Introduction to Category Theory

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Abstract

We introduce the basic notions of category theory at an undergraduate level. We cover categories, functors, natural transformations, universals, and Yoneda.

Contents

1 Introduction

Category theory arose in the 1940s due to Mac Lane and Eilenberg, who founded the subject to give a precise meaning to the notion of naturality. In the 80 years it has existed, category theory has found its way into nearly every branch of math, and is foundational to understanding modern research in numerous fields. In these notes, we will introduce the basic notions of the theory.

We will introduce the subject at a level comprehensible to the average undergraduate student. The main prerequisite to this talk is a knowledge of sets and functions.

2 Universality of product

The Cartesian product $X \times Y$ of sets X, Y has an extremely interesting property that motivates quite a bit of category theory. There are two very interesting functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, called the *natural projections* onto X and Y, respectively. They are defined by

$$\pi_X(x, y) = x, \qquad \pi_Y(x, y) = y.$$

These functions are interesting for the following reason. Let Z be any other set, and let $f : Z \to X$ and $g : Z \to Y$ be *any* functions. We write this information as a diagram

$$\begin{tikzcd}
& X \\
Z \arrow[ur, "f"] \arrow[dr, "g"'] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\
& Y
\end{tikzcd}$$

where arrows represent functions, and "vertices" represent sets. The interesting fact is that there is a *unique function* $\varphi : Z \to X \times Y$, as indicated by the dashed arrow in the diagram

$$\begin{tikzcd}
& X \\
Z \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, dashed, "\varphi"] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\
& Y
\end{tikzcd}$$

rendering the diagram *commutative*.

There is a little to unpack here. First, a diagram *commutes* when all paths between any two vertices are the same. For a simpler example, consider the diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

This diagram commutes when all paths between two sets are the same. The only pair of sets with more than 2 paths between them is A and D. We have that $g \circ h : A \to D$ and $k \circ f : A \to D$. Thus, the diagram commutes when $g \circ h = k \circ f$.

Going back to the diagram with the Cartesian product, we see that the diagram commutes if we have the equalities

$$\pi_X \circ \varphi = f, \qquad \pi_Y \circ \varphi = g.$$

The claim is that there is a *unique* morphism φ satisfying this property. This is easily proven.

Let $z \in Z$, and write $(x, y) = \varphi(z)$. Commutativity of the diagram gives that

$$f(z) = (\pi_X \circ \varphi)(z) = \pi_X(x, y) = x.$$

We thus have a definition for x. Similarly,

$$g(z) = (\pi_Y \circ \varphi)(z) = \pi_Y(x, y) = y.$$

Thus, we are forced to define $\varphi(z) = (f(z), g(z))$. The fact that this definition is *forced* means that the morphism is unique.

The reason we have done all of this is because

3 Categories

do i even bother with universality of product its like 2 pages of crap

update: yes, but motivate it with basically "the main philosophy of cat theory is viewing constructions via morphisms." then construct product with arrows, introduce categories, isomorphisms, how isomorphisms have basically the same stuffs, and moveon

4 Functors and representable functors

5 (Co)limits

6 Natural transformations and Yoneda