Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

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(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

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- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

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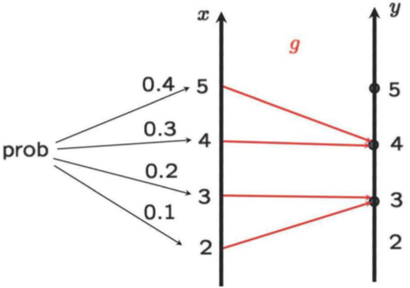

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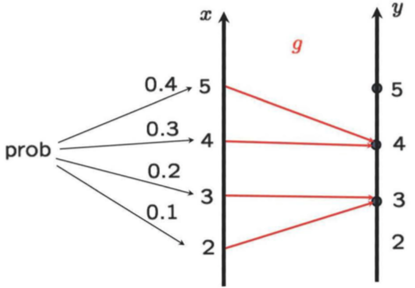
$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$, X: Continuous

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Therefore,

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Linear: $Y = aX + b$, when X is exponential

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if} \quad (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $b = 0$ and $a > 0$, Y is exponential with parameter $\frac{\lambda}{a}$, but generally not.

Linear: $Y = aX + b$, when X is normal

- Remember? Linear transformation preserves normality. Time to prove.

> If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 /2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\}$$

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$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$$\begin{aligned} f_Y(y) = &\frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \\ &\frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0 \end{aligned}$$

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Ex3. $X \sim \mathcal{U}[0,2]$. $Y = X^3$.

$$F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \frac{1}{2} y^{1/3}$$

$$f_Y(y) = \frac{1}{6} y^{-2/3}, \quad 0 \le y \le 8$$

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When $Y = g(X)$ is monotonic, a general formula can be drawn (see the textbook at pp 207)

Functions of multiple rvs: $Z = g(X, Y)$ (1)

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$$f_Z(z) = \begin{cases} 2z, & \text{if } 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

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VIDEO PAUSE

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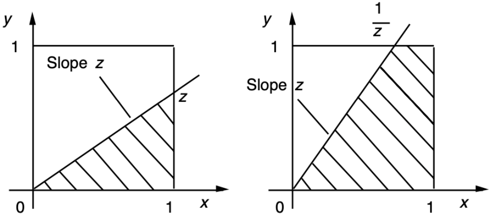
- Depending on the value of z, two cases need to be considered separately.



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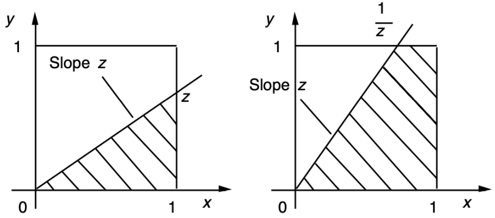

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$$f_Z(z) = \begin{cases} 1/2, & 0 \le z \le 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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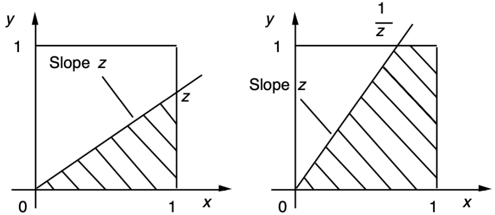

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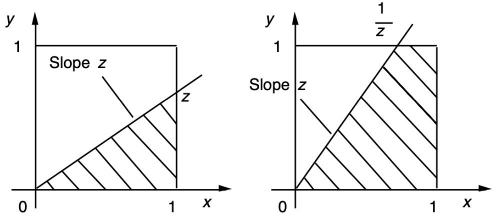

(Note) Sometimes, the problem is tricky, which requires careful case-by-case handing. :-)

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- $p_Z(z)$ is called $\boxed{\text{convolution}}$ of the PMFs of X and Y.

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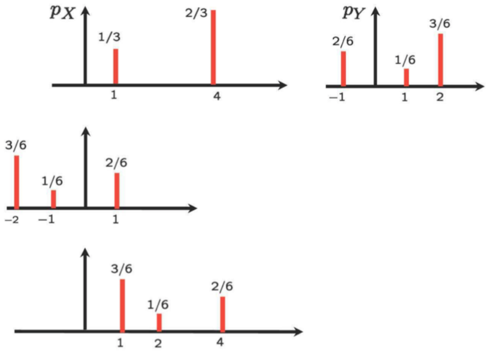
- Convolution: $p_Z(z) = \sum_x p_X(x) p_Y(z - x)$
- Interpretation for a given z:
  - (i) Flip (horizontally) the PMF of Y ($p_Y(-x)$)
  - (ii) Put it underneath the PMF of X
  - (iii) Right-shift the flipped PMF by z ($p_Y(-x + z)$)

Example. $z = 3$



$Y = X + Y$, $X \perp\!\!\!\perp Y$: Continuous

- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

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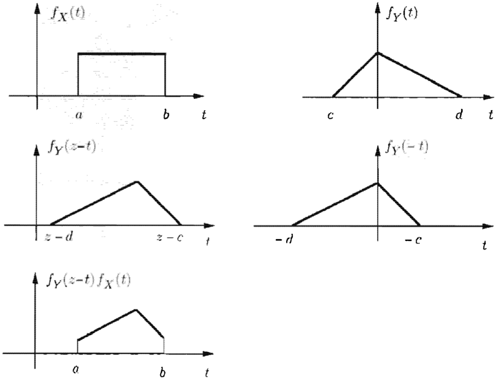

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- Youtube animation for convolution: `https://www.youtube.com/watch?v=C1N55M1VD2o`

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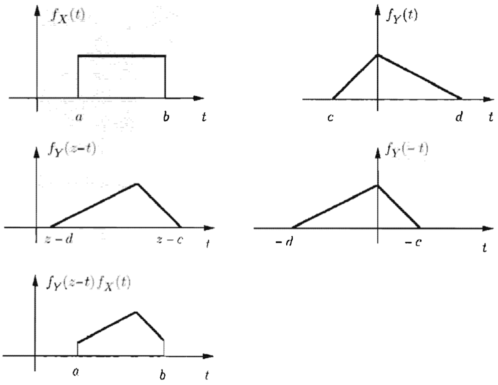

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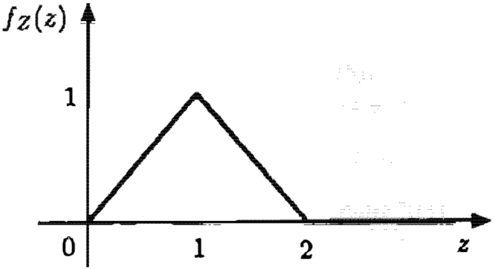

Convolution in Image Processing

https://www.youtube.com/watch?v=MQm6ZP1F6ms

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Sum of two independent normal rvs

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- Why normal rvs are used to model the sum of random noises.
- Extension. The sum of finitely many independent normals is also normal.

$Y = X + Y$, $X \perp\!\!\!\perp Y$, Normal (2)

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx
\end{aligned}
$$

- The details of integration is a little bit tedious. :-)

$$
f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right\}
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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

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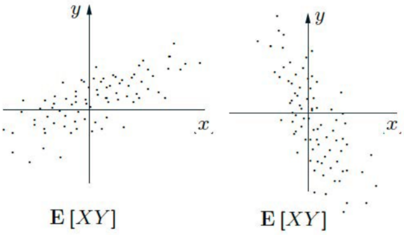

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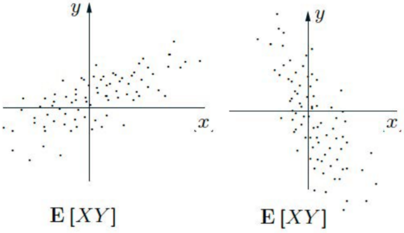

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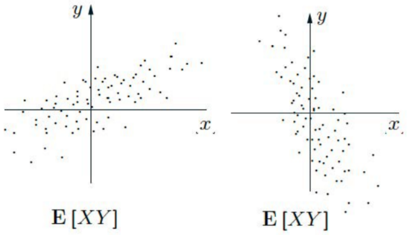

(Q) What about $\mathbb{E}[X+Y]$?
- When they are positively dependent, but have negative values?

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Example: $\text{cov}(X, Y) = 0$, but not independent

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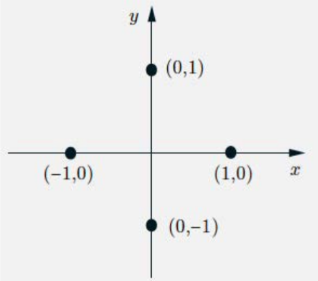

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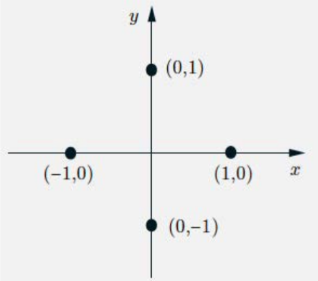

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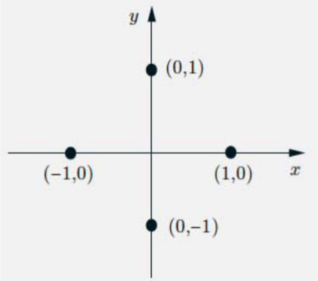
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
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- Theorem.
  1. $-1 \le \rho \le 1$ (proof at the next slide)
  2. $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$ for some constant c ($c > 0$ when $\rho = 1$ and $c < 0$ when $\rho = -1$). In other words, linear relation, meaning VERY related.

1. $-1 \leq \rho \leq 1$

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  Now, choose $a = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$. Then,

$$\mathbb{E}(X^2) - 2\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\mathbb{E}(XY) + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2}\mathbb{E}(Y^2) = \mathbb{E}(X^2) - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}(Y^2)} \geq 0$$

2. $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$

($\Rightarrow$) Suppose that $|\rho| = 1$. In the proof of CSI,

$$\mathbb{E}\left[\left(\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y}\right)^2\right] = \mathbb{E}(\tilde{X}^2) - \frac{(\mathbb{E}[\tilde{X}\tilde{Y}])^2}{\mathbb{E}(\tilde{Y}^2)} = \mathbb{E}(\tilde{X}^2)(1-\rho^2) = 0$$

$$\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}Y = 0 \leftrightarrow \tilde{X} = \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y} = \rho\sqrt{\frac{\mathbb{E}(\tilde{X}^2)}{\mathbb{E}(\tilde{Y}^2)}}\tilde{Y}$$

($\Leftarrow$) If $\tilde{Y} = c\tilde{X}$, then

$$\rho(X, Y) = \frac{\mathbb{E}(\tilde{X}c\tilde{X})}{\sqrt{\mathbb{E}[\tilde{X}^2]\mathbb{E}[(c\tilde{X})^2]}} = \frac{c}{|c|}$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

A Special Random Variable

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

Conditional Expectation

A random variable $g(Y) = \boxed{}$, called $\boxed{}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

Conditional Expectation $\mathbb{E}[X|Y]$

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- Thus, having a distribution, expectation, variance, all the things that a random variable has.
- Often confusing because of the notation.

Expectation of Conditional Expectation

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}
\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\
&= \mathbb{E}[X]
\end{aligned}$$

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$$\sum_{s=1}^{k} \frac{n_s}{n} m_s = \sum_{s=1}^{k} \frac{n_s}{n} \frac{1}{n_s} \sum_{i \in A_s} x_i = \frac{1}{n} \sum_{i=1}^{n} x_i = m$$

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$$m = \mathbb{E}(X) = \mathbb{E}\Big[\mathbb{E}[X|Y]\Big]$$

$$= \sum_{s=1}^k \mathbb{E}(X|Y=s)\mathbb{P}(Y=s)$$

$$=$$

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$$\begin{aligned} m = \mathbb{E}(X) &= \mathbb{E}\Big[\mathbb{E}[X|Y]\Big] \\ &= \sum_{s=1}^k \mathbb{E}(X|Y=s)\mathbb{P}(Y=s) \\ &= \sum_{s=1}^k \left(\frac{1}{n_s}\sum_{i\in A_s} x_i\right)\frac{n_s}{n} = \sum_{s=1}^k m_s \frac{n_s}{n} \end{aligned}$$

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Expectation and Variance of $\mathbb{E}[X|Y]$ and $\text{var}[X|Y]$

	$\mathbb{E}[X\|Y]$	$\text{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\text{var}(X\|Y)\right]$
Variance	$\text{var}\left[\mathbb{E}(X\|Y)\right]$	$\text{var}\left[\text{var}(X\|Y)\right]$

Law of total variance (LTV)

$$\mathrm{var}[X] =$$

Proof.

(1)

(2)

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Law of total variance (LTV)

$$\text{var}[X] = \mathbb{E}\Big[\text{var}(X|Y)\Big] + \text{var}[\mathbb{E}(X|Y)]$$

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$$\text{var}[X] = \mathbb{E}\Big[\text{var}(X|Y)\Big] + \text{var}[\mathbb{E}(X|Y)]$$

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$$\text{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

$$\tag{1}$$

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$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \mathrm{var}[X]$$

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- Same setting as that in page 36
- X: score of a randomly chosen student, Y: section of a student ($\in \{1, \ldots, k\}$)

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  - Weighted average of the section variances
  - average score variability within individual sections
- $\text{var}[\mathbb{E}(X|Y)]$: variability of the average of the differenct sections
  - $\mathbb{E}(X|Y = s)$: average score in section s
  - variability between sections

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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

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Questions?

Review Questions

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
2) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?
3) What are covariance and correlation coefficient? Why do we need them?
4) Please explain the concepts of conditional expectation and conditional variance.