

Lecture 3: Random Variable, Part I

Yi, Yung (이웅)

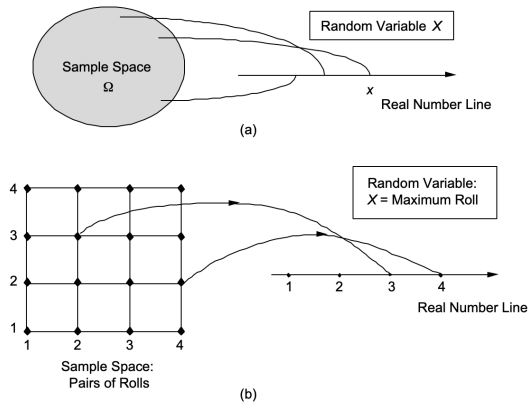
EE210: Probability and Introductory Random Processes
KAIST EE

September 14, 2022

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables

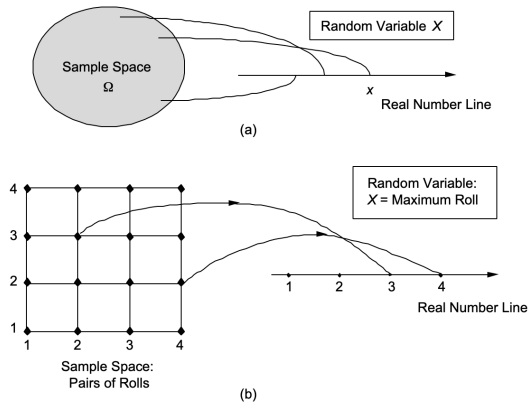
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- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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- For a fixed value x , we can associate an **event** that a random variable X has the value x , i.e., $\{\omega \in \Omega \mid X(\omega) = x\}$
- Assume that values x are discrete¹ such as $1, 2, 3, \dots$
For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$
- For a discrete random variable X , we call $p_X(x)$ **probability mass function** (PMF).

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- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$
- Remember that the random variable X is a **function** from Ω to \mathbb{R}

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- Only **binary** values

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$$X = \begin{cases} 0, & \text{w.p. } 1 - p, \\ 1, & \text{w.p. } p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an **indicator rv** of an event A . Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

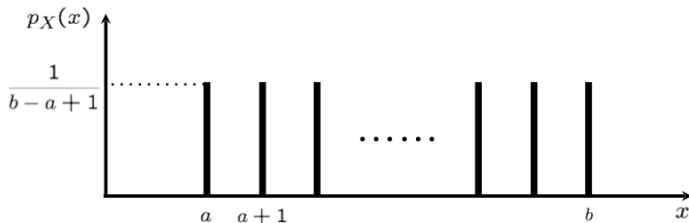
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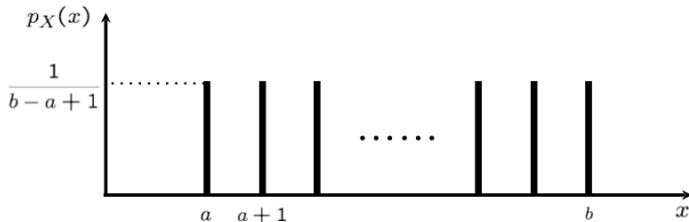
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- Models complete **ignorance** (I don't know anything about X)

$$^1 \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ which we read 'n choose k'.$$

L3(2)

- Models the number of **successes** in a given number of **independent** trials

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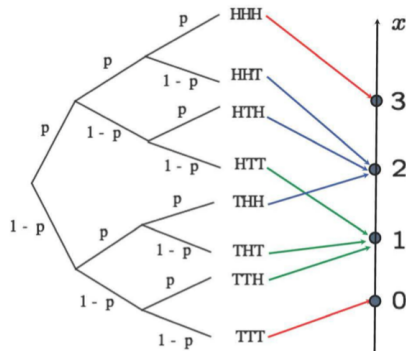
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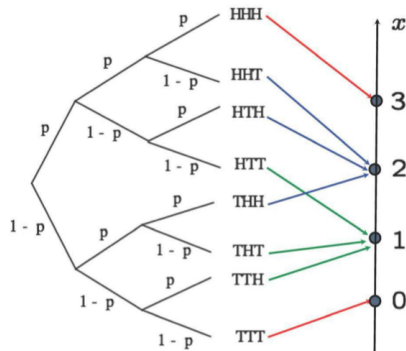


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$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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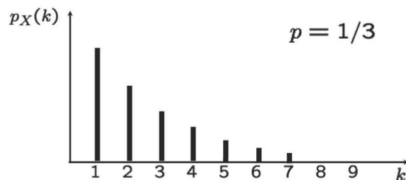
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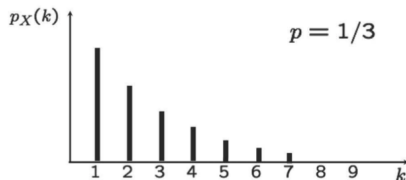
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- Models **waiting** times until something happens.



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Definition

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- **Example.** Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p = p_X(1)$$

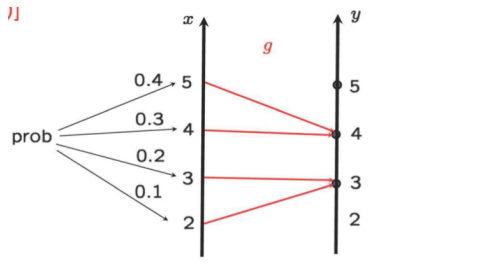
Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.
- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.
- For a constant c , $\mathbb{E}[c] = c$.

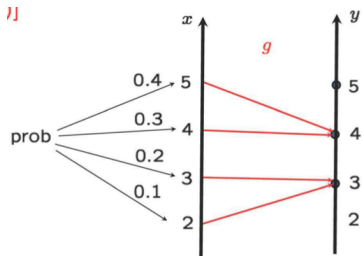
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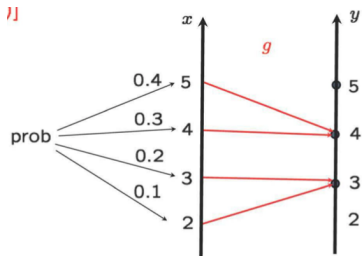


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Linearity of Expectation

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $Y = X + b, \text{var}[Y] = \text{var}[X]$
- $Y = aX, \text{var}[Y] = a^2\text{var}[X]$

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Example: Variance of a Bernoulli rv (p)

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Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}\mu &= \mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p \\ \mathbb{E}[X^2] &= 1 \times p + 0 \times (1 - p) = p \\ \text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\ &= p(1 - p)\end{aligned}$$

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- For two random variables X, Y , consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$\mathbb{P}(\{X = x\} \cap \{Y = y\})$$

- **Joint PMF.** For two random variables X, Y , consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$p_{X,Y}(x,y) \triangleq \mathbb{P}(\{X = x\} \cap \{Y = y\})$$

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$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

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- Marginal PMF.**

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Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
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$$p_{X,Y}(1,3) =$$

$$p_X(4) =$$

$$\mathbb{P}(X = Y) =$$

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y \ x	1	2	3	4
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3	2/20	4/20	1/20	2/20
2	0	1/20	3/20	1/20
1	0	1/20	0	0

$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y, X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

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- Y : number of successes in n Bernoulli trials with p

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- Example.** Mean of a binomial rv Y with (n, p)
- Y : number of successes in n Bernoulli trials with p
- $Y = X_1 + \dots + X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

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- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- Example.** Mean of a binomial rv Y with (n, p)

- Y : number of successes in n Bernoulli trials with p

- $Y = X_1 + \dots + X_n$, where X_i is a Bernoulli rv.

- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) **Conditioning for random variables**
- (6) Independence for random variables

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|--|--|

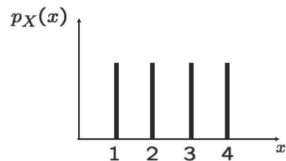
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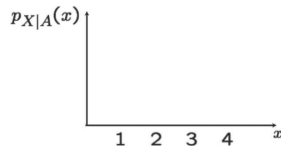
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 - (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

$$A = \{X \geq 2\}$$



$$\mathbb{E}[X] =$$

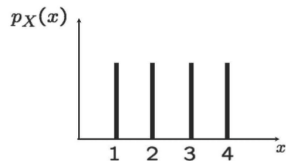
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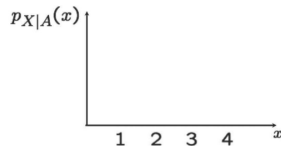
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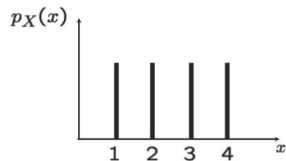
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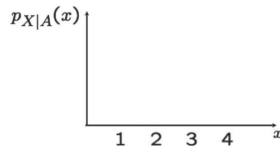
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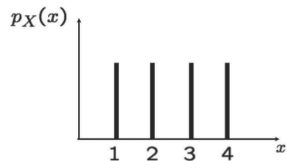
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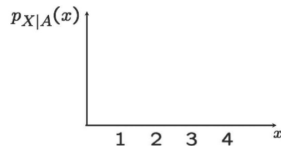
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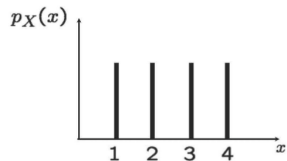
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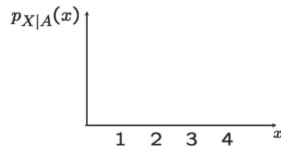
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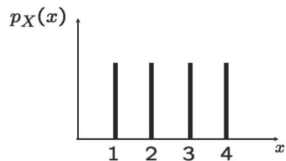
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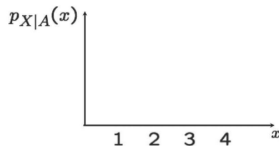
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- Multiplication rule

$$p_{X,Y}(x,y) =$$

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$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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VIDEO PAUSE

4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		
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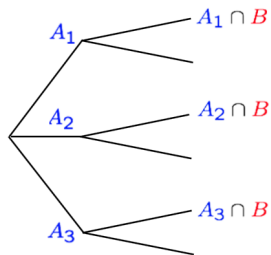
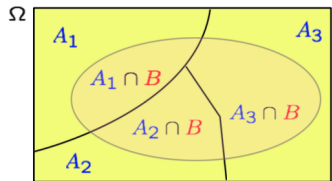
$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

$$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$$

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

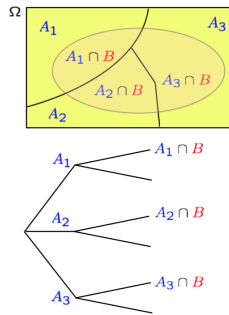
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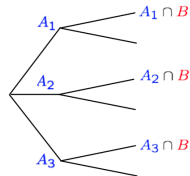
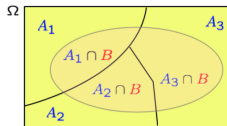
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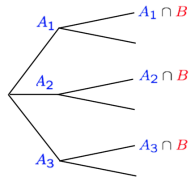
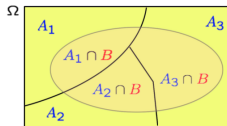
- Weighted average of expectations from A_i 's perspective



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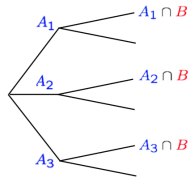
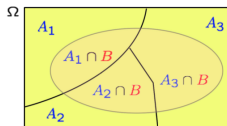
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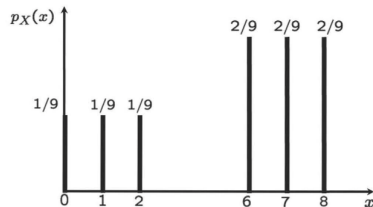
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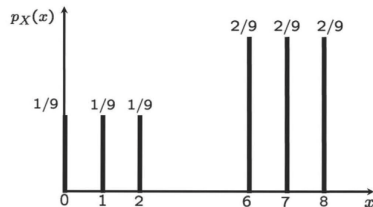
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$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{9}(0 + 1 + 2) + \frac{2}{9}(6 + 7 + 8) \\ &= \frac{3 + 12 + 14 + 16}{9} = 5\end{aligned}$$



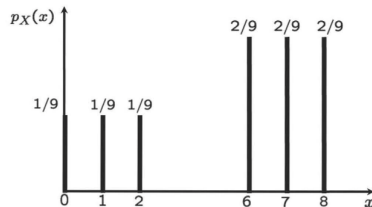
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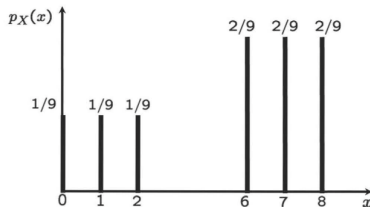
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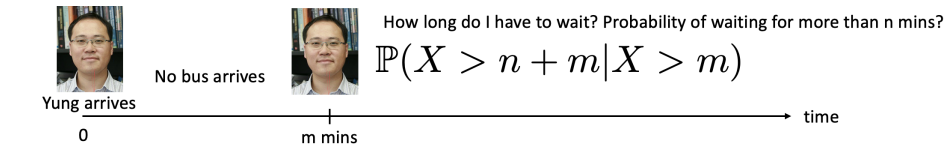
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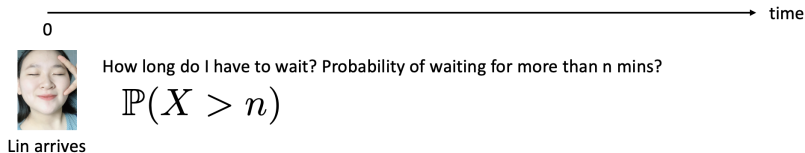
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- Total expectation theorem and a notion of **memorylessness** helps a lot.



A bus left at time 0



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- **Meaning.** Conditioned on $X > m$, $X - m$'s distribution is the same as the original X .

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

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- **Proof.**

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- **Theorem.** Any **geometric** random variable is **memoryless**.
- **Remind.** Geometric rv X with parameter p

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- **Meaning.** Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$\mathbb{E}[X] = 1 + \mathbb{E}[X - 1]$$

=

(from TET)

=

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- Thus, $\mathbb{E}[X] = \frac{1}{p}$

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

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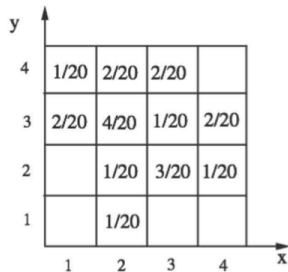
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- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?



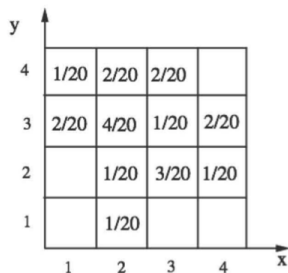
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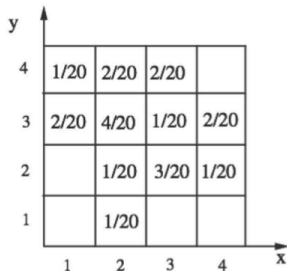
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VIDEO PAUSE

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$Y = 3$		
	$X = 1$	$X = 2$



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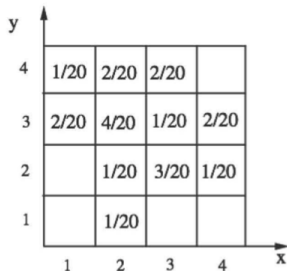
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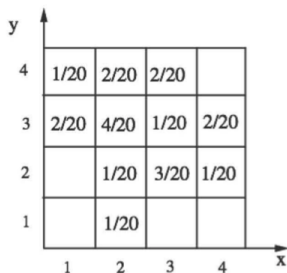
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- $X \perp\!\!\!\perp Y$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study **covariance**.

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- $\{X_i\}, i = 1, 2, \dots, n$: identically distributed (from symmetry)

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$$\mathbb{E}[X_i^2] = \mathbb{E}[X_1^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1$.
- **Key step 2.** Are X_i s are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \rightarrow X_2 = 1$, and $X_1 = 0 \rightarrow X_2 = 0$. Thus, **dependent**.

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\left[\sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j\right] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_1^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

- 1) What is a random variable? Why is it useful?
- 2) What is PMF (Probability Mass Function)?
- 3) Explain Bernoulli, Binomial, Geometric rvs. When are they useful and what are their PMFs?
- 4) Explain the memoryless property.
- 5) What are joint and marginal PMFs?
- 6) Describe and explain the total probability/expectation theorem for random variables? When is it useful to use total probability/expectation theorem?
- 7) Explain the definition and the meaning of expectation and variance. Why do we need them?
- 8) What is the difference between independence/conditional independence for events and those for random variables?