Lecture 8: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

- A lot of applications in engineering systems

- Basics on Random Process

- Bernoulli Process

- Poisson Process

- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split

- Markov Chain

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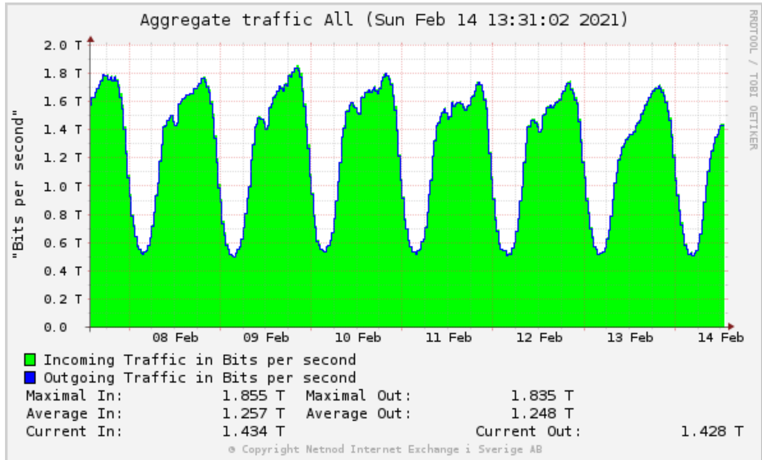
Things that evolve in time

- Many probabilistic experiments that evolve in time
    - Sequence of daily prices of a stock
    - Sequence of scores in football
    - Sequence of failure times of a machine
    - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



Random Process: A Sneak Peek

- A random process is a sequence of random variables indexed by time.
- Time: discrete or continuous
- Notation
  - $(X_t)_{t \in \mathcal{T}}$ or $\Big(X(t)\Big)_{t \in \mathcal{T}}$, where $\mathcal{T} = \mathbb{R}$ (continuous) or $\mathcal{T} = \{0, 1, 2, \ldots\}$ (discrete)
  - For the discrete case, we also often use $(X_n)_{n \in \mathbb{Z}_+}$.
  - We will use all of them, unless confusion arises.
- For a fixed time t, X_t is a random variable.
- The values that X_t can take: discrete or continuous

4 Types of Random Processes

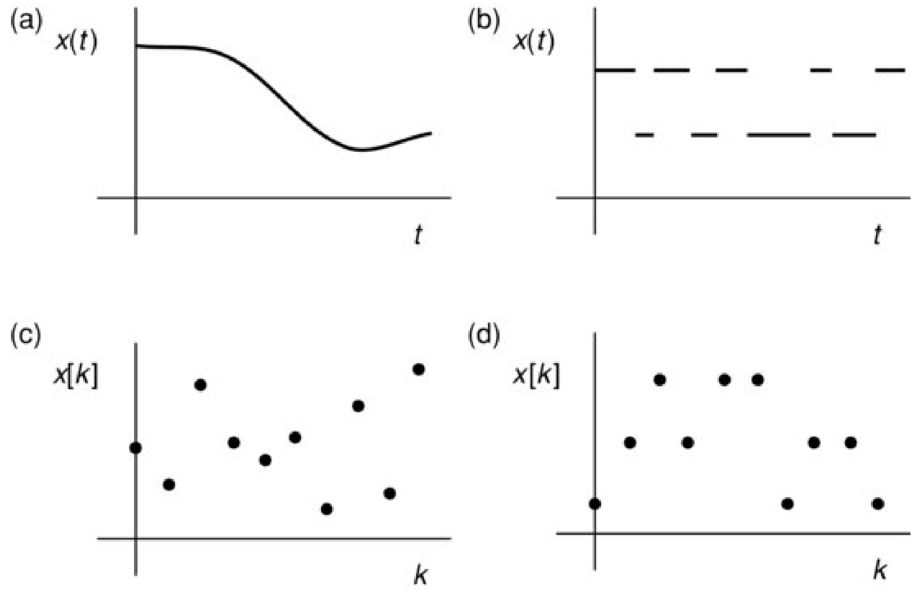
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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Bernoulli Process

- At each minute, we toss a coin with probability of head $0 < p < 1$.
  - Sequence of lottery wins/looses
  - Customers (each second) to a bank
  - Clicks (at each time slot) to server
- A sequence of independent Bernoulli trials $X_1, X_2, \ldots,$
  - We call index 1, 2, ... time slots (or simply slots)

|0 |0 |1 |0 |0 |0 |0 |1 |0 |1 |1 |0 |0 |

- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

Bernoulli Process as a Random Process

- Question. We've already studied a sequence of Bernoulli rvs $X_1, X_2, \ldots, X_n$. What's the difference?
- Physical difference: infinite sequence of $X_1, X_2, \ldots, .$
  - Sample space? set of all outcomes?
  - an outcome: an infinite sequence of sample values $x_1, x_2, \ldots,$ e.g., (0,1,1,0,0,1, ...)
- Semantic difference: Understand i in X_i as time. Also, interesting questions from the random process point of view.
  - Dependence: How $X_1, X_2, \ldots$ are related to each other as a time series
  - Long-term behavior: What is the fraction of times that a machine is idle?
  - Other interesting questions, depending on the target random process
- Next: Key questions and answers about Bernoulli process

(Q1) # of arrivals in n slots?

- $S_n = X_1 + X_2 + \cdots + X_n$
- $S_n \sim Bin(n, p)$
- $\mathbb{E}(S_n) = np$, $\text{var}(S_n) = np(1-p)$

(Q2) # of slots T_1 until the first arrival?

- $T_1 \sim Geom(p)$
- $\mathbb{E}(T_1) = 1/p$, $\text{var}(T_1) = \frac{1-p}{p^2}$

- T_1 is geometric? Memoryless
- Conditioned no first arrival at some time, what's the distribution of the remaining time until the first arrival?
- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

- Yes
- Because X_is are independent

(Q4) The process $(X_n)_{n=6}^{\infty}$?

- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

Fresh-start after Random N (1)

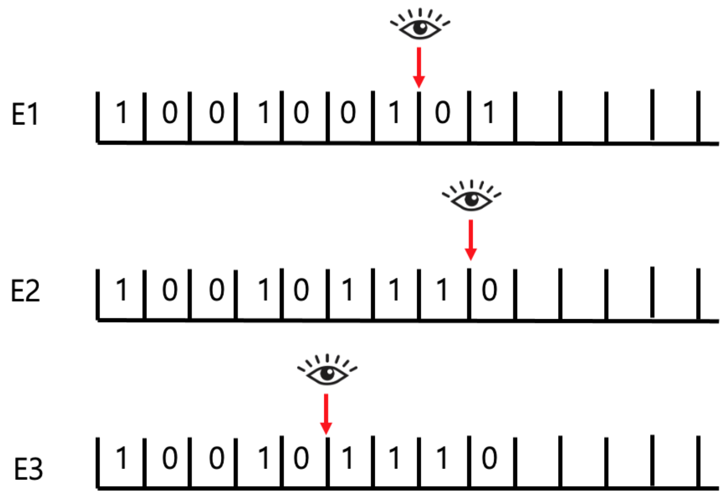
(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

- Examples of N

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



- Difference of N from n
  - The time when I watch the on-going Bernoulli process is random.

Fresh-start after Random N (2)

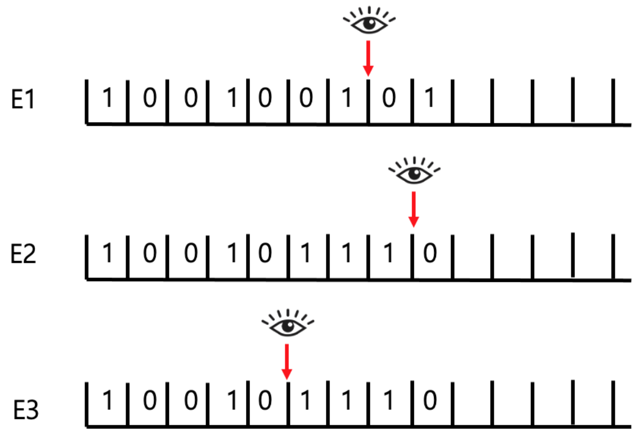
(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

- Examples of N

E1. Time of 3rd arrival

E2. First time when 3 consecutive arrivals have been observed

E3. Time just before 3 consecutive arrivals



E1. When I watch the process, N has been already determined. Yes

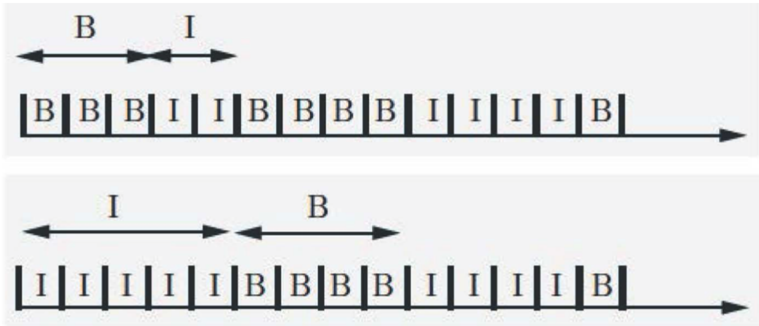
E2. Same as **E1.** Yes

E3. Need the future knowledge. '111' does not become random. No

- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

Distribution of Busy Periods

- Regard an arrival as business of a server
- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot



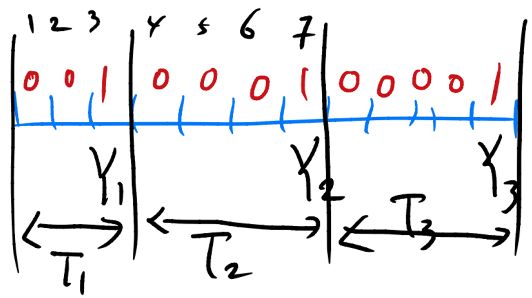
- (Q6) Distribution of B_1?
- N: time of the first busy slot. Fresh-start after N.
- B_1 is geometric with parameter $(1 - p)$
- Question: What about the second busy period B_2? B_3, B_4?

Time of k-th arrival

- Time of the first arrival $Y_1 \sim geom(p)$

  (Q7) Time of the k-th arrival Y_k?

- $T_k = Y_k - Y_{k-1}$: k-th inter-arrival ($k \geq 2,\ T_1 = Y_1$)
- $Y_k = T_1 + T_2 + \ldots + T_k$.



- After each T_k, the fresh-start occurs.
- $\{T_i\}$ are i.i.d. and $\sim geom(p)$
- $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$

PMF of Y_k

- $Y_k = T_1 + T_2 + \ldots + T_k$.
- $\{T_i\}$ are i.i.d. and $\sim geom(p)$

$$
\begin{aligned}
\mathbb{P}(Y_k = t) &= \mathbb{P}\Big(X_k = 1 \text{ and } k-1 \text{ arrivals during the first } t-1 \text{ slots}\Big) \\
&= \mathbb{P}\Big(X_k = 1\Big) \cdot \mathbb{P}\Big(k-1 \text{ arrivals during the first } t-1 \text{ slots}\Big) \\
&= p \times \binom{t-c}{k-1} p^{k-1}(1-p)^{t-k} = \binom{t-c}{k-1} p^{k}(1-p)^{t-k}, \quad t = k, k+1, \ldots
\end{aligned}
$$

- Y_k is called Pascal rv with parameter (k, p).
- $Pascal(1, p) = Geometric(p)$

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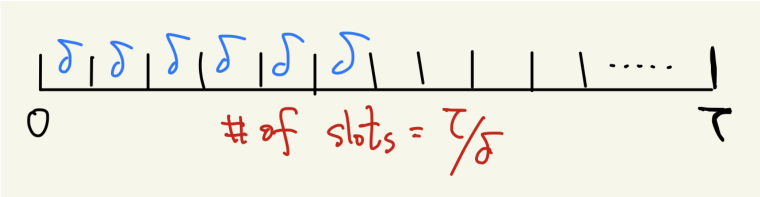
Design of Continuous Analog of Bernoulli Process

- Very useful to both continuous and discrete random processes that are "twins" and share the key properties.
  - Independence between what happens in a different time region
  - Memoryless and fresh-start property
- Choose one of discrete or continuous versions for our modeling convenience.
- Question. How do we design the continuous analog of Bernoulli process?
  - Key idea: Making it as a limiting system of a sequence of Bernoulli processes
- Need a "modeling sense" to make this possible. It's a good practice for engineers!

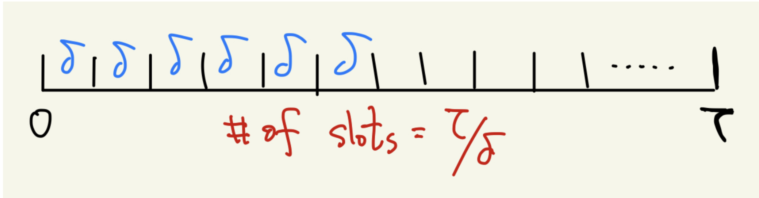
Key Design Idea to Develop a Continuous Twin (1)

- Continuous twin
  - Key point: Understand the number of arrivals over a given interval $[0, \tau]$.
  - Assume that it has some arrival rate λ ($\#$ of arrivals/unit time).
  - We know how to handle Bernoulli process with discrete time slots.
- Divide $[0, \tau]$ into slots whose length $= \delta$. Then, $n = \#$ of slots $= \frac{\tau}{\delta}$.



- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)

Key Design Idea to Develop a Continuous Twin (2)



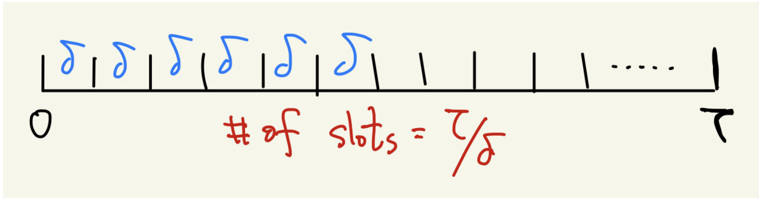
- Now, our design idea: during one time slot of length δ,

$$
\begin{aligned}
\mathbb{P}(1 \text{ arrival}) &\propto \text{arrival rate and slot length} \\
\mathbb{P}(\geq 2 \text{ arrivals}) &\propto \text{something, but very small} \\
&\quad\ \text{for small sloth length} \\
\mathbb{P}(0 \text{ arrival}) &= 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(1 \text{ arrival}) &= \lambda\delta + o(\delta) \\
\mathbb{P}(\geq 2 \text{ arrivals}) &= 0 o(\delta) \\
\mathbb{P}(0 \text{ arrival}) &= 1 - \lambda\delta + o(\delta)
\end{aligned}
$$

- $o(\delta)$: some function that goes to zero faster than δ goes to zero.
  - Thus, for very small δ, $o(\delta)$ becomes negligible.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

Key Design Idea to Develop a Continuous Twin (3)



- Our interest: Prob. of k arrivals over $[0, \tau]$
- Given "small" δ, # of arrivals $\sim$ *Binomial*(n, p), where $n = \tau/\delta$ and $p = \lambda\delta$
- As $\delta \to \infty$, $np = \tau/\delta \times \lambda\delta = \lambda\tau$.
- # of arrivals over $[0, \tau]$, $\sim$ *Poisson*$(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call Poisson process.

Poisson Process: Formalism

- N_s: number of arrivals over the interval $[0, s]$.
- (Independence) If $s < t$, the number $N_t - N_s$ of arrivals over $[s, t]$ is independent of the times of arrivals during $[0, s]$.
  - Thus, N_s can be a random variable over any interval of length s.
- (Small interval probability) The probabilities $\mathbb{P}(k, s)$ satisfy:

$$\mathbb{P}(0, s) = 1 - \lambda\tau + o(s)$$
$$\mathbb{P}(1, s) = \lambda s + o_1(s)$$
$$\mathbb{P}(k, s) = o_k(s) \quad \text{for } k = 2, 3, \ldots,$$

where

$$\lim_{s \to 0} \frac{o(s)}{s} = 0, \quad \lim_{s \to 0} \frac{o_k(s)}{s} = 0$$

Poisson Process: $\mathbb{P}(k,\tau)$, N_τ, and T

- (Q1) Number of arrivals of any interval with length $\tau \sim \textit{Poisson}(\lambda\tau)$, i.e.,

  $$\mathbb{P}(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

- $\mathbb{E}[N_\tau] = \lambda\tau$ and $\text{var}[N_\tau] = \lambda\tau$
- (Q2) Time of first arrival T

  $$F_T(t) = \mathbb{P}(T \le t) = 1 - \mathbb{P}(T > t) = 1 - P(0,t) = 1 - e^{-\lambda t}$$

  $$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \quad t \ge 0.$$

- $T \sim \textit{expo}(\lambda)$. Thus $\mathbb{E}[T] = 1/\lambda$ and $\text{var}[T] = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages per hour
- Mean and variance of mails received during a day
  - 5* 24 = 120
- $\mathbb{P}$[one new message in the next hour]
  - $\mathbb{P}(1, 1) = \frac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$
- $\mathbb{P}$[exactly two msgs during each of the next three hours]
  - $\left(\frac{5^2 e^{-5}}{2!}\right)^3$

Memoryless and Fresh-start Property

- Remind. Similar property for Bernoulli processes, but here no time slots.
- Fresh-start at determinsitic time: Start watching at time t, then you see the Poisson process, independent of what has happened in the past.
- Fresh-start at random time: Similarly holds. For example, when you start watching at random time T_1 (time of first arrival)
- (Q3) The k-th arrival time Y_k?
- k-th inter-arrival time $T_k = Y_k = Y_{k-1}$, $k \geq 2$, and $T_1 = Y_1$.
- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$

PDF of Y_k

- For a given δ, $\boxed{\delta \cdot f_{Y_k}(y)}$: prob of k-th arrival over $[y, y+\delta]$.
- When δ is small, only one arrival occurs. Thus,

$$\delta \cdot f_{Y_k}(y) = \mathbb{P}\Big(\text{an arrival over } [y, y+\delta]\Big) \times \mathbb{P}\Big(k-1 \text{ arrivals before } y\Big)$$

$$\approx \lambda\delta \times \mathbb{P}(k-1, y) = \lambda\delta \times \frac{\lambda^{k-1} y^{k-1} e^{-\lambda y}}{(k-1)!}$$

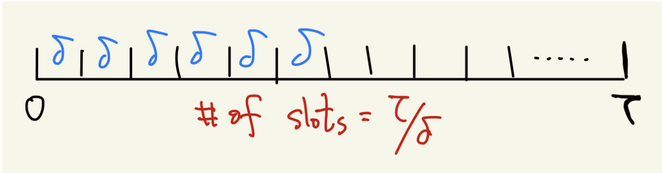
$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0.$$

- This is called Erlang rv.

- Time of first arrival: geometric / exponential
- Time of k-th arrivals: Pascal / Erlang

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6$/hour.

- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

(Q1) $\mathbb{P}$(fishing time $>$ 2 hours)

Method 1: $\mathbb{P}(0,2)$
Method 2: $\mathbb{P}(T_1 > 2)$

(Q2) $\mathbb{P}(\ 2 <$ fishing time $< 5)$

Method 1: $\mathbb{P}(0,2)(1 - \mathbb{P}(0,3))$
Method 2: $\mathbb{P}(2 < T_1 < 5)$

(Q3) $\mathbb{P}$(Catch at least two fish)

Method 1: $\sum_{k=2}^{\infty} = 1 - \mathbb{P}(0,2) - \mathbb{P}(1,2)$
Method 2: $\mathbb{P}(Y_k \leq 2)$

(Q4) $\mathbb{E}$[future fi. time|already fished for 3h]
Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

(Q5) $\mathbb{E}$[F=total fishing time]

$$\begin{aligned} 2 + \mathbb{E}[F-2] &= 2 + \mathbb{P}(F=2)\cdot 0 + \\ &\quad \mathbb{P}(F>2)\cdot \mathbb{E}[F-2|F>2] \\ &= 2 + \mathbb{P}(0,2)\cdot \frac{1}{\lambda} \end{aligned}$$

(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0,2)\cdot 1$

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Coding: Bernoulli Process and Poisson Process

- Inter-arrival times facilitates coding of both processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

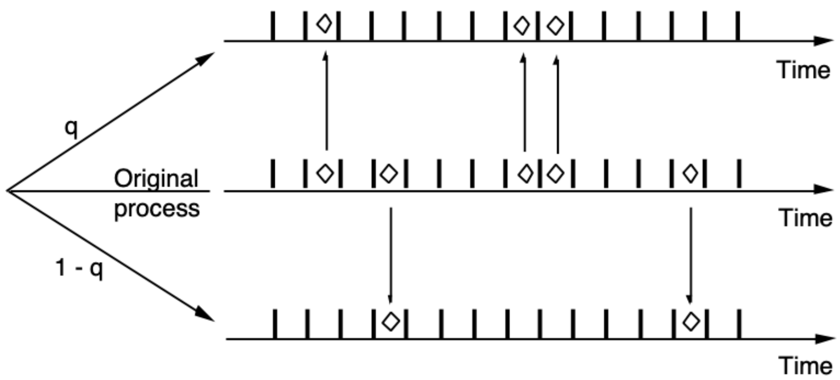
1. Start with a sequence of independent exponential random variables $T_1, T_2, \ldots$, with common parameter λ, and let these stand for the interarrival times.
2. Record an arrival at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Sum of Independent Poisson rvs

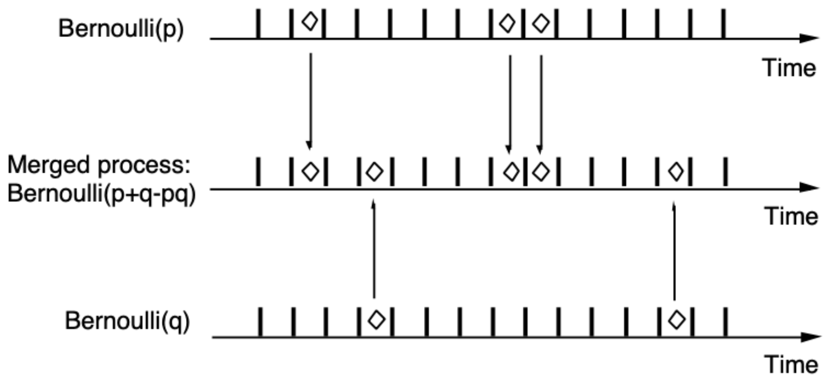
- $X \sim Poisson(\mu)$, $Y \sim Poisson(\nu)$,
- (Q1) $X \perp\!\!\!\perp Y$?
- (Q2) Distribution of $X + Y$?
  - Complex convolution, but any other easy way?
- X can be regarded as the number of arrivals of Poisson process with rate 1 over the time interval of length μ.
- Consecutive intervals of length μ and ν
- (Q1) $X \perp\!\!\!\perp Y$? Yes
- (Q2) Distribution of $X + Y$? $Poisson(\mu + \nu)$

Split and Merge: Bernoulli Process

- Split *Bernoulli*(p) into two processes with biased coin of head probability q
- Split decisions are independent of arrivals
- Split processes: also Bernoulli processes
- *Bernoulli*(pq) and *Bernoulli*$(p(1-q))$



- Merge *Bernoulli*(p) and *Bernoulli*(q) into one.
- Collided arrival is regarded just one arrival in the merged process
- Merged process: *Bernoulli*$(1-(1-p)(1-q))$



Split and Merge: Poisson Process

- Split Poisson process (λ) into two processes
  - Split based on the coin tossing with probability of head p
  - Poisson process ($p\lambda$) and Poisson process ($(1-p)\lambda$)
- Merge from Poisson process (λ_1) and Poisson process (λ_2)
  - Split based on the coin tossing with probability of head p
  - Poisson process ($\lambda_1 + \lambda_2$)
  - Bernoulli process of small interval δ

$$\mathbb{P}(0 \text{ arrivals in the merged process}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) = 1-(\lambda_1+\lambda_2)\delta + o(\delta)$$
$$\mathbb{P}(1 \text{ arrivals in the merged process}) \approx \lambda_1\delta(1-\lambda_2\delta) + \lambda_2\delta(1-\lambda_1\delta) = (\lambda_1+\lambda_2)\delta + o(\delta)$$

Competing Exponential

1. Two independent light bulbs have life times T_a and T_b of exponential distributions with λ_a and λ_b.

- (Q) Distribution of $Z = \min\{T_a, T_b\}$?

- T_a and T_b are the first arrival times of two Poisson processes of λ_a and λ_b.
- Z is the first arrival time of merged Poisson process $(\lambda_a + \lambda_b)$.
- Thus, $Z \sim exp(\lambda_a + \lambda_b)$

2. Three independent light bulbs have life times T of exponential distribution with λ.

- (Q) $\mathbb{E}$[time until the last bulb burns out]?

- Poisson process$(3\lambda) \xrightarrow{\text{1st burn out}}$ Poisson process$(2\lambda) \xrightarrow{\text{2st burn out}}$ Poisson process(λ)
- T_1: time until the first burn-out, T_2: time until the second burn-out, T_3: time until the third burn-out
- $T_1 \sim exp(3\lambda)$, $T_2 \sim exp(2\lambda)$, $T_3 \sim exp(\lambda)$

$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Questions?

Review Questions

1)