Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

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Outline

- Probabilistic Model
- Sample Space, Event, Probability Law
- Probability Axioms

What Do We Want?

Modeling: Approximate reality with a simple (mathematical) model

- Experiment — Flip two coins
- Observation: a random outcome — for example, (H, H)
- All outcomes — $\{(H,H),(H,T),(T,H),(T,T)\}$

- Our goal: Build up a $\boxed{\text{probabilistic model}}$ for an experiment with random outcomes
- Probabilistic model?
  - Assign a number to each outcome or a set of outcomes
  - Mathematical description of an uncertain situation
- Which model is good or bad?

Probabilistic Model

Goal: Build up a probabilistic model. Hmm... How?

The first thing: What are the *elements* of a probabilistic model?

Elements of Probabilistic Model

1. All outcomes of my interest: $\boxed{\text{Sample Space } \Omega}$
2. Assigned numbers to each outcome of Ω: $\boxed{\text{Probability Law } \mathbb{P}(\cdot)}$

Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

1. Sample Space Ω

The set of all outcomes of $\boxed{\text{my interest}}$

1. Mutually exclusive
2. Collectively exhaustive
3. At the $\boxed{\text{right granularity}}$ (not too concrete, not too abstract)

1. Toss a coin. What about this? $\Omega = \{H, T, HT\}$
2. Toss a coin. What about this? $\Omega = \{H\}$
3. (a) Just figuring out prob. of H or T. $\Longrightarrow \Omega = \{H, T\}$

   (b) The impact of the weather (rain or no rain) on the coin's behavior.

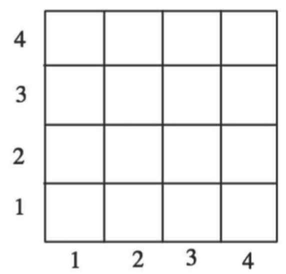
   $$\Longrightarrow \Omega = \{(H, R), (T, R), (H, NR), (T, NR)t\},$$

   where R(Rain), NR(No Rain).

Examples: Sample Space Ω

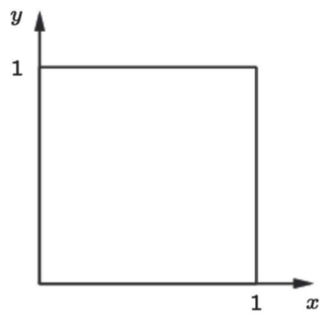
- *Discrete case:* Two rolls of a tetrahedral die

  - $\Omega = \{(1, 1), (1, 2), \ldots, (4, 4)\}$



- *Continuous case:* Dropping a needle in a plain

  - $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \le 1\}$



2. Probability Law

- Assign numbers to what? Each outcome?
- What is the probability of dropping a needle at $(0.5, 0.5)$ over the 1×1 plane?
- Assign numbers to each $\boxed{\text{subset}}$ of Ω
- A subset of Ω: $\boxed{\text{an event}}$
- $\mathbb{P}(A)$: Probability of an event A.
  - This is where probability meets set theory.
- Roll a dice. What is the probability of odd numbers?

  $\mathbb{P}(\{1,3,5\})$, where $\{1,3,5\} \subset \Omega$ is an event.

How should we construct $\mathbb{P}(\cdot)$?

- Need to construct $\mathbb{P}(\cdot)$ that naturally satisfies the intention of a probability theory designer just like you. What about the followings as starting points?
  - $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
  - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
  - For two disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
  - $\mathbb{P}(\Omega) = 1$ (Why not $\mathbb{P}(\Omega) = 10$?)
  - $\mathbb{P}(\emptyset) = 0$
  - If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$
  - many others

Probability Axioms

- Surprisingly, we need just the following three rules (called axioms):

Probability Axioms: Version 1

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. (Finite) additivity: For two disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

- No other things are necessary, and we can prove all other things from the above axioms.
- Note that coming up with the above axioms is far from trivial.

Examples

Prove the following properties using the axioms:

1. For any event A, $\mathbb{P}(A) \leq 1$

$$1 \overset{\text{A2}}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(A^c) \Longrightarrow \mathbb{P}(A) = 1 - \mathbb{P}(A^c) \overset{\text{A1}}{\leq} 1$$

2. $\mathbb{P}(\emptyset) = 0$

$$\mathbb{P}(\Omega \cup \emptyset) \overset{\text{A3}}{=} \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) \overset{\text{A2}}{=} 1 + \mathbb{P}(\emptyset) \overset{\text{from 1.}}{\Longrightarrow} \mathbb{P}(\emptyset) = 0$$

3. If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$

$$\mathbb{P}(B) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \overset{\text{A1}}{\geq} \mathbb{P}(A)$$

Probability Calculation Steps

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1. Specify the sample space
2. Specify a probability law
   - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4$, $\mathbb{P}(\{T\}) = 3/4$,
3. probability of head or tail
4. $1/4$, $3/4$

Discrete but infinite sample space

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- $\Omega = \{1, 2, 3, \ldots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \ldots$
- $\mathbb{P}(\text{even})$?

$$\begin{aligned}\mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \ldots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = 1/3\end{aligned}$$

- Is the above right? If not, why?

Probability Axioms Version 1 2

Probability Axioms: Version 1 2

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. (Finite) additivity: For two disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

A4. Countable additivity: If $A_1, A_2, A_3, \ldots$ is an infite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$.

Interpretation of Probability Theory

- A narrow view: A branch of math
  - axioms $\rightarrow$ theorems

- Frequencies: $\mathbb{P}(H) = 1/2$

- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$
  - Subjective, but providing numerical guidance

Questions?

Congratulations! You build up the very basics of a probabilistic model.

What else do we need to build up?

Review Questions

1) Please explain what a probabilistic model is and why we need it.
2) What is the mathematical definition of event?
3) What are the key elements of the probabilistic model?
4) Please list up the probability axioms and explain them.
5) Why do we need countable additivity in the probability axiom?