Lecture 3: Random Variable, Part I

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

August 25, 2021

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Roadmap

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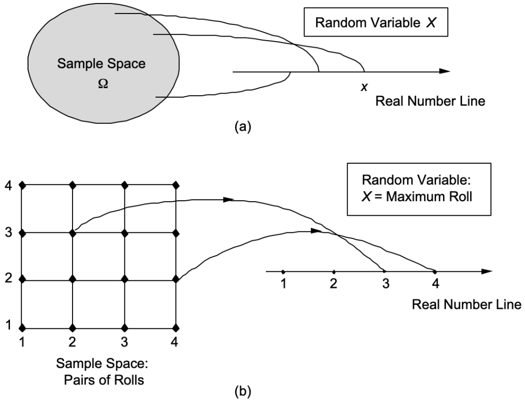
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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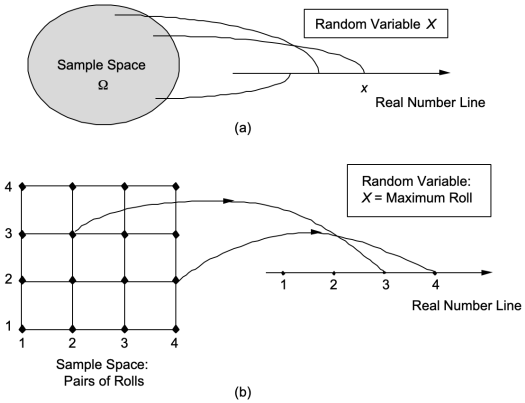

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Random Variable: More Formally

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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Example

- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$

- Remember that the random variable X is a **function** from Ω to $\mathbb{R}$

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Bernoulli X with parameter $p \in [0, 1]$

- Only binary values

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$$X = \begin{cases} 0, & \text{w.p.} \quad 1 - p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$

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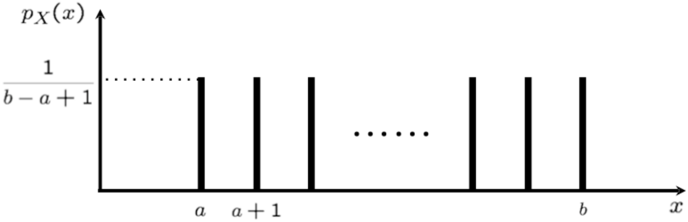

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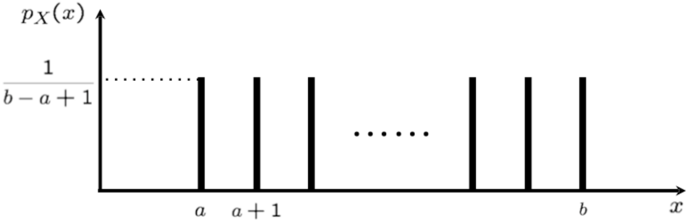

- Models complete ignorance (I don't know anything about X)

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Binomial X with parameter n, p

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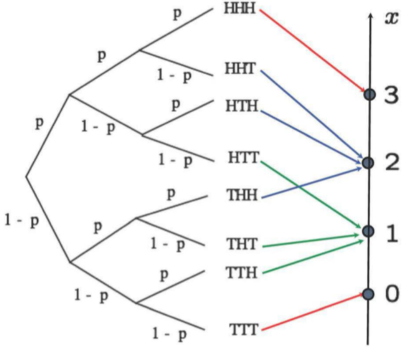

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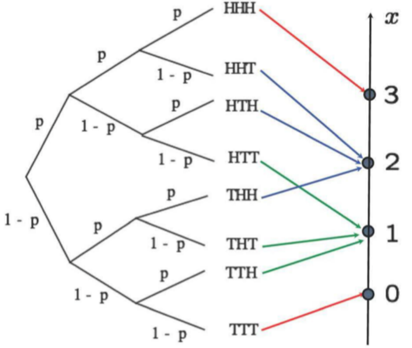
$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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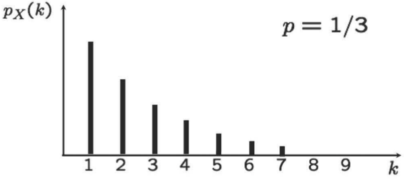
$$p_X(k) = (1-p)^{k-1} p$$



Geometric X with parameter p

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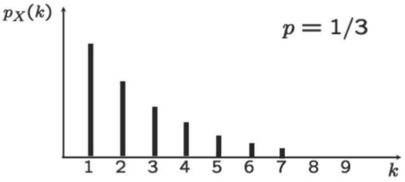
- Models waiting times until something happens.



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- Average

Definition

$$\mathbb{E}[X] = \sum_{x} x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)

Expectation/Mean

- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

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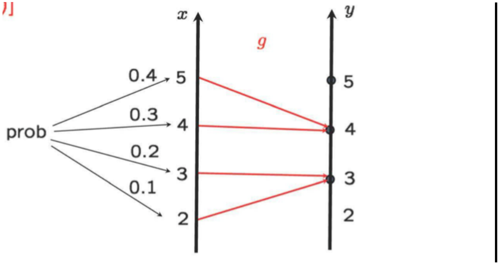

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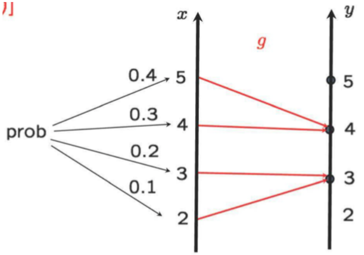

$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2)$$
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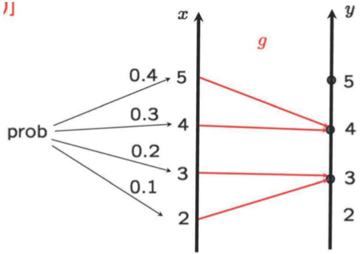

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Linearity of Expectation

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
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Example: Variance of a Bernoulli rv (p)

$$\mu = \mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p$$
$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1-p) = p$$
$$\text{var}[X] = \mathbb{E}[X^2] - \mu^2 = p - p^2$$
$$= p(1-p)$$

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- $\boxed{\phantom{\text{Marginal PMF.}}}$

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Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) =$

$p_X(4) =$

$\mathbb{P}(X = Y) =$

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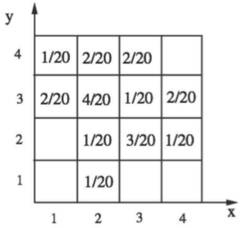
Example. VIDEO PAUSE



$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

Functions of Multiple RVs

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

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- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

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Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A.

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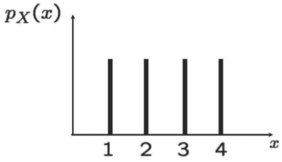
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$A = \{X \geq 2\}$



$\mathbb{E}[X] =$

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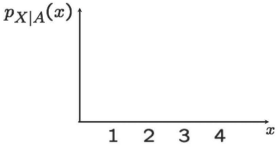

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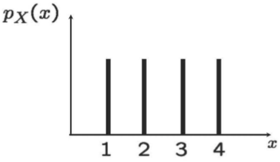

$$\mathbb{E}[X] = \frac{1}{4}(1+2+3+4) = 2.5$$

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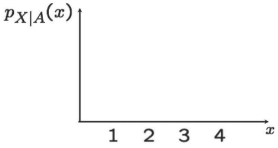

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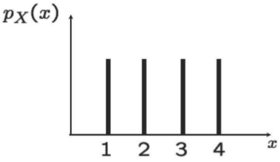

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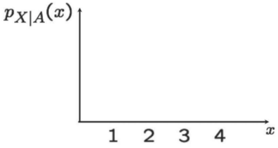

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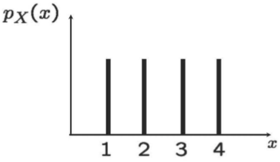

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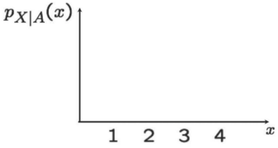

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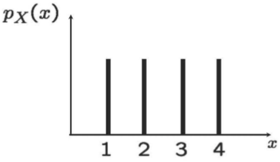

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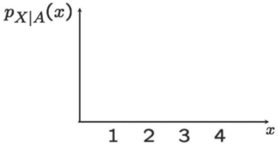

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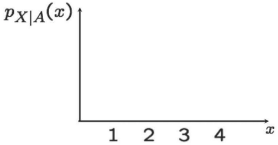

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Conditional PMF: Conditioning on a RV

KAIST EE

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- Conditional PMF

- Multiplication rule

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$

Conditional PMF

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$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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VIDEO PAUSE

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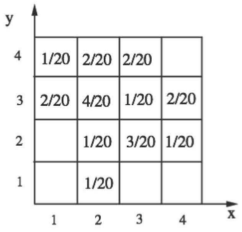

$p_{X|Y}(2|2) =$

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VIDEO PAUSE

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y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

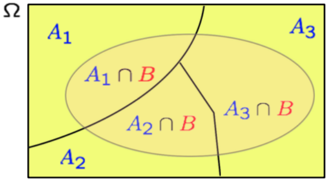
$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

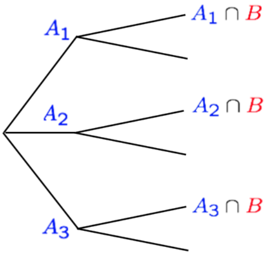
$$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$$

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$





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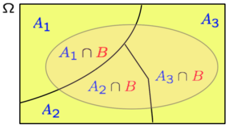
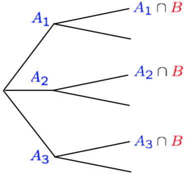




Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

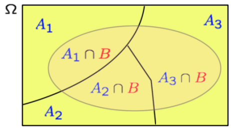
Total Probability Theorem

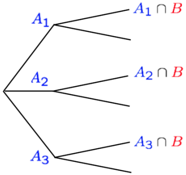
$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

- Weighted average of expectations from A_i's perspective





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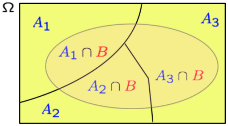

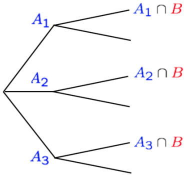

Total Expectation Theorem for $\{Y = y\}$

- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

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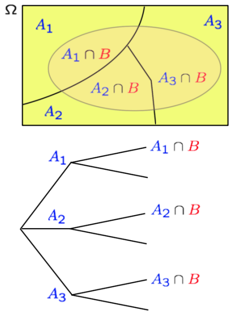
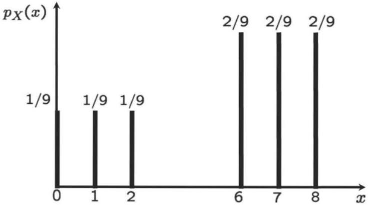


- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$$\mathbb{E}[X] =$$



Example 1: Total Expectation Theorem

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$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$
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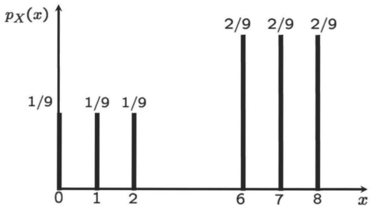

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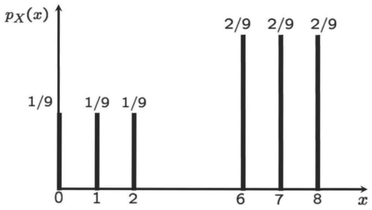
(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$



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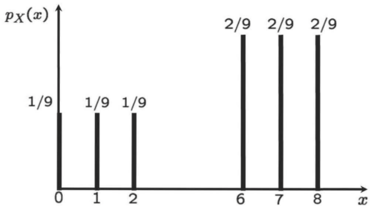
$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$
$$= \frac{3+12+14+16}{9} = 5$$

(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$
$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



Example 2: Mean of Geometric rv

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$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p + 2(1-p)p + 3(1-p)^2 p + \cdots$$

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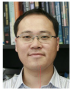
- Total expectation theorem and a notion of memorylessness helps a lot.

Memoryless Property: Motivating Example

Yung arrives

No bus arrives



How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n + m | X > m)$$

time

0

m mins

A bus left at time 0

time

0

Lin arrives

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n)$$

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Background: Memoryless Property

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$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$
- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.
$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

- Theorem. Any **geometric** random variable is memoryless.

Background: Memoryless Property of Geometric RVs

- Theorem. Any **geometric** random variable is memoryless.
- Remind. Geometric rv X with parameter p

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- Proof.

$$\mathbb{P}(X>n+m|X>m)=\frac{\mathbb{P}(X>n+m \text{ and } X>m)}{\mathbb{P}(X>m)}=\frac{\mathbb{P}(X>n+m)}{\mathbb{P}(X>m)}$$

$$=\frac{(1-p)^{n+m}}{(1-p)^m}=(1-p)^n=\mathbb{P}(X>n)$$

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- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$
\begin{aligned}
\mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\
&= && \text{(from TET)} \\
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- Thus, $\mathbb{E}[X] = \dfrac{1}{p}$

Roadmap

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

KAIST EE

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Independence, Conditional Independence

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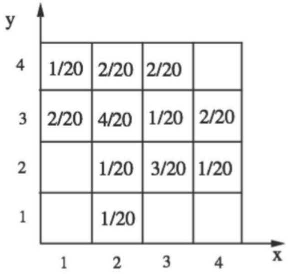
Example

- $X \perp\!\!\!\perp Y$?

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?



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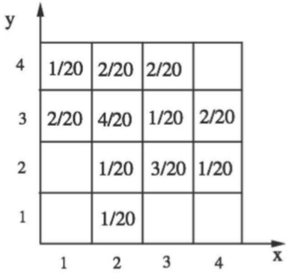

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VIDEO PAUSE

$Y = 4$		
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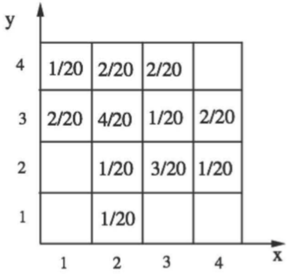

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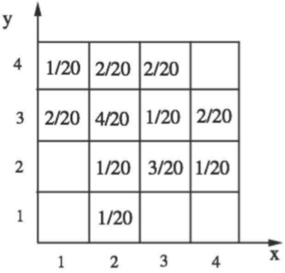

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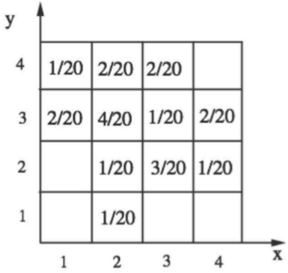
\- Yes.



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  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X-3Y] = \text{var}[X] + 9\text{var}[Y]$

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&= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)
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- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

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- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.

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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (from symmetry)

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Questions?

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFs?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?