Lecture 4: Random Variable, Part II

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Outline

- Continuous Random Variable
- PDF (Probability Density Function)
- CDF (Cumulative Distribution Function)
- Exponential and Normal Distribution
- Joint PDF, Conditional PDF
- Bayes' rule for continous and even mixed cases

Roadmap

- ○ Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- ○ Summarizing a random variable: Expectation and Variance
- ○ Functions of a single random variable, Functions of multiple random variables
- ○ Conditioning for random variables, Independence for random variables
- ○ Continuous random variables
  - Normal, Uniform, Exponential, etc.
- ○ Bayes' rule for random variables

Continuous RV and Probability Density Function

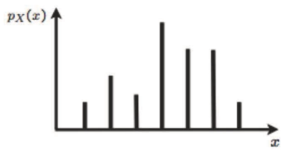
- Many cases when random variable have "continuous values", e.g., velocity of a car

Continuous Random Variable

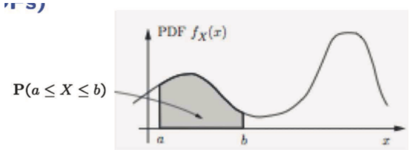
A rv X is continuous if $\exists$ a function f_X, called $\boxed{\text{probability density function (PDF)}}$, s.t.

$$\mathbb{P}(X \in B) = \int_B f_X(x)dx$$

- All of the concepts and methods (expectation, PMFs, and conditioning) for discrete rvs have continuous counterparts

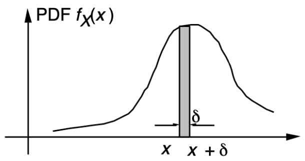


- $\mathbb{P}(a \le X \le b) = \sum_{x: a \le x \le b} p_X(x)$
- $p_X(x) \ge 0$, $\sum_x p_X(x) = 1$



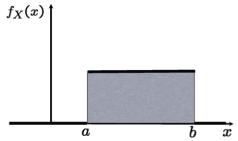
- $\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$
- $f_X(x) \ge 0$, $\int_{-\infty}^{\infty} f_X(x)dx = 1$

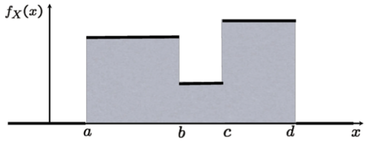
PDF and Examples



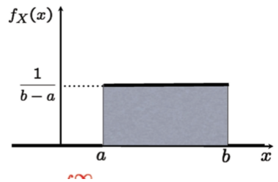
- $\mathbb{P}(a \le X \le a+\delta) \approx \boxed{f_X(a)\cdot\delta}$
- $\mathbb{P}(X=a)=0$

Examples





Expectation and Variance



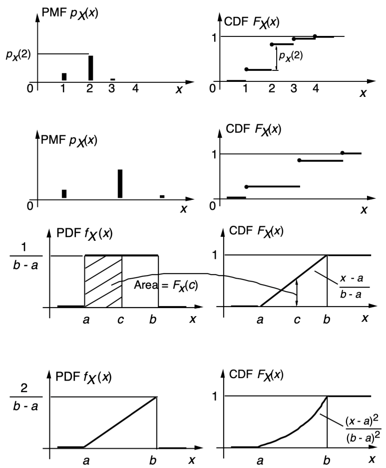
- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{b+a}{2}$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{b-a}\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3}$
- $\mathrm{var}[X] = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$

Cumulative Distribution Function (CDF)

- Discrete: PMF, Continuous: PDF
- Can we describe all rvs with a single mathematical concept?

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} \sum_{k \le x} p_X(k), & \text{discrete} \\ \int_{-\infty}^{x} f_X(t)dt, & \text{continuous} \end{cases}$$

- always well defined, because we can always compute the probability for the event $\{X \le x\}$
- CCDF (Complementary CDF): $\mathbb{P}(X > x)$



CDF Properties

- Non-decreasing
- $F_X(x)$ tends to 1, as $x \to \infty$
- $F_X(x)$ tends to 0, as $x \to -\infty$

Now, let's look at famous continuous random variables popularly used in our life.

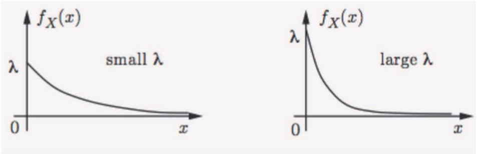
Exponential RV with parameter $\lambda > 0$: $\exp(\lambda)$

KAIST EE

- A rv X is called exponential with λ, if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{or} \quad F_X(x) = 1 - e^{-\lambda x}$$

- Models a waiting time
- CCDF $\mathbb{P}(X \geq x) = e^{-\lambda x}$ (waiting time decays exponentially)
- $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$, $\text{var}[X] = 1/\lambda^2$
- (Q) What is the discrete rv which models a waiting time?



Modeling Waiting Time? A Discrete Twin (1)

KAIST EE

- A discrete twin for modeling waiting times is geometric rvs.
- Models a system evolution over time: Continuous time vs. Discrete time. In many cases, continuous case is the some type of $\boxed{\text{limit}}$ of its corresponding discrete case.
- Can you make mathematical description, where geometric and exponential rvs meet each other in the limit?
- Key idea.
  - Continuous system: Discrete system with
    infinitely many slots whose duration is infinitely small.
- limiting system: $X_{exp}(\lambda)$ with CDF $F_{exp}(\cdot)$
- n-th system: $X^n_{geo}(p_n)$ with CDF $F^n_{geo}(\cdot)$

Modeling Waiting Time? A Discrete Twin (2)

KAIST EE

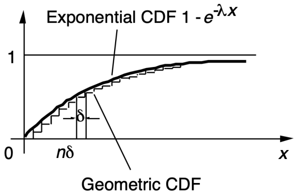
For a given $x > 0$,

- Define $\delta = \frac{x}{n}$ (a slot length in the n-th system)
- Remember

$$F_{exp}(x) = 1 - e^{-\lambda x}$$
$$F^n_{geo}(n) = 1 - (1 - p_n)^n$$

- Choose $p_n = 1 - e^{-\lambda\delta} = 1 - e^{-\lambda\frac{x}{n}}$.
- As $n \to \infty$, the slot length $\delta \to 0$ thus $p_n \to 0$
- The CDF values of exponential and n-th geometric rvs become equal whenever $x = \delta, 2\delta, 3\delta, \ldots$, i.e.,

$$F_{exp}(n\delta) = F^n_{geo}(n), \quad n = 1, 2, \ldots$$



- As n grows, the number of slots grows, but the success probability over one slot decreases, so that everything is balanced up.
- As n grows, $F^n_{geo}(n)$ approaches $F_{exp}(n\delta)$.

Normal (also called Gaussian) Random Variable

KAIST EE

Why important?

- Central limit theorem (중심극한정리)
  - One of the most remarkable findings in the probability theory
- Convenient analytical properties
- Modeling aggregate noise with many small, independent noise terms

Normal: PDF, Expectation, Variance

- Standard Normal $N(0,1)$

  $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

Need to check:

- a legitimate PDF or not

- expectation/variance

- General Normal $N(\mu, \sigma^2)$

  $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$
- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$

Normal: Useful Property

- Linear transformation preserves normality

Linear transformation of Normal

If $X \sim Norm(\mu, \sigma^2)$, then for $a \neq 0$ and b $Y = aX + b \sim Norm(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be $\boxed{\text{standardized}}$:

  If $X \sim Norm(\mu, \sigma^2)$, then $\boxed{Y = \frac{X-\mu}{\sigma}} \sim Norm(0,1)$
- Thus, we can make the table which records the following CDF values:

  $$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$$

Example

- Annual snowfall X is modeled as $Norm(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?
- $Y = \frac{X-60}{20}$.

$$\begin{aligned} \mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80-60}{20}) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587 \end{aligned}$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

** Continuous counterparts are intuitively understandable. So, we will be quick at reviewing them.

Continuous: Joint PDF and CDF (1)

Jointly Continuous

Two continuous rvs are $\boxed{\text{jointly continuous}}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for $\boxed{\text{every}}$ subset B of the two dimensional plane,

$$\mathbb{P}((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

1. The joint PDF is used to calculate probabilities
$$\mathbb{P}((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$
Our particular interest: $B = \{(x,y) \mid a \le x \le b, c \le y \le d\}$

Continuous: Joint PDF and CDF (2)

2. The marginal PDFs of X and Y are from the joint PDF as:
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$
3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$, and determines the joint PDF as:
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$
4. A function $g(X,Y)$ of X and Y defines a new random variable, and
$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy$$

Continuous: Conditional PDF given an event

* Conditional PDF, given an event

- $f_X(x)\cdot\delta \approx \mathbb{P}(x \le X \le x+\delta)$
  $f_{X|A}(x)\cdot\delta \approx \mathbb{P}(x \le X \le x+\delta | A)$
- $\mathbb{P}(X\in B) = \int_B f_X(x)dx$
  $\mathbb{P}(X\in B|A) = \int_B f_{X|A}(x)dx$

  Note: A is an event, but B is a subset that includes the possible values which can be taken by the rv X.
- $\int f_{X|A}(x) = 1$

* Conditional PDF, given $X \in B$

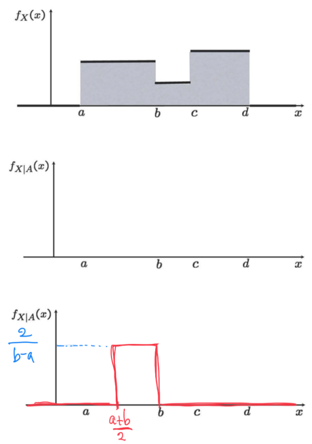
$$\mathbb{P}(x \le X \le x+\delta | X\in B) \approx f_{X|X\in B}(x)\cdot\delta$$

$$f_{X|X\in B}(x) = \begin{cases} 0, & \text{if } x \notin B \\ \frac{f_X(x)}{\mathbb{P}(B)}, & \text{if } x \in B \end{cases}$$

(Q) In the discrete, we consider the event $\{X=x\}$, not $\{X\in B\}$. Why?

Continuous: Conditional Expectation

$A = \{\frac{a+b}{2} \le X \le b\}$



- $\mathbb{E}[X] = \int x f_X(x)dx$
  $\mathbb{E}[X|A] = \int x f_{X|A}(x)dx$
- $\mathbb{E}[g(X)] = \int g(x)f_X(x)dx$
  $\mathbb{E}[g(X)|A] = \int g(x)f_{X|A}(x)dx$

$$\mathbb{E}[X|A] = \int_{(a+b)/2}^{b} x\frac{2}{b-a}dx = \frac{a}{4}+\frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^{b} x^2\frac{2}{b-a}dx =$$

Exponential RV: Memoryless

- Exponential rv is a continous counterpart of geometric rv.
- Thus, expected to be memeoryless.

Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

- **Proof.** Note that $\mathbb{P}(X > x) = e^{-\lambda x}$. Then,

$$\mathbb{P}(X > n + m | X > m) = \frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > m)} = \frac{e^{-\lambda(n+m)}}{e^{-\lambda m}} = e^{-\lambda n} = \mathbb{P}(X > n)$$

Total Probability/Expectation Theorem

Partition of Ω into $A_1, A_2, A_3, \ldots$

* Discrete case

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

* Continuous case

Total Probability Theorem

$$f_X(x) = \sum_i \mathbb{P}(A_i) f_{X|A_i}(x)$$

Total Expectation Theorem

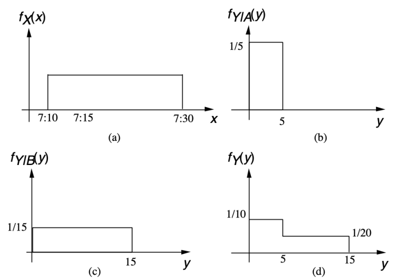
$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Example

- Your train's arrival every quarter hour (0, 15min, 30min, 45min).
- Your arrival $\sim$ uniform(7:10, 7:30) am.
- What is the PDF of waiting time for the first train?
- X : your arrival time, Y : waiting time.
- The value of X makes a different waiting time. So, consider two events:

$A = \{7{:}10 \leq X \leq 7{:}15\}$

$B = \{7{:}15 \leq X \leq 7{:}30\}$



$$f_Y(y) = \mathbb{P}(A) f_{Y|A}(y) + \mathbb{P}(B) f_{Y|B}(y)$$

$$f_Y(y) = \frac{1}{4}\frac{1}{5} + \frac{3}{4}\frac{1}{15} = \frac{1}{10}, \quad \text{for } 0 \leq y \leq 5$$

$$f_Y(y) = \frac{1}{4}0 + \frac{3}{4}\frac{1}{15} = \frac{1}{20}, \quad \text{for } 5 < y \leq 15$$

Continuous: Conditional PDF given a RV

- $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- Similarly, for $f_Y(y) > 0$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Remember: For a fixed event A, $\mathbb{P}(\cdot|A)$ is a legitimate probability law.
- Similarly, For a fixed y, $f_{X|Y}(x|y)$ is a legitimate PDF, since

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx}{f_Y(y)} = 1$$

- **Multiplication rule.**

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

- **Total prob./exp. theorem.**

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y)dy$$

$$\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y)\mathbb{E}[X|Y = y]dy$$

- **Independence.**

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

Example: Stick-breaking (Ch 3. Prob 21)

KAIST EE

- Break a stick of length l twice
  - first break at $X \sim uniform[0.l]$
  - second break at $Y \sim uniform[0, X]$
- (Q) What is $\mathbb{E}[Y]$?
- Since Y depends on X, the total expectation theorem seems useful.

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} f_X(x)\mathbb{E}[Y|X = x]dx$$

- Using the TET,

$$\mathbb{E}[Y] = \int_0^l \frac{1}{l}\mathbb{E}[Y|X = x]dx = \int_0^l \frac{1}{l}\frac{x}{2}dx = \frac{l}{4}$$

- $f_X(x)$ and $f_{Y|X}(y|x)$ seems easy to compute. Thus,

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x) = \frac{1}{l} \cdot \frac{1}{x}$$

You can do many other things with the joint PDF.

Roadmap

KAIST EE

- ○ Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- ○ Summarizing a random variable: Expectation and Variance
- ○ Functions of a single random variable, Functions of multiple random variables
- ○ Conditioning for random variables, Independence for random variables
- ○ Continuous random variables
  - Normal, Uniform, Exponential, etc.
- ○ Bayes' rule for random variables

Bayes Rule for Continuous

KAIST EE

- X: state/cause/original value $\rightarrow$ Y: result/resulting action/noisy measurement
- Model: $\mathbb{P}(X)$ (prior) and $\mathbb{P}(Y|X)$ (cause $\rightarrow$ result)
- Inference: $\mathbb{P}(X|Y)$?

$$p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$$

$$p_{X|Y}(x|y) = \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)}$$

$$p_Y(y) = \sum_{x'} p_X(x')p_{Y|X}(y|x')$$

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x) = f_Y(y)f_{X|Y}(x|y)$$

$$f_{X|Y}(x|y) = \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_Y(y) = \int f_X(x')f_{Y|X}(y|x')dx'$$

Bayes Rule for Mixed Case

KAIST EE

K: discrete, Y: continuous

- Inference of K given Y

$$p_{K|Y}(k|y) = \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)}$$

$$f_Y(y) = \sum_{k'} p_K(k')f_{Y|K}(y|k')$$

- Inference of Y given K

$$f_{Y|K}(y|k) = \frac{f_Y(y)p_{K|Y}(k|y)}{p_K(k)}$$

$$p_K(k) = \int f_Y(y')p_{K|Y}(k|y')dy'$$

Example: Signal Detection (1)

Inference of discrete K given continuous Y:

$$p_{K|Y}(k|y) = \frac{p_K(k) f_{Y|K}(y|k)}{f_Y(y)}, \quad f_Y(y) = \sum_{k'} p_K(k') f_{Y|K}(y|k')$$

- K: -1, +1, original signal, equally likely. $p_K(1) = 1/2, p_K(-1) = 1/2$.
- Y: measured signal with Gaussian noise, $Y = K + W$, $W \sim N(0,1)$

- Your received signal = 0.7. What's your guess about the original signal? +1
- Your received signal = -0.2. What's your guess about the original signal? -1

Example: Signal Detection (2)

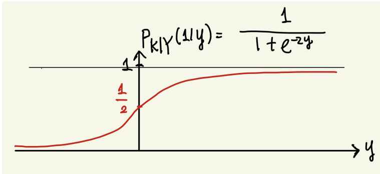
- $Y|K = 1 \sim N(1,1)$ and $Y|K = -1 \sim N(-1,1)$.

$$f_{Y|K}(y|k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}, \quad k = 1, -1$$

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2}$$

- Probability that $K = 1$, given $Y = y$? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$



Questions?

Review Questions

1) What is PDF and CDF?
2) Why do we need CDF?
3) What are joint/marginal/conditional PDFs?
4) Explain memorylessness of exponential random variables.
5) Explain the version of Bayes' rule for continuous and mixed random variables.