

Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

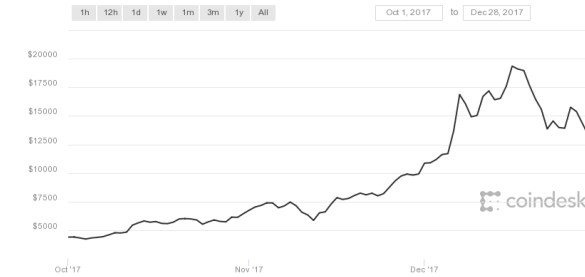
October 25, 2021

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes
- (6) Random Incidence Phenomenon

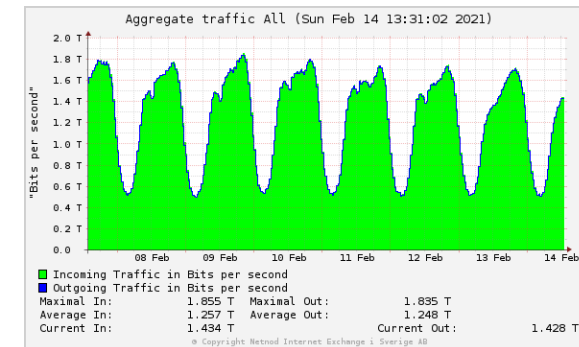
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Things that evolve in time

- Many probabilistic experiments that **evolve in time**
 - Sequence of daily prices of a stock
 - Sequence of scores in football
 - Sequence of failure times of a machine
 - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



(a) Prices of a cryptocurrency



(b) Internet traffic traces

- A random process is a **sequence** of random variables indexed by **time**.
- Time: discrete or continuous (a modeling choice in most cases)
- Notation
 - $(X_t)_{t \in \mathcal{T}}$ or $(X(t))_{t \in \mathcal{T}}$, where $\mathcal{T} = \mathbb{R}$ (continuous) or $\mathcal{T} = \{0, 1, 2, \dots\}$ (discrete)
 - For the discrete case, we also often use $(X_n)_{n \in \mathbb{Z}_+}$.
 - We will use all of them, unless confusion arises.
- For a **fixed** time t , X_t (or $X(t)$) is a random variable.
- The values that X_t (or $X(t)$) can take: discrete or continuous

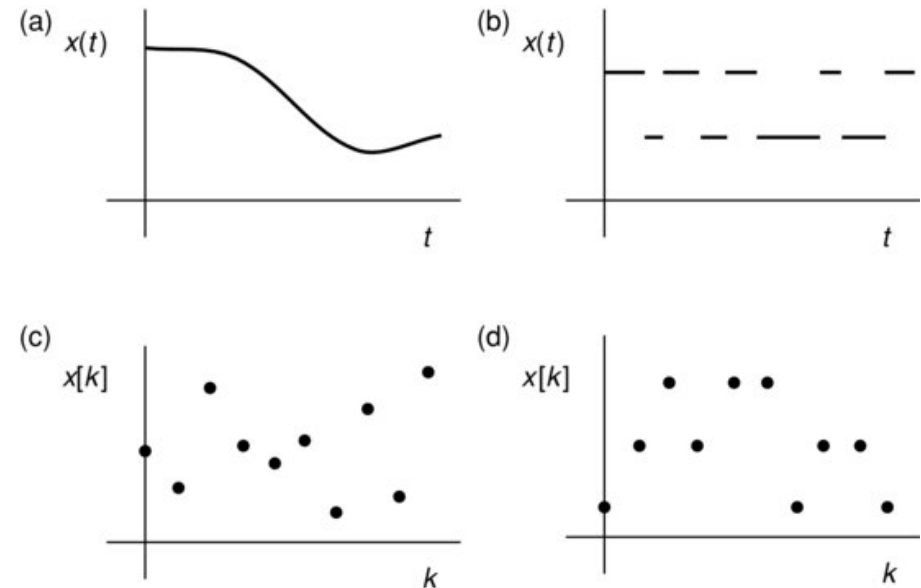
- **Example.** Discrete time RP. $\{X_1, X_2, X_3, \dots\}$.
 - X_i : # of Covid-19 infections at day i in South Korea, which is a random variable.
 - Then, $X_i : \Omega \mapsto \mathbb{R}$, where the sample space Ω is the set of all outcomes.
 - An outcome $\omega \in \Omega$ is a infinite sequence of infections
 - For example, $\omega_1 = (100, 150, 130, \dots)$, $\omega_2 = (200, 300, 400, \dots)$
 - $X_3(\omega_1) = 130$, $X_2(\omega_2) = 300$, $X_1(\omega_2) = 200$, etc.
- **Example.** Continuous time RP. $(X(t))_{t \in \mathbb{R}^+}$
 - $X(t)$: bitcoin price at time t , which is a random variable.
 - Then, $X(t) : \Omega \mapsto \mathbb{R}$.
 - An outcome $\omega \in \Omega$ is a trajectory of prices over $[0, \infty)$
 - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

- **Question.** Already studied a sequence (or a collection) of rvs X_1, X_2, \dots, X_n . What's the difference?
 - Assume a discrete time random process for our discussion.
- **Physical difference:** infinite sequence of X_1, X_2, \dots .
 - Sample space? set of all outcomes?
 - an outcome: an infinite sequence of sample values x_1, x_2, \dots ,
- **Semantic difference:** Understand i in X_i as time. Also, interesting questions are asked from the random process point of view.
 - **Dependence:** How X_1, X_2, \dots are related to each other as a time series. Prediction of values in the future.
 - **Long-term behavior:** What is the fraction of times that a stock price is above 3000?
 - Other interesting questions, depending on the target random process

4 Types of Random Processes

- Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



Random Processes in This Course

- The simplest RP
- discrete time
- $X_i \perp\!\!\!\perp \{X_{i-1}, X_{i-2}, \dots, X_1\}$
- **Bernoulli Process (BP)**
- “today” independent of “past”

Jacob Bernoulli (1654 - 1705),
Swiss



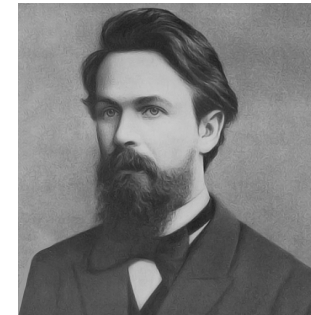
- The simplest RP
- Continuous time version of BP
- $[X(s)]_{s=0}^t \perp\!\!\!\perp [X(s)]_{s=t}^{t+a}$
- **Poisson Process (PP)**
- “today” independent of “past”

Simeon Denis Poisson (1781 -
1840), France



- One-step more general than BP/PP
- discrete time
- X_i depends on X_{i-1} , but $\perp\!\!\!\perp \{X_{i-2}, X_{i-2}, \dots, X_1\}$
- **Markov Chain (MC)**
- “today” depends only on “yesterday”

Andrey Markov (1856 - 1922),
Russia



- (1) Introduction of Random Processes
- (2) **Bernoulli Processes**
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- At each “minute”, we toss a coin with probability of head $0 < p < 1$.
 - Sequence of lottery wins/looses
 - Customers (each second) to a bank
 - Clicks (at each time slot) to server
- A sequence of **independent Bernoulli trials** X_1, X_2, \dots ,
 - We call index 1, 2, ... **time slots** (or simply slots)

| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | →

- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

Please pause the video and write down the questions that you want to ask about Bernoulli process.

VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

(Q1) # of arrivals in the first n slots?

- $S_n = X_1 + X_2 + \cdots + X_n$
- $S_n \sim \text{Binomial}(n, p)$
- $\mathbb{E}(S_n) = np$, $\text{var}(S_n) = np(1 - p)$
- This will hold for any n consecutive slots.

(Q2) # of slots T_1 until the first arrival?

- $T_1 \sim \text{Geom}(p)$
- $\mathbb{E}(T_1) = 1/p$, $\text{var}(T_1) = \frac{1-p}{p^2}$

- T_1 is geometric? [Memoryless](#)
- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots \implies the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

- Yes
- Because X_i s are independent

(Q4) After time $n = 6$, I start to look at the process $(X_n)_{n=6}^\infty$?

- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time n (doesn't matter what happened until $n = 5$).
- If you watch the on-going Bernoulli process(p) from some time n , you still see the same Bernoulli process(p).

Fresh-start after Random time N (1)

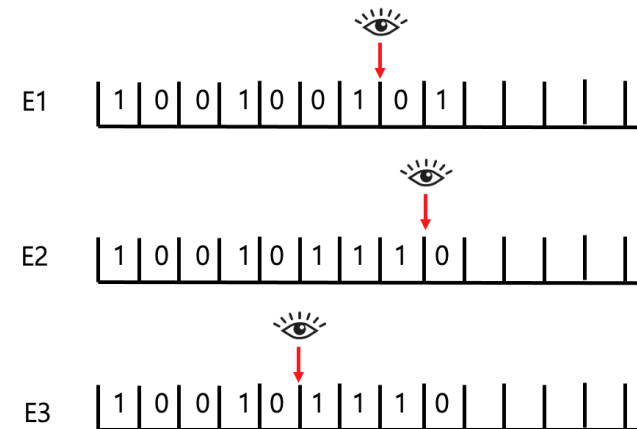
(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \dots)$? Fresh-start even after random N ?

- This means that the time I start to look at the process is a random variable.
- Examples of N

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



- Difference of N from n
 - The time when I watch the on-going Bernoulli process is **random**.
 - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω ?
- Do we experience the fresh-start for any N ? **E1, E2, and E3?**

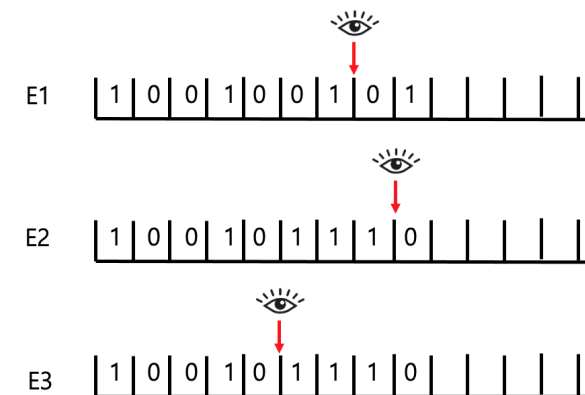
Fresh-start after Random N (2)

(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \dots)$? Fresh-start even after random N ?

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



E1. When I watch the process, N has been already determined. **Yes**

E2. Same as **E1**. **Yes**

E3. Need the future knowledge. '111' does not become random. **No**

- The question of $N = n$? can be answered just from the knowledge about X_1, X_2, \dots, X_n ? Then, Yes! (see pp. 301 for more formal description)

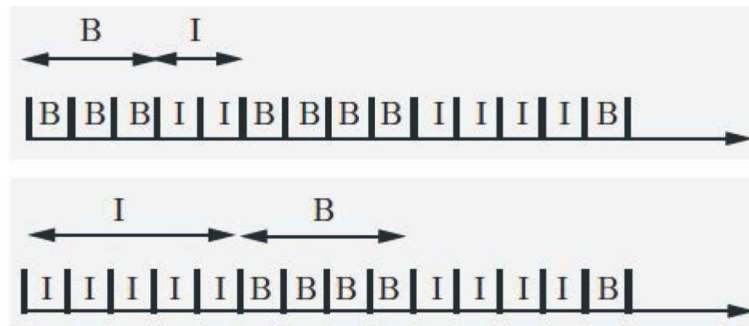
- In probability theory, a random time N is said to be a **stopping time**, if the question of “ $N = n$?” can be answered only from the present and the past knowledge of X_1, X_2, \dots, X_n .
- https://en.wikipedia.org/wiki/Stopping_time
- Fresh-start after N in Bernoulli process? **Yes, if N is a stopping time.**
- Please think about two examples of stopping time and not.

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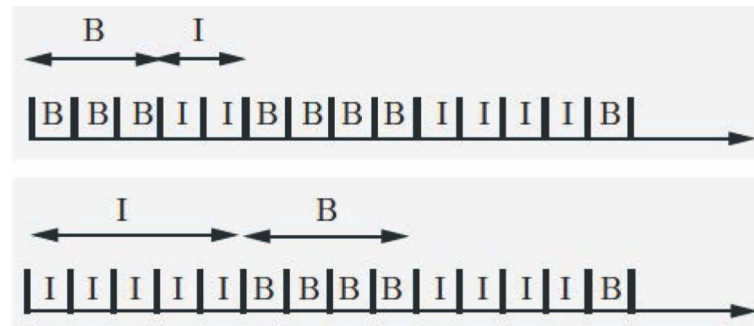
 - **Yes.** Time when 10 consecutive arrivals have been observed
 - **No.** Time of 2nd arrival in 10 consecutive arrivals

Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)
- **First busy period B_1** : starts with the first busy slot and ends just before the first subsequent idle slot



- **(Q6)** B_1 is a random variable. Distribution of B_1 ?
- N : time of the first busy slot. N is a stopping time?
 - **Yes**. Thus, **fresh-start after N** . Because we can answer the question of $N = n?$, just using X_1, X_2, \dots, X_n .
- B_1 is geometric with parameter $(1 - p)$



- **Question.** What about the second busy period B_2 ?
- N : time of the first busy slot of the second busy period. N is a stopping time?
- **Yes.** Thus, fresh-start after N .
- Then, B_1 and B_2 are **identically distributed** as $\text{Geom}(1 - p)$.
- B_3, B_4, \dots ?

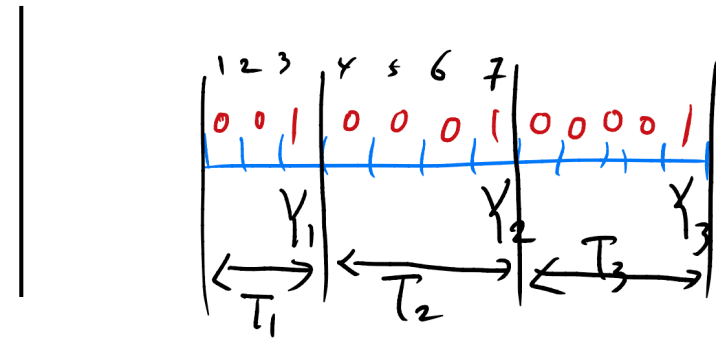
Time of k -th arrival

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

(Q7) Time of the k -th arrival Y_k ?

VIDEO PAUSE

- $T_k = Y_k - Y_{k-1}$: k -th inter-arrival ($k \geq 2$, $T_1 = Y_1$)
- $Y_k = T_1 + T_2 + \dots + T_k$.



- After each T_k , the fresh-start occurs.
- $\{T_i\}$ are i.i.d. and $\sim \text{Geom}(p)$
- We know Y_k 's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but **its distribution?**

- $Y_k = T_1 + T_2 + \dots + T_k$.
- $\{T_i\}$ are i.i.d. and $\sim \text{Geom}(p)$

$$\begin{aligned}\mathbb{P}(Y_k = t) &= \mathbb{P}(X_k = 1 \text{ and } k-1 \text{ arrivals during the first } t-1 \text{ slots}) \\ &= \mathbb{P}(X_k = 1) \cdot \mathbb{P}(k-1 \text{ arrivals during the first } t-1 \text{ slots}) \\ &= p \times \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \dots\end{aligned}$$

- In the sequence of Bernoulli trials, the time Y_k of k -th success
- PMF of Y_k

$$\mathbb{P}(Y_k = t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k}, & \text{if } t = k, k+1, \dots \\ 0, & \text{if } t = 1, 2, \dots, k-1 \end{cases}$$

- $\text{Pascal}(1, p) = \text{Geom}(p)$

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- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p .

$$p_S(k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

- Our interest: very large n and very small p , such that $np = \lambda$, i.e., $\lim_{n \rightarrow \infty} p_S(k)$?

$$\begin{aligned} p_S(k) &= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n}{n} \cdot \frac{(n-1)}{n} \cdots \frac{(n-k+1)}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \left(1 - \frac{\lambda}{n}\right)^{-k} \xrightarrow{n \rightarrow \infty} e^{-\lambda} \frac{\lambda^k}{k!} \end{aligned}$$

- A Poisson random variable Z with parameter λ takes nonnegative integer values, whose PMF is:

$$p_Z(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

- Infinitely many slots (n) with the infinitely small slot duration (thus infinitely small success probability $p = \lambda/n$)
- $\mathbb{E}(Z) = \lambda$ (because $\lambda = np$ is the mean of binomial rv)
- $\text{var}(Z) = \lambda$ (because $np(1 - p)$ is the variance of binomial rv)

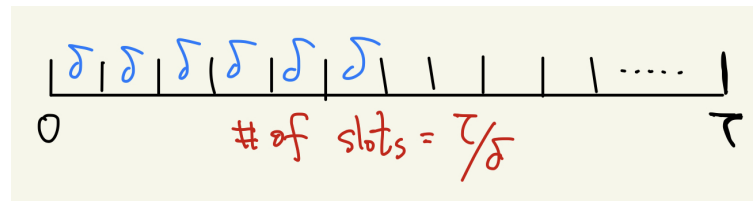
- A packet consisting of a string of n symbols is transmitted over a noisy channel.
- Each symbol: erroneous transmission with probability of 0.0001, independent of other symbols. Incorrect transmission is when at least one symbol is in error.
- **Question.** How small should n be in order for the probability of incorrect transmission to be less than 0.001?
- p is very small, and n is reasonably large \rightarrow Poisson approximation
- Prob. of incorrect transmission =
 $1 - \mathbb{P}(\text{no symbol error}) = 1 - e^{-\lambda} = 1 - e^{-0.0001n} < 0.0001$
- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

L4(3)

- **Remind.** Geometric vs. Exponential
 - Two rvs with memoryless property
 - continuous system = discrete system with infinitely many slots whose duration is infinitely small.
- Independence between what happens in a different time region
- Memoryless and fresh-start property
- Choose one of discrete or continuous versions for our modeling convenience.
- **Question.** How do we design the continuous analog of Bernoulli process?
 - Key idea: Making it as a **limit** of a sequence of Bernoulli processes
- Need a “modeling sense” to make this possible. It’s a good practice for engineers!
- **VIDEO PAUSE**

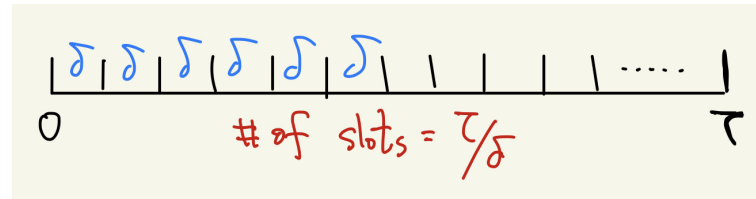
Key Design Idea to Develop a Continuous Twin (1)

- Continuous twin
 - Key point: Understand the number of arrivals over a given interval $[0, \tau]$.
 - Assume that it has some arrival rate λ (# of arrivals/unit time).
 - We know how to handle Bernoulli process with discrete time slots.
- Divide $[0, \tau]$ into slots whose length = δ . Then, $n = \#$ of slots = $\frac{\tau}{\delta}$.



- What's the limit as $\delta \rightarrow 0$ (equivalently, $n \rightarrow \infty$)

Key Design Idea to Develop a Continuous Twin (2)



- Now, our design idea: during one time slot of length δ ,

$\mathbb{P}(1 \text{ arrival}) \propto \text{arrival rate and slot length}$

$\mathbb{P}(\geq 2 \text{ arrivals}) \propto \text{something, but very small}$
for small slot length

$$\mathbb{P}(0 \text{ arrival}) = 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})$$

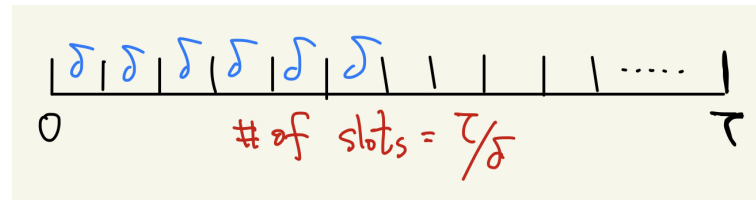
$$\mathbb{P}(1 \text{ arrival}) = \lambda\delta + o(\delta)$$

$$\mathbb{P}(\geq 2 \text{ arrivals}) = 0 + o(\delta)$$

$$\mathbb{P}(0 \text{ arrival}) = 1 - \lambda\delta + o(\delta)$$

- $o(\delta)$: some function that goes to zero faster than δ .
 - Thus, for very small δ , $o(\delta)$ becomes negligible, compared to δ .
 - Example: $o(\delta) = \delta^\alpha$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)



- Our interest: probability of k arrivals over $[0, \tau]$
- Given “small” δ , # of arrivals $\sim \text{Bin}(n, p)$, where $n = \tau/\delta$ and $p = \lambda\delta$
- As $\delta \rightarrow 0$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- # of arrivals over $[0, \tau]$, $\sim \text{Poisson}(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call **Poisson process**.

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An arrival process is called a **Poisson process** with rate λ , if the following are satisfied:

- **(Independence)** Let N_τ be the number of arrivals over the interval $[0, \tau]$. For any $\tau_1, \tau_2 > 0$, $N_{s+\tau_1} - N_s$ is independent of $N_{t+\tau_2} - N_t$, if $t > s + \tau_1$.
 - The number of arrivals over two disjoint intervals are independent.
- **(Time homogeneity)** For any s , the distribution of $N_{s+\tau} - N_s$ is equal to that of N_τ .
 - N_τ becomes the number of arrivals over any interval of length τ .
- **(Small interval probability)** Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \dots, \quad \text{where} \quad \lim_{\tau \rightarrow 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \rightarrow 0} \frac{o_k(\tau)}{\tau} = 0$$

An arrival process is called a **Poisson process** with rate λ , if the following are satisfied:

- **(Independence)** Let N_τ be the number of arrivals over the interval $[0, \tau]$. For any $\tau_1, \tau_2 > 0$, $N_{s+\tau_1} - N_s$ is independent of $N_{t+\tau_2} - N_t$, if $t > s + \tau_1$.
 - The number of arrivals over two disjoint intervals are independent.
- **(Time homogeneity)** For any s , the distribution of $N_{s+\tau} - N_s$ is equal to that of N_τ .
 - N_τ becomes the number of arrivals over any interval of length τ .
- **(Distribution of N_τ)** N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

(Q1) Number of arrivals of **any interval** with length $\tau \sim \text{Poisson}(\lambda\tau)$, i.e.,

$$\mathbb{P}(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

$$\mathbb{E}(N_\tau) = \lambda\tau \text{ and } \text{var}(N_\tau) = \lambda\tau$$

(Q2) Time of first arrival T

$$F_T(t) = \mathbb{P}(T \leq t) = 1 - \mathbb{P}(T > t) = 1 - P(0, t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0.$$

- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
 - Continuous twin of geometric rv in Bernoulli process
 - Memoryless

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
 - $5 \cdot 24 = 120$
- $\mathbb{P}[\text{one new message in the next hour}]$
 - $\mathbb{P}(1, 1) = \frac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$
- $\mathbb{P}[\text{exactly two msgs during each of the next three hours}]$
 - $\left(\frac{5^2 e^{-5}}{2!} \right)^3$

- **Remind.** Similar property for Bernoulli processes, but here no time slots.
- **Fresh-start at deterministic time t :** Start watching at time t , then you see the Poisson process, independent of what has happened in the past.
- **Fresh-start at random time T :** Similarly, it holds.
 - For example, when you start watching at random time T_1 (time of first arrival).
 - Generally, it holds when T is a **stopping time**.

(Q3) The k -th arrival time Y_k ?

- k -th inter-arrival time $T_k = Y_k - Y_{k-1}$, $k \geq 2$, and $T_1 = Y_1$.
- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k ?

- For a given δ , $\delta \cdot f_{Y_k}(y)$: prob. of k -th arrival over $[y, y + \delta]$.

- When δ is small, only one arrival occurs. Thus,

$$\begin{aligned}\delta \cdot f_{Y_k}(y) &= \mathbb{P}(\text{an arrival over } [y, y + \delta]) \times \mathbb{P}(k - 1 \text{ arrivals before } y) \\ &\approx \lambda \delta \times \mathbb{P}(k - 1, y) = \lambda \delta \times \frac{\lambda^{k-1} y^{k-1} e^{-\lambda y}}{(k - 1)!}\end{aligned}$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k - 1)!}, \quad y \geq 0.$$

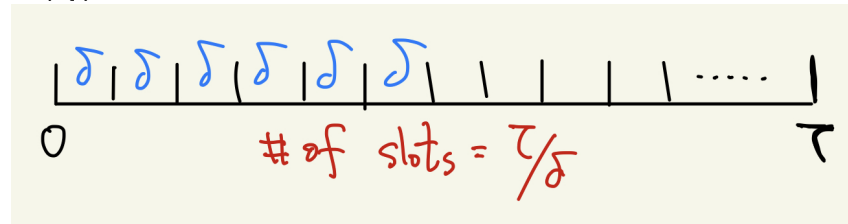
- This is called **Erlang** rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

$$f_Z(z) = \frac{\lambda^k z^{k-1} e^{-\lambda z}}{(k - 1)!}, \quad z \geq 0$$

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals	Binomial	Poisson
Interarrival time	Geometric	Exponential
Time of k -th arrival	Pascal	Erlang
Arrival rate	p /per slot	λ /unit time

Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6/\text{hour}$.
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

(Q1) $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1: $\mathbb{P}(0, 2)$

Method 2: $\mathbb{P}(T_1 > 2)$

(Q2) $\mathbb{P}(2 < \text{fishing time} < 5)$

Method 1: $\mathbb{P}(0, 2)(1 - \mathbb{P}(0, 3))$

Method 2: $\mathbb{P}(2 < T_1 < 5)$

(Q3) $\mathbb{P}(\text{Catch at least two fish})$

Method 1: $\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0, 2) - \mathbb{P}(1, 2)$

Method 2: $\mathbb{P}(Y_2 \leq 2)$

(Q4) $\mathbb{E}[\text{future fi. time} | \text{already fished for 3h}]$

Fresh-start. So,

$$\mathbb{E}[\exp(\lambda)] = 1/\lambda = 1/0.6$$

(Q5) $\mathbb{E}[F = \text{total fishing time}]$

$$\begin{aligned} 2 + \mathbb{E}[F - 2] &= 2 + \mathbb{P}(F = 2) \cdot 0 + \\ &\quad \mathbb{P}(F > 2) \cdot \mathbb{E}[F - 2 | F > 2] \\ &= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda} \end{aligned}$$

(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

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- (5) Playing with Bernoulli and Poisson Processes
- (6) Random Incidence Phenomenon

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, T_2, \dots , with common parameter p , and let these stand for the interarrival times.
2. Record a success (or arrival) at times $T_1, T_1 + T_2, T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

1. Start with a sequence of independent exponential random variables T_1, T_2, \dots , with common parameter λ , and let these stand for the interarrival times.
2. Record an arrival at times $T_1, T_1 + T_2, T_1 + T_2 + T_3$, etc.

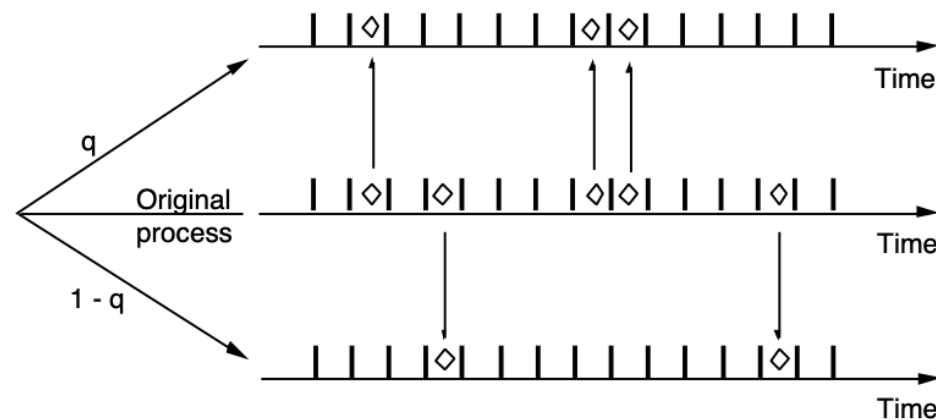
- It has been observed that after a rainy day, the number of days until the next rain $\sim \text{Geom}(p)$, independent of the past.
- **Question.** What is the probability that it rains on both the 5th and the 8th day of the month?
- **Approach 1:** Handling this problem directly with the geometric PMFs is very tedious and complex.
- **Approach 2:** Rainy days is a Bernoulli process with arrival probability p .
- Thus, the answer is p^2 .

- **Question.** How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- **Bernoulli process with p :** Obtain a sequence of random values following the **geometric distribution** with parameter p .
- **Poisson process with λ :** Obtain a sequence of random values following the **exponential distribution** with parameter λ .

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

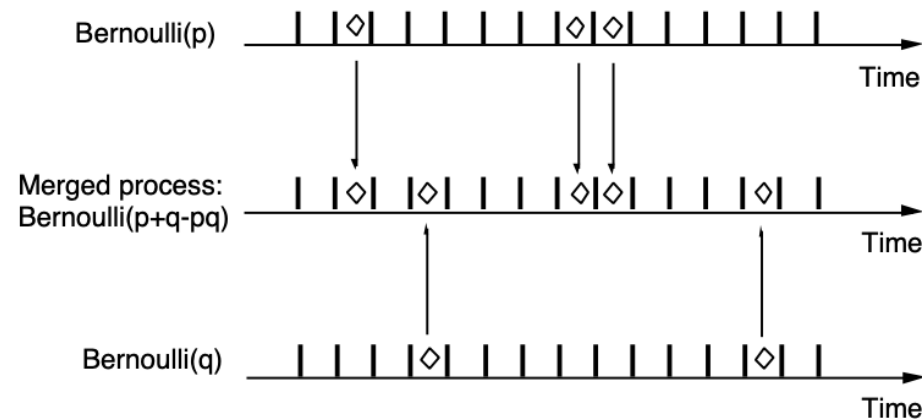
Split: Bernoulli Process

- Split $BP(p)$ into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1 - q)$.
- Split decisions are independent of arrivals.
- **Question.** What are the two split processes?
- $BP(pq)$ and $BP(p(1 - q))$. Why?
- Are they independent? **No.**



Merge: Bernoulli Process

- Merge $BP(p)$ and $BP(q)$ into one process.
- Collided arrival is regarded just one arrival in the merged process
- Probability of having at least one arrival: $1 - (1 - p)(1 - q) = p + q - pq$
- Merged process: $BP(p + q - pq)$
- $\mathbb{P}(\text{arrival from proc. 1} \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



- Split a Poisson process $PP(\lambda)$ into two processes by keeping each arrival w.p. p and discarding it w.p. $(1 - p)$
- **Question.** What are the split processes?
- Let's focus on the process that we keep
- Independence and time-homogeneity? **Yes**
- Small interval probability over δ -interval
 - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
 - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
 - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- **$PP(\lambda p)$ and $PP(\lambda(1 - p))$**

- Merge from $PP(\lambda_1)$ and $PP(\lambda_2)$
- Independence and time-homogeneity? Yes
- Small interval probability over δ -interval (ignoring $o(\delta)$ for small δ)

$$\mathbb{P}(0 \text{ arrival}) \approx (1 - \lambda_1 \delta)(1 - \lambda_2 \delta) \approx 1 - (\lambda_1 + \lambda_2)\delta$$

$$\mathbb{P}(1 \text{ arrival}) \approx (\lambda_1 \delta)(1 - \lambda_2 \delta) + \lambda_2 \delta(1 - \lambda_1 \delta) \approx (\lambda_1 + \lambda_2)\delta$$

- Merged process: $PP(\lambda_1 + \lambda_2)$

- **Red**: $PP(\lambda_1)$ and **Blue**: $PP(\lambda_2)$
- $\mathbb{P}(\text{from red} \mid \text{arrival at time } t \text{ in the merged proc.})? \frac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- Consider an event $A_k = \{k\text{-th arrival in the merged proc. is red}\}$
 - $\mathbb{P}(A_k)? \frac{\lambda_1}{\lambda_1 + \lambda_2}$
 - A_1, A_2, \dots are independent: origins (red or blue) of arrivals in the merged proc. are independent
- $\mathbb{P}(k \text{ out of first 10 arrivals are red})? \binom{10}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{10-k}$

1. Competing exponentials
2. Sum of independent Poisson rvs
3. Poisson arrivals during and Exponential interval

- Two independent light bulbs have life times $T_a \sim \text{Exp}(\lambda_a)$ and $T_b \sim \text{Exp}(\lambda_b)$.

(Q) Distribution of $Z = \min\{T_a, T_b\}$, the first time when a bulb burns out?

- Approach 1

- $\mathbb{P}(Z \geq z) = \mathbb{P}(T_a \geq z)\mathbb{P}(T_b \geq z) = e^{-\lambda_a z} e^{-\lambda_b z} = e^{-(\lambda_a + \lambda_b)z}$
- Thus, $Z \sim \text{Exp}(\lambda_a + \lambda_b)$

- Approach 2

- T_a and T_b are the first arrival times of two Poisson processes of λ_a and λ_b , respectively.
- Z is the first arrival time of merged Poisson process $(\lambda_a + \lambda_b)$.
- Thus, $Z \sim \text{Exp}(\lambda_a + \lambda_b)$

- Three independent light bulbs have life time $T \sim \text{Exp}(\lambda)$.

(Q) $\mathbb{E}[\text{time until the last bulb burns out}]?$

- Understanding from the merged Poisson process
 - T_1 : time until the first burn-out, T_2 : time until the second burn-out, T_3 : time until the third burn-out
 - $\mathbb{E}(T_1 + T_2 + T_3)?$
 - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
 - $T_1 \sim \text{Exp}(3\lambda)$, $T_2 \sim \text{Exp}(2\lambda)$, $T_3 \sim \text{Exp}(\lambda)$

$$\mathbb{E}[T_1 + T_2 + T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

- Two independent rvs X and Y , where $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\nu)$.
- **Question.** Distribution of $X + Y$? Complex convolution, but any other easy way?
- Poisson process perspective
 - X : number of arrivals of PP(1) over a time interval of length μ
 - Y : number of arrivals of PP(1) over a time interval of length ν
 - Two intervals do not overlap and located in a consecutive manner $\implies X \perp\!\!\!\perp Y$
- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$
- Thus, $X + Y \sim \text{Poisson}(\mu + \nu)$

- Problem 24, pp. 335
- Consider $PP(\lambda)$ and an independent rv $T \sim \text{Exp}(\nu)$
- **Question.** Distribution of N_T ?
- **Approach 1:** Total probability theorem

$$\mathbb{P}(N_T = k) = \int_0^\infty \mathbb{P}(N_T = k | T = \tau) f_T(\tau) d\tau = \int_0^\infty \mathbb{P}(N_\tau = k) f_T(\tau) d\tau$$

- Very tedious and not very intuitive.

- Consider $PP(\lambda)$ and an independent rv $T \sim \text{Exp}(\nu)$
- Consider another $PP(\nu)$, and let's view T as the first arrival time in $PP(\nu)$.
- Now, consider the merged process of $PP(\lambda)$ and $PP(\nu)$.
 - $\mathbb{P}[\text{from } PP(\lambda) | \text{arrival}] = \frac{\lambda}{\lambda + \nu}$ and $\mathbb{P}[\text{from } PP(\nu) | \text{arrival}] = \frac{\nu}{\lambda + \nu}$
- K : number of total arrivals until we get the first arrival from $PP(\nu)$.
 - Then, $K \sim \text{Geom}(\frac{\nu}{\lambda + \nu})$.
- Let L be the number of arrivals from $PP(\lambda)$ until we get the first arrival from $PP(\nu)$.

$$p_L(l) = p_K(l + 1) = \left(\frac{\nu}{\lambda + \nu} \right) \left(\frac{\lambda}{\lambda + \nu} \right)^l, \quad l = 0, 1, \dots$$

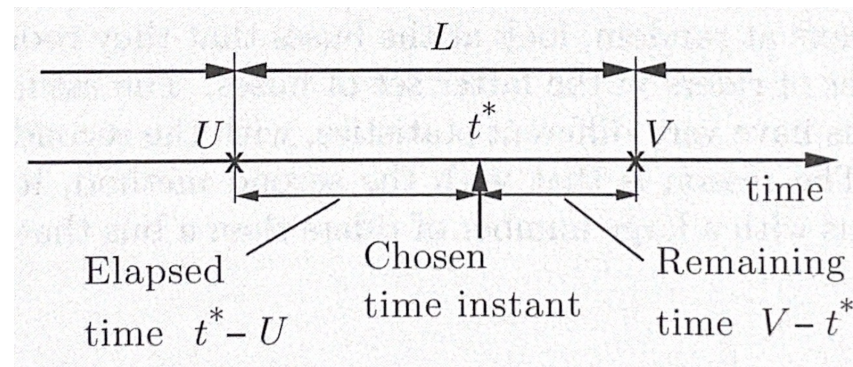
- Consider $S_n \sim \text{Binomial}(n, p)$. Then, $S_n = \sum_{i=1}^n X_i$, where $X_i \sim \text{Bern}(p)$.
- **Poisson approximation.** $\text{Poisson}(np)$ is a good approximation of S_n
 - Holds when $np = \lambda$ ($n \rightarrow \infty, p \rightarrow 0$), i.e., X_i 's behavior changes over n .
- **Normal approximation.** $\sigma\sqrt{n}Z + n\mu$ is a good approximation of S_n
 - Holds for a fixed p ($n \rightarrow \infty$), i.e., X_i 's behavior does not change over n .
- In practice, an actual numbers of n and p are given, so which approximation is good under what situation?
 - $p = 1/100, n = 100$: $np = 1$, very asymmetric X_i , small $p \implies$ **Poisson**
 - $p = 1/3, n = 100$: large, reasonably symmetric p , at least moderate $n \implies$ **Normal**
 - $p = 1/100, n = 10,000$: small p , but large $n \implies$ **Both Poisson and Normal**

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
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- (6) Random Incidence Phenomenon

- What we want to survey: How available are town buses in a city?
- Two approaches
 - M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus
 - M2. (1) select a few bus riders at random, (2) look at the buses that they rode, and (3) calculate the average number of riders in those buses
- Which is correct?
- (i) $M1 = M2$? (ii) $M1 > M2$? (iii) $M1 < M2$?

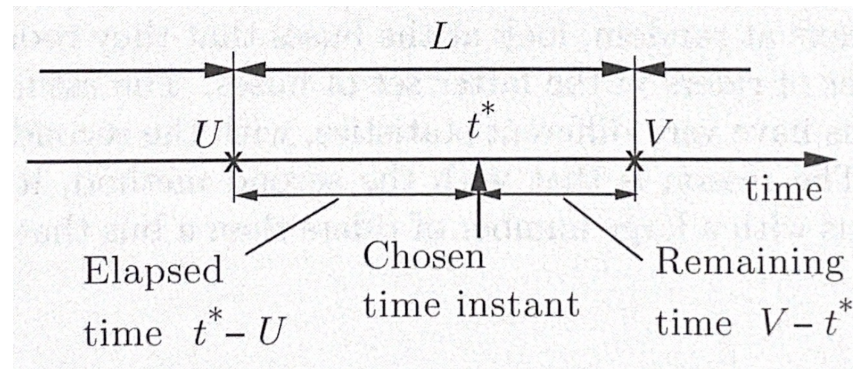
Random Incidence Phenomenon (1)

- **We know:** in $PP(\lambda)$, inter-arrival time $\sim \text{Exp}(\lambda)$
- Fix a time instant t^* , and consider the length L of the inter-arrival interval that **contains** t^* .



- **Practical context:** Yung shows up at the bus station at some arbitrary time t^* and records the time from the previous bus arrival (U) until the next bus arrival (V)
- **Question.** What is the distribution of L ?

VIDEO PAUSE



- t^* is **not** random, so “random incidence” may be confusing and misleading.
- **Assumption.** For simplicity, t^* is large enough that we must have an arrival before t^* ($U > 0$)
- One might superficially argue that $L \sim \text{Exp}(\lambda)$, but it is **NOT**.

$$L = (t^* - U) + (V - t^*)$$

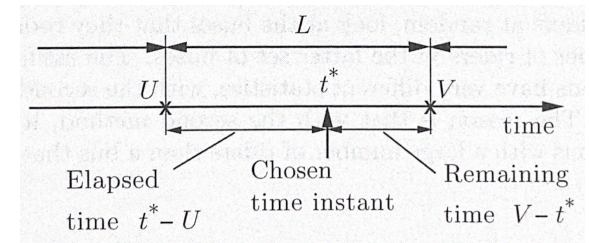
- $V - t^*$: Due to the **memoryless** property and the **fresh-restart**,

Thus, $V - t^* \sim \text{Exp}(\lambda)$

- $t^* - U$: If we run the PP(λ) backwards in time, it remains Poisson. Why? More formally,

$$\begin{aligned}\mathbb{P}(t^* - U > x) &= \mathbb{P}(\text{no arrivals over } [t^* - x, t^*]) \\ &= e^{-\lambda x} = \mathbb{P}(T_{\text{inter}} > x)\end{aligned}$$

Thus, $t^* - U \sim \text{Exp}(\lambda)$



$$L = (t^* - U) + (V - t^*)$$

- $L = X_1 + X_2$, where $X_1, X_2 \sim \text{Exp}(\lambda)$
- Time until we have two arrivals in $\text{PP}(\lambda)$
- Erlang random variable with parameter $(2, \lambda)$, i.e.,

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$$f_L(l) = \lambda^2 \cdot l \cdot e^{-\lambda l}, \quad l \geq 0$$

- Mean = $2/\lambda$
- Why not $\text{Exp}(\lambda)$? An observer who arrives at an arbitrary time is more likely to fall in a large rather than a small interarrival interval.

- Two Approaches
 - M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus
 - M2. (1) select a few bus riders at random, (2) look at the buses that they rode, and (3) calculate the average number of riders in those buses
- (i) $M1 = M2$? (ii) $M1 > M2$? (iii) $M1 < M2$?
- Answer: $M1 < M2$
- More likely to select a bus with a large number of riders than a bus that is near-empty.

Questions?

- 1) Explain what is random process. Why do we need such a concept?
- 2) Explain Bernoulli processes. When do we use Bernoulli processes?
- 3) Explain Poisson processes. When do we use Poisson processes?
- 4) What are the relations between those two processes? What features do they share?
- 5) In both processes, how do we compute (i) number of arrivals over a given interval of time, (ii) time until the first arrival, (iii) time until k -th arrival?
- 6) What is Pascal rvs and Erlang rvs?
- 7) What is the “stopping time” and how is it related to fresh-restart?
- 8) See many examples on how merging and splitting of BP and PP can be used for intuitive solving of many problems.