Lecture 5: Random Variable, Part III

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

April 27, 2021

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Roadmap

KAIST EE

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Derived Distribution: $Y = g(X)$

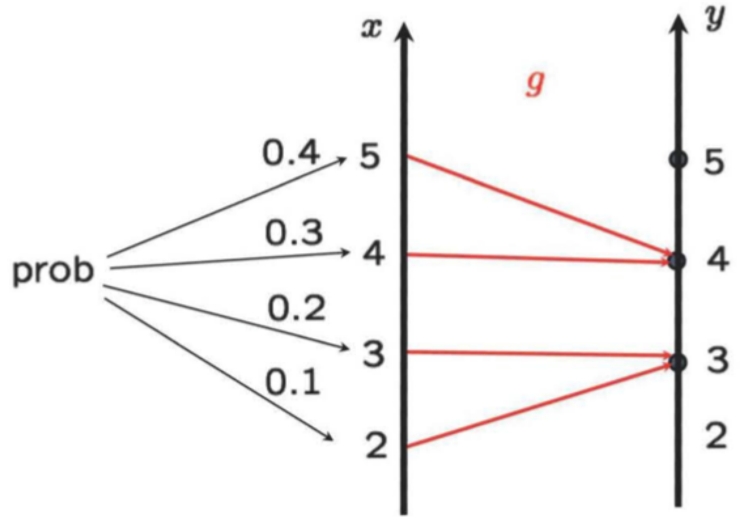
- Given the PDF of X, What is the PDF of $Y = g(X)$?
- Wait! Didn't we cover this topic? No. We covered just $\mathbb{E}[g(X)]$.
- Examples: $Y = X$, $Y = X + 1$, $Y = X^2$, etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x:g(x)=y} p_X(x) \end{aligned}$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$, X: Continuous

If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

$$\rightarrow f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

If $a < 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \geq \frac{y-b}{a}) = 1 - F_X(\frac{y-b}{a})$

$$\rightarrow f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Therefore,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Linear: $Y = aX + b$, when X is exponential

KAIST EE

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if} \quad (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $b = 0$ and $a > 0$, Y is exponential with parameter $\frac{\lambda}{a}$, but generally not.

Linear: $Y = aX + b$, when X is normal

- Remember? Linear transformation preserves normality. Time to prove.

> If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Proof.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 /2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\}$$

Generally, $Y = g(X)$, X: Continuous

Step 1. Find the CDF of Y:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$

Step 2. Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Ex1. $Y = X^2$.

$$F_Y(y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) +$$
$$\frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0$$

Ex2. $X \sim \mathcal{U}[0,1]$. $Y = \sqrt{X}$.

$$F_Y(y) = \mathbb{P}(\sqrt{X} \le y) = \mathbb{P}(X \le y^2) = y^2$$
$$f_Y(y) = 2y, \quad 0 \le y \le 1$$

Ex3. $X \sim \mathcal{U}[0,2]$. $Y = X^3$.

$$F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \frac{1}{2} y^{1/3}$$
$$f_Y(y) = \frac{1}{6} y^{-2/3}, \quad 0 \le y \le 8$$

When $Y = g(X)$ is monotonic, a general formula can be drawn (see the textbook at pp 207)

Functions of multiple rvs: $Z = g(X, Y)$ (1)

Basically, follow two-step approach: (i) CDF and (ii) differentiate.

Ex1. $X, Y \sim \mathcal{U}[0,1]$, and $X \perp\!\!\!\perp Y$. $Z = \max(X, Y)$.

* $\mathbb{P}(X \le z) = \mathbb{P}(Y \le z) = z, \;\; z \in [0,1]$.

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\max(X,Y) \le z) = \mathbb{P}(X \le z, Y \le z) \\ &= \mathbb{P}(X \le z)\mathbb{P}(Y \le z) = z^2 \qquad\qquad (\text{from } X \perp\!\!\!\perp Y) \end{aligned}$$

$$f_Z(z) = \begin{cases} 2z, & \text{if } 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

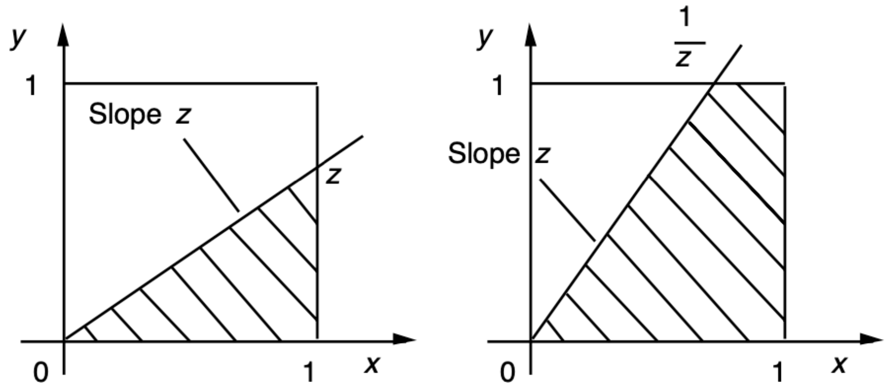
Basically, follow two step approach: (i) CDF and (ii) differentiate.

Ex2. $X, Y \sim \mathcal{U}[0,1]$, and $X \perp\!\!\!\perp Y$. $Z = Y/X$.
VIDEO PAUSE

$$F_Z(z) = \mathbb{P}(Y/X \le z)$$
$$= \begin{cases} z/2, & 0 \le z \le 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} 1/2, & 0 \le z \le 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

- Depending on the value of z, two cases need to be considered separately.



(Note) Sometimes, the problem is tricky, which requires careful case-by-case handing. :-)

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

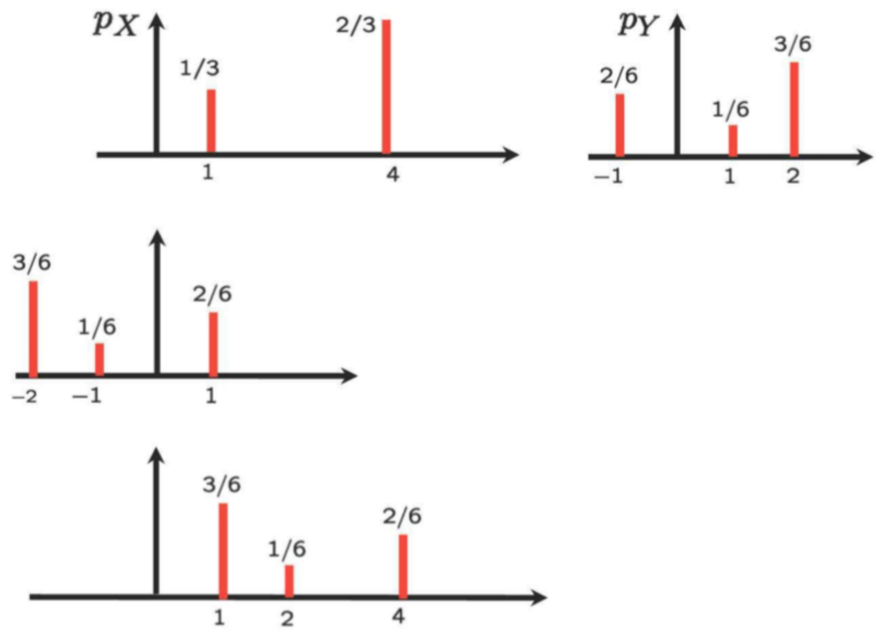
Functions of multiple rvs: $Z = X + Y$, $X \perp\!\!\!\perp Y$

- A very basic case with many applications
- Assume that $X, Y \in \mathbb{Z}$

$$
\begin{aligned}
p_Z(z) &= \mathbb{P}(X + Y = z) \\
&= \sum_{\{(x,y): x+y=z\}} \mathbb{P}(X = x, Y = y) \\
&= \sum_x \mathbb{P}(X = x, Y = z - x) \\
&= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
&= \sum_x p_X(x) p_Y(z - x)
\end{aligned}
$$

- $p_Z(z)$ is called $\boxed{\text{convolution}}$ of the PMFs of X and Y.

\- Interpretation (for a given z)
(i) Flip (horizontally) $p_Y(y)$ $(p_Y(-x))$
(ii) Put it underneath $p_X(x)$ $(p_Y(-x + z))$



$Y = X + Y$, $X \perp\!\!\!\perp Y$: Continuous

- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

- Example. $X, Y \sim \mathcal{U}[0,1]$ and $X \perp\!\!\!\perp Y$. What is the PDF of $Z = X + Y$?

-

$Y = X + Y$, $X \perp\!\!\!\perp Y$, Normal (1)

- Very special, but useful case

  \- X and Y are normal.

Sum of two independent normal rvs

$X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_x, \sigma_x^2)$ Then, $X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

- Why normal rvs are used to model the sum of random noises.
- Extension. The sum of finitely many independent normals is also normal.

KAIST EE

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx
\end{aligned}
$$

- The details of integration is a little bit tedious, but note where we use the independence condition.

$$
f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right\}
$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Dependence Degree: Motivating Example

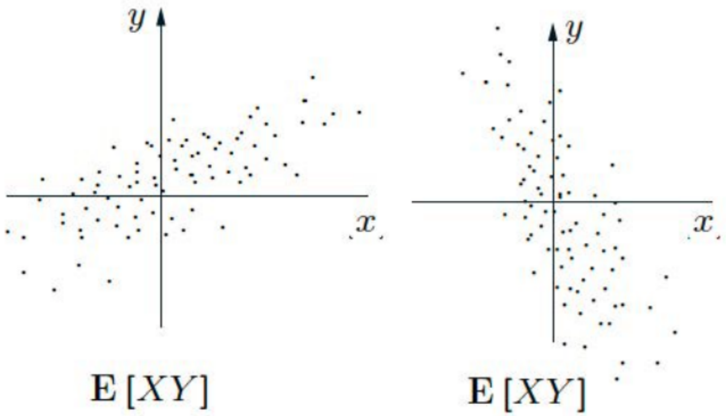
- covariance의 필요성을 이야기해주는 example을 찾아서 먼저 이야기를 해준다.

Making a Metric of Dependence Degree

- Goal: Given two rvs X and Y, assign some number that quantifies the degree of their dependence
- Reqs.
  a) Increases (resp. decreases) as they become more (resp. less) dependent.
  b) 0 when they are independent.
  c) Shows the direction of dependence by + and -
  d) Always bounded by some numbers, e.g., $[-1, 1]$
- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

OK. Let's Design!

- Simple case: $\mathbb{E}[X] = \mu_X = 0$ and $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If $X \uparrow$, $Y \uparrow$) or Negative (If $X \uparrow$, $Y \downarrow$)
- What about $\mathbb{E}[XY]$? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$ when $X \perp\!\!\!\perp Y$
  - More data points (thus increases) when $xy > 0$ (both positive or negative)



(Q) What about $\mathbb{E}[X + Y]$?

What If $\mu_X \neq 0, \mu_Y \neq 0$?

- Solution: Centering. $X \to X - \mu_X$ and $Y \to Y - \mu_Y$

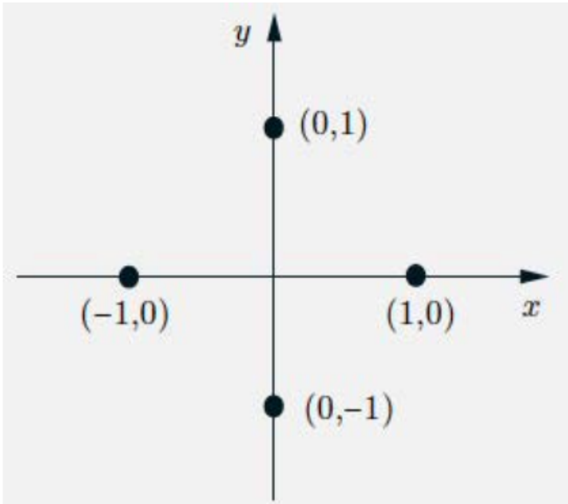
Covariance

$$\mathrm{cov}(X, Y) = \mathbb{E}\Big[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])\Big]$$

- After some algebra, $\mathrm{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \implies \mathrm{cov}(X, Y) = 0$
- $\mathrm{cov}(X, Y) = 0 \implies X \perp\!\!\!\perp Y$? NO.
- When $\mathrm{cov}(X, Y) = 0$, we say that X and Y are $\boxed{\text{uncorrelated.}}$

Example: $\text{cov}(X, Y) = 0$, but not independent

- $p_{X,Y}(1,0) = p_{X,Y}(0,1) = p_{X,Y}(-1,0) = p_{X,Y}(0,-1) = 1/4.$
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



Some Properties

$$\text{cov}(X, X) = 0$$

$$\text{cov}(aX + b, Y) = \mathbb{E}[(aX + b)Y] - \mathbb{E}[aX + b]\mathbb{E}[Y] = a \cdot \text{cov}(X, Y)$$

$$\text{cov}(X, Y + Z) = \mathbb{E}[X(Y + Z)] - \mathbb{E}[X]\mathbb{E}[Y + Z] = \text{cov}(X, Y) + \text{cov}(X, Z)$$

$$\text{var}[X + Y] = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 = \text{var}[X] + \text{var}[Y] - 2\text{cov}(X, Y)$$

Example: The hat problem in Lecture 3. Remember?

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- (Q) $\text{var}[X]$
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$.
- Key step 2. Are X_is are independent?
- $X_i \sim \mathit{Bernoulli}(1/n)$. Thus, $\mathbb{E}[X_i] = 1/n$ and $\text{var}[X_i] = \frac{1}{n}(1 - \frac{1}{n})$

  - For $i \neq j$,

$$\begin{aligned} \text{cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\ &= \mathbb{P}(X_i = 1 \text{ and } X_j = 1) - \frac{1}{n^2} \\ &= \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\ &= \frac{1}{n}\frac{1}{n-1} - \frac{1}{n^2} = \frac{1}{n^2(n-1)} \end{aligned}$$

$$\begin{aligned} \text{var}[X] &= \text{var}\left[\sum X_i\right] \\ &= \sum \text{var}[X_i] + \sum_{i \neq j} \text{cov}(X_i, X_j) \\ &= n\frac{1}{n}(1 - \frac{1}{n}) + n(n-1)\frac{1}{n^2(n-1)} = 1 \end{aligned}$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Bounding the metric: Correlation Coefficient

- Reqs. a), b), and c) satisfied.
  d) Always bounded by some numbers, e.g., $[-1, 1]$
- Dimensionless metric. How? $\boxed{\text{Normalization,}}$ but by what?

Correlation Coefficient

$$\rho(X, Y) = \mathbb{E}\left[\frac{(X - \mu_X)}{\sigma_X} \cdot \frac{(Y - \mu_Y)}{\sigma_Y}\right] = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$ (linear relation, VERY related)

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

A Special Random Variable

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

- If $h(y) = y^2$, then a new rv $h(Y)$ is:

$$h(Y) = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 4, & \text{w.p. } 1/2 \end{cases}$$

- Consider other rv X, such that

$$g(y) = \mathbb{E}[X|Y = y] = \begin{cases} 3, & \text{if } y = 0 \\ 8, & \text{if } y = 1 \\ 9, & \text{if } y = 2 \end{cases}$$

- Then, a rv $g(Y)$ is:

$$g(Y) = \begin{cases} 3, & \text{w.p. } 1/4 \\ 8, & \text{w.p. } 1/4 \\ 9, & \text{w.p. } 1/2 \end{cases}$$

- The rv $g(Y)$ looks special, so let's give a fancy notation to it.

- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

Conditional Expectation $\mathbb{E}[X|Y]$

KAIST EE

Conditional Expectation

A random variable $g(Y) = \boxed{\mathbb{E}[X|Y]}$, called $\boxed{\text{conditional expectation of } X \text{ given } Y}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

- A function of Y
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has.
- Often confusing because of the notation.

Expectation of $\mathbb{E}[X|Y]$

Expectation of Conditional Expectation

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\ &= \mathbb{E}[X]\end{aligned}$$

Examples and Meaning

- Stick of length l
- Uniformly break at point Y, and break what is left uniformly at point X.
- $\mathbb{E}[X|Y=y] = y/2$
- $\mathbb{E}[X|Y] = Y/2$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y/2] = \frac{1}{2}\frac{l}{2} = l/4$

- Forecasts on sales: calculating expected value, given any available information
- X : February sales
- Forecast in the beg. of the year: $\mathbb{E}[X]$
- End of Jan. new information $Y = y$ (Jan. sales)
  Revised forecast: $\mathbb{E}[X|Y=y]$
  Revised forecast $\neq \mathbb{E}[X]$
- Law of iterated expectations
  $\mathbb{E}[\text{revised forecast}] = \text{original one}$

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Conditional Variance $\mathrm{var}[X|Y]$

$$\mathrm{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\mathrm{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

Conditional Variance

A random variable $g(Y) = \boxed{\mathrm{var}[X|Y]}$ and called $\boxed{\text{conditional variance of } X \text{ given } Y}$, takes the value $g(y) = \mathrm{var}[X|Y = y]$, if Y happens to take the value y.

- A function of Y
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has

Expectation and Variance of $\mathbb{E}[X|Y]$ and $\text{var}[X|Y]$

	$\mathbb{E}[X\|Y]$	$\text{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\text{var}(X\|Y)\right]$
Variance	$\text{var}\left[\mathbb{E}(X\|Y)\right]$	$\text{var}\left[\text{var}(X\|Y)\right]$

Law of Total Variance

Law of total variance

$$\mathrm{var}[X] = \mathbb{E}\Big[\mathrm{var}(X|Y)\Big] + \mathrm{var}[\mathbb{E}(X|Y)]$$

Proof.

$$\mathrm{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

$$\mathbb{E}\Big[\mathrm{var}(X|Y)\Big] = \mathbb{E}[X^2] - \mathbb{E}\Big[(\mathbb{E}[X|Y])^2\Big] \tag{1}$$

$$\mathrm{var}\Big[\mathbb{E}(X|Y)\Big] = \mathbb{E}\Big[(\mathbb{E}[X|Y])^2\Big] - (\mathbb{E}\Big[\mathbb{E}(X|Y)\Big])^2 = \mathbb{E}\Big[(\mathbb{E}[X|Y])^2\Big] - (\mathbb{E}[X])^2 \tag{2}$$

$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \mathrm{var}[X]$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Sum of a random number of rvs

- N : number of stores visited (random)
- X_i: money spent in store i, independent of other X_j and N, X_is are identically distributed with $\mathbb{E}[X_i] = \mu$
- $Y = X_1 + X_2 + \ldots X_N$. What are $\mathbb{E}[Y]$ and var$[Y]$?
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N\mathbb{E}[X_i]] = \mathbb{E}[N]\mathbb{E}[X_i] = \mu\mathbb{E}[N]$
- $\text{var}[Y] = \mathbb{E}\Big[\text{var}(Y|N)\Big] + \text{var}[\mathbb{E}(Y|N)] = \mathbb{E}[N]\text{var}[X_i] - \mu^2\text{var}[N]$

$$\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$$
$$\text{var}[Y|N] = N\text{var}[X_i]$$
$$\mathbb{E}[\text{var}(Y|N)] = \mathbb{E}[N\text{var}[X_i]] = \mathbb{E}[N]\text{var}[X_i]$$

Questions?

Review Questions

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
2) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?
3) What are covariance and correlation coefficient? Why do we need them?
4) Please explain the concepts of conditional expectation and conditional variance.