Lecture 6: Statistical Inference

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

May 13, 2021

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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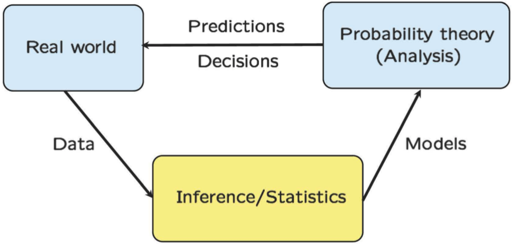
Inference: Big Picture



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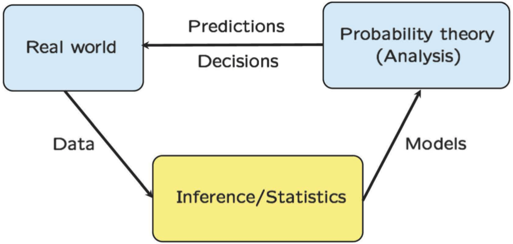

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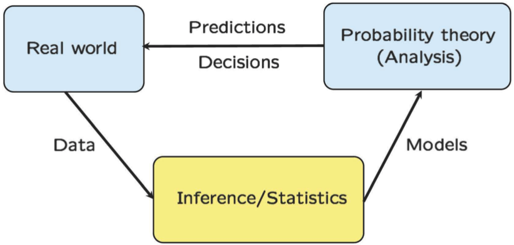

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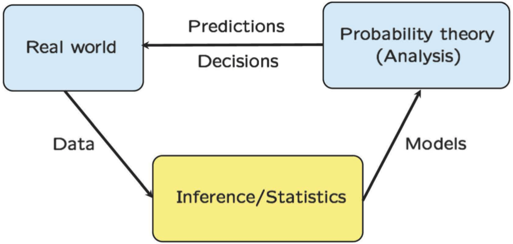

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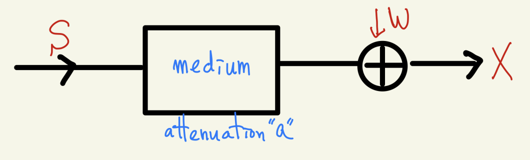
  - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building

What to Infer?: Unknown Model vs. Unknown Variable



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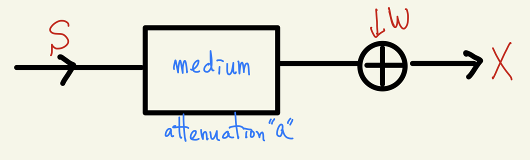

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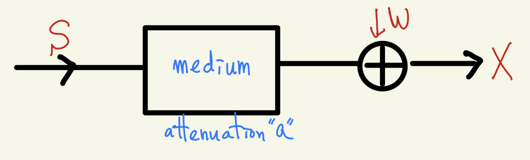

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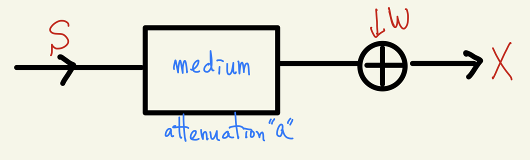

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- Same mathematical structure, because the parameters in models are variables in many cases

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  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

- Biased coin with parameter θ (probability of head). Assume that $\theta \in \{1/4, 3/4\}$.

Inference with Different Views: Bayesian vs. Classical (1)

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- *Classical approach* (Chapter 9)

Inference with Different Views: Bayesian vs. Classical (2)

Bayesian approach

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Inference with Different Views: Bayesian vs. Classical (2)

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- Observed data x gives: posterior distribution $p_{\Theta|X}(\theta|x)$

Classical approach

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- Who is the winner? A century-long debate (see p. 409 for discussion)

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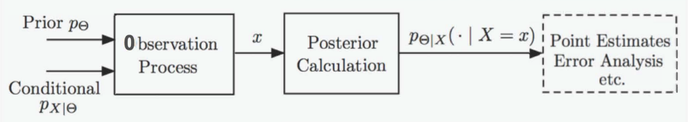

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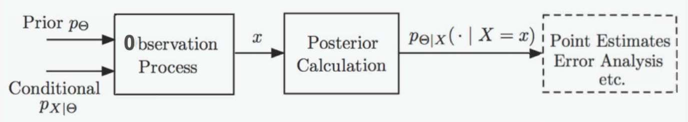

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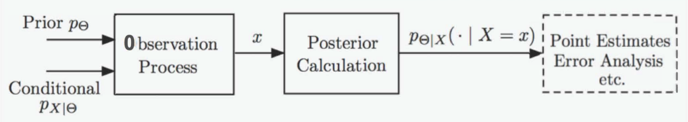

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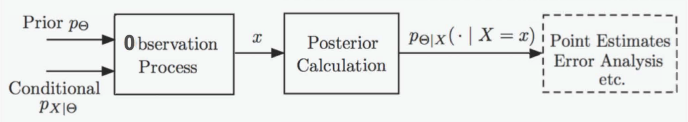

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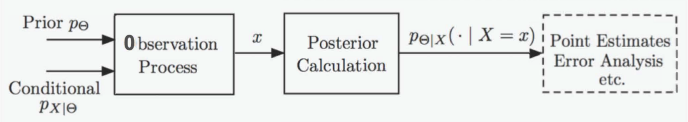

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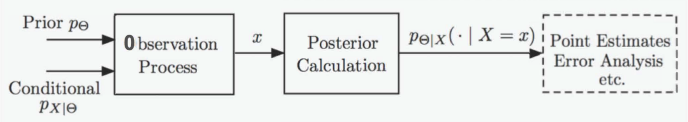

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- Find the posterior distribution $p_{X|\Theta}$ and $f_{X|\Theta}$.
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- Using the posterior distribution, apply one of the methods of choosing the final $\hat{\theta}$ for estimation and hypothesis testing.

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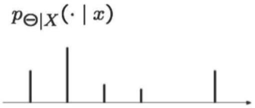

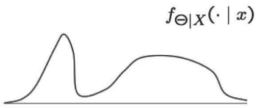

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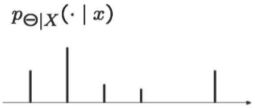

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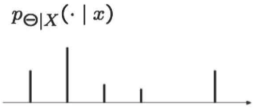

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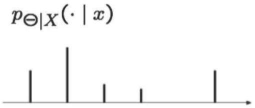

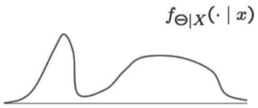

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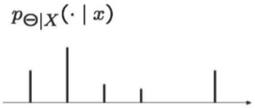

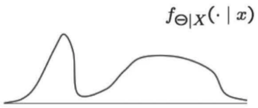

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- Why MAP and LMS are good? Not mathematically clear yet (later)

Estimator as a function

- Random observation: X

- Observation instance: x

- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\mathrm{MAP}} = g_{\mathrm{MAP}}(x), \quad \hat{\theta}_{\mathrm{LMS}} = g_{\mathrm{LMS}}(x)$$

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- MAP rule
  - Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$.
  - $\hat{\theta}_{\text{MAP}} = x$.

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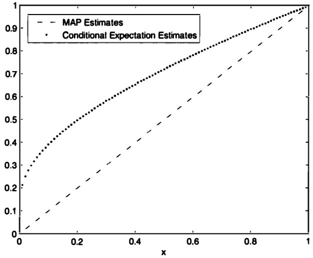

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- When $\alpha = \beta = 1$ (i.e., $U[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1)\prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_\Theta(2)\prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator

- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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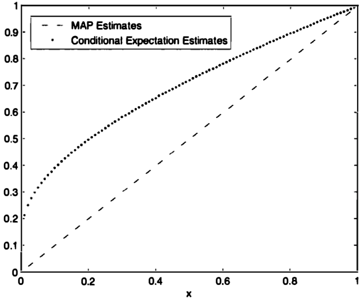

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- For $\alpha = \beta = 1$ ($\Theta = Uniform[0, 1]$),
$$\mathbb{E}[\Theta | X = k] = \frac{k + 1}{n + 2}$$

Example: Signal Recovery from Noisy Measurement (1)

- Unknown: $\Theta \sim \mathit{Uniform}[4, 10]$
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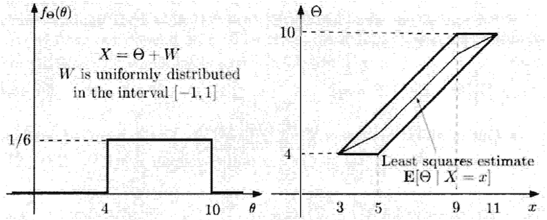

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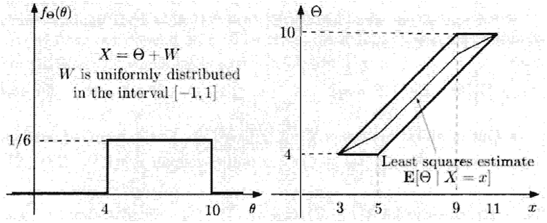

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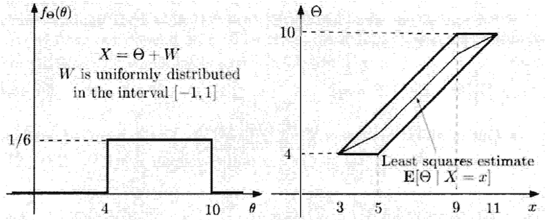
- $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$ = midpoint of the corresponding vertical section



Example: Signal Recovery from Noisy Measurement (2)

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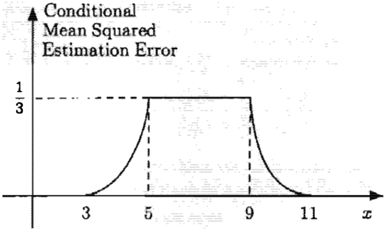
- Conditional MSE

$$\mathbb{E}\Big[(\Theta - \mathbb{E}[\Theta|X = x])^2 | X = x\Big]$$



Hardness of LMS Estimation

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{f_X(x)}$$

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- Any alternative to LMS estimator?

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

Linear LMS (LLMS) Estimator: Approach

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- Linear models are always the first choice for a simple design in engineering.

Linear LMS (LLMS) Estimator: Solution First

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LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)$$

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  - Baseline ($\mathbb{E}[\Theta]$) + correction term
  - If $X > \mathbb{E}[X] \implies \hat{\Theta}_L > \mathbb{E}[\Theta]$
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- If $\rho = 0$ (uncorrelated):
  - Just baseline ($\mathbb{E}[\Theta]$)
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data X

Linear LMS (LLMS) Estimator: Proof

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}\Big(X - \mathbb{E}(X)\Big) \tag{1}$$

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(3)

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- (3) is minimized when $a = \frac{\mathrm{cov}(\Theta,X)}{\mathrm{var}[X]}$. Then,

$$\hat{\Theta}_L = aX + b = aX + \mathbb{E}(\Theta) - a\mathbb{E}(X)$$

$$= (1)$$

- Using $\rho = \frac{\mathrm{cov}(\Theta,X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

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$$\begin{aligned}\mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6\end{aligned}$$

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$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

- LLMS estimator is:

$$\begin{aligned}\hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) \\ &= \frac{1}{2} + \frac{1/24}{7/144}(X - \frac{1}{4}) = \frac{6}{7}X + \frac{2}{7}\end{aligned}$$

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- Using $\mathbb{E}[\Theta] = 1/2$ and $\mathbb{E}[\Theta^2] = 1/3$,

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[\text{var}[X|\Theta]] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \frac{1}{12}\mathbb{E}[\Theta^2] + \frac{1}{4}\text{var}[\Theta] = \frac{7}{144}\end{aligned}$$

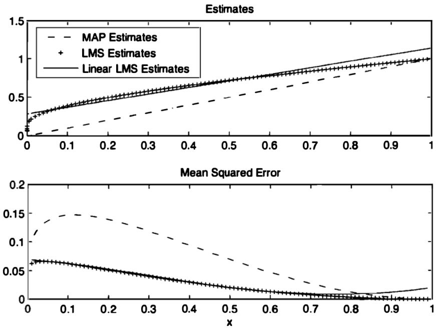
- $\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta]\mathbb{E}[X]$

$$\begin{aligned}\mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6\end{aligned}$$

$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

- LLMS estimator is:

$$\begin{aligned}\hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) \\ &= \frac{1}{2} + \frac{1/24}{7/144}(X - \frac{1}{4}) = \frac{6}{7}X + \frac{2}{7}\end{aligned}$$



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- Biased coin with probability of head θ
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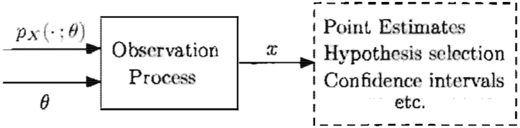
Yes, because the LMS esitmator was linear.

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

Framework of Classical Inference (1)



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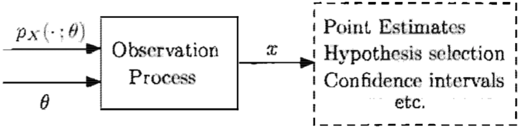

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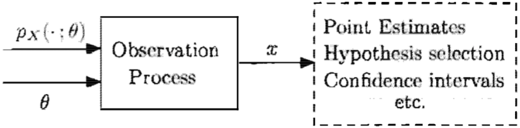

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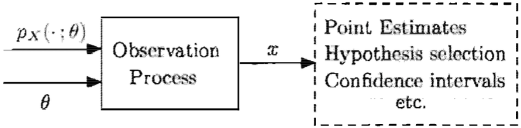
KAIST EE



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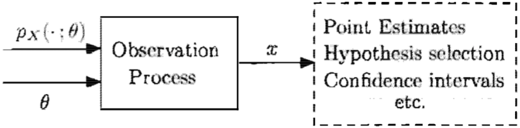

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- Choosing one among multiple probabilistic models
  - Each θ corresponds to a probabilistic model

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- Problem types
  - Estimation
  - Hypothesis testing
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  - ML (Maximum Likelihood) estimation
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- Just a taste in this course due to time constraint.

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KAIST EE

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- Very often, X_i are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \dots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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- When Θ is uniform (complete ignorance of Θ), MAP == ML

Example: Romeo and Juliet

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.
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KAIST EE

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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \mathbb{E}[X] = 1/\lambda$$

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$$\arg\max_{\theta} f_X(x;\theta) = \arg\max_{\theta} \prod_{i=1}^{n} \theta e^{-\theta x_i} = \arg\max_{\theta} \left(n \log \theta - \theta \sum_{i=1}^{n} x_i \right)$$

Questions?

1) What is statistical inference?
2) Draw the building blocks of Bayesian inference and explain how it works.
3) What are MAP and LMS estimators and their underlying philosophies?
4) What is LLMS estimator and why is it useful?
5) Compare the classical and Bayesian inference.
6) What is the ML estimator and how is it related to the MAP estimator?