Lecture 8: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- A lot of applications in engineering systems

- Basics on Random Process

- Bernoulli Process

- Poisson Process

- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split

- Markov Chain

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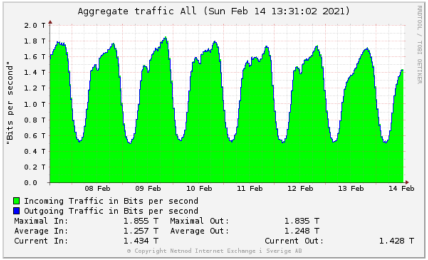
Things that evolve in time

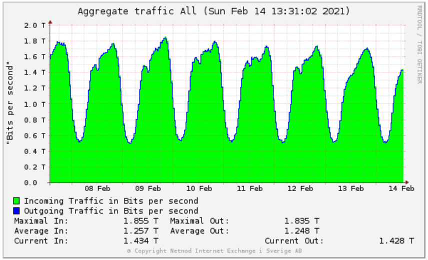
- Many probabilistic experiments that evolve in time

- Random process is a mathematical model for it.



Things that evolve in time

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



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- The values that X_t can take: discrete or continuous

4 Types of Random Processes

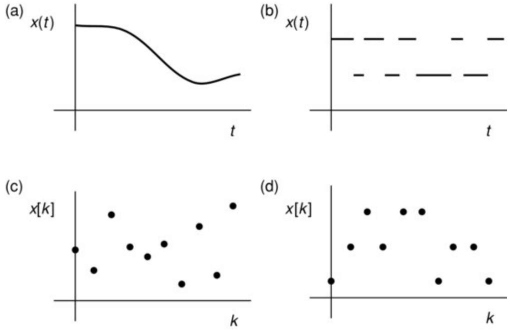
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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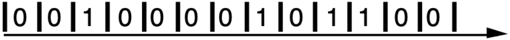

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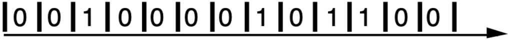

- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

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- Next: Key questions and answers about Bernoulli process

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- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

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- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

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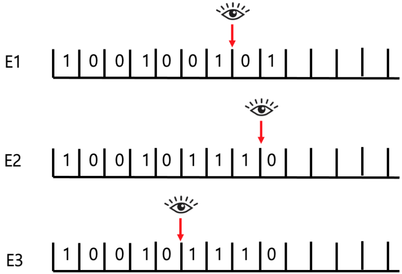
- Examples of N



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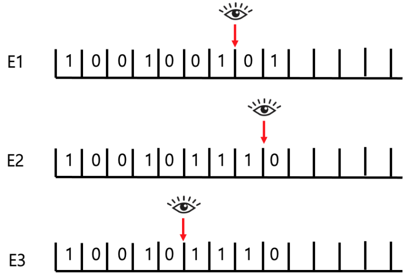

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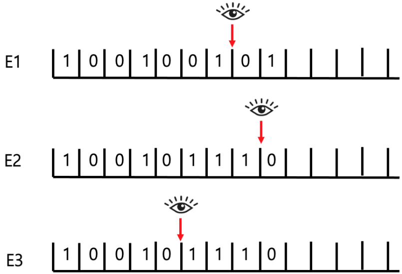

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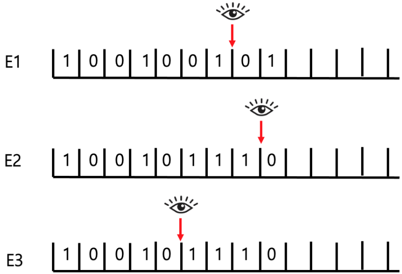

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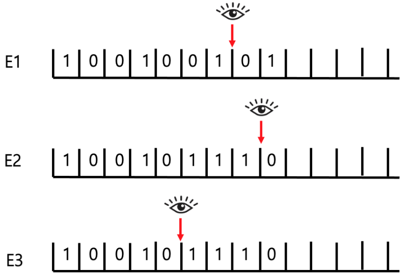

- Difference of N from n
  - The time when I watch the on-going Bernoulli process is random.

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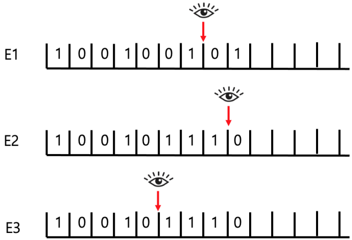

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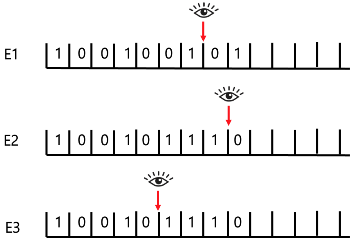

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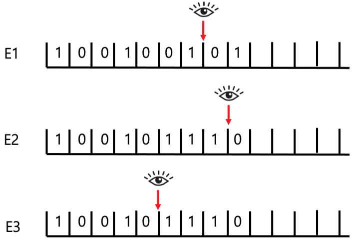

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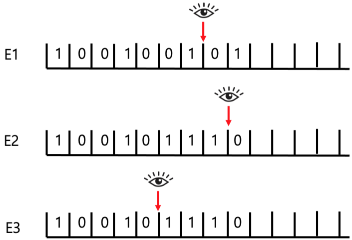

E1. When I watch the process, N has been already determined. Yes

E2. Same as **E1.** Yes

E3. Need the future knowledge. '111' does not become random. No

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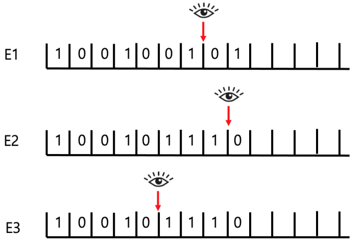

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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

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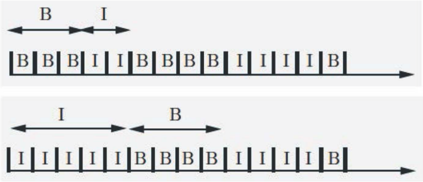

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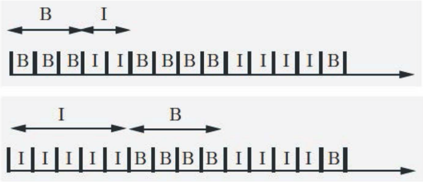

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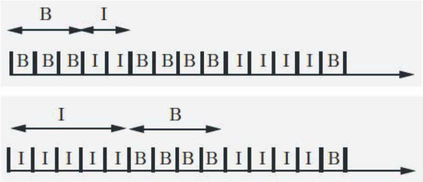

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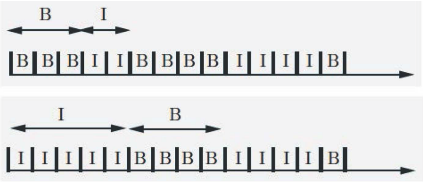

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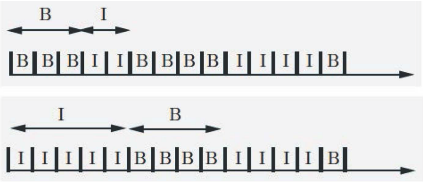

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- Question: What about the second busy period B_2? B_3, B_4?

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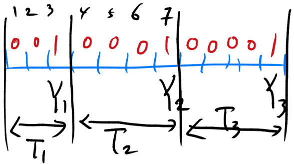

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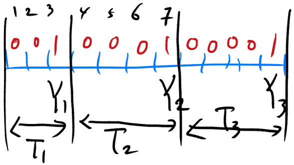

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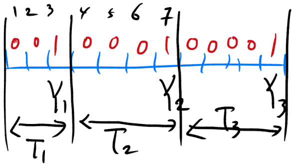

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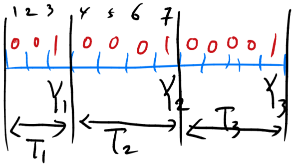

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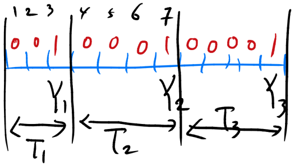

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- $Pascal(1, p) = Geometric(p)$

Roadmap

- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split
- Markov Chain

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!

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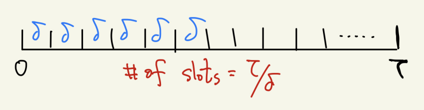

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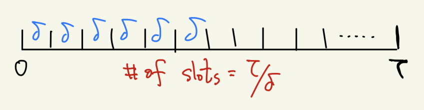
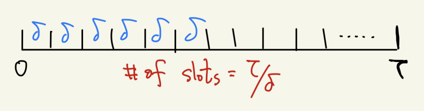


- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)

Key Design Idea to Develop a Continuous Twin (2)



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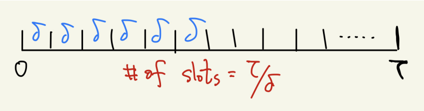

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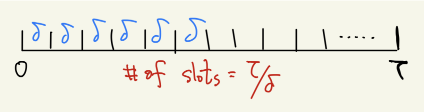

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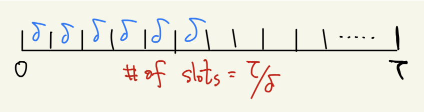

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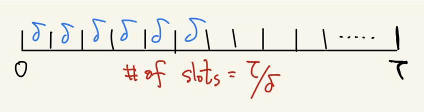

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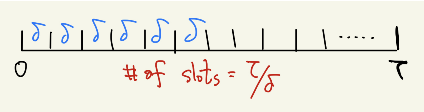

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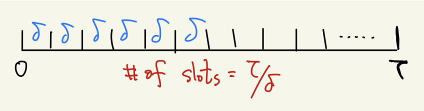

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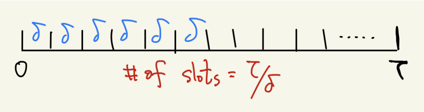

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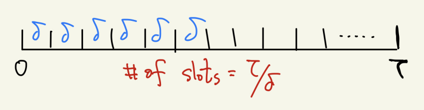

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- $o(\delta)$: some function that goes to zero faster than δ goes to zero.
  - Thus, for very small δ, $o(\delta)$ becomes negligible.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)

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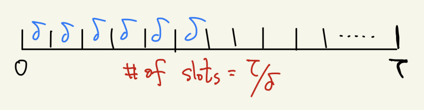

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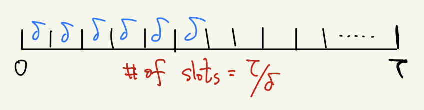

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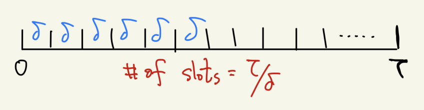

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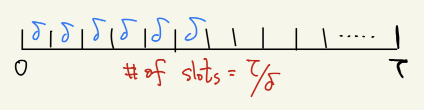

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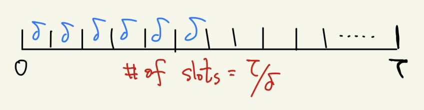

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- # of arrivals over $[0, \tau]$, $\sim$ *Poisson*$(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call Poisson process.

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  - Thus, N_s can be a random variable over any interval of length s.
- (Small interval probability) The probabilities $\mathbb{P}(k, s)$ satisfy:

$$\mathbb{P}(0, s) = 1 - \lambda\tau + o(s)$$
$$\mathbb{P}(1, s) = \lambda s + o_1(s)$$
$$\mathbb{P}(k, s) = o_k(s) \quad \text{for } k = 2, 3, \ldots,$$

where

$$\lim_{s\to 0} \frac{o(s)}{s} = 0, \quad \lim_{s\to 0} \frac{o_k(s)}{s} = 0$$

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Poisson Process: $\mathbb{P}(k, \tau)$, N_τ, and T

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  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

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- Mean and variance of mails received during a day
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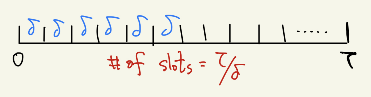
- Time of first arrival: geometric / exponential
- Time of k-th arrivals: Pascal / Erlang

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

Roadmap

- A lot of applications in engineering systems

- Basics on Random Process

- Bernoulli Process

- Poisson Process

- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split

- Markov Chain