Lecture 3: Random Variable, Part I

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

September 14, 2022

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

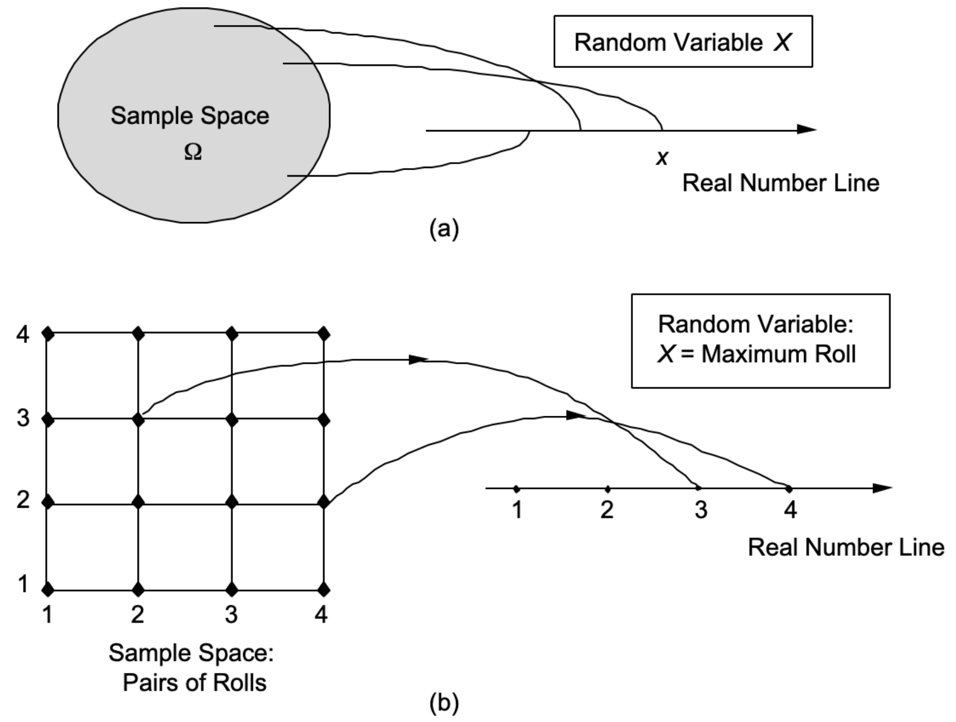
(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Random Variable: Idea

- In reality, many outcomes are numerical, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

Random Variable: More Formally

- Mathematically, a random variable X is a $\boxed{\text{function}}$ which maps from Ω to $\mathbb{R}$.
- Notation. Random variable X, numerical value x.
- Different random variables can be defined on the same sample space.
- For a fixed value x, we can associate an $\boxed{\text{event}}$ that a random variable X has the value x, i.e., $\boxed{\{\omega \in \Omega \mid X(\omega) = x\}}$
- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

- For a discrete random variable X, we call $p_X(x)$ $\boxed{\text{probability mass function}}$ (PMF).

[1] Finite or countably infinite.

Example

- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$

- Remember that the random variable X is a **function** from Ω to $\mathbb{R}$

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Bernoulli X with parameter $p \in [0, 1]$

- Only binary values
$$X = \begin{cases} 0, & \text{w.p.} \quad 1-p, \\ 1, & \text{w.p.} \quad p \end{cases}$$
In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

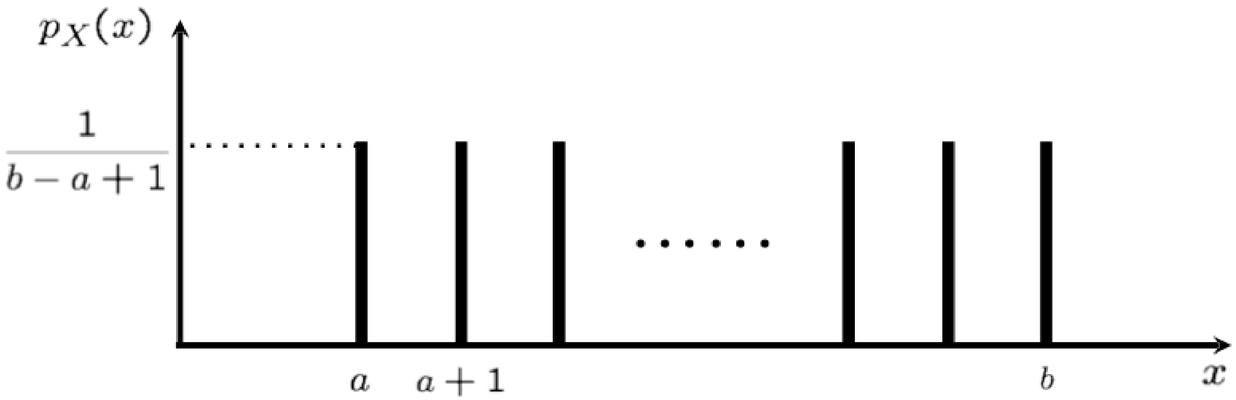
- Models a trial that results in binary results, e.g., success/failure, head/tail

- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:
$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$
- Choose a number out of $\Omega = \{a, a+1, \ldots, b\}$ uniformly at random.
- $p_X(i) = \frac{1}{b-a+1}, \;\; i \in \Omega$

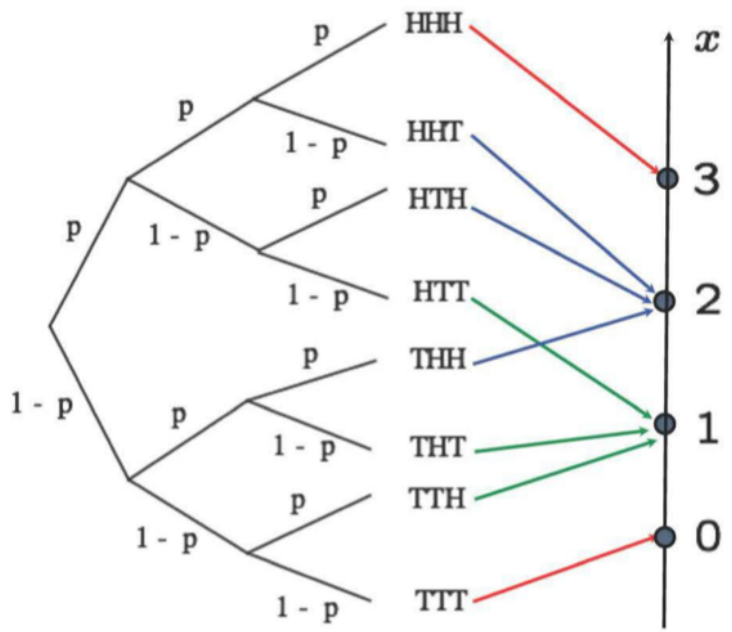


- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p

- Models the number of successes in a given number of independent trials
- n independent trials, where one trial has the success probability p.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



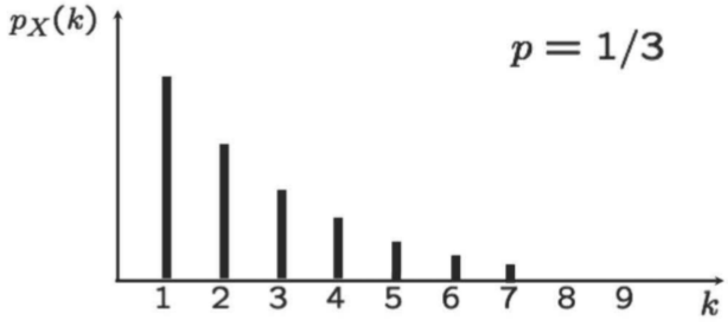
[1] $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, which we read '$n$ choose k'.

Geometric X with parameter p

- Infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the first success.

$$p_X(k) = (1-p)^{k-1}p$$

- Models waiting times until something happens.



Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Expectation/Mean

- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

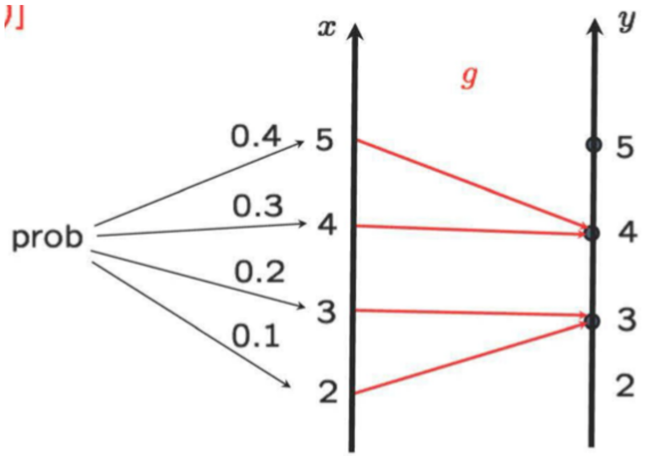
- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

Expectation of a function of a RV

- For a rv X, $Y = g(X)$ is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- Compute $\mathbb{E}[Y]$ for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2)$$
$$= 2.8 + 0.9 = 3.7$$

Linearity of Expectation

$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Variance

- Measures how much the $\boxed{\text{spread}}$ of a PMF is.
- What about $\mathbb{E}[X - \mu]$, where $\mu = \mathbb{E}[X]$? Zero
- Then, what about $\mathbb{E}[(X - \mu)^2]$?

Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

  $$\text{var}[X] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$
  $$= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$$

- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$

  $$\text{var}[Y] = \mathbb{E}[(X+b)^2] - (\mathbb{E}[X+b])^2$$

- $Y = aX$, $\text{var}[Y] = a^2\text{var}[X]$

  $$\text{var}[Y] = \mathbb{E}[a^2X^2] - (a\mathbb{E}[X])^2$$

Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}
\mu = \mathbb{E}[X] &= 1 \times p + 0 \times (1-p) = p \\
\mathbb{E}[X^2] &= 1 \times p + 0 \times (1-p) = p \\
\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\
&= p(1-p)
\end{aligned}$$

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Joint PMF

KAIST EE

- **Joint PMF.** For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

  $$p_{X,Y}(x, y) \triangleq \mathbb{P}(\{X = x\} \cap \{Y = y\})$$

- $\sum_x \sum_y p_{X,Y}(x, y) = 1$

- **Marginal PMF.**

  $$p_X(x) = \sum_y p_{X,Y}(x, y),$$

  $$p_Y(y) = \sum_x p_{X,Y}(x, y)$$

Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$$p_{X,Y}(1, 3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:
$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,
$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

Linearity of Expectation for Multiple RVs

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

  (easy to prove, using the definition.)

- $\mathbb{E}[X_1 \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X+3Y-Z] = 2\mathbb{E}[X]+3\mathbb{E}[Y]-\mathbb{E}[Z]$

- Example. Mean of a binomial rv Y with (n, p)
- Y: number of successes in n Bernoulli trials with p
- $Y = X_1 + \ldots X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Conditional PMF: Conditioning on an event

Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A.

- $p_X(x) \triangleq \mathbb{P}(X = x)$
- $\mathbb{E}[X] = \sum_x x p_X(x)$
- $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

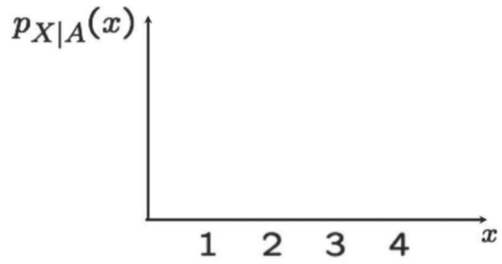
- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$$A = \{X \geq 2\}$$

$$\mathbb{E}[X] = \frac{1}{4}(1+2+3+4) = 2.5$$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{4}(1+2^2+3^2+4^2) - 2.5^2\end{aligned}$$



$$\mathbb{E}[X|A] = \frac{1}{3}(2+3+4) = 3$$

$$\begin{aligned}\text{var}[X|A] &= \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 \\ &= \frac{1}{3}(2^2+3^2+4^2) - 3^2 = 2/3\end{aligned}$$

Conditional PMF: Conditioning on a RV

What do we mean by "conditioning on a rv"? Consider $A = \{Y = y\}$ for a rv Y.

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$

- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
- $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
- $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
- $\text{var}[X|Y = y] \triangleq$ $\mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

KAIST EE

VIDEO PAUSE

- Conditional PMF

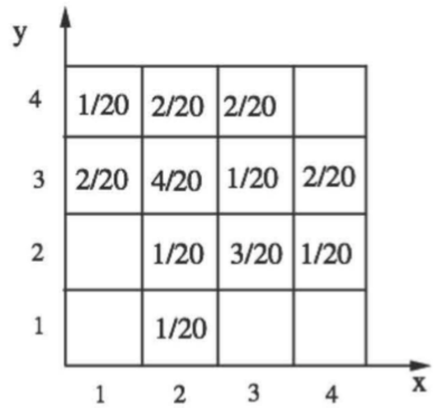
$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

- $\sum_x p_{X|Y}(x|y) = 1$
- Multiplication rule

$$\begin{aligned} p_{X,Y}(x,y) &= p_Y(y)p_{X|Y}(x|y) \\ &= p_X(x)p_{Y|X}(y|x) \end{aligned}$$

- $p_{X,Y,Z}(x,y,z) =$
$p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$



$p_{X|Y}(2|2) = \frac{1}{1+3+1}$

$p_{X|Y}(3|2) = \frac{3}{1+3+1}$

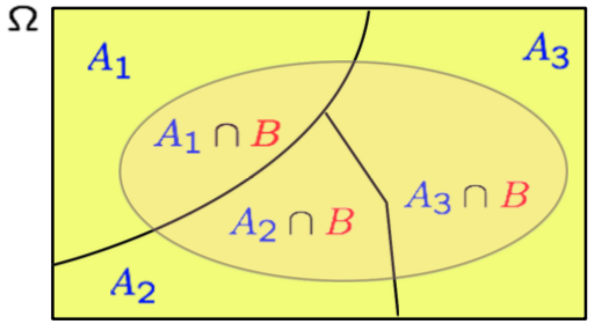
$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$

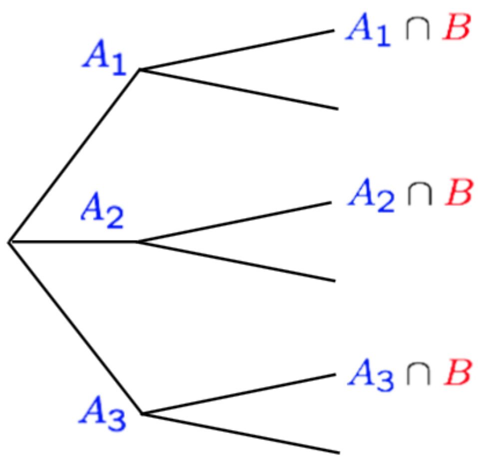
Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$



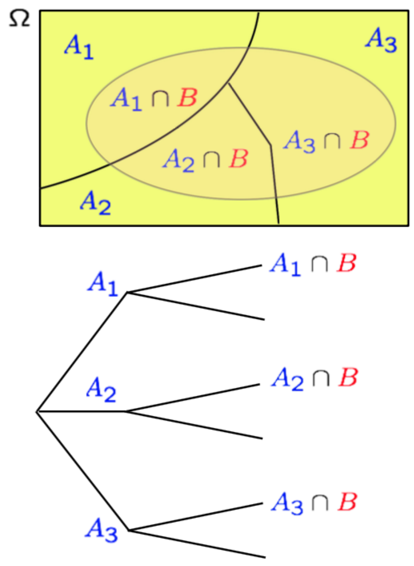


Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$



Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

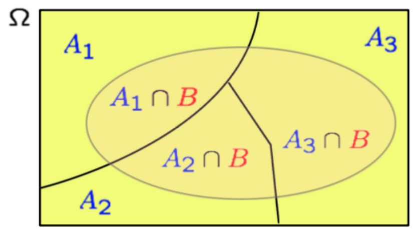
Total Probability Theorem

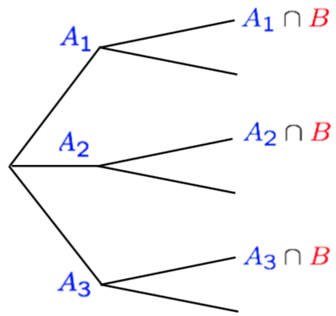
$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

- Weighted average of expectations from A_i's perspective





Total Expectation Theorem for $\{Y = y\}$

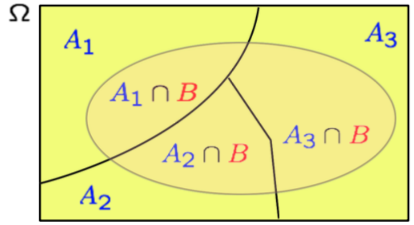
- Partition of Ω into A_1, A_2, A_3

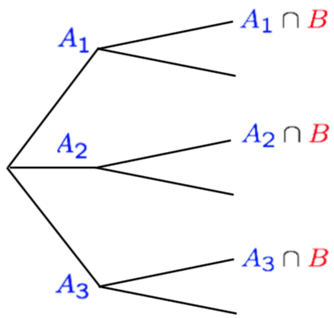
Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y] = \sum_y p_Y(y)\mathbb{E}[X|Y = y]$$





Example 1: Total Expectation Theorem

- Question. What is $\mathbb{E}(X)$?

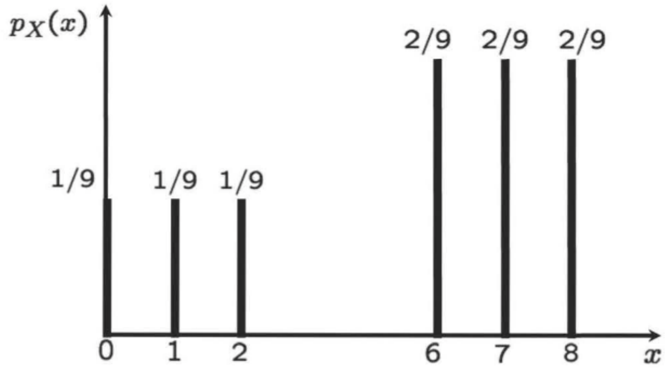
(1) Just using the definition of expectation,

$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$
$$= \frac{3+12+14+16}{9} = 5$$

(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\}, \ A_2 = \{X \in \{6,7,8\}\}$$

$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



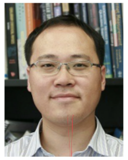
Example 2: Mean of Geometric rv

- Write softwares over and over, and each time w.p. p of working correctly (independent from previous programs).
- X: number of trials until the program works correctly.
- (Q) $\mathbb{E}(X)$?
- X is a geometric rv
- Direct computation is boring.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p + 2(1-p)p + 3(1-p)^2 p + \cdots$$

- Total expectation theorem and a notion of memorylessness helps a lot.

Memoryless Property: Motivating Example



Yung arrives

No bus arrives

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n + m | X > m)$$

time

0

m mins

A bus left at time 0

time

0

Lin arrives

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n)$$

- Some random variable often does not have memory.

- Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

Background: Memoryless Property of Geometric RVs

- Theorem. Any **geometric** random variable is memoryless.
- Remind. Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \quad \mathbb{P}(X > k) = \sum_{i=k+1}^{\infty} (1-p)^{i-1}p = (1-p)^k$$

- Proof.

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)}$$

$$= \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)$$

- Meaning. Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

Back to Example 2: Mean of Geometric rv

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$
\begin{aligned}
\mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\
&= 1 + \mathbb{P}(A_1)\mathbb{E}[X-1|X=1] + \mathbb{P}(A_2)\mathbb{E}[X-1|X>1] && \text{(from TET)} \\
&= 1 + (1-p)\mathbb{E}[X] && \text{(from memorylessness)}
\end{aligned}
$$

- Thus, $\mathbb{E}[X] = \dfrac{1}{p}$

Roadmap

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$
$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$
$$\mathbb{P}(\{X = x\} \cap B | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(B | C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$
$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

$$\mathbb{P}(\{X = x\} \cap \{Y = y\} | Z = z) = \mathbb{P}(X = x | Z = z) \cdot \mathbb{P}(Y = y | Z = z), \quad \text{for all } x, y$$
$$p_{X,Y|Z}(x, y | z) = p_{X|Z}(x | z) \cdot p_{Y|Z}(y | z)$$

Example

- $X \perp\!\!\!\perp Y$?

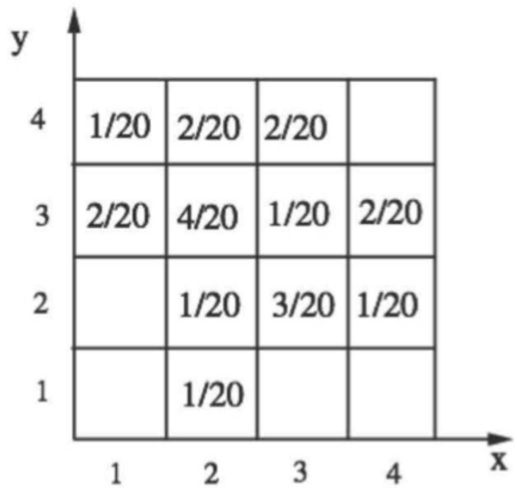
$$p_{X,Y}(1,1) = 0$$
$$p_X(1) = 3/20$$
$$p_Y(1) = 1/20$$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?

VIDEO PAUSE

$Y = 4$ $(1/3)$	$1/9$	$2/9$
$Y = 3$ $(2/3)$	$2/9$	$4/9$
	$X = 1$ $(1/3)$	$X = 2$ $(2/3)$

- Yes.



Expectation and Variance

- Always true.
  $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Generally, $\mathbb{E}[g(X,Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$
- However, if $X \perp\!\!\!\perp Y$,

$$\begin{aligned} \mathbb{E}[XY] &= \mathbb{E}[X]\mathbb{E}[Y] \\ \mathbb{E}[g(X)h(Y)] &= \mathbb{E}[g(X)]\mathbb{E}[g(Y)] \end{aligned}$$

- Proof.

$$\begin{aligned} \mathbb{E}[g(X)h(Y)] &= \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y) \\ &= \sum_x g(x)p_X(x) \sum_y h(y)p_Y(y) \end{aligned}$$

- Always true.
  $\text{var}[aX] = a^2\text{var}[X]$, $\text{var}[X+a] = \text{var}[X]$
- Generally, $\text{var}[X+Y] \neq \text{var}[X] + \text{var}[Y]$ (next slide)
- However, if $X \perp\!\!\!\perp Y$,

  $\text{var}[X+Y] = \text{var}[X] + \text{var}[Y]$
- Practice.
  - $X = Y \implies \text{var}[X+Y] = 4\text{var}[X]$
  - $X = -Y \implies \text{var}[X+Y] = 0$
  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

$\mathrm{var}[X+Y] \neq \mathrm{var}[X] + \mathrm{var}[Y]$

- Why not generally true?

$$\begin{aligned}\mathrm{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\ &= \mathbb{E}[X^2 + Y^2 + 2XY] - \left((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right) \\ &= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)\end{aligned}$$

- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

Example: The hat problem (1)

KAIST EE

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- $\mathbb{E}[X]$? var$[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- Key step 1. Define a rv $X_i = 1$ if i selects its own hat and 0 otherwise.
$$X = \sum_{i=1}^{n} X_i.$$
- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (from symmetry)

Example: The hat problem (2)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1.$
- Key step 2. Are X_is are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \to X_2 = 1$, and $X_1 = 0 \to X_2 = 0$. Thus, dependent.

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\Big[\sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j\Big] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_1^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 | X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

Review Questions

1) What is a random variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Geometric rvs. When are they useful and what are their PMFs?
4) Explain the memoryless property.
5) What are joint and marginal PMFs?
6) Describe and explain the total probability/expectation theorem for random variables? When is it useful to use total probability/expectation theorem?
7) Explain the definition and the meaning of expectation and variance. Why do we need them?
8) What is the difference between independence/conditional independence for events and those for random variables?