

## Lecture 3: Random Variable, Part I

Yi, Yung (이웅)

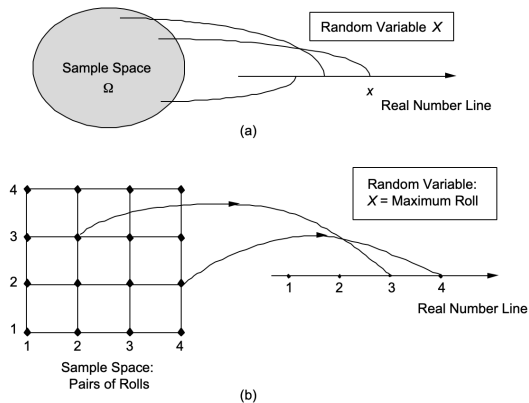
EE210: Probability and Introductory Random Processes  
KAIST EE

May 5, 2021

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables

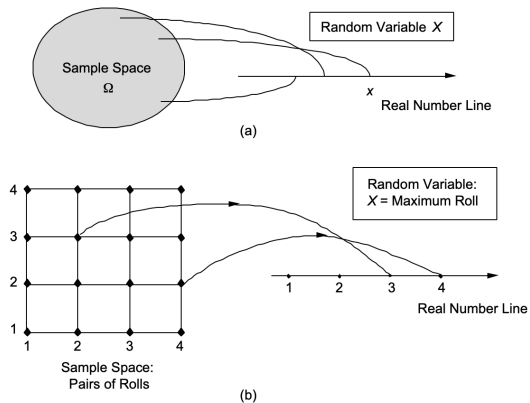
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- In reality, many outcomes are , e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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- For a fixed value  $x$ , we can associate an **event** that a random variable  $X$  has the value  $x$ , i.e.,  $\{\omega \in \Omega \mid X(\omega) = x\}$
- Assume that values  $x$  are discrete<sup>1</sup> such as  $1, 2, 3, \dots$ .  
For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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- Rolls a dice,  $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable  $X = 1$  for even numbers and  $X = 0$  for odd numbers
- Event  $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$ , but simply  $A_1 = \{X = 1\}$
- Event  $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$ , but simply  $A_0 = \{X = 0\}$
- Remember that the random variable  $X$  is a **function** from  $\Omega$  to  $\mathbb{R}$



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- Only **binary** values

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<sup>1</sup>w.p.: with probability

- Only **binary** values

$$X = \begin{cases} 0, & \text{w.p. } 1 - p, \\ 1, & \text{w.p. } p \end{cases}$$

In other words,  $p_X(0) = 1 - p$  and  $p_X(1) = p$  from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an **indicator rv** of an event  $A$ . Define a rv  $\mathbf{1}_A$  as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

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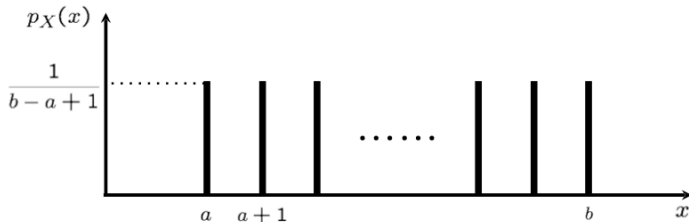
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- Choose a number out of  $\Omega = \{a, a + 1, \dots, b\}$  uniformly at random.

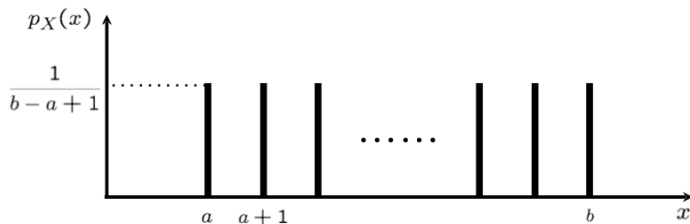


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- Models complete **ignorance** (I don't know anything about  $X$ )

---

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L3(2)

- Models the number of **successes** in a given number of **independent** trials

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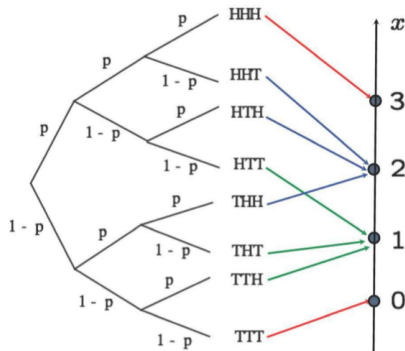
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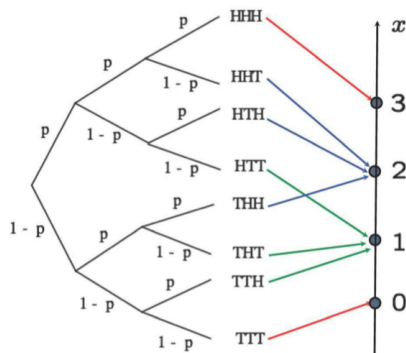
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$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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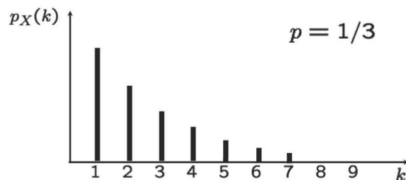
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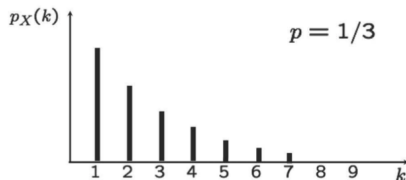
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- Models **waiting** times until something happens.



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- Average

### Definition

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- $p_X(x)$ : relative frequency of value  $x$  (trials with  $x$ /total trials)
- **Example.** Bernoulli rv with  $p$

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p = p_X(1)$$

Not very surprising. Easy to prove using the definition.

- If  $X \geq 0$ ,  $\mathbb{E}[X] \geq 0$ .
- If  $a \leq X \leq b$ ,  $a \leq \mathbb{E}[X] \leq b$ .
- For a constant  $c$ ,  $\mathbb{E}[c] = c$ .

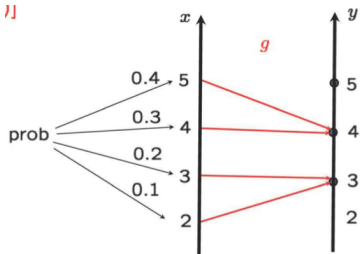


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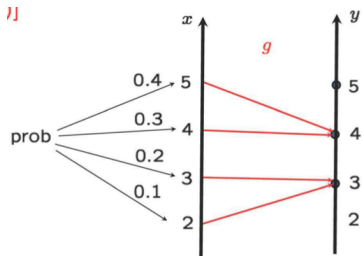
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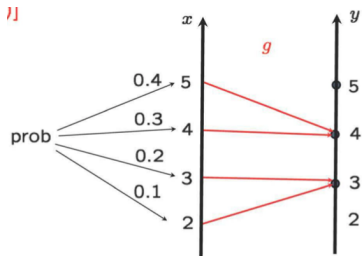


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## Linearity of Expectation

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

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### Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $Y = X + b, \text{var}[Y] = \text{var}[X]$
- $Y = aX, \text{var}[Y] = a^2\text{var}[X]$

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\ &= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2\end{aligned}$$

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Example: Variance of a Bernoulli rv ( $p$ )

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- $Y = X + b, \text{var}[Y] = \text{var}[X]$   
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Example: Variance of a Bernoulli rv ( $p$ )

$$\begin{aligned}\mu &= \mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p \\ \mathbb{E}[X^2] &= 1 \times p + 0 \times (1 - p) = p \\ \text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\ &= p(1 - p)\end{aligned}$$

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- For two random variables  $X, Y$ , consider two events  $\{X = x\}$  and  $\{Y = y\}$ , and

$$\mathbb{P}(\{X = x\} \cap \{Y = y\})$$

- **Joint PMF.** For two random variables  $X, Y$ , consider two events  $\{X = x\}$  and  $\{Y = y\}$ , and

$$p_{X,Y}(x,y) \triangleq \mathbb{P}(\{X = x\} \cap \{Y = y\})$$

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- Marginal PMF.**

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

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Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	0
3	2/20	4/20	1/20	2/20
2	0	1/20	3/20	1/20
1	0	1/20	0	0

$$p_{X,Y}(1,3) =$$

$$p_X(4) =$$

$$\mathbb{P}(X = Y) =$$

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1	0	1/20	0	0

$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv  $Z = g(X, Y)$ . (Ex)  $X + Y, X^2 + Y^2$ . Then, PMF of  $Z$  is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$



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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

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- $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$
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**Message.** When some rv  $X$  is written as a linear combination of other rvs,  $X$  becomes easy to handle.

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) **Conditioning for random variables**
- (6) Independence for random variables



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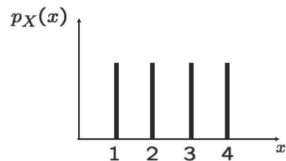
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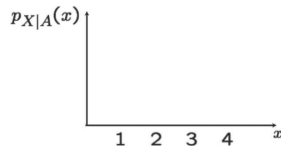
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  - (Note)  $p_{X|A}(x)$ ,  $\mathbb{E}[X|A]$ ,  $\mathbb{E}[g(X)|A]$ , and  $\text{var}[X|A]$  are all just notations!

$$A = \{X \geq 2\}$$



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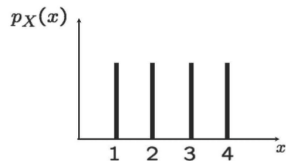
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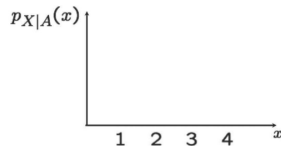
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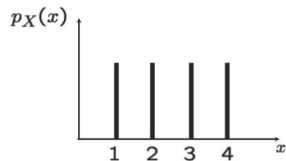


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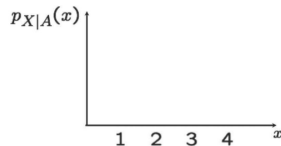


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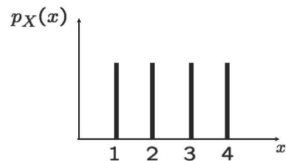
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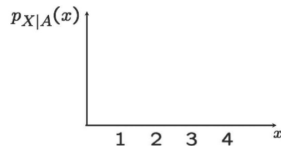
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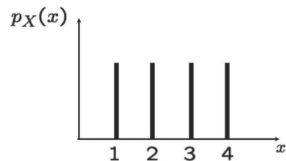
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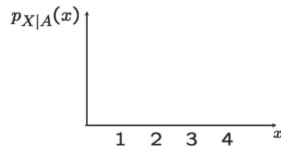
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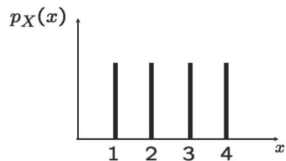
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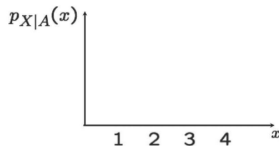
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- Conditional PMF

- Multiplication rule

$$p_{X,Y}(x,y) =$$

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VIDEO PAUSE

4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
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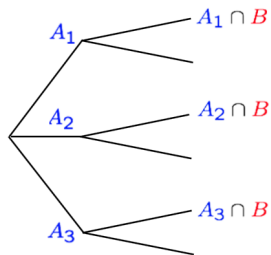
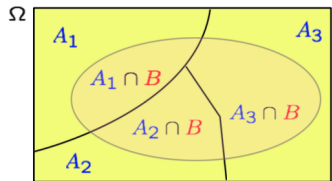
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$$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$$

- Partition of  $\Omega$  into  $A_1, A_2, A_3$
- Known:  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$
- What is  $\mathbb{P}(B)$ ?

## Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

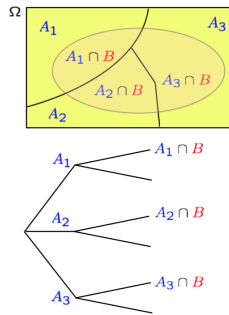




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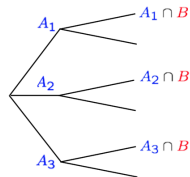
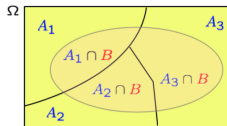
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$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

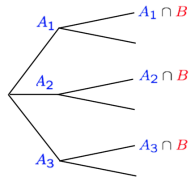
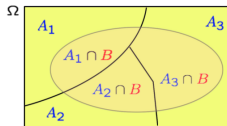
- Weighted average of expectations from  $A_i$ 's perspective



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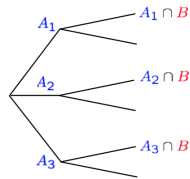
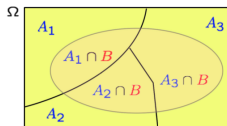
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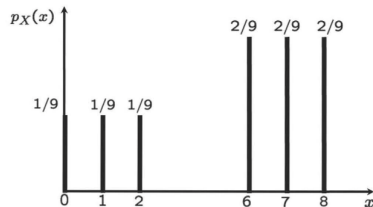
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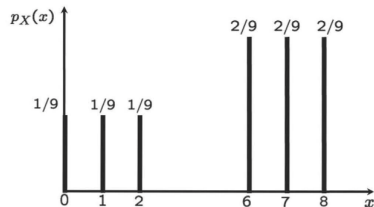
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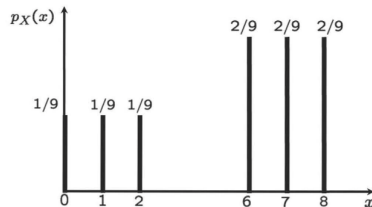
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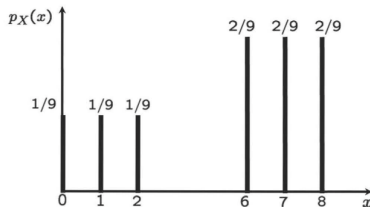
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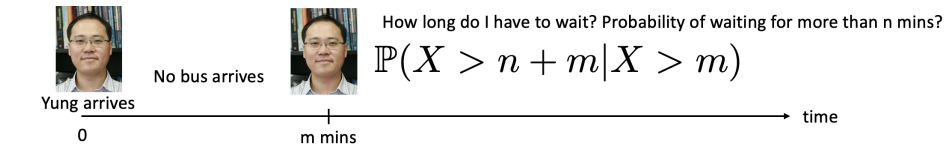
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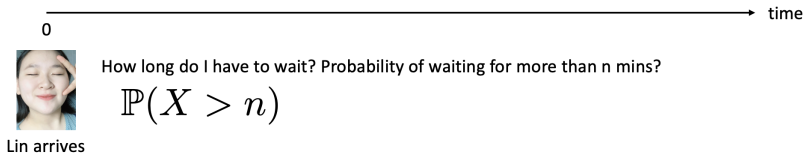
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- Total expectation theorem and a notion of **memorylessness** helps a lot.



A bus left at time 0



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=

(from TET)

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- Thus,  $\mathbb{E}[X] = \frac{1}{p}$

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables



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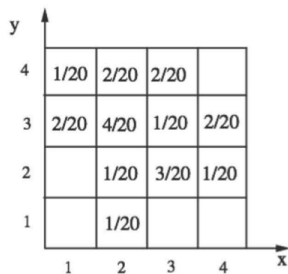
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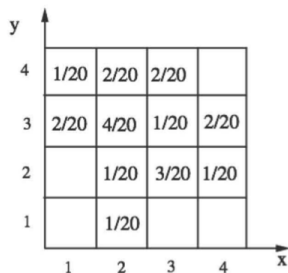
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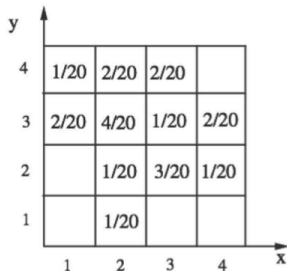
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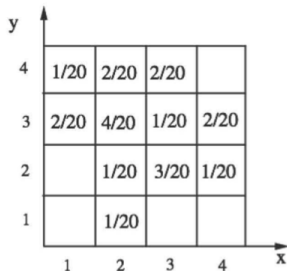
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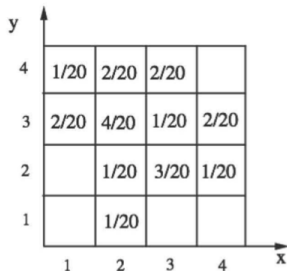
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- $X \perp\!\!\!\perp Y$  is a sufficient condition for  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study **covariance**.

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- $\{X_i\}, i = 1, 2, \dots, n$ : identically distributed (from symmetry)

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Questions?

- 1) What is Random Variable? Why is it useful?
- 2) What is PMF (Probability Mass Function)?
- 3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
- 4) What are joint and marginal PMFs?
- 5) Describe and explain the total probability/expectation theorem for random variables?
- 6) When is it useful to use total probability/expectation theorem?
- 7) What is conditional independence?