

## Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes  
KAIST EE

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### Roadmap

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes

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- (1) Introduction of Random Processes
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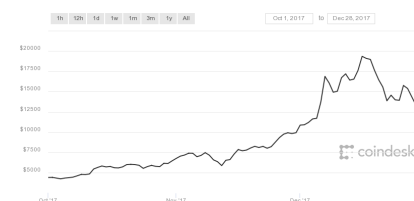
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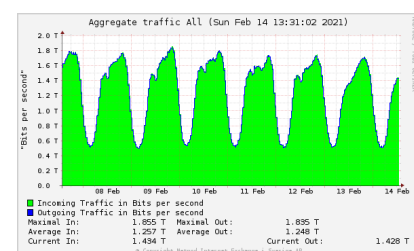
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## Things that evolve in time

- Many probabilistic experiments that **evolve in time**
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



(a) Prices of a cryptocurrency



(b) Internet traffic traces

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- A random process is a **sequence** of random variables indexed by **time**.
- Time: discrete or continuous (a modeling choice in most cases)
- Notation
  - $(X_t)_{t \in \mathcal{T}}$  or  $(X(t))_{t \in \mathcal{T}}$ , where  $\mathcal{T} = \mathbb{R}$  (continuous) or  $\mathcal{T} = \{0, 1, 2, \dots\}$  (discrete)
  - For the discrete case, we also often use  $(X_n)_{n \in \mathbb{Z}_+}$ .
  - We will use all of them, unless confusion arises.
- For a **fixed** time  $t$ ,  $X_t$  (or  $X(t)$ ) is a random variable.
- The values that  $X_t$  (or  $X(t)$ ) can take: discrete or continuous

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- **Example.** Discrete time RP.  $\{X_1, X_2, X_3, \dots\}$ .
  - $X_i$ : # of Covid-19 infections at day  $i$  in South Korea, which is a random variable.
  - Then,  $X_i : \Omega \mapsto \mathbb{R}$ , where the sample space  $\Omega$  is the set of all outcomes.
  - An outcome  $\omega \in \Omega$  is a infinite sequence of infections
  - For example,  $\omega_1 = (100, 150, 130, \dots)$ ,  $\omega_2 = (200, 300, 400, \dots)$
  - $X_3(\omega_1) = 130$ ,  $X_2(\omega_2) = 300$ ,  $X_1(\omega_2) = 200$ , etc.
- **Example.** Continuous time RP.  $(X(t))_{t \in \mathbb{R}^+}$ 
  - $X(t)$ : bitcoin price at time  $t$ , which is a random variable.
  - Then,  $X(t) : \Omega \mapsto \mathbb{R}$ .
  - An outcome  $\omega \in \Omega$  is a trajectory of prices over  $[0, \infty)$
  - $X(3.7, \omega_1) = 3409$ ,  $X(2, \omega_2) = 5000$ ,  $X(7.8, \omega_3) = 2800$ , etc.

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- **Question.** Already studied a sequence (or a collection) of rvs  $X_1, X_2, \dots, X_n$ . What's the difference?
  - Assume a discrete time random process for our discussion.
- **Physical difference:** infinite sequence of  $X_1, X_2, \dots$ .
  - Sample space? set of all outcomes?
  - an outcome: an infinite sequence of sample values  $x_1, x_2, \dots$ ,
- **Semantic difference:** Understand  $i$  in  $X_i$  as time. Also, interesting questions are asked from the random process point of view.
  - **Dependence:** How  $X_1, X_2, \dots$  are related to each other as a time series. Prediction of values in the future.
  - **Long-term behavior:** What is the fraction of times that a stock price is above 3000?
  - Other interesting questions, depending on the target random process

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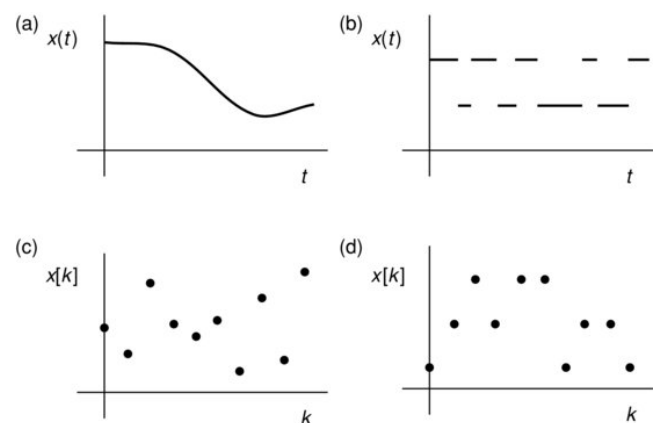
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## 4 Types of Random Processes

- Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



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- The simplest RP
- discrete time
- $X_i \perp\!\!\!\perp \{X_{i-1}, X_{i-2}, \dots, X_1\}$
- **Bernoulli Process (BP)**
- "today" independent of "past"

Jacob Bernoulli (1654 - 1705),  
Swiss



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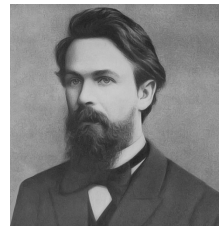
- The simplest RP
- Continuous time version of BP
- $[X(s)]_{s=0}^t \perp\!\!\!\perp [X(s)]_{s=t}^{t+a}$
- **Poisson Process (PP)**
- "today" independent of "past"

Simeon Denis Poisson (1781 -  
1840), France



- One-step more general than BP/PP
- discrete time
- $X_i$  depends on  $X_{i-1}$ , but  $\perp\!\!\!\perp \{X_{i-2}, X_{i-3}, \dots, X_1\}$
- **Markov Chain (MC)**
- "today" depends only on "yesterday"

Andrey Markov (1856 - 1922),  
Russia



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- (1) Introduction of Random Processes
- (2) **Bernoulli Processes**
- (3) Poisson Processes: Poisson RV and Basic Idea
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- At each “minute”, we toss a coin with probability of head  $0 < p < 1$ .
  - Sequence of lottery wins/looses
  - Customers (each second) to a bank
  - Clicks (at each time slot) to server
- A sequence of **independent Bernoulli trials**  $X_1, X_2, \dots$ ,
  - We call index 1, 2, ... **time slots** (or simply slots)

| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | →

- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

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Please pause the video and write down the questions that you want to ask about Bernoulli process.

**VIDEO PAUSE**

Q1.

Q2.

Q3.

Q4.

Q5.

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(Q1) # of arrivals in the first  $n$  slots?

- $S_n = X_1 + X_2 + \dots + X_n$
- $S_n \sim \text{Binomial}(n, p)$
- $\mathbb{E}(S_n) = np$ ,  $\text{var}(S_n) = np(1 - p)$
- This will hold for any  $n$  consecutive slots.

(Q2) # of slots  $T_1$  until the first arrival?

- $T_1 \sim \text{Geom}(p)$
- $\mathbb{E}(T_1) = 1/p$ ,  $\text{var}(T_1) = \frac{1-p}{p^2}$

- $T_1$  is geometric? [Memoryless](#)
- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots  $\implies$  the fresh-start anytime when I look at the process?

(Q3)  $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$ ?

- Yes
- Because  $X_i$ s are independent

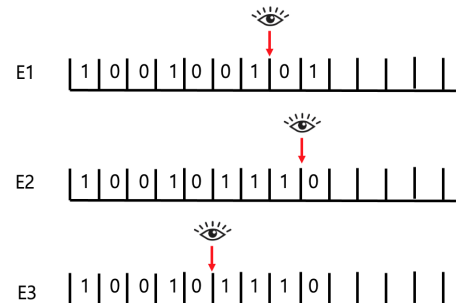
(Q4) After time  $n = 6$ , I start to look at the process  $(X_n)_{n=6}^\infty$ ?

- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time  $n$  (doesn't matter what happened until  $n = 5$ ).
- If you watch the on-going Bernoulli process( $p$ ) from some time  $n$ , you still see the same Bernoulli process( $p$ ).

(Q5) The process  $(X_N, X_{N+1}, X_{N+2}, \dots)$ ? Fresh-start even after random  $N$ ?

- This means that the time I start to look at the process is a random variable.
- Examples of  $N$

- E1.** Time of 3rd arrival
- E2.** First time when 3 consecutive arrivals have been observed
- E3.** Time just before 3 consecutive arrivals



- Difference of  $N$  from  $n$ 
  - The time when I watch the on-going Bernoulli process is **random**.
  - $N$  is a random variable, i.e.,  $N : \Omega \mapsto \mathbb{R}$ . What is  $\Omega$ ?
- Do we experience the fresh-start for any  $N$ ? **E1, E2, and E3?**

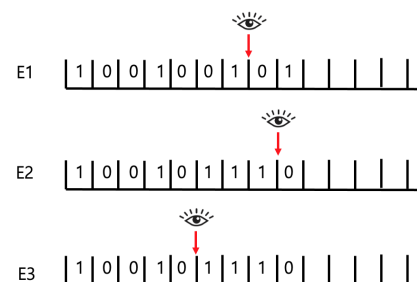
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# Fresh-start after Random $N$ (2)

(Q5) The process  $(X_N, X_{N+1}, X_{N+2}, \dots)$ ? Fresh-start even after random  $N$ ?

- E1.** Time of 3rd arrival
- E2.** First time when 3 consecutive arrivals have been observed
- E3.** Time just before 3 consecutive arrivals



- E1.** When I watch the process,  $N$  has been already determined. **Yes**
- E2.** Same as **E1**. **Yes**
- E3.** Need the future knowledge. '111' does not become random. **No**

- The question of  $N = n$ ? can be answered just from the knowledge about  $X_1, X_2, \dots, X_n$ ? Then, Yes! (see pp. 301 for more formal description)

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- In probability theory, a random time  $N$  is said to be a **stopping time**, if the question of “ $N = n?$ ” can be answered only from the present and the past knowledge of  $X_1, X_2, \dots, X_n$ .
- [https://en.wikipedia.org/wiki/Stopping\\_time](https://en.wikipedia.org/wiki/Stopping_time)
- Fresh-start after  $N$  in Bernoulli process? **Yes, if  $N$  is a stopping time.**
- Please think about two examples of stopping time and not. **VIDEO PAUSE**
  - **Yes.** Time when 10 consecutive arrivals have been observed
  - **No.** Time of 2nd arrival in 10 consecutive arrivals

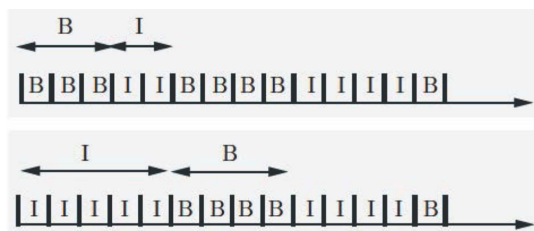
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## Distribution of Busy Periods (1)

- Regard an arrival as a server being busy (just for our easy understanding)
- **First busy period  $B_1$** : starts with the first busy slot and ends just before the first subsequent idle slot

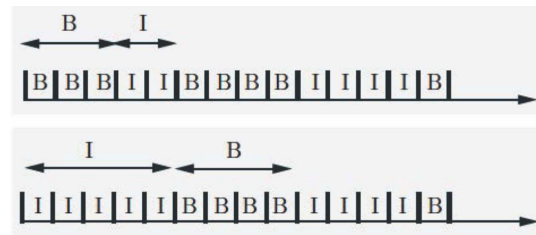


- **(Q6)**  $B_1$  is a random variable. Distribution of  $B_1$ ?
- $N$ : time of the first busy slot.  $N$  is a stopping time?
  - **Yes.** Thus, **fresh-start after  $N$** . Because we can answer the question of  $N = n?$ , just using  $X_1, X_2, \dots, X_n$ .
- $B_1$  is geometric with parameter  **$(1 - p)$**

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- **Question.** What about the second busy period  $B_2$ ?
- $N$ : time of the first busy slot of the second busy period.  $N$  is a stopping time?
- **Yes.** Thus, fresh-start after  $N$ .
- Then,  $B_1$  and  $B_2$  are **identically distributed** as  $\text{Geom}(1 - p)$ .
- $B_3, B_4, \dots$ ?

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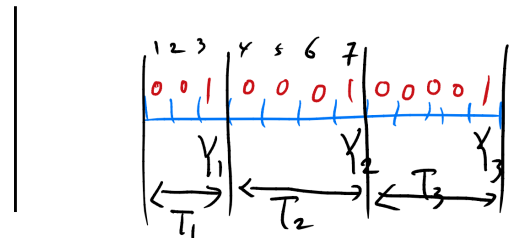
## Time of $k$ -th arrival

- Time of the first arrival  $Y_1 \sim \text{Geom}(p)$

(Q7) Time of the  $k$ -th arrival  $Y_k$ ?

VIDEO PAUSE

- $T_k = Y_k - Y_{k-1}$ :  $k$ -th inter-arrival ( $k \geq 2$ ,  $T_1 = Y_1$ )
- $Y_k = T_1 + T_2 + \dots + T_k$ .



- After each  $T_k$ , the fresh-start occurs.
- $\{T_i\}$  are i.i.d. and  $\sim \text{Geom}(p)$
- We know  $Y_k$ 's expectation and variance:  $\mathbb{E}[Y_k] = \frac{k}{p}$ ,  $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$ , but **its distribution?**

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- $Y_k = T_1 + T_2 + \dots + T_k.$

- $\{T_i\}$  are i.i.d. and  $\sim \text{Geom}(p)$

$$\begin{aligned}\mathbb{P}(Y_k = t) &= \mathbb{P}(X_k = 1 \text{ and } k-1 \text{ arrivals during the first } t-1 \text{ slots}) \\ &= \mathbb{P}(X_k = 1) \cdot \mathbb{P}(k-1 \text{ arrivals during the first } t-1 \text{ slots}) \\ &= p \times \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \dots\end{aligned}$$

- In the sequence of Bernoulli trials, the time  $Y_k$  of  $k$ -th success

- PMF of  $Y_k$

$$\mathbb{P}(Y_k = t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k}, & \text{if } t = k, k+1, \dots \\ 0, & \text{if } t = 1, 2, \dots, k-1 \end{cases}$$

- $\text{Pascal}(1, p) = \text{Geom}(p)$

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Background: Poisson rv  $X$  with parameter  $\lambda$  (1)

- A random variable  $S \sim \text{Bin}(n, p)$ : Models the number of successes in a given number  $n$  of independent trials with success probability  $p$ .

$$p_S(k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

- Our interest: very large  $n$  and very small  $p$ , such that  $np = \lambda$ , i.e.,  $\lim_{n \rightarrow \infty} p_S(k)$ ?

$$\begin{aligned} p_S(k) &= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{n}{n} \cdot \frac{(n-1)}{n} \cdots \frac{(n-k+1)}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \left(1 - \frac{\lambda}{n}\right)^{-k} \xrightarrow{n \rightarrow \infty} e^{-\lambda} \frac{\lambda^k}{k!} \end{aligned}$$

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- A Poisson random variable  $Z$  with parameter  $\lambda$  takes nonnegative integer values, whose PMF is:

$$p_Z(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

- Infinitely many slots ( $n$ ) with the infinitely small slot duration (thus infinitely small success probability  $p = \lambda/n$ )
- $\mathbb{E}(Z) = \lambda$  (because  $\lambda = np$  is the mean of binomial rv)
- $\text{var}(Z) = \lambda$  (because  $np(1-p)$  is the variance of binomial rv)

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## Example: Poisson Approximation

- A packet consisting of a string of  $n$  symbols is transmitted over a noisy channel.
- Each symbol: erroneous transmission with probability of 0.0001, independent of other symbols. Incorrect transmission is when at least one symbol is in error.
- **Question.** How small should  $n$  be in order for the probability of incorrect transmission to be less than 0.001?
- $p$  is very small, and  $n$  is reasonably large  $\rightarrow$  Poisson approximation
- Prob. of incorrect transmission =  
 $1 - \mathbb{P}(\text{no symbol error}) = 1 - e^{-\lambda} = 1 - e^{-0.0001n} < 0.0001$
- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

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L4(3)

- **Remind.** Geometric vs. Exponential
  - Two rvs with memoryless property
  - continuous system = discrete system with infinitely many slots whose duration is infinitely small.
- Independence between what happens in a different time region
- Memoryless and fresh-start property
- Choose one of discrete or continuous versions for our modeling convenience.
- **Question.** How do we design the continuous analog of Bernoulli process?
  - Key idea: Making it as a **limit** of a sequence of Bernoulli processes
- Need a “modeling sense” to make this possible. It’s a good practice for engineers!
- **VIDEO PAUSE**

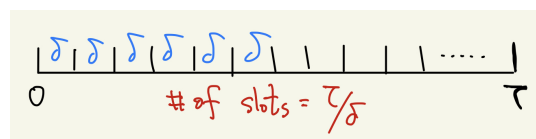
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## Key Design Idea to Develop a Continuous Twin (1)

- Continuous twin
  - Key point: Understand the number of arrivals over a given interval  $[0, \tau]$ .
  - Assume that it has some arrival rate  $\lambda$  (# of arrivals/unit time).
  - We know how to handle Bernoulli process with discrete time slots.
- Divide  $[0, \tau]$  into slots whose length =  $\delta$ . Then,  $n = \#$  of slots =  $\frac{\tau}{\delta}$ .

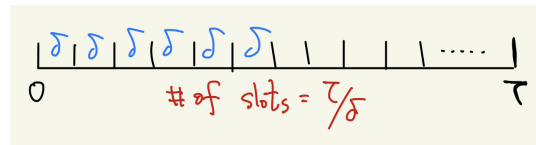


- What's the limit as  $\delta \rightarrow 0$  (equivalently,  $n \rightarrow \infty$ )

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- Now, our design idea: during one time slot of length  $\delta$ ,

$\mathbb{P}(1 \text{ arrival}) \propto \text{arrival rate and slot length}$   
 $\mathbb{P}(\geq 2 \text{ arrivals}) \propto \text{something, but very small}$   
 for small slot length  
 $\mathbb{P}(0 \text{ arrival}) = 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})$

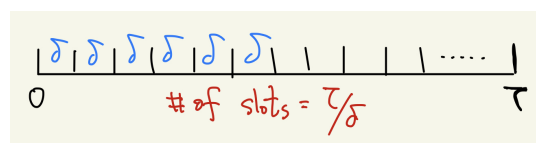
$$\begin{aligned}
 \mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\
 \mathbb{P}(\geq 2 \text{ arrivals}) &= o(\delta) \\
 \mathbb{P}(0 \text{ arrival}) &= 1 - \lambda\delta + o(\delta)
 \end{aligned}$$

- $o(\delta)$ : some function that goes to zero faster than  $\delta$ .
  - Thus, for very small  $\delta$ ,  $o(\delta)$  becomes negligible, compared to  $\delta$ .
  - Example:  $o(\delta) = \delta^\alpha$ , where any  $\alpha > 1$

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- Our interest: probability of  $k$  arrivals over  $[0, \tau]$
- Given “small”  $\delta$ , # of arrivals  $\sim \text{Bin}(n, p)$ , where  $n = \tau/\delta$  and  $p = \lambda\delta$
- As  $\delta \rightarrow 0$ ,  $np = \tau/\delta \times \lambda\delta = \lambda\tau$ .
- # of arrivals over  $[0, \tau]$ ,  $\sim \text{Poisson}(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call **Poisson process**.

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## Poisson Process: Definition (1)

An arrival process is called a **Poisson process** with rate  $\lambda$ , if the following are satisfied:

- **(Independence)** Let  $N_\tau$  be the number of arrivals over the interval  $[0, \tau]$ . For any  $\tau_1, \tau_2 > 0$ ,  $N_{s+\tau_1} - N_s$  is independent of  $N_{t+\tau_2} - N_t$ , if  $t > s + \tau_1$ .
  - The number of arrivals over two disjoint intervals are independent.
- **(Time homogeneity)** For any  $s$ , the distribution of  $N_{s+\tau} - N_s$  is equal to that of  $N_\tau$ .
  - $N_\tau$  becomes the number of arrivals over any interval of length  $\tau$ .
- **(Small interval probability)** Let  $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$ , which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \dots, \quad \text{where} \quad \lim_{\tau \rightarrow 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \rightarrow 0} \frac{o_k(\tau)}{\tau} = 0$$

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An arrival process is called a **Poisson process** with rate  $\lambda$ , if the following are satisfied:

- **(Independence)** Let  $N_\tau$  be the number of arrivals over the interval  $[0, \tau]$ . For any  $\tau_1, \tau_2 > 0$ ,  $N_{s+\tau_1} - N_s$  is independent of  $N_{t+\tau_2} - N_t$ , if  $t > s + \tau_1$ .
  - The number of arrivals over two disjoint intervals are independent.
- **(Time homogeneity)** For any  $s$ , the distribution of  $N_{s+\tau} - N_s$  is equal to that of  $N_\tau$ .
  - $N_\tau$  becomes the number of arrivals over any interval of length  $\tau$ .
- **(Distribution of  $N_\tau$ )**  $N_\tau$  is the Poisson rv with parameter  $\lambda\tau$ , i.e., if we let  $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$ , we have:

$$\mathbb{P}(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

**(Q1)** Number of arrivals of any interval with length  $\tau \sim \text{Poisson}(\lambda\tau)$ , i.e.,

$$\mathbb{P}(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

$$\mathbb{E}(N_\tau) = \lambda\tau \text{ and } \text{var}(N_\tau) = \lambda\tau$$

**(Q2)** Time of first arrival  $T$

$$F_T(t) = \mathbb{P}(T \leq t) = 1 - \mathbb{P}(T > t) = 1 - P(0, t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0.$$

- $T \sim \text{Exp}(\lambda)$ . Thus,  $\mathbb{E}(T) = 1/\lambda$  and  $\text{var}(T) = 1/\lambda^2$ 
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

- Receive emails according to a Poisson process at rate  $\lambda = 5$  messages/hour
- Mean and variance of mails received during a day
  - $5 \cdot 24 = 120$
- $\mathbb{P}[\text{one new message in the next hour}]$ 
  - $\mathbb{P}(1, 1) = \frac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$
- $\mathbb{P}[\text{exactly two msgs during each of the next three hours}]$ 
  - $\left( \frac{5^2 e^{-5}}{2!} \right)^3$

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## Memoryless and Fresh-start Property

- **Remind.** Similar property for Bernoulli processes, but here no time slots.
- **Fresh-start at deterministic time  $t$ :** Start watching at time  $t$ , then you see the Poisson process, independent of what has happened in the past.
- **Fresh-start at random time  $T$ :** Similarly, it holds.
  - For example, when you start watching at random time  $T_1$  (time of first arrival).
  - Generally, it holds when  $T$  is a **stopping time**.

(Q3) The  $k$ -th arrival time  $Y_k$ ?

- $k$ -th inter-arrival time  $T_k = Y_k - Y_{k-1}$ ,  $k \geq 2$ , and  $T_1 = Y_1$ .
- $Y_k = T_1 + T_2 + \dots + T_k$  is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$  and  $\text{var}[Y_k] = k/\lambda^2$ , but what is the distribution of  $Y_k$ ?

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- For a given  $\delta$ ,  $\delta \cdot f_{Y_k}(y)$ : prob. of  $k$ -th arrival over  $[y, y + \delta]$ .

- When  $\delta$  is small, only one arrival occurs. Thus,

$$\delta \cdot f_{Y_k}(y) = \mathbb{P}(\text{an arrival over } [y, y + \delta]) \times \mathbb{P}(k-1 \text{ arrivals before } y)$$

$$\approx \lambda \delta \times \mathbb{P}(k-1, y) = \lambda \delta \times \frac{\lambda^{k-1} y^{k-1} e^{-\lambda y}}{(k-1)!}$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0.$$

- This is called **Erlang** rv.

An Erlang random variable  $Z$  with parameter  $(k, \lambda)$  has the following pdf:

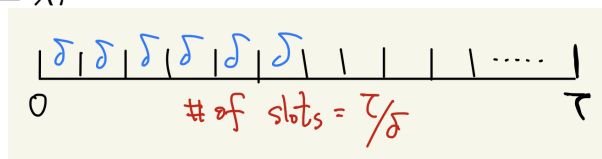
$$f_Z(z) = \frac{\lambda^k z^{k-1} e^{-\lambda z}}{(k-1)!}, \quad z \geq 0$$

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## Poisson Process vs. Bernoulli Process

$$- n = \tau/\delta, \quad p = \lambda\delta, \quad np = \lambda\tau$$



	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals	Binomial	Poisson
Interarrival time	Geometric	Exponential
Time of $k$ -th arrival	Pascal	Erlang
Arrival rate	$p$ /per slot	$\lambda$ /unit time

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- Catching fish: Poisson process  $\lambda = 0.6/\text{hour}$ .
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

(Q1)  $\mathbb{P}(\text{fishing time} > 2 \text{ hours})$

Method 1:  $\mathbb{P}(0, 2)$

Method 2:  $\mathbb{P}(T_1 > 2)$

(Q2)  $\mathbb{P}(2 < \text{fishing time} < 5)$

Method 1:  $\mathbb{P}(0, 2)(1 - \mathbb{P}(0, 3))$

Method 2:  $\mathbb{P}(2 < T_1 < 5)$

(Q3)  $\mathbb{P}(\text{Catch at least two fish})$

Method 1:  $\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0, 2) - \mathbb{P}(1, 2)$

Method 2:  $\mathbb{P}(Y_2 \leq 2)$

(Q4)  $\mathbb{E}[\text{future fi. time} | \text{already fished for 3h}]$

Fresh-start. So,

$$\mathbb{E}[\exp(\lambda)] = 1/\lambda = 1/0.6$$

(Q5)  $\mathbb{E}[F = \text{total fishing time}]$

$$2 + \mathbb{E}[F - 2] = 2 + \mathbb{P}(F = 2) \cdot 0 +$$

$$\mathbb{P}(F > 2) \cdot \mathbb{E}[F - 2 | F > 2]$$

$$= 2 + \mathbb{P}(0, 2) \cdot \frac{1}{\lambda}$$

(Q6)  $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

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- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes

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**Alternative Description of the Bernoulli Process**

1. Start with a sequence of independent geometric random variables  $T_1, T_2, \dots$ , with common parameter  $p$ , and let these stand for the interarrival times.
2. Record a success (or arrival) at times  $T_1, T_1 + T_2, T_1 + T_2 + T_3$ , etc.

**Alternative Description of the Poisson Process**

1. Start with a sequence of independent exponential random variables  $T_1, T_2, \dots$ , with common parameter  $\lambda$ , and let these stand for the interarrival times.
2. Record an arrival at times  $T_1, T_1 + T_2, T_1 + T_2 + T_3$ , etc.

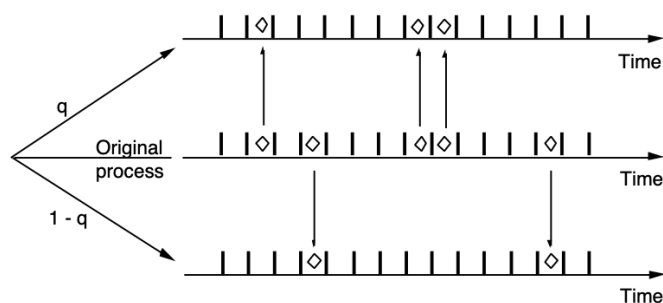
## Example

- It has been observed that after a rainy day, the number of days until the next rain  $\sim \text{Geom}(p)$ , independent of the past.
- **Question.** What is the probability that it rains on both the 5th and the 8th day of the month?
- **Approach 1:** Handling this problem directly with the geometric PMFs is very tedious and complex.
- **Approach 2:** Rainy days is a Bernoulli process with arrival probability  $p$ .
- Thus, the answer is  $p^2$ .

- **Question.** How to make software codes of Bernoulli process with  $p$  and Poisson process with  $\lambda$
- Inter-arrival times are very handy.
- **Bernoulli process with  $p$ :** Obtain a sequence of random values following the **geometric distribution** with parameter  $p$ .
- **Poisson process with  $\lambda$ :** Obtain a sequence of random values following the **exponential distribution** with parameter  $\lambda$ .

- Bernoulli random variable:  $\text{Bern}(p)$
- Bernoulli process:  $\text{BP}(p)$
- Poisson random variable:  $\text{Poisson}(\lambda)$
- Poisson process:  $\text{PP}(\lambda)$

- Split  $BP(p)$  into two processes. Whenever there is an arrival, keep it w.p.  $q$  and discard it w.p.  $(1 - q)$ .
- Split decisions are independent of arrivals.
- **Question.** What are the two split processes?
- $BP(pq)$  and  $BP(p(1 - q))$ . Why?
- Are they independent? **No.**

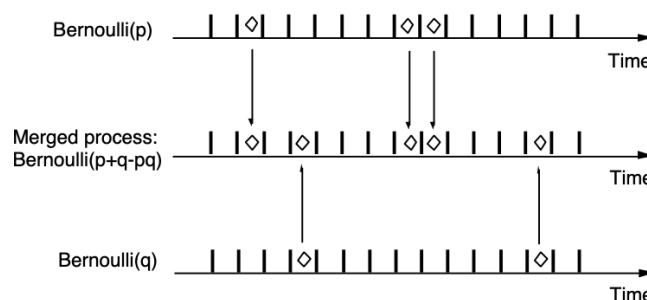


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- Merge  $BP(p)$  and  $BP(q)$  into one process.
- Collided arrival is regarded just one arrival in the merged process
- Probability of having at least one arrival:  $1 - (1 - p)(1 - q) = p + q - pq$
- Merged process:  $BP(p + q - pq)$
- $\mathbb{P}(\text{arrival from proc. 1} \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



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- Split a Poisson process  $PP(\lambda)$  into two processes by keeping each arrival w.p.  $p$  and discarding it w.p.  $(1 - p)$
- **Question.** What are the split processes?
- Let's focus on the process that we keep
- Independence and time-homogeneity? **Yes**
- Small interval probability over  $\delta$ -interval
  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- **$PP(\lambda p)$  and  $PP(\lambda(1 - p))$**

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- Merge from  $PP(\lambda_1)$  and  $PP(\lambda_2)$
- Independence and time-homogeneity? **Yes**
- Small interval probability over  $\delta$ -interval (ignoring  $o(\delta)$  for small  $\delta$ )
 
$$\mathbb{P}(0 \text{ arrival}) \approx (1 - \lambda_1\delta)(1 - \lambda_2\delta) \approx 1 - (\lambda_1 + \lambda_2)\delta$$

$$\mathbb{P}(1 \text{ arrival}) \approx (\lambda_1\delta)(1 - \lambda_2\delta) + \lambda_2\delta(1 - \lambda_1\delta) \approx (\lambda_1 + \lambda_2)\delta$$
- Merged process:  **$PP(\lambda_1 + \lambda_2)$**

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- **Red**:  $PP(\lambda_1)$  and **Blue**:  $PP(\lambda_2)$
- $\mathbb{P}(\text{from red} \mid \text{arrival at time } t \text{ in the merged proc.})? \frac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- Consider an event  $A_k = \{k\text{-th arrival in the merged proc. is red}\}$ 
  - $\mathbb{P}(A_k)? \frac{\lambda_1}{\lambda_1 + \lambda_2}$
  - $A_1, A_2, \dots$  are independent: origins (red or blue) of arrivals in the merged proc. are independent
- $\mathbb{P}(k \text{ out of first 10 arrivals are red})? \binom{10}{k} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{10-k}$

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1. Competing exponentials
2. Sum of independent Poisson rvs
3. Poisson arrivals during and Exponential interval

L8(5)

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- Two independent light bulbs have life times  $T_a \sim \text{Exp}(\lambda_a)$  and  $T_b \sim \text{Exp}(\lambda_b)$ .

(Q) Distribution of  $Z = \min\{T_a, T_b\}$ , the first time when a bulb burns out?

- Approach 1

- $\mathbb{P}(Z \geq z) = \mathbb{P}(T_a \geq z)\mathbb{P}(T_b \geq z) = e^{-\lambda_a z} e^{-\lambda_b z} = e^{-(\lambda_a + \lambda_b)z}$
- Thus,  $Z \sim \text{Exp}(\lambda_a + \lambda_b)$

- Approach 2

- $T_a$  and  $T_b$  are the first arrival times of two Poisson processes of  $\lambda_a$  and  $\lambda_b$ , respectively.
- $Z$  is the first arrival time of merged Poisson process  $(\lambda_a + \lambda_b)$ .
- Thus,  $Z \sim \text{Exp}(\lambda_a + \lambda_b)$

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- Three independent light bulbs have life time  $T \sim \text{Exp}(\lambda)$ .

(Q)  $\mathbb{E}[\text{time until the last bulb burns out}]?$

- Understanding from the merged Poisson process

- $T_1$ : time until the first burn-out,  $T_2$ : time until the second burn-out,  $T_3$ : time until the third burn-out
- $\mathbb{E}(T_1 + T_2 + T_3)?$
- $\text{PP}(3\lambda) \xrightarrow{\text{1st burn out}} \text{PP}(2\lambda) \xrightarrow{\text{2nd burn out}} \text{PP}(\lambda)$
- $T_1 \sim \text{Exp}(3\lambda)$ ,  $T_2 \sim \text{Exp}(2\lambda)$ ,  $T_3 \sim \text{Exp}(\lambda)$

$$\mathbb{E}[T_1 + T_2 + T_3] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

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- Two independent rvs  $X$  and  $Y$ , where  $X \sim \text{Poisson}(\mu)$  and  $Y \sim \text{Poisson}(\nu)$ .
- **Question.** Distribution of  $X + Y$ ? Complex convolution, but any other easy way?
- Poisson process perspective
  - $X$ : number of arrivals of PP(1) over a time interval of length  $\mu$
  - $Y$ : number of arrivals of PP(1) over a time interval of length  $\nu$
  - Two intervals do not overlap and located in a consecutive manner  $\implies X \perp\!\!\!\perp Y$
- Distribution of  $X + Y$ : the number of arrivals of PP(1) over a time interval of length  $\mu + \nu$
- Thus,  $X + Y \sim \text{Poisson}(\mu + \nu)$

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- Problem 24, pp. 335
- Consider PP( $\lambda$ ) and an independent rv  $T \sim \text{Exp}(\nu)$
- **Question.** Distribution of  $N_T$ ?
- **Approach 1:** Total probability theorem

$$\mathbb{P}(N_T = k) = \int_0^\infty \mathbb{P}(N_T = k | T = \tau) f_T(\tau) d\tau = \int_0^\infty \mathbb{P}(N_\tau = k) f_T(\tau) d\tau$$

- Very tedious and not very intuitive.

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- Consider  $PP(\lambda)$  and an independent rv  $T \sim \text{Exp}(\nu)$
- Consider another  $PP(\nu)$ , and let's view  $T$  as the first arrival time in  $PP(\nu)$ .
- Now, consider the merged process of  $PP(\lambda)$  and  $PP(\nu)$ .
  - $\mathbb{P}[\text{from } PP(\lambda)|\text{arrival}] = \frac{\lambda}{\lambda+\nu}$  and  $\mathbb{P}[\text{from } PP(\nu)|\text{arrival}] = \frac{\nu}{\lambda+\nu}$
- $K$ : number of total arrivals until we get the first arrival from  $PP(\nu)$ .
  - Then,  $K \sim \text{Geom}(\frac{\nu}{\lambda+\nu})$ .
- Let  $L$  be the number of arrivals from  $PP(\lambda)$  until we get the first arrival from  $PP(\nu)$ .

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right) \left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \dots$$

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- Consider  $S_n \sim \text{Binomial}(n, p)$ . Then,  $S_n = \sum_{i=1}^n X_i$ , where  $X_i \sim \text{Bern}(p)$ .
- **Poisson approximation.**  $\text{Poisson}(np)$  is a good approximation of  $S_n$ 
  - Holds when  $np = \lambda$  ( $n \rightarrow \infty, p \rightarrow 0$ ), i.e.,  $X_i$ 's behavior changes over  $n$ .
- **Normal approximation.**  $\sigma\sqrt{n}Z + n\mu$  is a good approximation of  $S_n$ 
  - Holds for a fixed  $p$  ( $n \rightarrow \infty$ ), i.e.,  $X_i$ 's behavior does not change over  $n$ .
- In practice, an actual numbers of  $n$  and  $p$  are given, so which approximation is good under what situation?
  - $p = 1/100, n = 100$ :  $np = 1$ , very asymmetric  $X_i$ , small  $p \implies$  **Poisson**
  - $p = 1/3, n = 100$ : large, reasonably symmetric  $p$ , at least moderate  $n \implies$  **Normal**
  - $p = 1/100, n = 10,000$ : small  $p$ , but large  $n \implies$  **Both Poisson and Normal**

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## Questions?

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## Review Questions

1)

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