

Lecture 8: Random Processes, Part I

Yi, Yung (이윤)

EE210: Probability and Introductory Random Processes
KAIST EE

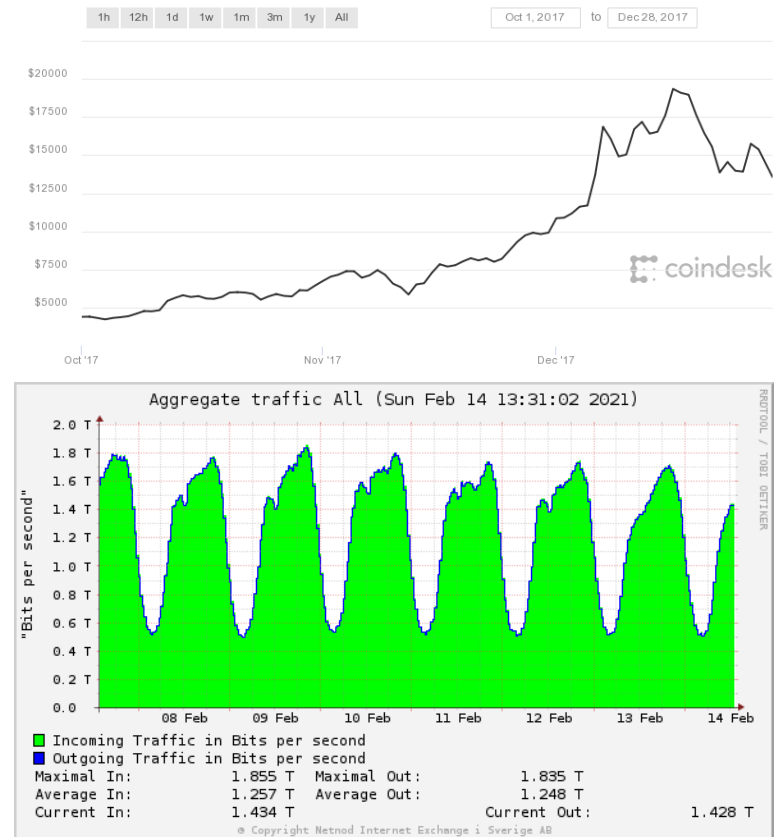
MONTH DAY, 2021

- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
 - Coding of both processes
 - Merge and Split
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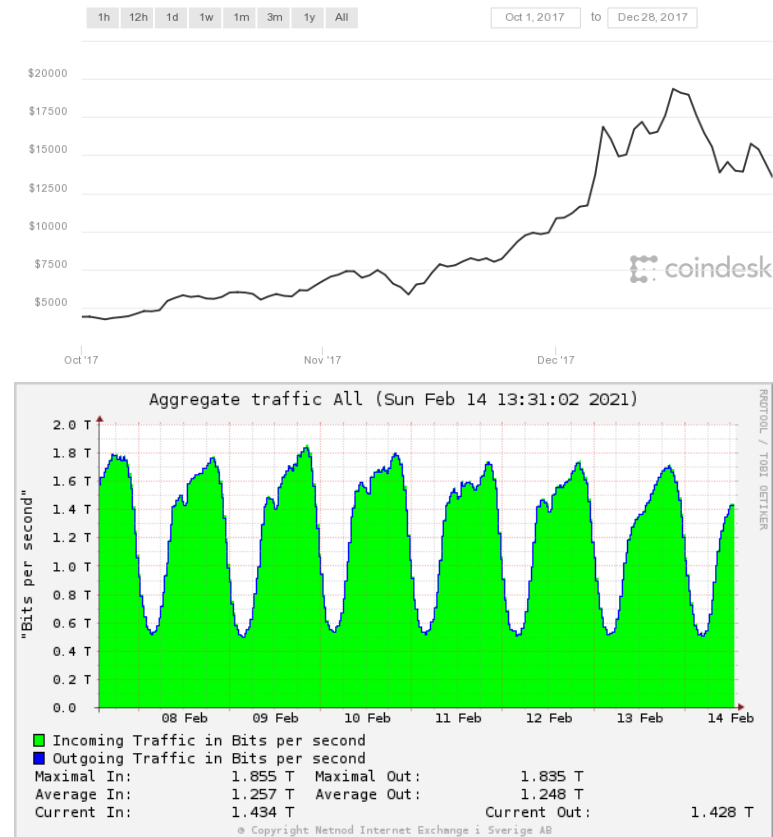
Things that evolve in time

- Many probabilistic experiments that **evolve in time**
- Random process is a mathematical model for it.



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- Many probabilistic experiments that **evolve in time**
 - Sequence of daily prices of a stock
 - Sequence of scores in football
 - Sequence of failure times of a machine
 - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



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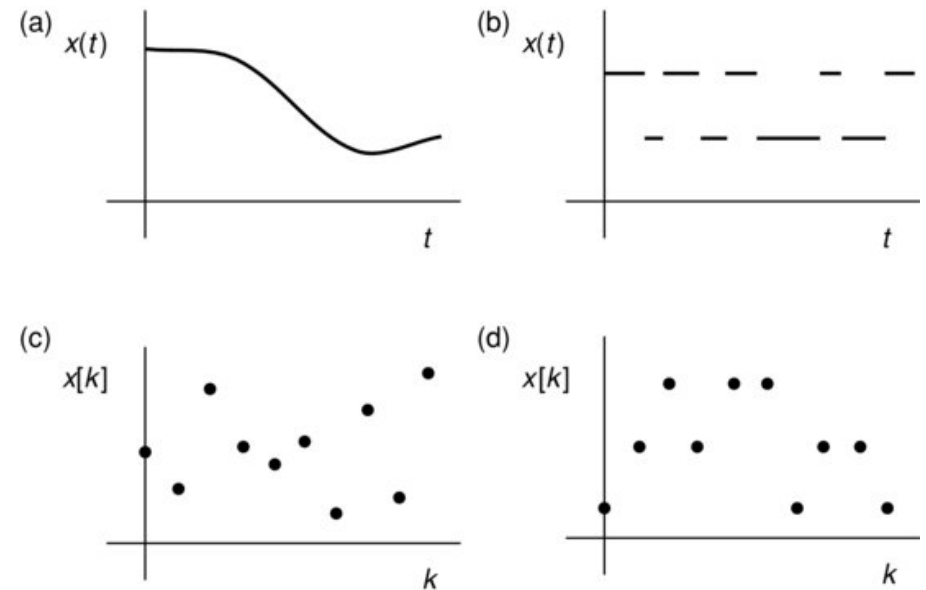
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- The values that X_t can take: discrete or continuous

4 Types of Random Processes

- Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



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- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

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- Next: Key questions and answers about Bernoulli process

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- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

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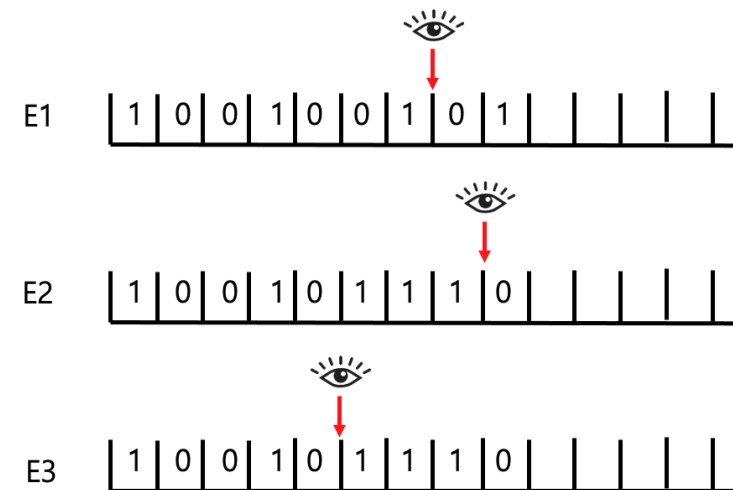
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- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time n
- If you watch the on-going Bernoulli process(p) from some time n , you still see the same Bernoulli process(p).

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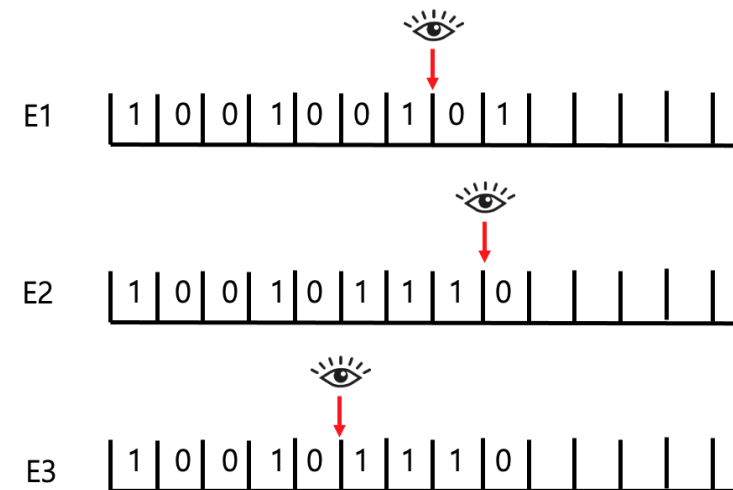
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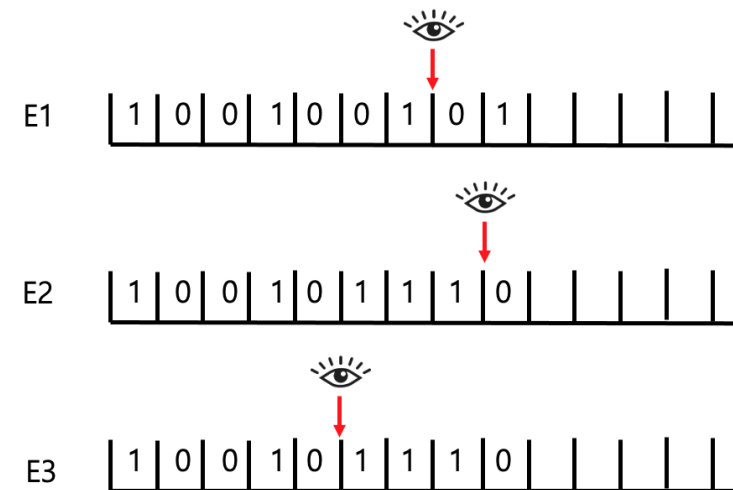


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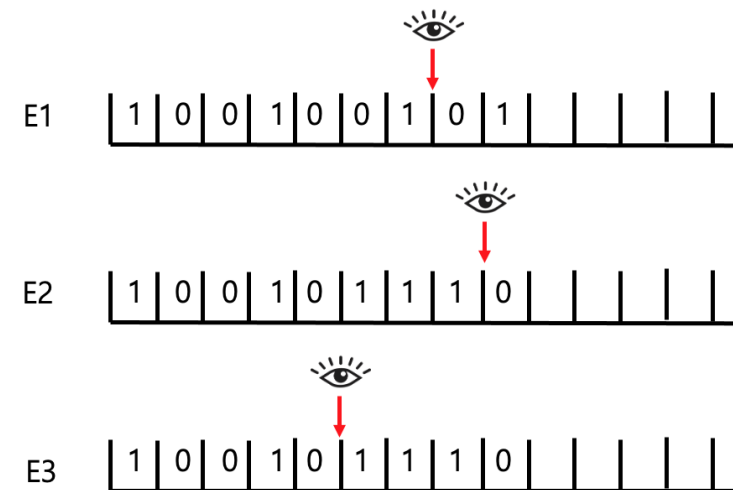
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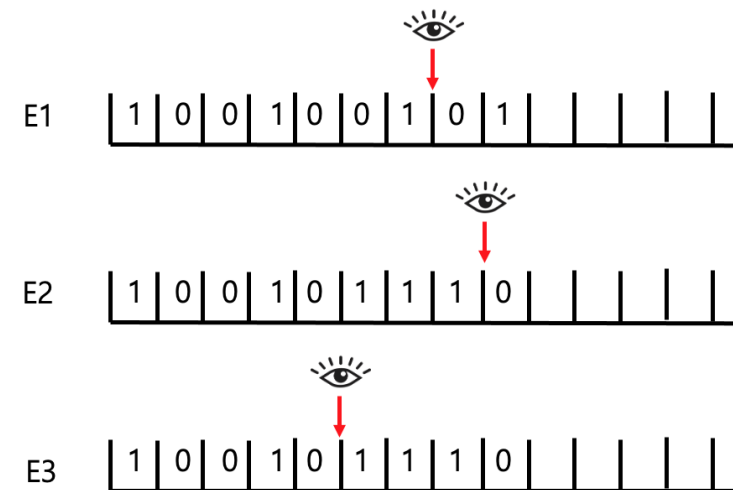
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- Difference of N from n
 - The time when I watch the on-going Bernoulli process is **random**.

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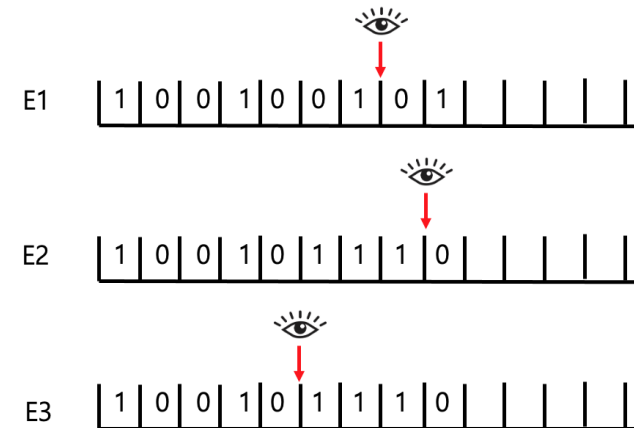
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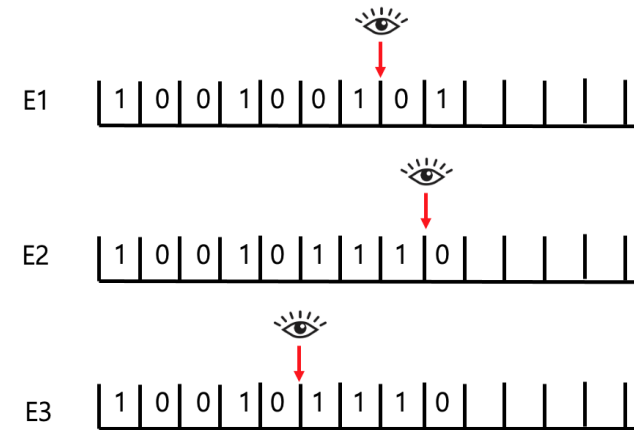
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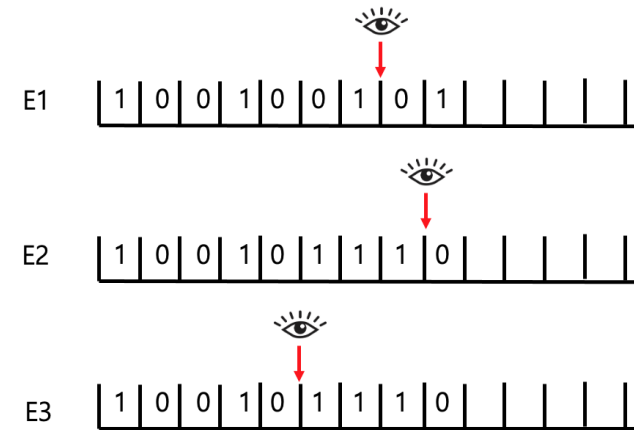
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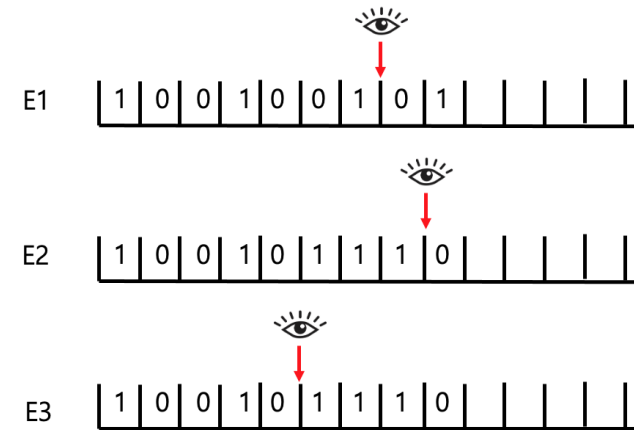
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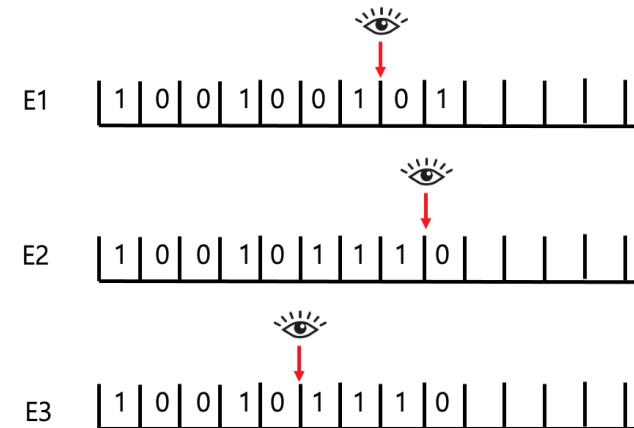
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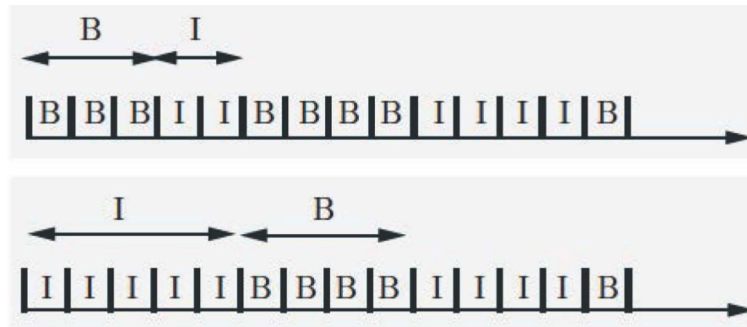
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- The question of $N = n$? can be answered just from the knowledge about X_1, X_2, \dots, X_n ? Then, Yes! (see pp. 301 for more formal description)

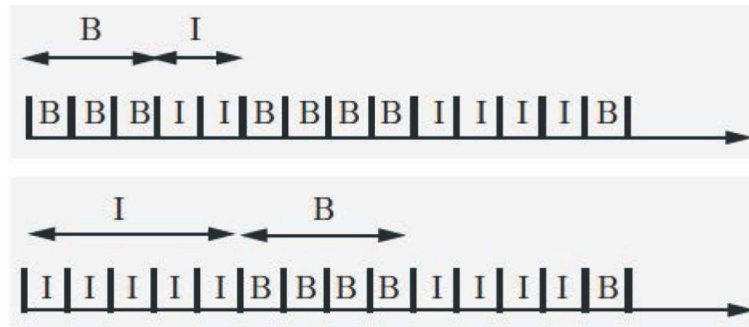
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Distribution of Busy Periods

- Regard an arrival as business of a server
- First busy period B_1 : starts with the first busy slot and ends just before the first subsequent idle slot

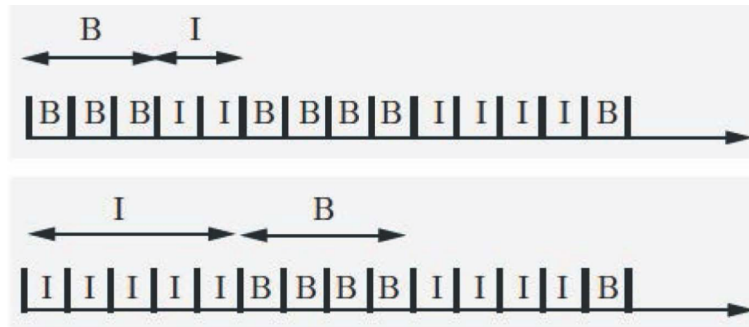


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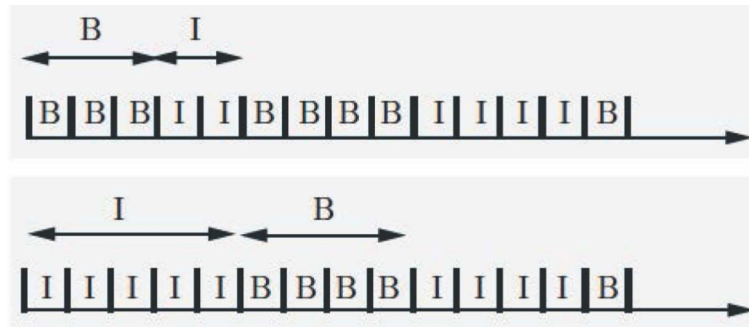
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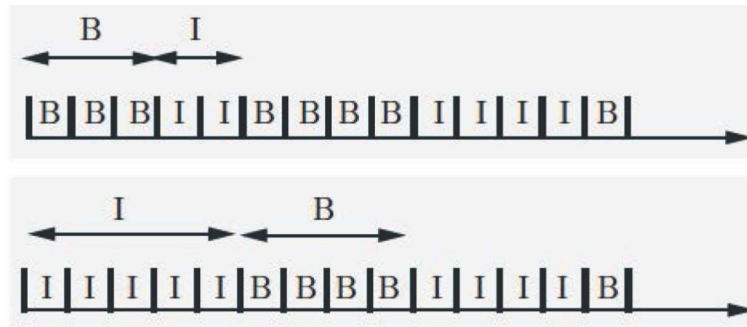
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- (Q6) Distribution of B_1 ?
- N : time of the first busy slot. Fresh-start after N .
- B_1 is geometric with parameter $(1 - p)$
- Question: What about the second busy period B_2 ? B_3 , B_4 ?

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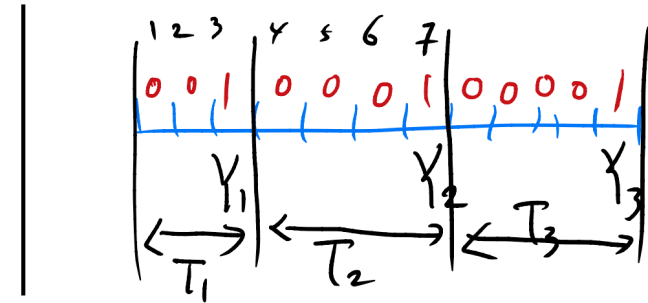
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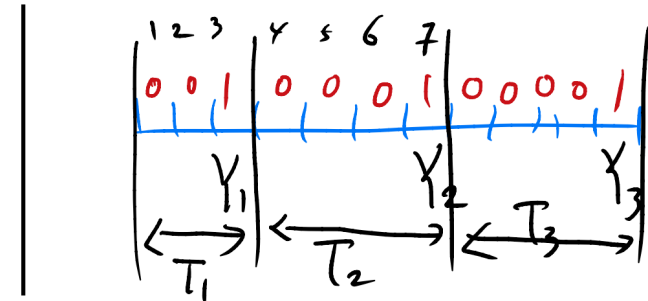


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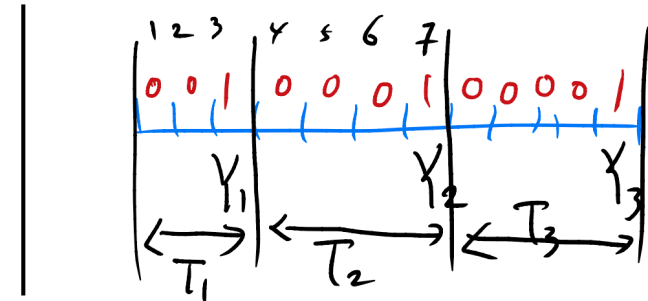


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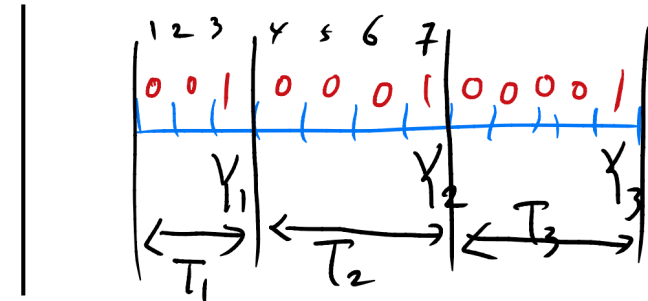
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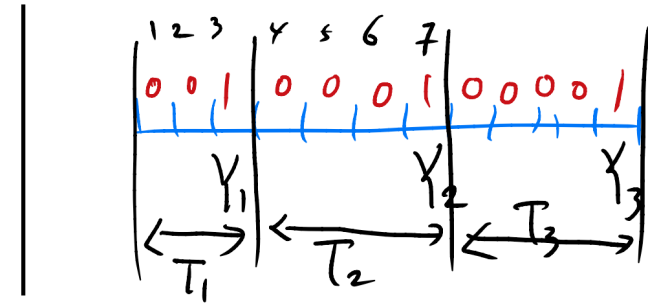
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- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
 - Coding of both processes
 - Merge and Split
- Markov Chain

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- Need a “modeling sense” to make this possible. It’s a good practice for engineers!

- Continuous twin

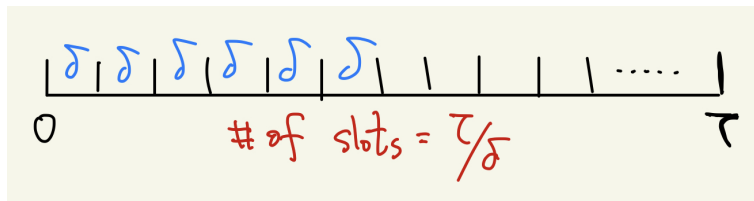
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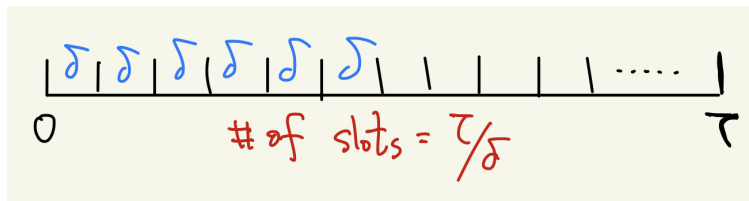
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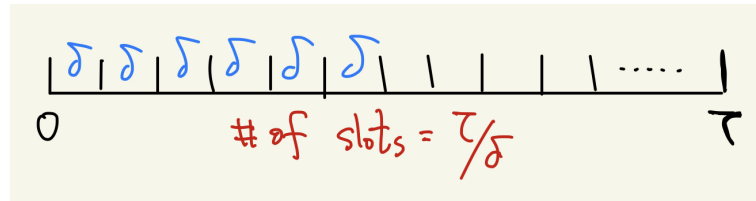
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- What's the limit as $\delta \rightarrow 0$ (equivalently, $n \rightarrow \infty$)

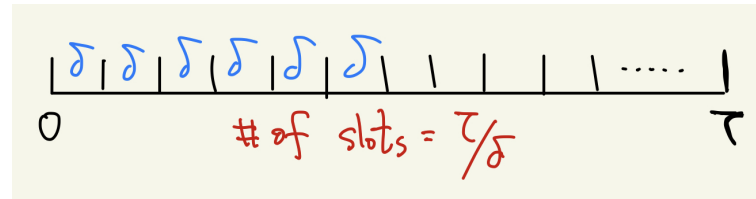
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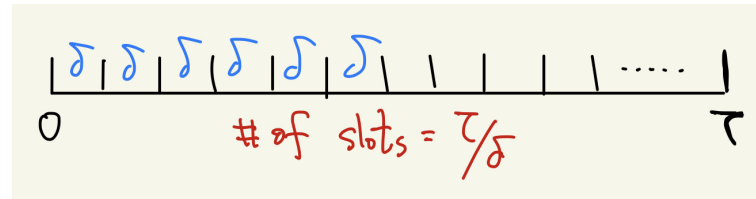
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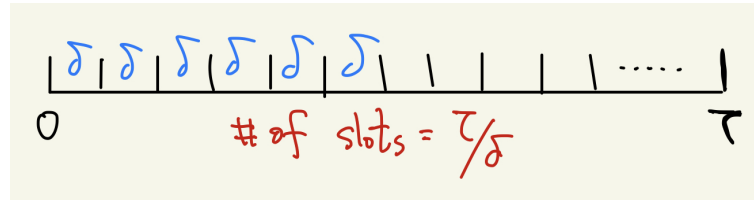


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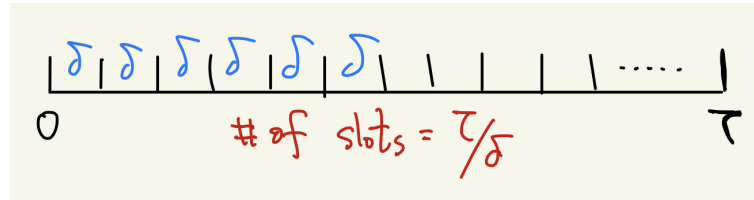
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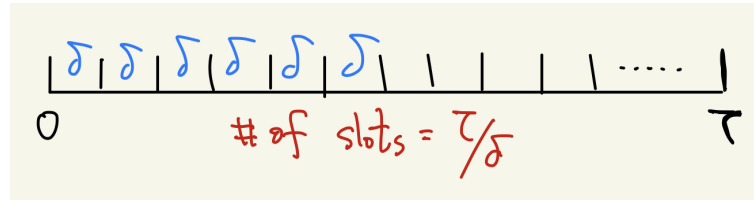
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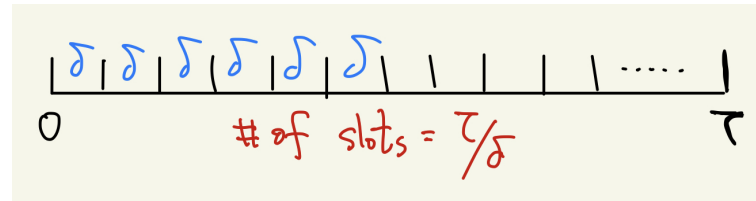
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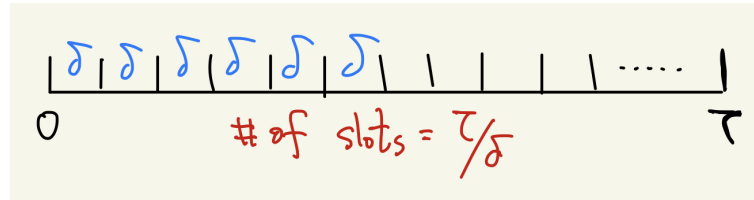
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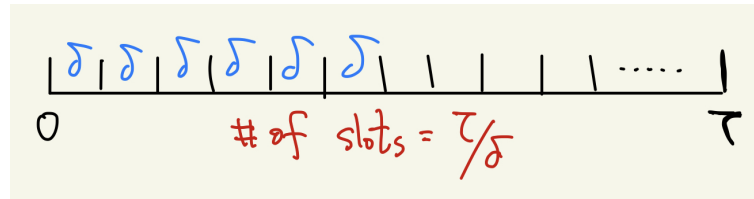


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$$\begin{aligned}\mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\ \mathbb{P}(\geq 2 \text{ arrivals}) &= o(\delta) \\ \mathbb{P}(0 \text{ arrival}) &= 1 - \lambda\delta + o(\delta)\end{aligned}$$

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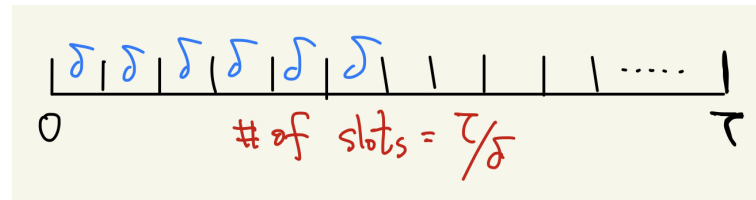
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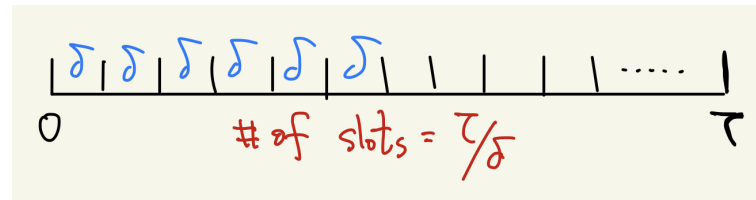
- $o(\delta)$: some function that goes to zero faster than δ goes to zero.
 - Thus, for very small δ , $o(\delta)$ becomes negligible.
 - Example: $o(\delta) = \delta^\alpha$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)



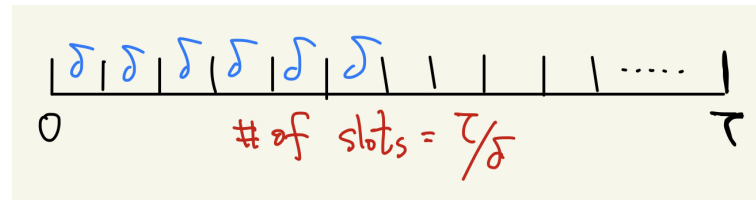
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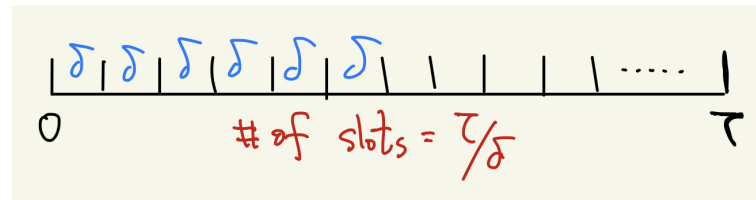
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- As $\delta \rightarrow \infty$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- # of arrivals over $[0, \tau]$, $\sim \text{Poisson}(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call **Poisson process**.

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- **(Small interval probability)** The probabilities $\mathbb{P}(k, s)$ satisfy:

$$\mathbb{P}(0, s) = 1 - \lambda s + o(s)$$

$$\mathbb{P}(1, s) = \lambda s + o_1(s)$$

$$\mathbb{P}(k, s) = o_k(s) \quad \text{for } k = 2, 3, \dots,$$

where

$$\lim_{s \rightarrow 0} \frac{o(s)}{s} = 0, \quad \lim_{s \rightarrow 0} \frac{o_k(s)}{s} = 0$$

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- $T \sim \text{expo}(\lambda)$. Thus $\mathbb{E}[T] = 1/\lambda$ and $\text{var}[T] = 1/\lambda^2$
 - Continuous twin of geometric rv in Bernoulli process
 - Memoryless

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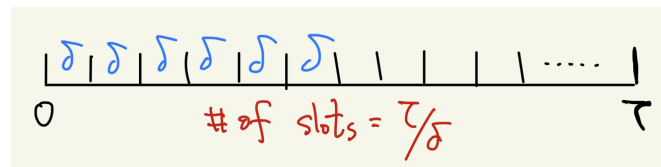
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- Time of first arrival: geometric / exponential
- Time of k -th arrivals: Pascal / Erlang

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ /unit time	p /per trial

Example: Poisson Fishing (Problem 10, page 329)

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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
 - Coding of both processes
 - Merge and Split
- Markov Chain

- Inter-arrival times facilitates coding of both processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, T_2, \dots , with common parameter p , and let these stand for the interarrival times.
2. Record a success (or arrival) at times $T_1, T_1 + T_2, T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

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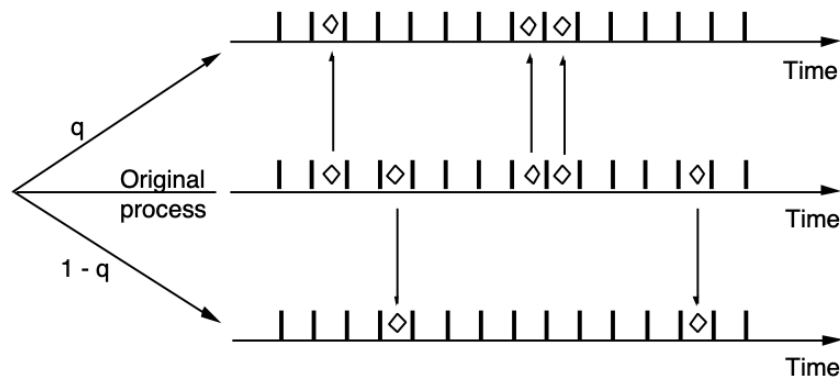
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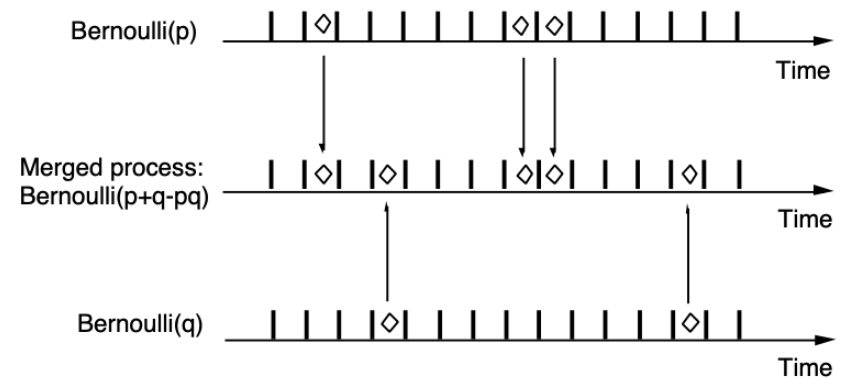
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Split and Merge: Bernoulli Process

- Split $Bernoulli(p)$ into two processes with biased coin of head probability q

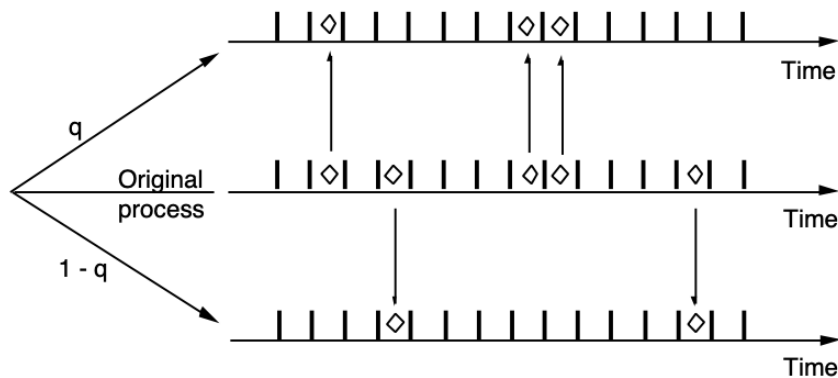


- Merge $Bernoulli(p)$ and $Bernoulli(q)$ into one.

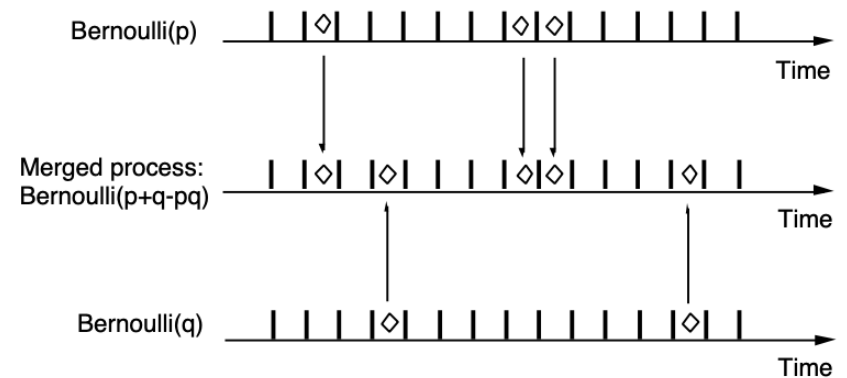


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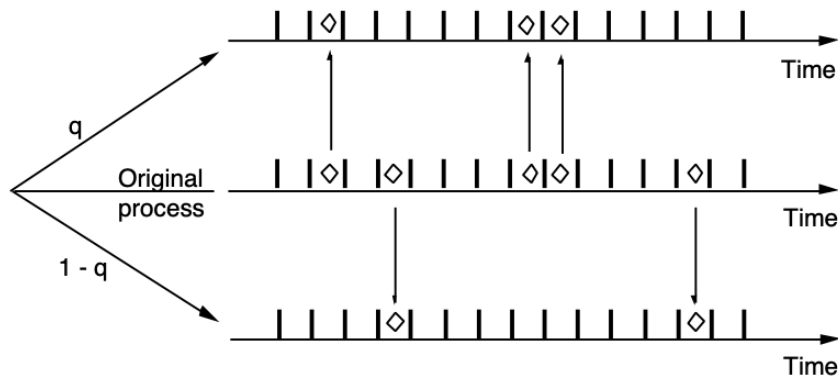


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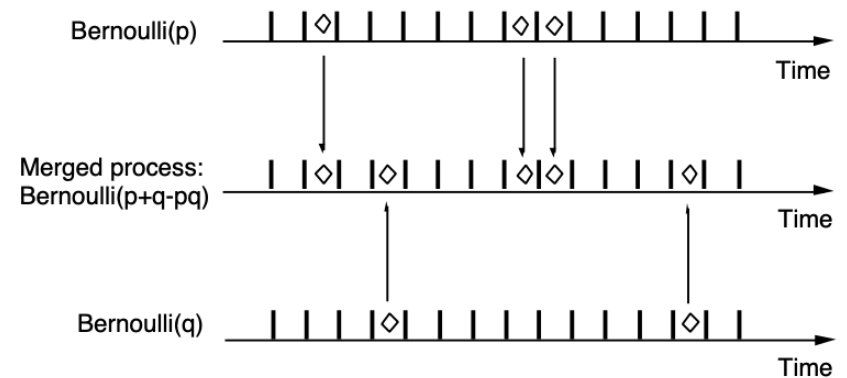


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- Split processes: also Bernoulli processes
- $Bernoulli(pq)$ and $Bernoulli(p(1 - q))$

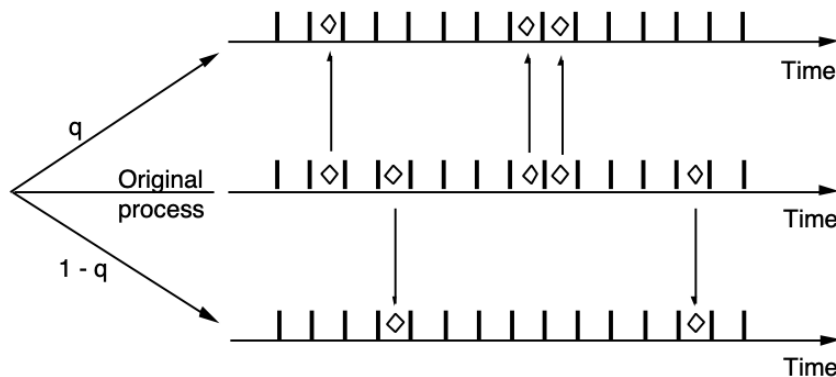


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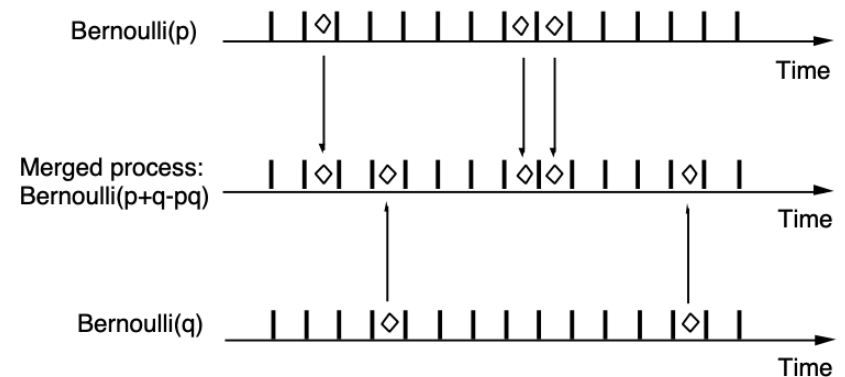


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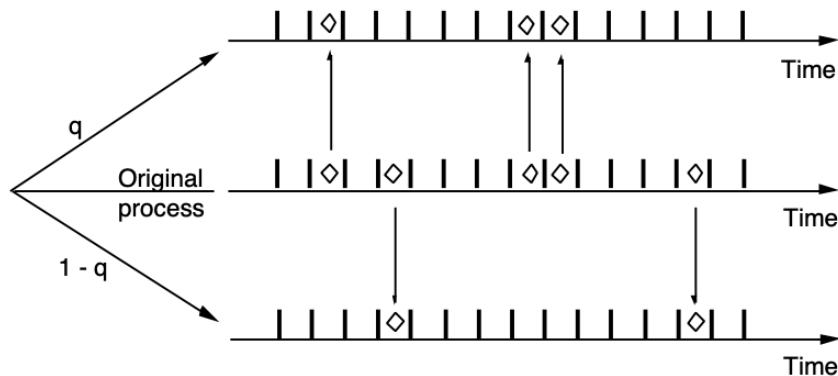


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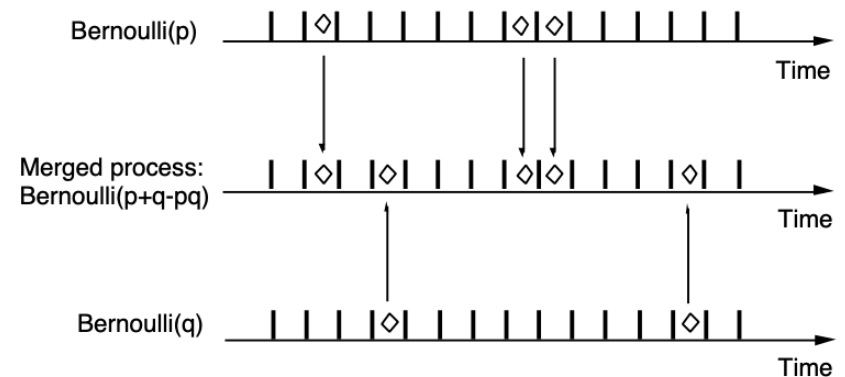


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$$\mathbb{P}(1 \text{ arrivals in the merged process}) \approx \lambda_1\delta(1 - \lambda_2\delta) + \lambda_2\delta(1 - \lambda_1\delta) = (\lambda_1 + \lambda_2)\delta + o(\delta)$$

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$$\mathbb{E}[T_1 + T_2 + T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Questions?

1)