Lecture 4: Random Variable, Part II

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

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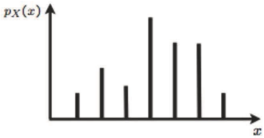

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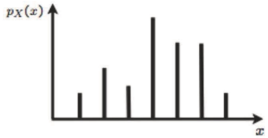

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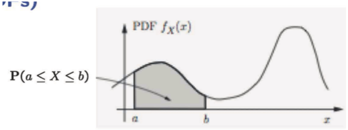
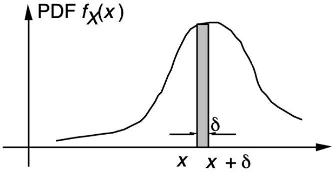


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$
- $f_X(x) \geq 0$, $\int_{-\infty}^{\infty} f_X(x)dx = 1$



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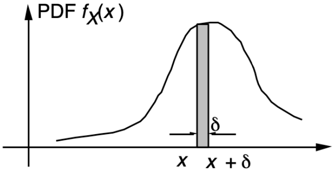
Examples

PDF and Examples



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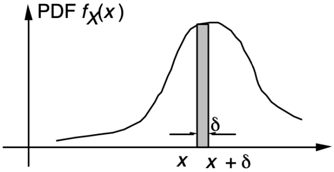

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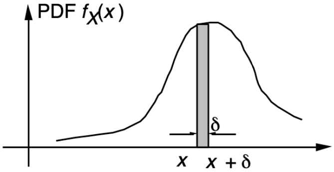

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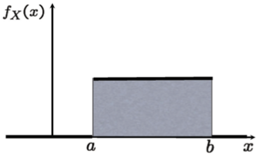

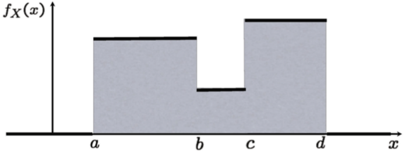

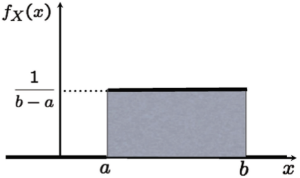

- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx =$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx =$
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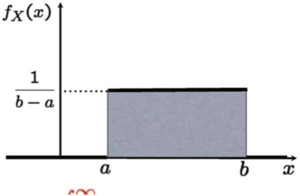

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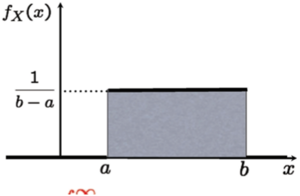
Expectation and Variance



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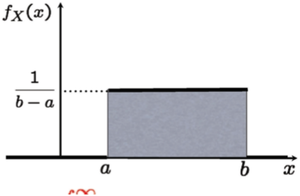

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- $\text{var}[X] = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$

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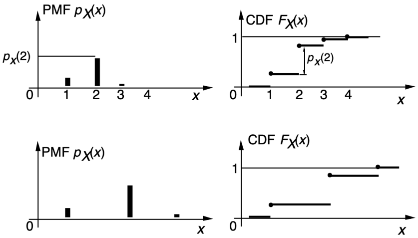

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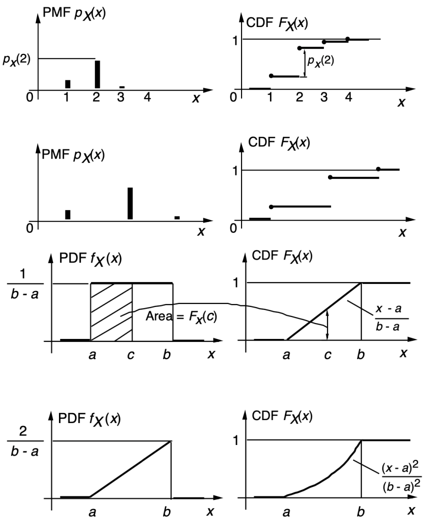

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- If X is continuous
  - $F_X(x)$ is a continuous function of x.
  - $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ and $f_X(x) = \frac{dF_X}{dx}(x)$

Example: Maximum of Random Variables

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$, and $X_i \in \{1, 2, \cdots, 10\}$ uniformly at random
- Question. $p_X(x)$?
- Approach 1: $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$?
- Approach 2

$$
\begin{aligned}
F_X(x) = \mathbb{P}(\max\{X_1, X_2, X_3\} \le x) &= \mathbb{P}(X_1 \le x, X_2 \le x, X_3 \le x) \\
&= \mathbb{P}(X_1 \le x) \cdot \mathbb{P}(X_2 \le x) \cdot \mathbb{P}(X_3 \le x) = \left(\frac{x}{10}\right)^3
\end{aligned}
$$

Thus,

$$
p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \cdots, 10
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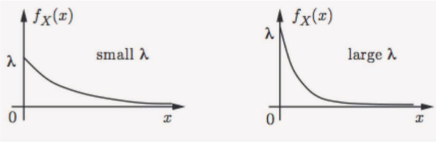

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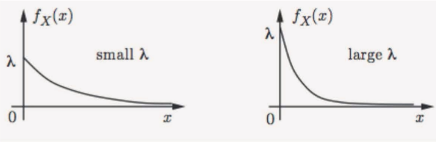

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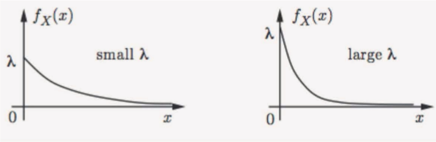

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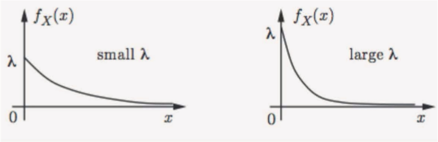

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- (Check) $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$, $\text{var}[X] = 1/\lambda^2$

Exponential RV: Mean and Variance

- $\mathbb{E}(X) = 1/\lambda$. Use integration by parts: $\int udv = uv - \int vdu$

$$\int_0^\infty x\lambda e^{-\lambda x} dx = (-xe^{-\lambda x})\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^\infty x^2\lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx = 0 + \frac{2}{\lambda}\mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

- $\mathbb{P}(X > x) = e^{-\lambda x}$

Exponential RV: Model of Continuous Waiting Time

- $\mathbb{P}(X > x) = e^{-\lambda x}$
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- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

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  - 6 a.m. from midnight = 1/4 day, 6 p.m. from midnight = 3/4 day

$$\mathbb{P}(1/4 \le X \le 3/4) = \mathbb{P}(X \ge 1/4) - \mathbb{P}(X \ge 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

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- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

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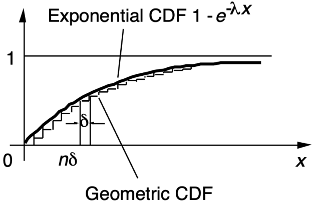
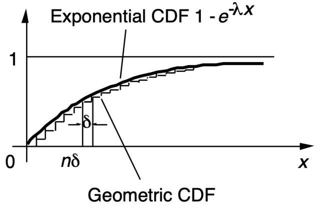
  - $\mathbb{P}(X_\delta^{geo} \leq n) = 1 - (1 - p_\delta)^n = 1 - e^{-\lambda\delta n}$

Geometric vs. Exponential (3)

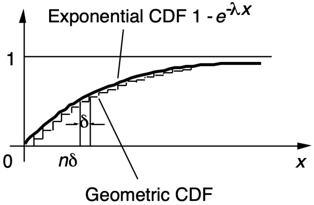




- Note that $\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda x}$. Then, when $x = n\delta$, $n = 1, 2, \dots$

$$\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda \delta n} =$$

Geometric vs. Exponential (3)



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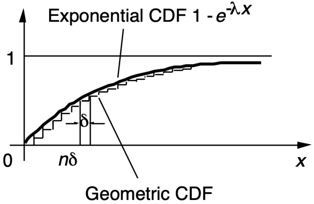
$$\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda \delta n} = \mathbb{P}(X_\delta^{geo} \le n)$$



- Note that $\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda x}$. Then, when $x = n\delta$, $n = 1, 2, \ldots$

$$\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda \delta n} = \mathbb{P}(X_\delta^{geo} \le n)$$

- If we choose sufficiently small δ, the slot length $\downarrow$ and $p_\delta \downarrow$

$$\mathbb{P}(X_\delta^{geo} \le n) \xrightarrow{\delta \to 0} \mathbb{P}(X^{exp} \le x),\ x = n\delta$$

Roadmap

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

Normal: PDF, Expectation, Variance

- Standard Normal $\mathcal{N}(0,1)$

  $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

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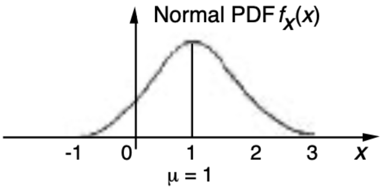
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

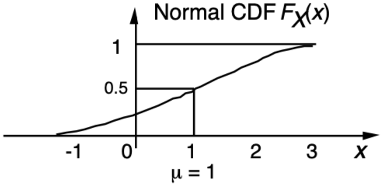
- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

- General Normal $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$





Check: PDF, Expectation, Variance

- PDF's normalization property: $\dfrac{1}{\sigma\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = 1$

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- Expectation
  - $f_X(x)$ is symmetric in terms of $x=\mu$. Thus, we should have $\mathbb{E}(X)=\mu$.
- Variance

$$
\begin{aligned}
\text{var}(X) &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}dx \overset{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2}dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\left(-ye^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = \sigma^2
\end{aligned}
$$

$\int udv = uv - \int vdu$: $u=y$ and $dv = ye^{-y^2/2} \rightarrow du = dy$ and $v = -e^{-y^2/2}$

Normality: Preserved under Linear Transformation

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- Linear transformation preserves normality (we will verify this in Lecture 5)

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Linear transformation preserves normality (we will verify this in Lecture 5)

> If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be $\boxed{}$:

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{} \sim \mathcal{N}(0, 1)$

- Linear transformation preserves normality (we will verify this in Lecture 5)

$$\boxed{\text{If } X \sim \mathcal{N}(\mu, \sigma^2), \text{ then for } a \neq 0 \text{ and } b,\ Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2).}$$

- Thus, every normal rv can be $\boxed{\text{standardized}}$:

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{Y = \frac{X-\mu}{\sigma}} \sim \mathcal{N}(0, 1)$

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  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{Y = \frac{X-\mu}{\sigma}} \sim \mathcal{N}(0, 1)$

- Thus, we can make the table which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$$

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

KAIST EE

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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
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2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
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2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
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- $Y = \frac{X-60}{20}$.

$$\mathbb{P}(X \geq 80) = \mathbb{P}(Y \geq \frac{80-60}{20})$$
$$= \mathbb{P}(Y \geq 1) = 1 - \Phi(1)$$
$$= 1 - 0.8413 = 0.1587$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Normal RVs: Why Important?

- Central limit theorem
  - One of the most remarkable findings in the probability theory
  - Sum of any random variables $\approx$ Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

[0] Central limit theorem: 중심극한정리

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

Continuous: Joint PDF and CDF (1)

Two continuous rvs are $\boxed{}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy,$$

Two continuous rvs are $\boxed{\text{jointly continuous}}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

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Two continuous rvs are $\boxed{\text{jointly continuous}}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}\Big[(X,Y) \in B\Big] = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

Our particular interest: $B = \{(x,y) \mid a \le x \le b, c \le y \le d\}$

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Continuous: Joint PDF and CDF (2)

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

2. The marginal PDFs of X and Y are from the joint PDF as:
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$
3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$, and determines the joint PDF as:
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$
4. A function $g(X,Y)$ of X and Y defines a new random variable, and
$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$$

Continuous: Conditional PDF given an event

* Conditional PDF, given an event A

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

Continuous: Conditional PDF given an event

* Conditional PDF, given an event A

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

Continuous: Conditional PDF given an event

* Conditional PDF, given an event A

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
  $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

Continuous: Conditional PDF given an event

* Conditional PDF, given an event A

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta)$
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$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if} \quad x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if} \quad x \in C \end{cases}$$

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(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

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Continuous: Conditional Expectation

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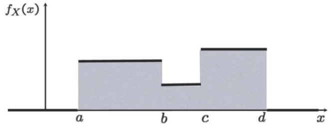
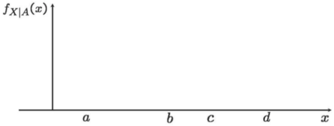
$A = \{\frac{a+b}{2} \leq X \leq b\}$





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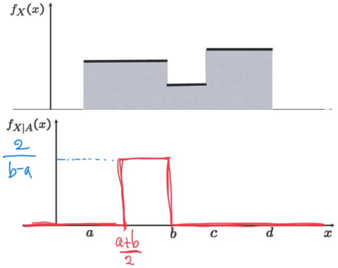

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$$\mathbb{E}[X|A] = \int_{(a+b)/2}^{b} x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

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$$\mathbb{P}(X > n + m | X > m) = \frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > m)} = \frac{e^{-\lambda(n+m)}}{e^{-\lambda m}} = e^{-\lambda n} = \mathbb{P}(X > n)$$

Total Probability/Expectation Theorem

Partition of Ω into $A_1, A_2, A_3, \ldots$

* Discrete case	* Continuous case

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Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i)$$
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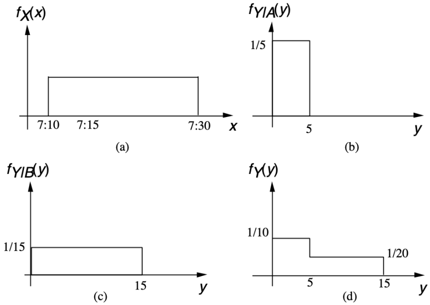
- The train's arrival every quarter hour (0, 15min, 30min, 45min).
- Your arrival $\sim \mathcal{U}(7{:}10, 7{:}30)$ am.



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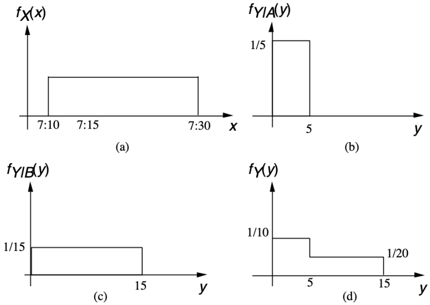

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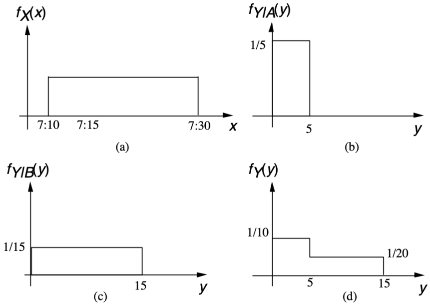

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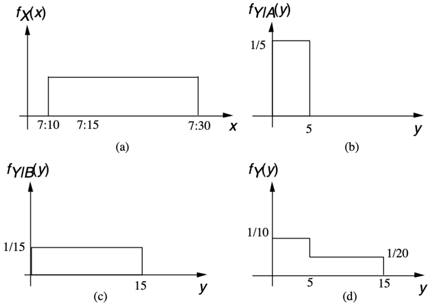

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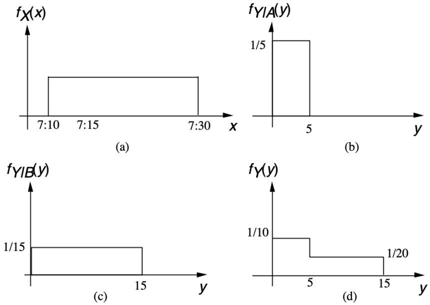

VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A) f_{Y|A}(y) + \mathbb{P}(B) f_{Y|B}(y)$$

for $0 \le y \le 5$

for $5 < y \le 15$

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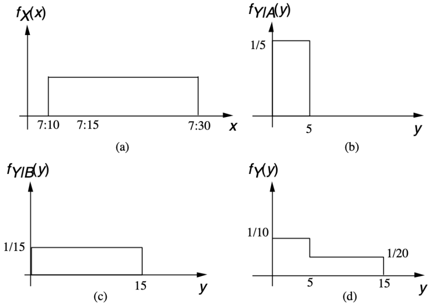

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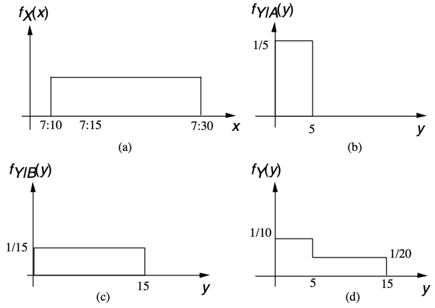

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- Independence

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x \text{ and } y$$

Example: Stick-breaking (1)

(Prob 21 at pp. 191)

- Break a stick of length l twice
  - first break at $Y \sim \mathcal{U}[0, l]$
  - second break at $X \sim \mathcal{U}[0, Y]$

[0] $\mathcal{U}[a, b]$: continuous uniform random variable over the interval $[a, b]$

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(a) joint PDF $f_{X,Y}(x, y)$?

$$f_Y(y) = \frac{1}{l}, \quad 0 \le y \le 1$$

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Using $f_{X,Y}(x, y) = f_Y(y) f_{X|Y}(x|y)$,

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{l} \cdot \frac{1}{y}, & 0 \le x \le y \le l, \\ 0, & \text{otherwise} \end{cases}$$

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(b) marginal PDF $f_X(x)$?

$$f_X(x) = \int f_{X,Y}(x, y) dy = \int_x^l \frac{1}{ly} dy$$

$$= \frac{1}{l} \ln(l/x), \quad 0 \le x \le l$$

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Example: Stick-breaking (2)

(c) Evaluate $\mathbb{E}(X)$, using $f_X(x)$

(d) Evaluate $\mathbb{E}(X)$, using $X = Y \cdot (X/Y)$

If $Y \perp\!\!\!\perp X/Y$, it becomes easy, but true?

(e) Evaluate $\mathbb{E}(X)$, using TET

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If $Y \perp\!\!\!\perp X/Y$, it becomes easy, but true?
Yes, because whatever Y is, the fraction X/Y does not depend on it.

$$\mathbb{E}(X) = \mathbb{E}(Y)\mathbb{E}(X/Y) = \frac{l}{2} \cdot \frac{1}{2} = \frac{l}{4}$$

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- Message. There are many ways to rearch our goal. Of crucial importance is how to find the best way!

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

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Using Bayes Rule for Parameter Learning

- X: **parameter** $\rightarrow$ Y: result of **my model**
- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (parameter $\rightarrow$ model)
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1. Light bulb's lifetime $Y \sim \exp(\lambda)$. Given the lifetime y, the modified belief about λ?
2. Romeo and Juliet start dating, but Romeo will be late by a random variable $Y \sim \mathcal{U}[0, \theta]$. Given the time of being late y, the modified belief about θ?

K: discrete, Y: continuous

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- Inference of K given Y

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Bayes Rule for Mixed Case

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- Wait! $p_{K|Y}(k|y)$? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

- For small δ (in other words, taking the limit as $\delta \to 0$).

  Let $A = \{K = k\}$.

$$
\begin{aligned}
p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \le Y \le y + \delta) \\
&= \frac{\mathbb{P}(A)\mathbb{P}(y \le Y \le y + \delta|A)}{\mathbb{P}(y \le Y \le y + \delta)} \\
&\approx \frac{\mathbb{P}(A) f_{Y|A}(y)\delta}{f_Y(y)\delta} \\
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- Y: measured signal with Gaussian noise, $Y = K + W$, $W \sim \mathcal{N}(0,1)$
- Your received signal = 0.7. What's your guess about the original signal? +1
- Your received signal = -0.2. What's your guess about the original signal? -1
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

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- $Y|\{K=1\} \sim \mathcal{N}(1,1)$ and $Y|\{K=-1\} \sim \mathcal{N}(-1,1)$.
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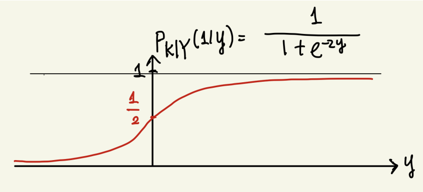

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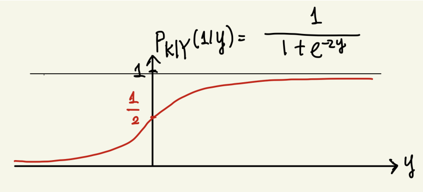

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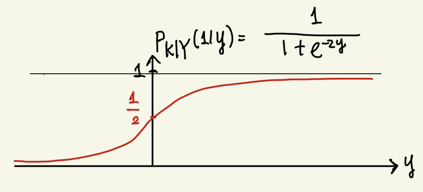
  - Similarly, compute $p_{K|Y}(-1|y)$ and then do the inference



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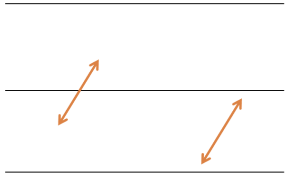

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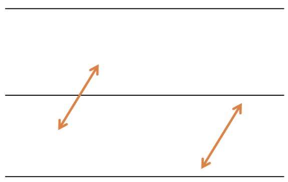

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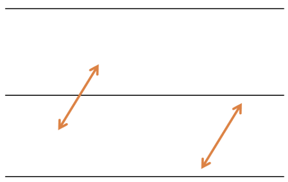

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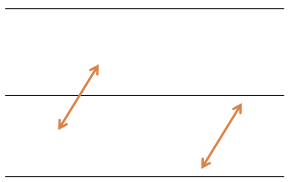

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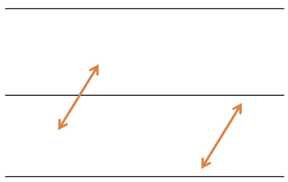

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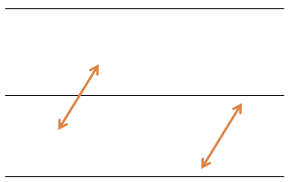

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X: vertical distance from the midpoint of the needle to the nearest of the parallel lines

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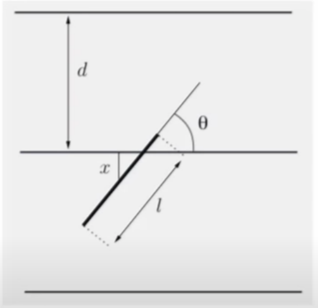

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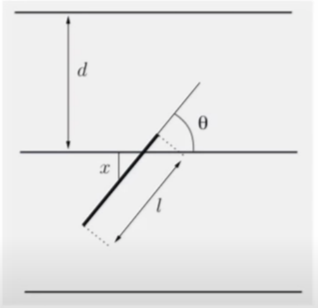
Θ: acute angle formed by the axis of the needle and the parallel lines



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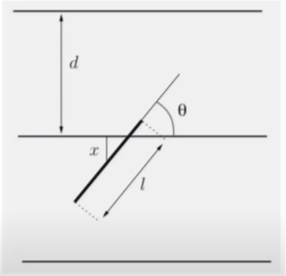

- A random vector (X, Θ) with a uniform joint PDF over $\{(x, \theta) | 0 \le x \le d/2,\ 0 \le \theta \le \pi/2\}$ with

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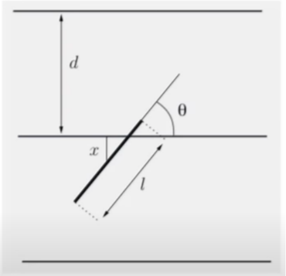

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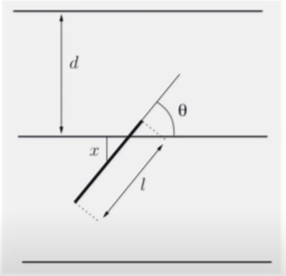

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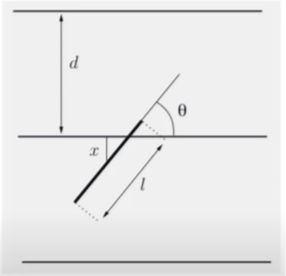

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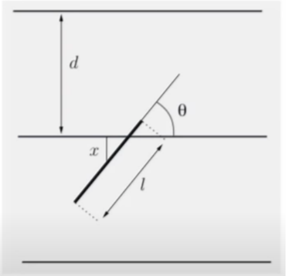

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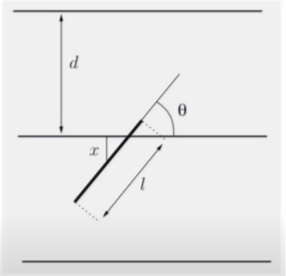
- When does the needle intersect a line? $X \le \frac{l}{2} \sin \Theta$
- The probability of intersection

$$\mathbb{P}(X \le \frac{l}{2} \sin \Theta) = \iint_{x \le (l/2) \sin \theta} f_{X,\Theta}(x, \theta)\ dx\ d\theta$$



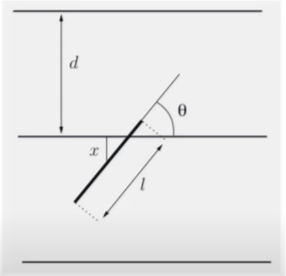
- A random vector (X, Θ) with a uniform joint PDF over $\{(x,\theta) | 0 \le x \le d/2,\ 0 \le \theta \le \pi/2\}$ with

$$f_{X,\Theta}(x,\theta) = \begin{cases} 4/(\pi d), & \text{if } x \in [0, d/2] \text{ and } \theta \in [0, \pi/2], \\ 0, & \text{otherwise} \end{cases}$$

- When does the needle intersect a line? $X \le \frac{l}{2} \sin \Theta$
- The probability of intersection

$$\mathbb{P}(X \le \frac{l}{2} \sin \Theta) = \iint_{x \le (l/2) \sin \theta} f_{X,\Theta}(x,\theta)\ dx\ d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \int_0^{(l/2)\sin\theta} dx d\theta = \frac{4}{\pi d} \int_0^{\pi/2} \frac{l}{2} \sin\theta d\theta = \frac{2l}{\pi d} (-\cos\theta) \Big|_0^{\pi/2} = \frac{2l}{\pi d}$$



Example: Buffon's Needle (2)

- A random vector (X, Θ) with a uniform joint PDF over $\{(x,\theta) | 0 \le x \le d/2,\ 0 \le \theta \le \pi/2\}$ with

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- When does the needle intersect a line? $X \le \frac{l}{2} \sin \Theta$
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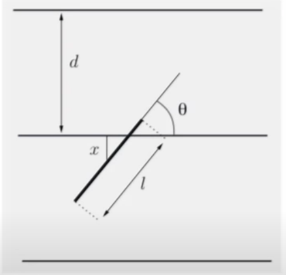
- In history, a method for the experimental evaluation of π.



Questions?

Review Questions

1) What do we mean by "continuous" in continuous random variables?
2) Explain PDF and CDF. Why do we need CDF?
3) What are joint/marginal/conditional PDFs?
4) Explain the relationship between Geometric rvs and Exponential rvs.
5) Explain how normality is preserved under linear transformaion for Normal (Gaussian) rvs.
6) Explain how we can use Bayes' rule for parameter learning.
7) Explain the version of Bayes' rule for continuous and mixed random variables.