Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

June 12, 2021

Roadmap

(1) Random variable: Idea and formal definition
(2) Popular discrete random variables
(3) Summarizing random variables: Expectation and Variance
(4) (Functions of) multiple random variables
(5) Conditioning for random variables
(6) Independence for random variables

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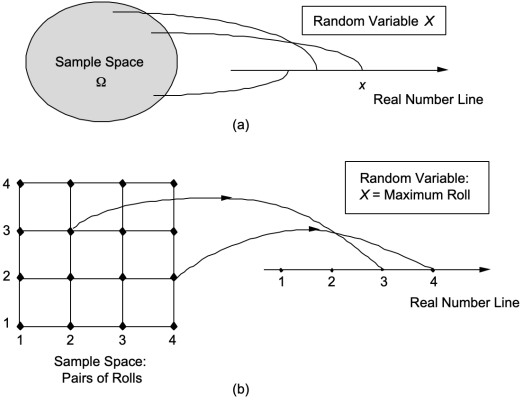
Random Variable: Idea

- In reality, many outcomes are $\boxed{}$, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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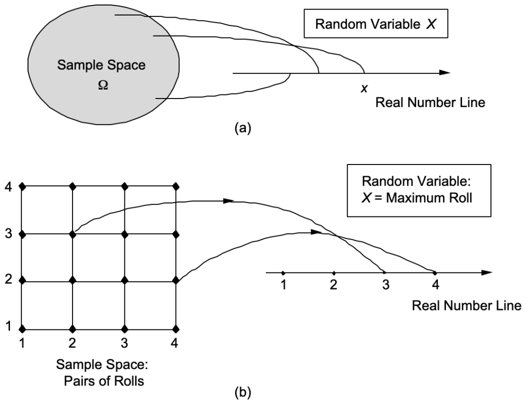

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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Example

- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$

- Remember that the random variable X is a **function** from Ω to $\mathbb{R}$

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Bernoulli X with parameter $p \in [0, 1]$

- Only binary values

[1] w.p.: with probability

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$$X = \begin{cases} 0, & \text{w.p.} \quad 1 - p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail

- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$

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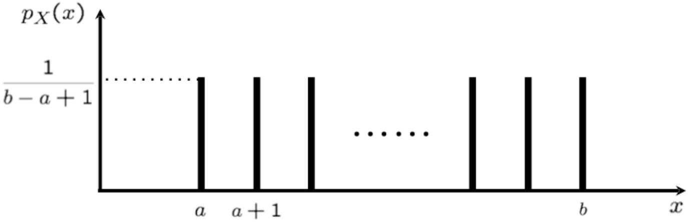

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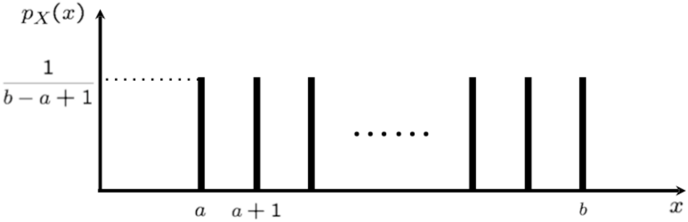

- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p

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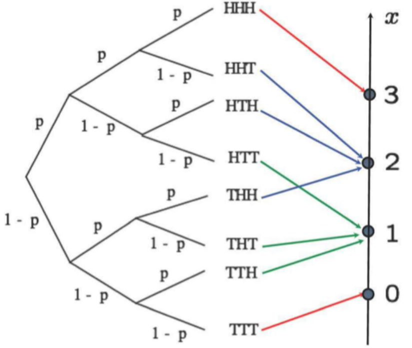

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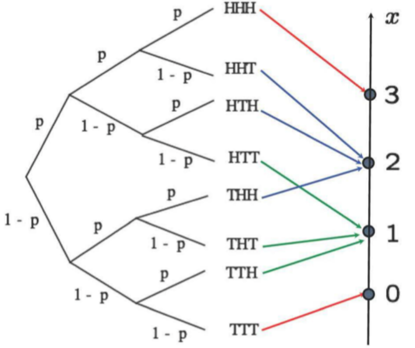
$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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Geometric X with parameter p

- Infinitely many independent Bernoulli trials, where each trial has success probability p

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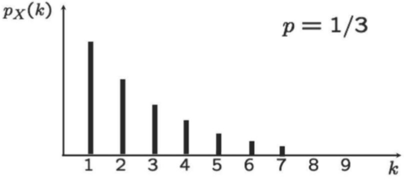
$$p_X(k) = (1-p)^{k-1} p$$



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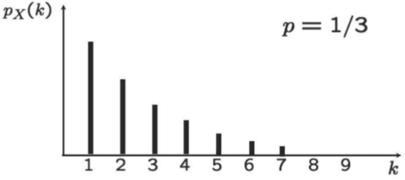
- Models waiting times until something happens.



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- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)

Expectation/Mean

- Average

Definition

$$\mathbb{E}[X] = \sum_{x} x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

- For a rv X, $Y = g(X)$ is also a r.v.

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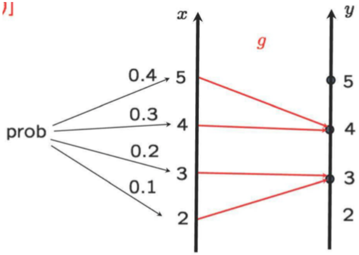

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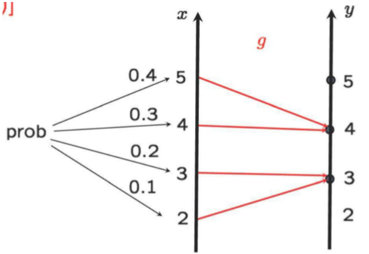

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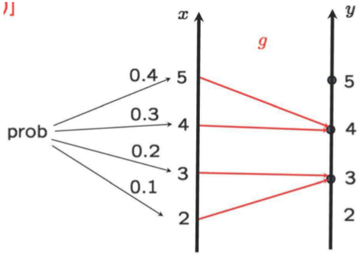

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Linearity of Expectation

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- Measures how much the $\boxed{\text{spread}}$ of a PMF is.

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$

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Example: Variance of a Bernoulli rv (p)

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Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}
\mu = \mathbb{E}[X] &= 1 \times p + 0 \times (1-p) = p \\
\mathbb{E}[X^2] &= 1 \times p + 0 \times (1-p) = p \\
\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\
&= p(1-p)
\end{aligned}$$

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$$\mathbb{P}\left(\{X = x\} \cap \{Y = y\}\right)$$

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- $\boxed{\text{Joint PMF.}}$ For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

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- **Marginal PMF.**

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Example. VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) =$

$p_X(4) =$

$\mathbb{P}(X = Y) =$

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- $\boxed{\text{Marginal PMF.}}$

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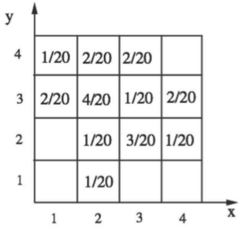

$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

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$$p_Z(z) = \mathbb{P}(g(X,Y)=z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x,y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

Linearity of Expectation for Multiple RVs

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Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

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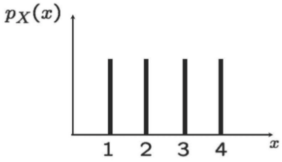
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$A = \{X \geq 2\}$



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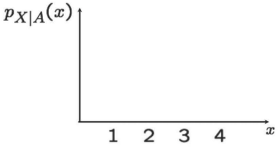

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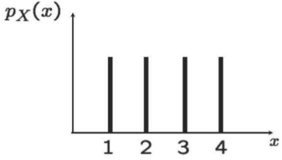

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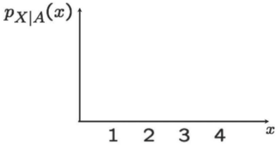

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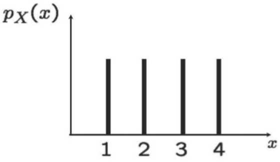

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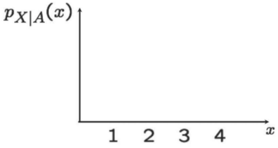

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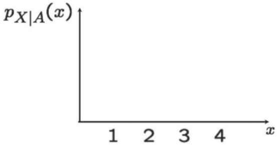

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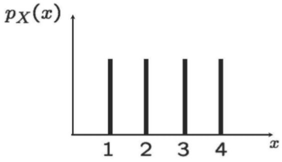

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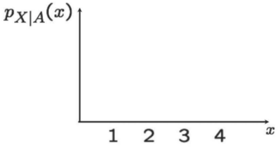

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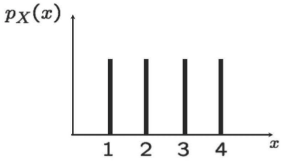

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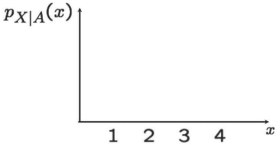

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Conditional PMF

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- Multiplication rule

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$

Conditional PMF

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for y such that $p_Y(y) > 0$.

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Conditional PMF

VIDEO PAUSE

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y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

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$\mathbb{E}[X | Y = 3] =$

KAIST EE

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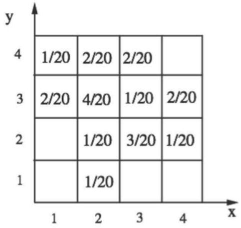


$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

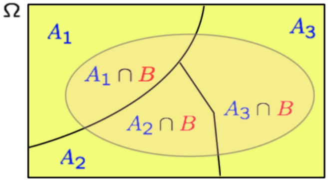
$$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$$

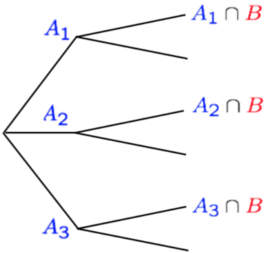
KAIST EE

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$





Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

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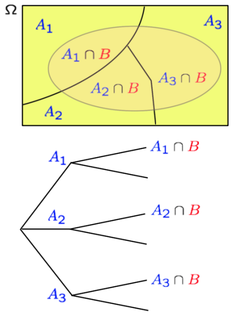

Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

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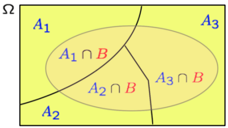
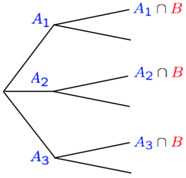
Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

- Weighted average of expectations from A_i's perspective





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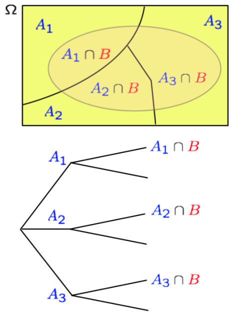

Total Expectation Theorem for $\{Y = y\}$

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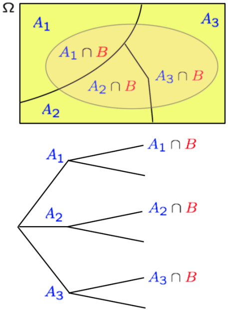
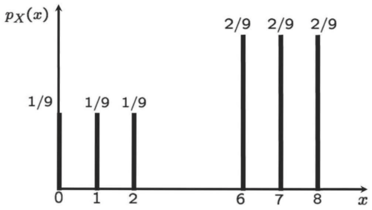


Example 1: Total Expectation Theorem

- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$$\mathbb{E}[X] =$$



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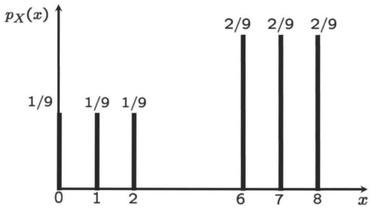

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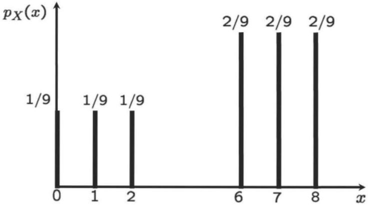

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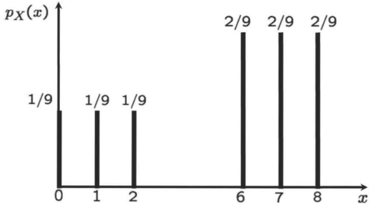
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



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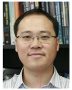
- Total expectation theorem and a notion of memorylessness helps a lot.

No bus arrives



How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n + m | X > m)$$

Yung arrives

time

0

m mins

A bus left at time 0

time

0

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n)$$

Lin arrives

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$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

Background: Memoryless Property of Geometric RVs

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- Remind. Geometric rv X with parameter p

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- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$\begin{aligned}\mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\ &= && \text{(from TET)} \\ &= && \text{(from memorylessness)}\end{aligned}$$

Back to Example 2: Mean of Geometric rv

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- Thus, $\mathbb{E}[X] = \dfrac{1}{p}$

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

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Example

- $X \perp\!\!\!\perp Y$?

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

KAIST EE

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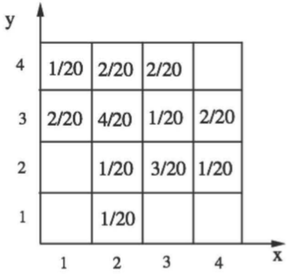

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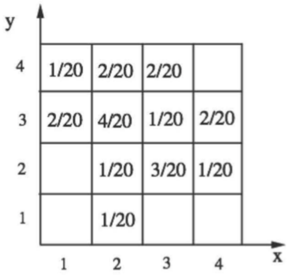
VIDEO PAUSE

$Y = 4$		
$Y = 3$		
	$X = 1$	$X = 2$



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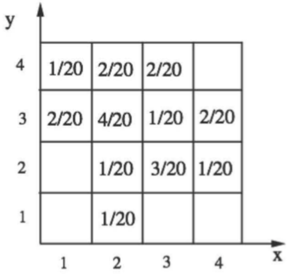

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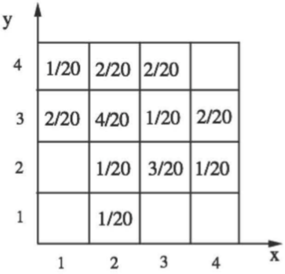

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  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X-3Y]=\text{var}[X]+9\text{var}[Y]$

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- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

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- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- Key step 1. Define a rv $X_i = 1$ if i selects its own hat and 0 otherwise.

$$X = \sum_{i=1}^{n} X_i.$$

Example: The hat problem (1)

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (from symmetry)

Example: The hat problem (2)

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- $\mathrm{var}(X) = 2 - 1 = 1$

Questions?

Review Questions

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFs?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?