Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

August 25, 2021

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Roadmap

(1) Introduction of Random Processes

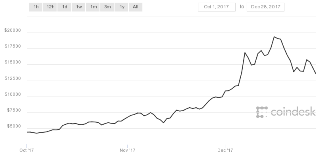
(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

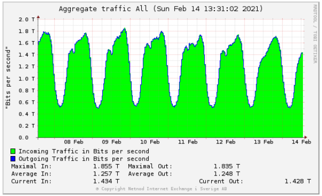
(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

- Many probabilistic experiments that evolve in time

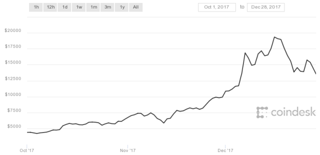


(a) Prices of a crytocurrency



(b) Internet traffic traces

- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet



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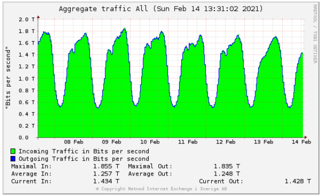

(b) Internet traffic traces

- Many probabilistic experiments that evolve in time
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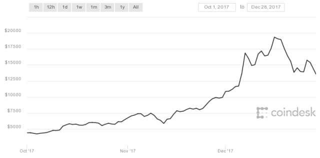
- Random process is a mathematical model for it.



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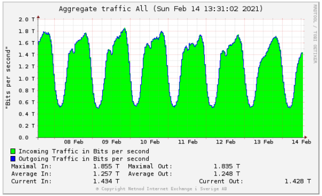

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- The values that X_t (or $X(t)$) can take: discrete or continuous

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  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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  - Other interesting questions, depending on the target random process

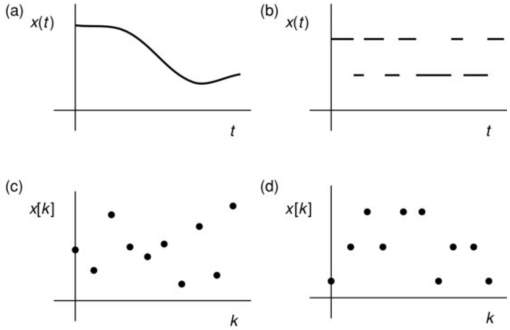
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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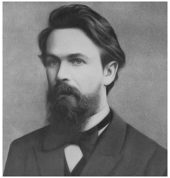

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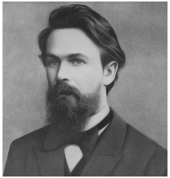

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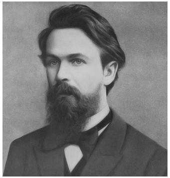

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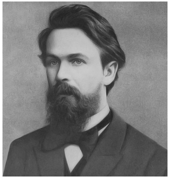

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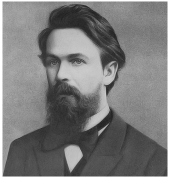

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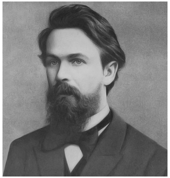

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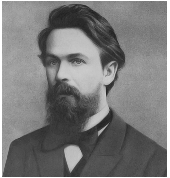

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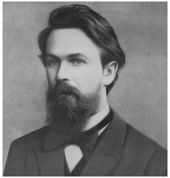

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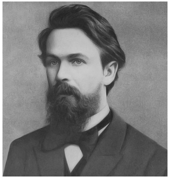

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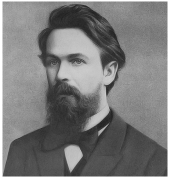

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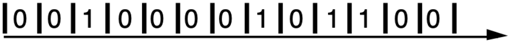

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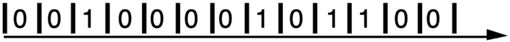

- Discrete time, discrete value

Bernoulli Process

- At each "minute", we toss a coin with probability of head $0 < p < 1$.
  - Sequence of lottery wins/looses
  - Customers (each second) to a bank
  - Clicks (at each time slot) to server
- A sequence of independent Bernoulli trials $X_1, X_2, \ldots,$
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| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 |

- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Please pause the video and write down the questions that you want to ask about Bernoulli process. VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

(Q1) # of arrivals in the first n slots?

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- T_1 is geometric? Memoryless
- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Memoryless and Fresh-start after Deterministic time n

Independence across slots $\Longrightarrow$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

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- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

Fresh-start after Random time N (1)

(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

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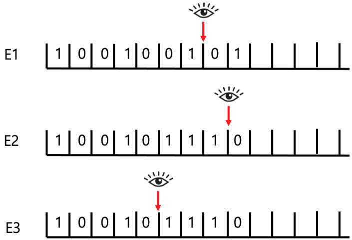

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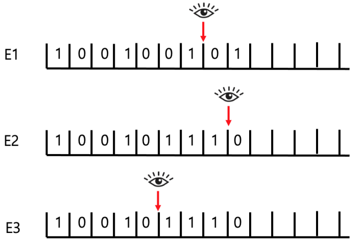

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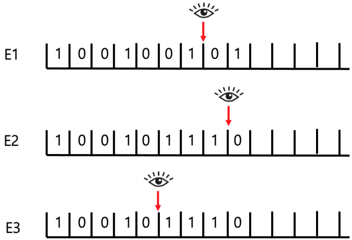

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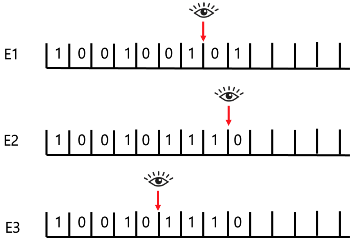

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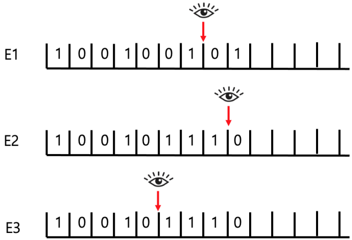

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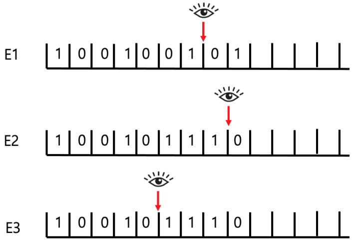

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- Do we experience the fresh-start for any N? **E1, E2, and E3?**

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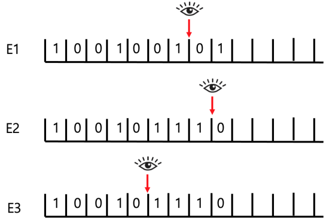

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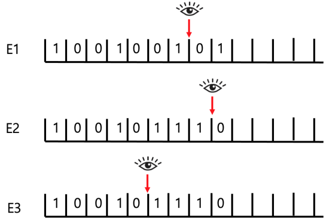

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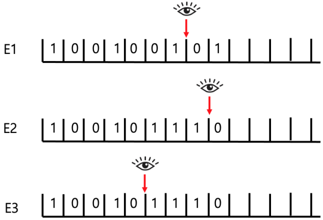

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E2. Same as **E1.** Yes

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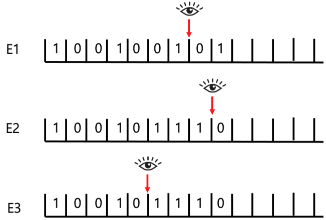

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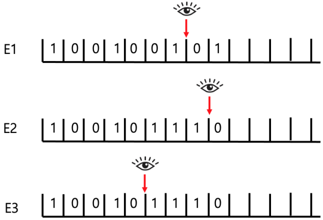

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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

Stopping Time

- In probability theory, a random time N is said to be a stopping time, if the question of "$N = n$?" can be answered only from the present and the past knowledge of $X_1, X_2, \ldots, X_n$.

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  - Yes. Time when 10 consecutive arrivals have been observed
  - No. Time of 2nd arrival in 10 consecutive arrivals

Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)

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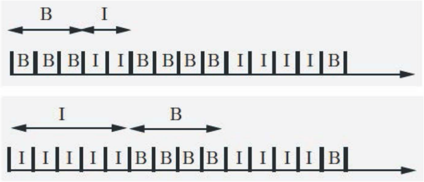
- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot



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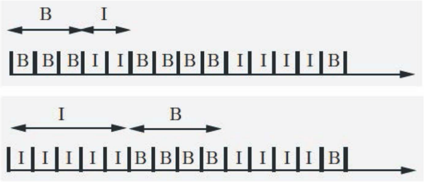

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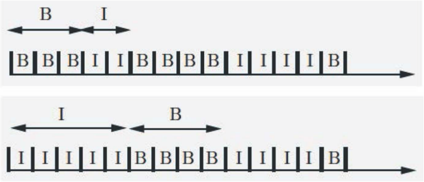

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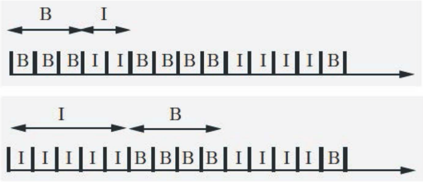

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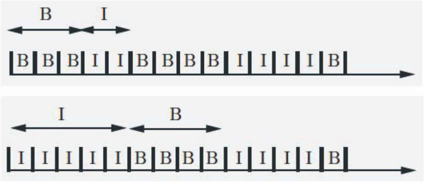

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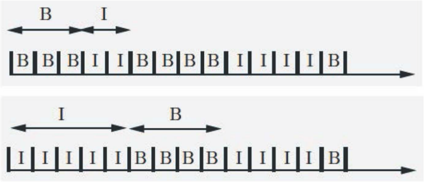
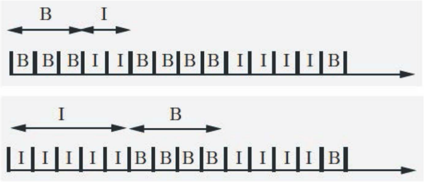
- B_1 is geometric with parameter $(1 - p)$

Distribution of Busy Periods (2)



- Question. What about the second busy period B_2?



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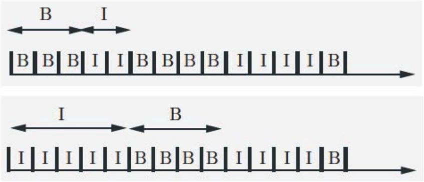
Distribution of Busy Periods (2)



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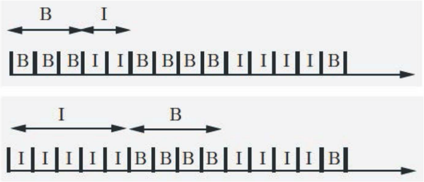

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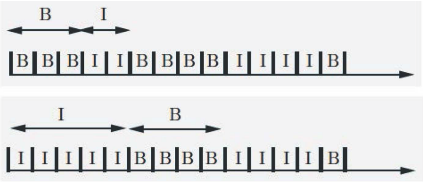

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- $B_3, B_4, \ldots$?

- Time of the first arrival $Y_1 \sim \mathrm{Geom}(p)$

Time of k-th arrival

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VIDEO PAUSE

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VIDEO PAUSE

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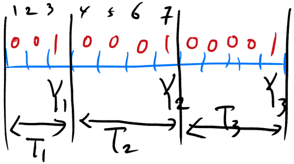

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VIDEO PAUSE

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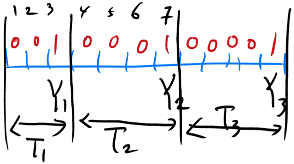

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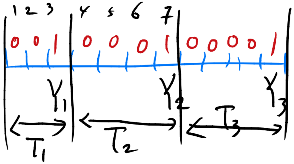

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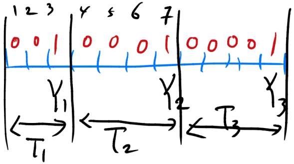

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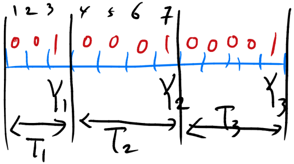

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Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Background: Poisson rv X with parameter λ (1)

- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

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- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

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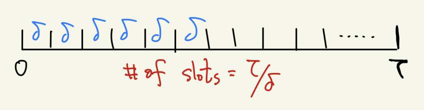

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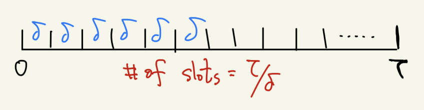
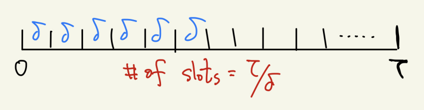


- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)



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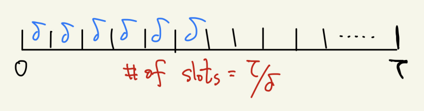

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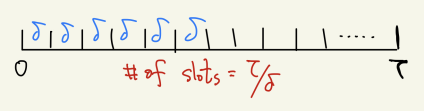
KAIST EE



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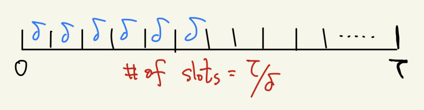

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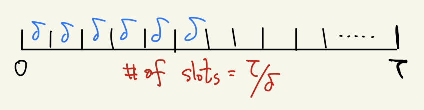

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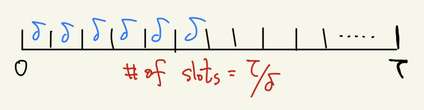

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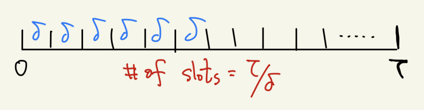

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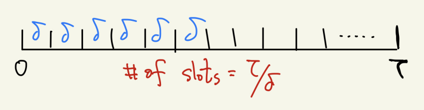

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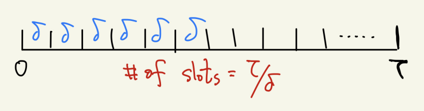

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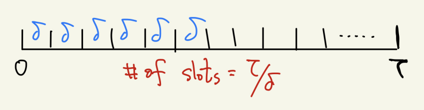
- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)



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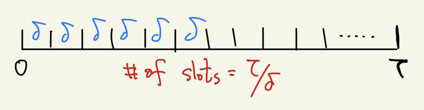

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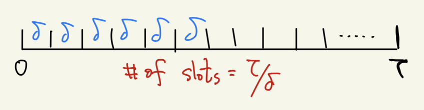

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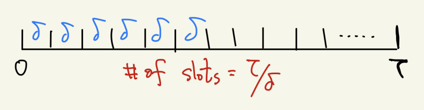
Key Design Idea to Develop a Continuous Twin (3)



- Our interest: probability of k arrivals over $[0, \tau]$
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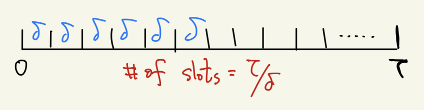

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- This is a continuous twin process of Bernous process, which we call Poisson process.

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

An arrival process is called a Poisson process with rate λ, if the following are satisfied:

Poisson Process: Definition (1)

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- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \ldots, \quad \text{where} \quad \lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

Poisson Process: Definition (2)

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k, \tau) = e^{\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Poisson Process: $\mathbb{P}(k, \tau)$, N_τ, and T

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KAIST EE

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
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- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

PDF of Y_k

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- For a given δ, $\boxed{}$: prob. of k-th arrival over $[y, y+\delta]$.

PDF of Y_k

- For a given δ, $\boxed{\delta \cdot f_{Y_k}(y)}$: prob. of k-th arrival over $[y, y+\delta]$.

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- This is called Erlang rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

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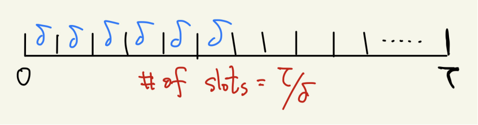
Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals		
Interarrival time		
Time of k-th arrival		
Arrival rate		

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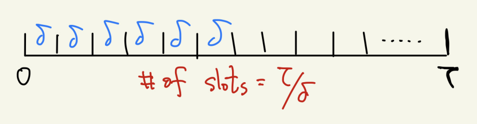

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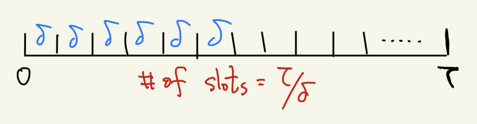

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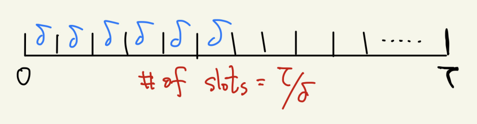

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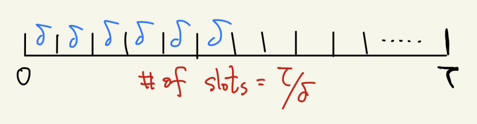

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Arrival rate	p/per slot	λ/unit time

Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6/\text{hour}$.
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

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Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

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(Q1) $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1: $\mathbb{P}(0, 2)$
Method 2: $\mathbb{P}(T_1 > 2)$

(Q2) $\mathbb{P}(2 < \text{fishing time} < 5)$

Method 1: $\mathbb{P}(0, 2)(1 - \mathbb{P}(0, 3))$
Method 2:$\mathbb{P}(2 < T_1 < 5)$

(Q3) $\mathbb{P}(\text{Catch at least two fish})$

Method 1:$\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0, 2) - \mathbb{P}(1, 2)$
Method 2: $\mathbb{P}(Y_2 \leq 2)$

(Q4) $\mathbb{E}[\text{future fi. time}|\text{already fished for 3h}]$
Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

(Q5) $\mathbb{E}[\text{F=total fishing time}]$

Example: Poisson Fishing (Problem 10, page 329)

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$$
\begin{aligned}
2 + \mathbb{E}[F-2] &= 2 + \mathbb{P}(F=2)\cdot 0 + \\
&\quad \mathbb{P}(F>2)\cdot \mathbb{E}[F-2|F>2] \\
&= 2 + \mathbb{P}(0,2)\cdot \frac{1}{\lambda}
\end{aligned}
$$

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&\quad \mathbb{P}(F > 2) \cdot \mathbb{E}[F - 2 | F > 2] \\
&= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda}
\end{aligned}
$$

(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.

2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

1. Start with a sequence of independent exponential random variables $T_1, T_2, \ldots$, with common parameter λ, and let these stand for the interarrival times.

2. Record an arrival at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Example

- It has been observed that after a rainy day, the number of days until the next rain $\sim \text{Geom}(p)$, independent of the past.

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- Question. What is the probability that it rains on both the 5th and the 8th day of the month?
- Approach 1: Handling this problem directly with the geometric PMFs is very tedious and complex.
- Approach 2: Rainy days is a Bernoulli process with arrival. probability p.
- Thus, the answer is p^2.

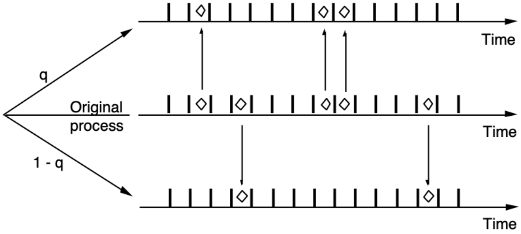
Coding of Random Arrivals

- Question. How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.
- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

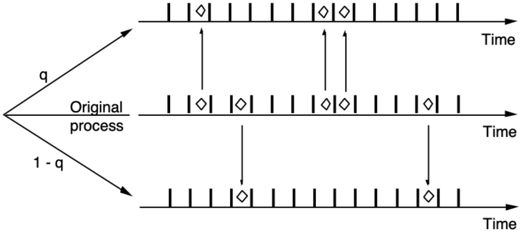
Notations In the Rest of These Slides

- Bernoulli random variable: $\mathrm{Bern}(p)$
- Bernoulli process: $\mathrm{BP}(p)$
- Poisson random variable: $\mathrm{Poisson}(\lambda)$
- Poisson process: $\mathrm{PP}(\lambda)$

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.



- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.
- Split decisions are independent of arrivals.



- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.
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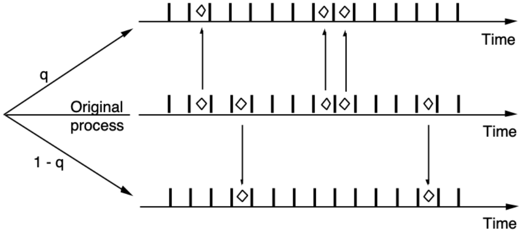
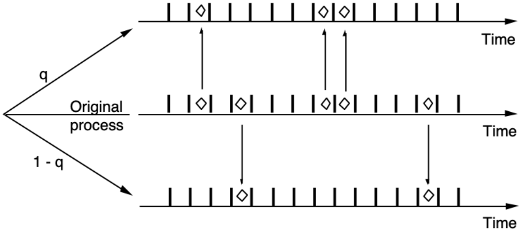
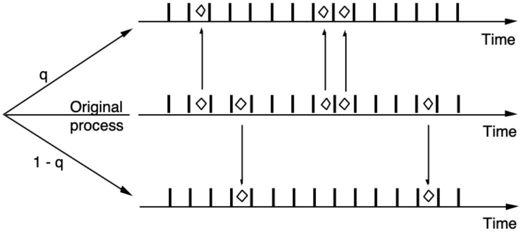


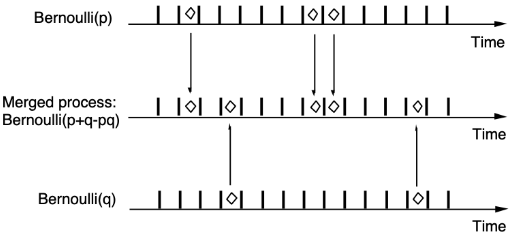
- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.
- Split decisions are independent of arrivals.
- Question. What are the two split processes?
- BP(pq) and BP($p(1-q)$). Why?



- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.
- Split decisions are independent of arrivals.
- Question. What are the two split processes?
- BP(pq) and BP($p(1-q)$). Why?
- Are they independent? No.



- Merge BP(p) and BP(q) into one process.



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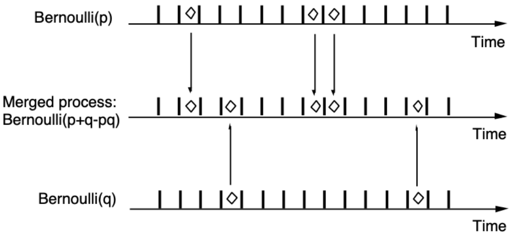

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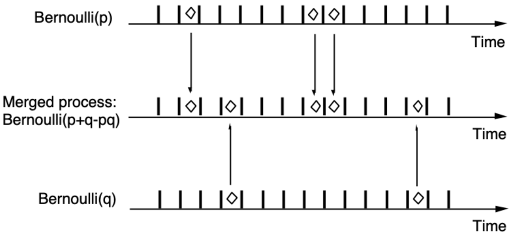
- Probability of having at least one arrival: $1-(1-p)(1-q)=p+q-pq$



Merge: Bernoulli Process

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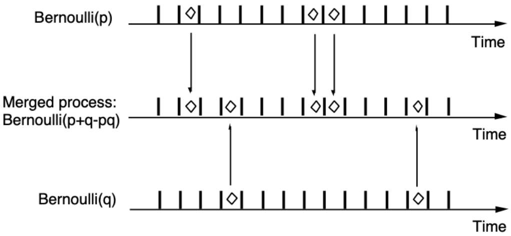
- Collided arrival is regarded just one arrival in the merged process
- Probability of having at least one arrival: $1 - (1-p)(1-q) = p + q - pq$
- Merged process: BP($p + q - pq$)



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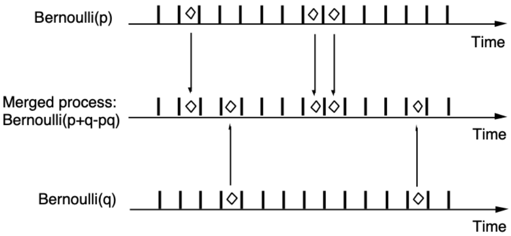
- Collided arrival is regarded just one arrival in the merged process
- Probability of having at least one arrival: $1-(1-p)(1-q) = p+q-pq$
- Merged process: $\mathrm{BP}(p+q-pq)$
- $\mathbb{P}(\text{arrival from proc. 1} \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



- Split a Poisson process PP(λ) into two processes by keeping each arrival w.p. p and discarding it w.p. $(1 - p)$

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- Question. What are the split processes?
- Let's focus on the process that we keep
- Independence and time-homogeneity? Yes
- Small interval probability over δ-interval
  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- PP(λp) and PP($\lambda(1-p)$)

Merge: Poisson Process (1)

- Merge from PP(λ_1) and PP(λ_2)

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- Small interval probabilty over δ-interval (ignoring $o(\delta)$ for small δ)

  $\mathbb{P}(0 \text{ arrival}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1-(\lambda_1+\lambda_2)\delta$

  $\mathbb{P}(1 \text{ arrival}) \approx (\lambda_1\delta)(1-\lambda_2\delta)+\lambda_2\delta(1-\lambda_1\delta) \approx (\lambda_1+\lambda_2)\delta$

Merge: Poisson Process (1)

- Merge from $\mathrm{PP}(\lambda_1)$ and $\mathrm{PP}(\lambda_2)$
- Independence and time-homogeneity? Yes
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- Merged process: $\mathrm{PP}(\lambda_1+\lambda_2)$

Merge: Poisson Process (2)

- Red: $\text{PP}(\lambda_1)$ and Blue: $\text{PP}(\lambda_2)$

Merge: Poisson Process (2)

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Merge: Poisson Process (2)

- Red: PP(λ_1) and Blue: PP(λ_2)
- $\mathbb{P}$(from red | arrival at time t in the merged proc.)? $\dfrac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

Merge: Poisson Process (2)

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- Consider an event $A_k = \{k\text{-th arrival in the merged proc. is red}\}$

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- $\mathbb{P}$(k out of first 10 arrivals are red)? $\binom{10}{k}\left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)^{10-k}$

1. Competing exponentials
2. Sum of independent Poisson rvs
3. Poisson arrivals during and Exponential interval

Competing Exponential (1)

- Two independent light bulbs have life times $T_a \sim \text{Exp}(\lambda_a)$ and $T_b \sim \text{Exp}(\lambda_b)$.

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(Q) Distribution of $Z = \min\{T_a, T_b\}$, the first time when a bulb burns out?

- Approach 1
    - $\mathbb{P}(Z \geq z) = \mathbb{P}(T_a \geq z)\mathbb{P}(T_b \geq z) = e^{-\lambda_a z} e^{-\lambda_b z} = e^{-(\lambda_a + \lambda_b)z}$

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  - Z is the first arrival time of merged Poisson process $(\lambda_a + \lambda_b)$.
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- Three independent light bulbs have life time $T \sim \text{Exp}(\lambda)$.

(Q) $\mathbb{E}$[time until the last bulb burns out]?

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  - T_1: time until the first burn-out, T_2: time until the second burn-out, T_3: time until the third burn-out
  - $\mathbb{E}(T_1 + T_2 + T_3)$?

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  - $\mathbb{E}(T_1 + T_2 + T_3)$?
  - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
  - $T_1 \sim \text{Exp}(3\lambda),\ T_2 \sim \text{Exp}(2\lambda),\ T_3 \sim \text{Exp}(\lambda)$

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  - $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2st burn out}} \text{PP}(\lambda)$
  - $T_1 \sim \text{Exp}(3\lambda)$, $T_2 \sim \text{Exp}(2\lambda)$, $T_3 \sim \text{Exp}(\lambda)$

$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Sum of Independent Poisson rvs

- Two independent rvs X and Y, where $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\nu)$.

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- Poisson process perspective
  - X: number of arrivals of PP(1) over a time interval of length μ
  - Y: number of arrivals of PP(1) over a time interval of length ν
  - Two intervals do not overlap and located in a consecutive manner $\implies X \perp\!\!\!\perp Y$

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    - X: number of arrivals of PP(1) over a time interval of length μ
    - Y: number of arrivals of PP(1) over a time interval of length ν
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- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$

Sum of Independent Poisson rvs

- Two independent rvs X and Y, where $X \sim \text{Poisson}(\mu)$ and $Y \sim \text{Poisson}(\nu)$.
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- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$
- Thus, $X + Y \sim \text{Poisson}(\mu + \nu)$

Poisson arrivals during Exponential Interval (1)

- Problem 24, pp. 335
- Consider PP(λ) and an independent rv $T \sim \text{Exp}(\nu)$

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- Very tedious and not very intuitive.

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Poisson arrivals during Exponential Interval (2)

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- K: number of total arrivals until we get the first arrival from $\text{PP}(\nu)$.
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- Now, consider the merged process of PP(λ) and PP(ν).
  - $\mathbb{P}\Big[\text{from PP}(\lambda)|\text{arrival}\Big] = \frac{\lambda}{\lambda+\nu}$ and $\mathbb{P}\Big[\text{from PP}(\nu)|\text{arrival}\Big] = \frac{\nu}{\lambda+\nu}$
- K: number of total arrivals until we get the first arrival from PP(ν).
  - Then, $K \sim \text{Geom}(\frac{\nu}{\lambda+\nu})$.
- Let L be the number of arrivals from PP(λ) until we get the first arrival from PP(ν).

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \ldots$$

- Consider $S_n \sim \text{Binomial}(n, p)$. Then, $S_n = \sum_{i=1}^{n} X_i$, where $X_i \sim \text{Bern}(p)$.

Approximation of Binomial: Poisson vs. Normal

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  - $p = 1/100$, $n = 10,000$: small p, but large n $\implies$ Both Poisson and Normal

Questions?

1)