

Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
 - Definition, Transition Probability Matrix, State Transition Diagram
 - Classification of States
 - Steady-state Behaviors and Stationary Distribution
 - Transient Behaviors

- Assume discrete times $n = 1, 2, \dots$
- Random process: A sequence of X_1, X_2, X_3, \dots

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- **Markov chain**
- One of the most popular random processes in engineering

- A machine: working or broken down on a given day.
 - If working, break down in the next day w.p. b , and continue working w.p. $1 - b$.
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- What will happen at $(n + 1)$ -th day depends only on what happens at n -th day?

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Thus, for any $n \geq 0$, we introduce a simple notation p_{ij}

$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

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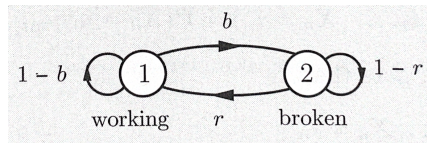
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$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$

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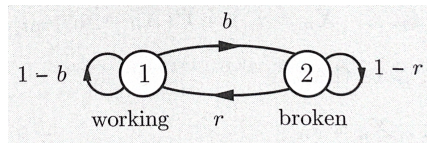
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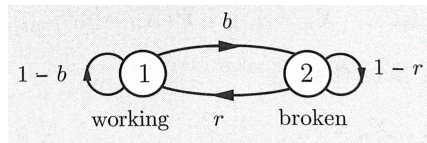
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- $\sum_{j=1}^m p_{ij} = 1$ (for each row i , the column sum = 1)

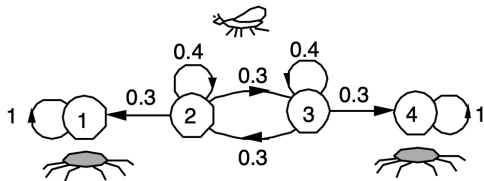
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	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

P_{ij}

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$$\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} = \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2$$

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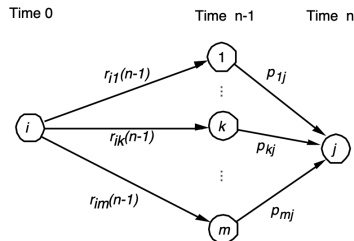
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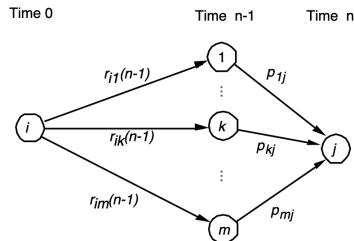
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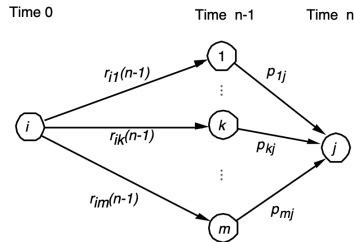
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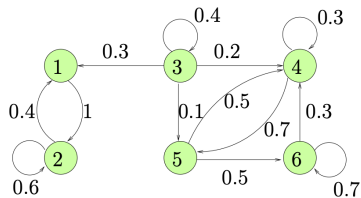
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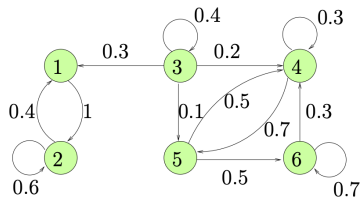


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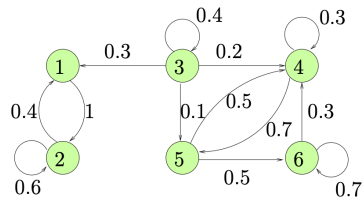
- Classes



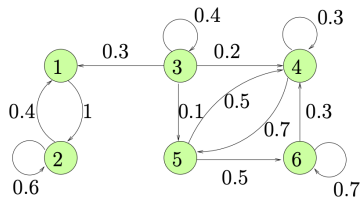
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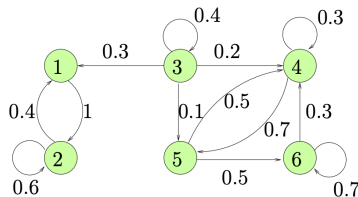
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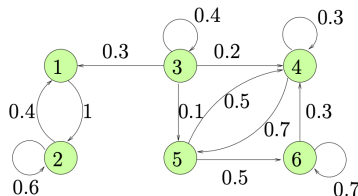
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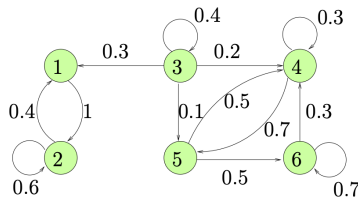
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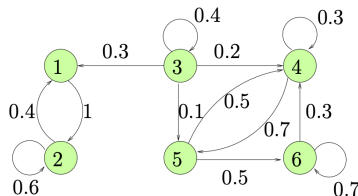
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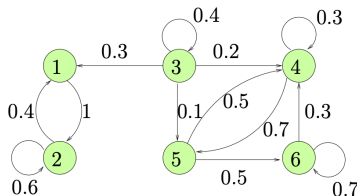
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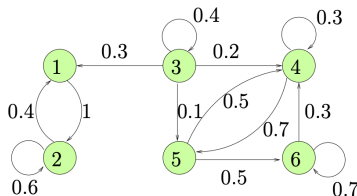
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- Difference between 1 and 3
 - 1: If I start from 1, visit 1 infinite times.



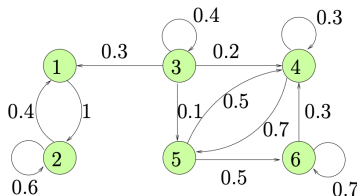
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 - 1 and 2 can reach each other but no other state
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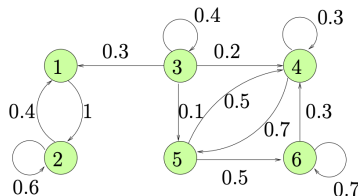


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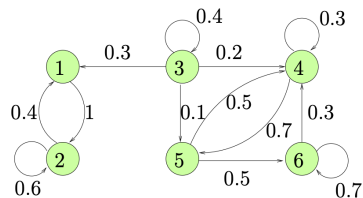
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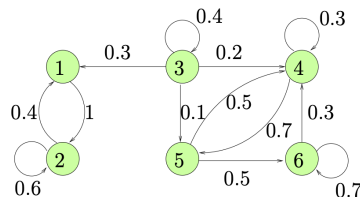


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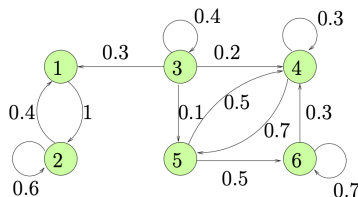
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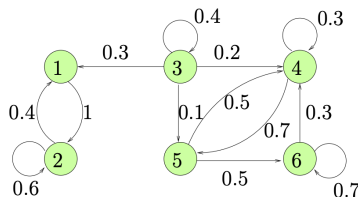
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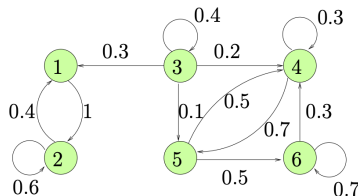
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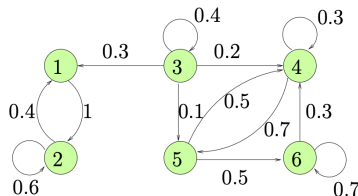
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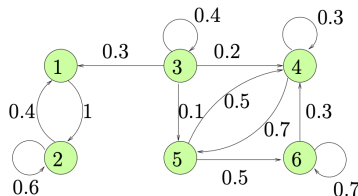
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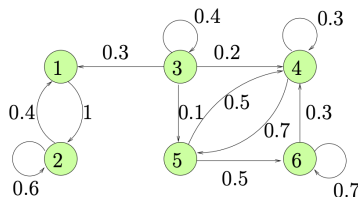
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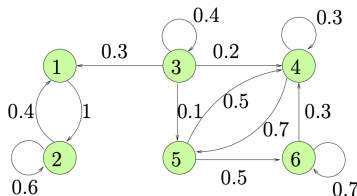
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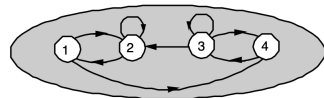


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 - If we start from a recurrent state i , then there is always some probability of returning to i . It means that, given enough time, it is certain that it returns to i .

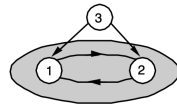


Classification of States (2)

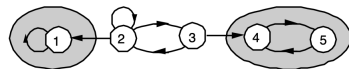
- A set of recurrent states which communicate with each other form a **class**.



Single class of recurrent states



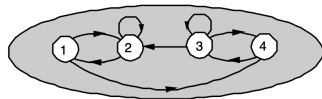
Single class of recurrent states (1 and 2)
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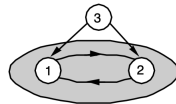
Two classes of recurrent states
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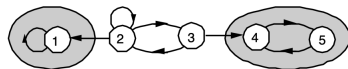
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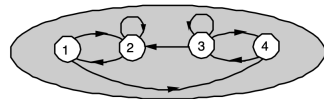


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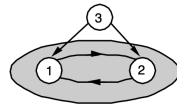


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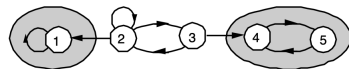
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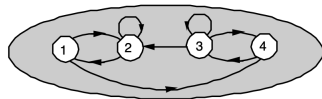


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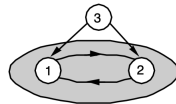


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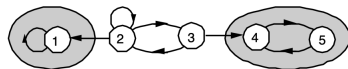
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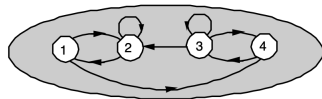


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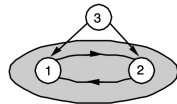


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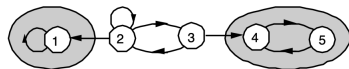
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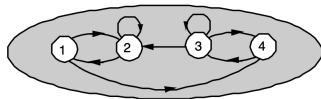


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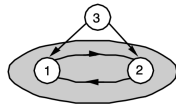


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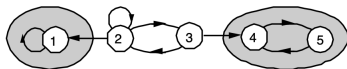
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- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



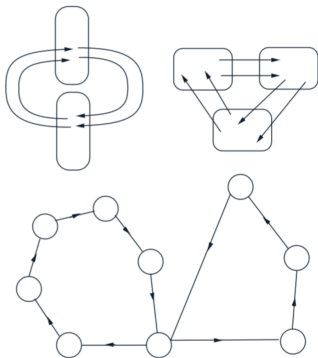
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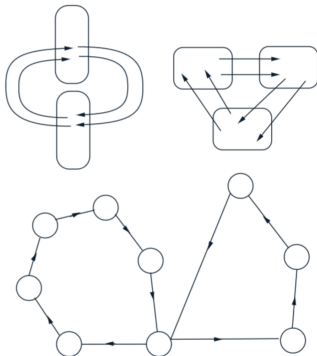
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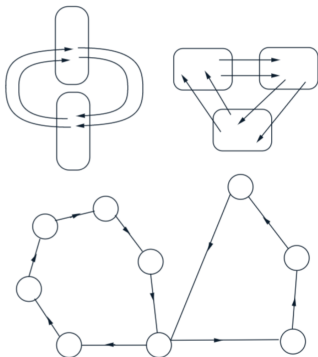
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- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.

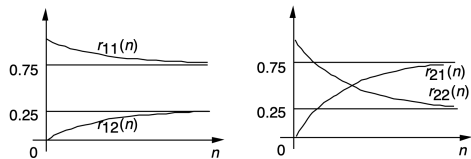


- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
- A recurrent class that is not periodic is said to be aperiodic.



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
 - Definition, Transition Probability Matrix, State Transition Diagram
 - Classification of States
 - **Steady-state Behaviors and Stationary Distribution**
 - Transient Behaviors

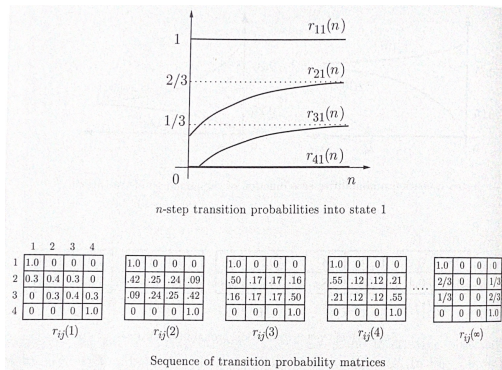
n -step transition prob.: $r_{ij}(n)$ for large n



n -step transition probabilities as a function of the number n of transitions

	UpD	B						
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$	
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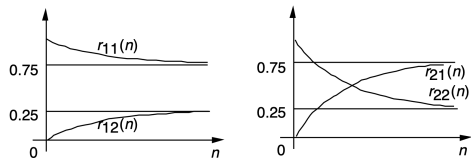
Sequence of n -step transition probability matrices



Sequence of transition probability matrices

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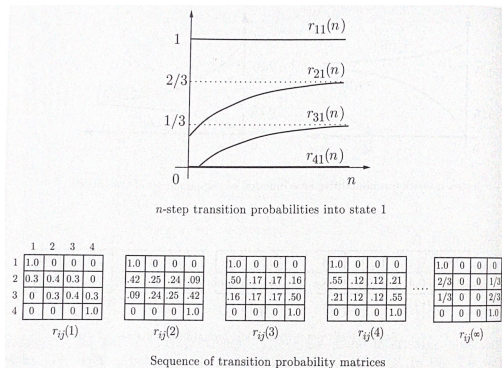
- Convergence irrespective of the starting state



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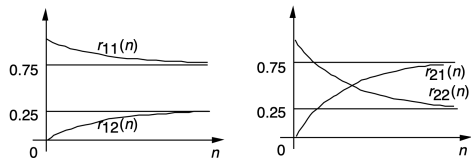
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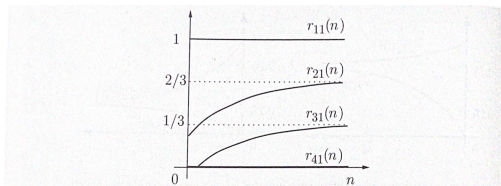


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Sequence of n -step transition probability matrices

- Convergence depending on the starting state



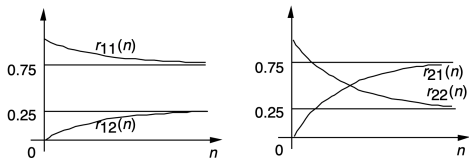
n -step transition probabilities into state 1

	1	2	3	4													
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21	2/3
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55	1/3
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$				$r_{ij}(\infty)$

Sequence of transition probability matrices

n -step transition prob.: $r_{ij}(n)$ for large n

- Convergence irrespective of the starting state

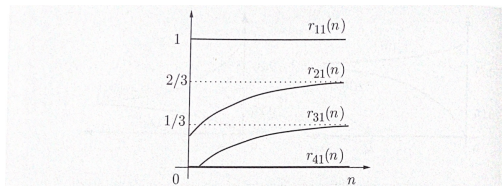


n -step transition probabilities as a function of the number n of transitions

	UpD	B								
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496	.7501	.2499
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	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(5)$	

Sequence of n -step transition probability matrices

- Convergence depending on the starting state



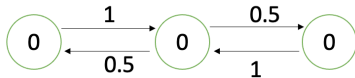
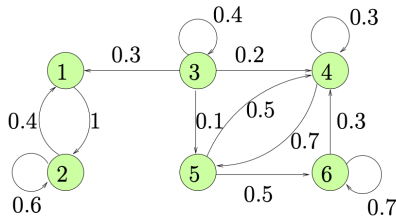
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4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0					
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$				$r_{ij}(\infty)$			

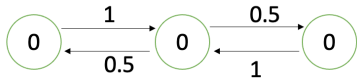
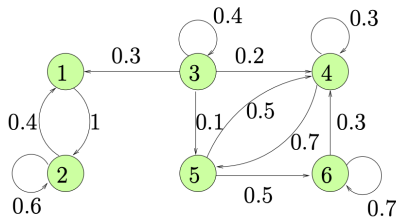
Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?

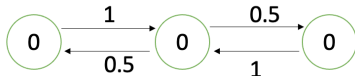
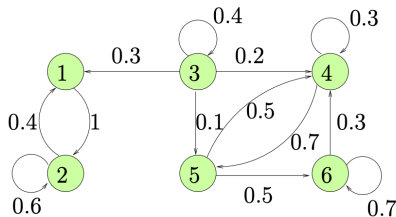


- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
- Convergence occurs, independent of the starting state, if:



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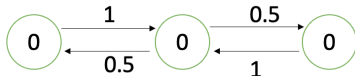
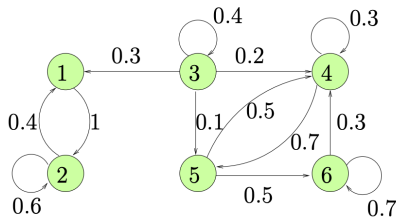
C1. Only a single recurrent class



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C1. Only a single recurrent class

C1. For the case of multiple recurrent classes, one stays at the class including the starting state.

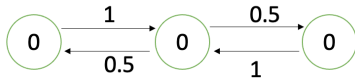
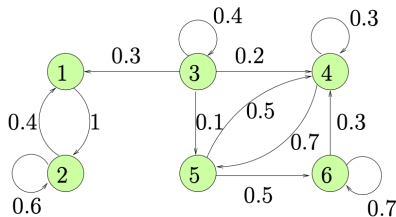


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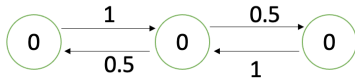
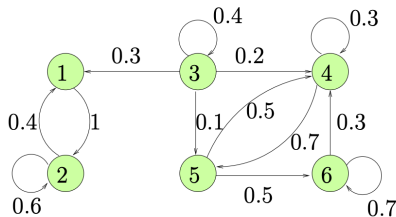
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C2. Divergent behavior for periodic recurrent classes.



- If $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$,

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- Balance equation + Normalization equation \implies Finding the steady-state probabilities $\{\pi_i\}$.

- A two-state MC with:

$$p_{11} = 0.8, \quad p_{12} = 0.2,$$

$$p_{21} = 0.6, \quad p_{22} = 0.4.$$

- Balance equation:

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$$

- Normalization equation: $\pi_1 + \pi_2 = 1$
- The stationary distribution is: $\pi_1 = 0.25, \pi_2 = 0.75$.

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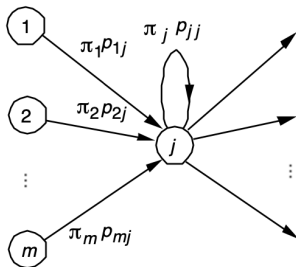
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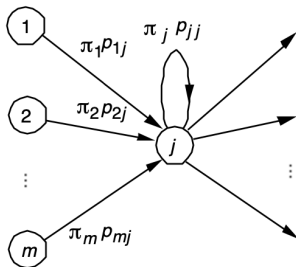
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 - If the initial state is chosen according to $\{\pi_j\}$, the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution is equal to the stationary distribution"

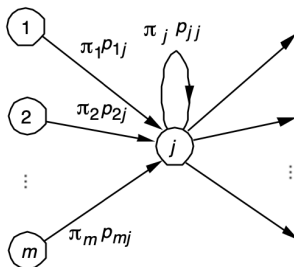
- π_j : the long-term **expected fraction of time** that the state is equal to j .



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 - The expected frequency π_j of visits to j is equal to the sum of the expected frequencies $\pi_k p_{kj}$ of transitions that lead to j .



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
 - Definition, Transition Probability Matrix, State Transition Diagram
 - Classification of States
 - Steady-state Behaviors and Stationary Distribution
 - **Transient Behaviors**

Absorption Probability

- **Definition.** A state k is **absorbing**, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.

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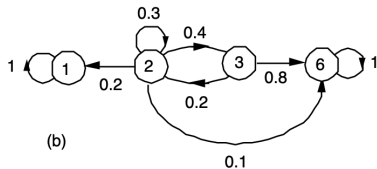
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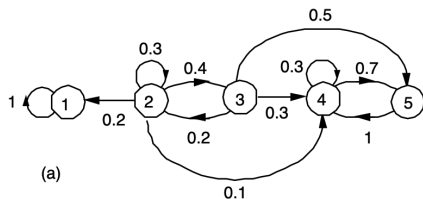
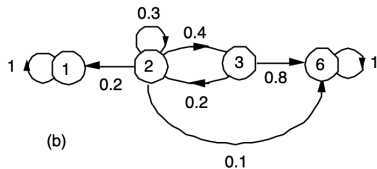
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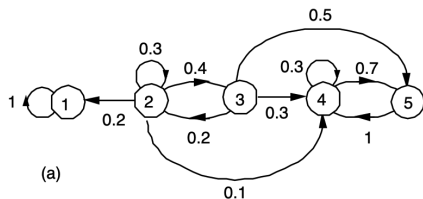
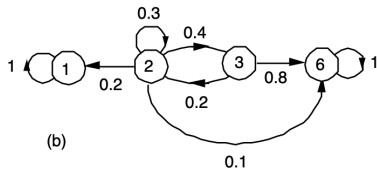
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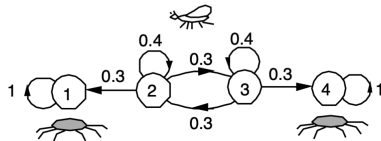


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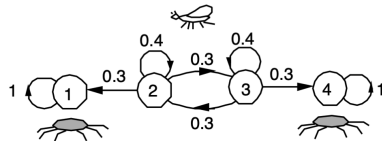
- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

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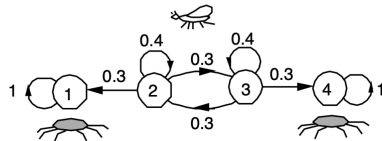


- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

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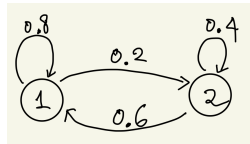
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- For generalized description, please see the textbook (pp. 367).

Expected time to a particular recurrent state s

- Assume a single recurrent class

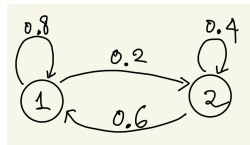


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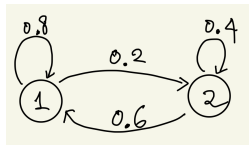


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$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

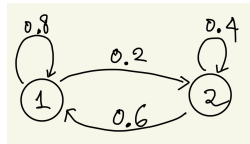
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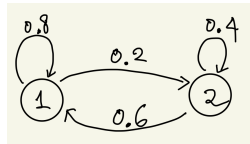
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- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

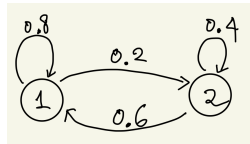
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Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?