

## Lecture 7: Random Processes, Part I

Yi, Yung (이웅)

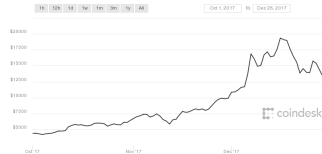
EE210: Probability and Introductory Random Processes  
KAIST EE

August 26, 2021

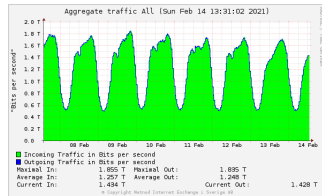
- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
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- Many probabilistic experiments that **evolve in time**

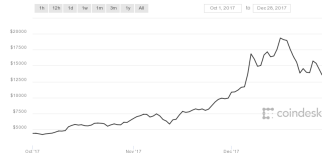


(a) Prices of a cryptocurrency

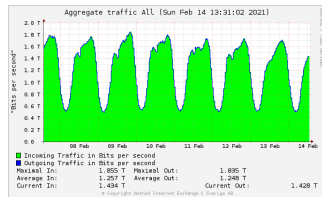


(b) Internet traffic traces

- Many probabilistic experiments that **evolve in time**
  - Sequence of daily prices of a stock
  - Sequence of scores in football
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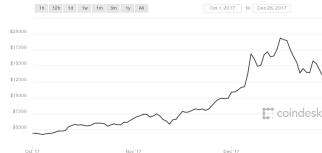


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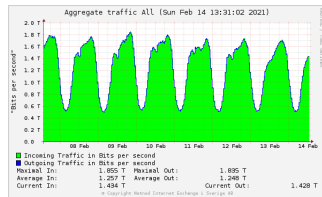


(b) Internet traffic traces

- Many probabilistic experiments that **evolve in time**
  - Sequence of daily prices of a stock
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- Random process is a mathematical model for it.



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(b) Internet traffic traces



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- The values that  $X_t$  (or  $X(t)$ ) can take: discrete or continuous

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  - $X(3.7, \omega_1) = 3409$ ,  $X(2, \omega_2) = 5000$ ,  $X(7.8, \omega_3) = 2800$ , etc.



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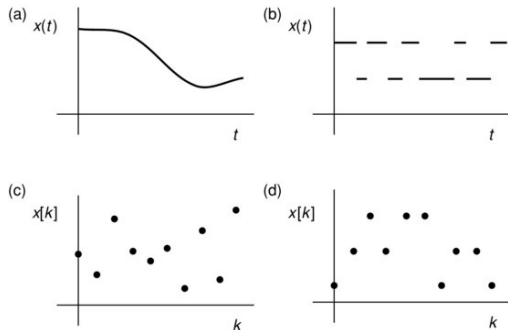


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  - Other interesting questions, depending on the target random process

## - Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



- The simplest RP
- discrete time

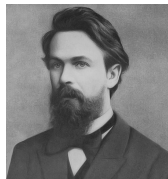
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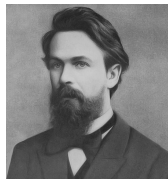
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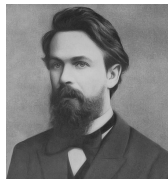
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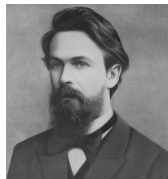


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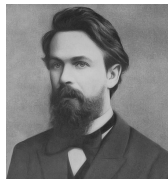


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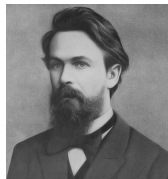
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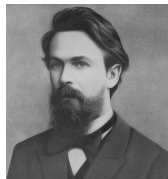
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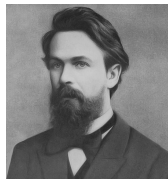
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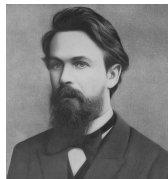
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- A sequence of **independent Bernoulli trials**  $X_1, X_2, \dots$ ,
  - We call index 1, 2, ... **time slots** (or simply slots)

| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | →

- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

Please pause the video and write down the questions that you want to ask about Bernoulli process.

**VIDEO PAUSE**

Q1.

Q2.

Q3.

Q4.

Q5.

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- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots  $\implies$  the fresh-start anytime when I look at the process?

(Q3)  $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6?$

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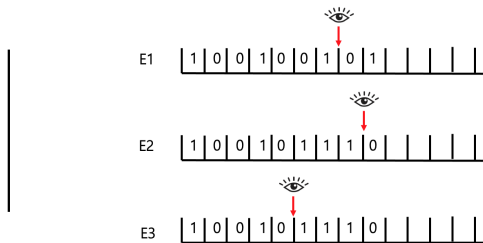
- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time  $n$  (doesn't matter what happened until  $n = 5$ ).
- If you watch the on-going Bernoulli process( $p$ ) from some time  $n$ , you still see the same Bernoulli process( $p$ ).

(Q5) The process  $(X_N, X_{N+1}, X_{N+2}, \dots)$ ? Fresh-start even after **random**  $N$ ?

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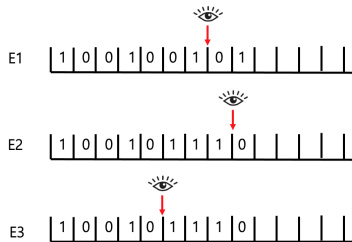
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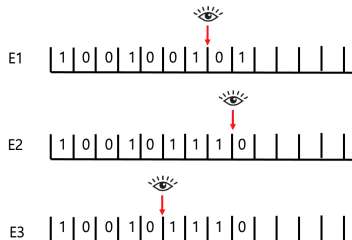


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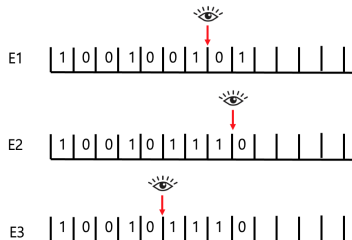
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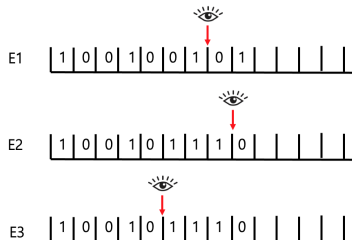
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- Difference of  $N$  from  $n$ 
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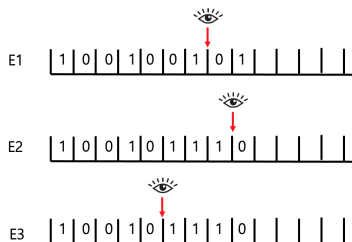
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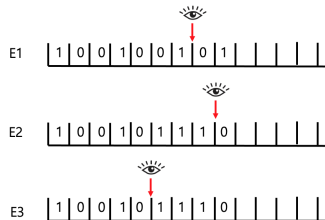
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- Do we experience the fresh-start for any  $N$ ? **E1, E2, and E3?**

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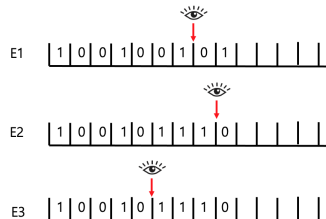


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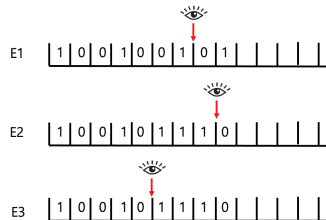
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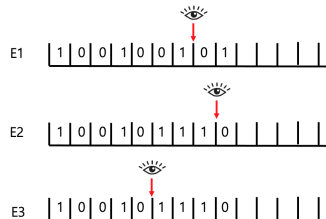
**E2.** Same as **E1**. **Yes**

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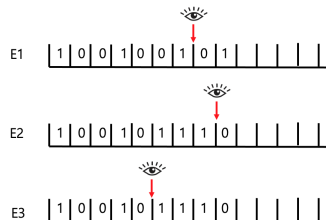
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- The question of  $N = n$ ? can be answered just from the knowledge about  $X_1, X_2, \dots, X_n$ ? Then, Yes! (see pp. 301 for more formal description)





- In probability theory, a random time  $N$  is said to be a **stopping time**, if the question of “ $N = n?$ ” can be answered only from the present and the past knowledge of  $X_1, X_2, \dots, X_n$ .

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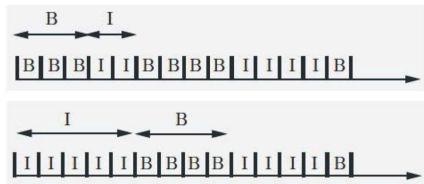
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  - **Yes.** Time when 10 consecutive arrivals have been observed
  - **No.** Time of 2nd arrival in 10 consecutive arrivals

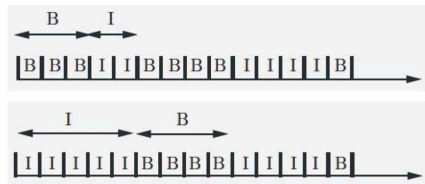
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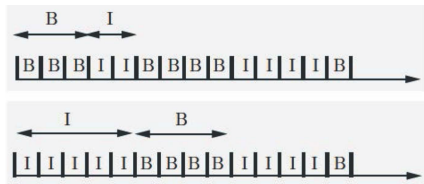


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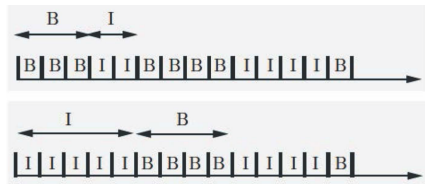
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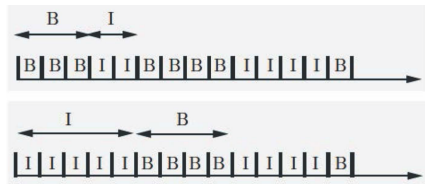
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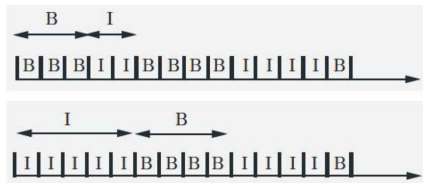


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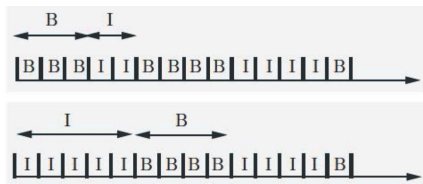
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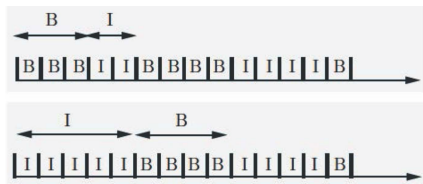
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- **Question.** What about the second busy period  $B_2$ ?

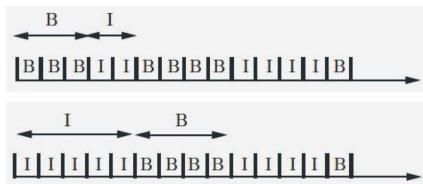


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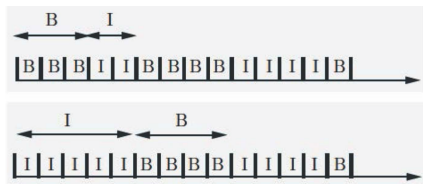


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- $B_3, B_4, \dots$ ?

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(Q7) Time of the  $k$ -th arrival  $Y_k$ ?

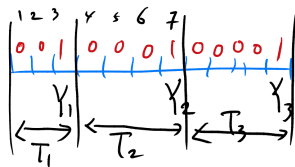
VIDEO PAUSE

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VIDEO PAUSE

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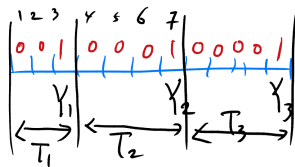


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VIDEO PAUSE

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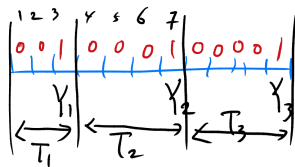


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VIDEO PAUSE

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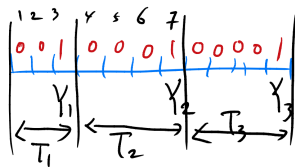
- After each  $T_k$ , the fresh-start occurs.

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VIDEO PAUSE

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- $Y_k = T_1 + T_2 + \dots + T_k$ .



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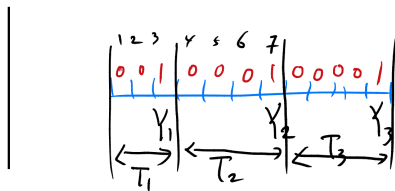


- Time of the first arrival  $Y_1 \sim \text{Geom}(p)$

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- A random variable  $S \sim \text{Bin}(n, p)$ : Models the number of successes in a given number  $n$  of independent trials with success probability  $p$ .

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- Need a “modeling sense” to make this possible. It’s a good practice for engineers!
- **VIDEO PAUSE**

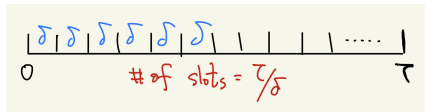
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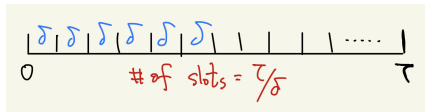
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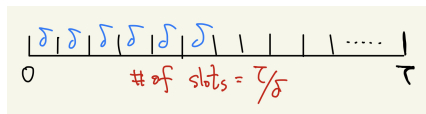
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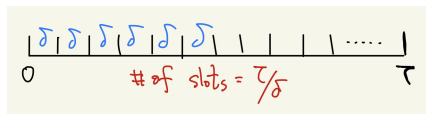
- What's the limit as  $\delta \rightarrow 0$  (equivalently,  $n \rightarrow \infty$ )



- Now, our design idea: during one time slot of length  $\delta$ ,

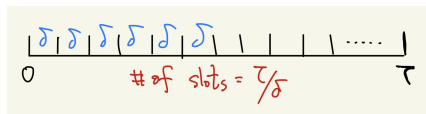
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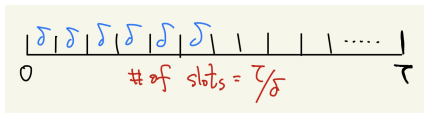
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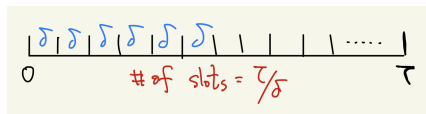
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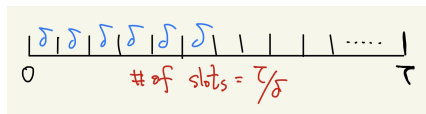
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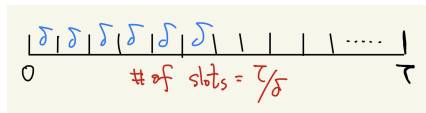
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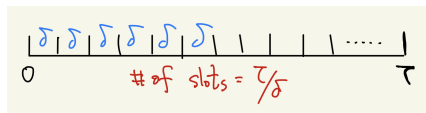
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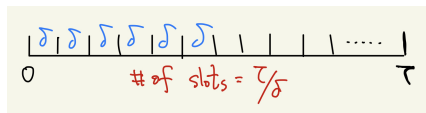
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$$\begin{aligned} \mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\ \mathbb{P}(\geq 2 \text{ arrivals}) &= o(\delta) \\ \mathbb{P}(0 \text{ arrival}) &= 1 - \lambda\delta + o(\delta) \end{aligned}$$



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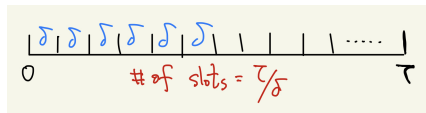
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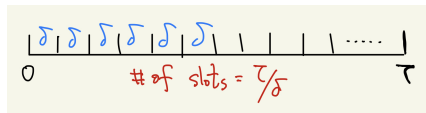
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- $o(\delta)$ : some function that goes to zero faster than  $\delta$ .
  - Thus, for very small  $\delta$ ,  $o(\delta)$  becomes negligible, compared to  $\delta$ .
  - Example:  $o(\delta) = \delta^\alpha$ , where any  $\alpha > 1$

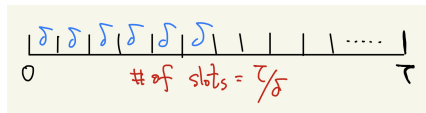




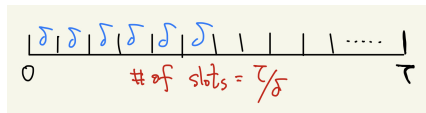
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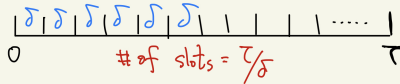
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  - The number of arrivals over two disjoint intervals are independent.
- **(Time homogeneity)** For any  $s$ , the distribution of  $N_{s+\tau} - N_s$  is equal to that of  $N_\tau$ .
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- **(Independence)** Let  $N_\tau$  be the number of arrivals over the interval  $[0, \tau]$ . For any  $\tau_1, \tau_2 > 0$ ,  $N_{s+\tau_1} - N_s$  is independent of  $N_{t+\tau_2} - N_t$ , if  $t > s + \tau_1$ .
  - The number of arrivals over two disjoint intervals are independent.
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- **(Small interval probability)** Let  $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$ , which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \dots, \quad \text{where} \quad \lim_{\tau \rightarrow 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \rightarrow 0} \frac{o_k(\tau)}{\tau} = 0$$

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- $T \sim \text{Exp}(\lambda)$ . Thus,  $\mathbb{E}(T) = 1/\lambda$  and  $\text{var}(T) = 1/\lambda^2$ 
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

- Receive emails according to a Poisson process at rate  $\lambda = 5$  messages/hour
- Mean and variance of mails received during a day
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- $\mathbb{E}[Y_k] = k/\lambda$  and  $\text{var}[Y_k] = k/\lambda^2$ , but what is the distribution of  $Y_k$ ?



- For a given  $\delta$ , : prob. of  $k$ -th arrival over  $[y, y + \delta]$ .

- For a given  $\delta$ ,  $\delta \cdot f_{Y_k}(y)$ : prob. of  $k$ -th arrival over  $[y, y + \delta]$ .



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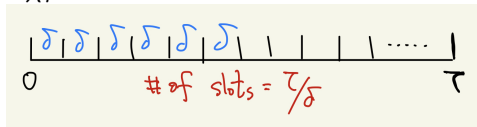
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- This is called **Erlang** rv.

An Erlang random variable  $Z$  with parameter  $(k, \lambda)$  has the following pdf:

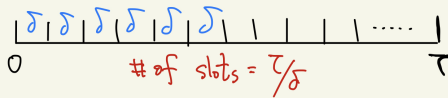
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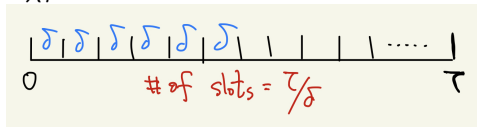
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time of arrival	Discrete	Continuous
PMF of # of arrivals		
Interarrival time		
Time of $k$ -th arrival		
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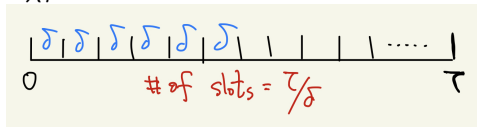
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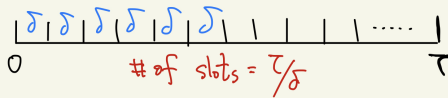


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Arrival rate	$p$ /per slot	$\lambda$ /unit time

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$$\begin{aligned} 2 + \mathbb{E}[F - 2] &= 2 + \mathbb{P}(F = 2) \cdot 0 + \\ &\quad \mathbb{P}(F > 2) \cdot \mathbb{E}[F - 2 | F > 2] \\ &= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda} \end{aligned}$$



- Catching fish: Poisson process  $\lambda = 0.6/\text{hour}$ .
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

**(Q1)**  $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1:  $\mathbb{P}(0, 2)$

Method 2:  $\mathbb{P}(T_1 > 2)$

**(Q2)**  $\mathbb{P}(2 < \text{fishing time} < 5)$

Method 1:  $\mathbb{P}(0, 2)(1 - \mathbb{P}(0, 3))$

Method 2:  $\mathbb{P}(2 < T_1 < 5)$

**(Q3)**  $\mathbb{P}(\text{Catch at least two fish})$

Method 1:  $\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0, 2) - \mathbb{P}(1, 2)$

Method 2:  $\mathbb{P}(Y_2 \leq 2)$

**(Q4)**  $\mathbb{E}[\text{future fi. time} | \text{already fished for 3h}]$

Fresh-start. So,

$$\mathbb{E}[\exp(\lambda)] = 1/\lambda = 1/0.6$$

**(Q5)**  $\mathbb{E}[F = \text{total fishing time}]$

$$\begin{aligned} 2 + \mathbb{E}[F - 2] &= 2 + \mathbb{P}(F = 2) \cdot 0 + \\ &\quad \mathbb{P}(F > 2) \cdot \mathbb{E}[F - 2 | F > 2] \\ &= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda} \end{aligned}$$

**(Q6)**  $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes

## Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables  $T_1, T_2, \dots$ , with common parameter  $p$ , and let these stand for the interarrival times.
2. Record a success (or arrival) at times  $T_1, T_1 + T_2, T_1 + T_2 + T_3$ , etc.

## Alternative Description of the Poisson Process

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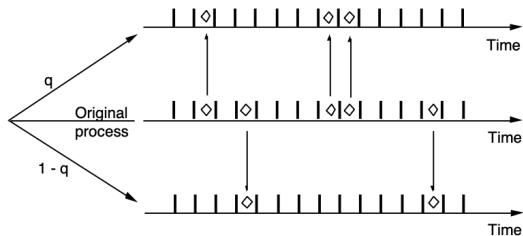
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- Thus, the answer is  $p^2$ .



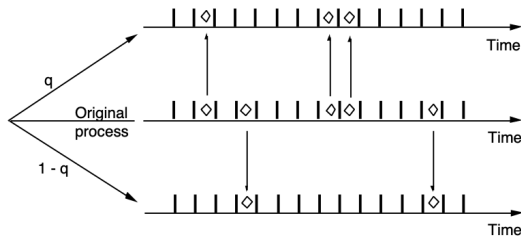
- **Question.** How to make software codes of Bernoulli process with  $p$  and Poisson process with  $\lambda$
- Inter-arrival times are very handy.
- **Bernoulli process with  $p$ :** Obtain a sequence of random values following the **geometric distribution** with parameter  $p$ .
- **Poisson process with  $\lambda$ :** Obtain a sequence of random values following the **exponential distribution** with parameter  $\lambda$ .

- Bernoulli random variable:  $\text{Bern}(p)$
- Bernoulli process:  $\text{BP}(p)$
- Poisson random variable:  $\text{Poisson}(\lambda)$
- Poisson process:  $\text{PP}(\lambda)$

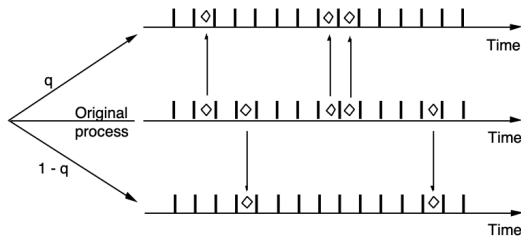
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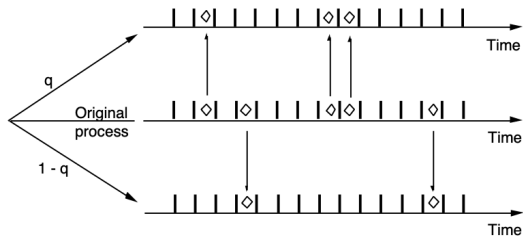
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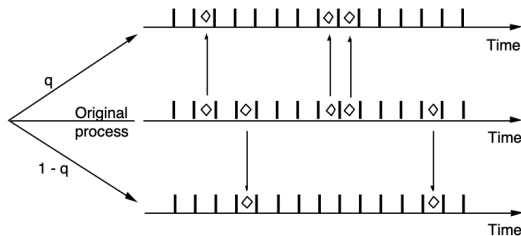
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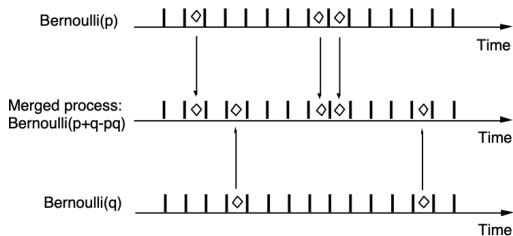
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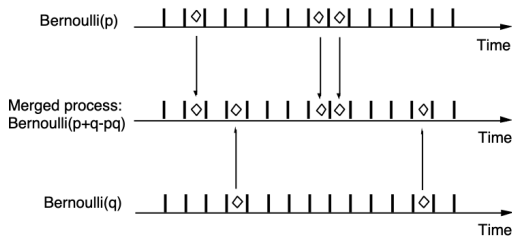


- Merge  $BP(p)$  and  $BP(q)$  into one process.

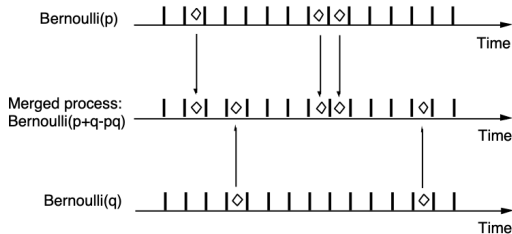




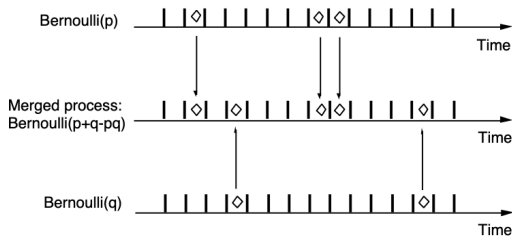
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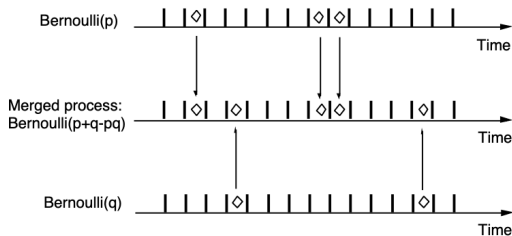
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  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- **$PP(\lambda p)$  and  $PP(\lambda(1 - p))$**

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1. Competing exponentials
2. Sum of independent Poisson rvs
3. Poisson arrivals during and Exponential interval



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$$\mathbb{E}[T_1 + T_2 + T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

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- Distribution of  $X + Y$ : the number of arrivals of PP(1) over a time interval of length  $\mu + \nu$
- Thus,  $X + Y \sim \text{Poisson}(\mu + \nu)$

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$$\mathbb{P}(N_T = k) = \int_0^\infty \mathbb{P}(N_T = k | T = \tau) f_T(\tau) d\tau = \int_0^\infty \mathbb{P}(N_\tau = k) f_T(\tau) d\tau$$

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- Consider  $PP(\lambda)$  and an independent rv  $T \sim \text{Exp}(\nu)$
- **Question.** Distribution of  $N_T$ ?
- **Approach 1:** Total probability theorem

$$\mathbb{P}(N_T = k) = \int_0^\infty \mathbb{P}(N_T = k | T = \tau) f_T(\tau) d\tau = \int_0^\infty \mathbb{P}(N_\tau = k) f_T(\tau) d\tau$$

- Very tedious and not very intuitive.

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- Let  $L$  be the number of arrivals from  $PP(\lambda)$  until we get the first arrival from  $PP(\nu)$ .

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right) \left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \dots$$

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  - $p = 1/100, n = 10,000$ : small  $p$ , but large  $n \implies$  **Both Poisson and Normal**

Questions?

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