Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

May 13, 2021

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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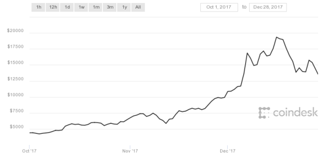
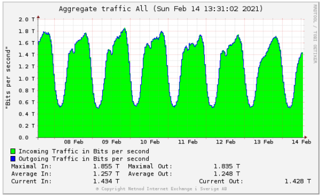
(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

- Many probabilistic experiments that evolve in time



(a) Prices of a crytocurrency



(b) Internet traffic traces

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
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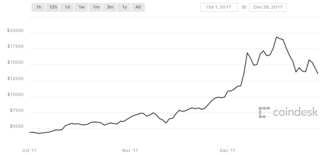

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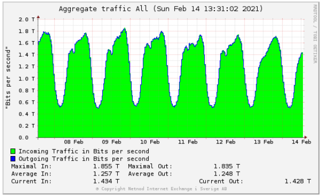

(b) Internet traffic traces

- Many probabilistic experiments that evolve in time
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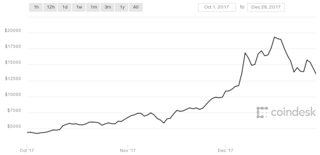
- Random process is a mathematical model for it.



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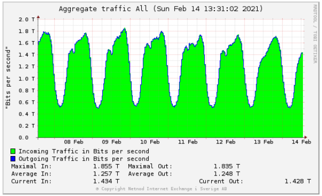

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- The values that X_t (or $X(t)$) can take: discrete or continuous

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  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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  - Other interesting questions, depending on the target random process

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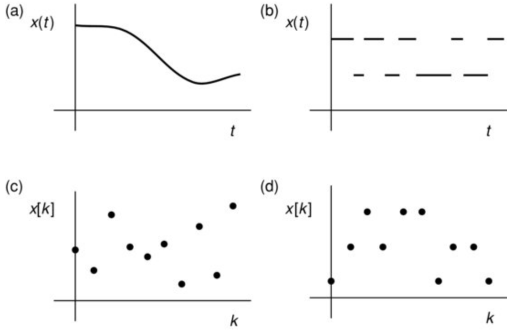
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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- discrete time

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Simeon Denis Poisson (1781 - 1840), France

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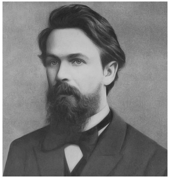

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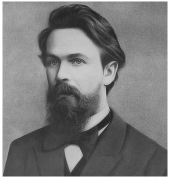

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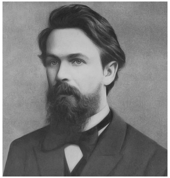

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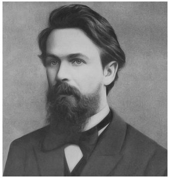

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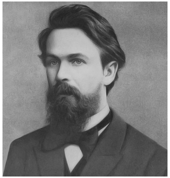

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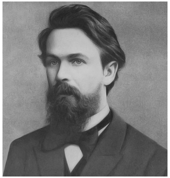

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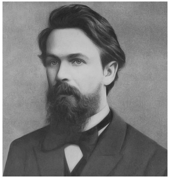

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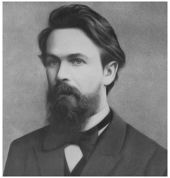

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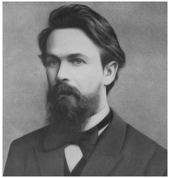

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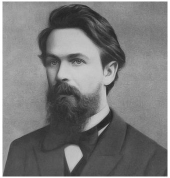

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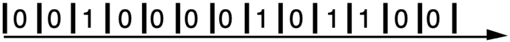

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- Discrete time, discrete value

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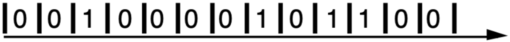

- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Please pause the video and write down the questions that you want to ask about Bernoulli process. VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

(Q1) # of arrivals in the first n slots?

Number of arrivals and Time until the first arrival

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- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots $\Longrightarrow$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

Memoryless and Fresh-start after Deterministic time n

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- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

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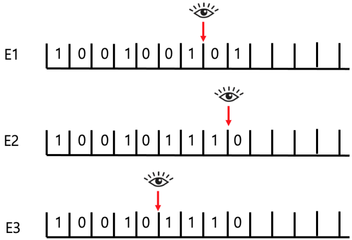

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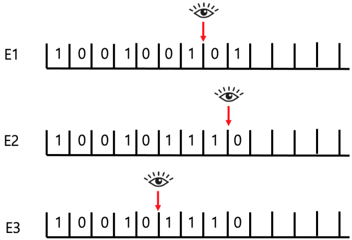

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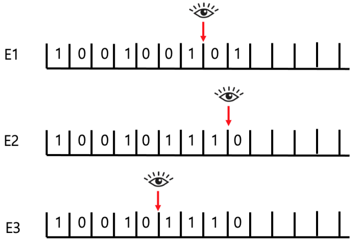

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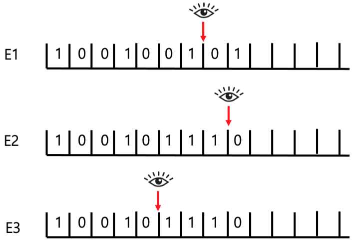

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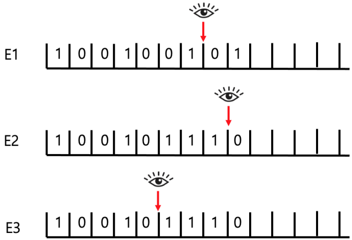

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  - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω?

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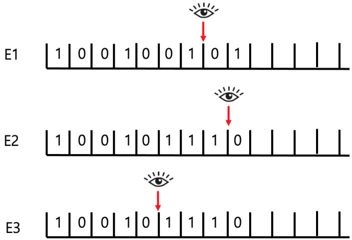

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- Do we experience the fresh-start for any N? **E1, E2, and E3?**

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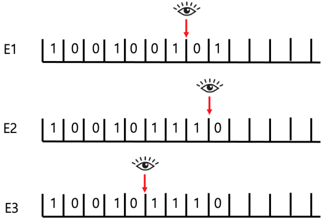

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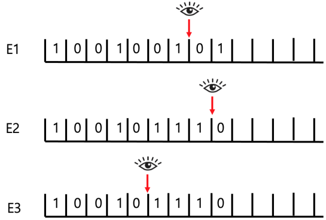

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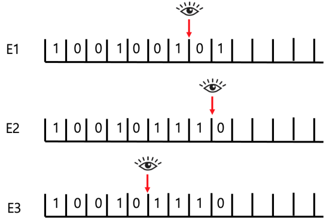

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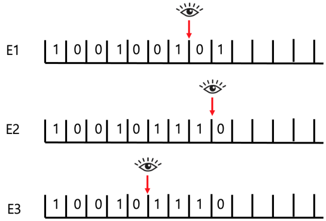

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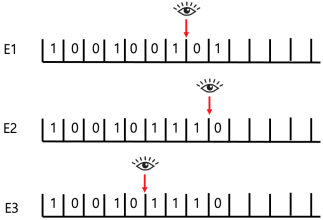

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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

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  - No. Time of 2nd arrival in 10 consecutive arrivals

Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)

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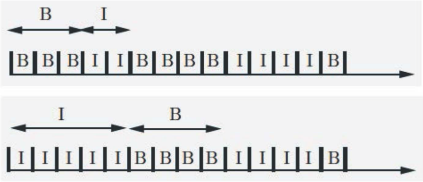

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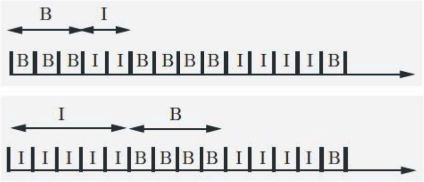

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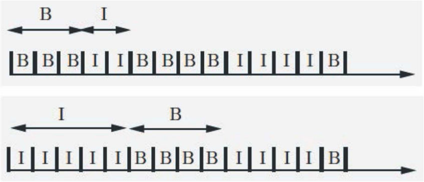

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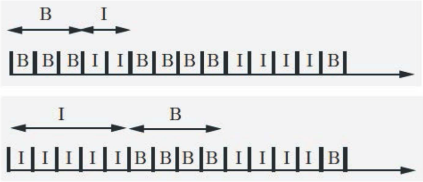

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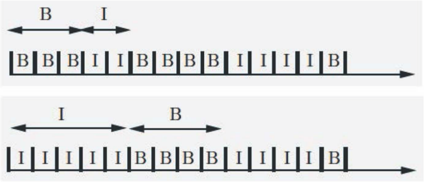

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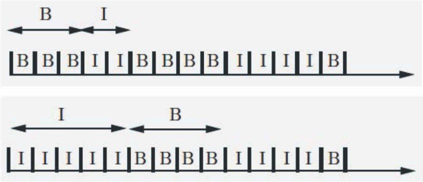
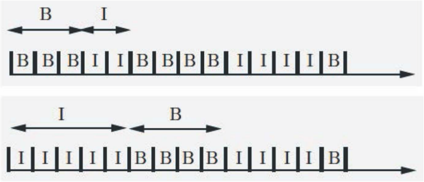
- B_1 is geometric with parameter $(1 - p)$

Distribution of Busy Periods (2)



- Question. What about the second busy period B_2?

Distribution of Busy Periods (2)



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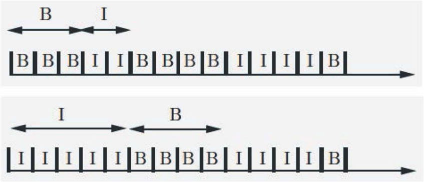

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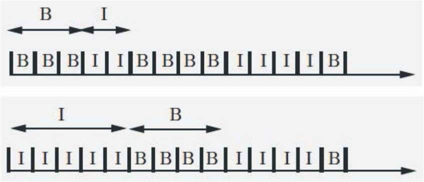

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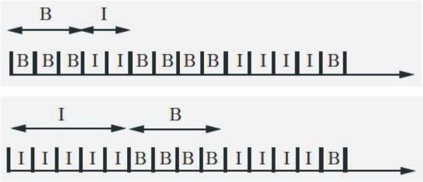

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- $B_3, B_4, \ldots$?

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VIDEO PAUSE

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VIDEO PAUSE

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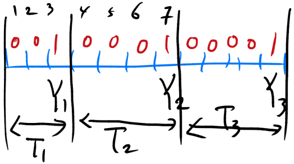

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VIDEO PAUSE

- $T_k = Y_k - Y_{k-1}$: k-th inter-arrival ($k \geq 2$, $T_1 = Y_1$)
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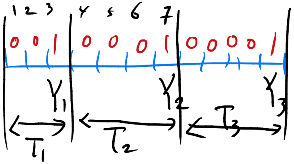

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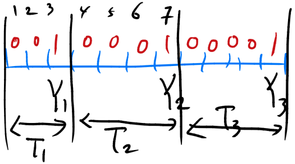

- After each T_k, the fresh-start occurs.

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VIDEO PAUSE

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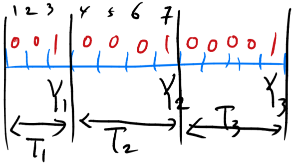

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Time of k-th arrival

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

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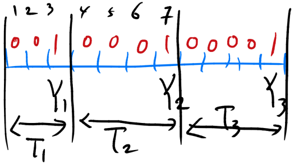

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- We know Y_k's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but its distribution?

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PMF of Y_k

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- $\text{Pascal}(1, p) = \text{Geom}(p)$

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Background: Poisson rv X with parameter λ (1)

- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

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- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

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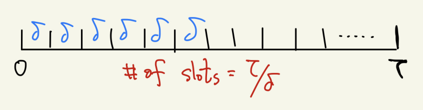

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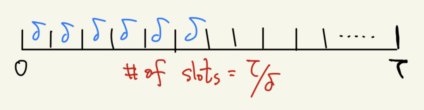
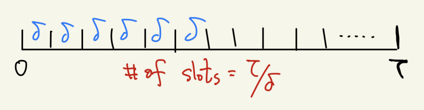


- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)



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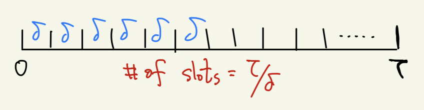

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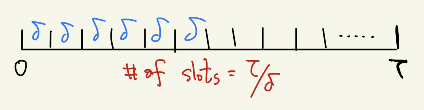

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$\mathbb{P}$(1 arrival) $\propto$ arrival rate and slot length

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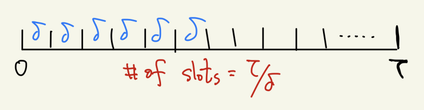

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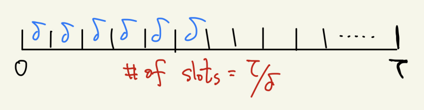

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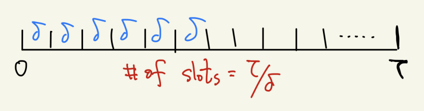

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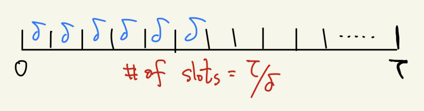

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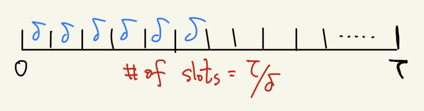

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\mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\
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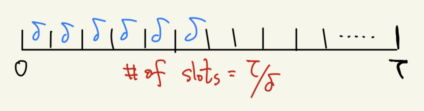

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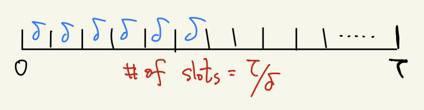
- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$



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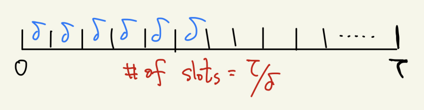

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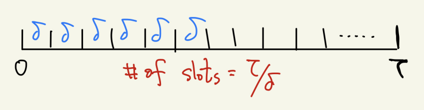

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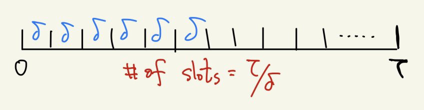

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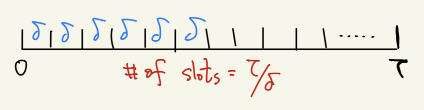
- # of arrivals over $[0, \tau]$, $\sim$ Poisson$(\lambda\tau)$



- Our interest: probability of k arrivals over $[0, \tau]$
- Given "small" δ, # of arrivals $\sim$ Bin(n, p), where $n = \tau/\delta$ and $p = \lambda\delta$
- As $\delta \to 0$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- # of arrivals over $[0, \tau]$, $\sim$ Poisson$(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call Poisson process.

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$
$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$
$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \ldots, \quad \text{where} \quad \lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

Poisson Process: Definition (2)

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k, \tau) = e^{\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
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- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

PDF of Y_k

- For a given δ, $\boxed{}$: prob. of k-th arrival over $[y, y+\delta]$.

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- This is called Erlang rv.

> An Erlang random variable Z with parameter (k, λ) has the following pdf:
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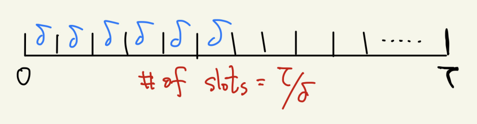

	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals		
Interarrival time		
Time of k-th arrival		
Arrival rate		

Poisson Process vs. Bernoulli Process

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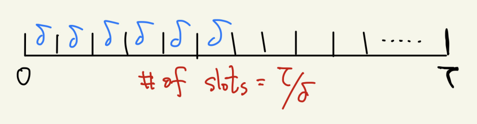

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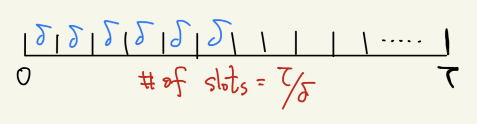

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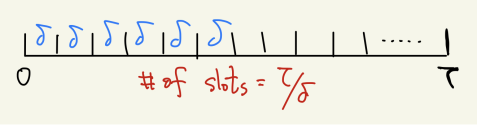

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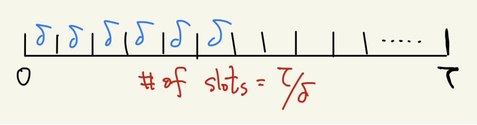

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Arrival rate	p/per slot	λ/unit time

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KAIST EE

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Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

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(Q4) $\mathbb{E}[\text{future fi. time}|\text{already fished for 3h}]$
Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

(Q5) $\mathbb{E}[\text{F=total fishing time}]$

Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6/\text{hour}$.
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

(Q1) $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1: $\mathbb{P}(0, 2)$
Method 2: $\mathbb{P}(T_1 > 2)$

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$$= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda}$$

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\end{aligned}
$$

(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

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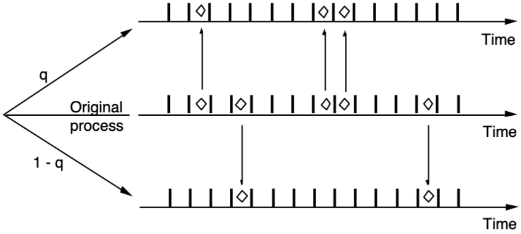
- Thus, the answer is p^2.

- Question. How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.
- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

Notations In the Rest of These Slides

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1 - q)$.



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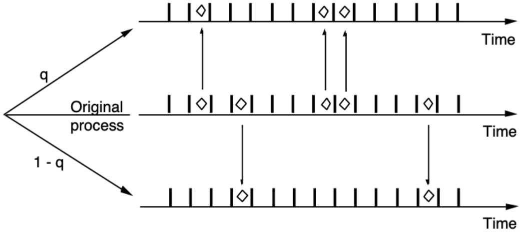

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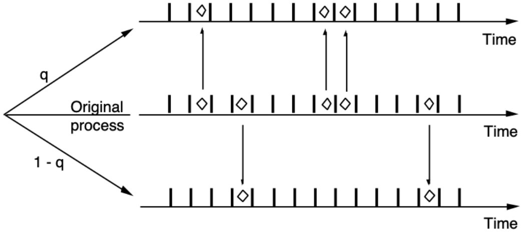

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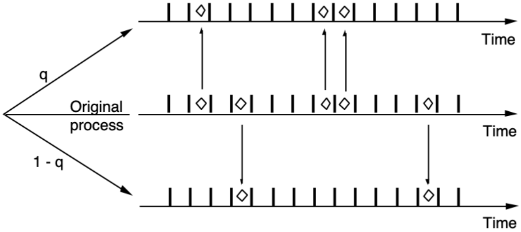

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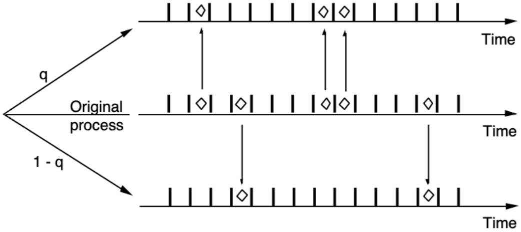
- Are they independent? No.



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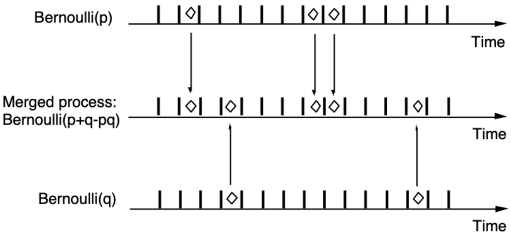

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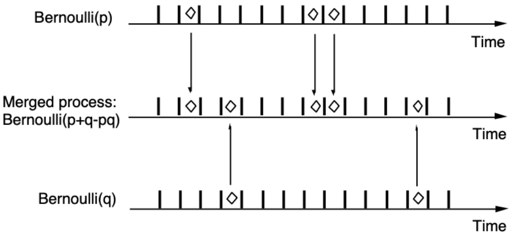

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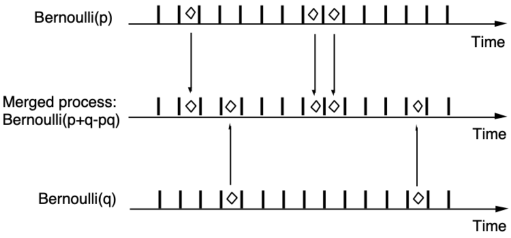

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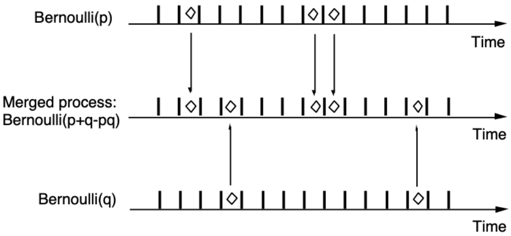

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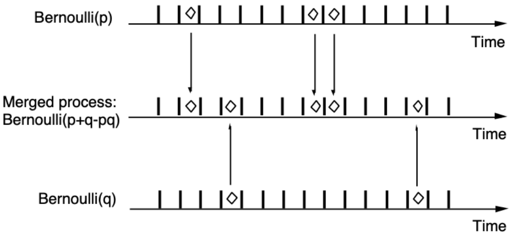

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  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
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1. Competing exponentials

2. Sum of independent Poisson rvs

3. Poisson arrivals during and Exponential interval

Competing Exponential (1)

- Two independent light bulbs have life times $T_a \sim \text{Exp}(\lambda_a)$ and $T_b \sim \text{Exp}(\lambda_b)$.

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  - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
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$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Sum of Independent Poisson rvs

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  - Two intervals do not overlap and located in a consecutive manner $\implies X \perp\!\!\!\perp Y$
- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$
- Thus, $X + Y \sim \text{Poisson}(\mu + \nu)$

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- Very tedious and not very intuitive.

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- K: number of total arrivals until we get the first arrival from PP(ν).
  - Then, $K \sim \text{Geom}(\frac{\nu}{\lambda+\nu})$.
- Let L be the number of arrivals from PP(λ) until we get the first arrival from PP(ν).

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \dots$$

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  - $p = 1/100$, $n = 10,000$: small p, but large $n \implies$ Both Poisson and Normal

Questions?

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