

## Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes  
KAIST EE

September 5, 2021

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

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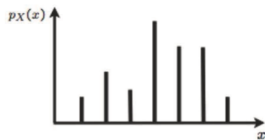


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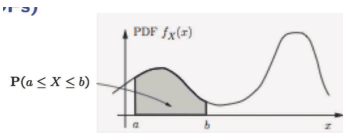
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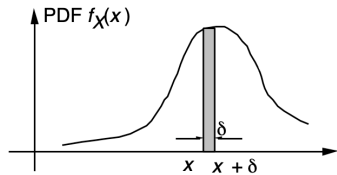
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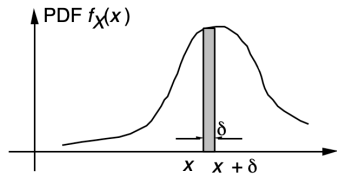


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x) dx = 1$



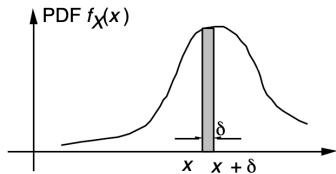
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Examples



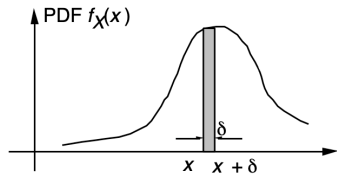
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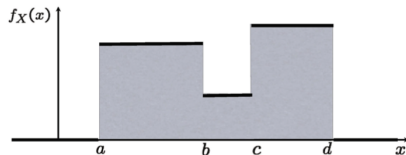
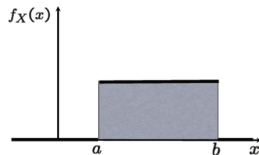
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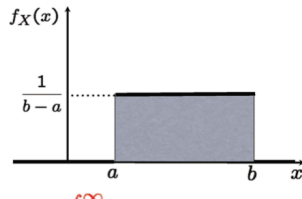
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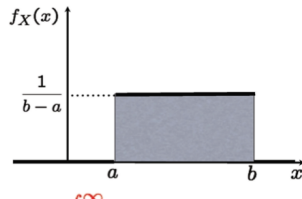
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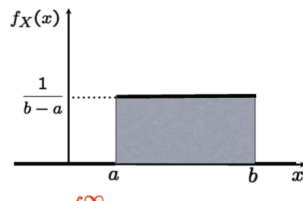




- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx =$
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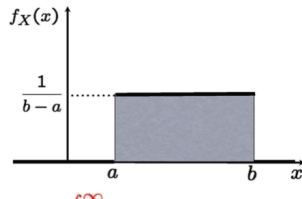


- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{b+a}{2}$
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- $\text{var}[X] = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

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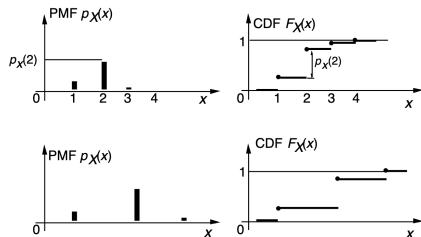
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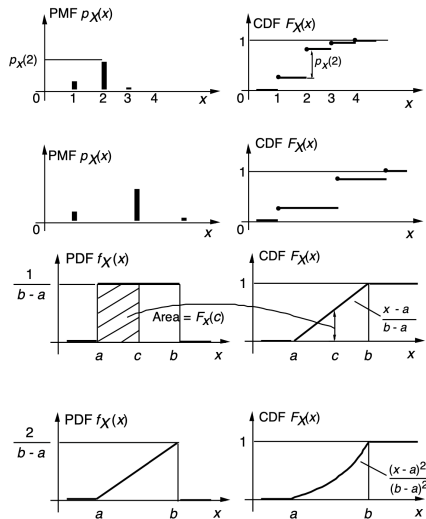


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  - $F_X(x)$  is a continuous function of  $x$ .
  - $F_X(x) = \int_{-\infty}^x f_X(t)dt$  and  $f_X(x) = \frac{dF_X}{dx}(x)$

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$ , and  $X_i \in \{1, 2, \dots, 10\}$  uniformly at random
- **Question.**  $p_X(x)$ ?
- Approach 1:  $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$ ?
- Approach 2

$$\begin{aligned} F_X(x) &= \mathbb{P}(\max\{X_1, X_2, X_3\} \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x) \\ &= \mathbb{P}(X_1 \leq x) \cdot \mathbb{P}(X_2 \leq x) \cdot \mathbb{P}(X_3 \leq x) = \left(\frac{x}{10}\right)^3 \end{aligned}$$

Thus,

$$p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \dots, 10$$

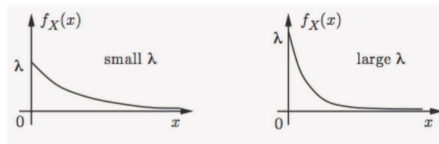
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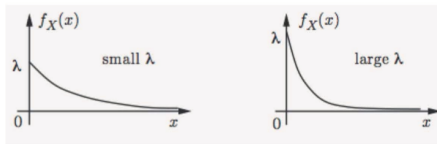
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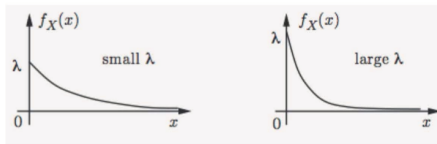
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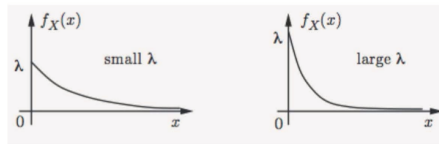
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- (Check)  $\mathbb{E}[X] = 1/\lambda$ ,  $\mathbb{E}[X^2] = 2/\lambda^2$ ,  $\text{var}[X] = 1/\lambda^2$

- $\mathbb{E}(X) = 1/\lambda$ . Use **integration by parts**:  $\int u dv = uv - \int v du$

$$\int_0^{\infty} x \lambda e^{-\lambda x} dx = (-x e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx = 0 + \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

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- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

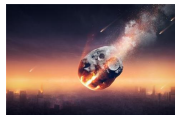
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- (Solution)
  - $\mathbb{E}(X) = 1/\lambda = 10$ . Thus,  $\lambda = \frac{1}{10}$ .
  - 6 a.m. from midnight = 1/4 day, 6 p.m. from midnight = 3/4 day

$$\mathbb{P}(1/4 \leq X \leq 3/4) = \mathbb{P}(X \geq 1/4) - \mathbb{P}(X \geq 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

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- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

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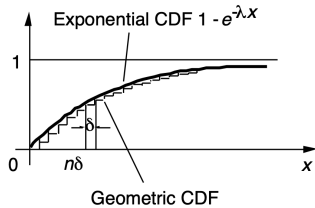
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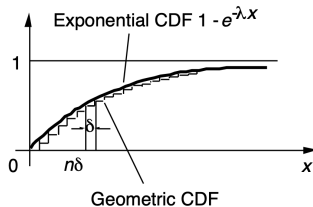
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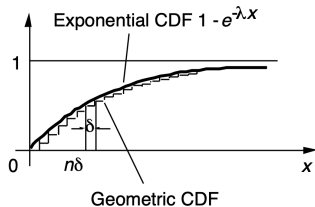
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  - $\mathbb{P}(X_{\delta}^{geo} \leq n) = 1 - (1 - p_{\delta})^n = 1 - e^{-\lambda\delta n}$





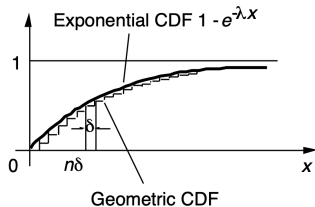
- Note that  $\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda x}$ . Then, when  $x = n\delta$ ,  $n = 1, 2, \dots$

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- If we choose sufficiently small  $\delta$ , the slot length  $\downarrow$  and  $p_{\delta} \downarrow$

$$\mathbb{P}(X_{\delta}^{geo} \leq n) \xrightarrow{\delta \rightarrow 0} \mathbb{P}(X^{exp} \leq x), \quad x = n\delta$$

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- (6) Bayes' rule for RVs

- **Standard** Normal  $\mathcal{N}(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$



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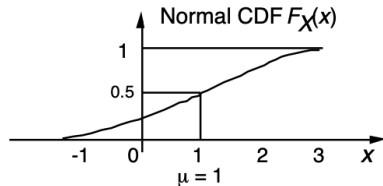
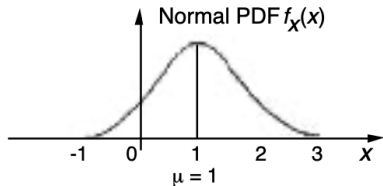
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

- General Normal  $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$



- PDF's normalization property:  $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

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- Variance

$$\begin{aligned} \text{var}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \stackrel{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} (-ye^{-y^2/2}) \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sigma^2 \end{aligned}$$

$$\int u dv = uv - \int v du: u = y \text{ and } dv = ye^{-y^2/2} \rightarrow du = dy \text{ and } v = -e^{-y^2/2}$$



- Linear transformation preserves normality (we will verify this in Lecture 5)

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for  $a \neq 0$  and  $b$ ,  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

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- Thus, every normal rv can be :

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then   $\sim \mathcal{N}(0, 1)$



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- Thus, every normal rv can be **standardized**:

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- Thus, we can make the **table** which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

- Annual snowfall  $X$  is modeled as  $\mathcal{N}(60, 20^2)$ . What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
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$$\begin{aligned}\mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80 - 60}{20}) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587\end{aligned}$$

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2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

- Central limit theorem
  - One of the most remarkable findings in the probability theory
  - Sum of **any** random variables  $\approx$  Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

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<sup>0</sup>Central limit theorem: 중심극한정리

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

Two continuous rvs are  if a non-negative function  $f_{X,Y}(x,y)$  (called joint PDF) satisfies: for **every subset**  $B$  of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy,$$



Two continuous rvs are **jointly continuous** if a non-negative function  $f_{X,Y}(x,y)$  (called joint PDF) satisfies: for **every subset**  $B$  of the two dimensional plane,

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1. The joint PDF is used to calculate probabilities

$$\mathbb{P}[(X, Y) \in B] = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy$$

Our particular interest:  $B = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$

2. The **marginal** PDFs of  $X$  and  $Y$  are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

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$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The **joint CDF** is defined by  $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$ , and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

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$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

4. A function  $g(X, Y)$  of  $X$  and  $Y$  defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

\* Conditional PDF, given an event  $A$

\* Conditional PDF, given  $\{X \in C\}$

**Notation:**  $A$  is an event, but  $B$  and  $C$  is a subset that includes the possible values which can be taken by the rv  $X$ . Sorry for the confusion, if any.

\* Conditional PDF, given an event  $A$

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$   
 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

\* Conditional PDF, given  $\{X \in C\}$

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- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$   
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

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- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$   
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$
- $\int f_{X|A}(x) dx = 1$

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$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$$

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- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$   
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- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$   
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$
- $\int f_{X|A}(x) dx = 1$

## \* Conditional PDF, given $\{X \in C\}$

$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$$

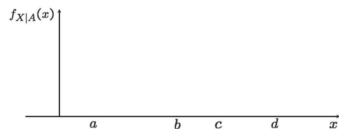
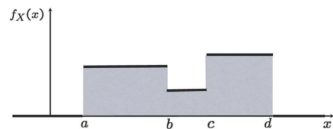
(Q) In the discrete, we consider the event  $\{X = x\}$ , not  $\{X \in B\}$ . Why?

**Notation:**  $A$  is an event, but  $B$  and  $C$  is a subset that includes the possible values which can be taken by the rv  $X$ . Sorry for the confusion, if any.

- $\mathbb{E}[X] = \int x f_X(x) dx$   
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$$A = \left\{ \frac{a+b}{2} \leq X \leq b \right\}$$

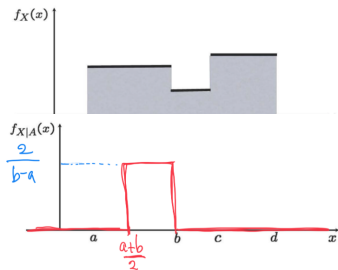


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$$\mathbb{E}[X|A] = \int_{(a+b)/2}^b x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^b x^2 \frac{2}{b-a} dx =$$

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Partition of  $\Omega$  into  $A_1, A_2, A_3, \dots$

\* Discrete case

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## Total Probability Theorem

$$\begin{aligned} p_X(x) &= \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) \\ &= \sum_i \mathbb{P}(A_i) p_{X|A_i}(x) \end{aligned}$$

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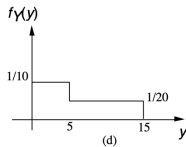
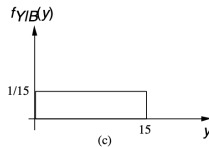
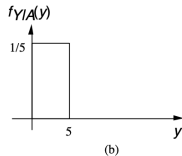
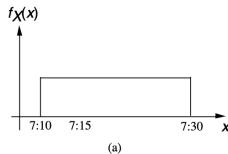
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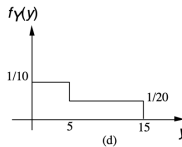
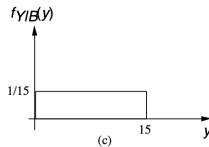
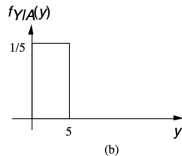
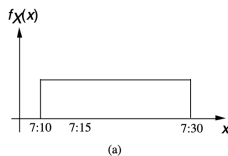
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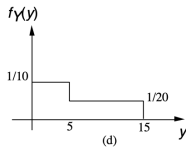
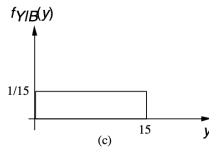
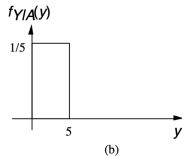
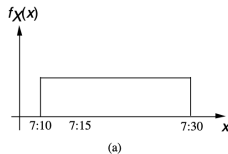
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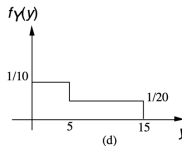
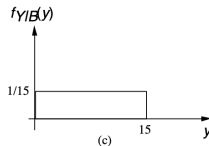
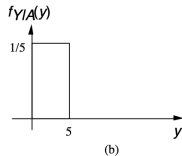
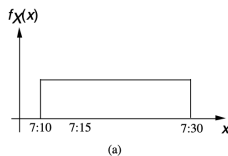
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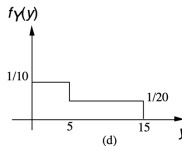
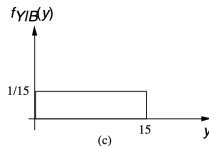
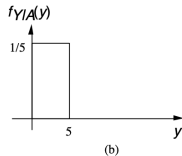
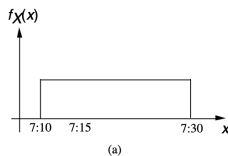
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VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A)f_{Y|A}(y) + \mathbb{P}(B)f_{Y|B}(y)$$

$$\text{for } 0 \leq y \leq 5$$

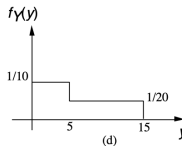
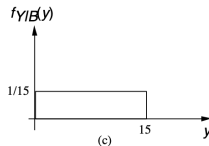
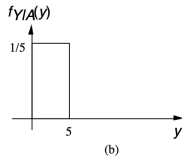
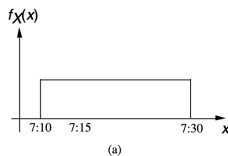
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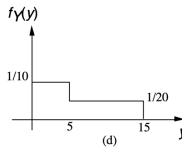
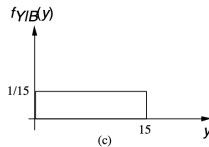
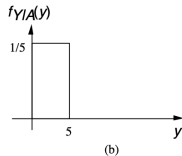
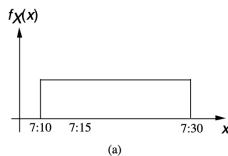
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$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

(Prob 21 at pp. 191)

- Break a stick of length  $l$  twice
  - first break at  $Y \sim \mathcal{U}[0, l]$
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---

<sup>0</sup> $\mathcal{U}[a, b]$ : continuous uniform random variable over the interval  $[a, b]$

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$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x, y) dy = \int_x^l \frac{1}{ly} dy \\ &= \frac{1}{l} \ln(l/x), \quad 0 \leq x \leq l \end{aligned}$$

---

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(c) Evaluate  $\mathbb{E}(X)$ , using  $f_X(x)$

(d) Evaluate  $\mathbb{E}(X)$ , using  $X = Y \cdot (X/Y)$

If  $Y \perp\!\!\!\perp X/Y$ , it becomes easy, but true?

(e) Evaluate  $\mathbb{E}(X)$ , using TET

(c) Evaluate  $\mathbb{E}(X)$ , using  $f_X(x)$

$$\begin{aligned}\mathbb{E}(X) &= \int_0^l x f_X(x) dx = \int_0^l \frac{x}{l} \ln(l/x) dx \\ &= \frac{l}{4}\end{aligned}$$

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- **Message.** There are many ways to reach our goal. Of crucial importance is how to find the best way!

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

- $X$ : state/cause/original value  $\rightarrow$   $Y$ : result/resulting action/noisy measurement
- Given:  $\mathbb{P}(X)$  and  $\mathbb{P}(Y|X)$  (cause  $\rightarrow$  result)
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2. Romeo and Juliet start dating, but Romeo will be late by a random variable  $Y \sim \mathcal{U}[0, \theta]$ . Given the time of being late  **$y$** , the modified belief about  **$\theta$** ?



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- Wait!  $p_{K|Y}(k|y)$ ? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

- For small  $\delta$  (in other words, taking the limit as  $\delta \rightarrow 0$ ).

Let  $A = \{K = k\}$ .

$$\begin{aligned} p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \leq Y \leq y + \delta) \\ &= \frac{\mathbb{P}(A)\mathbb{P}(y \leq Y \leq y + \delta|A)}{\mathbb{P}(y \leq Y \leq y + \delta)} \\ &\approx \frac{\mathbb{P}(A)f_{Y|A}(y)\delta}{f_Y(y)\delta} \\ &= \frac{\mathbb{P}(A)f_{Y|A}(y)}{f_Y(y)} \end{aligned}$$

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- Your received signal = 0.7. What's your guess about the original signal? **+1**
- Your received signal = -0.2. What's your guess about the original signal? **-1**
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

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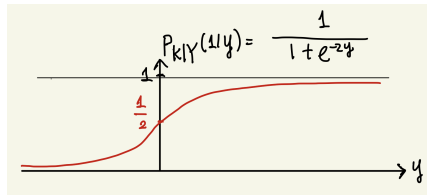
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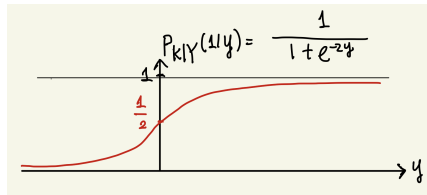
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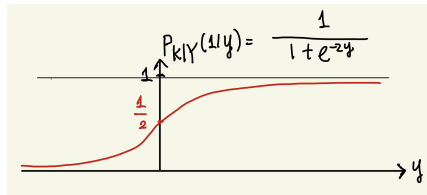
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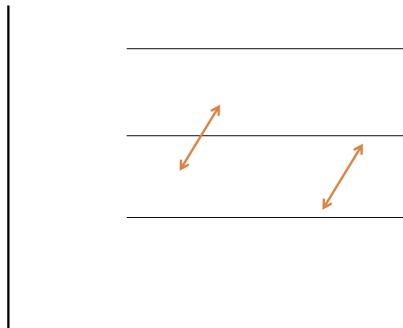
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- Similarly, compute  $p_{K|Y}(-1|y)$  and then do the inference



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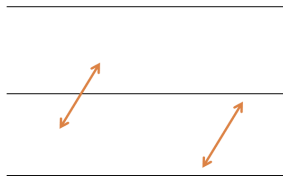
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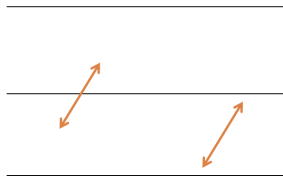
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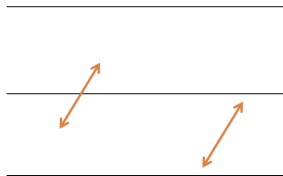
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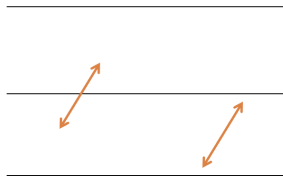
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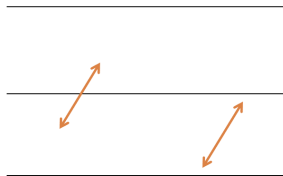
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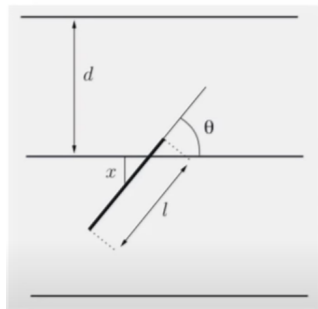


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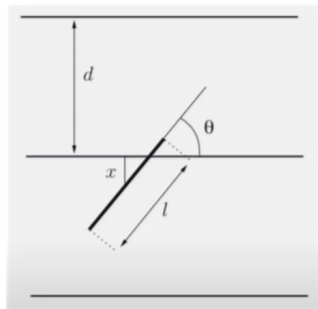
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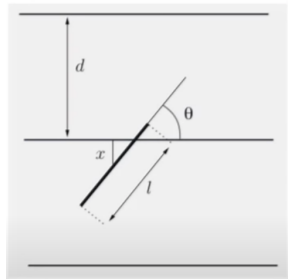
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- $\Theta$** : acute angle formed by the axis of the needle and the parallel lines



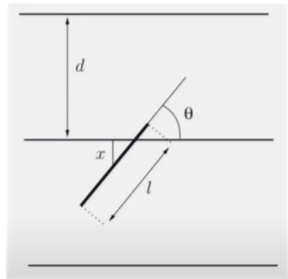
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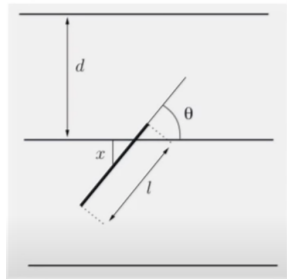
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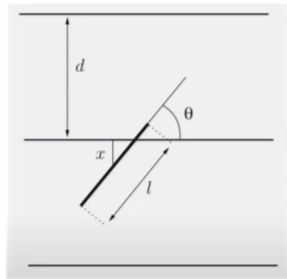


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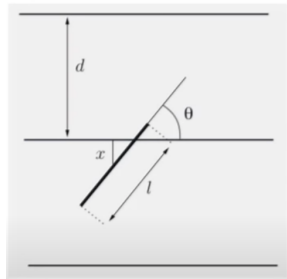


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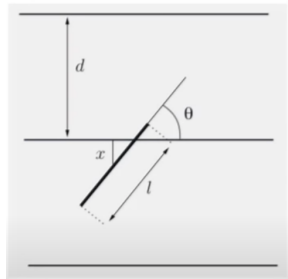
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- When does the needle intersect a line?  $X \leq \frac{l}{2} \sin \Theta$
- The probability of intersection

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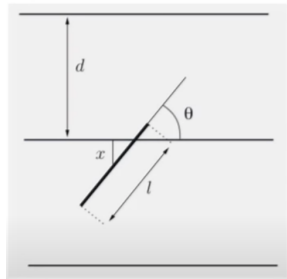
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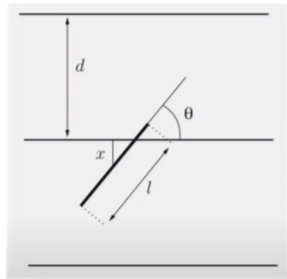
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- In history, a method for the experimental evaluation of  $\pi$ .



Questions?

- 1) What do we mean by “continuous” in continuous random variables?
- 2) Explain PDF and CDF. Why do we need CDF?
- 3) What are joint/marginal/conditional PDFs?
- 4) Explain the relationship between Geometric rvs and Exponential rvs.
- 5) Explain how normality is preserved under linear transformation for Normal (Gaussian) rvs.
- 6) Explain how we can use Bayes' rule for parameter learning.
- 7) Explain the version of Bayes' rule for continuous and mixed random variables.