Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

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(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

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- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

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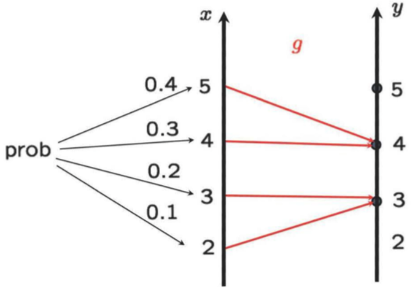

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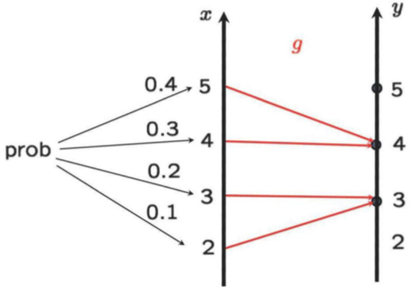
$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



If $a > 0$,

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If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

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Linear: $Y = aX + b$, $a \neq 0$, X: Continuous

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Therefore,

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Linear: $Y = aX + b$, when X is exponential

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if} \quad (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $b = 0$ and $a > 0$, Y is exponential with parameter $\frac{\lambda}{a}$, but generally not.

Linear: $Y = aX + b$, when X is normal

- Remember? Linear transformation preserves normality. Time to prove.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\}$$

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$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0$$

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When $Y = g(X)$ is monotonic, a general formula can be drawn (see the textbook at pp 207)

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VIDEO PAUSE

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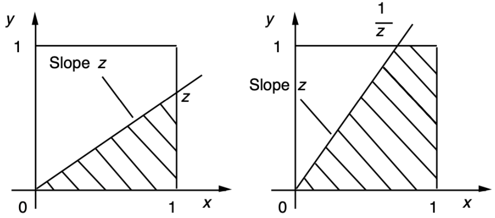
- Depending on the value of z, two cases need to be considered separately.



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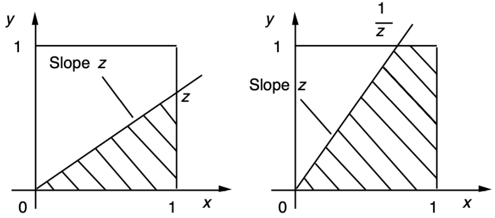

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$$f_Z(z) = \begin{cases} 1/2, & 0 \le z \le 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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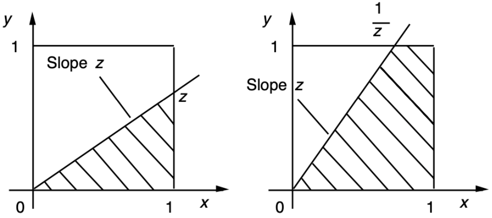

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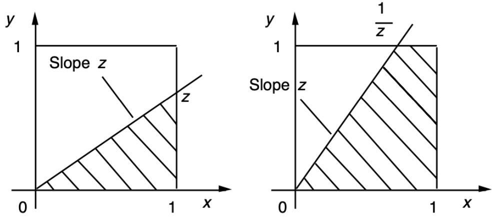

(Note) Sometimes, the problem is tricky, which requires careful case-by-case handing. :-)

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$$= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) = \sum_x p_X(x) p_Y(z - x)$$

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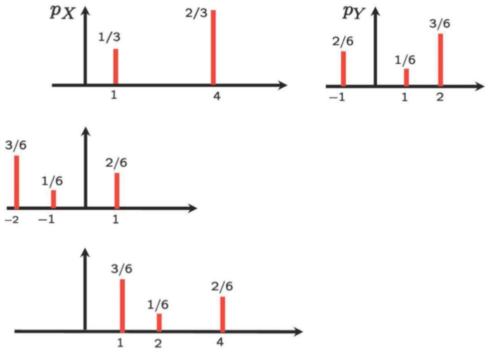
- $p_Z(z)$ is called $\boxed{\text{convolution}}$ of the PMFs of X and Y.

- Convolution: $p_Z(z) = \sum_x p_X(x) p_Y(z - x)$
- Interpretation for a given z:
  (i) Flip (horizontally) the PMF of Y ($p_Y(-x)$)
  (ii) Put it underneath the PMF of X
  (iii) Right-shift the flipped PMF by z ($p_Y(-x + z)$)

Example. $z = 3$



- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

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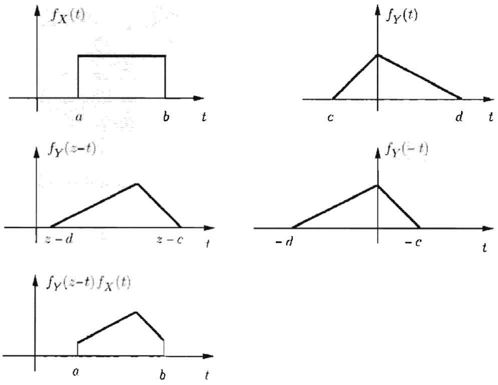

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- Youtube animation for convolution: `https://www.youtube.com/watch?v=C1N55M1VD2o`

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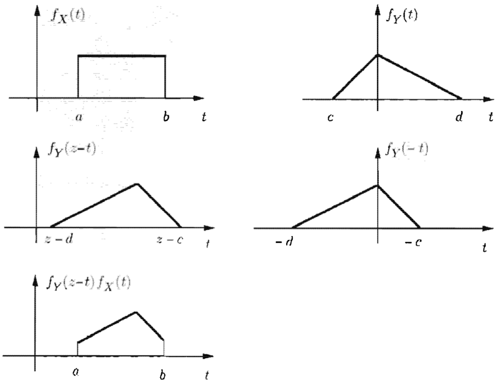

Example

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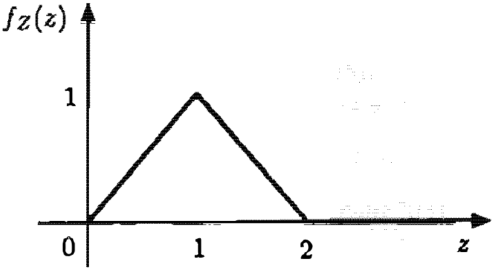

https://www.youtube.com/watch?v=MQm6ZP1F6ms

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Sum of two independent normal rvs

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- Why normal rvs are used to model the sum of random noises.
- Extension. The sum of finitely many independent normals is also normal.

$Y = X + Y$, $X \perp\!\!\!\perp Y$, Normal (2)

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx
\end{aligned}
$$

- The details of integration is a little bit tedious. :-)

$$
f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right\}
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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

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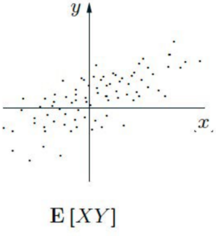

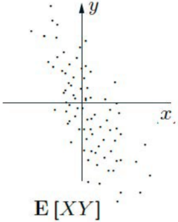

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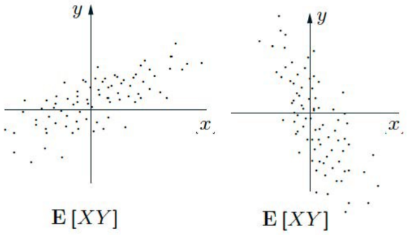

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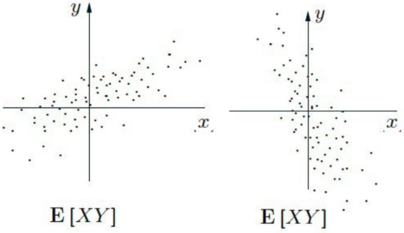

(Q) What about $\mathbb{E}[X + Y]$?

- When they are positively dependent, but have negative values?

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Example: $\mathrm{cov}(X, Y) = 0$, but not independent

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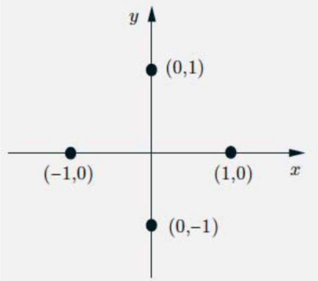

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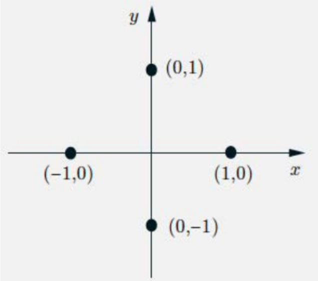

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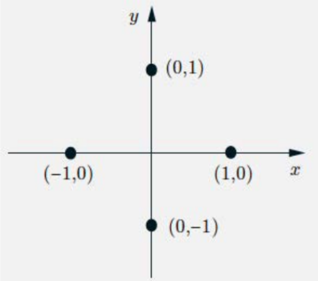
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
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- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$.
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  1. $-1 \le \rho \le 1$ (proof at the next slide)

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- Theorem.
  1. $-1 \le \rho \le 1$ (proof at the next slide)
  2. $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$ for some constant c ($c > 0$ when $\rho = 1$ and $c < 0$ when $\rho = -1$). In other words, linear relation, meaning VERY related.

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- Proof of $-1 \leq \rho \leq 1$:

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  Now, choose $a = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$. Then,

$$\mathbb{E}(X^2) - 2\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\mathbb{E}(XY) + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2}\mathbb{E}(Y^2) = \mathbb{E}(X^2) - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}(Y^2)} \ge 0$$

2. $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$

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$(\Rightarrow)$ Suppose that $|\rho| = 1$. In the proof of CSI,

$$\mathbb{E}\left[\left(\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y}\right)^2\right] = \mathbb{E}(\tilde{X}^2) - \frac{(\mathbb{E}[\tilde{X}\tilde{Y}])^2}{\mathbb{E}(\tilde{Y}^2)} = \mathbb{E}(\tilde{X}^2)(1 - \rho^2) = 0$$

$$\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}Y = 0 \leftrightarrow \tilde{X} = \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y} = \rho\sqrt{\frac{\mathbb{E}(\tilde{X}^2)}{\mathbb{E}(\tilde{Y}^2)}}\tilde{Y}$$

$(\Leftarrow)$ If $\tilde{Y} = c\tilde{X}$, then

$$\rho(X, Y) = \frac{\mathbb{E}(\tilde{X}c\tilde{X})}{\sqrt{\mathbb{E}[\tilde{X}^2]\mathbb{E}[(c\tilde{X})^2]}} = \frac{c}{|c|}$$

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A Special Random Variable

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

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A random variable $g(Y) = \boxed{\phantom{\mathbb{E}[X|Y]}}$, called $\boxed{\phantom{\text{conditional expectation of } X \text{ given } Y}}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

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Conditional Expectation $\mathbb{E}[X|Y]$

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- Often confusing because of the notation.

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Expectation of Conditional Expectation

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y=y]p_Y(y) \\ &= \mathbb{E}[X]\end{aligned}$$

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$$=$$

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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

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Conditional Variance var$[X|Y]$

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	$\mathbb{E}[X\|Y]$	$\text{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\text{var}(X\|Y)\right]$
Variance	$\text{var}\left[\mathbb{E}(X\|Y)\right]$	$\text{var}\left[\text{var}(X\|Y)\right]$

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Law of total variance (LTV)

$$\mathrm{var}[X] =$$

Proof.

$$(1)$$

$$(2)$$

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$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \mathrm{var}[X]$$

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- Same setting as that in page 36
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  - average score variability within individual sections
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  - $\mathbb{E}(X|Y = s)$: average score in section s
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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

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Questions?

Review Questions

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
2) How does CDF help in computing the derived distributions?
3) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?
4) What are covariance and correlation coefficient? Why do we need those concepts?
5) Explain the concepts of conditional expectation and conditional variance.
6) Explain law of iterative expectations and law of total variance
7) How can we apply the above two law to handle a case of random number of sum of random variables?