

## Lecture 2: Conditioning, Bayes' Rule, and Independence

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EE210: Probability and Introductory Random Processes  
KAIST EE

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## (1) Conditional Probability

- How should I change my belief about event  $A$ , if I come to know that event  $B$  occurs?

## (2) Bayes' Rule and Bayesian Inference

- prob. of  $A$  given that  $B$  occurs vs. prob. of  $B$  given that  $A$  occurs

## (3) Independence, Conditional Independence

- Can I ignore my knowledge about event  $B$ , when I consider event  $A$ ?

- (1) Conditional Probability
- (2) Bayes' Rule and Bayesian Inference
- (3) Independence, Conditional Independence

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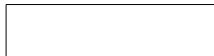
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- From now on, given  $B$ ,  $\mathbb{P}(\cdot|B)$  should be a new **probability law**.
  - Three axioms<sup>1</sup> should be satisfied.

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  - $\mathbb{P}(\Omega|B) = 1$ ?
  - $\mathbb{P}(B|B) = 1$  from our common sense.
  - True?

- How to fix this?

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- (1) Conditional Probability
- (2) Bayes' Rule and Bayesian Inference
- (3) Independence, Conditional Independence

From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference* (추론).



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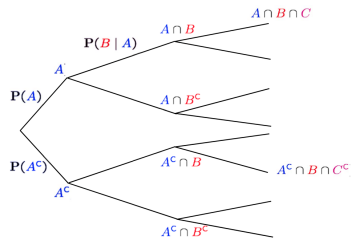
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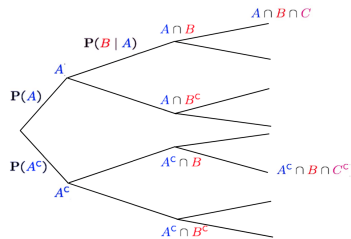
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We will study this topic rigorously later in this class (chapter 8).

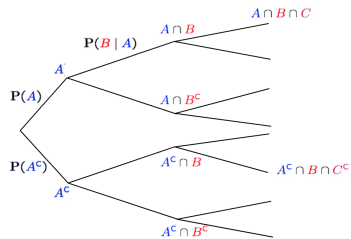
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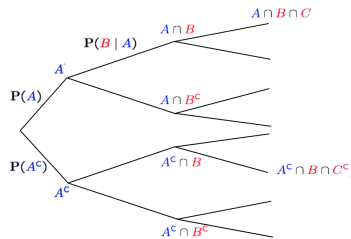


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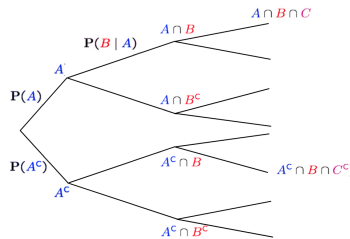




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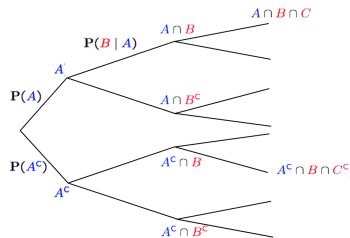
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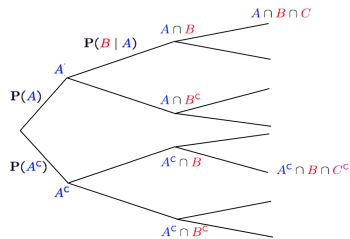
Generally,

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) =$$



**VIDEO PAUSE**

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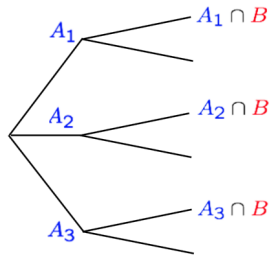
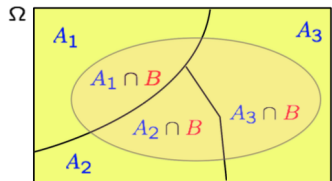


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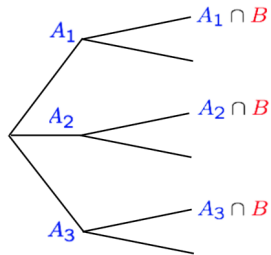
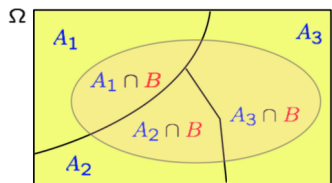
$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, A_2, \dots, A_{n-1})$$

- Partition of  $\Omega$  into  $A_1, A_2, A_3$



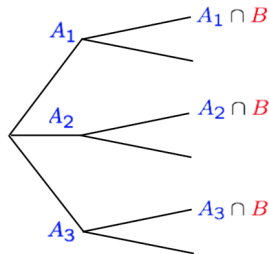
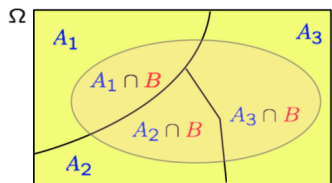
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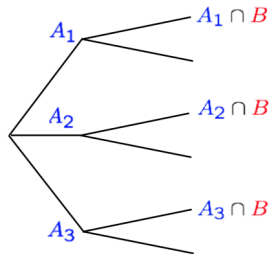
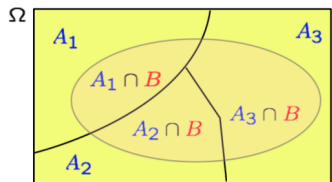
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## Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i) \mathbb{P}(B|A_i)$$

- $\mathbb{P}(A_i \cap B) = \mathbb{P}(A_i) \mathbb{P}(B|A_i)$



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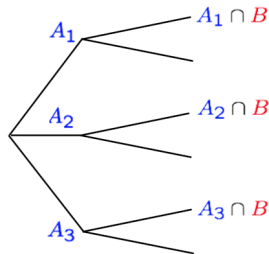
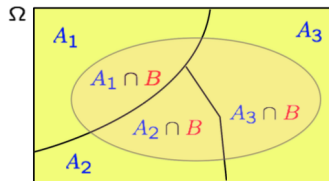


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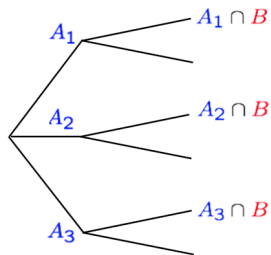
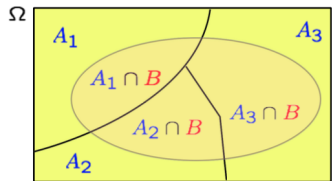
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- Weighted average from the point of  $A_i$  knowledge.

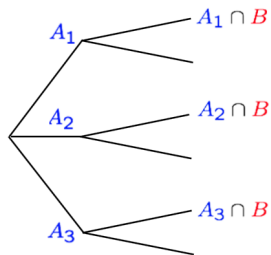
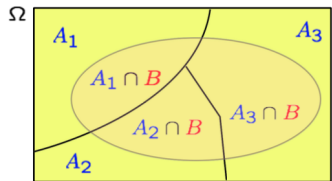


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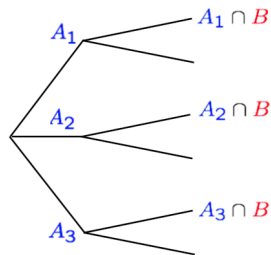
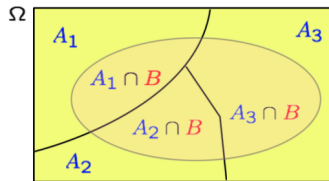
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- revised belief about  $A_i$ , given  $B$  occurs



- Partition of  $\Omega$  into  $A_1, A_2, A_3$
- We know:  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$
- What is  $\mathbb{P}(A_i|B)$ ?
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## Bayes' Rule

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j)\mathbb{P}(B|A_j)}$$



# Example 1: Airplane-Radar

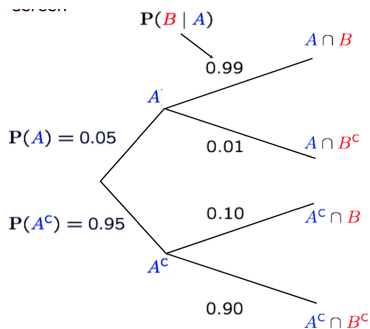
## VIDEO PAUSE

- $A$  : Airplane is flying above
- $B$  : Something on radar screen

$$\mathbb{P}(A \cap B) =$$
$$=$$

$$\mathbb{P}(B) =$$
$$=$$

$$\mathbb{P}(A|B) =$$
$$=$$



# Example 1: Airplane-Radar

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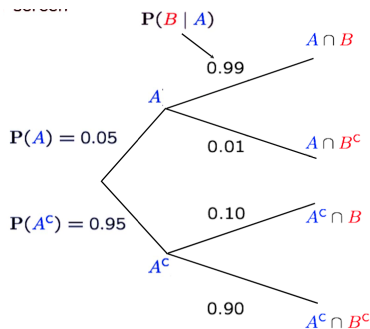
=

$$\mathbb{P}(B) =$$

=

$$\mathbb{P}(A|B) =$$

=



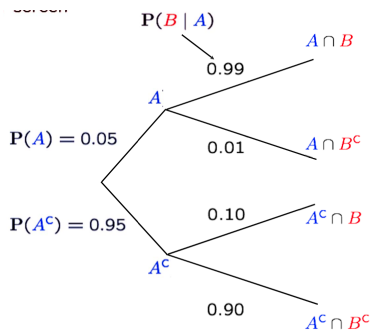
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$$\mathbb{P}(A|B) = \quad =$$



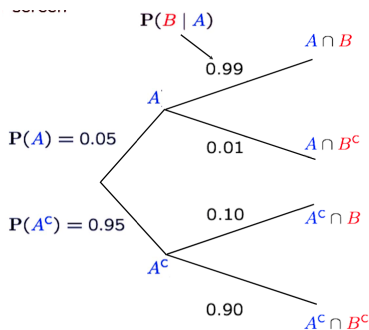
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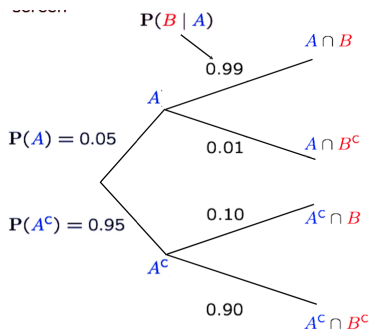
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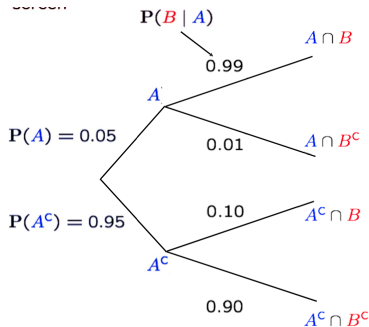
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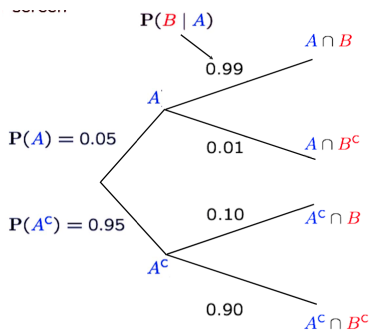
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$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.0495}{0.1445} \approx 0.34$$



- $A_1$ : you are happy,  $A_2$ : you are sad
- $B$ : you shout.
- Assume:

$$\mathbb{P}(A_1) = 0.7, \mathbb{P}(A_2) = 0.3,$$

$$\mathbb{P}(B|A_1) = 0.3, \mathbb{P}(B|A_2) = 0.5.$$

- Calculate  $\mathbb{P}(A_1|B)$  and  $\mathbb{P}(A_2|B)$ .

VIDEO PAUSE

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- Calculate  $\mathbb{P}(A_1|B)$  and  $\mathbb{P}(A_2|B)$ .

$$\mathbb{P}(A_1)\mathbb{P}(B|A_1) = 0.7 \times 0.3 = 0.21$$

$$\mathbb{P}(A_2)\mathbb{P}(B|A_2) = 0.3 \times 0.5 = 0.15$$

$$\mathbb{P}(B) = 0.21 + 0.15 = 0.36$$

$$\mathbb{P}(A_1|B) = \frac{0.21}{0.36} \approx 0.583$$

$$\mathbb{P}(A_2|B) = \frac{0.15}{0.36} \approx 0.417$$

- (1) Conditional Probability
- (2) Bayes' Rule and Bayesian Inference
- (3) Independence, Conditional Independence

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.





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- Independence makes our analysis and modeling **much simpler**, because I can remove independent events in the analysis of what I am interested in.

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- Using  $\mathbb{P}(A \cap B|C) = \frac{\mathbb{P}[B \cap (A \cap C)]}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C) \mathbb{P}(B|A \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|C) \mathbb{P}(B|C)$ ,

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(Q1)  $A \perp\!\!\!\perp B \rightarrow A \perp\!\!\!\perp B|C?$

- Suppose that  $A$  and  $B$  are independent. If you heard that  $C$  occurred,  $A$  and  $B$  are still independent?

VIDEO PAUSE

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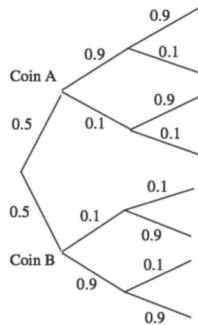
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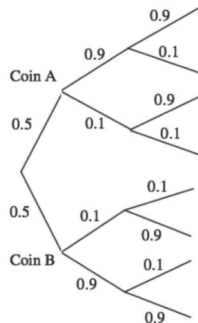
- Two coins: Blue and Red. Choose one uniformly at random, and proceed with two independent tosses.





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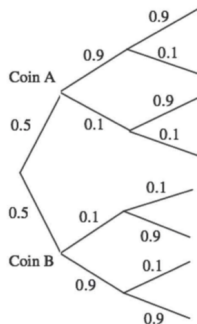
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- $\mathbb{P}(\text{head of blue}) = 0.9$  and  $\mathbb{P}(\text{head of red}) = 0.1$   
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- $H_1 \perp\!\!\!\perp H_2|B$ ? Yes

$$\mathbb{P}(H_1 \cap H_2|B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1|B)\mathbb{P}(H_2|B) = 0.9 \times 0.9$$

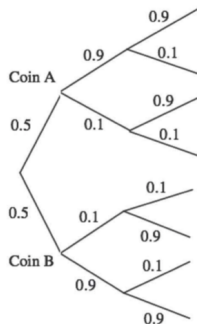


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- $H_1 \perp\!\!\!\perp H_2 \mid B$ ? Yes

$$\mathbb{P}(H_1 \cap H_2 \mid B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1 \mid B)\mathbb{P}(H_2 \mid B) = 0.9 \times 0.9$$

- $H_1 \perp\!\!\!\perp H_2$ ? No



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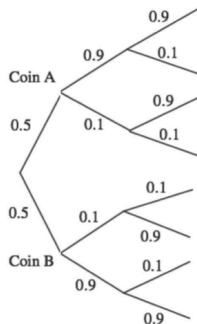
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- $H_1 \perp\!\!\!\perp H_2$ ? **No**

$$\mathbb{P}(H_1) = \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c)$$

$$= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2}$$

$$\mathbb{P}(H_2) = \mathbb{P}(H_1) \quad (\text{because of symmetry})$$



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- $H_1 \perp\!\!\!\perp H_2 | B$ ? **Yes**

$$\mathbb{P}(H_1 \cap H_2 | B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1 | B) \mathbb{P}(H_2 | B) = 0.9 \times 0.9$$

- $H_1 \perp\!\!\!\perp H_2$ ? **No**

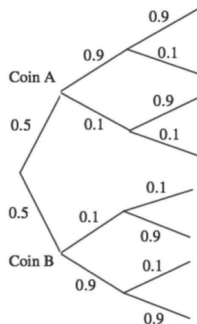
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$$\mathbb{P}(H_1 \cap H_2) = \mathbb{P}(B) \mathbb{P}(H_1 \cap H_2 | B) + \mathbb{P}(B^c) \mathbb{P}(H_1 \cap H_2 | B^c)$$

$$= \frac{1}{2} (0.9 \times 0.9) + \frac{1}{2} (0.1 \times 0.1) \neq \frac{1}{2}$$



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- We need both.

## Independence of Multiple Events

The events  $A_1, A_2, \dots, A_n$  are said to be independent if

$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \dots, n\}$$

Questions?

- 1) What is conditional probability? Why do we need it?
- 2) What is the definition of conditional probability? Are you happy about the definition?
- 3) What is the meaning that the conditional probability is a new probability law?
- 4) What is Bayes' rule? What does it give us?
- 5) Explain the overall framework of Bayesian inference.
- 6) What is the total probability theorem?
- 7) What's the difference between independence and conditional independence?