

Lecture 6: Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

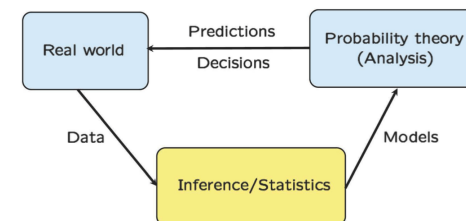
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- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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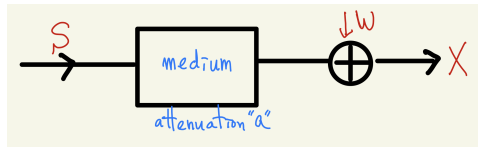
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- Inference
 - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
 - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
 - Connecting big data and big model building

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- $X = aS + W$
- Modeling building
 - know the original signal S , observe X
 - infer the model parameter a
- Variable estimation
 - know a , observe X
 - infer the original signal S
- Same mathematical structure, because the parameters in models are variables in many cases

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- Hypothesis testing
 - Unknown: a few possible ones
 - Goal: small probability of incorrect decision
 - (Ex) Something detected on the radar. Is it a bird or an airplane?
- Estimation
 - Unknown: a value included in an infinite, typically continuous set
 - Goal: Finding the value close to the true value
 - (Ex) Biased coin with unknown probability of head $\theta \in [0, 1]$. Data of heads and tails. What is θ ?
 - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

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- Biased coin with parameter θ (probability of head). Assume that $\theta \in \{1/4, 3/4\}$.
- Throw the coin 3 times and get (H, H, H) . Goal: infer θ , 1/4 or 3/4?
- Distribution of θ (prior) e.g.,

$$\mathbb{P}(\theta = \frac{3}{4}) = 1/2, \quad \mathbb{P}(\theta = \frac{1}{4}) = 1/2$$
- Use Bayes' rule and find the posterior:

$$\mathbb{P}[\theta = \frac{3}{4} | (HHH)] = \frac{27}{28}, \quad \mathbb{P}[\theta = \frac{1}{4} | (HHH)] = \frac{1}{28}$$
- Choose θ with larger posterior probability.
- Bayesian approach (Chapter 8)
- Find the probability of (H, H, H) , if $\theta = \frac{1}{4}$ or $\frac{3}{4}$ (likelihood)

$$\mathbb{P}[(HHH) | \theta = \frac{3}{4}] = \left(\frac{3}{4}\right)^3$$

$$\mathbb{P}[(HHH) | \theta = \frac{1}{4}] = \left(\frac{1}{4}\right)^3$$
- Choose θ with a larger likelihood.
- Classical approach (Chapter 9)

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Bayesian approach

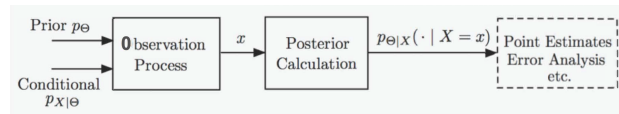
- Unknown: random variable with some distribution (prior)
- Unknown model as chosen randomly from a given model class
- Observed data x gives: posterior distribution $p_{\theta|x}(\theta|x)$
- Choose θ with larger posterior probability (other methods exist)

Classical approach

- Unknown: deterministic value
- Unknown model as one of multiple probabilistic models
- Observed data x gives: likelihood $p(X; \theta)$
- Choose θ with larger likelihood (other methods exist)

- Who is the winner? A century-long debate (see p. 409 for discussion)

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- Unknown Θ
 - physical quantity or model parameter
 - random variable
 - prior distribution p_Θ and f_Θ
 - Observations or measurements X
 - observation model $p_{X|\Theta}$ and $f_{X|\Theta}$
 - That is, the joint distribution of X and Θ , $p_{X,\Theta}$ and $f_{X,\Theta}$, is given
- Find the posterior distribution $p_{X|\Theta}$ and $f_{X|\Theta}$.
 - Use Bayes' rule
 - Using the posterior distribution, apply one of the methods of choosing the final $\hat{\theta}$ for estimation and hypothesis testing.

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- Given observation x , which θ are you going to choose?

M1. Choose the largest: Maximum a posteriori probability (MAP) rule

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} p_{\Theta|X}(\theta|x), \quad \hat{\theta}_{\text{MAP}} = \arg \max_{\theta} f_{\Theta|X}(\theta|x)$$

M2. Choose the mean: Conditional expectation, aka LMS (Least Mean Square)

$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta|X=x]$$

- Why MAP and LMS are good? Not mathematically clear yet (later)

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- Random observation: X

- Observation instance: x

- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\text{MAP}} = g_{\text{MAP}}(x), \quad \hat{\theta}_{\text{LMS}} = g_{\text{LMS}}(x)$$

- Estimator as a mapping from X to a random variable

$$\hat{\Theta} = g(X), \quad \hat{\Theta}_{\text{MAP}} = g_{\text{MAP}}(X), \quad \hat{\Theta}_{\text{LMS}} = g_{\text{LMS}}(X)$$

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Example 1: Romeo and Juliet

- Romeo and Juliet start dating.
 - Romeo: late by $X \sim U[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'}$$

$$= \frac{1/\theta}{\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta|\log x|}, \quad x \leq \theta \leq 1,$$

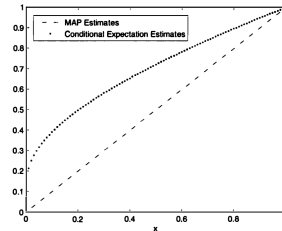
and $f_{\Theta|X}(\theta|x) = 0$, $\theta < x$ or $\theta > 1$.

- MAP rule
 - Given x , $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$.
 - $\hat{\theta}_{\text{MAP}} = x$.

- Conditional expectation estimator

$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\theta|X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta$$

$$= (1-x)/|\log x|$$



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Example 2: Biased Coin with Beta Prior (1)

- Biased coin with probability of head θ
- Unknown θ : modeled by Θ with some prior $f_{\Theta}(\theta)$
- Observation X : number of heads out of n tosses
- Posterior PDF

$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k},$$
 c the normalizing constant
- If $\Theta \sim \text{Beta}(\alpha, \beta)$, what is $\hat{\theta}_{\text{MAP}}$?
- What is $\text{Beta}(\alpha, \beta)$?

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Example 2: Biased Coin with Beta Prior (2)

Beta distribution

A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $B(\alpha, \beta)$, called Beta function, is a normalizing constant, given by

$$B(\alpha, \beta) = \int_0^1 \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

- A special case of $\text{Beta}(1, 1)$ is $\text{Uniform}[0, 1]$

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Example 2: Biased Coin with Beta Prior (3)

- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta|X=k \sim \text{Beta}(k+\alpha, n-k+\beta)$
 - Very useful: Beta prior \implies Beta posterior

- Proof.** For $\text{Beta}(\alpha, \beta)$ prior,

$$f_{\Theta}(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$f_{\Theta|X}(\theta|k) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k} = \frac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}$$

where $d = c \binom{n}{k}$.

- Taking the logarithm,

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} [(\alpha+k-1) \log \theta + (\beta+n-k+1) \log(1-\theta)] = \frac{\alpha+k-1}{\alpha+\beta-2+n}$$

- When $\alpha = \beta = 1$ (i.e., $U[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_{\Theta}(1)$ and $p_{\Theta}(2)$.
- $\{w_1, w_2, \dots, w_n\}$: a collection of words which suggest "spam".
- For each i , a Bernoulli $X_i = 1$ if w_i appears and 0 otherwise.
- Observation model $p_{X_i|\Theta(x_i|1)}$ and $p_{X_i|\Theta(x_i|2)}$ are known. Conditioned on Θ , X_i are independent.
- Posterior PMF

$$\mathbb{P}(\Theta = m | (x_1, \dots, x_n)) = \frac{p_{\Theta}(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_{\Theta}(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_{\Theta}(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_{\Theta}(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

- MAP estimate is intuitive, but we need more mathematical support.
- Claim 1.** For a given x , the MAP rule minimizes the probability of an incorrect decision.
- Claim 2.** The MAP rule minimizes the overall probability of an incorrect decision, averaged over x .
- Proof.** Let I and I_{map} be the indicator rv, representing the correct decision by any general estimator and the MAP, respectively.

$$\mathbb{E}[I|X=x] = \mathbb{P}[g(X) = \Theta|X=x] \leq \mathbb{P}[g_{map}(X) = \Theta|X=x] = \mathbb{E}[I_{map}|X=x]$$

Thus, **Claim 1** holds. We now take the expectation of the above equations, the law of iterated expectations leads to **Claim 2**.

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- Unknown: θ modeled by Θ with prior $f_{\Theta}(\cdot)$. Assume $\Theta \sim \text{Uniform}[4, 10]$.
- No observations available
- MAP estimate
 - Any value $\hat{\theta}_{map} \in [4, 10]$ (why? posterior = prior), not very useful
- What is your other choice?
 - Expectation: $\hat{\theta} = \mathbb{E}[\Theta] = 7$
 - looks reasonable, but why?
- Because it minimizes mean squared error (MSE)

$$\min_{\hat{\theta}} \mathbb{E}[(\Theta - \hat{\theta})^2] = \min_{\hat{\theta}} \left(\text{var}(\Theta - \hat{\theta}) + (\mathbb{E}[\Theta - \hat{\theta}])^2 \right) = \min_{\hat{\theta}} \left(\text{var}(\Theta) + (\mathbb{E}[\Theta - \hat{\theta}])^2 \right)$$

- minimized when $\hat{\theta} = \mathbb{E}[\Theta]$.

- Unknown: θ modeled by Θ with prior $f_{\Theta}(\cdot)$.
- Observation $X = x$ with model $f_{X|\Theta}(x|\theta)$
- Minimizing conditional mean squared error

$$\min_{\hat{\theta}} \mathbb{E}[(\Theta - \hat{\theta})^2 | X = x]$$

- minimized when $\hat{\theta} = \mathbb{E}[\Theta | X = x]$.
- LMS estimator $\hat{\Theta} = \mathbb{E}[\Theta | X]$
- Performance (MSE: Mean Squared Error)
 - When $X = x$, $\mathbb{E}[(\Theta - \mathbb{E}[\Theta | X = x])^2 | X = x] = \text{var}(\Theta | X = x)$
 - Averaged over X : $\mathbb{E}[(\Theta - \mathbb{E}[\Theta | X])^2] = \mathbb{E}[\text{var}(\Theta | X = x)]$

- Romeo and Juliet start dating.
 - Romeo: late by $X \sim U[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

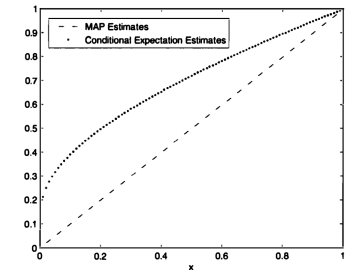
$$f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_{\Theta|X}(\theta|x) &= \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'} \\ &= \frac{1/\theta}{\int_x^1 \frac{1}{\theta'}d\theta'} = \frac{1}{\theta|\log x|}, \quad x \leq \theta \leq 1, \end{aligned}$$

and $f_{\Theta|X}(\theta|x) = 0$, $\theta < x$ or $\theta > 1$.

- MAP rule
 - $\hat{\theta}_{\text{MAP}} = x$.
- LMS estimator

$$\begin{aligned} \hat{\theta}_{\text{LMS}} &= \mathbb{E}[\theta | X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta \\ &= (1-x)/|\log x| \end{aligned}$$



- Remind.** If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

- Fact.** If $\Theta \sim \text{Beta}(\alpha, \beta)$,

$$\mathbb{E}[\Theta] = \frac{1}{B(\alpha, \beta)} \int_0^1 \theta \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\alpha}{\alpha + \beta}$$

- Using the above fact,

$$\mathbb{E}[\Theta | X = k] = \frac{k + \alpha}{k + \alpha + n - k + \beta} = \frac{k + \alpha}{\alpha + \beta + n}$$

- For $\alpha = \beta = 1$ ($\Theta = \text{Uniform}[0, 1]$),

$$\mathbb{E}[\Theta | X = k] = \frac{k+1}{n+2}$$

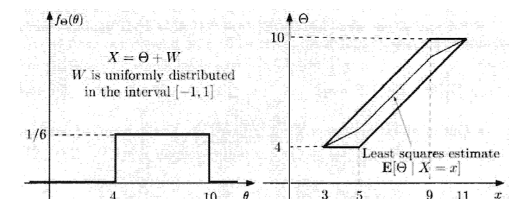
- Unknown: $\Theta \sim \text{Uniform}[4, 10]$
- Observe Θ with random error W as X . $W \sim \text{Uniform}[-1, 1]$

$$X = \Theta + W$$

- Given $\Theta = \theta$, $X = \theta + W \sim \text{Uniform}[\theta - 1, \theta + 1]$.

$$f_{\Theta, X}(\theta, x) = f_{\Theta}(\theta)f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}, & \text{if } 4 \leq \theta \leq 10, \theta - 1 \leq x \leq \theta + 1, \\ 0, & \text{otherwise} \end{cases}$$

- $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$ = midpoint of the corresponding vertical section



- Unknown: $\Theta \sim \text{Uniform}[4, 10]$
- Observe Θ with random error W as X . $W \sim \text{Uniform}[-1, 1]$

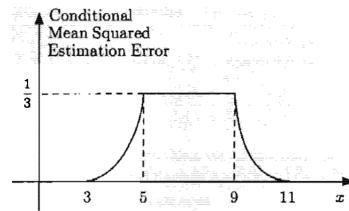
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- Conditional MSE

$$\mathbb{E}[(\Theta - \mathbb{E}[\Theta|X = x])^2 | X = x]$$



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$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{f_X(x)}$$

$$f_X(x) = \int f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'$$

- Observation model $f_{X|\Theta}(x|\theta)$ may not be always available
- Finding the posterior distribution is hard for multi-dimensional Θ
- Θ is very often high-dimensional, especially in the era of big data and deep learning
 - AlexNet in image recognition: 61M parameters (though not a Bayesian inference)
- Any alternative to LMS estimator?

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- Give up optimality, but choose a simple, but good one.
- General estimators $\hat{\Theta} = g(X)$, LMS estimator $\hat{\Theta}_{LMS} = \mathbb{E}[\Theta|X]$
- We consider a restricted class of $g(X)$: $\hat{\Theta} = aX + b$.
- Our goal is:

$$\min_{a,b} \mathbb{E}[(\Theta - aX - b)^2]$$

- Linear models are always the first choice for a simple design in engineering.

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LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) = \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X} (X - \mathbb{E}(X))$$

- No distributions on Θ and X : only means, variances, and covariances
- MSE $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$? Assume $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$. $\mathbb{E}[(\Theta - \rho \frac{\sigma_\Theta}{\sigma_X} X)^2] = (1 - \rho^2) \text{var}[\Theta]$
 - Uncertainty about Θ **decreases** by the factor of $1 - \rho^2$
 - What happens if $|\rho| = 1$ or $\rho = 0$?
- If $\rho > 0$:
 - Baseline ($\mathbb{E}[\Theta]$) + correction term
 - If $X > \mathbb{E}[X] \Rightarrow \hat{\Theta}_L > \mathbb{E}[\Theta]$
 - If $X < \mathbb{E}[X] \Rightarrow \hat{\Theta}_L < \mathbb{E}[\Theta]$
- If $\rho = 0$ (uncorrelated):
 - Just baseline ($\mathbb{E}[\Theta]$)
 - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
 - No use of data X

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) \quad (1)$$

$$= \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X} (X - \mathbb{E}(X)) \quad (2)$$

$$\min_{a,b} \text{ERR}(a, b) = \min_{a,b} \mathbb{E}[(\Theta - aX - b)^2]$$

- Assume a was found.

$$\mathbb{E}[(Y - b)^2], \quad Y = \Theta - aX$$

- Minimized when $b = \mathbb{E}(Y) = \mathbb{E}(\Theta) - a\mathbb{E}(X)$.

$$\begin{aligned} \text{ERR}(a, b) &= \mathbb{E}[(Y - \mathbb{E}(Y))^2] = \text{var}(Y) \\ &= \text{var}[\Theta] + a^2 \text{var}[X] - 2a \text{cov}(\Theta, X) \end{aligned} \quad (3)$$

- (3) is minimized when $a = \frac{\text{cov}(\Theta, X)}{\text{var}[X]}$. Then,

$$\begin{aligned} \hat{\Theta}_L &= aX + b = aX + \mathbb{E}(\Theta) - a\mathbb{E}(X) \\ &= (1) \end{aligned}$$

- Using $\rho = \frac{\text{cov}(\Theta, X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho \sigma_\Theta \sigma_X}{\sigma_X^2} = \frac{\rho \sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

Example: Romeo and Juliet

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[\theta/2] = 1/4$
- Using $\mathbb{E}[\Theta] = 1/2$ and $\mathbb{E}[\Theta^2] = 1/3$,

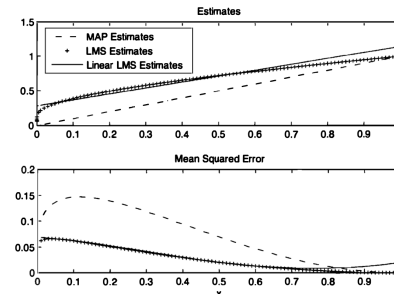
$$\begin{aligned} \text{var}[X] &= \mathbb{E}[\text{var}[X|\Theta]] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \frac{1}{12} \mathbb{E}[\Theta^2] + \frac{1}{4} \text{var}[\Theta] = \frac{7}{144} \end{aligned}$$
- $\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta] \mathbb{E}[X]$

$$\begin{aligned} \mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6 \end{aligned}$$

$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

- LLMS estimator is:

$$\begin{aligned} \hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) \\ &= \frac{1}{2} + \frac{1/24}{7/144} (X - \frac{1}{4}) = \frac{6}{7} X + \frac{2}{7} \end{aligned}$$



Example: Biased Coin with Uniform Prior

- Biased coin with probability of head θ
- Unknown $\Theta \sim \text{uniform}[0, 1]$,
 - $\mathbb{E}[\Theta] = 1/2$, $\text{var}[X] = 1/12$
- n tosses, X : number of heads.
- $p_{X|\Theta}(k|\theta)$: $\text{Binomial}(n, \theta)$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[n\Theta] = n/2$

$$\begin{aligned} \text{var}(X) &= \mathbb{E}[\text{var}(X|\Theta)] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[n\Theta(1 - \Theta)] + \text{var}[n\Theta] \\ &= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12} = \frac{n(n+2)}{12} \end{aligned}$$

$$\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta] \mathbb{E}[X] = \mathbb{E}[\Theta X] - n/4$$

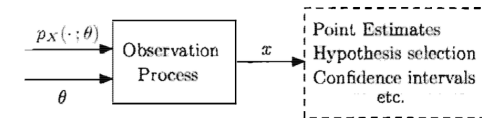
$$\begin{aligned} \mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[n\Theta^2] = n/3 \end{aligned}$$

$$\text{cov}(\Theta, X) = \frac{n}{3} - \frac{n}{4} = \frac{12}{n}$$

$$\hat{\Theta}_L = \frac{1}{2} + \frac{n/12}{n(n+2)/12} (X - \frac{n}{2}) = \frac{X+1}{n+2}$$

- What was the LMS estimator? $\frac{X+1}{n+2}$
 - Same! Intuitive?
 Yes, because the LMS estimator was linear.

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- Unknown θ
 - **deterministic (not random)** quantity (thus, no prior distribution)
 - No prior, No posterior probabilities
- Observations or measurements X
 - Random observation X 's distribution just depends on θ
 - Notation: $p_X(x; \theta)$ and $f_X(x; \theta)$, θ -parameterized distribution of observations
- Choosing one among multiple probabilistic models
 - Each θ corresponds to a probabilistic model

- Problem types
 - **Estimation**
 - Hypothesis testing
 - Significance testing
- Key inference methods
 - **ML (Maximum Likelihood) estimation**
 - Linear regression
 - Likelihood ratio test
 - Significant testing
- Just a taste in this course due to time constraint.

- Random observation $x = (x_1, x_2, \dots, x_n)$ of $X = (X_1, X_2, \dots, X_n)$
 - Assume a scalar θ and a vector of observation in this lecture.
- **Likelihood $p_X(x_1, x_2, \dots, x_n; \theta)$**
 - $p_X(x_1, x_2, \dots, x_n; \theta)$
 - NOT the probability that the unknown parameter is equal to θ .
 - but, the probability that **the observed value x arises when the parameter is θ** .
 - ML (Maximum Likelihood) estimation

$$\hat{\theta}_{ml} = \arg \max_{\theta} p_X(x_1, x_2, \dots, x_n; \theta)$$

- Very often, X_i are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \dots, x_n; \theta) = \log \prod_{i=1}^n p_{X_i}(x_i; \theta) = \sum_{i=1}^n \log p_{X_i}(x_i; \theta)$$

- ML and MAP: How are they related?

- MAP in the Bayesian inference

$$\hat{\theta}_{map} = \arg \max_{\theta} p_{\Theta|X}(\theta|x) = \arg \max_{\theta} \frac{p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{p_X(x)} = \frac{1}{p_X(x)} \arg \max_{\theta} p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)$$

- ML in the classical inference

$$\hat{\theta}_{ml} = \arg \max_{\theta} p_X(x; \theta)$$

- $p_{X|\Theta}(x|\theta)$ in the Bayesian setting corresponds to $p_X(x; \theta)$ in the classical setting.
- When Θ is **uniform** (complete ignorance of Θ), MAP == ML

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.

- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.

- MAP: $\hat{\theta}_{MAP} = x$

- LMS: $\hat{\theta}_{LMS} = (1 - x)/|\log x|$

- LLMS: $\hat{\theta}_L = \frac{6}{7}x + \frac{2}{7}$

- ML: $\hat{\theta}_{ML} = \hat{\theta}_{MAP} = x$

- n identical, independent exponential rvs, X_1, X_2, \dots, X_n with parameter θ .

- Observation x_1, x_2, \dots, x_n

- What is the ML estimate of θ ?

- **Reminder.** $X \sim \exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \mathbb{E}[X] = 1/\lambda$$

- Any guess? $\hat{\theta}_{ML} = \frac{n}{x_1 + x_2 + \dots + x_n}$

$$\arg \max_{\theta} f_X(x; \theta) = \arg \max_{\theta} \prod_{i=1}^n \theta e^{-\theta x_i} = \arg \max_{\theta} \left(n \log \theta - \theta \sum_{i=1}^n x_i \right)$$

Questions?

- 1) What is statistical inference?
- 2) Draw the building blocks of Bayesian inference and explain how it works.
- 3) What are MAP and LMS estimators and their underlying philosophies?
- 4) What is LLMS estimator and why is it useful?
- 5) Compare the classical and Bayesian inference.
- 6) What is the ML estimator and how is it related to the MAP estimator?