

Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

- Probabilistic Model
- Sample Space, Event, Probability Law
- Probability Axioms

What Do We Want?

Modeling: Approximate reality with a simple (mathematical) model

- Experiment
 - Flip two coins
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 - Assign a number to each outcome or a set of outcomes
 - Mathematical description of an uncertain situation

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- **Our goal:** Build up a **probabilistic model** for an experiment with random outcomes
 - **Probabilistic model?**
 - Assign a number to each outcome or a set of outcomes
 - Mathematical description of an uncertain situation
 - Which model is good or bad?

Goal: Build up a probabilistic model. Hmm... How?

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Elements of Probabilistic Model

1. All outcomes of my interest:

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Elements of Probabilistic Model

1. All outcomes of my interest: Sample Space Ω
2. Assigned numbers to each outcome of Ω : Probability Law $\mathbb{P}(\cdot)$

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Elements of Probabilistic Model

1. All outcomes of my interest: Sample Space Ω
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Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

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1. Sample Space Ω

The set of all outcomes of my interest

1. Mutually exclusive
2. Collectively exhaustive
3. At the right granularity (not too concrete, not too abstract)

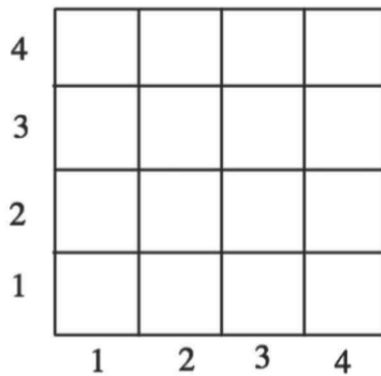
1. Toss a coin. What about this?
 $\Omega = \{H, T, HT\}$
2. Toss a coin. What about this? $\Omega = \{H\}$
3. (a) Just figuring out prob. of H or T.
 $\implies \Omega = \{H, T\}$

(b) The impact of the weather (rain or no rain) on the coin's behavior.

 $\implies \Omega = \{(H, R), (T, R), (H, NR), (T, NR)\}$,
where R(Rain), NR(No Rain).

- *Discrete case:* Two rolls of a tetrahedral die

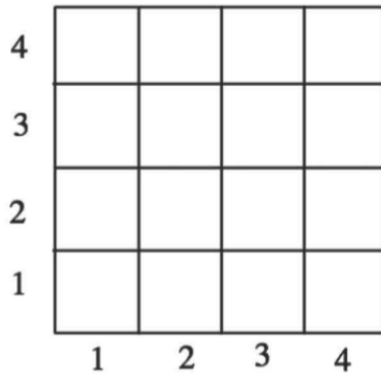
- $\Omega = \{(1, 1), (1, 2), \dots, (4, 4)\}$



4				
3				
2				
1				
	1	2	3	4

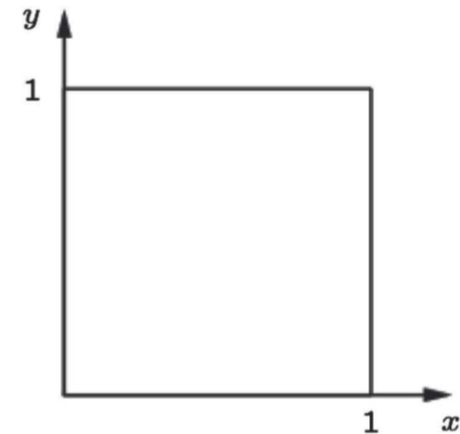
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- $\Omega = \{(1, 1), (1, 2), \dots, (4, 4)\}$



- *Continuous case:* Dropping a needle in a plain

- $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$



2. Probability Law

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- Roll a dice. What is the probability of odd numbers?

$\mathbb{P}(\{1, 3, 5\})$, where $\{1, 3, 5\} \subset \Omega$ is an event.

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 - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
 - For two disjoint events A and B , $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

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 - many others

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- Note that coming up with the above axioms is far from trivial.

Prove the following properties using the axioms:

1. For any event A , $\mathbb{P}(A) \leq 1$

2. $\mathbb{P}(\emptyset) = 0$

3. If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$

Prove the following properties using the axioms:

1. For any event A , $\mathbb{P}(A) \leq 1$

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$$\mathbb{P}(B) \stackrel{A3}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \stackrel{A1}{\geq} \mathbb{P}(A)$$

1. Specify the sample space
2. Specify a probability law
 - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4, \mathbb{P}(\{T\}) = 3/4,$
3. probability of head or tail
4. $1/4, 3/4$

- $\Omega = \{1, 2, 3, \dots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \dots$
- $\mathbb{P}(\text{even})?$
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- Is the above right? If not, why?

- $\Omega = \{1, 2, 3, \dots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \dots$

- $\mathbb{P}(\text{even})$?

$$\begin{aligned}\mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \dots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots = 1/3\end{aligned}$$

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Probability Axioms: Version 2

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. **Countable additivity:** If A_1, A_2, A_3, \dots is an infinite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$.

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- Frequencies: $\mathbb{P}(H) = 1/2$
- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$
 - Subjective, but providing numerical guidance

Questions?

Congratulations! You build up the very basics of a probabilistic model.

What else do we need to build up?

- 1) Please explain what a probabilistic model is and why we need it.
- 2) What is the mathematical definition of event?
- 3) What are the key elements of the probabilistic model?
- 4) Please list up the probability axioms and explain them.
- 5) Why do we need countable additivity in the probability axiom?