Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

September 10, 2021

Roadmap

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

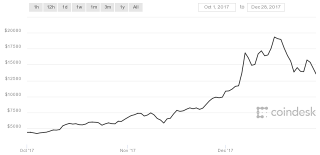
(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

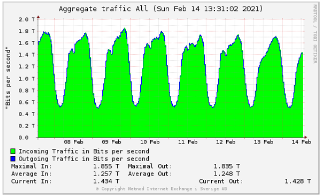
(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

- Many probabilistic experiments that evolve in time

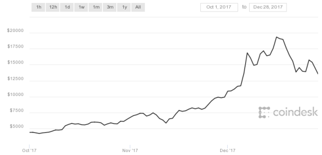


(a) Prices of a crytocurrency



(b) Internet traffic traces

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet



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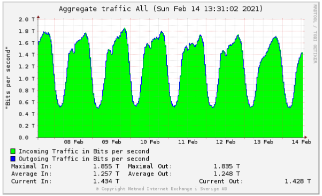

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Things that evolve in time

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
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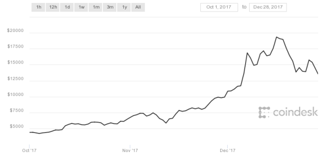
- Random process is a mathematical model for it.



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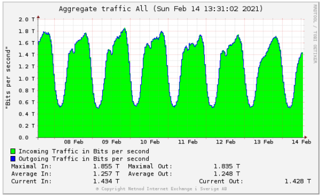

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- The values that X_t (or $X(t)$) can take: discrete or continuous

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  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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  - Other interesting questions, depending on the target random process

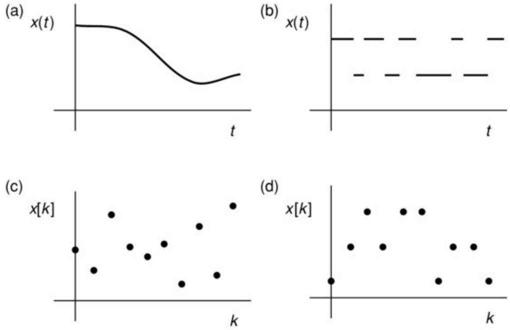
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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Simeon Denis Poisson (1781 - 1840), France

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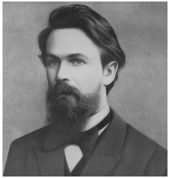

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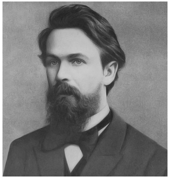

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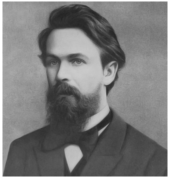

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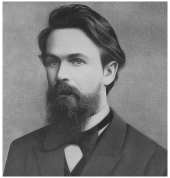

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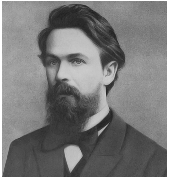

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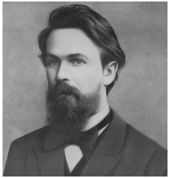

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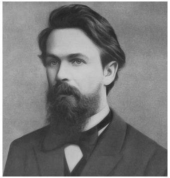

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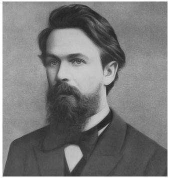

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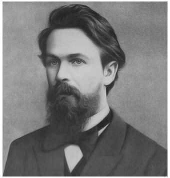

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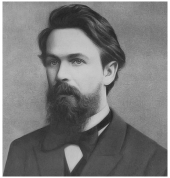

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| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | $\rightarrow$

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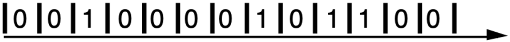

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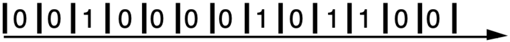

- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Please pause the video and write down the questions that you want to ask about Bernoulli process. VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

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- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots $\implies$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

Memoryless and Fresh-start after Deterministic time n

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- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

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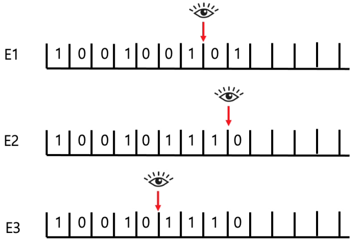

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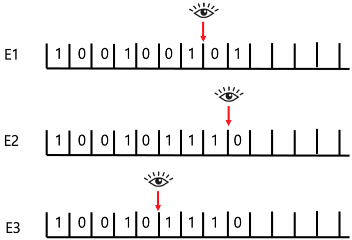

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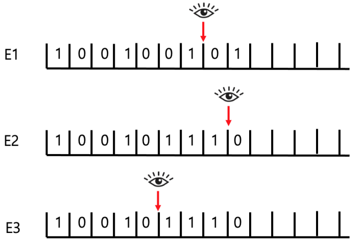

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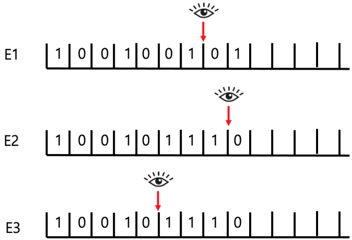

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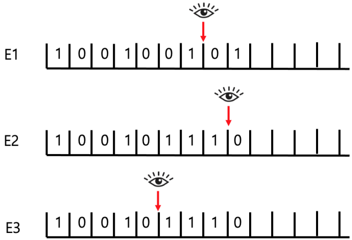

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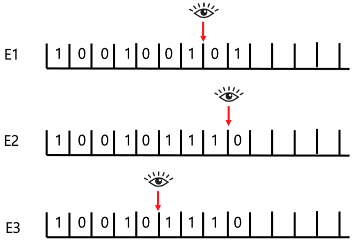

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- Do we experience the fresh-start for any N? **E1, E2, and E3?**

KAIST EE

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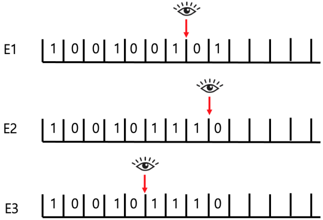

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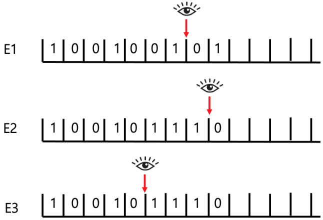

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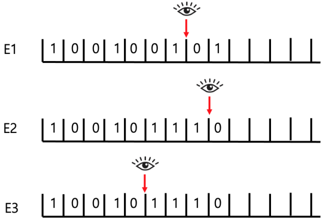

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Fresh-start after Random N (2)

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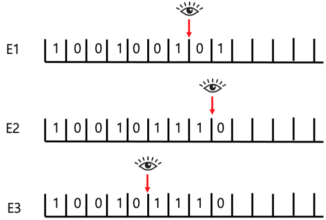

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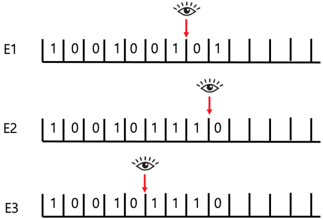

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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

- In probability theory, a random time N is said to be a stopping time, if the question of "$N = n$?" can be answered only from the present and the past knowledge of $X_1, X_2, \ldots, X_n$.

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Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)

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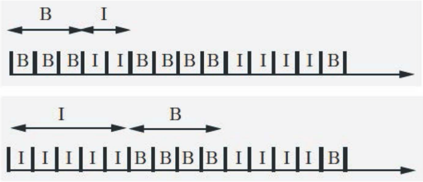

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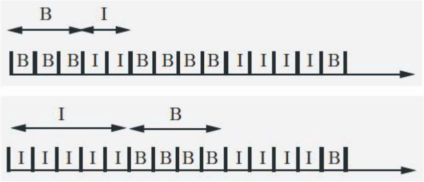

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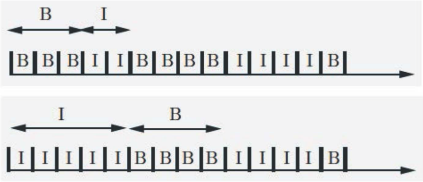

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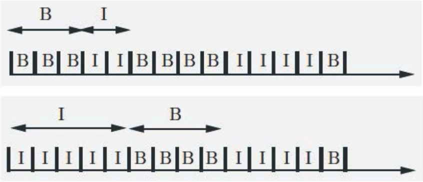

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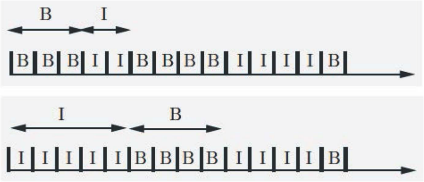

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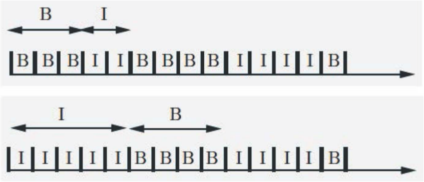
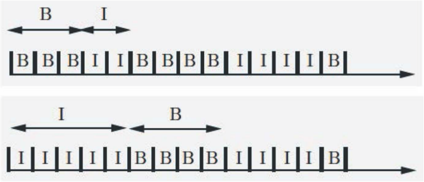
- B_1 is geometric with parameter $(1-p)$

Distribution of Busy Periods (2)



- Question. What about the second busy period B_2?

Distribution of Busy Periods (2)



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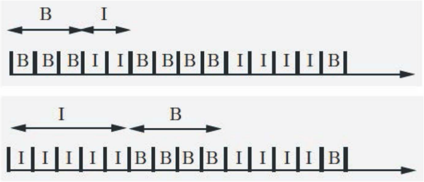
Distribution of Busy Periods (2)



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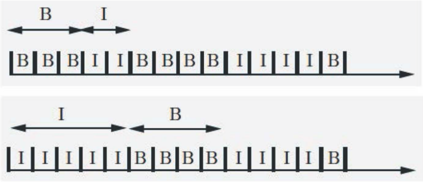

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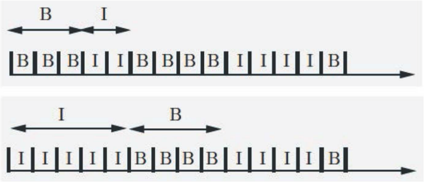

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- $B_3, B_4, \ldots$?

Time of k-th arrival

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

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VIDEO PAUSE

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VIDEO PAUSE

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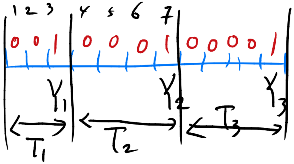

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VIDEO PAUSE

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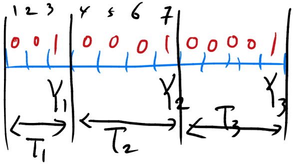

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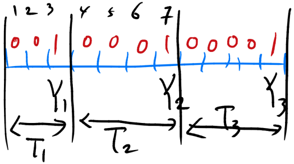

- After each T_k, the fresh-start occurs.

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

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VIDEO PAUSE

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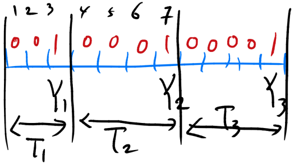

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Time of k-th arrival

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

  (Q7) Time of the k-th arrival Y_k?

VIDEO PAUSE

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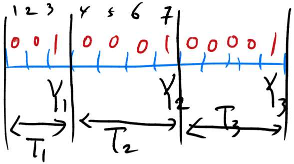

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- We know Y_k's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but its distribution?

PMF of Y_k

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\end{aligned}
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Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Background: Poisson rv X with parameter λ (1)

- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

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KAIST EE

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

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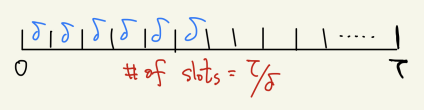

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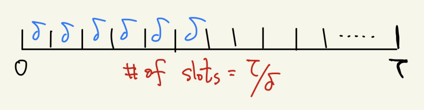
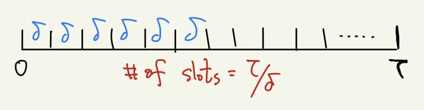


- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)



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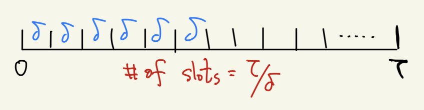

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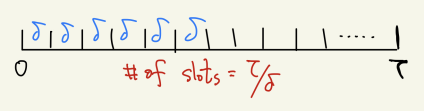

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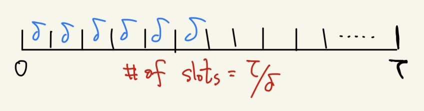
$\mathbb{P}(\geq 2 \text{ arrivals}) \propto$ something, but very small



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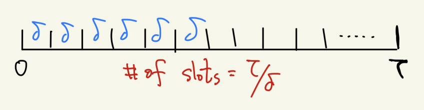

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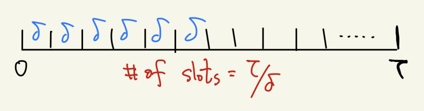

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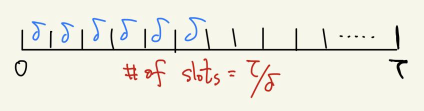

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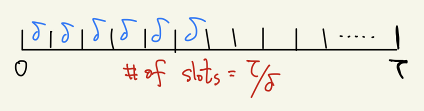

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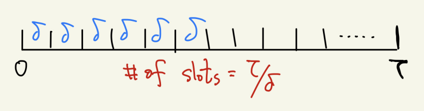

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$$\mathbb{P}(0 \text{ arrival}) = 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})$$

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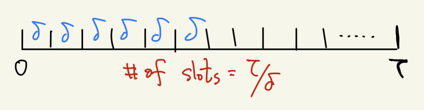
- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$



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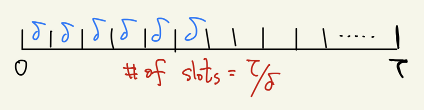

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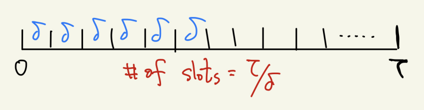

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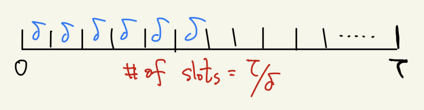

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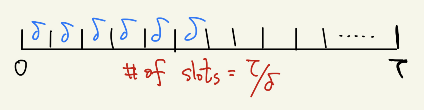
- # of arrivals over $[0, \tau]$, $\sim$ Poisson$(\lambda\tau)$



- Our interest: probability of k arrivals over $[0, \tau]$
- Given "small" δ, # of arrivals $\sim$ Bin(n, p), where $n = \tau/\delta$ and $p = \lambda\delta$
- As $\delta \to 0$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- # of arrivals over $[0, \tau]$, $\sim$ Poisson$(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call Poisson process.

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

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Poisson Process: Definition (1)

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \ldots, \quad \text{where} \quad \lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k,\tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k,\tau) = e^{\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Poisson Process: $\mathbb{P}(k, \tau)$, N_τ, and T

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
- $\mathbb{P}$[one new message in the next hour]
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- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

PDF of Y_k

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- For a given δ, $\boxed{}$: prob. of k-th arrival over $[y, y+\delta]$.

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- For a given δ, $\boxed{\delta \cdot f_{Y_k}(y)}$: prob. of k-th arrival over $[y, y+\delta]$.

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- This is called Erlang rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

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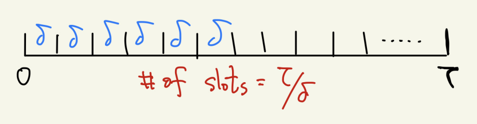

	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals		
Interarrival time		
Time of k-th arrival		
Arrival rate		

Poisson Process vs. Bernoulli Process

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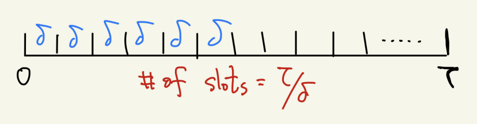

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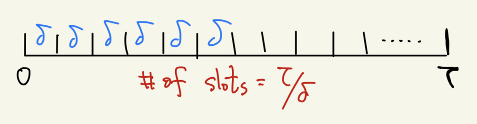

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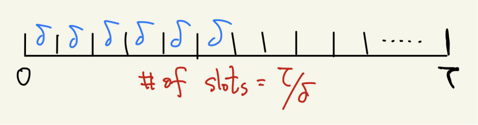

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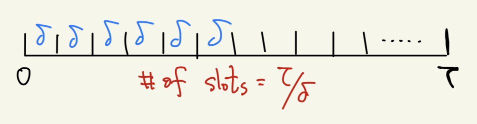

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Arrival rate	p/per slot	λ/unit time

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- Catching fish: Poisson process $\lambda = 0.6/\text{hour}$.
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(Q1) $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1: $\mathbb{P}(0,2)$
Method 2: $\mathbb{P}(T_1 > 2)$

(Q2) $\mathbb{P}(2 < \text{fishing time} < 5)$

Method 1: $\mathbb{P}(0,2)(1 - \mathbb{P}(0,3))$
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(Q3) $\mathbb{P}(\text{Catch at least two fish})$

Method 1:$\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0,2) - \mathbb{P}(1,2)$
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(Q4) $\mathbb{E}[\text{future fi. time}|\text{already fished for 3h}]$
Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

(Q5) $\mathbb{E}[\text{F=total fishing time}]$

$$
\begin{aligned}
2 + \mathbb{E}[F-2] &= 2 + \mathbb{P}(F=2)\cdot 0 + \\
&\quad \mathbb{P}(F>2)\cdot \mathbb{E}[F-2|F>2] \\
&= 2 + \mathbb{P}(0,2)\cdot \frac{1}{\lambda}
\end{aligned}
$$

Example: Poisson Fishing (Problem 10, page 329)

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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0,2)\cdot 1$

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

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- Thus, the answer is p^2.

Coding of Random Arrivals

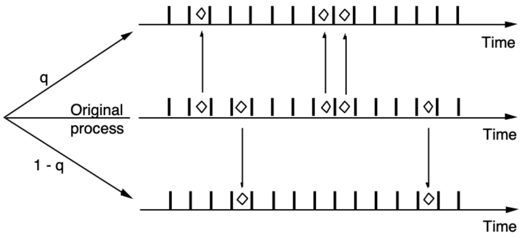
- Question. How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.
- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

Notations In the Rest of These Slides

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

Split: Bernoulli Process

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.



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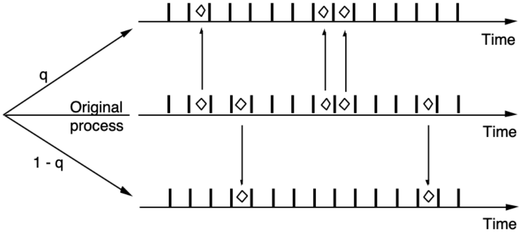

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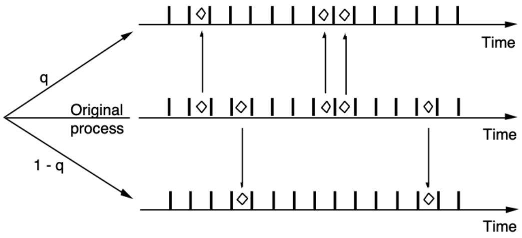

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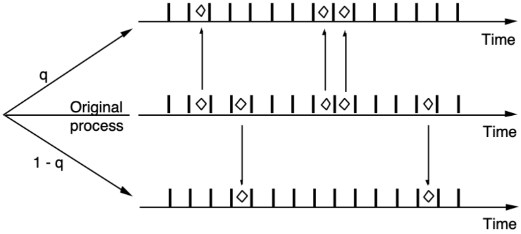

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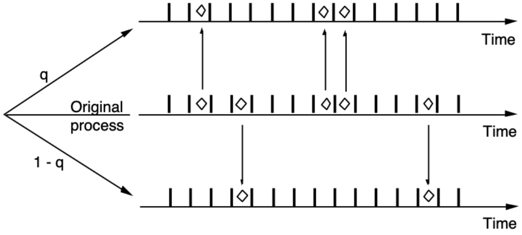
- Are they independent? No.



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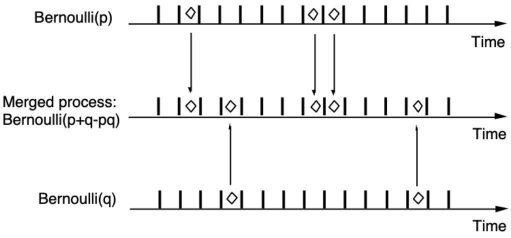

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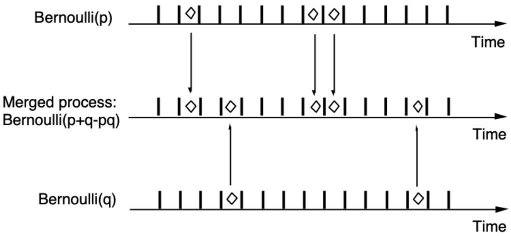

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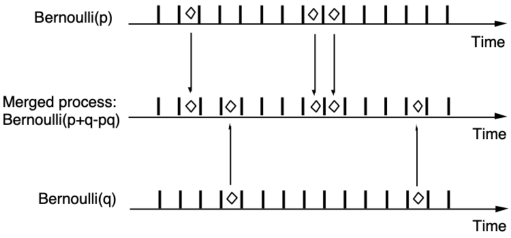

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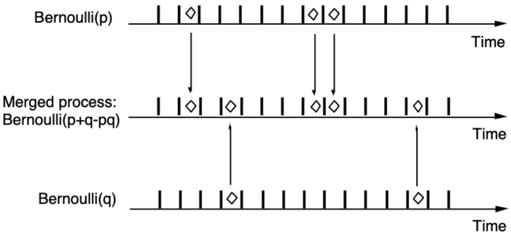

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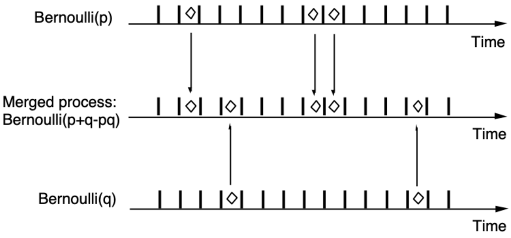
- $\mathbb{P}(\text{arrival from proc. } 1 \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



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  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- PP(λp) and PP($\lambda(1-p)$)

Merge: Poisson Process (1)

- Merge from PP(λ_1) and PP(λ_2)

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- Small interval probabilty over δ-interval (ignoring $o(\delta)$ for small δ)

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  $\mathbb{P}(1 \text{ arrival}) \approx (\lambda_1\delta)(1-\lambda_2\delta) + \lambda_2\delta(1-\lambda_1\delta) \approx (\lambda_1+\lambda_2)\delta$

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- Red: PP(λ_1) and Blue: PP(λ_2)
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Using Poisson Processes for Intuitive Problem Solving

1. Competing exponentials

2. Sum of independent Poisson rvs

3. Poisson arrivals during and Exponential interval

Competing Exponential (1)

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- Approach 1
  - $\mathbb{P}(Z \geq z) = \mathbb{P}(T_a \geq z)\mathbb{P}(T_b \geq z) = e^{-\lambda_a z} e^{-\lambda_b z} = e^{-(\lambda_a + \lambda_b)z}$

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  - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
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  - $T_1 \sim \text{Exp}(3\lambda)$, $T_2 \sim \text{Exp}(2\lambda)$, $T_3 \sim \text{Exp}(\lambda)$

$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Sum of Independent Poisson rvs

- Two independent rvs X and Y, where $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\nu)$.

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- Thus, $X+Y \sim \text{Poisson}(\mu+\nu)$

Poisson arrivals during Exponential Interval (1)

- Problem 24, pp. 335
- Consider PP(λ) and an independent rv $T \sim \text{Exp}(\nu)$

Poisson arrivals during Exponential Interval (1)

- Problem 24, pp. 335
- Consider PP(λ) and an independent rv $T \sim \text{Exp}(\nu)$
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- Problem 24, pp. 335
- Consider PP(λ) and an independent rv $T \sim \text{Exp}(\nu)$
- Question. Distribution of N_T?
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Poisson arrivals during Exponential Interval (1)

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- Very tedious and not very intuitive.

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- K: number of total arrivals until we get the first arrival from $\mathrm{PP}(\nu)$.
  - Then, $K \sim \mathrm{Geom}(\frac{\nu}{\lambda+\nu})$.
- Let L be the number of arrivals from $\mathrm{PP}(\lambda)$ until we get the first arrival from $\mathrm{PP}(\nu)$.

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \ldots$$

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  - $p = 1/100$, $n = 10,000$: small p, but large $n \implies$ Both Poisson and Normal

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

Example: Survey of Utilization of Town Buses

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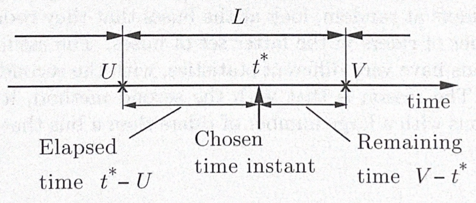

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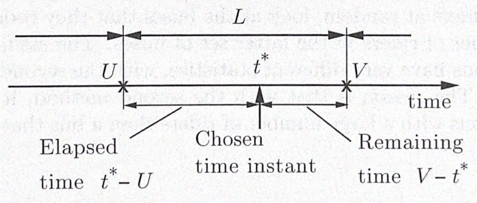

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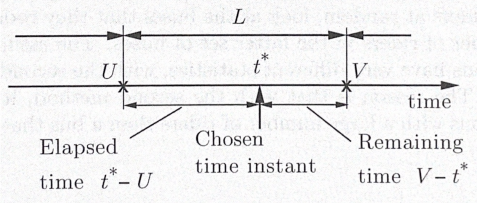
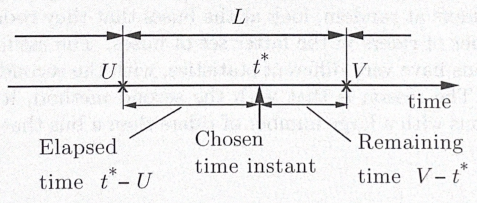


- Practical context: Yung shows up at the bus station at some arbitrary time $t^\star$ and records the time from the previous bus arrival (U) until the next bus arrival (V)
- Question. What is the distribution of L?

VIDEO PAUSE

Random Incidence Phenomemon (2)



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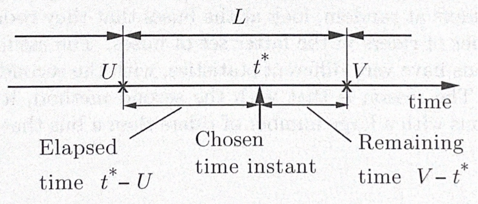

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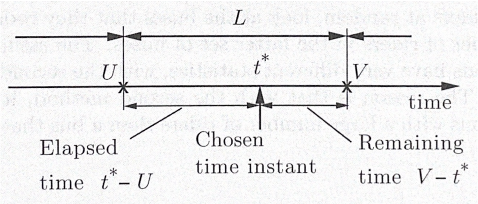

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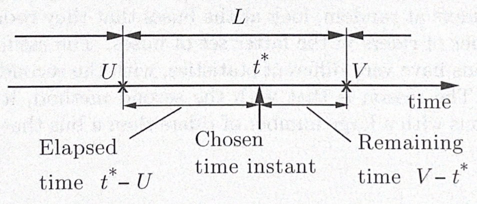

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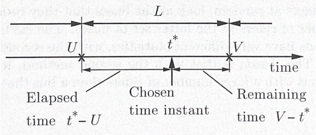
- One might superficially argue that $L \sim \text{Exp}(\lambda)$, but it is NOT.

Distribution of L (1)

$$L = (t^\star - U) + (V - t^\star)$$

- $V - t^\star$:

- $t^\star - U$:



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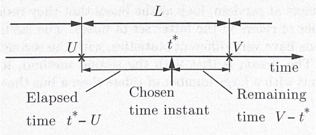

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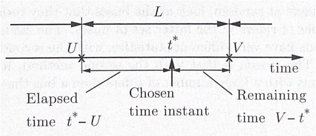

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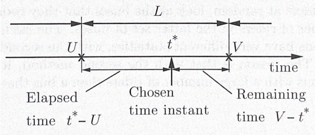

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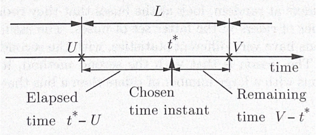

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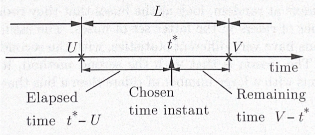

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$$f_L(l) = \lambda^2 \cdot l \cdot e^{-\lambda l}, \quad l \geq 0$$

- Mean $= 2/\lambda$
- Why not Exp(λ)? An observer who arrives at an arbitrary time is more likely to fall in a large rather than a small interarrival interval.

Back to Survey of Utilization of Town Buses

- Two Approaches
  M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus

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- (i) M1 $=$ M2? (ii) M1 $>$ M2? (iii) M1 $<$ M2?

- Answer: M1 $<$ M2

- More likely to select a bus with a large number of riders than a bus that is near-empty.

Questions?

1) Explain what is random process. Why do we need such a concept?
2) Explain Bernoulli processes. When do we use Bernoulli processes?
3) Explain Poisson processes. When do we use Poisson processes?
4) What are the relations between those two processces? What features do they share?
5) In both processces, ho do we compute (i) number of arrivals over a given interval of time, (ii) time until the first arrival, (iii) time until k-th arrival?
6) What is Pascal rvs and Erlang rvs?
7) What is the "stopping time" and how is it related to fresh-restart?
8) See many examples on how merging and splitting of BP and PP can be used for intuitive soloving of many problems.