Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

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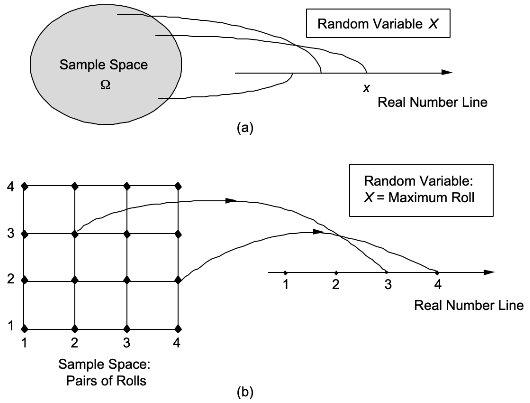
Random Variable: Idea

- In reality, many outcomes are ___, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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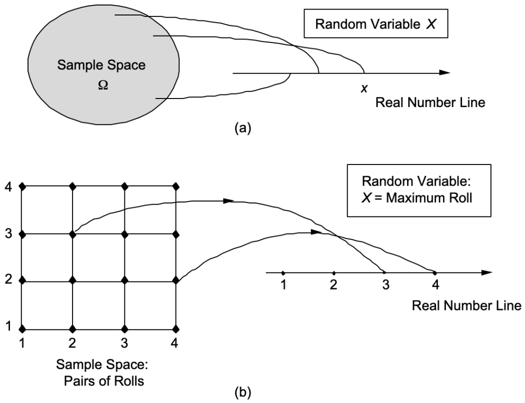

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}\Big(\{\omega \in \Omega \mid X(\omega) = x\}\Big)$$

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Roadmap

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Bernoulli X with parameter $p \in [0, 1]$

- Only binary values

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$$X = \begin{cases} 0, & \text{w.p.} \quad 1 - p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail

- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$

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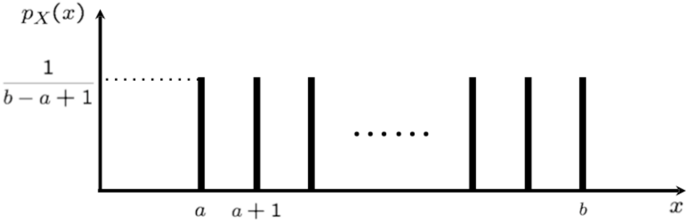

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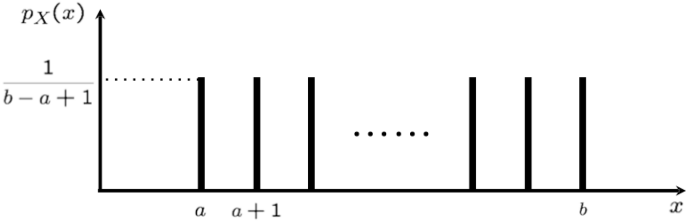

- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p

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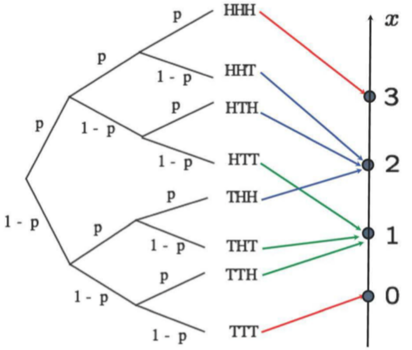

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L3(2)

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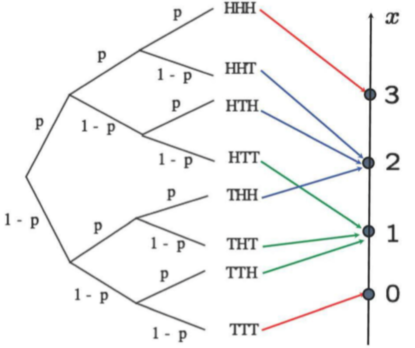
$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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Geometric X with parameter p

- Infinitely many independent Bernoulli trials, where each trial has success probability p

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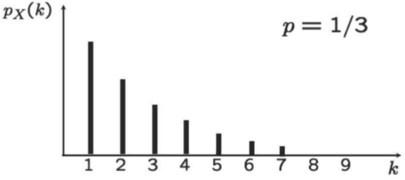
$$p_X(k) = (1-p)^{k-1}p$$



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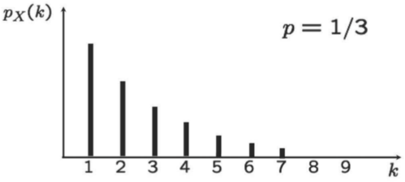
- Models waiting times until something happens.



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- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)

Expectation/Mean

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Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

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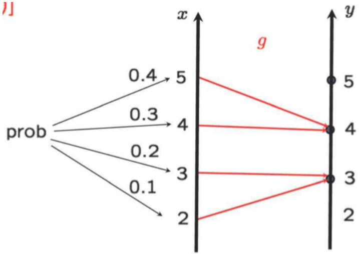

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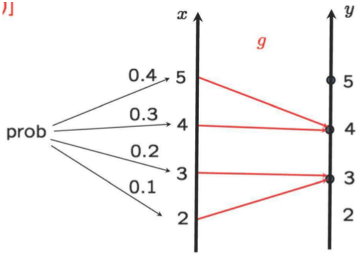

$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2)$$
$$= 2.8 + 0.9 = 3.7$$

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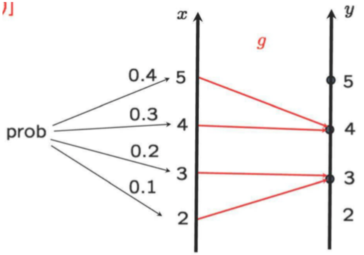

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Linearity of Expectation

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

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Example: Variance of a Bernoulli rv (p)

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Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}
\mu = \mathbb{E}[X] &= 1 \times p + 0 \times (1-p) = p \\
\mathbb{E}[X^2] &= 1 \times p + 0 \times (1-p) = p \\
\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\
&= p(1-p)
\end{aligned}$$

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Joint PMF

- $\boxed{\phantom{\text{Joint PMF}}}$ For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$\mathbb{P}\Big(\{X = x\} \cap \{Y = y\}\Big)$$

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$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

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Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) =$

$p_X(4) =$

$\mathbb{P}(X = Y) =$

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2		1/20	3/20	1/20
1		1/20		

$$p_{X,Y}(1, 3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

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(easy to prove, using the definition.)

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- $\mathbb{E}[X_1 \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

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Message. When some rv X is write as a linear combination of other rvs, it is often easy to handle X.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

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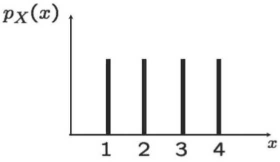
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$A = \{X \geq 2\}$



$\mathbb{E}[X] =$

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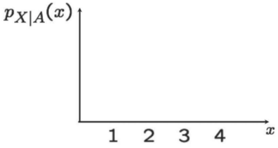

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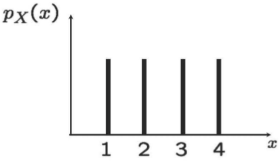

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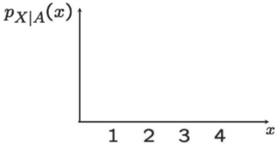

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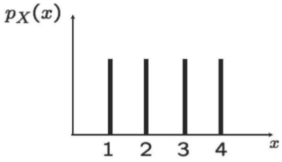

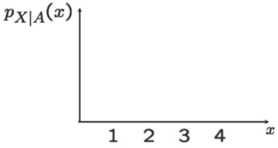

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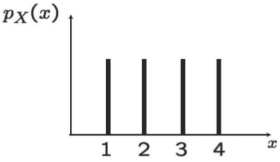

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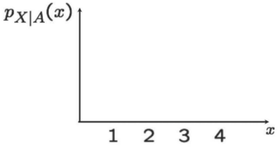

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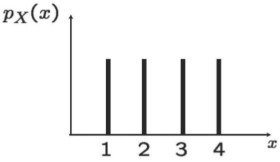

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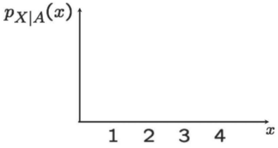

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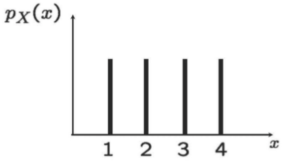

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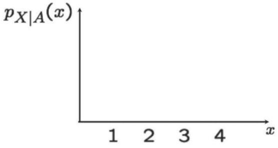

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Conditional PMF

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- Multiplication rule.

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$

Conditional PMF

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$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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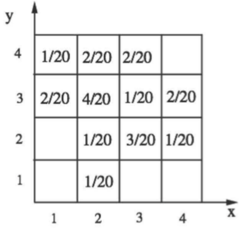
VIDEO PAUSE



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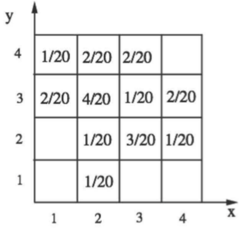

$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

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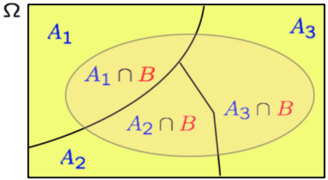
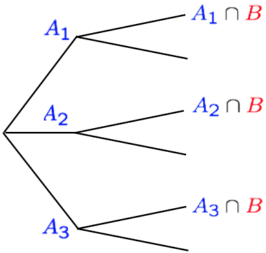
$$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$$

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$? (probability of result)

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$





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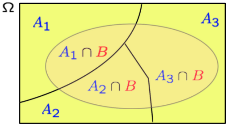

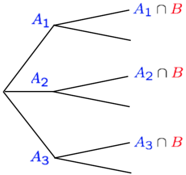

KAIST EE

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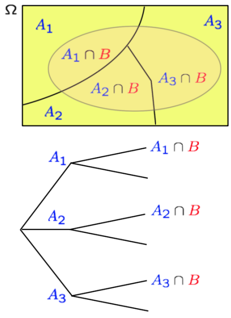

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Total Expectation Theorem for $\{Y = y\}$

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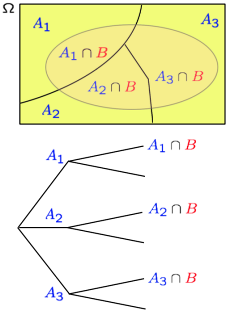

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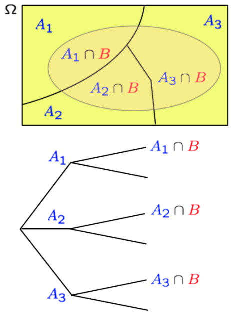
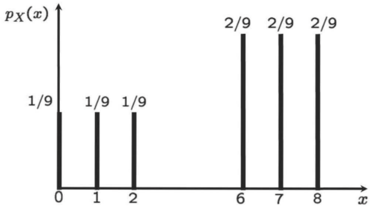


Example 1: Total Expectation Theorem

- Using the definition of expectation,

$$\mathbb{E}[X] =$$



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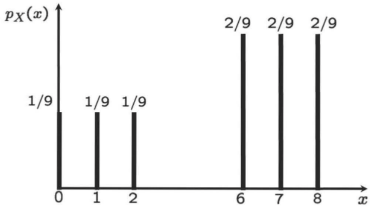

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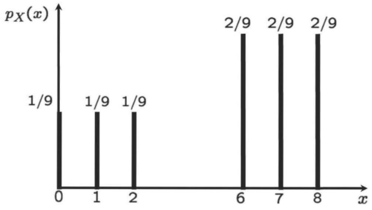

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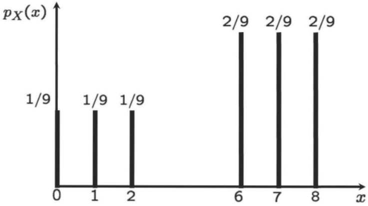

Background: Memoryless Property

- Some random variable often does not have memory.

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- Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

Background: Memoryless Property

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- Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

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- Suppose that X is the time of waiting for a bus and X is memoryless. At the bus stop, I have waited for the bus for 10 mins. Then, the time until the bus arrival does not depend on how much I have waited for a bus. No memory.

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- Proof.

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)}$$

$$= \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)$$

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$$\mathbb{E}[X] = 1 + (1-p)\frac{1}{p} = 1/p.$$

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Independence, Conditional Independence

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Example

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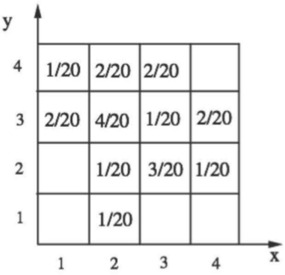
- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?



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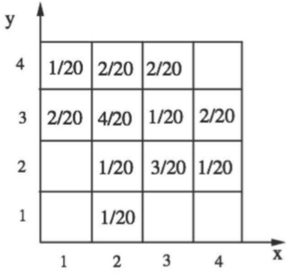

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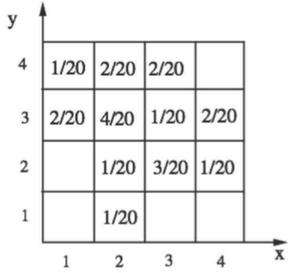

$Y = 4\ (1/3)$	$1/9$	$2/9$
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  - $X \perp\!\!\!\perp Y \implies$
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- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (symmetry)

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- Assume $n = 2$. Then, $X_1 = 1 \to X_2 = 1$, and $X_1 = 0 \to X_2 = 0$. Thus, dependent.

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- Assume $n = 2$. Then, $X_1 = 1 \to X_2 = 1$, and $X_1 = 0 \to X_2 = 0$. Thus, dependent.

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\begin{aligned}
\text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \mathbb{E}\Big[\sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j\Big] - (\mathbb{E}[X])^2
\end{aligned}
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Questions?

Review Questions

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFS?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?