

## Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes  
KAIST EE

MONTH DAY, 2021

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

- Assume discrete times  $n = 1, 2, \dots$
- Random process: A sequence of  $X_1, X_2, X_3, \dots$

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- **Markov chain**
- One of the most popular random processes in engineering

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p.  $b$ , and continue working w.p.  $1 - b$ .
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$$\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$
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- What will happen at  $(n + 1)$ -th day depends only on what happens at  $n$ -th day?



- **Definition.** Let  $X_1, \dots, X_n, \dots$  be a sequence of random variables taking values in some finite space  $\mathcal{S} = \{1, 2, \dots, m\}$ , such that for all  $i, j \in \mathcal{S}$ ,  $n \geq 0$ , the following **Markov property** is satisfied:

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Thus, for any  $n \geq 0$ , we introduce a simple notation  $p_{ij}$

$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$



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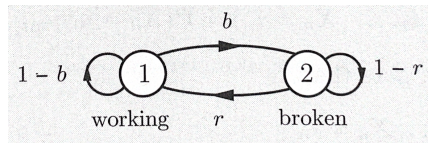
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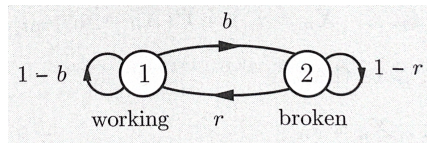
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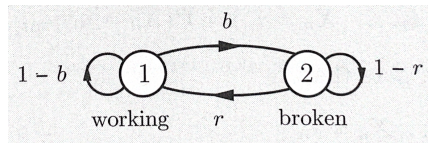
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- $\sum_{j=1}^m p_{ij} = 1$  (for each row  $i$ , the column sum = 1)



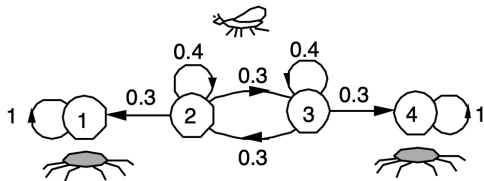
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	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$P_{ij}$

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$$\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} = \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2$$

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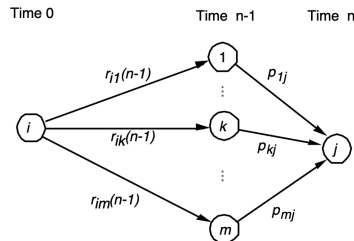
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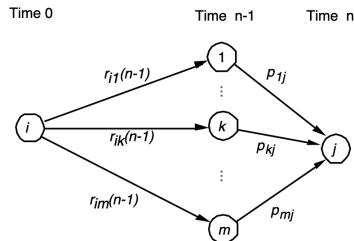
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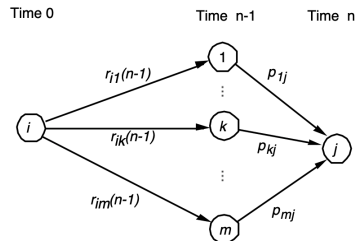
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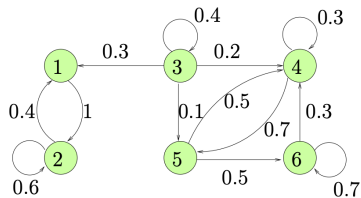
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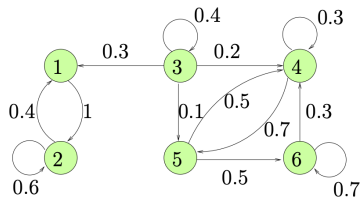


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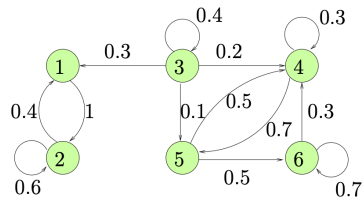
- Classes



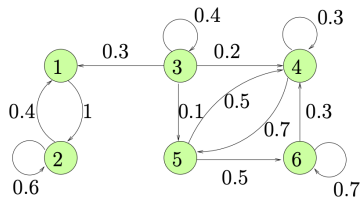
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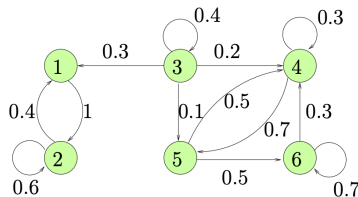
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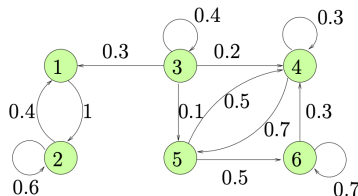
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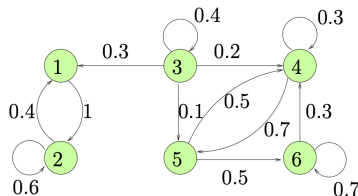
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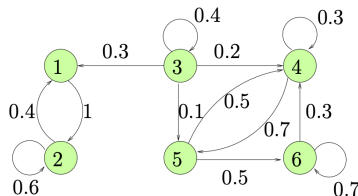


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- Difference between 1 and 3

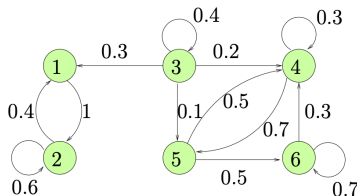




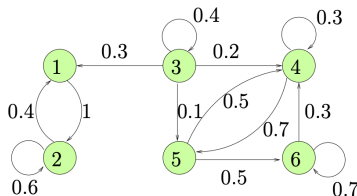
- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
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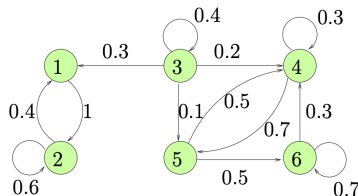


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  - **Insight 2.** Some states are visited infinite times, but some states are not.



- Classes

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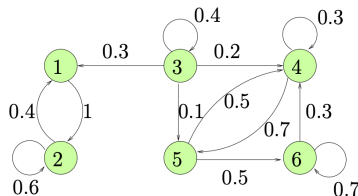


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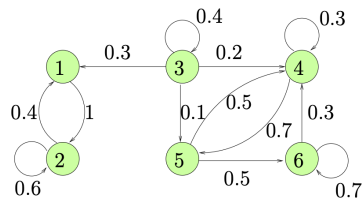
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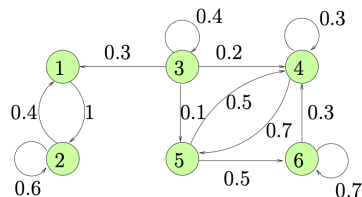
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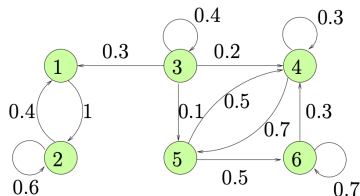
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- State 2 will share the above properties with 1 (similarly, 4, 5, and 6)
- **Insight 3.** States in the same class share some properties.



- **Definition.** State  $j$  is **accessible** from state  $i$ , if for some  $n$   $r_{ij}(n) > 0$ .

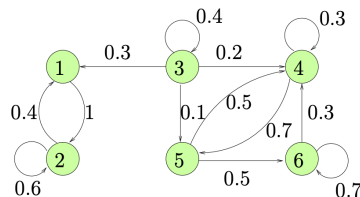


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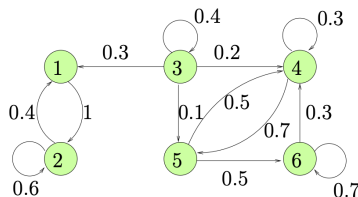




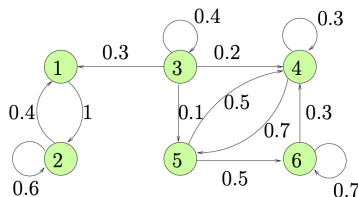
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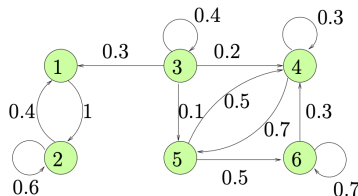
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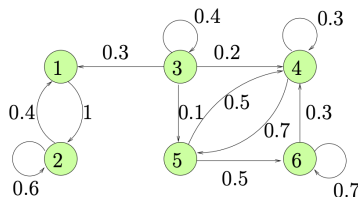
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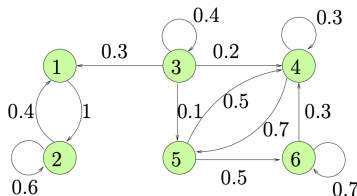
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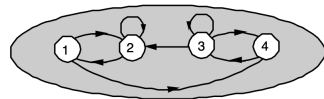


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  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state  $i$ , then there is always some probability of returning to  $i$ . It means that, given enough time, it is certain that it returns to  $i$ .

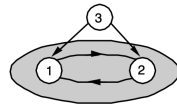


## Classification of States (2)

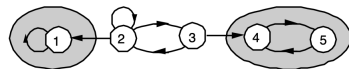
- A set of recurrent states which communicate with each other form a **class**.



Single class of recurrent states



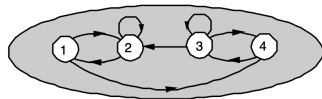
Single class of recurrent states (1 and 2)  
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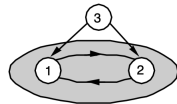
Two classes of recurrent states  
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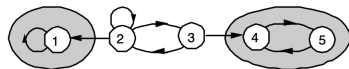
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  - A MC can be decomposed into one or more recurrent classes, plus possibly some transient states.



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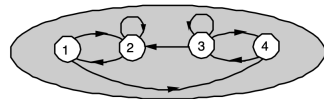
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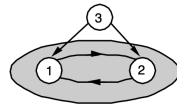
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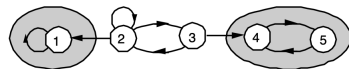
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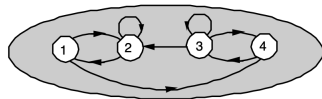


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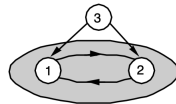


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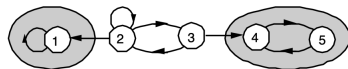
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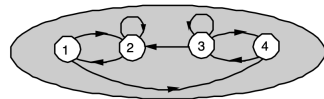


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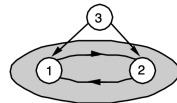


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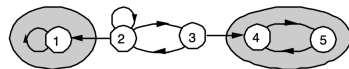
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Single class of recurrent states

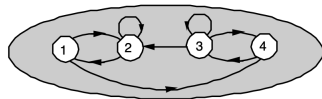


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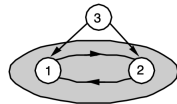


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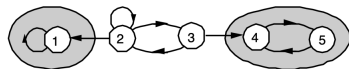
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- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



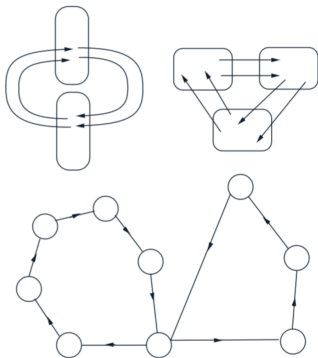
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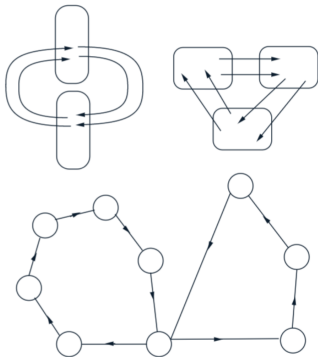
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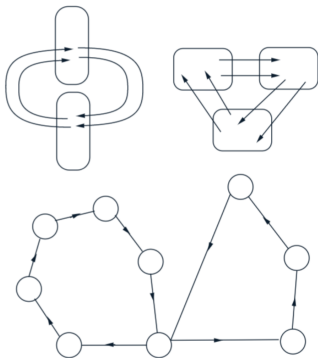
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- The states in a recurrent class are periodic if they can be grouped into  $d > 1$  groups so that all transitions from one group lead to the next group.



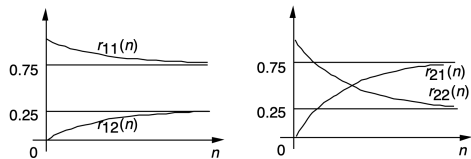
- The states in a recurrent class are periodic if they can be grouped into  $d > 1$  groups so that all transitions from one group lead to the next group.
- A recurrent class that is not periodic is said to be aperiodic.



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - **Steady-state Behaviors and Stationary Distribution**
  - Transient Behaviors



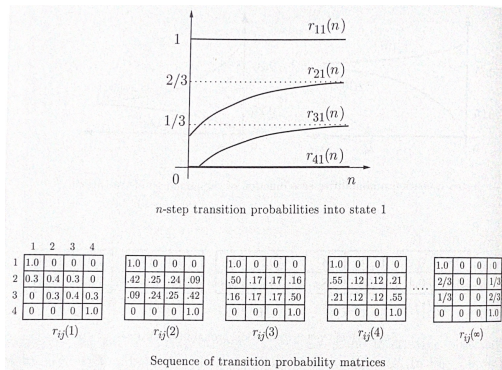
# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$



$n$ -step transition probabilities as a function of the number  $n$  of transitions

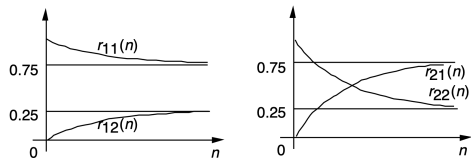
	UpD	B						
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$	
							$r_{ij}(5)$	

Sequence of  $n$ -step transition probability matrices



# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$

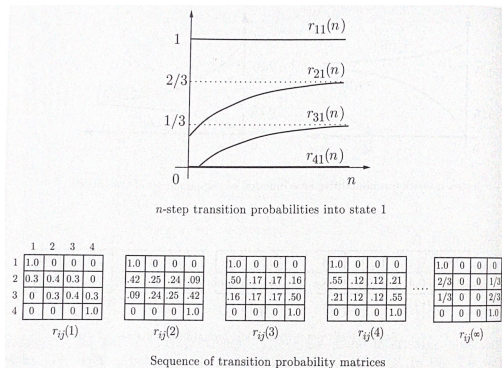
- Convergence irrespective of the starting state



$n$ -step transition probabilities as a function of the number  $n$  of transitions

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	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$	
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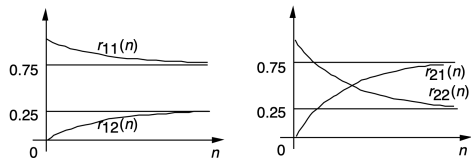
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Sequence of transition probability matrices

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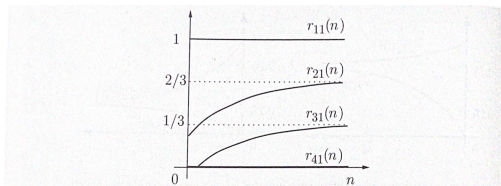


$n$ -step transition probabilities as a function of the number  $n$  of transitions

	UpD	B								
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496	.7501	.2499
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512	.7498	.2502
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(5)$	

Sequence of  $n$ -step transition probability matrices

- Convergence depending on the starting state



$n$ -step transition probabilities into state 1

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
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2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

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1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$			

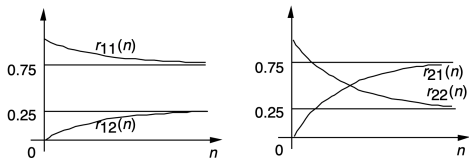
....

	1	2	3	4												
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21

Sequence of transition probability matrices

# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$

- Convergence irrespective of the starting state

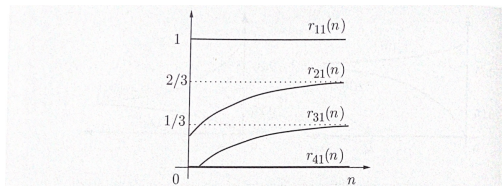


$n$ -step transition probabilities as a function of the number  $n$  of transitions

	UpD	B								
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496	.7501	.2499
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512	.7498	.2502
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(5)$	

Sequence of  $n$ -step transition probability matrices

- Convergence depending on the starting state



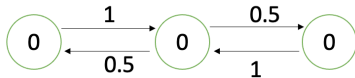
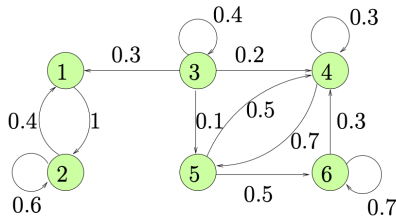
$n$ -step transition probabilities into state 1

	1	2	3	4													
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.55	.12	.12	.21	2/3	0	0	1/3	
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.21	.12	.12	.55	1/3	0	0	2/3	
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$				$r_{ij}(\infty)$

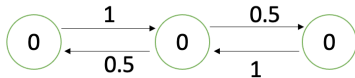
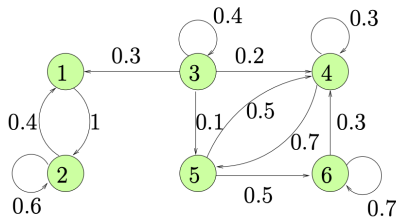
Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?

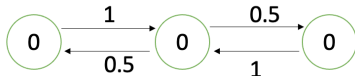
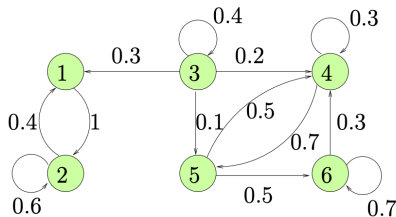


- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Convergence occurs, independent of the starting state, if:



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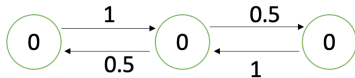
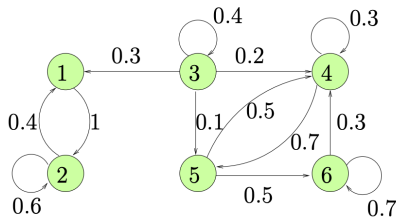
**C1.** Only a single recurrent class



- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
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**C1.** Only a single recurrent class

**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.



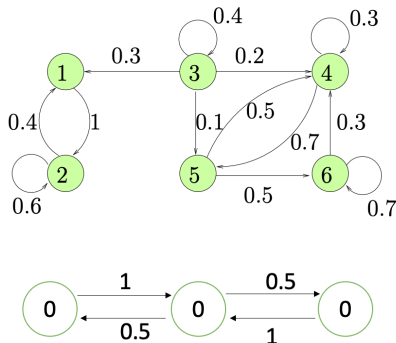


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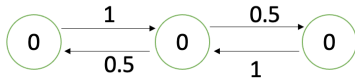
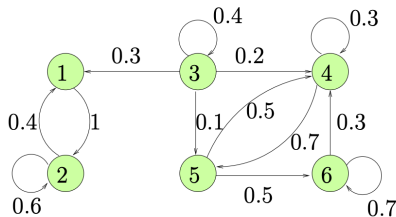
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**C1.** Only a single recurrent class

**C2.** such recurrent class is aperiodic

**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.

**C2.** Divergent behavior for periodic recurrent classes.



- If  $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ,

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies$$

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$$\sum_{i=1}^m \pi_i = 1$$

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- Normalization equation

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- Balance equation + Normalization equation  $\implies$  Finding the steady-state probabilities  $\{\pi_i\}$ .

- A two-state MC with:

$$p_{11} = 0.8, \quad p_{12} = 0.2,$$

$$p_{21} = 0.6, \quad p_{22} = 0.4.$$

- Balance equation:

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$$

- Normalization equation:  $\pi_1 + \pi_2 = 1$
- The stationary distribution is:  $\pi_1 = 0.25, \pi_2 = 0.75$ .





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- If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).

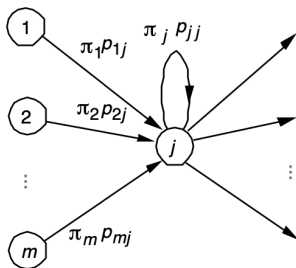


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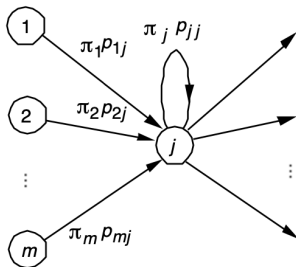
$$\mathbb{P}(X_0 = j) = \pi_j, \quad j = 1, \dots, m \implies \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Then,  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .
  - If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution is equal to the stationary distribution"

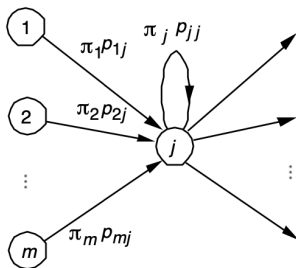
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  - The expected frequency  $\pi_j$  of visits to  $j$  is equal to the sum of the expected frequencies  $\pi_k p_{kj}$  of transitions that lead to  $j$ .



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - **Transient Behaviors**

# Absorption Probability

- **Definition.** A state  $k$  is **absorbing**, if  $p_{kk} = 1$ , and  $p_{kj} = 0$  for all  $j \neq k$ .

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<sup>0</sup>The notation  $a_i$  should have dependence on  $s$ , but we omit it for simplicity.

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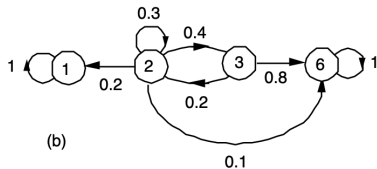
(Q) For a fixed absorbing state  $s$ , the probability  $a_i$  of reaching  $s$ , starting from a transient state  $i$ ?

- Fix  $s = 6$ .

$$a_1 = 0, \quad a_6 = 1$$

$$a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6$$

$$a_3 = 0.2a_2 + 0.8a_6$$



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# Absorption Probability

- Definition.** A state  $k$  is **absorbing**, if  $p_{kk} = 1$ , and  $p_{kj} = 0$  for all  $j \neq k$ .  
- states 1 and 6 are absorbing

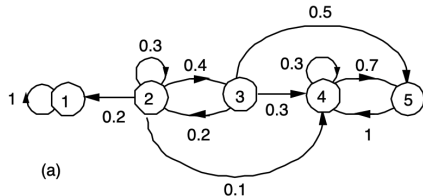
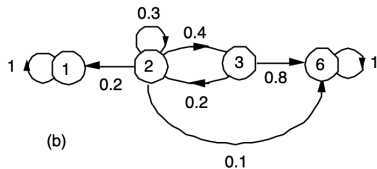
(Q) For a fixed absorbing state  $s$ , the probability  $a_i$  of reaching  $s$ , starting from a transient state  $i$ ?

- Fix  $s = 6$ .

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(Q) What if there are some non-absorbing recurrent state?

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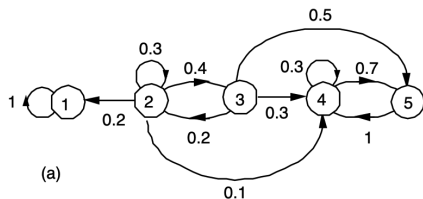
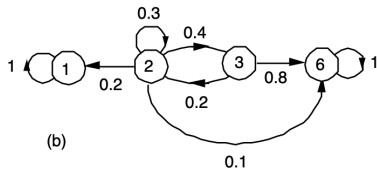
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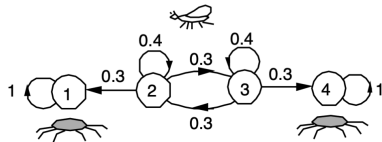


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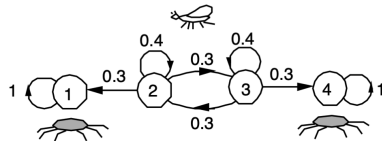
- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

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(Q) Starting from a transient state  $i$ , expected number of transitions  $\mu_i$  until absorption to any absorbing state?



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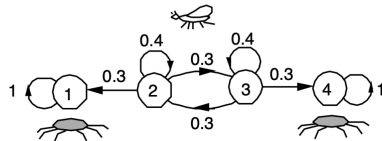


- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

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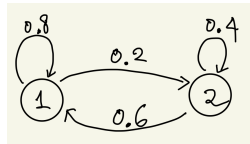
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- For generalized description, please see the textbook (pp. 367).

# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

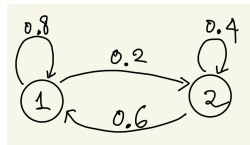


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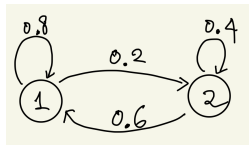
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$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

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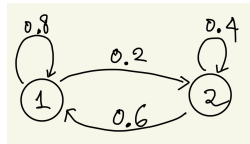
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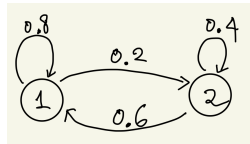
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$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

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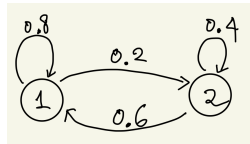
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- For generalized description, please see the textbook (pp. 368)

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Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?