Lecture 2: Conditioning, Bayes' Rule, and Independence

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EE210: Probability and Introductory Random Processes
KAIST EE

April 27, 2021

(1) Conditional Probability
   ◦ How should I change my belief about event A, if I come to know that event B occurs?

(2) Bayes' Rule and Bayesian Inference
   ◦ prob. of A given that B occurs vs. prob. of B given that A occurs

(3) Independence, Conditional Independence
   ◦ Can I ignore my knowledge about event B, when I consider event A?

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

(3) Independence, Conditional Independence

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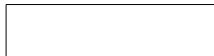
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- From now on, given B, $\mathbb{P}(\cdot|B)$ should be a new $\boxed{\text{probability law.}}$
  - Three axioms[1] should be satisfied.

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  - $\mathbb{P}(B|B) = 1$ from our common sense.

  - True?

• How to fix this?

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$$\mathbb{P}(A \cup C \mid B) = \frac{\mathbb{P}\big[(A \cup C) \cap B\big]}{\mathbb{P}(B)} = \frac{\mathbb{P}\big[(A \cap B) \cup (C \cap B)\big]}{\mathbb{P}(B)} = \mathbb{P}(A \mid B) + \mathbb{P}(C \mid B)$$

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(1) Conditional Probability

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(3) Independence, Conditional Independence

From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference (추론).*

- A_1: Happy (:-)), A_2: Sad (:-()

- B: Shout

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- A_i: state/cause/original value

- B: result/resulting action/noisy measurement

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- Assume that somebody gives you the following information:

  $\mathbb{P}(A_1), \quad \mathbb{P}(A_2), \quad \mathbb{P}(B|A_1), \quad \mathbb{P}(B|A_2).$

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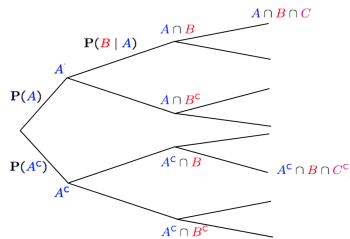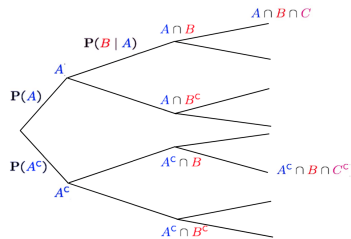
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- Inference: $\mathbb{P}$(cause | result)?

We will study this topic rigorously later in this class (chapter 8).

Multiplication Rule

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- $\mathbb{P}(B|A) = \boxed{}$

- $\mathbb{P}(A \cap B) = \boxed{}$

- $\mathbb{P}(A^c \cap B \cap C^c) = \boxed{}$

  $= \boxed{}$

- $\mathbb{P}(B|A) = \boxed{\dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}}$

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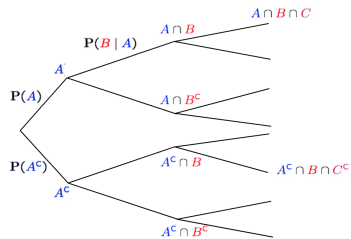
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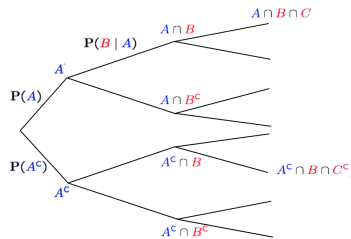
- $\mathbb{P}(A^c \cap B \cap C^c) = \boxed{\mathbb{P}(A^c \cap B) \cdot \mathbb{P}(C^c|A^c \cap B)}$

  $=$

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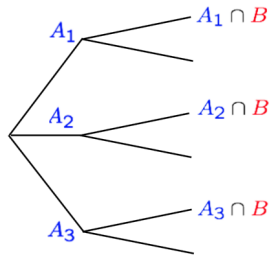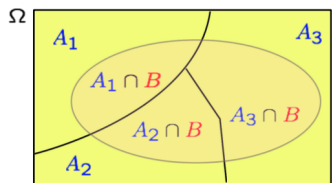 $= \boxed{\mathbb{P}(A^c) \cdot \mathbb{P}(B|A^c) \cdot \mathbb{P}(C^c|A^c \cap B)}$

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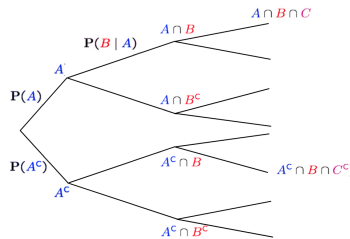
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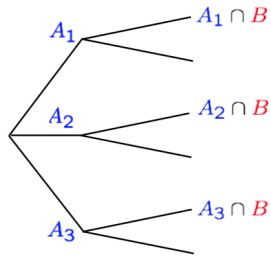 $= \boxed{\mathbb{P}(A^c) \cdot \mathbb{P}(B|A^c) \cdot \mathbb{P}(C^c|A^c \cap B)}$

Generally,

$\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) =$

$\mathbf{P}(A)$

$\mathbf{P}(A^c)$

A

A^c

$\mathbf{P}(B \mid A)$

$A \cap B$

$A \cap B^c$

$A^c \cap B$

$A^c \cap B^c$

$A \cap B \cap C$
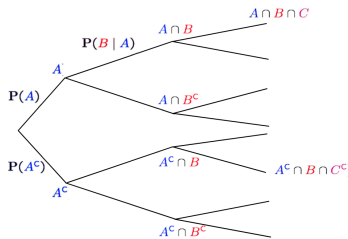
$A^c \cap B \cap C^c$

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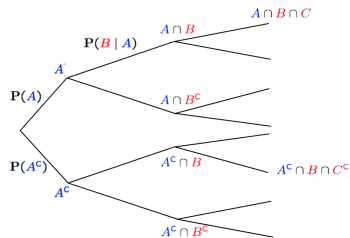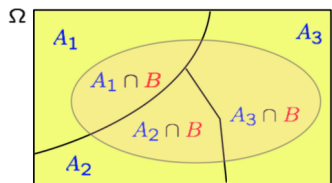

Generally,

<mark>VIDEO PAUSE</mark>

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, A_2, \ldots, A_{n-1})$$

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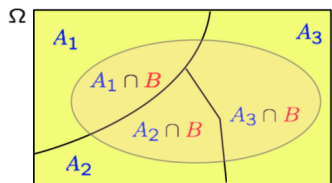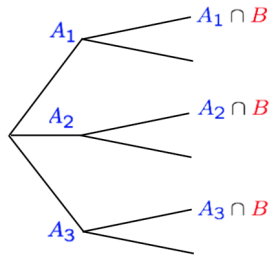- **Partition** of Ω into A_1, A_2, A_3

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[1]Partition: A_1, A_2, A_3 are mutually exclusive and $\Omega = A_1 \cup A_2 \cup A_3$

- Partition of Ω into A_1, A_2, A_3
- We know: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$

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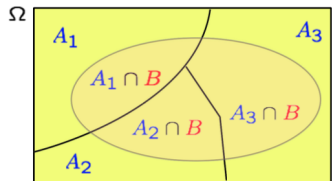
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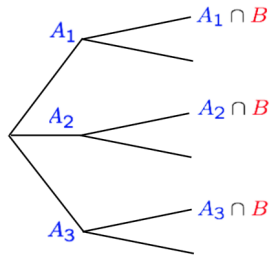
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Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

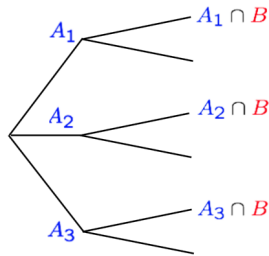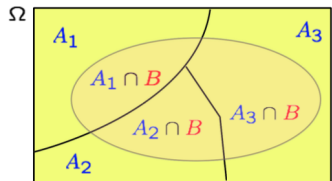- $\mathbb{P}(A_i \cap B) = \mathbb{P}(A_i)\mathbb{P}(B|A_i)$

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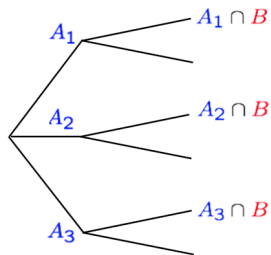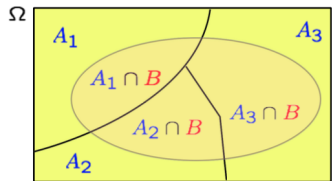- What is $\mathbb{P}(B)$? (probability of result)

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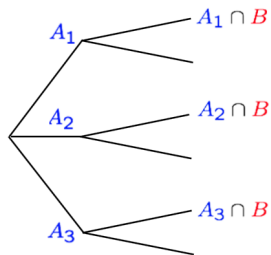- Weighted average from the point of A_i knowledge.
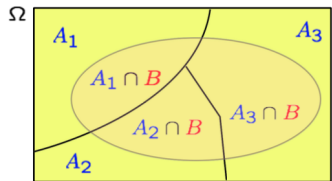

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- revised belief about A_i, given B occurs
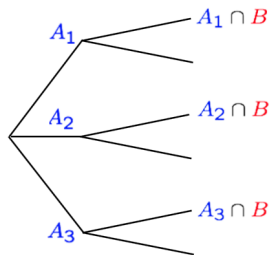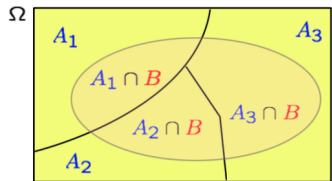
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Bayes' Rule

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j)\mathbb{P}(B|A_j)}$$

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- A : Airplane is flying above
- B : Something on radar screen

$\mathbb{P}(A \cap B) =$

$\quad\quad =$

$\mathbb{P}(B) =$

$\quad =$

$\mathbb{P}(A|B) = \quad\quad\quad =$

screen

$\mathbf{P}(B \mid A)$

$A \cap B$

0.99

A

$\mathbf{P}(A) = 0.05$

0.01 $A \cap B^{\mathsf{c}}$

0.10 $A^{\mathsf{c}} \cap B$

$\mathbf{P}(A^{\mathsf{c}}) = 0.95$

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Example 1: Airplane-Radar

- A : Airplane is flying above
- B : Something on radar screen

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$\qquad =$

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$\qquad =$

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$A \cap B$

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A

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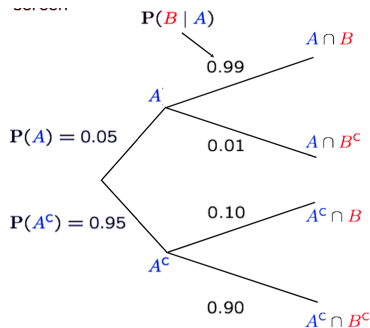
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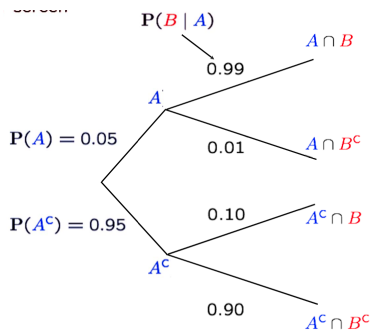## Example 1: Airplane-Radar

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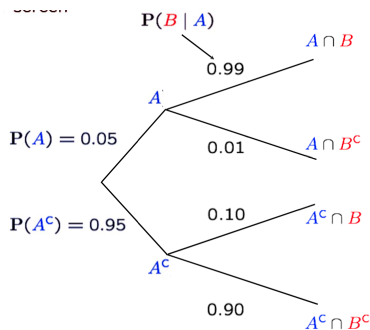
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A

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$A \cap B^c$

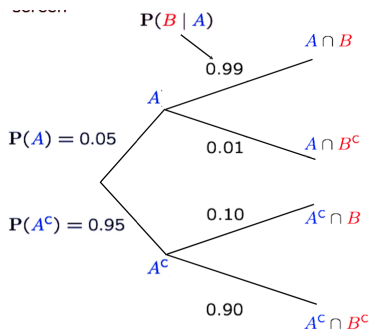$\mathbf{P}(A^c) = 0.95$

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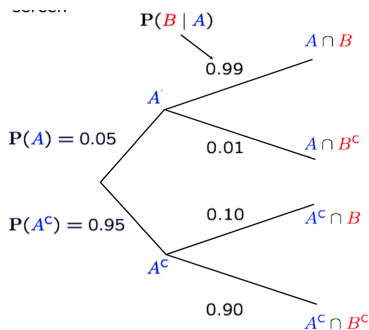## Example 1: Airplane-Radar

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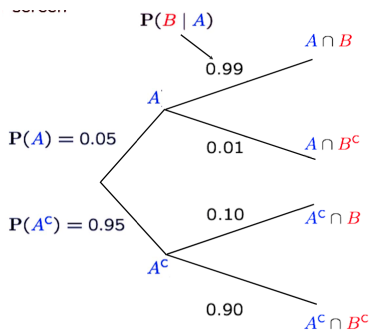
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VIDEO PAUSE

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$$\mathbb{P}(A_1)\mathbb{P}(B|A_1) = 0.7 \times 0.3 = 0.21$$
$$\mathbb{P}(A_2)\mathbb{P}(B|A_2) = 0.3 \times 0.5 = 0.15$$
$$\mathbb{P}(B) = 0.21 + 0.15 = 0.36$$

$$\mathbb{P}(A_1|B) = \frac{0.21}{0.36} \approx 0.583$$
$$\mathbb{P}(A_2|B) = \frac{0.15}{0.36} \approx 0.417$$

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

(3) Independence, Conditional Independence

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.

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- Independence makes our analysis and modeling much simpler, because I can remove independent events in the analysis of what I am interested in.

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- Using $\mathbb{P}(A \cap B|C) = \frac{\mathbb{P}[B \cap (A \cap C)]}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C)\mathbb{P}(B|A \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|C)\mathbb{P}(B|C),$

Conditional Independence of A and B given C, $\boxed{A \perp\!\!\!\perp B|C}$

$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \times \mathbb{P}(B|C)$

(Q1) $A \perp\!\!\!\perp B \rightarrow A \perp\!\!\!\perp B | C$?

KAIST EE

- Suppose that A and B are independent. If you heard that C occurred, A and B are still independent? <mark>VIDEO PAUSE</mark>

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- Suppose that A and B are independent. If you heard that C occurred, A and B are still independent?

- Two independent coin tosses
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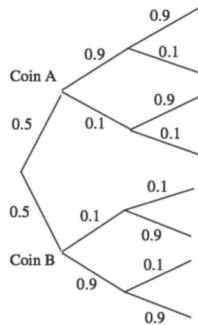
- $\mathbb{P}(H_1|D) = 1/2$, $\mathbb{P}(H_2|D) = 1/2$

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  - H_1: 1st toss is a head
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- $\mathbb{P}(H_1|D) = 1/2$, $\mathbb{P}(H_2|D) = 1/2$

- $\mathbb{P}(H_1 \cap H_2 | D) = 0$,

- No.

(Q2) $A \perp\!\!\!\perp B | C \rightarrow A \perp\!\!\!\perp B$?

KAIST EE

- Two coins: Blue and Red. Choose one uniformly at random, and proceed with two independent tosses.
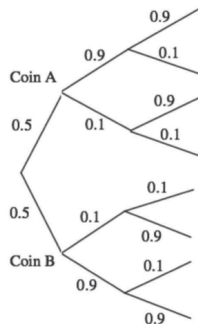
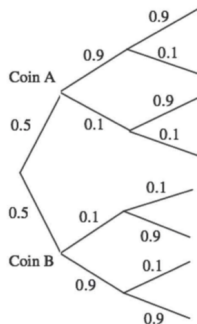# (Q2) $A \perp\!\!\!\perp B | C \rightarrow A \perp\!\!\!\perp B$?

- Two coins: Blue and Red. Choose one uniformly at random, and proceed with two independent tosses.
- $\mathbb{P}(\text{head of blue}) = 0.9$ and $\mathbb{P}(\text{head of red}) = 0.1$
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KAIST EE

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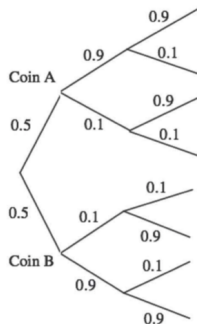
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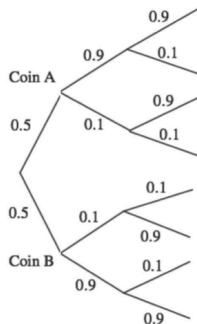- $H_1 \perp\!\!\!\perp H_2 | B$? Yes

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- $H_1 \perp\!\!\!\perp H_2$? No
  $$\mathbb{P}(H_1) = \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c)$$
  $$= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2}$$
  $$\mathbb{P}(H_2) = \mathbb{P}(H_1) \quad \text{(because of symmetry)}$$

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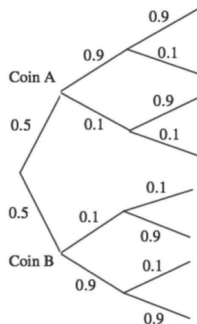 $$\mathbb{P}(H_1) = \mathbb{P}(B)\mathbb{P}(H_1 | B) + \mathbb{P}(B^c)\mathbb{P}(H_1 | B^c)$$
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KAIST EE

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Independence of Multiple Events

The events $A_1, A_2, \ldots, A_n$ ar said to be independent if

$$\mathbb{P}\Big(\bigcap_{i \in S} A_i\Big) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}$$

Questions?

1) What is conditional probability? Why do we need it?

2) Explain the overall framework of Bayesian inference.

3) What is the total probability theorem?

4) What is Bayes' rule? What does it can give us?

5) What's the difference between independence and conditional independence?