_{\texttt{state}}, \overbrace{10 \times (2+3)}^{\texttt{program}}) \longrightarrow (\varnothing, 10 \times 5) \longrightarrow (\varnothing, 50)$$

State: none

Program: arithmetic expression

Example: (Fragment of) OCaml

$$\underbrace{(\varnothing}_{\texttt{state}}\;,\;\overbrace{\texttt{let x = 3 in if x > 10 then 4 else 5}}^{\texttt{program}}) \longrightarrow (\varnothing\;,\;\texttt{if 3 > 10 then 4 else 5})$$

$$\longrightarrow (\varnothing\;,\;\texttt{if false then 4 else 5})$$

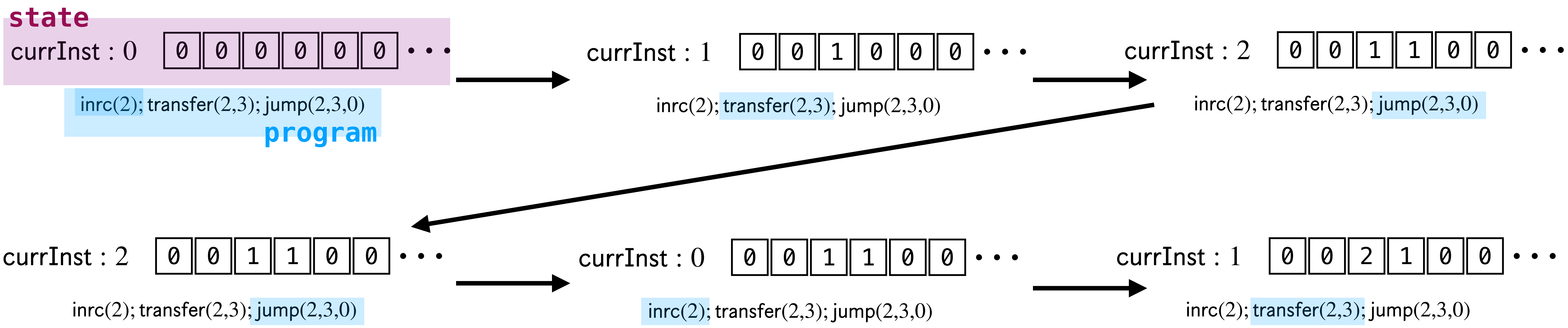
$$\longrightarrow (\varnothing\;,\;\texttt{5})$$

`State:` none

`Program:` OCaml expression

For purely functional languages
there is no state

Example: Unlimited Register Machines



<u>State:</u> (current instruction pointer) + (collection of number registers)

<u>Program:</u> sequence of commands for updating registers values and current instruction

Example: Stack-Oriented Language

state program

$(\varnothing, \texttt{push 2; push 3; add}) \longrightarrow$

$(2 :: \varnothing, \texttt{push 3; add}) \longrightarrow$

$(3 :: 2 :: \varnothing, \texttt{add}) \longrightarrow$

$(5 :: \varnothing, \epsilon)$

<u>State:</u> stack (i.e., list) of values

<u>Program:</u> sequence of commands for manipulating the stack

High-Level

program

transformed program

$$(S, p) \longrightarrow (S', p')$$

state

transformed state

High-Level

$$(\underbrace{S}_{\text{state}}, \overbrace{p}^{\text{program}}) \longrightarrow (\underbrace{S'}_{\text{transformed state}}, \overbrace{p'}^{\text{transformed program}})$$

When we define the small-step semantics of PL, we need to define two things:

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When we define the small-step semantics of PL, we need to define two things:

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When we define the small-step semantics of PL, we need to define two things:

» What kind of **state** are we manipulating?

» What **rules** describe how to transform configurations?

Example

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<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
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$$\frac{e_2 \longrightarrow e_2'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

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Reduction is a Relation

$$\begin{tikzcd}[column sep=small] & (1+2)+(2+3) \arrow[dl] \arrow[dr] & \\ (1+2)+5 \arrow[dr] & & 3+(2+3) \arrow[dl] \\ & 8 & \end{tikzcd}$$

It's important to recognize that **reduction is a *relation***

This means there may be multiple choices of reductions

When possible, we try do design our rules to avoid this

Reduction is a Relation

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There are two reductions from $(\textsf{add}\ (\textsf{add}\ 1\ 2)\ (\textsf{add}\ 2\ 3))$ in our current rule set

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There are two reductions from $(\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3))$ in our current rule set

We can avoid this by *breaking symmetry*. We will enforce that the right argument can reduced only when the left argument is completely reduced

Example: Addition

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
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<int>  ::= ...
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$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left}$$

$$\frac{v\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{add}\ v\ e_2) \longrightarrow (\texttt{add}\ v\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

Enforcing an Evaluation Order

$$\frac{\mathtt{add}\ 1\ 2 \longrightarrow 3}{(\mathtt{add}\ (\mathtt{add}\ 1\ 2)\ (\mathtt{add}\ 2\ 3)) \longrightarrow (\mathtt{add}\ 3\ (\mathtt{add}\ 2\ 3))}\ \texttt{add-left}$$

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The new rule enforces that arguments of **add** are evaluated from left to right

Practice Problem

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Write down the reduction rules for ***eq*** *(to the best of your ability) so that the left argument is evaluated before the right argument*

Answer

```
<expr> ::= ( <op> <expr> <expr> )
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- Show that $C \longrightarrow C'$
- Given C, determine a configuration C' such that $C \longrightarrow C'$ (and show that it holds)

Recall: Derivations

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We've done this!

Recall: Building Derivations

$$\mathsf{sub}\ 10\ (\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3)) \longrightarrow \mathsf{sub}\ 10\ (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))$$

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Two Questions

Once we have a small-step semantics, there are two questions we can ask (as PL designers and on the final exam):

» Show that $C \longrightarrow C'$

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Single-Step Evaluation

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The more "realistic" situation is to be given a program and then try to figure out what it evaluates to in a single step

This is why we want to be careful about how we design our rules: *we don't want to get too caught up on which rule to apply*

Example

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left} \qquad \frac{n\ \texttt{is a number} \qquad e_2 \longrightarrow e_2'}{(\texttt{add}\ n\ e_2) \longrightarrow (\texttt{add}\ n\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

$$(\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3))) \longrightarrow ???$$

$$\cfrac{\cfrac{\cfrac{}{\text{add } 1\ 2 \to 3}\ \text{add-ok}}{\text{add } (\text{add } 1\ 2)\ (\text{add } 2\ 3) \to \text{add } 3\ (\text{add } 2\ 3)}\ \text{add-left}}{\text{sub } 10\ (\text{add } (\text{add } 1\ 2)\ (\text{add } 2\ 3)) \to \text{sub } 10\ (\text{add } 3\ (\text{add } 2\ 3))}\ \text{sub-right}$$

Practice Problem

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1'\ e_2)}\ \texttt{add-left} \qquad \frac{e_2 \longrightarrow e_2'}{(\texttt{add}\ e_1\ e_2) \longrightarrow (\texttt{add}\ e_1\ e_2')}\ \texttt{add-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{add}\ n_1\ n_2) \longrightarrow n_1 + n_2}\ \texttt{add-ok}$$

$$\frac{e_1 \longrightarrow e_1'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1'\ e_2)}\ \texttt{sub-left} \qquad \frac{e_2 \longrightarrow e_2'}{(\texttt{sub}\ e_1\ e_2) \longrightarrow (\texttt{sub}\ e_1\ e_2')}\ \texttt{sub-right}$$

$$\frac{n_1\ \texttt{is a number} \qquad n_2\ \texttt{is a number}}{(\texttt{sub}\ n_1\ n_2) \longrightarrow n_1 - n_2}\ \texttt{sub-ok}$$

$$(\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3))) \longrightarrow (\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5))$$

Give a derivation of the above reduction

Answer

$$(\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ (\mathsf{add}\ 2\ 3))) \longrightarrow (\mathsf{sub}\ 10\ (\mathsf{add}\ 3\ 5))$$

Multi-Step Reduction Relation

$$\frac{}{C \longrightarrow^{\star} C}\ \texttt{refl} \qquad \frac{C \longrightarrow C' \qquad C' \longrightarrow^{\star} D}{C \longrightarrow^{\star} D}\ \texttt{trans}$$

Given any single-step reduction relation, we can derive the **multi-step reduction relation:**

» Every configuration reduces to itself **(reflexivity)**

» Every $\longrightarrow^{\star}$ reduction can be extended by a single step **(transitivity)**

Two Questions (Again)

Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

- Show that $C \longrightarrow^{\star} C'$
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How To: Derivations of Multi-Step Reductions

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2$$

want to show

» Derive all necessary single-step evaluations

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$$\texttt{sub}\ 10\ (\texttt{add}\ 3\ (\texttt{add}\ 2\ 3)) \longrightarrow \texttt{sub}\ 10\ (\texttt{add}\ 3\ 5)$$ (you did this)

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$\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5) \longrightarrow \texttt{sub}\ 10\ 8$ (exercise)

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$\texttt{sub}\ 10\ (\texttt{add}\ 3\ 5) \longrightarrow \texttt{sub}\ 10\ 8$ (exercise)

$\texttt{sub}\ 10\ 8 \longrightarrow 2$ (easy)

» Derive all necessary single-step evaluations

How To: Derivations of Multi-Step Reductions

$$\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2$$

- » Derive all necessary single-step evaluations
- » Combine them with the **transitivity rule**

How To: Derivations of Multi-Step Reductions

(we did this)

$$\dfrac{\dfrac{\vdots}{\texttt{s}\ 10\ (\texttt{a}\ (\texttt{a}\ 1\ 2)\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3))} \qquad \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow^{\star} 2}{\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^{\star} 2}\ \texttt{trans}$$

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$$\dfrac{\begin{array}{c}\texttt{(we did this)}\\ \vdots\\ \texttt{s}\ 10\ (\texttt{a}\ (\texttt{a}\ 1\ 2)\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3))\end{array} \qquad \dfrac{\begin{array}{c}\texttt{(you did this)}\\ \vdots\\ \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ 5)\end{array} \qquad \texttt{s}\ 10\ (\texttt{a}\ 3\ 5) \longrightarrow^\star 2}{\texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3)) \longrightarrow^\star 2}\,\texttt{trans}}{\texttt{sub}\ 10\ (\texttt{add}\ (\texttt{add}\ 1\ 2)\ (\texttt{add}\ 2\ 3)) \longrightarrow^\star 2}\,\texttt{trans}$$

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$$
\dfrac{
\dfrac{\texttt{(we did this)}}{\vdots}\atop \mathtt{s}\ 10\ (\mathtt{a}\ (\mathtt{a}\ 1\ 2)\ (\mathtt{a}\ 2\ 3)) \longrightarrow \mathtt{s}\ 10\ (\mathtt{a}\ 3\ (\mathtt{a}\ 2\ 3))
\qquad
\dfrac{
\dfrac{\texttt{(you did this)}}{\vdots}\atop \mathtt{s}\ 10\ (\mathtt{a}\ 3\ (\mathtt{a}\ 2\ 3)) \longrightarrow \mathtt{s}\ 10\ (\mathtt{a}\ 3\ 5)
\qquad
\dfrac{
\dfrac{\texttt{(an exercise)}}{\vdots}\atop \mathtt{s}\ 10\ (\mathtt{a}\ 3\ 5) \longrightarrow \mathtt{s}\ 10\ 8
\qquad
\mathtt{s}\ 10\ 8 \longrightarrow^{\star} 2
}{
\mathtt{s}\ 10\ (\mathtt{a}\ 3\ 5) \longrightarrow^{\star} 2
}\ \texttt{trans}
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\mathtt{s}\ 10\ (\mathtt{a}\ 3\ (\mathtt{a}\ 2\ 3)) \longrightarrow^{\star} 2
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\dfrac{\texttt{(easy)}}{\vdots \atop \texttt{s}\ 10\ 8 \longrightarrow 2}
\qquad
2 \longrightarrow^{\star} 2
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$$
\dfrac{
\begin{matrix}\texttt{(we did this)}\\ \vdots \\ \texttt{s}\ 10\ (\texttt{a}\ (\texttt{a}\ 1\ 2)\ (\texttt{a}\ 2\ 3)) \longrightarrow \texttt{s}\ 10\ (\texttt{a}\ 3\ (\texttt{a}\ 2\ 3))\end{matrix}
\qquad
\dfrac{
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Two Questions (Again)

Once we have an operational semantics, there are two questions we can ask (as PL designers and on the final exam):

» Show that $C \longrightarrow^\star C'$

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If our rules are well defined, then should be easy:

Solve this single-step evaluation problem until you reach a configuration that cannot be further reduced

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When are we done?

$e_1 \to e_2 \to e_3 \; \dots \; \boxed{e_k}$

value

When evaluating, there are **three** "end" cases:

a chosen set of well-formed exprs.

» **value:** we reach the end of our computation and the value of our program

» **stuck:** we reach an expression that cannot be reduced, but that is not a value

"two" && true

» **diverge:** the computation never reaches a point where the expression is not reducible

let rec f x = f x in f 0

$\Downarrow$

f 0 $\Rightarrow$ f 0 $\Rightarrow$ f 0 ...

moving onto big-step...

Big-Step Semantics

$$(\mathsf{sub}\ 10\ (\mathsf{add}\ (\mathsf{add}\ 1\ 2)\ (\mathsf{add}\ 2\ 3))) \Downarrow 2$$

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Notation: We write $e \Downarrow v$ to mean that e evaluates to the value v

Big-Step Semantics

$$\texttt{(sub 10 (add (add 1 2) (add 2 3)))} \Downarrow 2$$

Big-step semantics deals only with a program and its value

Notation: We write $e \Downarrow v$ to mean that e evaluates to the value v

This is what we've been doing in this course so far

Example

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

$$\frac{n \texttt{ is a number}}{n \Downarrow n}\ \texttt{numEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{add}\ e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\texttt{sub}\ e_1\ e_2) \Downarrow v_1 - v_2}\ \texttt{subEval}$$

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we'll remove these side conditions once we have type-checking

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Practice Problem

Write the rule for ***eq***

Answer

```
<expr> ::= ( <op> <expr> <expr> )
         | <bool> | <int>
<op>   ::= add | sub | eq
<bool> ::= true | false
<int>  ::= ...
```

Relation to Small-Step

$$e \longrightarrow^{\star} v \qquad \approx \qquad e \Downarrow v$$

The big-step relation **"cuts out the middle steps"** of a small-step relation

This means fewer and clearer rules, but less fine-grain control of the evaluation sequence

Note: We can't always have both small-step and big-step!

Order of Evaluation

order of evaluation

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v_1 \texttt{ is a number} \qquad v_2 \texttt{ is a number}}{(\mathsf{add}\ e_1\ e_2) \Downarrow v_1 + v_2}\ \texttt{addEval}$$

With small-step semantics, we can choose the order of evaluations based on the rules

With big-step semantics, we can't because our relation only deals with the *final* value, nothing intermediate

We will take the order of operations to be from left to right

Taking Stock

big-step

$$e \Downarrow v$$

e evaluates to v

single-step

$$e \longrightarrow e'$$

e reduces to e' in a single step

multi-step

$$e \longrightarrow^{\star} e'$$

e reduces to e' in many steps