Lecture 9: Introduction to Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

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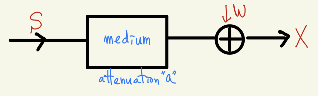

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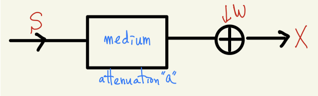

- Process of extracting information about an unknown variable or an unknown model from noisy available data

Statistical Inference: Three Main Ideas

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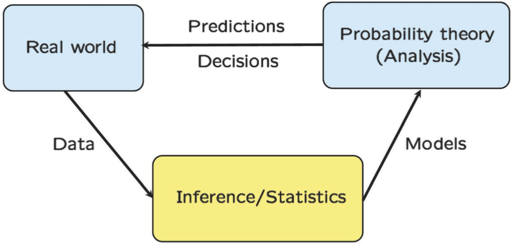
3. How to obtain samples has impact on inference (e.g., when we need to pay for online surveys)

Inference, Real World, Probability Theory



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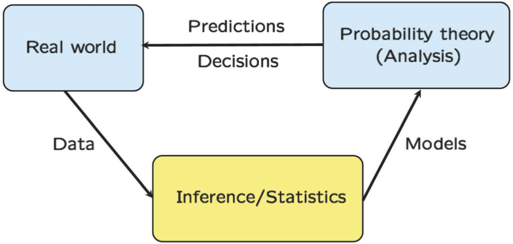

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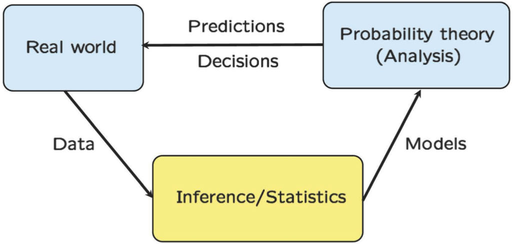

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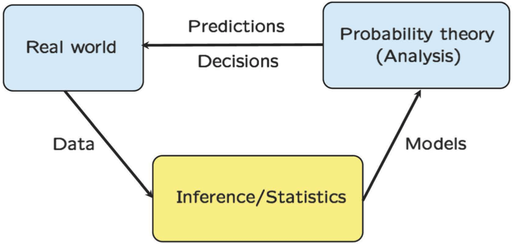

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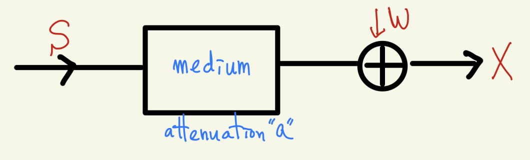
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building



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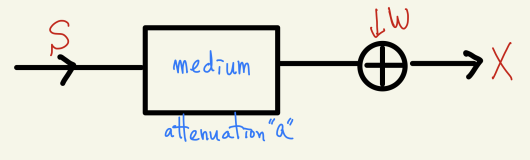
What to Infer?: Unknown Model vs. Unknown Variable



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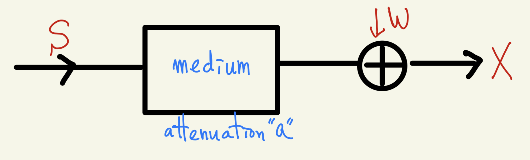

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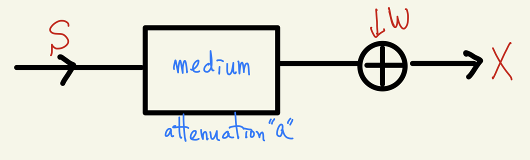

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- Same mathematical structure, because the parameters in models are variables in many cases

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  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

Different Views: Bayesian vs. Classical (1)

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$$\mathbb{P}\left[\theta = \frac{3}{4} \middle| (HHH)\right] = \frac{27}{28}, \ \mathbb{P}\left[\theta = \frac{1}{4} \middle| (HHH)\right] = \frac{1}{28}$$

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(Note) There are other inference methods, and here we just show examples.

Different Views: Bayesian vs. Classical (2)

Bayesian approach

Classical approach

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Different Views: Bayesian vs. Classical (3)

- Fundamental difference about the nature of unknown models or variables
- Random variable or deterministic quantity
- Who is the winner? A century-long debate
- Example of debate: mass of the electron by noisy measurement
  - Classical. while unknown, it is a constant and there is no justification for modeling it as a random variable.
  - Bayesian. Prior distribution reflects our state of knowledge, e.g., some range of candidate values from our previous noisy measurements.
- Particular prior? too arbitrary vs. every statistical procedure's hidden choices
- Pratical issues: Bayesian approach is often computationally intractable (multi-dimensional integrals)

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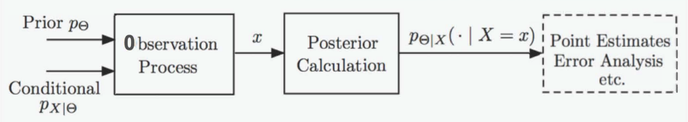

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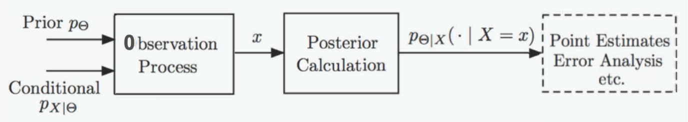

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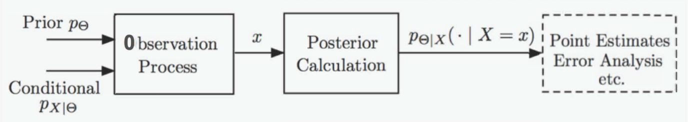

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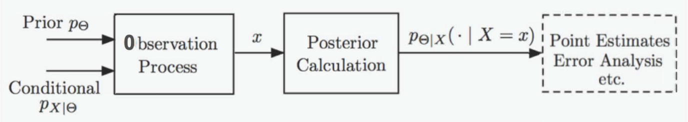

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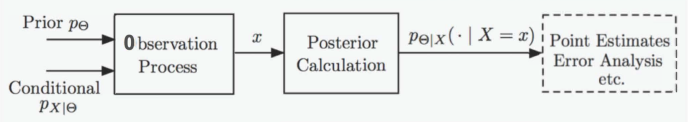

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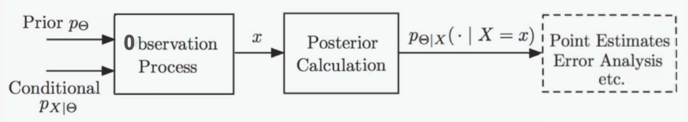

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Framework of Bayesian Inference

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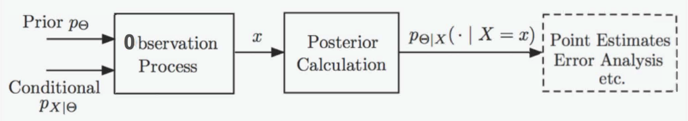

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- However, one may use it for further processing, depending on what he/she wants, e.g., point estimation.
- Multiple observations and multiple parameters are possible
  - $X = (X_1, \ldots, X_n)$
  - $\Theta = (\Theta_1, \ldots, \Theta_n)$

Remind: Bayes' Rule: 4 Versions

- Θ: discrete, X: discrete

$$p_{\Theta|X}(\theta|x) = \frac{p_\Theta(\theta)p_{X|\Theta}(x|\theta)}{p_X(x)}$$

$$p_X(x) = \sum_{\theta'} p_\Theta(\theta')p_{X|\Theta}(x|\theta')$$

- Θ: continuous, X: continuous

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- Prior

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- Prior and observation model (likelihood)

$$f_\Theta(\theta) = \begin{cases} 1, & 0 \le \theta \le 1 \\ 0, & \text{otherwise} \end{cases}, \qquad f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

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$$f_\Theta(\theta) = \begin{cases} 1, & 0 \le \theta \le 1 \\ 0, & \text{otherwise} \end{cases}, \qquad f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

- Posterior

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} =$$

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.
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$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} = \begin{cases} \frac{1/\theta}{\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta |\log x|}, & x \le \theta \le 1, \\ 0, & \theta < x \text{ or } \theta > 1 \end{cases}$$

◦ What happens if we have more observation samples?

- Romeo was late n times by $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $X_i \sim \mathcal{U}[0, \theta]$.
- $X_1, \ldots, X_n$ are conditionally independent, given $\Theta = \theta$.
- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.
- Observation: Romeo was late n times by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$
- See Example 8.2 at pp. 414 for more detailed treatment.

Example: Spam Filtering

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.

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- Posterior PMF

$$\mathbb{P}\Big[\Theta = m|(x_1, \ldots, x_n)\Big] = \frac{p_\Theta(m)\prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j)\prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

Example: Biased Coin with Beta Prior (1)

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- Question. Suppose that you have freedom to choose the form of the prior distribution. What prior will you choose? Requirement of "good" priors?
- We will look at the prior whose distribution is something called the Beta distribution.

Background: Beta Distribution

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Beta distribution

A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

$$f_\Theta(\theta) = \begin{cases} \theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise}, \end{cases}$$

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A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

$$f_\Theta(\theta) = \begin{cases} \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise}, \end{cases}$$

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- See `https://youtu.be/8yaRt24qA1M` for the integration in the Beta function formula.
- A special case of Beta$(1, 1)$ is $\mathcal{U}[0, 1]$

- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
- In other words, Beta prior $\Longrightarrow$ Beta posterior (why useful?)

Proof.

(a) First, the posterior pdf is given by:

$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k}, \quad c \text{ the normalizing constant}$$

(b) Next, for $\text{Beta}(\alpha, \beta)$ prior, $f_{\Theta}(\theta) = \dfrac{1}{B(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$.

(c) Then, $f_{\Theta|X}(\theta|k) = c \dbinom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k} = \dfrac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1}(1-\theta)^{\beta+n-k-1}$,

where $d = c\binom{n}{k}$.

Example: Parameter Inference with Normal Prior (1)

- Inference of a parameter θ

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- X: noisy observation of θ, modeled as:
  $X = \theta + W$, where $W \sim \mathcal{N}(0, \sigma^2)$

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X_1 &= \theta + W_1, \quad W_1 \sim \mathcal{N}(0, \sigma_1^2) \\
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Background: The PDF Form of Gaussian

Lemma. Up to recaling, the pdf of the form $e^{-\frac{1}{2}(ax^2-2bx+c)}$ is $\mathcal{N}(\frac{b}{a}, \frac{1}{a})$.

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  - Thus, $\sigma^2 = \dfrac{1}{a}$ and $\dfrac{\mu}{\sigma^2} = b \implies \mu = b\sigma^2 = \dfrac{b}{a}$

Background: Product of Two Gaussian Densities

Theorem. The product of two Gaussian pdfs $\mathcal{N}(\mu_0, \nu_0)$ and $\mathcal{N}(\mu_1, \nu_1)$ is $\mathcal{N}\left(\frac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \frac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$.

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Proof. Using the Lemma in the previous slide, i.e., up to recaling, the pdf of the form $e^{-\frac{1}{2}(ax^2-2bx+c)}$ is $\mathcal{N}(\frac{b}{a}, \frac{1}{a})$,

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$$\Longrightarrow \mathcal{N}\left(\nu\left(\frac{\mu_0}{\nu_0} + \frac{\mu_1}{\nu_1}\right), \overbrace{\frac{1}{\nu_0^{-1} + \nu_1^{-1}}}^{=\nu}\right) = \mathcal{N}\left(\frac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \frac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$$

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Example: Parameter Inference with Normal Prior (2)

- n observations of θ: $W_i \sim \mathcal{N}(0, \sigma_i^2)$, and θ with the normal prior $\Theta \sim \mathcal{N}(x_0, \sigma_0^2)$

$$\boxed{X_i = \theta + W_i, \quad W_i \sim \mathcal{N}(0, \sigma_i^2), \quad i = 1, \ldots, n}$$

- $\Theta, W_1, \ldots, W_n$ are indendent and let $X = (X_1, \ldots, X_n)$, $x = (x_1, \ldots, x_n)$.
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- Prior. $f_{\Theta}(\theta) = c_1 \cdot \exp\left\{-\frac{(\theta - x_0)^2}{2\sigma_0^2}\right\}$

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- Observation model. Noting that $X_1, X_2, \dots, X_n$ are indpendent,

$$f_{X|\Theta}(x|\theta) = c_2 \cdot \exp\left\{-\frac{(\theta - x_1)^2}{2\sigma_1^2}\right\} \cdots \exp\left\{-\frac{(\theta - x_n)^2}{2\sigma_n^2}\right\}$$

Example: Parameter Inference with Normal Prior (3)

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'}$$

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- Numerator: $f_\Theta(\theta) f_{X|\Theta}(x|\theta) = c_1 c_2 \cdot \exp\left\{-\sum_{i=0}^{n} \frac{(x_i-\theta)^2}{2\sigma_i^2}\right\}$, which can be reexpressed as the following, using the product of $n+1$ Gaussians:

$$c_1 c_2 \cdot \exp\left\{-\sum_{i=0}^{n} \frac{(x_i-\theta)^2}{2\sigma_i^2}\right\} = d \cdot \exp\left\{-\frac{(\theta-m)^2}{2v}\right\},$$

where $m = \dfrac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \dfrac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$

$$\boxed{f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'}}$$

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- Denominator: just a constant, not a function of θ

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'}$$

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Example: Parameter Inference with Normal Prior (4)

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- Prior: Normal, Posterior: Normal
- Special case when $\sigma^2 = \sigma_0^2 = \sigma_1^2 = \cdots = \sigma_n^2$. Then,

$$m = \frac{x_0 + x_1 + \ldots x_n}{n+1}, \qquad v = \frac{\sigma^2}{n+1}$$

Example: Parameter Inference with Normal Prior (4)

- Thus, the posterior pdf $f_{\Theta|X}(\theta|x) = \ a \cdot \exp\left\{-\frac{(\theta-m)^2}{2v}\right\}$, where

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- Prior: Normal, Posterior: Normal
- Special case when $\sigma^2 = \sigma_0^2 = \sigma_1^2 = \cdots = \sigma_n^2$. Then,

$$m = \frac{x_0 + x_1 + \ldots x_n}{n+1}, \qquad v = \frac{\sigma^2}{n+1}$$

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  - the standard deviation of the posterior goes to 0, at the rough rate of $1/\sqrt{n}$.

Why Prior == Posteror is Useful?

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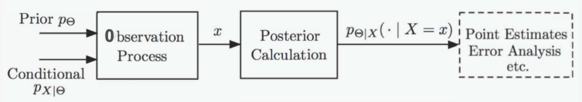
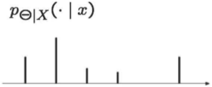
$$\text{mean} = \frac{(m/v) + (x_{n+1}/\sigma^2_{n+1})}{(1/v) + (1/\sigma^2_{n+1})}, \qquad \text{variance} = \frac{1}{(1/v) + (1/\sigma^2_{n+1})}$$

Roadmap

1. Overview on Statistical Inference
2. Bayesian Inference: Framework
3. Examples
4. MAP (Maximum A Posteriori) Estimator
5. LMS (Least Mean Squares) Estimator
6. LLMS (Linear LMS) Estimator
7. Classical Inference: ML Estimator


Prior p_Θ
Conditional $p_{X|\Theta}$
Observation Process
x
Posterior Calculation
$p_{\Theta|X}(\cdot \mid X = x)$
Point Estimates
Error Analysis
etc.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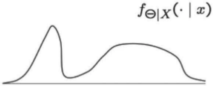
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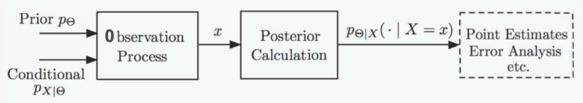
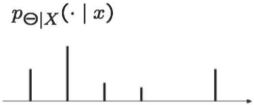
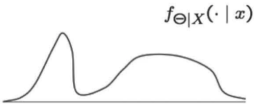


- Point Estimate
  - Given observation x, which single value θ are you going to choose as your inference result? People often want just the summary and a simple answer.
  - Very often, θ, our inference target, is by nature a single value, i.e., mass of the electron.

Two Natural Point Estimates





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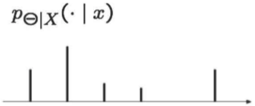

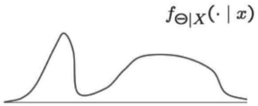

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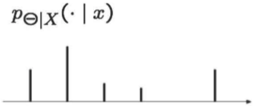

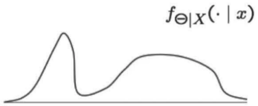

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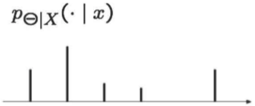

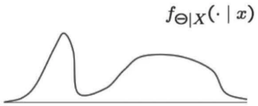

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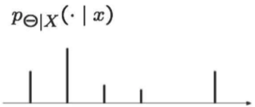

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- Why MAP and LMS are good? Not mathematically clear yet (We will discuss later)
- Notation: The community uses $\hat{\theta}$ to mean the estiamted value, i.e., hat for estimated value.

Estimator as a Function

- Random observation: X

- Observation instance: x

- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\mathsf{MAP}} = g_{\mathsf{MAP}}(x), \quad \hat{\theta}_{\mathsf{LMS}} = g_{\mathsf{LMS}}(x)$$

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From now on we focus on the MAP estimate, mainly based on the examples that we've discussed in the previous section.

Example: Romeo and Juliet

Slide 16 for more details

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.
- Observation: Romeo was late by x.
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- Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$. $\implies$ $\hat{\theta}_{\text{MAP}} = x$.

Example: Spam Filtering

Slide 18 for more details

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.
- $\{w_1, w_2, \ldots, w_n\}$: a collection of words which suggest "spam".
- For each i, a Bernoulli $X_i = 1$ if w_i appears and 0 otherwise.
- Assumption: Conditioned on Θ, X_i are independent.
- Posterior PMF

$$\mathbb{P}\left[\Theta = m | (x_1, \ldots, x_n)\right] = \frac{p_\Theta(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|1) > p_\Theta(2) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|2)$$

Example: Biased Coin with Beta Prior

Slide 21 for more details

- Biased coin with probability of head θ
- Unknown θ: modeled by Θ with some prior $f_\Theta(\theta)$
- Observation X: number of heads out of n tosses
  - If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
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- When $\alpha = \beta = 1$ (i.e., $\mathcal{U}[0,1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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Slide 27 for more details

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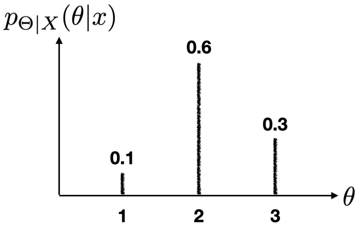

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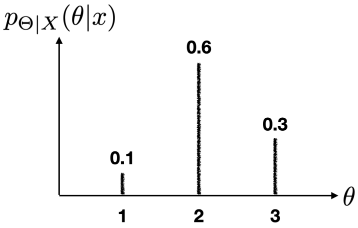

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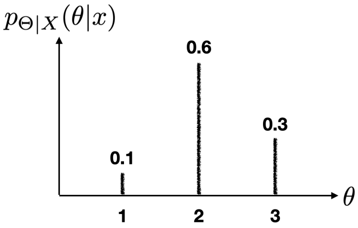

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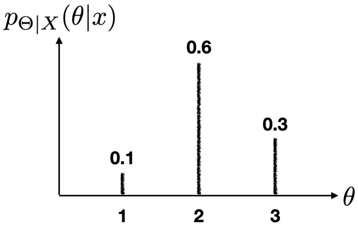

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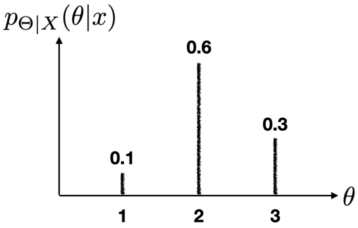

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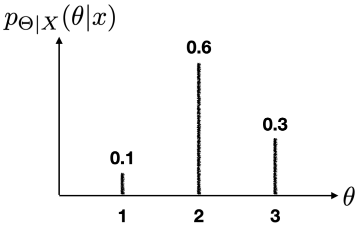

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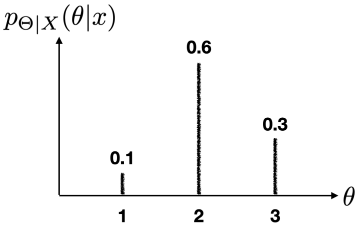

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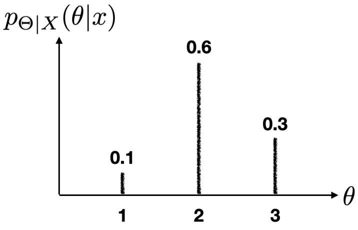

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$$\begin{aligned}\mathbb{P}(\hat{\Theta} \neq \Theta) &= \sum_x \mathbb{P}(\hat{\Theta} \neq \Theta | X = x) p_X(x) \\ &= \sum_x \mathbb{P}(\hat{\theta} \neq \Theta | X = x) p_X(x) \\ &\geq \sum_x \mathbb{P}(\hat{\theta}_{\mathsf{MAP}} \neq \Theta | X = x) p_X(x)\end{aligned}$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

Roadmap

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

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  - Averaged over X: $\mathbb{E}\left[(\Theta - \mathbb{E}[\Theta|X])^2\right] = \mathbb{E}\left[\mathrm{var}(\Theta|X)\right]$

Example: Romeo and Juliet

Slides 17 and 35 for more details

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
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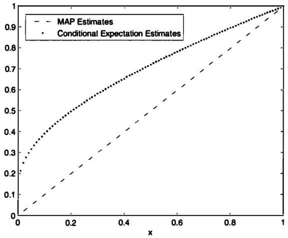

Example: Biased Coin with Beta Prior

Slides 21 and 37 for more details

- Biased coin with prob. of head θ. Unknown θ modeled by Θ with prior $f_\Theta(\theta)$.
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Example: Parameter Inference with Normal Prior

Slides 27 and 38 for more details

- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\frac{(\theta - m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

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Example: Signal Recovery from Noisy Measurement (1)

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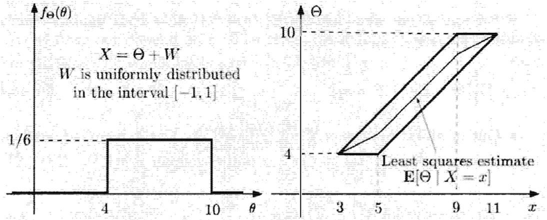

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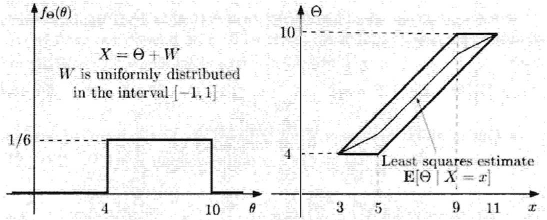

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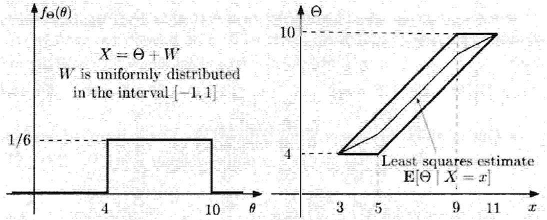
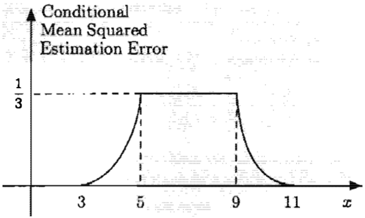
$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta|X=x]$: midpoint of the corresponding vertical section



- What is conditional MSE? $\mathbb{E}\Big[(\Theta - \mathbb{E}[\Theta|X=x])^2|X=x\Big]$
- Given $X=3$, it's the variance of $\mathcal{U}[4,4]=0$
- Given $X=5$, it's the variance of $\mathcal{U}[4,6]=(6-4)^2/12=1/3$
- The rising pattern between $X=3$ and $X=5$ is quadratic. This is because the expectation increases linearly, where the variance increases in a quadratic manner.



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- Any alternative to LMS estimator?

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- Linear models are always the first choice for a simple design in engineering.

KAIST EE

LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}\Big(X - \mathbb{E}(X)\Big) = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big),$$

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LLMS Estimator: Result and Interpretation

KAIST EE

LLMS

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◦ If $\rho > 0$:
- Baseline ($\mathbb{E}[\Theta]$) + correction term
- If $X > \mathbb{E}[X] \Longrightarrow \hat{\Theta}_L > \mathbb{E}[\Theta]$
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KAIST EE

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- If $\rho = 0$ (uncorrelated):
  - Just baseline ($\mathbb{E}[\Theta]$)
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data X

- MSE $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$?

$$\boxed{\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)}$$

KAIST EE

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\mathbb{E}\left[(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X)^2\right] &= \text{var}(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X) \\
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- Uncertainty about Θ after observation decreases by the factor of $1 - \rho^2$

$$\boxed{\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)}$$

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- Uncertainty about Θ after observation decreases by the factor of $1 - \rho^2$
- What happens if $|\rho| = 1$ or $\rho = 0$?

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Linear LMS (LLMS) Estimator: Proof

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}\Big(X - \mathbb{E}(X)\Big) \tag{1}$$

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(3)

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Slide pp. 43

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Slide pp. 43

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Slide pp. 43

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- Using $\rho = \frac{\mathrm{cov}(\Theta,X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

Slides 17, 35, and 45 for more details

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.
- Random observation: X
- $\hat{\Theta}_{\mathsf{MAP}} = X$, and $\hat{\Theta}_{\mathsf{LMS}} = (1 - X)/|\log X|$.

Example: Romeo and Juliet (1)

Slides 17, 35, and 45 for more details

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- Question. What is the LLMS estimator $\hat{\Theta}_{\mathsf{L}}$?

Example: Romeo and Juliet (2)

$$\hat{\Theta}_{\mathsf{L}} = \mathbb{E}(\Theta) + \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}\Big(X - \mathbb{E}(X)\Big)$$

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$$\begin{aligned}\mathsf{var}[X] &= \mathbb{E}[\mathsf{var}[X|\Theta]] + \mathsf{var}[\mathbb{E}[X|\Theta]] \\ &= \frac{1}{12}\mathbb{E}[\Theta^2] + \frac{1}{4}\mathsf{var}[\Theta] = \frac{7}{144}\end{aligned}$$

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$$\begin{aligned}\mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6\end{aligned}$$

$$\mathrm{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

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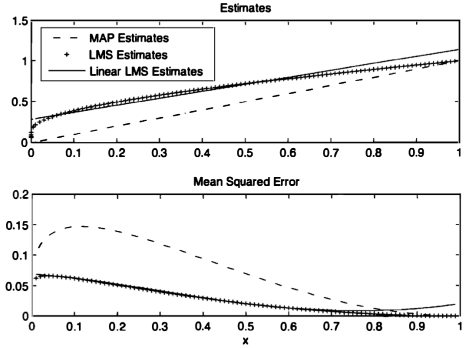

Example: Biased Coin with Uniform Prior

- Biased coin with probability of head θ
- Unknown $\Theta \sim \mathcal{U}[0,1]$,
  - $\mathbb{E}[\Theta] = 1/2$, $\text{var}[\Theta] = 1/12$
- n tosses, X: number of heads.
- $p_{X|\Theta}(k|\theta) \sim \text{Binomial}(n, \theta)$

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$$
\begin{aligned}
\text{var}(X) &= \mathbb{E}[\text{var}(X|\Theta)] + \text{var}(\mathbb{E}[X|\Theta]) \\
&= \mathbb{E}[n\Theta(1-\Theta)] + \text{var}[n\Theta] \\
&= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12} = \frac{n(n+2)}{12}
\end{aligned}
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- Yes, because the LMS esitmator was linear.

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

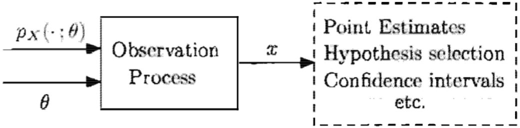
(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator



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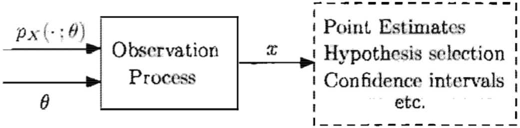

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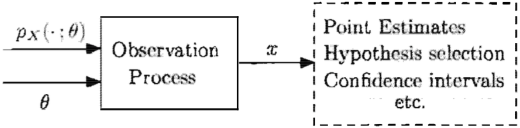
Framework of Classical Inference (1)



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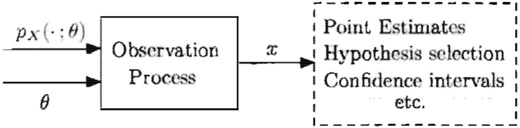

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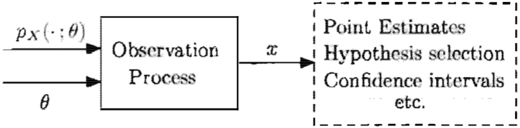

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- Choosing one among multiple probabilistic models
  - Each θ corresponds to a probabilistic model

Framework of Classical Inference (2)

- Problem types
  - Estimation: θ: prob. of head?
  - Hypothesis testing: $\theta = 1/2$ or $\theta = 1/4$?
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  - ML (Maximum Likelihood) estimation
  - Linear regression
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- Just a taste in this course.

Maximum Likelihood Estimation

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- Random observation $x = (x_1, x_2, \ldots, x_n)$ of $X = (X_1, X_2, \ldots, X_n)$
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- Very often, X_is are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \ldots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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KAIST EE

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- Thus, when Θ is uniform (complete ignorance of Θ) in MAP, MAP == ML

Example: Romeo and Juliet

Slides 17, 35, 45, and 56 for more details

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.
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Questions?

Review Questions

1) What is statistical inference?
2) Draw the building blocks of Bayesian inference and explain how it works.
3) What are MAP and LMS estimators and their underlying philosophies?
4) What is LLMS estimator and why is it useful?
5) Compare the classical and Bayesian inference.
6) What is the ML estimator and how is it related to the MAP estimator?