Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

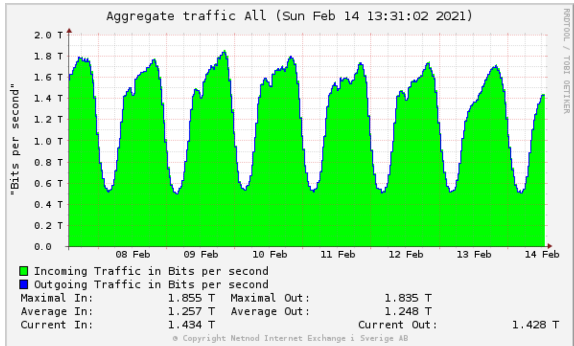
(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

Things that evolve in time

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.

(a) Prices of a crytocurrency



(b) Internet traffic traces

Random Process: Basics (1)

- A random process is a sequence of random variables indexed by time.
- Time: discrete or continuous (a modeling choice in most cases)
- Notation
  - $(X_t)_{t\in\mathcal{T}}$ or $\Big(X(t)\Big)_{t\in\mathcal{T}}$, where $\mathcal{T}=\mathbb{R}$ (continuous) or $\mathcal{T}=\{0,1,2,\dots\}$ (discrete)
  - For the discrete case, we also often use $(X_n)_{n\in\mathbb{Z}_+}$.
  - We will use all of them, unless confusion arises.
- For a fixed time t, X_t (or $X(t)$) is a random variable.
- The values that X_t (or $X(t)$) can take: discrete or continuous

Random Processes: Basics (2)

- Example. Discrete time RP. $\{X_1, X_2, X_3, \ldots, \}$.
  - X_i: # of Covid-19 infections at day i in South Korea, which is a random variable.
  - Then, $X_i : \Omega \mapsto \mathbb{R}$, where the sample space Ω is the set of all outcomes.
  - An outcome $\omega \in \Omega$ is a infinite sequence of infections
  - For example, $\omega_1 = (100, 150, 130, \ldots)$, $\omega_2 = (200, 300, 400, \ldots)$
  - $X_3(\omega_1) = 130$, $X_2(\omega_2) = 300$, $X_1(\omega_2) = 200$, etc.
- Example. Continuous time RP. $(X(t))_{t \in \mathbb{R}^+}$
  - $X(t)$: bitcoin price at time t, which is a random variable.
  - Then, $X(t) : \Omega \mapsto \mathbb{R}$.
  - An outcome $\omega \in \Omega$ is a trajectory of prices over $[0, \infty)$
  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

Random Processes: Our Interest

- Question. Already studied a sequence (or a collection) of rvs $X_1, X_2, \ldots, X_n$. What's the difference?
  - Assume a discrete time random process for our discussion.
- Physical difference: infinite sequence of $X_1, X_2, \ldots, .$
  - Sample space? set of all outcomes?
  - an outcome: an infinite sequence of sample values $x_1, x_2, \ldots,$
- Semantic difference: Understand i in X_i as time. Also, interesting questions are asked from the random process point of view.
  - Dependence: How $X_1, X_2, \ldots$ are related to each other as a time series. Prediction of values in the future.
  - Long-term behavior: What is the fraction of times that a stock price is above 3000?
  - Other interesting questions, depending on the target random process

4 Types of Random Processes

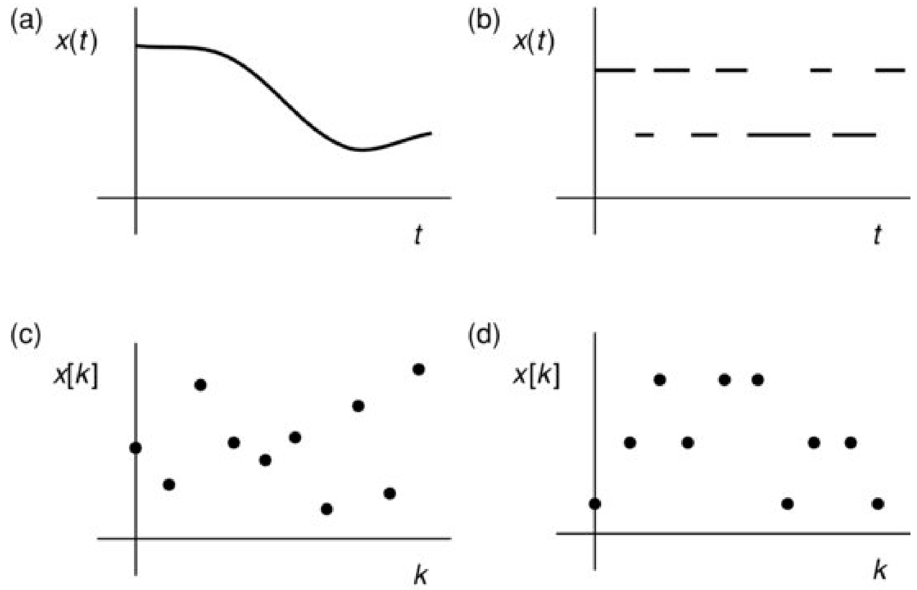
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



Random Processes in This Course

- The simplest RP
- discrete time
- $X_i \perp\!\!\!\perp \{X_{i-1}, X_{i-2}, \ldots, X_1\}$
- Bernoulli Process (BP)
- "today" independent of "past"

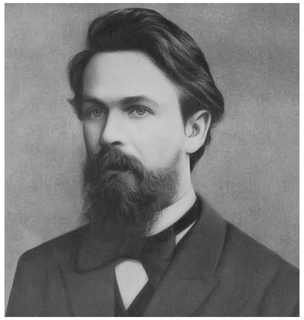
Jacob Bernoulli (1654 - 1705), Swiss

- The simplest RP
- Continous time version of BP
- $[X(s)]_{s=0}^{t} \perp\!\!\!\perp [X(s)]_{s=t}^{t+a}$
- Poisson Process (PP)
- "today" independent of "past"

Simeon Denis Poisson (1781 - 1840), France

- One-step more general than BP/PP
- discrete time
- X_i depends on X_{i-1}, but $\perp\!\!\!\perp \{X_{i-2}, X_{i-2}, \ldots, X_1\}$
- Markov Chain (MC)
- "today" depends only on "yesterday"

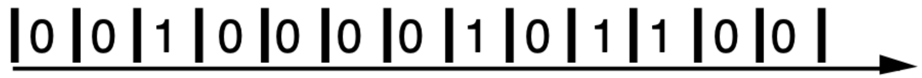
Andrey Markov (1856 - 1922), Russia



Roadmap

Bernoulli Process

- At each "minute", we toss a coin with probability of head $0 < p < 1$.
  - Sequence of lottery wins/looses
  - Customers (each second) to a bank
  - Clicks (at each time slot) to server
- A sequence of independent Bernoulli trials $X_1, X_2, \ldots,$
  - We call index 1, 2, ... time slots (or simply slots)



- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Bernoulli Process: Questions

Please pause the video and write down the questions that you want to ask about Bernoulli process. VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

Number of arrivals and Time until the first arrival

(Q1) # of arrivals in the first n slots?

- $S_n = X_1 + X_2 + \cdots + X_n$
- $S_n \sim \text{Binomial}(n, p)$
- $\mathbb{E}(S_n) = np$, $\text{var}(S_n) = np(1-p)$
- This will hold for any n consecutive slots.

(Q2) # of slots T_1 until the first arrival?

- $T_1 \sim \text{Geom}(p)$
- $\mathbb{E}(T_1) = 1/p$, $\text{var}(T_1) = \frac{1-p}{p^2}$

- T_1 is geometric? Memoryless
- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Memoryless and Fresh-start after Deterministic time n

Independence across slots $\Longrightarrow$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

- Yes
- Because X_is are independent

(Q4) After time $n = 6$, I start to look at the process $(X_n)_{n=6}^{\infty}$?

- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

Fresh-start after Random time N (1)

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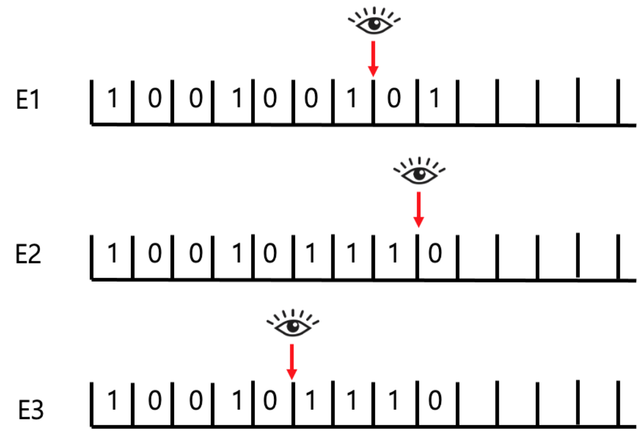
(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

- This means that the time I start to look at the process is a random variable.
- Examples of N

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



- Difference of N from n
  - The time when I watch the on-going Bernoulli process is random.
  - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω?
- Do we experience the fresh-start for any N? **E1, E2, and E3?**

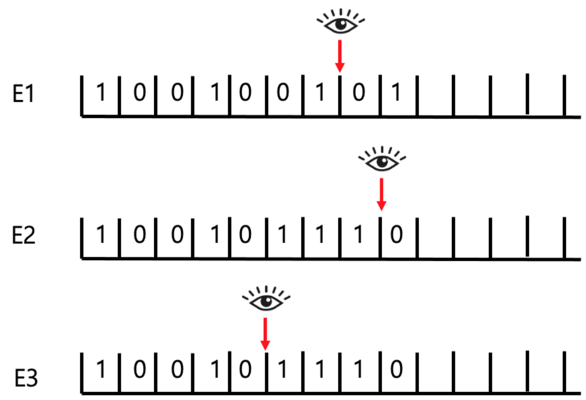
Fresh-start after Random N (2)

(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



E1. When I watch the process, N has been already determined. Yes

E2. Same as **E1.** Yes

E3. Need the future knowledge. '111' does not become random. No

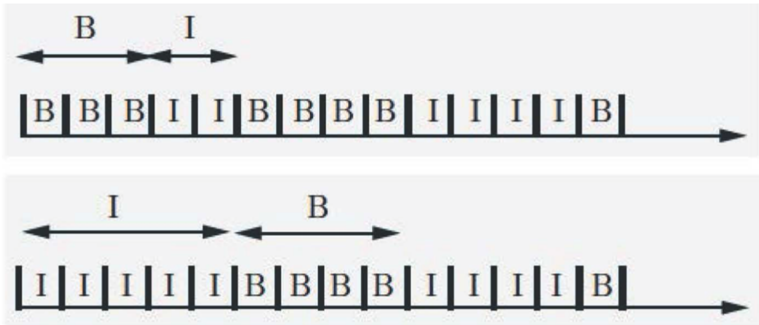
- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

Stopping Time

- In probability theory, a random time N is said to be a stopping time, if the question of "$N = n$?" can be answered only from the present and the past knowledge of $X_1, X_2, \ldots, X_n$.
- https://en.wikipedia.org/wiki/Stopping_time
- Fresh-start after N in Bernoulli process? Yes, if N is a stopping time.
- Please think about two examples of stopping time and not. VIDEO PAUSE
  - Yes. Time when 10 consecutive arrivals have been observed
  - No. Time of 2nd arrival in 10 consecutive arrivals

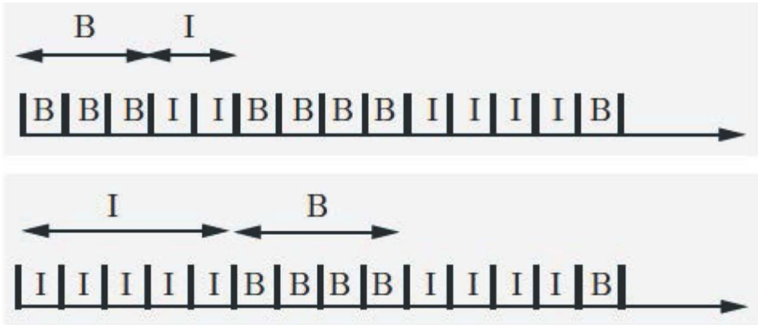
Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)
- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot



- (Q6) B_1 is a random variable. Distribution of B_1?
- N: time of the first busy slot. N is a stopping time?
  - Yes. Thus, fresh-start after N. Because we can answer the question of $N = n$?, just using $X_1, X_2, \dots, X_n$.
- B_1 is geometric with parameter $(1 - p)$

Distribution of Busy Periods (2)



- Question. What about the second busy period B_2?
- N: time of the first busy slot of the second busy period. N is a stopping time?
- Yes. Thus, fresh-start after N.
- Then, B_1 and B_2 are identically distributed as $\text{Geom}(1-p)$.
- $B_3, B_4, \ldots$?

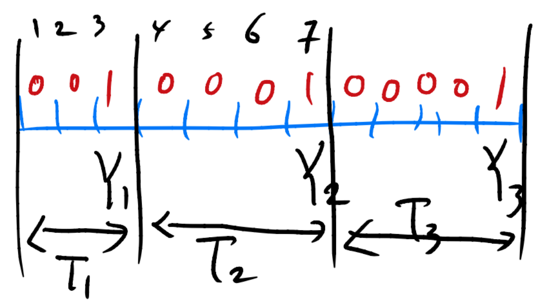
Time of k-th arrival

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

  (Q7) Time of the k-th arrival Y_k?

VIDEO PAUSE

- $T_k = Y_k - Y_{k-1}$: k-th inter-arrival $(k \geq 2,\ T_1 = Y_1)$
- $Y_k = T_1 + T_2 + \ldots + T_k$.



- After each T_k, the fresh-start occurs.
- $\{T_i\}$ are i.i.d. and $\sim \text{Geom}(p)$
- We know Y_k's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but its distribution?

PMF of Y_k

- $Y_k = T_1 + T_2 + \ldots + T_k$.
- $\{T_i\}$ are i.i.d. and $\sim \text{Geom}(p)$

$$
\begin{aligned}
\mathbb{P}(Y_k = t) &= \mathbb{P}\left(X_k = 1 \text{ and } k-1 \text{ arrivals during the first } t-1 \text{ slots}\right) \\
&= \mathbb{P}\left(X_k = 1\right) \cdot \mathbb{P}\left(k-1 \text{ arrivals during the first } t-1 \text{ slots}\right) \\
&= p \times \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} = \binom{t-1}{k-1} p^{k}(1-p)^{t-k}, \quad t = k, k+1, \ldots
\end{aligned}
$$

Pascal Random Variable with Parameter (k, p)

- In the sequence of Bernoulli trials, the time Y_k of k-th success
- PMF of Y_k

$$\mathbb{P}(Y_k = t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k}, & \text{if} \quad t = k, k+1, \ldots \\ 0, & \text{if} \quad t = 1, 2, \ldots k-1 \end{cases}$$

- $\text{Pascal}(1, p) = \text{Geom}(p)$

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Background: Poisson rv X with parameter λ (1)

- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

$$p_S(k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

- Our interest: very large n and very small p, such that $np = \lambda$, i.e., $\lim_{n \to \infty} p_S(k)$?

$$\begin{aligned} p_S(k) &= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n}{n} \cdot \frac{(n-1)}{n} \cdots \frac{(n-k+1)}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \left(1 - \frac{\lambda}{n}\right)^{-k} \xrightarrow{n \to \infty} e^{-\lambda} \frac{\lambda^k}{k!} \end{aligned}$$

Background: Poisson rv X with parameter λ (2)

- A Poisson random variable Z with parameter λ takes nonnegative integer values, whose PMF is:

$$p_Z(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

- Infinitely many slots (n) with the infinitely small slot duration (thus infinitely small success probabilty $p = \lambda/n$)

- $\mathbb{E}(Z) = \lambda$ (because $\lambda = np$ is the mean of binomial rv)

- $\text{var}(Z) = \lambda$ (because $np(1-p)$ is the variance of binomial rv)

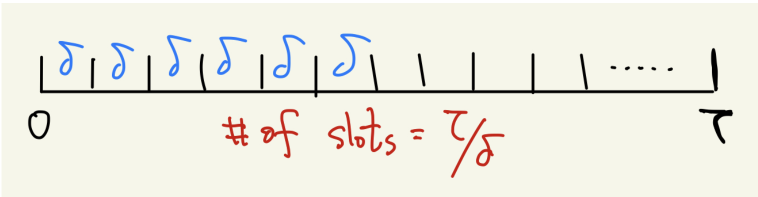
Example: Poisson Approximation

- A packet consisting of a string of n symbols is transmitted over a noisy channel.
- Each symbol: errorneous transimission with probability of 0.0001, independent of other symbols. Incorrect transmission is when at least one symbol is in error.
- Question. How small should n be in order for the probability of incorrect transmission to be less than 0.001?
- p is very small, and n is reasonably large $\rightarrow$ Poisson approximation
- Prob. of incorrect transmission $=$ $1 - \mathbb{P}(\text{no symbol error}) = 1 - e^{-\lambda} = 1 - e^{-0.0001n} < 0.0001$
- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

Design of Continuous Analogue of Bernoulli Process

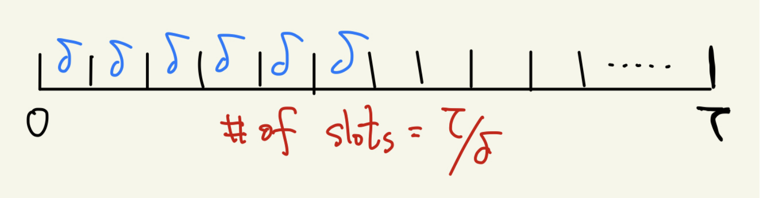
- Remind. Geometric vs. Exponential L4(3)
  - Two rvs with memoryless property
  - continuous system = discrete system with infintely many slots whose duration is infinitley small.
- Independence between what happens in a different time region
- Memoryless and fresh-start property
- Choose one of discrete or continuous versions for our modeling convenience.
- Question. How do we design the continuous analog of Bernoulli process?
  - Key idea: Making it as a $\boxed{\text{limit}}$ of a sequence of Bernoulli processes
- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

- Continuous twin
  - Key point: Understand the number of arrivals over a given interval $[0, \tau]$.
  - Assume that it has some arrival rate λ ($\#$ of arrivals/unit time).
  - We know how to handle Bernoulli process with discrete time slots.
- Divide $[0, \tau]$ into slots whose length $= \delta$. Then, $n = \#$ of slots $= \frac{\tau}{\delta}$.



- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)

Key Design Idea to Develop a Continuous Twin (2)



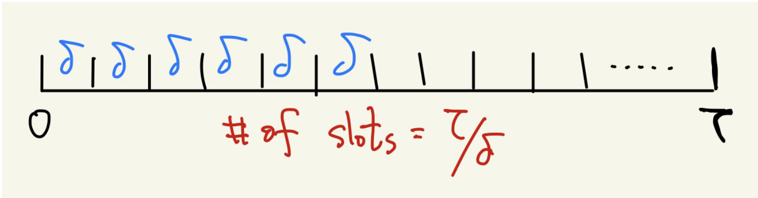
- Now, our design idea: during one time slot of length δ,

$$
\begin{aligned}
\mathbb{P}(1 \text{ arrival}) &\propto \text{arrival rate and slot length} \\
\mathbb{P}(\geq 2 \text{ arrivals}) &\propto \text{something, but very small} \\
&\quad\ \text{for small sloth length} \\
\mathbb{P}(0 \text{ arrival}) &= 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\
\mathbb{P}(\geq 2 \text{ arrivals}) &= 0 o(\delta) \\
\mathbb{P}(0 \text{ arrival}) &= 1 - \lambda\delta + o(\delta)
\end{aligned}
$$

- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)



- Our interest: probability of k arrivals over $[0, \tau]$
- Given "small" δ, $\#$ of arrivals $\sim$ Bin(n, p), where $n = \tau/\delta$ and $p = \lambda\delta$
- As $\delta \to 0$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- $\#$ of arrivals over $[0, \tau]$, $\sim$ Poisson($\lambda\tau$)
- This is a continuous twin process of Bernous process, which we call Poisson process.

Roadmap

Poisson Process: Definition (1)

An arrival process is called a Poisson process with rate λ, if the following are satisfied:

- (Independence) Let N_τ be the number of arrivals over the interval $[0, \tau]$. For any $\tau_1, \tau_2 > 0$, $N_{s+\tau_1} - N_s$ is independent of $N_{t+\tau_2} - N_t$, if $t > s + \tau_1$.
    - The number of arrivals over two disjoint intervals are independent.
- (Time homogeneity) For any s, the distribution of $N_{s+\tau} - N_s$ is equal to that of N_τ.
    - N_τ becomes the number of arrivals over any interval of length τ.
- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$
$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$
$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \ldots, \quad \text{where} \quad \lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

Poisson Process: Definition (2)

An arrival process is called a Poisson process with rate λ, if the following are satisfied:

- (Independence) Let N_τ be the number of arrivals over the interval $[0, \tau]$. For any $\tau_1, \tau_2 > 0$, $N_{s+\tau_1} - N_s$ is independent of $N_{t+\tau_2} - N_t$, if $t > s + \tau_1$.
  - The number of arrivals over two disjoint intervals are independent.
- (Time homogeneity) For any s, the distribution of $N_{s+\tau} - N_s$ is equal to that of N_τ.
  - N_τ becomes the number of arrivals over any interval of length τ.
- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k, \tau) = e^{\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

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(Q1) Number of arrivals of any interval with length $\tau \sim \text{Poisson}(\lambda\tau)$, i.e.,

$$\mathbb{P}(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\mathbb{E}(N_\tau) = \lambda\tau \text{ and } \text{var}(N_\tau) = \lambda\tau$$

(Q2) Time of first arrival T

$$F_T(t) = \mathbb{P}(T \le t) = 1 - \mathbb{P}(T > t) = 1 - P(0,t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \quad t \ge 0.$$

- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
  - 5* 24 = 120
- $\mathbb{P}$[one new message in the next hour]
  - $\mathbb{P}(1,1) = \dfrac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$
- $\mathbb{P}$[exactly two msgs during each of the next three hours]
  - $\left(\dfrac{5^2 e^{-5}}{2!}\right)^3$

- Remind. Similar property for Bernoulli processes, but here no time slots.
- Fresh-start at determinsitic time t: Start watching at time t, then you see the Poisson process, independent of what has happened in the past.
- Fresh-start at random time T: Similarly, it holds.
  - For example, when you start watching at random time T_1 (time of first arrival).
  - Generally, it holds when T is a stopping time.

(Q3) The k-th arrival time Y_k?

- k-th inter-arrival time $T_k = Y_k - Y_{k-1}$, $k \geq 2$, and $T_1 = Y_1$.
- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

PDF of Y_k

- For a given δ, $\boxed{\delta \cdot f_{Y_k}(y)}$: prob. of k-th arrival over $[y, y+\delta]$.
- When δ is small, only one arrival occurs. Thus,

$$\begin{aligned}\delta \cdot f_{Y_k}(y) &= \mathbb{P}\left(\text{an arrival over } [y, y+\delta]\right) \times \mathbb{P}\left(k-1 \text{ arrivals before } y\right) \\ &\approx \lambda\delta \times \mathbb{P}(k-1, y) = \lambda\delta \times \frac{\lambda^{k-1} y^{k-1} e^{-\lambda y}}{(k-1)!}\end{aligned}$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0.$$

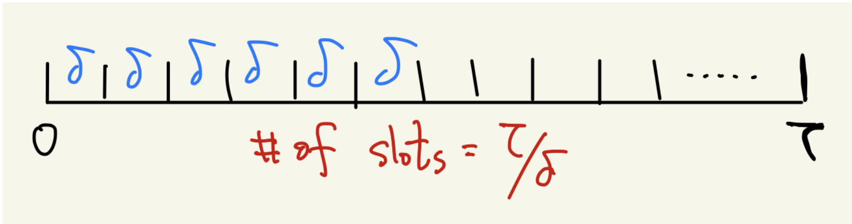
- This is called Erlang rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

$$f_Z(z) = \frac{\lambda^k z^{k-1} e^{-\lambda z}}{(k-1)!}, \quad z \geq 0$$

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals	Binomial	Poisson
Interarrival time	Geometric	Exponential
Time of k-th arrival	Pascal	Erlang
Arrival rate	p/per slot	λ/unit time

Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6$/hour.
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

(Q1) $\mathbb{P}$(fishing time $>$ 2 hours)

Method 1: $\mathbb{P}(0,2)$
Method 2: $\mathbb{P}(T_1 > 2)$

(Q2) $\mathbb{P}(2 <$ fishing time $< 5)$

Method 1: $\mathbb{P}(0,2)(1-\mathbb{P}(0,3))$
Method 2:$\mathbb{P}(2 < T_1 < 5)$

(Q3) $\mathbb{P}$(Catch at least two fish)

Method 1:$\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0,2) - \mathbb{P}(1,2)$
Method 2: $\mathbb{P}(Y_2 \leq 2)$

(Q4) $\mathbb{E}$[future fi. time|already fished for 3h]
Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

(Q5) $\mathbb{E}$[F=total fishing time]

$$2 + \mathbb{E}[F-2] = 2 + \mathbb{P}(F=2)\cdot 0 + \mathbb{P}(F>2)\cdot\mathbb{E}[F-2|F>2]$$
$$= 2 + \mathbb{P}(0,2)\cdot\frac{1}{\lambda}$$

(Q6) $\mathbb{E}$[number of fish] $= \lambda \cdot 2 + \mathbb{P}(0,2)\cdot 1$

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

Description via Inter-arrival Times

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

1. Start with a sequence of independent exponential random variables $T_1, T_2, \ldots$, with common parameter λ, and let these stand for the interarrival times.
2. Record an arrival at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Example

- It has been observed that after a rainy day, the number of days until the next rain $\sim \text{Geom}(p)$, independent of the past.
- Question. What is the probability that it rains on both the 5th and the 8th day of the month?
- Approach 1: Handling this problem directly with the geometric PMFs is very tedious and complex.
- Approach 2: Rainy days is a Bernoulli process with arrival. probability p.
- Thus, the answer is p^2.

Coding of Random Arrivals

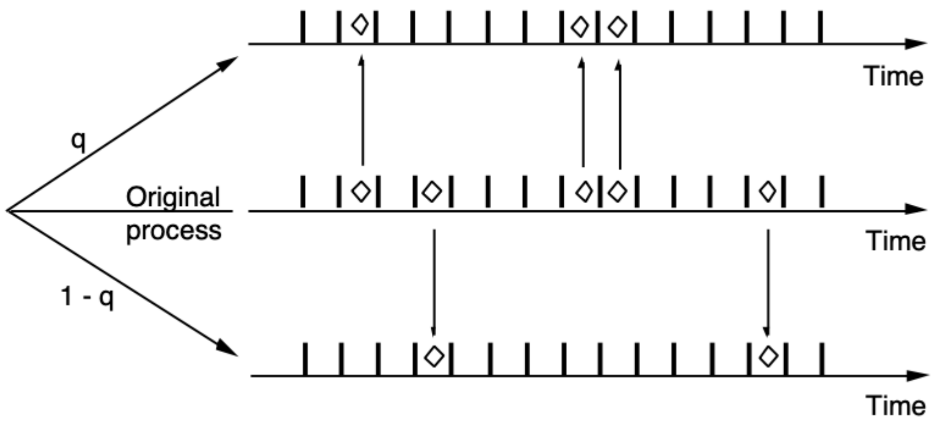
- Question. How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.
- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

Notations In the Rest of These Slides

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

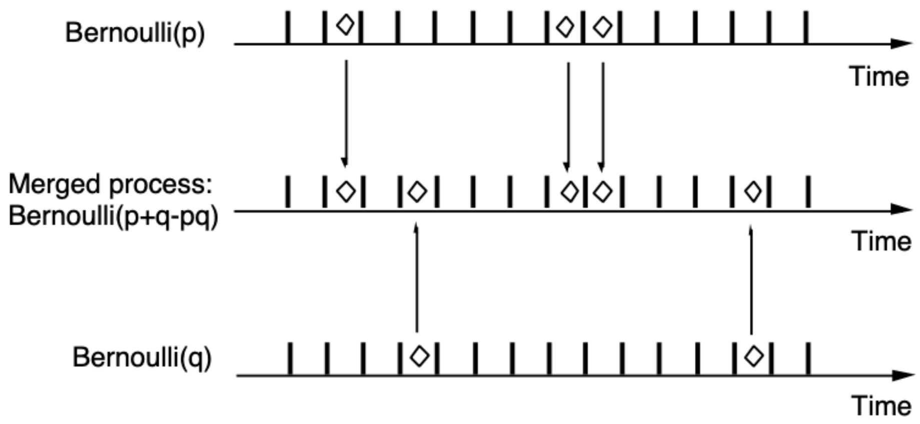
Split: Bernoulli Process

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1 - q)$.
- Split decisions are independent of arrivals.
- Question. What are the two split processes?
- BP(pq) and BP($p(1 - q)$). Why?
- Are they independent? No.



Merge: Bernoulli Process

- Merge BP(p) and BP(q) into one process.
- Collided arrival is regarded just one arrival in the merged process
- Probability of having at least one arrival: $1-(1-p)(1-q) = p+q-pq$
- Merged process: BP($p+q-pq$)
- $\mathbb{P}(\text{arrival from proc. } 1 \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



Split: Poisson Process

- Split a Poisson process PP(λ) into two processes by keeping each arrival w.p. p and discarding it w.p. $(1 - p)$
- Question. What are the split processes?
- Let's focus on the process that we keep
- Independence and time-homogeneity? Yes
- Small interval probability over δ-interval
  - $\mathbb{P}(\text{1 arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(\text{2 arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(\text{0 arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- PP(λp) and PP($\lambda(1 - p)$)

Merge: Poisson Process (1)

- Merge from PP(λ_1) and PP(λ_2)
- Independence and time-homogeneity? Yes
- Small interval probabilty over δ-interval (ignoring $o(\delta)$ for small δ)

  $\mathbb{P}(0 \text{ arrival}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1-(\lambda_1+\lambda_2)\delta$

  $\mathbb{P}(1 \text{ arrival}) \approx (\lambda_1\delta)(1-\lambda_2\delta)+\lambda_2\delta(1-\lambda_1\delta) \approx (\lambda_1+\lambda_2)\delta$
- Merged process: PP($\lambda_1+\lambda_2$)

Merge: Poisson Process (2)

- Red: PP(λ_1) and Blue: PP(λ_2)
- $\mathbb{P}$(from red | arrival at time t in the merged proc.)? $\dfrac{\lambda_1\delta}{\lambda_1\delta + \lambda_2\delta} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$
- Consider an event $A_k = \{k\text{-th arrival in the merged proc. is red}\}$
  - $\mathbb{P}(A_k)$? $\dfrac{\lambda_1}{\lambda_1 + \lambda_2}$
  - $A_1, A_2, \ldots$ are independent: origins (red or blue) of arrivals in the merged proc. are indepdnent
- $\mathbb{P}$(k out of first 10 arrivals are red)? $\dbinom{10}{k}\left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{10-k}$

Using Poisson Processes for Intuitive Problem Solving

1. Competing exponentials
2. Sum of independent Poisson rvs
3. Poisson arrivals during and Exponential interval

Competing Exponential (1)

- Two independent light bulbs have life times $T_a \sim \text{Exp}(\lambda_a)$ and $T_b \sim \text{Exp}(\lambda_b)$.

(Q) Distribution of $Z = \min\{T_a, T_b\}$, the first time when a bulb burns out?

- Approach 1
  - $\mathbb{P}(Z \geq z) = \mathbb{P}(T_a \geq z)\mathbb{P}(T_b \geq z) = e^{-\lambda_a z} e^{-\lambda_b z} = e^{-(\lambda_a+\lambda_b)z}$
  - Thus, $Z \sim \text{Exp}(\lambda_a + \lambda_b)$
- Approach 2
  - T_a and T_b are the first arrival times of two Poisson processes of λ_a and λ_b, respectively.
  - Z is the first arrival time of merged Poisson process $(\lambda_a + \lambda_b)$.
  - Thus, $Z \sim \text{Exp}(\lambda_a + \lambda_b)$

Competing Exponential (2)

- Three independent light bulbs have life time $T \sim \text{Exp}(\lambda)$.

(Q) $\mathbb{E}$[time until the last bulb burns out]?

- Understanding from the merged Poisson process
  - T_1: time until the first burn-out, T_2: time until the second burn-out, T_3: time until the third burn-out
  - $\mathbb{E}(T_1 + T_2 + T_3)$?
  - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
  - $T_1 \sim \text{Exp}(3\lambda),\ T_2 \sim \text{Exp}(2\lambda),\ T_3 \sim \text{Exp}(\lambda)$

$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

- Two independent rvs X and Y, where $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\nu)$.
- Question. Distribution of $X + Y$? Complex convolution, but any other easy way?
- Poisson process perspective
  - X: number of arrivals of PP(1) over a time interval of length μ
  - Y: number of arrivals of PP(1) over a time interval of length ν
  - Two intervals do not overlap and located in a consecutive manner $\Longrightarrow X \perp\!\!\!\perp Y$
- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$
- Thus, $X + Y \sim \text{Poisson}(\mu + \nu)$

Poisson arrivals during Exponential Interval (1)

- Problem 24, pp. 335
- Consider PP(λ) and an independent rv $T \sim \text{Exp}(\nu)$
- Question. Distribution of N_T?
- Approach 1: Total probability theorem

$$\mathbb{P}(N_T = k) = \int_0^\infty \mathbb{P}(N_T = k | T = \tau) f_T(\tau) d\tau = \int_0^\infty \mathbb{P}(N_\tau = k) f_T(\tau) d\tau$$

- Very tedious and not very intuitive.

Poisson arrivals during Exponential Interval (2)

- Consider PP(λ) and an independent rv $T \sim$ Exp(ν)
- Consider another PP(ν), and let's view T as the first arrival time in PP(ν).
- Now, consider the merged process of PP(λ) and PP(ν).
  - $\mathbb{P}\Big[\text{from PP}(\lambda)|\text{arrival}\Big] = \frac{\lambda}{\lambda+\nu}$ and $\mathbb{P}\Big[\text{from PP}(\nu)|\text{arrival}\Big] = \frac{\nu}{\lambda+\nu}$
- K: number of total arrivals until we get the first arrival from PP(ν).
  - Then, $K \sim$ Geom($\frac{\nu}{\lambda+\nu}$).
- Let L be the number of arrivals from PP(λ) until we get the first arrival from PP(ν).

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \ldots$$

Approximation of Binomial: Poisson vs. Normal

- Consider $S_n \sim \text{Binomial}(n, p)$. Then, $S_n = \sum_{i=1}^{n} X_i$, where $X_i \sim \text{Bern}(p)$.
- Poisson approximation. $\text{Poisson}(np)$ is a good approximation of S_n
  - Holds when $np = \lambda$ ($n \to \infty$, $p \to 0$), i.e., X_i's behavior changes over n.
- Normal approximation. $\sigma\sqrt{n}Z + n\mu$ is a good approximation of S_n
  - Holds for a fixed p ($n \to \infty$), i.e., X_i's behavior does not change over n.
- In practice, an actual numbers of n and p are given, so which approximation is good under what situation?
  - $p = 1/100$, $n = 100$: $np = 1$, very asymmetric X_i, small p $\implies$ Poisson
  - $p = 1/3$, $n = 100$: large, reasonly symmetric p, at least moderate n $\implies$ Normal
  - $p = 1/100$, $n = 10,000$: small p, but large n $\implies$ Both Poisson and Normal

Roadmap

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(6) Random Incidence Phenomenon

Example: Survey of Utilization of Town Buses

- What we want to survey: How available are town buses in a city?
- Two approaches

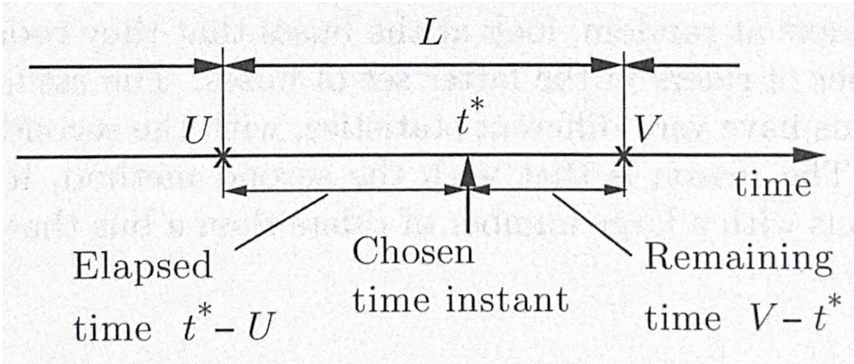
  M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus

  M2. (1) select a few bus riders at random, (2) look at the buses that they rode, and (3) calculate the average number of riders in those buses

- Which is correct?
- (i) M1 $=$ M2? (ii) M1 $>$ M2? (iii) M1 $<$ M2?

Random Incidence Phenomemon (1)

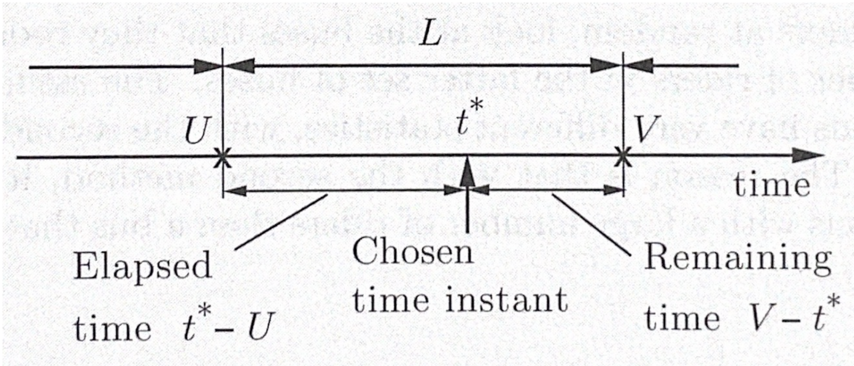
- We know: in PP(λ), inter-arrival time $\sim$ Exp(λ)
- Fix a time instant $t^\star$, and consider the length L of the inter-arrival interval that constains $t^\star$.



- Practical context: Yung shows up at the bus station at some arbitrary time $t^\star$ and records the time from the previous bus arrival (U) until the next bus arrival (V)
- Question. What is the distribution of L?

VIDEO PAUSE

Random Incidence Phenomemon (2)



- $t^\star$ is not random, so "random incidence" may be confusing and misleading.
- Assumption. For simplicity, $t^\star$ is large enough that we must have an arrival before $t^\star$ $(U > 0)$
- One might superficially argue that $L \sim \text{Exp}(\lambda)$, but it is NOT.

Distribution of L (1)

$$\boxed{L = (t^\star - U) + (V - t^\star)}$$

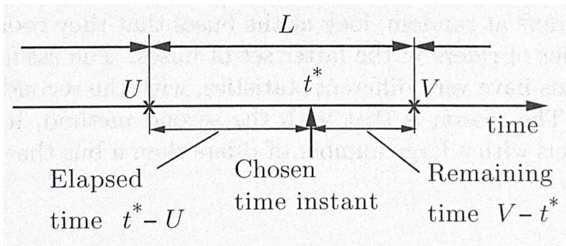
- $V - t^\star$: Due to the memoryless property and the fresh-restart,

  Thus, $\quad V - t^\star \sim \text{Exp}(\lambda)$

- $t^\star - U$: If we run the PP(λ) backwards in time, it remains Poisson. Why? More formally,

  $$\mathbb{P}(t^\star - U > x) = \mathbb{P}(\text{no arrivals over } [t^\star - x, t^\star])$$
  $$= e^{-\lambda x} = \mathbb{P}(T_{\text{inter}} > x)$$

  Thus, $\quad t^\star - U \sim \text{Exp}(\lambda)$



Distribution of L (2)

$$\boxed{L = (t^\star - U) + (V - t^\star)}$$

- $L = X_1 + X_2$, where $X_1, X_2 \sim \text{Exp}(\lambda)$
- Time until we have two arrivals in $\text{PP}(\lambda)$
- Erlang random variable with parameter $(2, \lambda)$, i.e., page 37

$$f_L(l) = \lambda^2 \cdot l \cdot e^{-\lambda l}, \quad l \geq 0$$

- Mean $= 2/\lambda$
- Why not $\text{Exp}(\lambda)$? An observer who arrives at an arbitrary time is more likely to fall in a large rather than a small interarrival interval.

Back to Survey of Utilization of Town Buses

- Two Approaches

  M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus

  M2. (1) select a few bus riders at random, (2) look at the buses that they rode, and (3) calculate the average number of riders in those buses

- (i) M1 $=$ M2? (ii) M1 $>$ M2? (iii) M1 $<$ M2?
- Answer: M1 $<$ M2
- More likely to select a bus with a large number of riders than a bus that is near-empty.

Questions?

Review Questions

1) Explain what is random process. Why do we need such a concept?
2) Explain Bernoulli processes. When do we use Bernoulli processes?
3) Explain Poisson processes. When do we use Poisson processes?
4) What are the relations between those two processces? What features do they share?
5) In both processces, ho do we compute (i) number of arrivals over a given interval of time, (ii) time until the first arrival, (iii) time until k-th arrival?
6) What is Pascal rvs and Erlang rvs?
7) What is the "stopping time" and how is it related to fresh-restart?
8) See many examples on how merging and splitting of BP and PP can be used for intuitive soloving of many problems.