

## Lecture 5: Random Variable, Part III

Yi, Yung (이육)

EE210: Probability and Introductory Random Processes  
KAIST EE

MONTH DAY, 2021

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables
- (Derived) Distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

- Given the PDF of  $X$ , What is the PDF of  $Y = g(X)$ ?
- Wait! Didn't we cover this topic? No. We covered just  $\mathbb{E}[g(X)]$ .
- Examples:  $Y = X$ ,  $Y = X + 1$ ,  $Y = X^2$ , etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear:  $Y = aX + b$

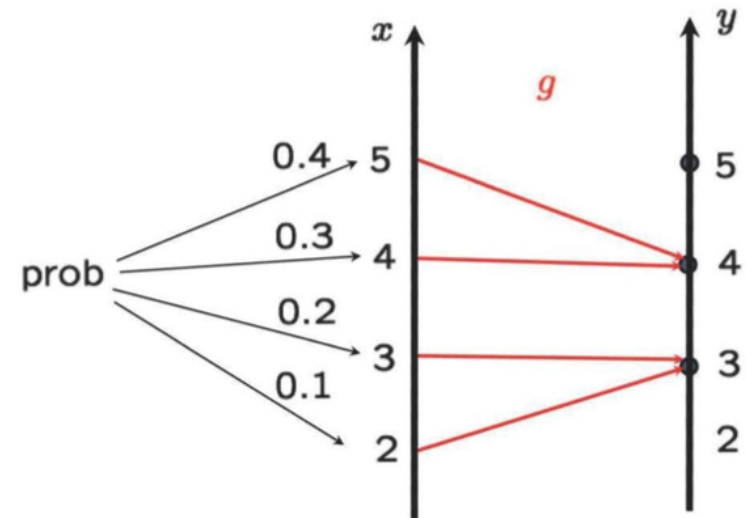
## Discrete Case

- Take all values of  $x$  such that  $g(x) = y$ , i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x: g(x)=y} p_X(x) \end{aligned}$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$

$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear:  $Y = aX + b, a \neq 0$

$$\text{If } a > 0, \quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\rightarrow f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\text{If } a < 0, \quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X > \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\rightarrow f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\text{Therefore, } f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

**Special case.**  $X$  is normal. Then,  $Y$  is also normal, i.e.,  $Y \sim N(a\mu + b, a^2\sigma^2)$

## Generally, $Y = g(X)$

Step 1. Find the CDF of  $Y$ :

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y)$$

Step 2. Differentiate:  $f_Y(y) = \frac{dF_Y}{dy}(y)$

\*\* When  $Y = g(X)$  is monotonic, a **general formula** can be drawn (see the textbook at pp 207)

Ex1.  $X \sim \text{uniform}[0, 1]$ .  $Y = \sqrt{X}$ .

$$F_Y(y) = \mathbb{P}(\sqrt{X} \leq y) = \mathbb{P}(X \leq y^2) = y^2$$

$$f_Y(y) = 2y, \quad 0 \leq y \leq 1$$

Ex2.  $X \sim \text{uniform}[0, 2]$ .  $Y = X^3$ .

$$F_Y(y) = \mathbb{P}(X^3 \leq y) = \mathbb{P}(X \leq \sqrt[3]{y}) = \frac{1}{2}y^{1/3}$$

$$f_Y(y) = \frac{1}{6}y^{-2/3}, \quad 0 \leq y \leq 8$$

Ex3.  $X$  with  $f_X(x)$ .  $Y = X^2$ .

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \\ &\quad \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \geq 0 \end{aligned}$$

## Functions of multiple rvs: $Y = g(X, Y)$ (1)

Basically, follow two step approach: (i) CDF and (ii) differentiate.

**Ex1.**  $X, Y \sim \text{uniform}[0, 1]$ , and  $X \perp\!\!\!\perp Y$ .  $Z = \max(X, Y)$ .

\*  $\mathbb{P}(X \leq z) = \mathbb{P}(Y \leq z) = z, z \in [0, 1]$ .

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\max(X, Y) \leq z) = \mathbb{P}(X \leq z, Y \leq z) \\ &= \mathbb{P}(X \leq z)\mathbb{P}(Y \leq z) = z^2 \end{aligned}$$

$$f_Z(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

## Functions of multiple rvs: $Y = g(X, Y)$ (2)

Basically, follows two step approach: (i) CDF and (ii) differentiate.

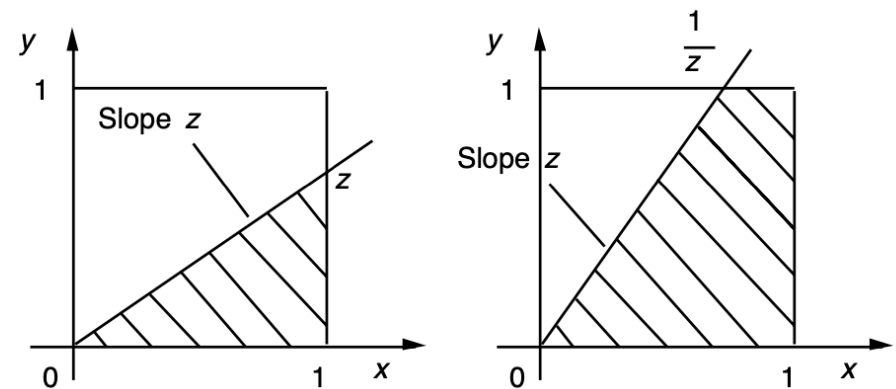
**Ex2.**  $X, Y \sim \text{uniform}[0, 1]$ , and  $X \perp\!\!\!\perp Y$ .  
 $Z = Y/X$ .

$$F_Z(z) = \mathbb{P}(Y/X \leq z)$$

$$= \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} 1/2, & 0 \leq z \leq 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

- Depending on the value of  $z$ , two cases need to be considered separately.





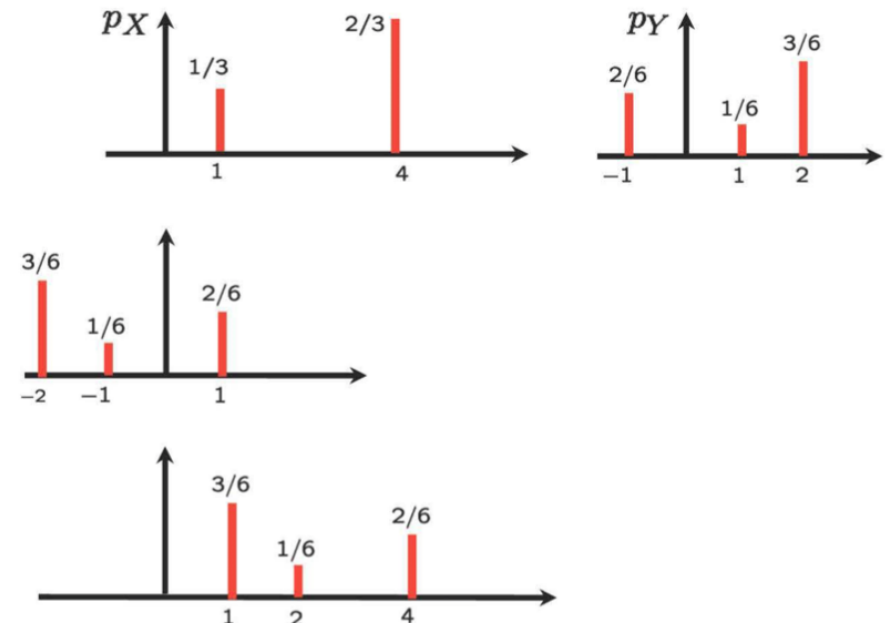
# Functions of multiple rvs: $Z = X + Y$ , $X \perp\!\!\!\perp Y$

- A very basic case with many applications
- Assume that  $X, Y \in \mathbb{Z}$

$$\begin{aligned} p_Z(z) &= \mathbb{P}(X + Y = z) \\ &= \sum_{\{(x,y): x+y=z\}} \mathbb{P}(X = x, Y = y) \\ &= \sum_x \mathbb{P}(X = x, Y = z - x) \\ &= \sum_x \mathbb{P}(X = x) \mathbb{P}(Y = z - x) \\ &= \sum_x p_X(x) p_Y(z - x) \end{aligned}$$

- $p_Z(z)$  is called **convolution** of the PMFs of  $X$  and  $Y$ .

- Interpretation (for a given  $z$ )
  - (i) Flip (horizontally)  $p_Y(y)$  ( $p_Y(-x)$ )
  - (ii) Put it underneath  $p_X(x)$  ( $p_Y(-x + z)$ )



## $Z = X + Y, X \perp\!\!\!\perp Y$ : Continuous

- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

- Very special, but useful case
  - $X$  and  $Y$  are **normal**.

### Sum of two independent normal rvs

$X \sim N(\mu_x, \sigma_x^2)$  and  $Y \sim N(\mu_y, \sigma_y^2)$

Then,  $X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

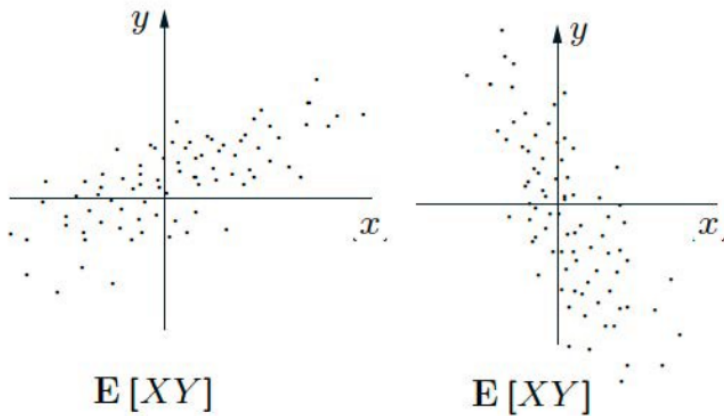
- Why normal rvs are used to model the sum of random noises.
- (Extension) The sum of **finitely many** independent normals is also normal.

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables
  
- (Derived) Distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

- Goal: Given two rvs  $X$  and  $Y$ , assign some number that quantifies the degree of their dependence
- Reqs.
  - a) Increases (resp. decreases) as they become more (resp. less) dependent.
  - b) 0 when they are independent.
  - c) Shows the direction of dependence by  $+$  and  $-$
  - d) Always bounded by some numbers, e.g.,  $[-1, 1]$
- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball):  
<http://m.mlb.com/glossary/advanced-stats>

## OK. Let's Design!

- Simple case:  $\mathbb{E}[X] = \mu_X = 0$  and  $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If  $X \uparrow$ ,  $Y \uparrow$ ) or Negative (If  $X \uparrow$ ,  $Y \downarrow$ )
- What about  $\mathbb{E}[XY]$ ? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$  when  $X \perp\!\!\!\perp Y$
  - More data points (thus increases) when  $xy > 0$  (both positive or negative)



(Q) What about  $\mathbb{E}[X + Y]$ ?

What If  $\mu_X \neq 0, \mu_Y \neq 0$ ?

- Solution: Centering.  $X \rightarrow X - \mu_X$  and  $Y \rightarrow Y - \mu_Y$

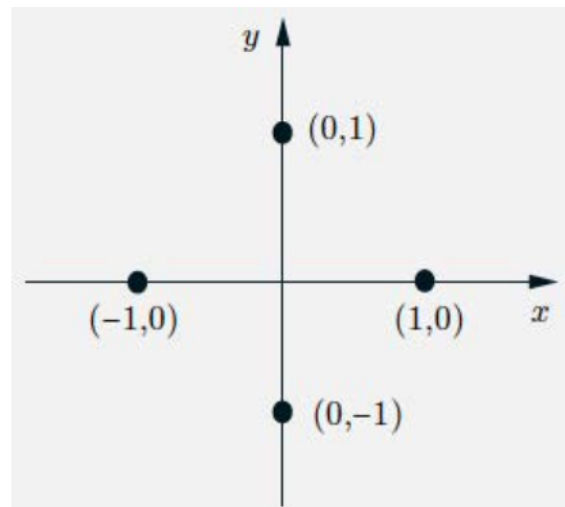
### Covariance

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])]$$

- After some algebra,  $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \implies \text{cov}(X, Y) = 0$
- $\text{cov}(X, Y) = 0 \implies X \perp\!\!\!\perp Y$ ? NO.
- When  $\text{cov}(X, Y) = 0$ , we say that  $X$  and  $Y$  are **uncorrelated**.

## Example: $\text{cov}(X, Y) = 0$ , but not independent

- $p_{X,Y}(1, 0) = p_{X,Y}(0, 1) = p_{X,Y}(-1, 0) = p_{X,Y}(0, -1) = 1/4$ .
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ , and  $\mathbb{E}[XY] = 0$ . So,  $\text{cov}(X, Y) = 0$
- Are they independent? No, because if  $X = 1$ , then we should have  $Y = 0$ .



$$\text{cov}(X, X) = 0$$

$$\text{cov}(aX + b, Y) = \mathbb{E}[(aX + b)Y] - \mathbb{E}[aX + b]\mathbb{E}[Y] = a \cdot \text{cov}(X, Y)$$

$$\text{cov}(X, Y + Z) = \mathbb{E}[X(Y + Z)] - \mathbb{E}[X]\mathbb{E}[Y + Z] = \text{cov}(X, Y) + \text{cov}(X, Z)$$

$$\text{var}[X + Y] = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 = \text{var}[X] + \text{var}[Y] - 2\text{cov}(X, Y)$$



## Example: The hat problem in Lecture 3. Remember?

- $n$  people throw their hats in a box and then pick one at random
- $X$ : number of people with their own hat
- (Q)  $\text{var}[X]$
- Key step 1. Define a rv  $X_i = 1$  if  $i$  selects own hat and 0 otherwise. Then,  $X = \sum_{i=1}^n X_i$ .
- Key step 2. Are  $X_i$ s are independent?
- $X_i \sim \text{Bernoulli}(1/n)$ . Thus,  $\mathbb{E}[X_i] = 1/n$  and  $\text{var}[X_i] = \frac{1}{n}(1 - \frac{1}{n})$

- For  $i \neq j$ ,

$$\begin{aligned}\text{cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] \\ &= \mathbb{P}(X_i = 1 \text{ and } X_j = 1) - \frac{1}{n^2} \\ &= \mathbb{P}(X_i = 1) \mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\ &= \frac{1}{n} \frac{1}{n-1} - \frac{1}{n^2} = \frac{1}{n^2(n-1)}\end{aligned}$$

$$\begin{aligned}\text{var}[X] &= \text{var}\left[\sum X_i\right] \\ &= \sum \text{var}[X_i] + \sum_{i \neq j} \text{cov}(X_i, X_j) \\ &= n \frac{1}{n} \left(1 - \frac{1}{n}\right) + n(n-1) \frac{1}{n^2(n-1)} = 1\end{aligned}$$

- Reqs. a), b), and c) satisfied.
  - d) Always bounded by some numbers, e.g.,  $[-1, 1]$
- Dimensionless metric. How? Normalization, but by what?

## Correlation Coefficient

$$\rho(X, Y) = \mathbb{E} \left[ \frac{(X - \mu_X)}{\sigma_X} \cdot \frac{Y - \mu_Y}{\sigma_Y} \right] = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$  (linear relation, VERY related)

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables
  
- (Derived) Distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

# A Special Random Variable

- Consider a rv  $Y$ , such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

- If  $h(y) = y^2$ , then a new rv  $h(Y)$  is:

$$h(Y) = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 4, & \text{w.p. } 1/2 \end{cases}$$

- Consider other rv  $X$ , such that

$$g(y) = \mathbb{E}[X|Y = y] = \begin{cases} 3, & \text{if } y = 0 \\ 8, & \text{if } y = 1 \\ 9, & \text{if } y = 2 \end{cases}$$

- Then, a rv  $g(Y)$  is:

$$g(Y) = \begin{cases} 3, & \text{w.p. } 1/4 \\ 8, & \text{w.p. } 1/4 \\ 9, & \text{w.p. } 1/2 \end{cases}$$

- The rv  $g(Y)$  looks special, so let's notate it with some fancy one.
- What about?  $X_{\text{exp}}(Y)$ ,  $\mathbb{E}[X_Y]$ ,  $\mathbb{E}_X[Y]$ ?

## Conditional Expectation

A random variable  $g(Y) = \mathbb{E}[X|Y]$ , called **conditional expectation of  $X$  given  $Y$** , takes the value  $g(y) = \mathbb{E}[X|Y = y]$ , if  $Y$  happens to take the value  $y$ .

- A function of  $Y$
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has
- Often confusing because of the notation

## Expectation of Conditional Expectation

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

**Proof.**

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X|Y]] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\ &= \mathbb{E}[X] \end{aligned}$$

- Stick of length  $l$
  - Uniformly break at point  $Y$ , and break what is left uniformly at point  $X$ .
  - $\mathbb{E}[X|Y = y] = y/2$
  - $\mathbb{E}[X|Y] = Y/2$
  - $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y/2] = \frac{1}{2} \frac{l}{2} = l/4$
- Forecasts on sales: calculating expected value, given any available information
  - $X$  : February sales
  - Forecast in the beg. of the year:  $\mathbb{E}[X]$
  - End of Jan. new information  $Y = y$  (Jan. sales)  
Revised forecast:  $\mathbb{E}[X|Y = y]$   
Revised forecast  $\neq \mathbb{E}[X]$
  - Law of iterated expectations  
 $\mathbb{E}[\text{revised forecast}] = \text{original one}$

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\text{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

## Conditional Variance

A random variable  $g(Y) = \text{var}[X|Y]$  and called **conditional variance of  $X$  given  $Y$** , takes the value  $g(y) = \text{var}[X|Y = y]$ , if  $Y$  happens to take the value  $y$ .

- A function of  $Y$
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has



	$\mathbb{E}[X Y]$	$\text{var}[X Y]$
Expectation	$\mathbb{E}[\mathbb{E}(X Y)]$	$\mathbb{E}[\text{var}(X Y)]$
Variance	$\text{var}[\mathbb{E}(X Y)]$	$\text{var}[\text{var}(X Y)]$

## Law of total variance

$$\text{var}[X] = \mathbb{E}[\text{var}(X|Y)] + \text{var}[\mathbb{E}(X|Y)]$$

**Proof.**

$$\text{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

$$\mathbb{E}[\text{var}(X|Y)] = \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \quad (1)$$

$$\text{var}[\mathbb{E}(X|Y)] = \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}(X|Y)])^2 = \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2 \quad (2)$$

$$(1) + (2) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}[X]$$

- $N$  : number of stores visited (**random**)
- $X_i$ : money spent in store  $i$ , independent of other  $X_j$  and  $N$ ,  $X_i$ s are identically distributed with  $\mathbb{E}[X_i] = \mu$
- $Y = X_1 + X_2 + \dots X_N$ . What are  $\mathbb{E}[Y]$  and  $\text{var}[Y]$ ?
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N\mathbb{E}[X_i]] = \mathbb{E}[N]\mathbb{E}[X_i] = \mu\mathbb{E}[N]$
- $\text{var}[Y] = \mathbb{E}[\text{var}(Y|N)] + \text{var}[\mathbb{E}(Y|N)] = \mathbb{E}[N]\text{var}[X_i] - \mu^2\text{var}[N]$

$$\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$$

$$\text{var}[Y|N] = N\text{var}[X_i]$$

$$\mathbb{E}[\text{var}(Y|N)] = \mathbb{E}[N\text{var}[X_i]] = \mathbb{E}[N]\text{var}[X_i]$$

Questions?

- 1) What are the key steps to get the derived distributions of  $Y = g(X)$  or  $Z = g(X, Y)$ ?
- 2) How can we compute the distribution of  $Z + X + Y$  when  $X$  and  $Y$  are independent?
- 3) What are covariance and correlation coefficient? Why do we need them?
- 4) Please explain the concepts of conditional expectation and conditional variance.