

## Lecture 8: Random Processes, Part II

Yi, Yung (이육)

EE210: Probability and Introductory Random Processes  
KAIST EE

MONTH DAY, 2021

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- Markov Chain
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

- Assume discrete times  $n = 1, 2, \dots$
- Random process: A sequence of  $X_1, X_2, X_3, \dots$

- “Simplest” random process
  - Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

- Bernoulli process
- A random process that is a little more complex than the above?
  - Process that depends only on “yesterday”, not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

- Markov chain
- One of the most popular random processes in engineering

# Example: Machine Failure, Repair, and Replacement

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p.  $b$ , and continue working w.p.  $1 - b$ .
  - If broken down, it will be repaired and be working in the next day w.p.  $r$ , and continue to be broken down w.p.  $1 - r$ .
- $X_n \in \{1, 2\}$ : status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$ : A random process satisfying: for any  $n \geq 1$ ,  
$$\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$
$$\mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$
- What will happen at  $(n + 1)$ -th day depends only on what happens at  $n$ -th day?

- **Definition.** Let  $X_1, \dots, X_n, \dots$  be a sequence of random variables taking values in some finite space  $\mathcal{S} = \{1, 2, \dots, m\}$ , such that for all  $i, j \in \mathcal{S}$ ,  $n \geq 0$ , the following **Markov property** is satisfied:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0),$$

- For any fixed  $n$ , the future of the process after  $n$  is **independent** of  $\{X_1, \dots, X_n\}$ , **given**  $X_n$  (i.e., depends only on  $X_n$ )
- The value that  $X_n$  can take is called '**state**'. Thus, the space  $\mathcal{S}$  is called **state space**.
- **Time homogeneity.** The probability  $\mathbb{P}(X_{n+1} = j | X_n = i)$  does NOT depends on  $n$ .

Thus, for any  $n \geq 0$ , we introduce a simple notation  $p_{ij}$

$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

- **Transition Probability Matrix.** Consider a  $m \times m$  matrix  $\mathbf{P} = [p_{ij}]$ , where

$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

- Machine example.

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

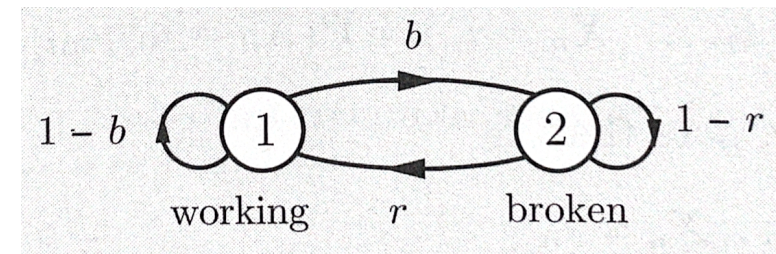
$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

$$p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- Transition probability matrix

$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$

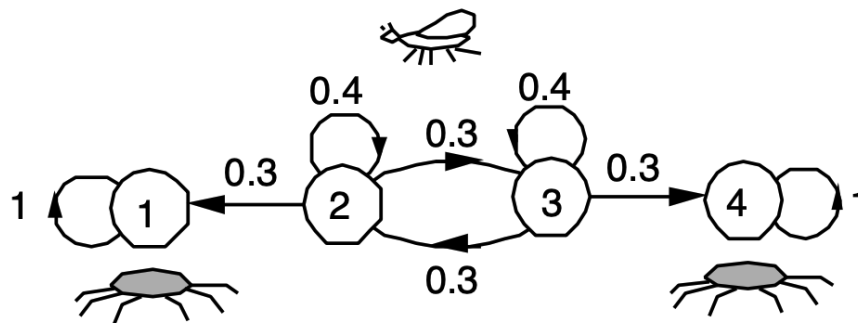
- State transition diagram



- Both are the complete description of Markov chain.
- $\sum_{j=1}^m p_{ij} = 1$  (for each row  $i$ , the column sum = 1)

# Spider-Fly example

- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- $X_n$ : position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$p_{ij}$

(Q) What is the probability of a sample path in a Markov chain?

$$\begin{aligned} & \mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n) \\ &= \mathbb{P}(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) \\ &= p_{i_{n-1}i_n} \cdot \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0i_1} \cdot p_{i_1i_2} \cdots p_{i_{n-1}i_n} \end{aligned}$$

- Spider-Fly example

$$\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} = \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2$$



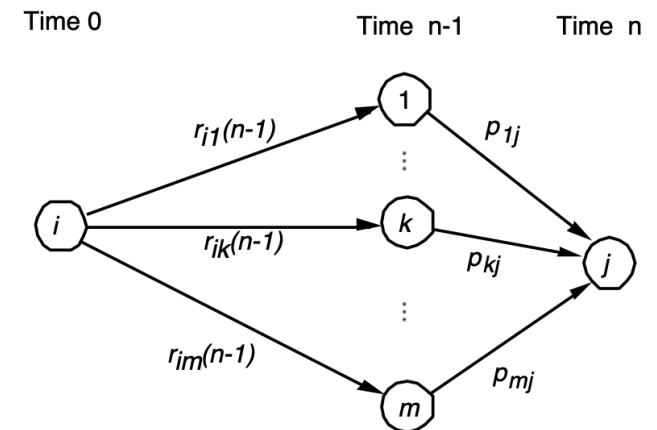
(Q) What is the probability that my state is  $i$ , starting from  $i$  after  $n$  steps?

- $n$ -step transition probability

$$r_{ij}(n) \triangleq \mathbb{P}(X_n = j \mid X_0 = i)$$

- Recursive formula, starting with  $r_{ij}(1) = p_{ij}$

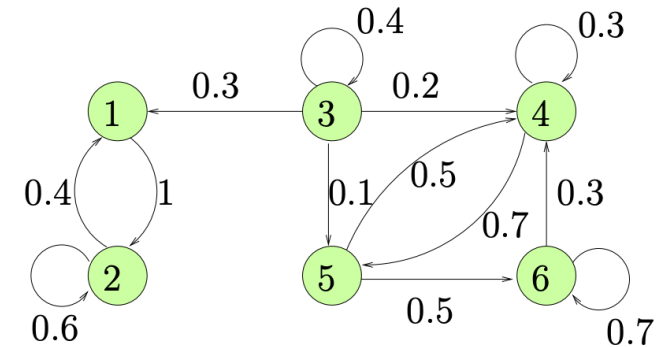
$$\begin{aligned} r_{ij}(n) &= \mathbb{P}(X_n = j \mid X_0 = i) = \\ &\sum_{k=1}^m \mathbb{P}(X_{n-1} = k \mid X_0 = i) \mathbb{P}(X_n = j \mid X_{n-1} = k, X_0 = i) \\ &= \sum_{k=1}^m r_{ik}(n-1) p_{kj} \end{aligned}$$



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - **Classification of States**
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

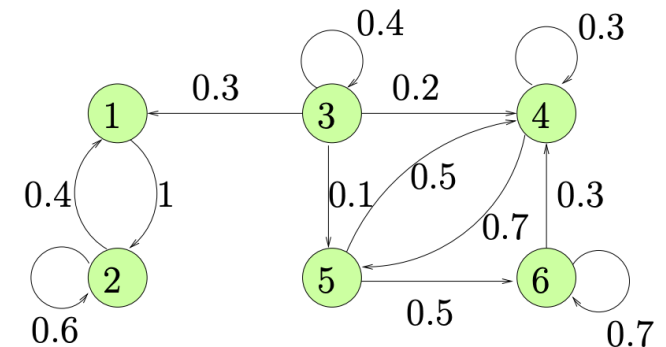
# Examples: Different States and Classes

- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
  - Divide into three classes:  $\{3\}$ ,  $\{1, 2\}$ ,  $\{4, 5, 6\}$
  - **Insight 1.** Multiple classes may exist.
- Difference between 1 and 3
  - 1: If I start from 1, visit 1 infinite times.
  - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
  - **Insight 2.** Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly, 4, 5, and 6)
- **Insight 3.** States in the same class share some properties.



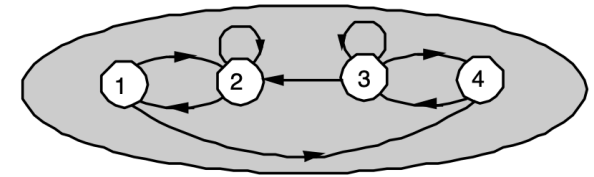
# Classification of States (1)

- **Definition.** State  $j$  is **accessible** from state  $i$ , if for some  $n$   $r_{ij}(n) > 0$ .
  - 6 is accessible from 3, but not the other way around.
- **Definition.** If  $i$  is accessible from  $j$  and  $j$  is accessible from  $i$ , we say that  $i$  communicates with  $j$ .
  - $1 \leftrightarrow 2$ , but 3 does not communicate with 5.
- **Definition.** Let  $A(i) = \{\text{states accessible from } i\}$ . State  $i$  is **recurrent**, if  $\forall j \in A(i)$ ,  $i$  is also accessible from  $j$ . In other words, “I communicate with all of my neighbors!”
  - A state that is not recurrent is **transient**.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state  $i$ , then there is always some probability of returning to  $i$ . It means that, given enough time, it is certain that it returns to  $i$ .

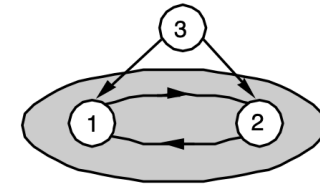


# Classification of States (2)

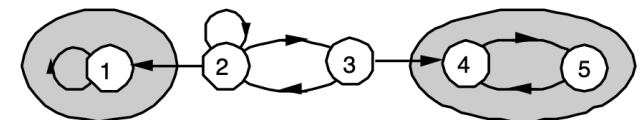
- A set of recurrent states which communicate with each other form a **class**.
- Markov chain decomposition
  - A MC can be decomposed into one or more recurrent classes, plus possibly some transient states.
  - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
  - A transient state is not accessible from any recurrent state.
  - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



Single class of recurrent states

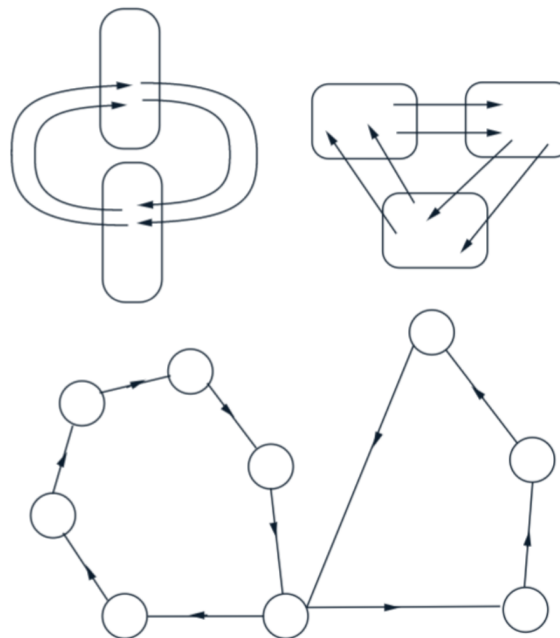


Single class of recurrent states (1 and 2)  
and one transient state (3)



Two classes of recurrent states  
(class of state 1 and class of states 4 and 5)  
and two transient states (2 and 3)

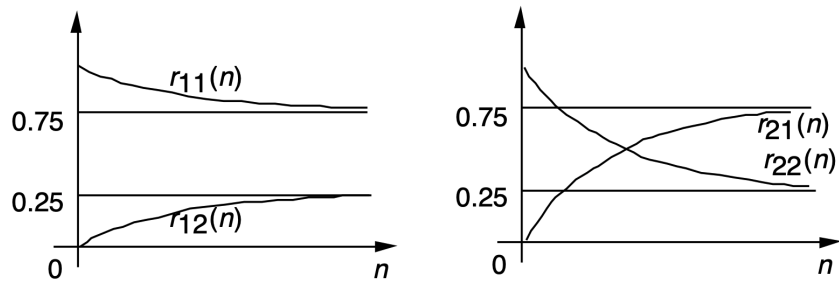
- The states in a recurrent class are periodic if they can be grouped into  $d > 1$  groups so that all transitions from one group lead to the next group.
- A recurrent class that is not periodic is said to be aperiodic.



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - **Steady-state Behaviors and Stationary Distribution**
  - Transient Behaviors

# $n$ -step transition prob.: $r_{ij}(n)$ for large $n$

- Convergence irrespective of the starting state

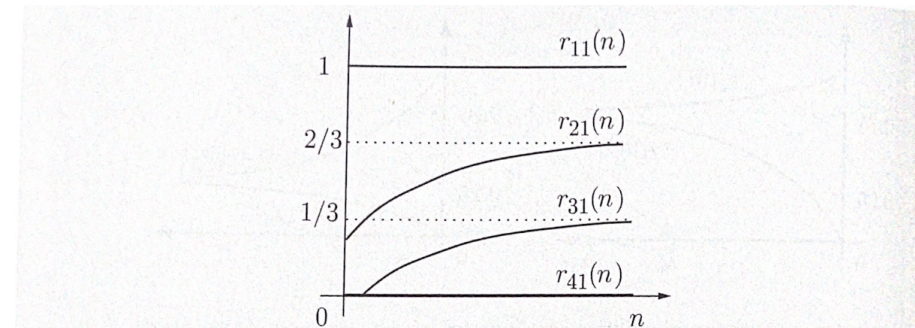


$n$ -step transition probabilities as a function of the number  $n$  of transitions

	UpD	B										
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496	.7501	.2499		
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512	.7498	.2502		
	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(5)$			

Sequence of  $n$ -step transition probability matrices

- Convergence depending on the starting state



$n$ -step transition probabilities into state 1

	1	2	3	4																
1	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0
2	0.3	0.4	0.3	0	.42	.25	.24	.09	.50	.17	.17	.16	.55	.12	.12	.21	2/3	0	0	1/3
3	0	0.3	0.4	0.3	.09	.24	.25	.42	.16	.17	.17	.50	.21	.12	.12	.55	1/3	0	0	2/3
4	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0	0	0	0	1.0
	$r_{ij}(1)$				$r_{ij}(2)$				$r_{ij}(3)$				$r_{ij}(4)$				$r_{ij}(\infty)$			

Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?



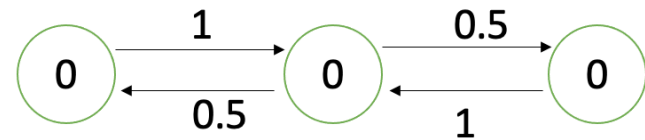
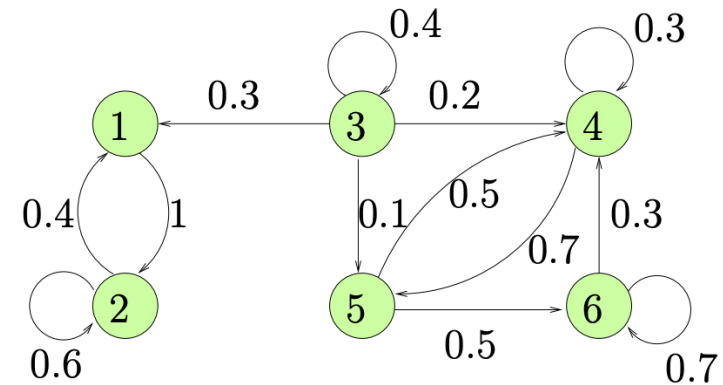
- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ?
- Convergence occurs, independent of the starting state, if:

**C1.** Only a single recurrent class

**C2.** such recurrent class is aperiodic

**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.

**C2.** Divergent behavior for periodic recurrent classes.



- If  $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$ , for some  $\pi_j \leq 1$ ,

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj} \implies \pi_j = \sum_{k=1}^m \pi_k p_{kj} \text{ (Balance equation)}$$

- Normalization equation

$$\sum_{i=1}^m \pi_i = 1$$

- Balance equation + Normalization equation  $\implies$  Finding the steady-state probabilities  $\{\pi_i\}$ .

- A two-state MC with:

$$p_{11} = 0.8, \quad p_{12} = 0.2,$$

$$p_{21} = 0.6, \quad p_{22} = 0.4.$$

- Balance equation:

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$

$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$$

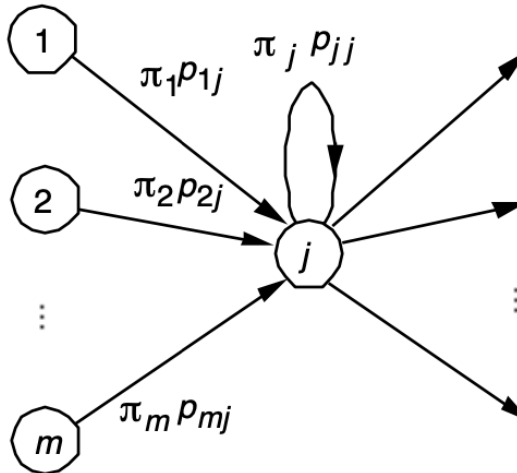
- Normalization equation:  $\pi_1 + \pi_2 = 1$
- The stationary distribution is:  $\pi_1 = 0.25, \pi_2 = 0.75$ .

- $\{\pi_j\}$  is also called a **stationary distribution**. Why?
- **Distribution**, because  $\sum_{j=1}^m \pi_j = 1$ .
- **Stationary**, because, if you choose the starting state according to  $\{\pi_j\}$ , then

$$\mathbb{P}(X_0 = j) = \pi_j, \quad j = 1, \dots, m \implies \mathbb{P}(X_1 = j) = \sum_{k=1}^m \mathbb{P}(X_0 = k) p_{kj} = \sum_{k=1}^m \pi_k p_{kj} = \pi_j$$

- Then,  $\mathbb{P}(X_n = j) = \pi_j$ , for all  $n$  and  $j$ .
  - If the initial state is chosen according to  $\{\pi_j\}$ , the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution is equal to to the stationary distribution"

- $\pi_j$ : the long-term **expected fraction of time** that the state is equal to  $j$ .
- Balance equation:  $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$  means:
  - The expected frequency  $\pi_j$  of visits to  $j$  is equal to the sum of the expected frequencies  $\pi_k p_{kj}$  of transitions that lead to  $j$ .



- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- **Markov Chain**
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - **Transient Behaviors**

# Absorption Probability

- **Definition.** A state  $k$  is **absorbing**, if  $p_{kk} = 1$ , and  $p_{kj} = 0$  for all  $j \neq k$ .  
- states 1 and 6 are absorbing

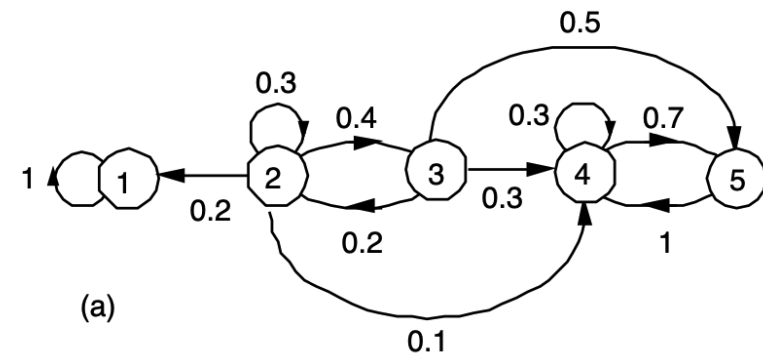
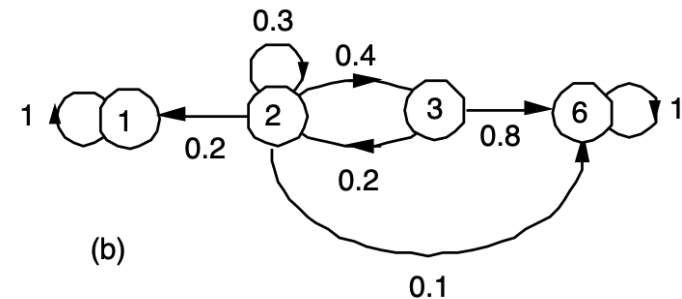
(Q) For a fixed absorbing state  $s$ , the probability  $a_i$  of reaching  $s$ , starting from a transient state  $i$ ?

- Fix  $s = 6$ .

$$a_1 = 0, \quad a_6 = 1$$

$$a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6$$

$$a_3 = 0.2a_2 + 0.8a_6$$

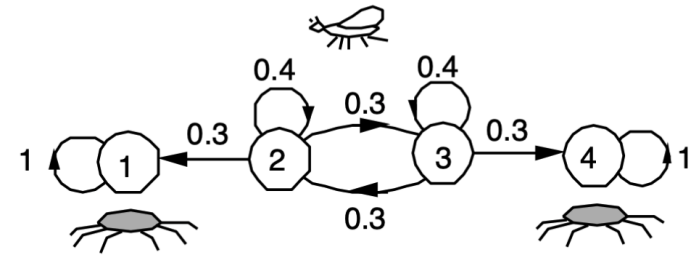


(Q) What if there are some non-absorbing recurrent state?

- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

<sup>0</sup>The notation  $a_i$  should have dependence on  $s$ , but we omit it for simplicity.

(Q) Starting from a transient state  $i$ , expected number of transitions  $\mu_i$  until absorption to any absorbing state?



- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad (\text{for recurrent states})$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad (\text{for transient states})$$

- For generalized description, please see the textbook (pp. 367).

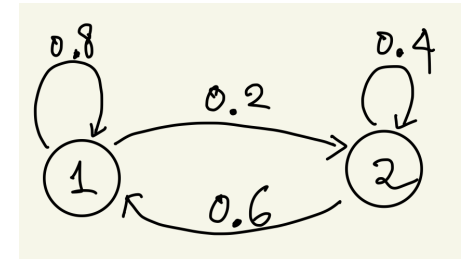


# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

(Q) **First passage time.** Starting from a  $i$ , expected number of transitions  $t_i$  to reach  $s$  for the first time?

(Q) **First recurrence time.** Starting from a  $s$ , expected number of transitions  $t_s^*$  to reach  $s$  for the first time?



- Mean first passage time from 2 to 1

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

- For generalized description, please see the textbook (pp. 368)

<sup>0</sup>The notation  $t_i$  should have the dependence on  $s$ , but we omit it for simplicity.

Questions?

- 1) Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are the limiting distribution and the stationary distribution of MCs?
- 4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
- 5) What are recurrent and transient states in MC?