Lecture 6: Law of Large Numbers and Central Limit Theorem

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

- Two most remarkable findings in probability theory

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- $X_1, X_2, \ldots, X_n$: i.i.d (independent and identically distributed) random variables
- $\mathbb{E}[X_i] = \mu$, $\text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

$$\boxed{S_n = X_1 + X_2 + \ldots + X_n}$$

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- Possible apporach. Take a certain scaling with respect to n that corresponds to a new glass, and investigate the system for large n

Sample Mean

- Consider the sample mean, and try to understand how S_n behaves:

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- We call this law of large numbers (LLN).

Let's Establish Mathematically

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- Need to build up the new concept of convergence for the sequence of rvs.

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  - For any $\epsilon > 0$, $\mathbb{P}\left(\{|Y_n - Y| \geq \epsilon\}\right) \xrightarrow{n\to\infty} 0$.

Convergence in Probability (2)

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- A special case: when $Y = a$ for some constant a: $Y_n \xrightarrow{\text{in prob.}} a$
- `https://youtu.be/Ajar_6MAOLw?t=248`

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- Proof. For any $\epsilon > 0$,
$$\begin{aligned}\mathbb{P}(|Y_n - 0| \geq \epsilon) = \mathbb{P}(X_1 \geq \epsilon, \ldots, X_n \geq \epsilon) &= \mathbb{P}(X_1 \geq \epsilon) \times \cdots \times \mathbb{P}(X_n \geq \epsilon)\\ &= (1-\epsilon)^n \xrightarrow{n\to\infty} 0\end{aligned}$$

Example 2: Convergence in Probability

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Example 3: Convergence in Probability

- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

- Consider a sequence of rvs Y_n with the following distribution:

$$\mathbb{P}(Y_n = y) = \begin{cases} 1 - \frac{1}{n}, & \text{for } y = 0 \\ \frac{1}{n}, & \text{for } y = n^2 \\ 0, & \text{otherwise} \end{cases}$$

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- Thus, Y_n converges to 0 in probability.

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- The proof requires some knowledge about useful inequalities, which we will cover later.

WLLN: Why Useful?

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- For example, assume that a large number of identically distributed noises come to you. Then, you can roughly approximate it as ($n \times$ average noise)
- Provides an interpretation of expectations (as well as probabilities) in terms of a long sequence of identical independent experiments. For example, what is the probability of head of a coin? Toss 1000 times, and count the number of heads.

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- The answer is $\frac{1}{2}$

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- Interestingly, it converges to some well-known random variable.

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- Let $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$. Then, $\mathbb{E}[Z_n] = 0$ and $\text{var}(Z_n) = 1$.
- Z_n is well-centered with a variance irrespective of n.
- We expect that Z_n converges to something meaningful as $n \to \infty$, but what?
- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.
    - Need a new concept of convergence: "convergence in distribution"

- Consider a sequence of rvs $(Y_n)_{n=1,2,\ldots}$ and a rv Y.

Convergence in Distribution: $Y_n \xrightarrow{\text{in dist.}} Y$

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- Another type of convergence of rvs
- Comparison with convergence in probability?
  - Convergence in probability $\Longrightarrow$ Convergence in distribution, but the reverse is not true.
  - The proof is beyond what this class covers, but it will be interesting to find an example that shows convergence in distribution, which is not convergence in probability.

- $X_n \sim \text{Bernoulli}(1/2)$, for all $n \geq 1$.

Example: in Distribution, but not in Probability

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$$\begin{aligned}\mathbb{P}(|X_n - X| \geq \epsilon) &= \mathbb{P}(|X_n - 1 + X_n| \geq \epsilon) = \mathbb{P}(|2X_n - 1| \geq \epsilon) \\ &= \mathbb{P}(1 \geq \epsilon) \qquad (\text{because } |2X_n - 1| = 1)\end{aligned}$$

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- We can find ϵ small enough so that the above does not go to zero.

- $S_n = X_1 + X_2 + \cdots + X_n, \qquad Z_n = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$

Central Limit Theorem

Z_n convergens to Z in distribution, where $Z \sim \mathcal{N}(0,1)$.

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Central Limit Theorem

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- Very surprising!
- Irrespecitive of the distribution of X_i, Z is normal.

- For simplicity, assume that $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$, $i = 1, 2, \ldots, n$

LLG vs. CLT: Different Scaling Glasses

- For simplicity, assume that $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$, $i = 1, 2, \ldots, n$
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> Scaling S_n by $1/n$, you go to a deterministic world.

- Central Limit Theorem

> Scaling S_n by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each X_i.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \qquad \mathbb{P}(Z_n \le z) \xrightarrow{n\to\infty} \mathbb{P}(Z \le z),\ Z \sim \mathcal{N}(0,1)$$

[1] Only unique mode. A single maximum or minimum.

Practical Use of CLT

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- Can approximate S_n with a normal rv $\sim (n\mu, n\sigma^2)$
  - $S_n = n\mu + Z_n\sigma\sqrt{n}$

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  - If X_i resembles a normal rv more, smaller n works: symmetry and unimodality[1]

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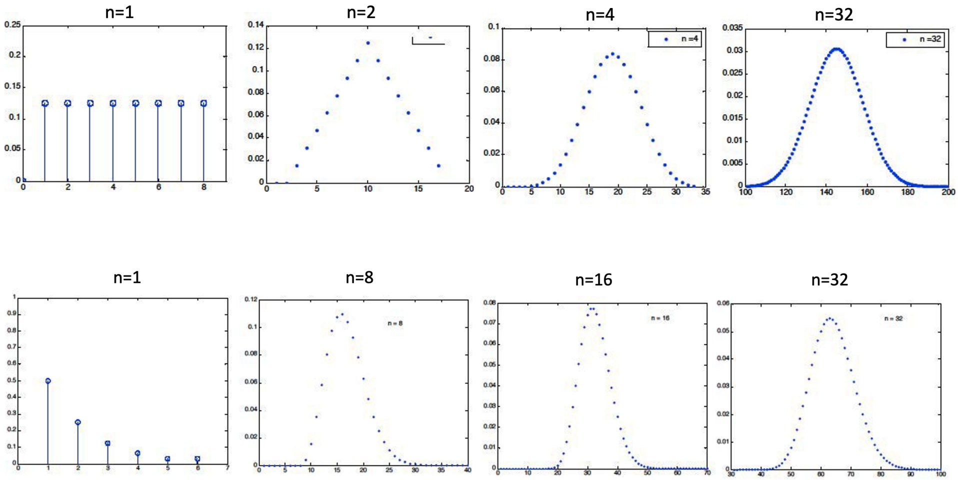

n=1
n=2
n=4
n =4
n=32
n =32
n=1
n=8
n = 8
n=16
n = 16
n=32
n = 32

$\mathbb{P}(S_n \leq a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)

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- Load container with $n = 100$ packages

$$\mathbb{P}(S_{100} \ge 210) = \mathbb{P}\left[\frac{S_{100} - 100 \cdot 2}{2\sqrt{100}} \ge \frac{210 - 200}{20}\right] = \mathbb{P}(Z_{100} \ge 0.5)$$

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$$\approx \mathbb{P}(Z \ge 0.5) = 1 - \mathbb{P}(Z \le 0.5) = 1 - \Phi(0.5)$$

Examples of CLT (2)

$\mathbb{P}(S_n \leq a) \approx b$: Given two parameters, find the third

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- The value of a such that $\Phi(\frac{a-200}{20}) = 0.95$? $\frac{a-200}{20} = 1.645$ and $a = 232.9$

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- The value of n such that $\frac{210-2n}{2\sqrt{n}} = 1.645$? $n = 89$

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(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

- (Q) Knowing $\mathbb{E}(X)$, can we say something about the distribution of X?

Markov Inequality

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Proof. For any $a > 0$, define Y_a as:

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Thus, $a \cdot \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$. □

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- For reasonably large a, CI provides much better bound.
- Knowing the variance helps
- Both bounds are the ones that bound the probability of rare events.

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

Weak law of large numbers

M_n converges to μ in probability.

Proof. For any given $\epsilon > 0$,

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Weak law of large numbers

M_n converges to μ in probability.

Proof. For any given $\epsilon > 0$,

$$\mathbb{P}\left(|M_n - \mu| \geq \epsilon\right) \leq \frac{\text{var}(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{n \to \infty} 0$$

Comparison: WLLN vs. CLT

We ask the same question, and try to answer it, using WLLN or CLT.

See how the answers become different.

Example: Polling using WLLN

- p: fraction of voters who support "Yung".

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Example: Polling using CLT

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- Compare: 50,000 from LLN vs. 9604 from CLT

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

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- If the context is clear, we omit X and use just $M(s)$.

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$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2 \\ 1/6, & \text{if } x = 3 \\ 1/3, & \text{if } x = 5 \end{cases}$$

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- Question. MGF of $\mathcal{N}(\mu, \sigma^2)$?

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

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- Thus, $\text{var}(X) = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

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Inversion Property

The transform $M_X(s)$ associated with a random variable X uniquely determines the CDF of X, assuming that $M_X(s)$ is finite for all s in some interval $[-a, a]$, where a is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own unique MGF.

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$$p_X(-1) = \frac{1}{4},\ p_X(0) = \frac{1}{2},\ p_X(4) = \frac{1}{8},\ p_X(5) = \frac{1}{8}$$

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- X is a geometric rv with parameter p

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Back to CLT Proof (1)

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$
- $Z_n = \dfrac{S_n}{\sqrt{n}} = \dfrac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$
- We will show: MGF of Z_n converges to MFG of $\mathcal{N}(0,1)$ (using inversion property)
- Proof.

$$\mathbb{E}\left[e^{sS_n/\sqrt{n}}\right] = \mathbb{E}\left[e^{sX_1/\sqrt{n}}\right] \times \cdots \times \mathbb{E}\left[e^{sX_n/\sqrt{n}}\right]$$
$$= \left(\mathbb{E}\left[e^{sX_1/\sqrt{n}}\right]\right)^n = \left(M_{X_1}\left(\frac{s}{\sqrt{n}}\right)\right)^n$$

- For simplicity, let $M(\cdot) = M_{X_1}(\cdot)$

Back to CLT Proof (2)

- $M(0) = 1, \; M'(0) = 0, \; M''(0) = 1$

$\square$

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- If we apply l'hopital's rule twice (please check), we get

$$\lim_{y\to 0} \frac{\log M(ys)}{y^2} = \frac{s^2}{2}$$

$\square$

Questions?

1) Explain the meaning of Markov inequality and Chebyshev inequality.
2) What are the practical values of LLN and CLT?
3) Explain LLN and CLT from the *scaling* perspective.
4) Why do we need different concepts of convergence for random variables?
5) Explain what is convergence in probability.
6) Explain what is convergence in distribution.
7) Why is MGF (Moment Generating Function) useful?
8) Prove CLT using MGF.