

Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

August 25, 2021

- (1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$
- (2) Derived distribution of $Z = X + Y$
- (3) Covariance: Degree of dependence between two rvs.
- (4) Correlation coefficient
- (5) Conditional expectation and law of iterative expectations
- (6) Conditional variance and law of total variance
- (7) Random number of sum of random variables

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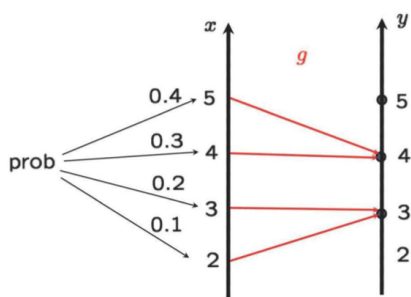
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- What are easy or difficult cases?
- Easy cases
 - Discrete
 - Linear: $Y = aX + b$

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x:g(x)=y} p_X(x) \end{aligned}$$

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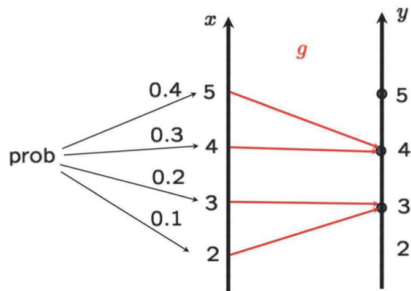


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$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$

$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$, X : Continuous

If $a > 0$,

If $a < 0$,

$$\text{If } a > 0, \quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$$

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Therefore,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if } (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $b = 0$ and $a > 0$, Y is exponential with parameter $\frac{\lambda}{a}$, but generally not.

- Remember? Linear transformation preserves normality. Time to prove.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b , $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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$$\begin{aligned} f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\} \\ &= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\} \end{aligned}$$

Generally, $Y = g(X)$, X : Continuous

Step 1. Find the CDF of Y :

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When $Y = g(X)$ is monotonic, a **general formula** can be drawn (see the textbook at pp 207)

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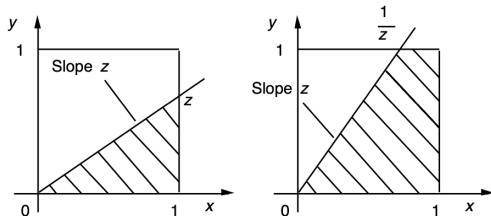
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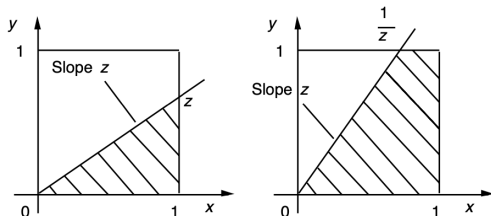
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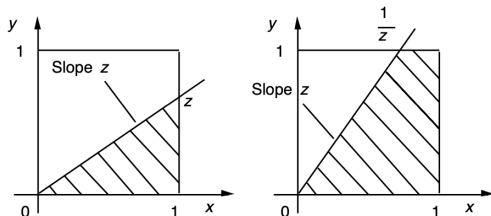
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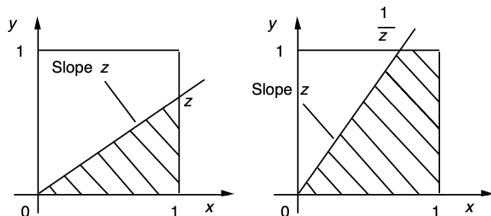
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(Note) Sometimes, the problem is tricky, which requires careful case-by-case handling. :-)

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- $p_Z(z)$ is called of the PMFs of X and Y .

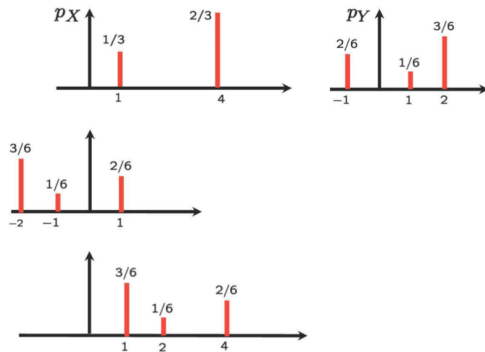
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- $p_Z(z)$ is called convolution of the PMFs of X and Y .

- Convolution: $p_Z(z) = \sum_x p_X(x)p_Y(z-x)$
- Interpretation for a given z :
 - (i) Flip (horizontally) the PMF of Y ($p_Y(-x)$)
 - (ii) Put it underneath the PMF of X
 - (iii) Right-shift the flipped PMF by z ($p_Y(-x+z)$)

Example. $z = 3$



- Same logic as the discrete case

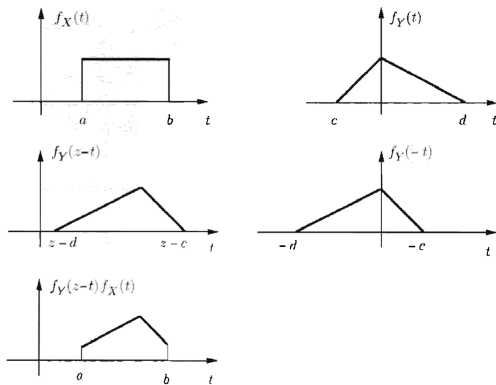
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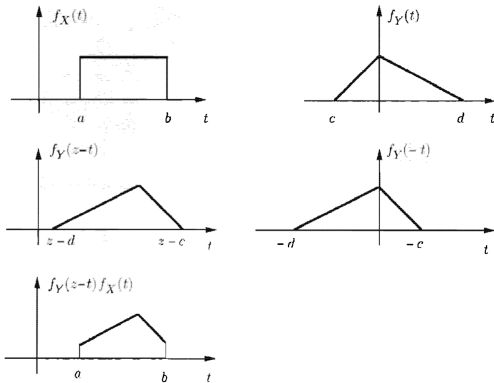


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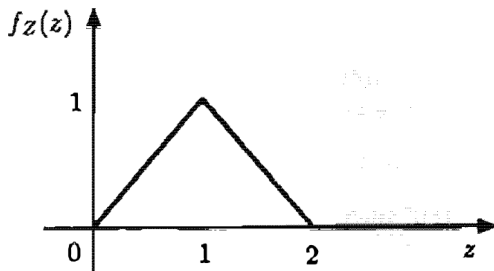
- Youtube animation for convolution:
<https://www.youtube.com/watch?v=C1N55M1VD2o>

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- **Example.** $X, Y \sim \mathcal{U}[0, 1]$ and $X \perp\!\!\!\perp Y$. What is the PDF of $Z = X + Y$? Draw the PDF of Z .

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- Why normal rvs are used to model the **sum of random noises**.
- **Extension**. The sum of **finitely many** independent normals is also normal.

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx \end{aligned}$$

- The details of integration is a little bit tedious. :-)

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} \exp\left\{-\frac{(z - \mu_x - \mu_y)^2}{2(\sigma_x^2 + \sigma_y^2)}\right\}$$

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 - R3.** Always bounded by some numbers (i.e., dimensionless metric). For example, $[-1, 1]$

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 - R2.** Shows the 'direction' of dependence by $+$ and $-$
 - R3.** Always bounded by some numbers (i.e., dimensionless metric). For example, $[-1, 1]$
- Good engineers: Good at making good metrics
 - Metric of how our society is economically polarized
 - Cybermetrics in MLB (Major League Baseball):
<http://m.mlb.com/glossary/advanced-stats>

- Simple case: $\mathbb{E}[X] = \mu_x = 0$ and $\mathbb{E}[Y] = \mu_Y = 0$

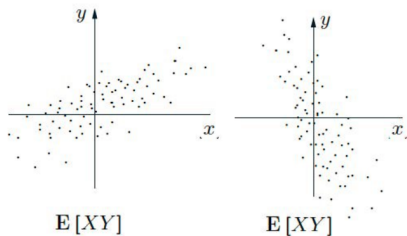
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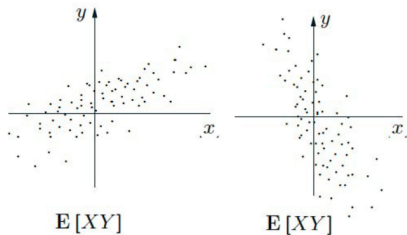


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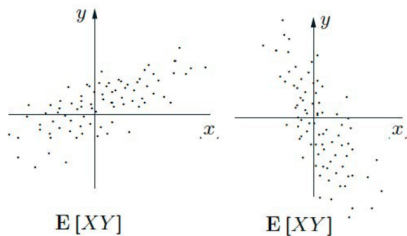


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- When they are positively dependent, but have negative values?

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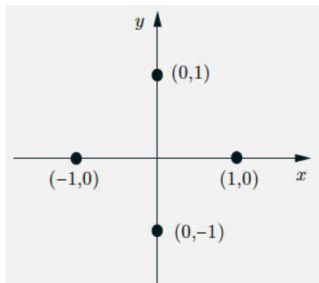
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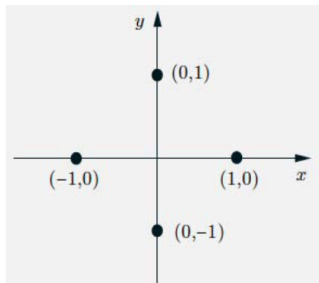
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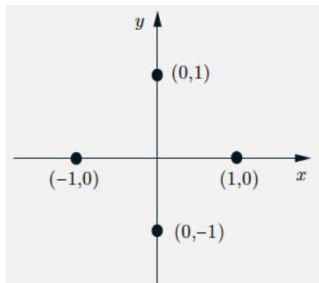
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- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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- Theorem.**
 - $-1 \leq \rho \leq 1$ (proof at the next slide)
 - $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$ for some constant c ($c > 0$ when $\rho = 1$ and $c < 0$ when $\rho = -1$). In other words, linear relation, meaning VERY related.

$$1. -1 \leq \rho \leq 1$$

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Now, choose $a = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$. Then,

$$\mathbb{E}(X^2) - 2\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\mathbb{E}(XY) + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2}\mathbb{E}(Y^2) = \mathbb{E}(X^2) - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}(Y^2)} \geq 0$$

$$2. |\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$$

(\Rightarrow) Suppose that $|\rho| = 1$. In the proof of CSI,

$$\mathbb{E} \left[\left(\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)} \tilde{Y} \right)^2 \right] = \mathbb{E}(\tilde{X}^2) - \frac{(\mathbb{E}[\tilde{X}\tilde{Y}])^2}{\mathbb{E}(\tilde{Y}^2)} = \mathbb{E}(\tilde{X}^2)(1 - \rho^2) = 0$$

$$\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)} \tilde{Y} = 0 \Leftrightarrow \tilde{X} = \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)} \tilde{Y} = \rho \sqrt{\frac{\mathbb{E}(\tilde{X}^2)}{\mathbb{E}(\tilde{Y}^2)}} \tilde{Y}$$

(\Leftarrow) If $\tilde{Y} = c\tilde{X}$, then

$$\rho(X, Y) = \frac{\mathbb{E}(\tilde{X}c\tilde{X})}{\sqrt{\mathbb{E}[\tilde{X}^2]\mathbb{E}[(c\tilde{X})^2]}} = \frac{c}{|c|}$$

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- Consider a rv Y , such that

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- Consider other rv X , which, we assume, has:

$$\mathbb{E}[X|Y = y] = \begin{cases} 3, & \text{if } y = 0 \\ 8, & \text{if } y = 1 \\ 9, & \text{if } y = 2 \end{cases}$$

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- Consider other rv X , which, we assume, has:

$$g(y) = \mathbb{E}[X|Y = y] = \begin{cases} 3, & \text{if } y = 0 \\ 8, & \text{if } y = 1 \\ 9, & \text{if } y = 2 \end{cases}$$

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- The rv $g(Y)$ looks special, so let's give a fancy notation to it.
- What about? $X_{\text{exp}}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

A random variable $g(Y) = \boxed{}$, called $\boxed{}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y .

Conditional Expectation

A random variable $g(Y) = \mathbb{E}[X|Y]$, called **conditional expectation of X given Y** , takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y .

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- Often confusing because of the notation.

Expectation of Conditional Expectation

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|Y]] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\ &= \mathbb{E}[X]\end{aligned}$$



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- whole average: (i) taking the average m_s of each section and (ii) forming a weighted average

$$\sum_{s=1}^k \frac{n_s}{n} m_s = \sum_{s=1}^k \frac{n_s}{n} \frac{1}{n_s} \sum_{i \in A_s} x_i = \frac{1}{n} \sum_{i=1}^n x_i = m$$

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- (1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$
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- (4) Correlation coefficient
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- (6) **Conditional variance and law of total variance**
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	$\mathbb{E}[X Y]$	$\text{var}[X Y]$
Expectation	$\mathbb{E}[\mathbb{E}(X Y)]$	$\mathbb{E}[\text{var}(X Y)]$
Variance	$\text{var}[\mathbb{E}(X Y)]$	$\text{var}[\text{var}(X Y)]$

Law of total variance (LTV)

$$\text{var}[X] =$$

Proof.

(1)

(2)

Law of total variance (LTV)

$$\text{var}[X] = \mathbb{E}[\text{var}(X|Y)] + \text{var}[\mathbb{E}(X|Y)]$$

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Proof.

$$\text{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2 \tag{1}$$

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$$(1) + (2) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}[X]$$

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- $\text{var}[\mathbb{E}(X|Y)]$: variability of the average of the different sections
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 - Weighted average of the section variances
 - **average score variability within individual sections**
- $\text{var}[\mathbb{E}(X|Y)]$: variability of the average of the different sections
 - $\mathbb{E}(X|Y = s)$: average score in section s
 - **variability between sections**

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- Since $X \sim \mathcal{U}[0, Y]$, $\text{var}(X|Y) = \frac{Y^2}{12} \rightarrow \mathbb{E}[\text{var}[X|Y]] = \frac{1}{12} \int_0^l \frac{1}{l} y^2 dy = \frac{l^2}{36}$

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- $\mathbb{E}(X|Y) = Y/2 \rightarrow \text{var}(\mathbb{E}[X|Y]) = \frac{1}{4} \text{var}[Y] = \frac{1}{4} \frac{l^2}{12} = \frac{l^2}{48}$

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- $\mathbb{E}(X|Y) = Y/2 \rightarrow \text{var}(\mathbb{E}[X|Y]) = \frac{1}{4} \text{var}[Y] = \frac{1}{4} \frac{l^2}{12} = \frac{l^2}{48}$
- $\text{var}(X) = \frac{l^2}{36} + \frac{l^2}{48} = \frac{7l^2}{144}$

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- $Y = X_1 + X_2 + \dots X_N$. What are $\mathbb{E}[Y]$ and $\text{var}[Y]$?
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- $\text{var}[Y] = \mathbb{E}[\text{var}(Y|N)] + \text{var}[\mathbb{E}(Y|N)]$
 $\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$
 $\text{var}[Y|N] = N\text{var}[X_i]$

- N : number of stores visited (**random**)
- X_i : money spent in store i , independent of other X_j and N , X_i s are identically distributed with $\mathbb{E}[X_i] = \mu$
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$$\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$$

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Questions?

- 1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
- 2) How can we compute the distribution of $Z = X + Y$ when X and Y are independent?
- 3) What are covariance and correlation coefficient? Why do we need them?
- 4) Please explain the concepts of conditional expectation and conditional variance.