Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

May 5, 2021

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

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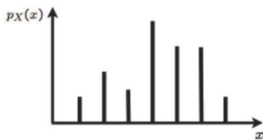$$\mathbb{P}(X \in B) = \int_B f_X(x)dx, \qquad \text{every subset } B \in \mathbb{R}$$

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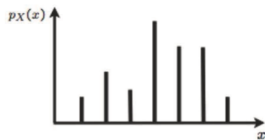

- $\mathbb{P}(a \leq X \leq b) = \sum_{x:a \leq x \leq b} p_X(x)$
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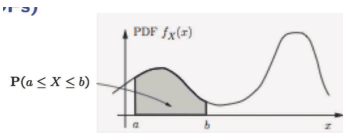
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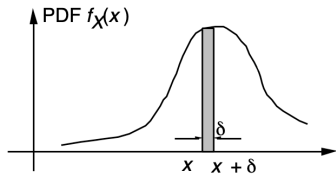
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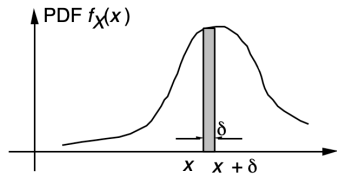

- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x)dx = 1$

PDF $f_X(x)$

x $x + \delta$

δ

- $\mathbb{P}(a \leq X \leq a + \delta) \approx \boxed{}$

Examples

PDF $f_X(x)$

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$x \quad x + \delta$

- $\mathbb{P}(a \leq X \leq a + \delta) \approx \boxed{f_X(a) \cdot \delta}$

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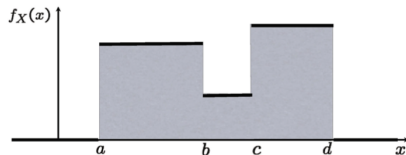δ

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Examples

$f_X(x)$

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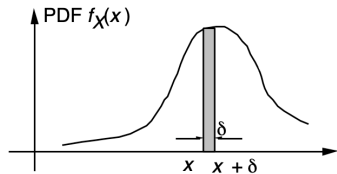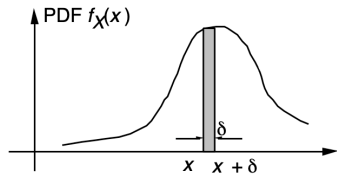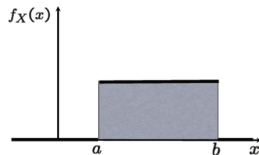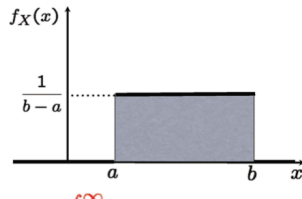
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- $\text{var}[X] =$
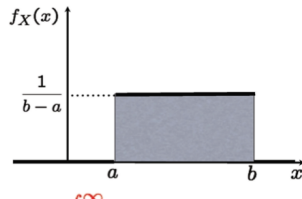
- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} \frac{b^2-a^2}{2} = \frac{b+a}{2}$

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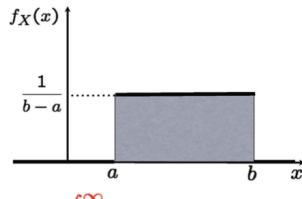
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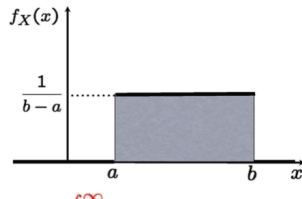
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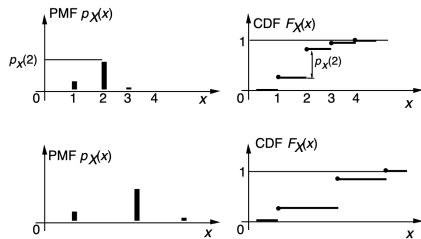
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  - $F_X(x)$ is a continuous function of x.
  - $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ and $f_X(x) = \dfrac{dF_X}{dx}(x)$

Example: Maximum of Random Variables

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$, and $X_i \in \{1, 2, \cdots, 10\}$ uniformly at random
- Question. $p_X(x)$?
- Approach 1: $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$?
- Approach 2

$$F_X(x) = \mathbb{P}(\max\{X_1, X_2, X_3\} \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x)$$
$$= \mathbb{P}(X_1 \leq x) \cdot \mathbb{P}(X_2 \leq x) \cdot \mathbb{P}(X_3 \leq x) = \left(\frac{x}{10}\right)^3$$

Thus,

$$p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \cdots, 10$$

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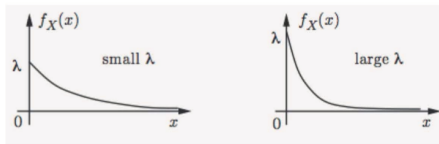
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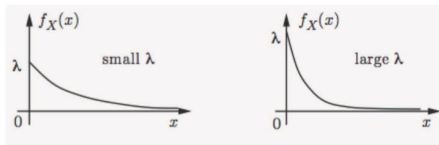
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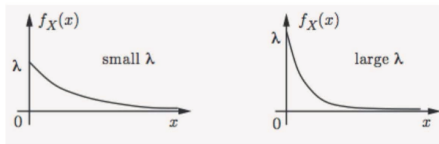
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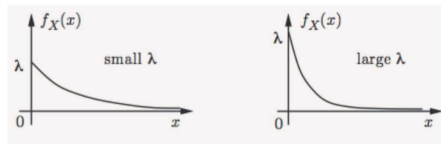
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- (Check) $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$, $\text{var}[X] = 1/\lambda^2$

- $\mathbb{E}(X) = 1/\lambda$. Use integration by parts: $\int u dv = uv - \int v du$

$$\int_0^\infty x\lambda e^{-\lambda x} dx = (-xe^{-\lambda x})\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^\infty x^2\lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx = 0 + \frac{2}{\lambda}\mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

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- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

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- (Solution)
  - $\mathbb{E}(X) = 1/\lambda = 10$. Thus, $\lambda = \frac{1}{10}$.
  - 6 a.m. from midnight $= 1/4$ day, 6 p.m. from midnight $= 3/4$ day

  $$\mathbb{P}(1/4 \leq X \leq 3/4) = \mathbb{P}(X \geq 1/4) - \mathbb{P}(X \geq 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

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- In many cases, continuous case is some type of limit of its corresponding discrete case.
- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

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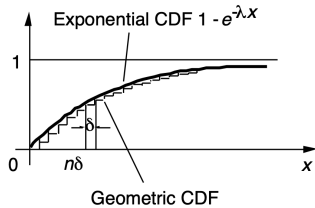 - Set the length of a slot to be δ, which is a parameter.

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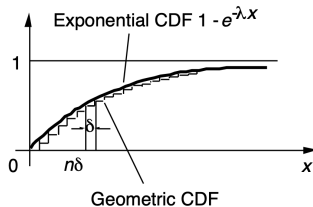
Geometric vs. Exponential (2)

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  - $\mathbb{P}(X_\delta^{geo} \leq n) = 1 - (1 - p_\delta)^n = 1 - e^{-\lambda \delta n}$
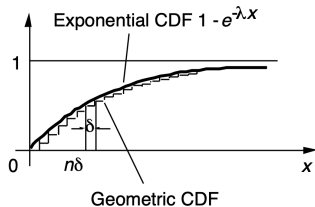
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$$\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda \delta n} = \mathbb{P}(X^{geo}_{\delta} \leq n)$$

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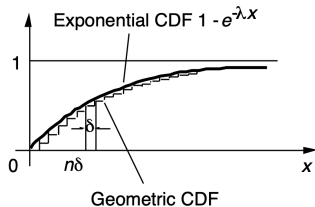
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- If we choose sufficiently small δ, the slot length $\downarrow$ and $p_{\delta} \downarrow$

$$\mathbb{P}(X^{geo}_{\delta} \leq n) \xrightarrow{\delta \to 0} \mathbb{P}(X^{exp} \leq x), \, x = n\delta$$

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(6) Bayes' rule for RVs

- Standard Normal $\mathcal{N}(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

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$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

- General Normal $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$
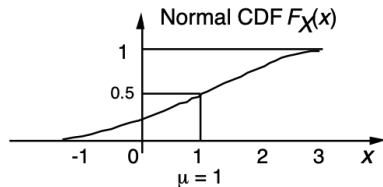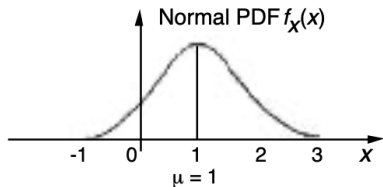
Normal PDF $f_X(x)$

Normal CDF $F_X(x)$

- PDF's normalization property: $\dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

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  - A little bit boring :-). See Problem 14 at pp 189.

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  ◦ A little bit boring :-). See Problem 14 at pp 189.

- Expectation

  ◦ $f_X(x)$ is symmetric in terms of $x = \mu$. Thus, we should have $\mathbb{E}(X) = \mu$.

- PDF's normalization property: $\dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$
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- Expectation
  - $f_X(x)$ is symmetric in terms of $x = \mu$. Thus, we should have $\mathbb{E}(X) = \mu$.

- Variance

$$\text{var}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \overset{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}(-ye^{-y^2/2})\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sigma^2$$

$\int u\,dv = uv - \int v\,du$: $u = y$ and $dv = ye^{-y^2/2} \rightarrow du = dy$ and $v = -e^{-y^2/2}$

- Linear transformation preserves normality (we will verify this in Lecture 5)

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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- Thus, every normal rv can be $\boxed{}$:

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{} \sim \mathcal{N}(0, 1)$

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- Thus, every normal rv can be $\boxed{\text{standardized}}$:

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{Y = \frac{X - \mu}{\sigma}} \sim \mathcal{N}(0, 1)$

KAIST EE

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> If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be standardized :

  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

- Thus, we can make the table which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$$

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

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2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Example

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

- $Y = \frac{X-60}{20}$.

$$\mathbb{P}(X \geq 80) = \mathbb{P}(Y \geq \frac{80-60}{20})$$
$$= \mathbb{P}(Y \geq 1) = 1 - \Phi(1)$$
$$= 1 - 0.8413 = 0.1587$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

- Central limit theorem
  - One of the most remarkable findings in the probability theory

  - Sum of any random variables $\approx$ Normal random variable

- Modeling aggregate noise with many small, independent noise terms

- Convenient analytical properties, allowing closed forms in many cases

- Highly popular in communication and machine learning areas

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[0]Central limit theorem: 중심극한정리

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

Two continuous rvs are _____ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y)dxdy,$$

Two continuous rvs are jointly continuous if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y)dxdy,$$

Two continuous rvs are $\boxed{\text{jointly continuous}}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y)dxdy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}\Big[(X,Y) \in B\Big] = \iint_{(x,y) \in B} f_{X,Y}(x,y)dxdy$$

Our particular interest: $B = \{(x,y) \mid a \le x \le b, c \le y \le d\}$

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

4. A function $g(X, Y)$ of X and Y defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

* Conditional PDF, given an event A

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

* Conditional PDF, given an event A

* $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

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- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
  $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

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  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
  $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

- $\int f_{X|A}(x) dx = 1$

* Conditional PDF, given $\{X \in C\}$

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* Conditional PDF, given an event A

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  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
  $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

- $\int f_{X|A}(x) dx = 1$

* Conditional PDF, given $\{X \in C\}$

$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \le X \le x + \delta | X \in C)$

$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

* Conditional PDF, given an event A

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
  $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

- $\int f_{X|A}(x) dx = 1$

* Conditional PDF, given $\{X \in C\}$

$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$

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(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

- $\mathbb{E}[X] = \int x f_X(x) dx$
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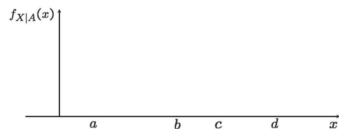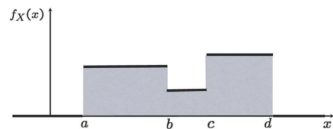
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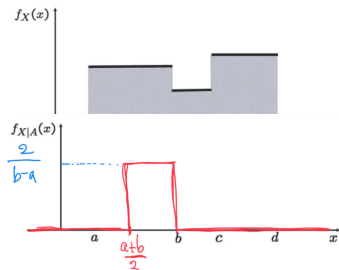
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$$\mathbb{E}[X|A] = \int_{(a+b)/2}^{b} x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

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Partition of Ω into $A_1, A_2, A_3, \ldots$

* Discrete case

* Continuous case

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Total Probability Theorem
$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i)$$
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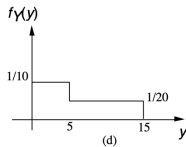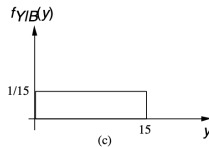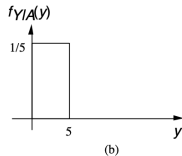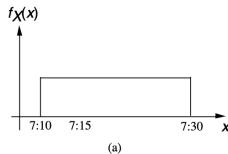
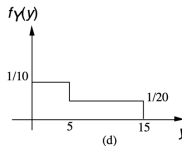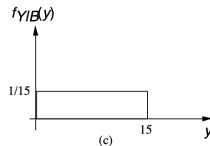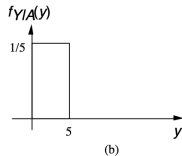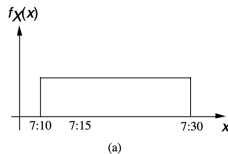- Your arrival $\sim \mathcal{U}(7{:}10, 7{:}30)$ am.

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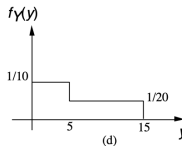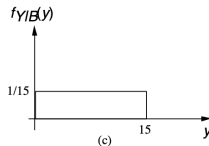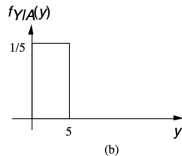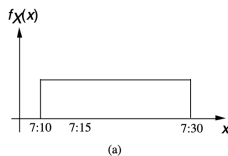
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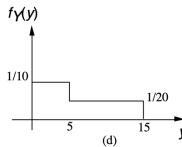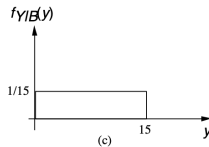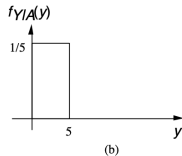- X : your arrival time, Y : waiting time.

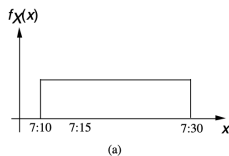## Example: Train Arrival

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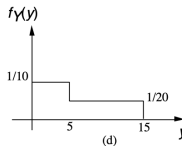
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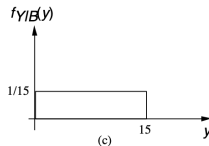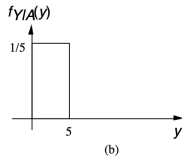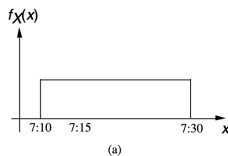
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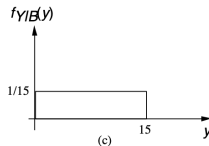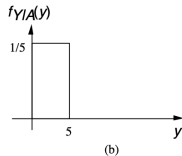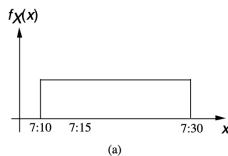
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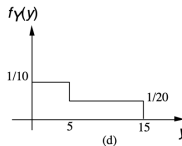$$f_Y(y) = \frac{1}{4}0 + \frac{3}{4}\frac{1}{15} = \frac{1}{20}, \quad \text{for } 5 < y \leq 15$$

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- Independence

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \quad \text{for all } x \text{ and } y$$

(Prob 21 at pp. 191)

- Break a stick of length l twice
  - first break at $Y \sim \mathcal{U}[0, l]$
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$^0\mathcal{U}[a, b]$: continuous uniform random variable over the interval $[a, b]$

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Using $f_{X,Y}(x, y) = f_Y(y) f_{X|Y}(x|y)$,

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(b) marginal PDF $f_X(x)$?

$$f_X(x) = \int f_{X,Y}(x, y)dy = \int_x^l \frac{1}{ly} dy$$
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---

[0]$\mathcal{U}[a, b]$: continuous uniform random variable over the interval $[a, b]$

(c) Evaluate $\mathbb{E}(X)$, using $f_X(x)$

(d) Evaluate $\mathbb{E}(X)$, using $X = Y \cdot (X/Y)$

If $Y \perp\!\!\!\perp X/Y$, it becomes easy, but true?

(e) Evaluate $\mathbb{E}(X)$, using TET

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Example: Stick-breaking (2)

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- Message. There are many ways to rearch our goal. Of crucial importance is how to find the best way!

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

Bayes Rule for Continuous

- X: state/cause/original value $\rightarrow$ Y: result/resulting action/noisy measurement

- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (cause $\rightarrow$ result)

- Inference: $\mathbb{P}(X|Y)$?

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$$p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y|x)$$
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- X: **parameter** $\rightarrow$ Y: result of **my model**

- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (parameter $\rightarrow$ model)

- Inference: $\mathbb{P}(X|Y)$? Probabilistic feature of the parameter given the result of the model?

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1. Light bulb's lifetime $Y \sim \exp(\lambda)$. Given the lifetime y, the modified belief about λ?

2. Romeo and Juliet start dating, but Romeo will be late by a random variable $Y \sim \mathcal{U}[0, \theta]$. Given the time of being late y, the modified belief about θ?

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- Wait! $p_{K|Y}(k|y)$? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

- For small δ (in other words, taking the limit as $\delta \to 0$).

  Let $A = \{K = k\}$.

$$
\begin{aligned}
p_{K|Y}(k|y) &\approx \mathbb{P}(A | y \le Y \le y + \delta) \\
&= \frac{\mathbb{P}(A)\mathbb{P}(y \le Y \le y + \delta | A)}{\mathbb{P}(y \le Y \le y + \delta)} \\
&\approx \frac{\mathbb{P}(A) f_{Y|A}(y) \delta}{f_Y(y) \delta} \\
&= \frac{\mathbb{P}(A) f_{Y|A}(y)}{f_Y(y)}
\end{aligned}
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- Your received signal = 0.7. What's your guess about the original signal? +1
- Your received signal = -0.2. What's your guess about the original signal? -1

- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

- $Y|\{K = 1\} \sim \mathcal{N}(1, 1)$ and $Y|\{K = -1\} \sim \mathcal{N}(-1, 1)$.
  (Remind: linear transformation preserves normality.)

Example: Signal Detection (2)

- $Y|\{K=1\} \sim \mathcal{N}(1,1)$ and $Y|\{K=-1\} \sim \mathcal{N}(-1,1)$.
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$$f_{Y|K}(y|k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}, \quad k = 1, -1$$

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- Probability that $K = 1$, given $Y = y$? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$
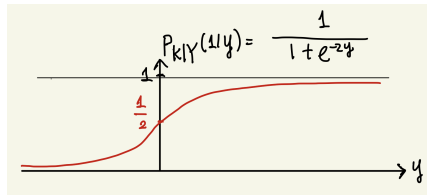
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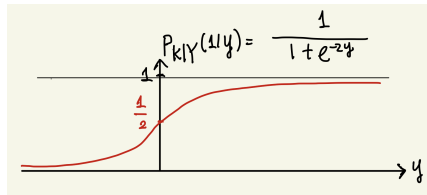 (Remind: linear transformation preserves normality.)

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◦ If $y > 0$, the inference probability for $K = 1$
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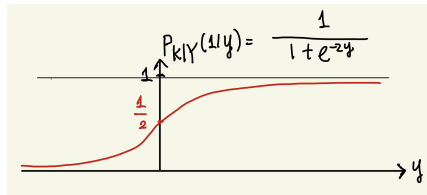 (Remind: linear transformation preserves normality.)

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- If $y > 0$, the inference probability for $K = 1$ exceeds $\frac{1}{2}$. So, original signal = 1.
- Similarly, compute $p_{K|Y}(-1|y)$ and then do the inference

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$

Questions?

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1) What is PDF and CDF?

2) Why do we need CDF?

3) What are joint/marginal/conditional PDFs?

4) Explain memorylessness of exponential random variables.

5) Explain the version of Bayes' rule for continuous and mixed random variables.