

Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

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Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

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- Markov chain
- One of the most popular random processes in engineering!

- A machine: working or broken down on a given day.
 - If working, break down in the next day w.p. b , and continue working w.p. $1 - b$.
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- $X_n \in \{1, 2\}$: status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$: A random process satisfying: for any $n \geq 1$,
 $\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$
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- What will happen at $(n + 1)$ -th day depends only on what happens at n -th day?

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- **Definition.** Let X_1, \dots, X_n, \dots be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1, 2, \dots, m\}$, such that for all $i, j \in \mathcal{S}$, $n \geq 0$, the following
[red box] is satisfied:

for all $n \geq 0$, all $i, j \in \mathcal{S}$, and all possible sequences i_0, \dots, i_{n-1} of earlier states,

$$\boxed{\text{[red box]}} = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

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- **Alternate definition via conditional independence.** For any fixed n , the future of the process after n is **independent** of $\{X_1, \dots, X_{n-1}\}$, **given** X_n .

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- Thus, for any $n \geq 0$, we introduce a simple notation p_{ij}
$$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$
- (Q) Any convenient way of describing a MC for intuitive understanding?

- Machine example: $\mathcal{S} = \{1, 2\}$

$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b,$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r,$$

$$p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

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The $m \times m$ matrix $\mathbf{P} = [p_{ij}]$, where $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

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- Property.

$$\sum_{j=1}^m p_{ij} = 1 \text{ (for each row } i, \text{ the column sum} = 1\text{)}$$

- Machine example.

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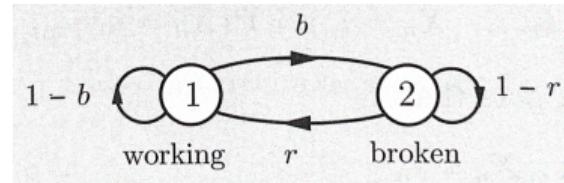
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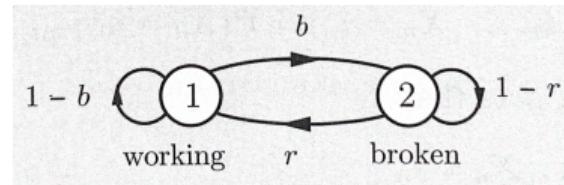
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- Any other way? **State Transition Diagram**



- Transition probability matrix and state transition diagram are the two ways of completely describing a given Markov chain.

- A fly moves along a line in unit increments.

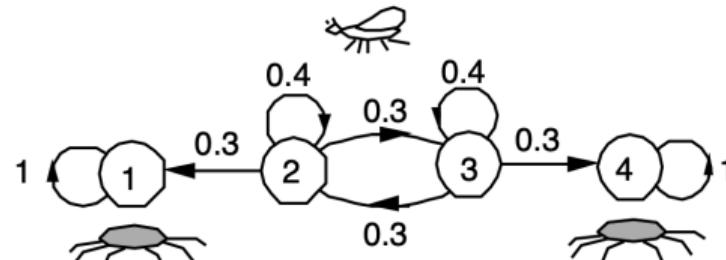
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- X_n : position of the fly. Please draw the state transition diagram and find the transition probability matrix.

Spider-Fly Example (Example 7.2)

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	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$$p_{ij}$$

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- $(Y_n : n \geq 0)$ is not a MC.

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 - When F meets with N , then N becomes infected, following **infection model**.

- X_n : number of infectious (F) persons at the beginning of time slot n .
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Q1. Is $(X_n : n \geq 0)$ a MC?

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 - Being successful in good modeling depends on the choice of “state” (good modeling sense).
 - Markov chain can be used widely if we choose the state space appropriately.

Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) *n*-step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

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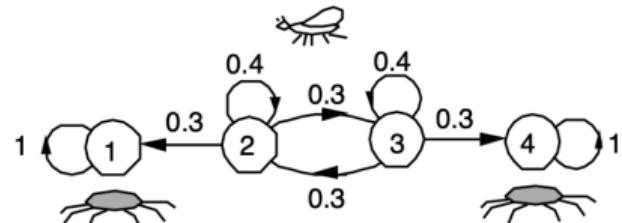
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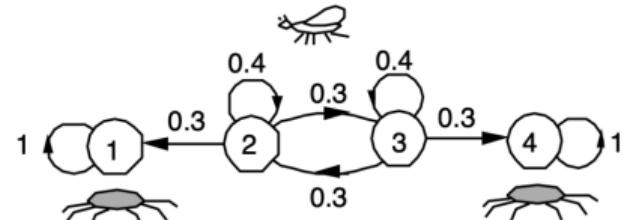


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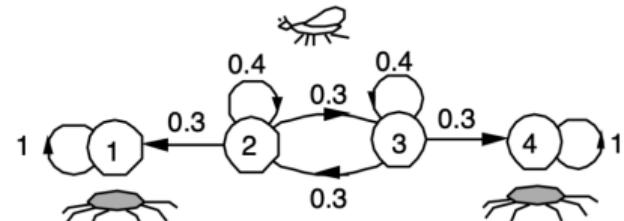


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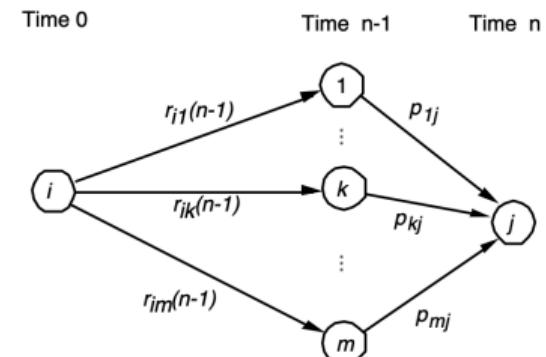
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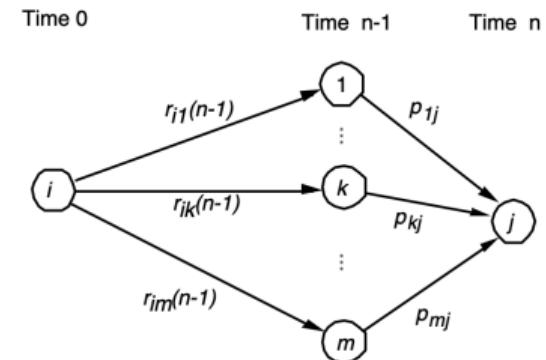
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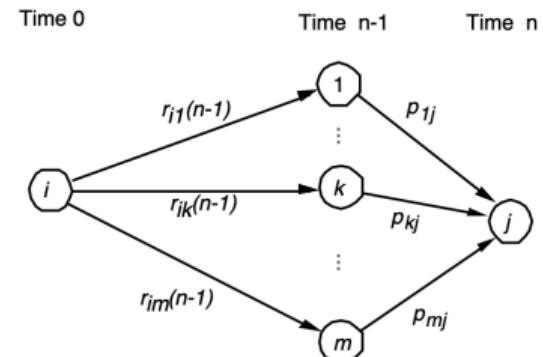
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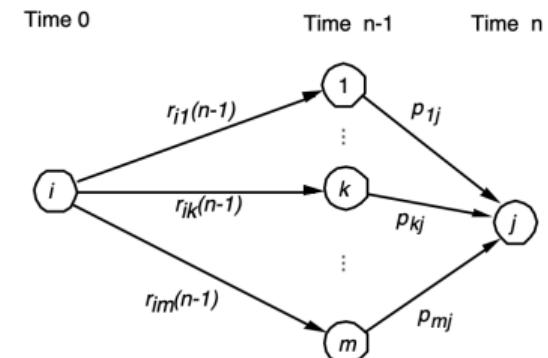
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- Possible to compute $r_{ij}(n)$ recursively. This is called **Chapman-Kolmogorov equation**.

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- Let $\mathbf{P}^{(n)}$ be the matrix of n -step transition probability, i.e., $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- (Q) What is the relation between $\mathbf{P}^{(n)}$ and \mathbf{P} ? Can we express $\mathbf{P}^{(n)}$ with \mathbf{P} ?

- $r_{ij}(n+1) = \sum_{k=1}^m r_{ik}(n)r_{kj}(1)$ and $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$

- $r_{ij}(n + l) = \sum_{k=1}^m r_{ik}(n)r_{kj}(l)$ and $\mathbf{P}^{(n)} \triangleq [r_{ij}(n)]$
- Then, by letting $n = 1, l = 1$,
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- In other words, n -step transition probability matrix is just a **n -time multiplication** of the transition probability matrix \mathbf{P} .

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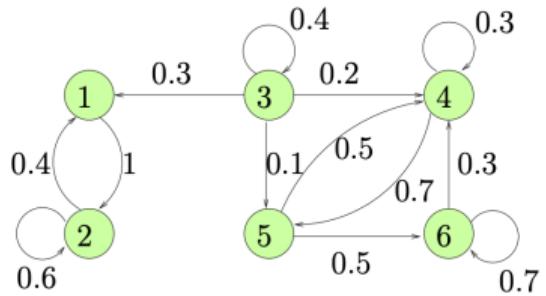
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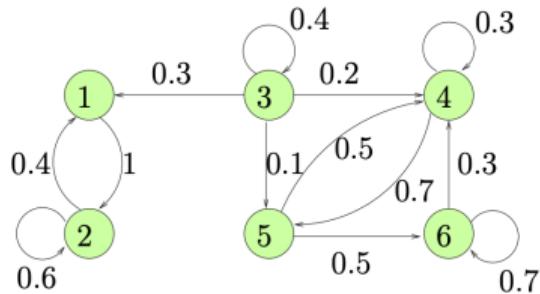
Different States and Classes?

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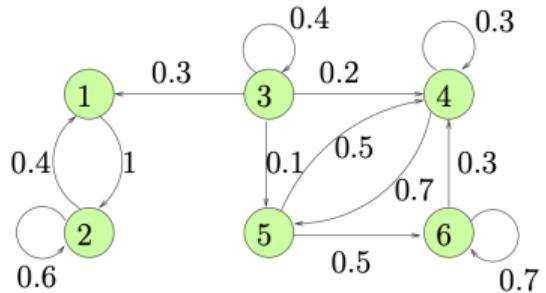
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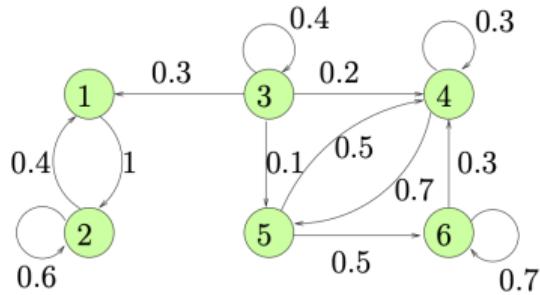
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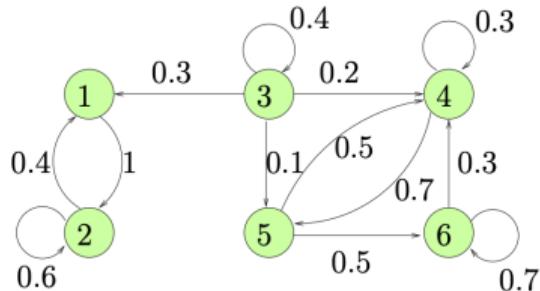
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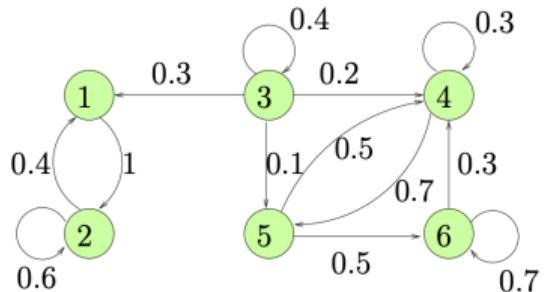
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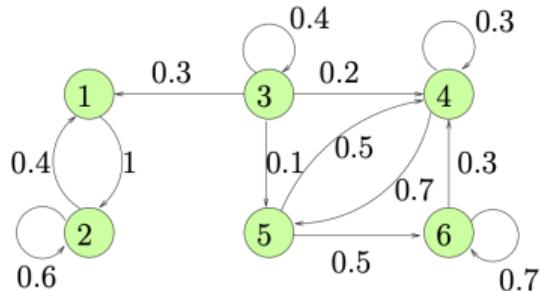
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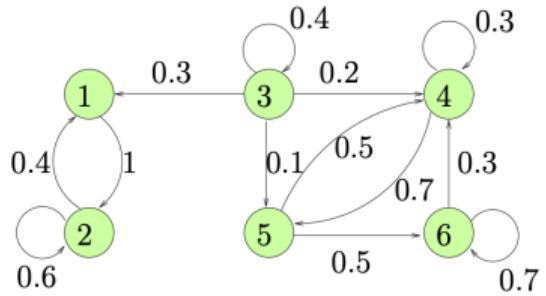
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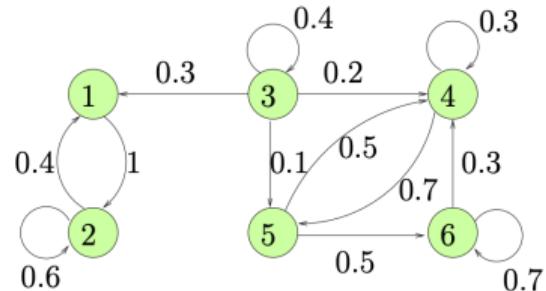
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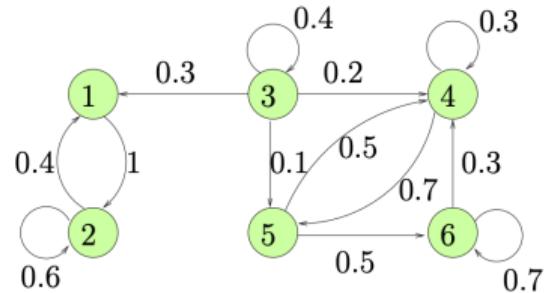
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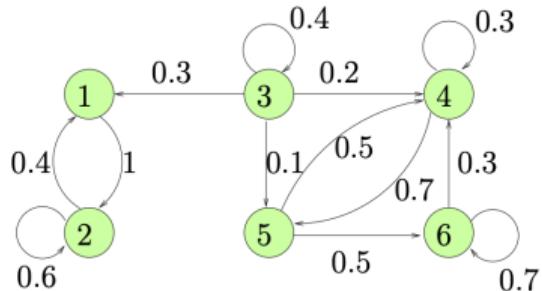
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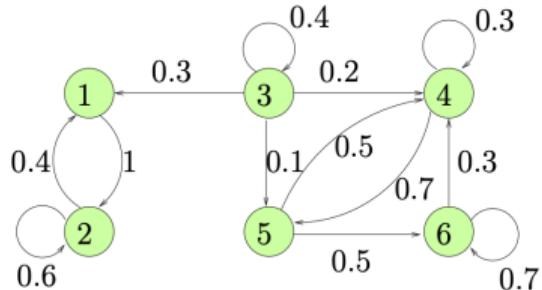
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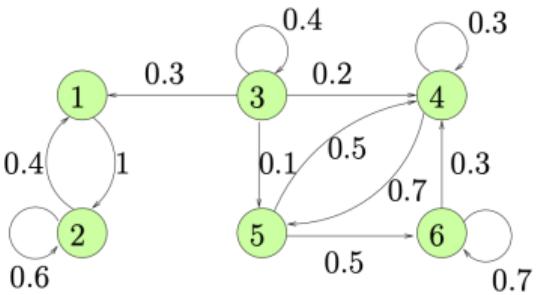


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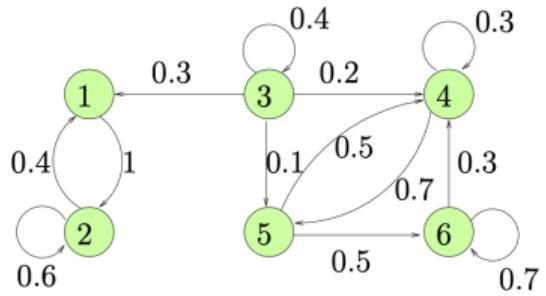


Classification of States (1)



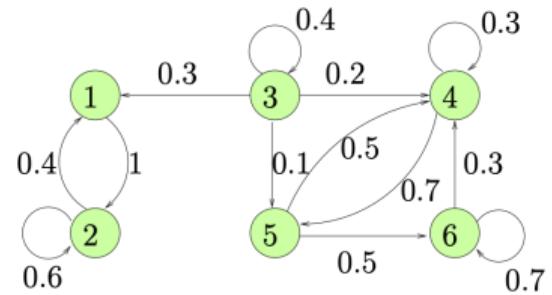
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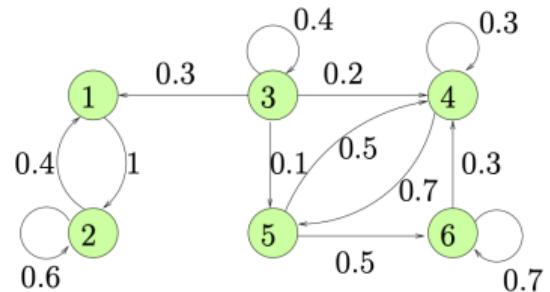
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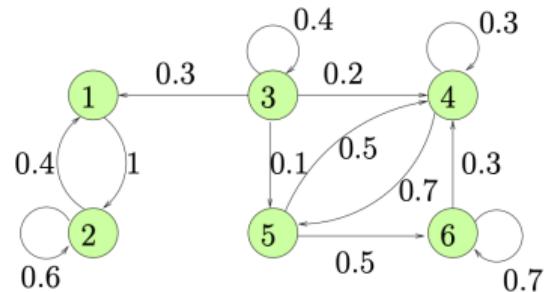
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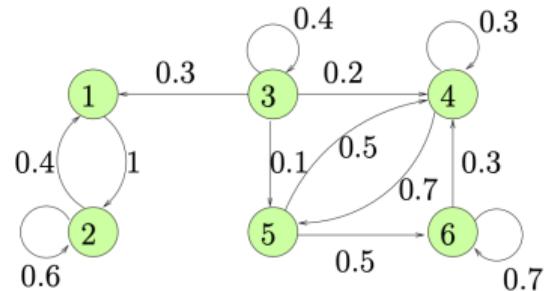
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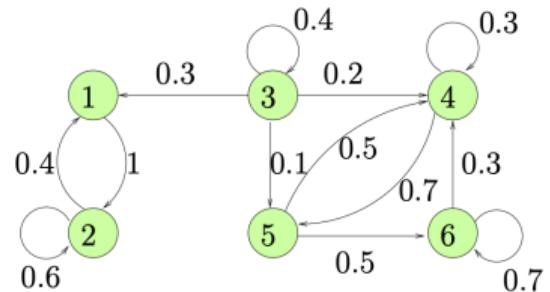
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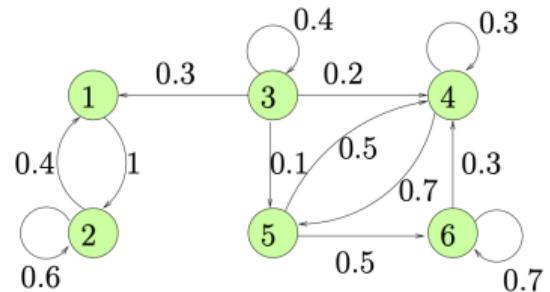
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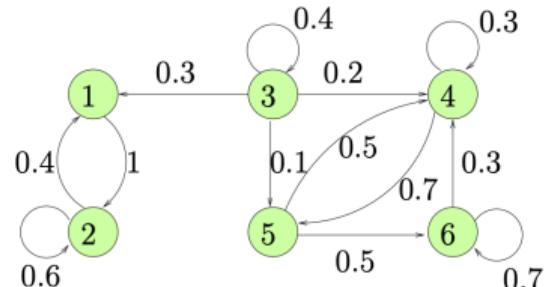
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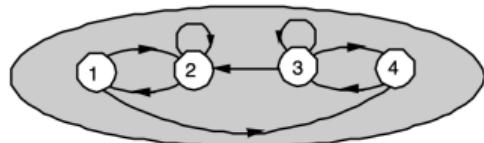
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 - If we start from a recurrent state i , then there is always some probability of returning to i . It means that, given enough time, it is certain that it returns to i .

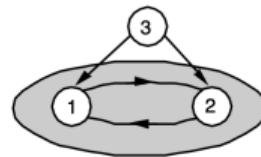


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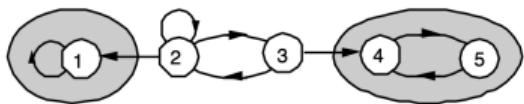
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Single class of recurrent states



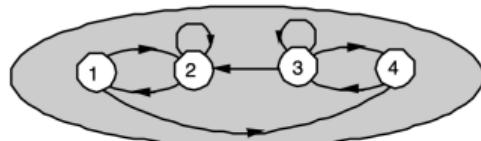
Single class of recurrent states (1 and 2)
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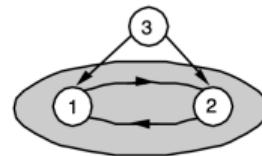
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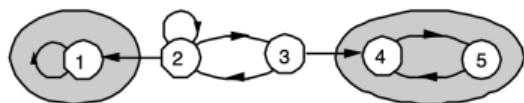
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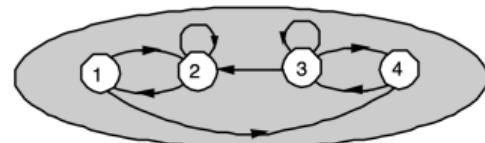
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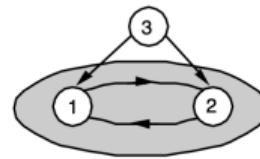
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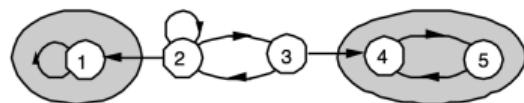
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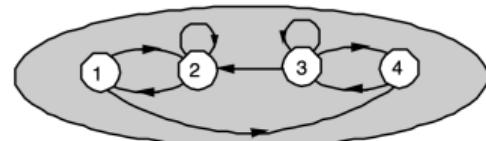
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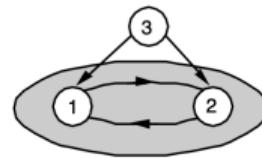
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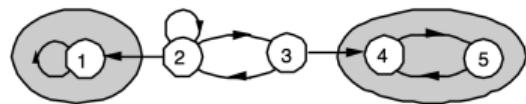
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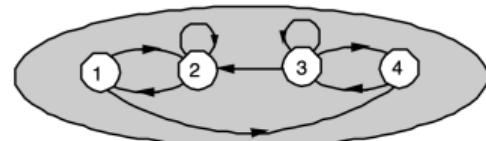
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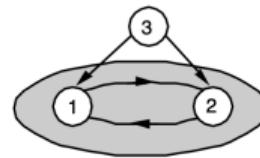
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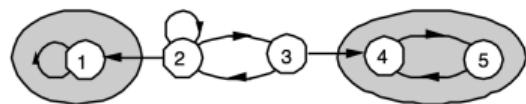
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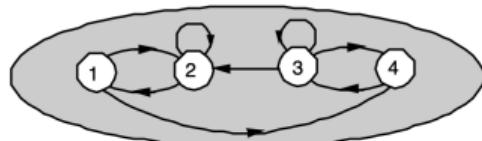
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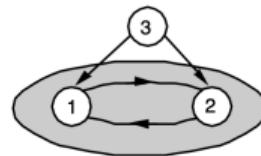
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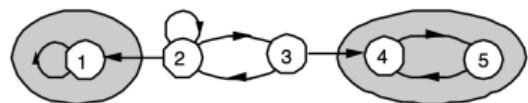
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- The MC with only a single recurrent class is said to be **irreducible** (더이상 분해할 수 없는).



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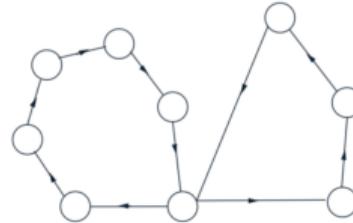
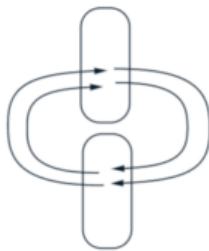
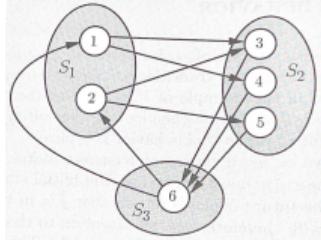


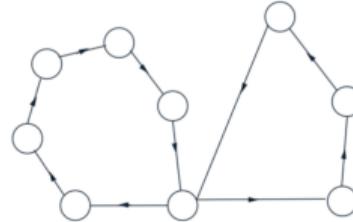
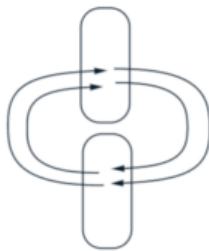
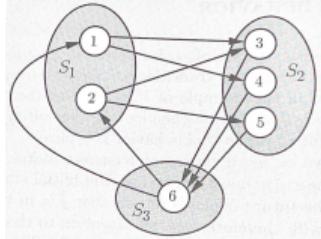
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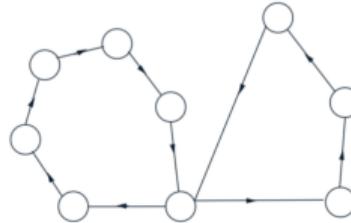
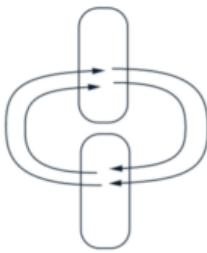
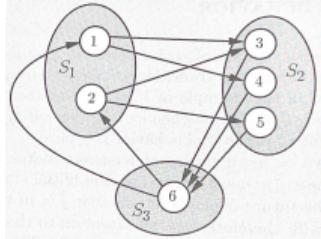
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Periodicity

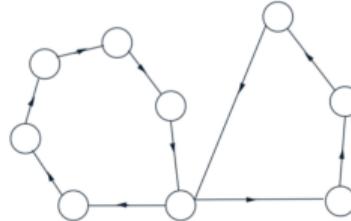
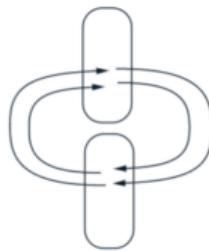
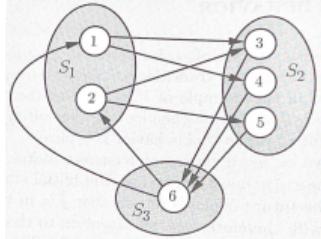




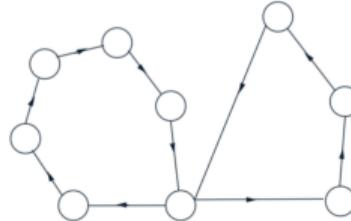
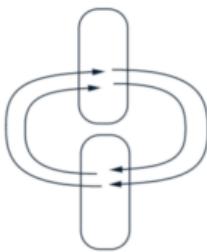
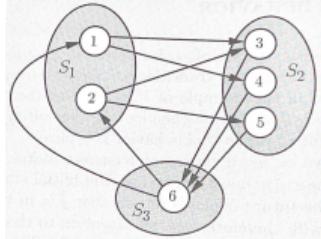
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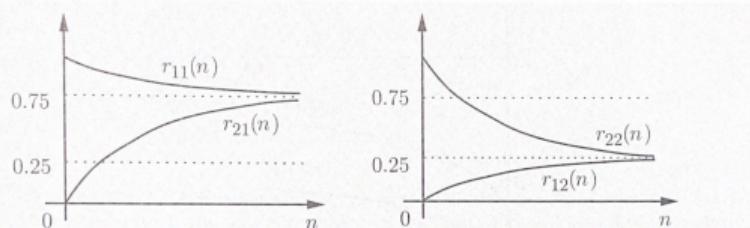


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- Often, it is not easy to see some MC is periodic or not. But, one easy way is to check whether there exists a self-transition or not. **An MC with a self-transition must be aperiodic.**

Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

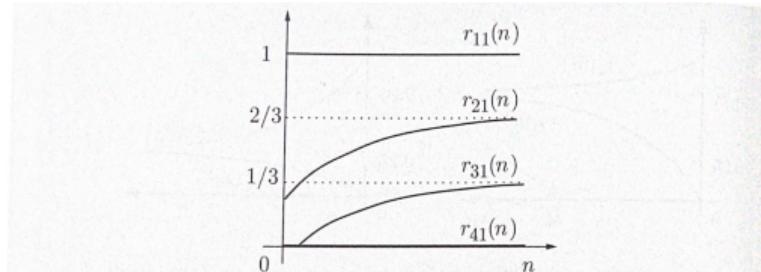
n -step transition prob.: $r_{ij}(n)$ for large n



n -step transition probabilities as a function of the number n of transitions

	U	B				
U	0.8	0.2	.76	.24	.752	.248
B	0.6	0.4	.72	.28	.744	.256
$r_{ij}(1)$			$r_{ij}(2)$		$r_{ij}(3)$	
						$r_{ij}(4)$
						$r_{ij}(5)$

Sequence of n -step transition probability matrices



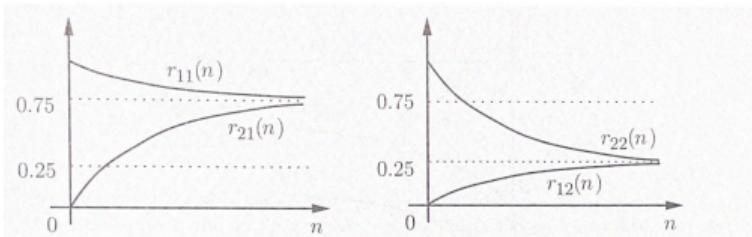
n -step transition probabilities into state 1

	1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4		1	2	3	4	
1	1.0	0	0	0		1.0	0	0	0		1.0	0	0	0		1.0	0	0	0		1.0	0	0	0	
2	0.3	0.4	0.3	0		.42	.25	.24	.09		.50	.17	.17	.16		.55	.12	.12	.21		2/3	0	0	1/3	
3	0	0.3	0.4	0.3		.09	.24	.25	.42		.16	.17	.17	.50		.21	.12	.12	.55		1/3	0	0	2/3	
4	0	0	0	1.0		0	0	0	1.0		0	0	0	1.0		0	0	0	1.0		0	0	0	1.0	
						$r_{ij}(1)$					$r_{ij}(2)$					$r_{ij}(3)$					$r_{ij}(4)$				

Sequence of transition probability matrices

n -step transition prob.: $r_{ij}(n)$ for large n

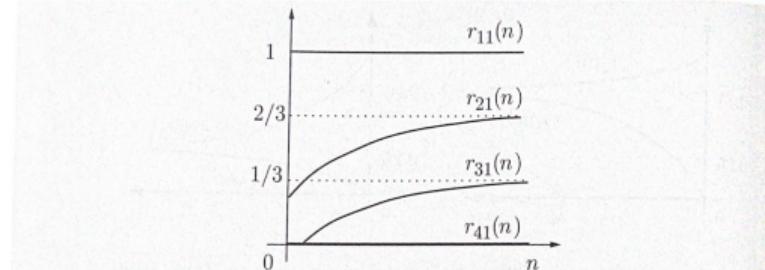
- Convergence **irrespective** of the start state



n -step transition probabilities as a function of the number n of transitions

	U	B				
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Sequence of n -step transition probability matrices

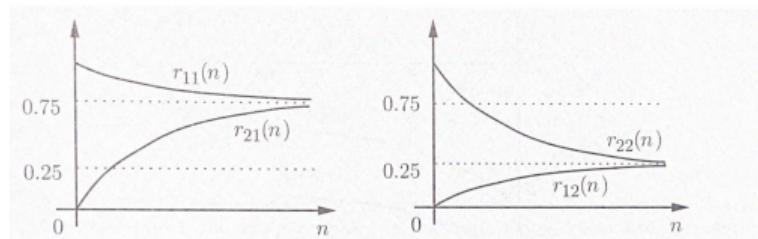


n -step transition probabilities into state 1

	1	2	3	4					
1	1.0	0	0	0	$r_{ij}(1)$	$r_{ij}(2)$	$r_{ij}(3)$	$r_{ij}(4)$	$r_{ij}(\infty)$
2	0.3	0.4	0.3	0					
3	0	0.3	0.4	0.3					
4	0	0	0	1.0					

Sequence of transition probability matrices

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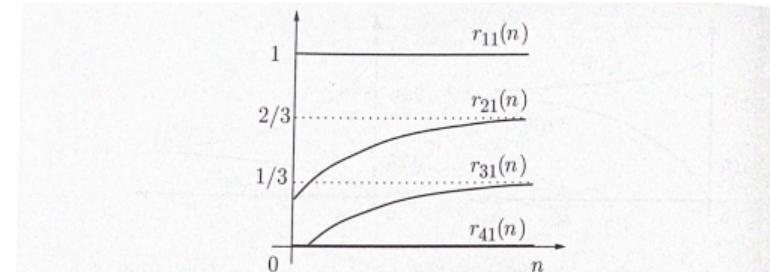


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Sequence of *n*-step transition probability matrices

- Convergence **depending on** the start state

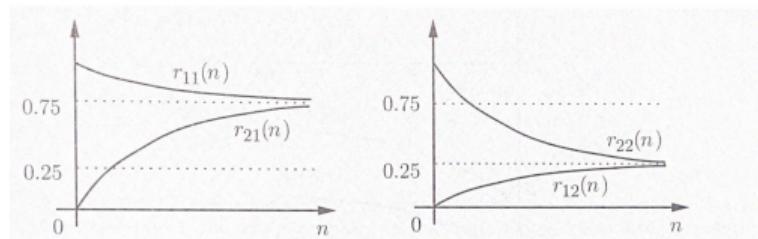


n-step transition probabilities into state 1

	1	2	3	4					
1	1.0	0	0	0					
2	0.3	0.4	0.3	0	.42	.25	.24	.09	
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Sequence of transition probability matrices

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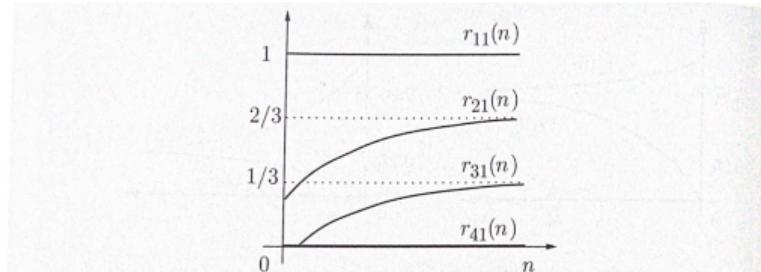


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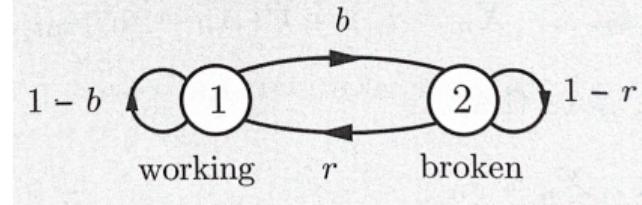
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	$r_{ij}(1)$		$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(\infty)$

Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs, **independent of** the start state? If so, how does it depend on the start state and the shape of the MC?

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?

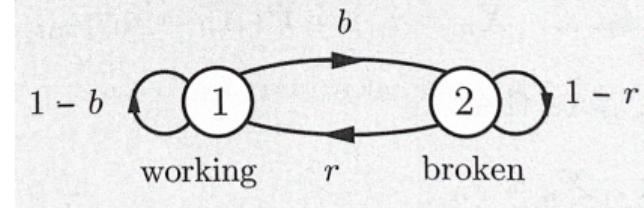


$$\pi_{\text{working}} = \alpha$$

$$\pi_{\text{broken}} = \beta$$

- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
- Interpretation.

$\pi_j \approx \mathbb{P}(X_n = j)$ for large n



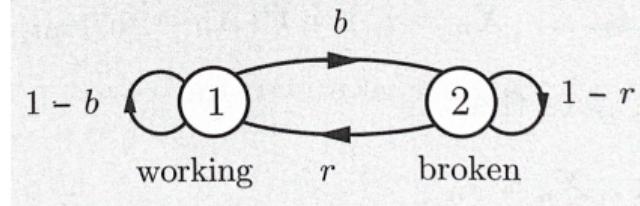
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- After running the MC for a long time, we see how long the MC will stay at which state on average.



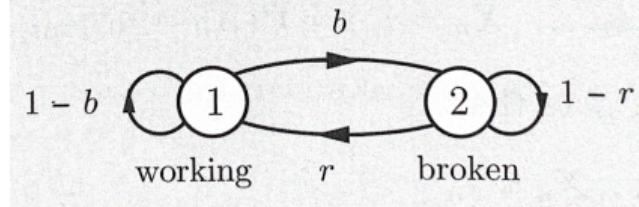
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- After running the MC for a long time, we see how long the MC will stay at which state on average.
- Helps in understanding how this MC behaves.



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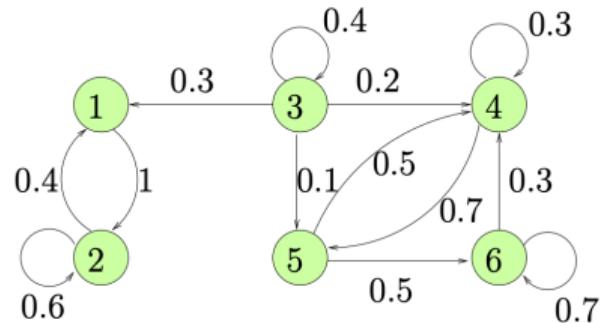
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- $r_{ij}(n) \xrightarrow{n \rightarrow \infty} \pi_j$, for some $\pi_j \leq 1$?
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C1. Only a **single recurrent class**

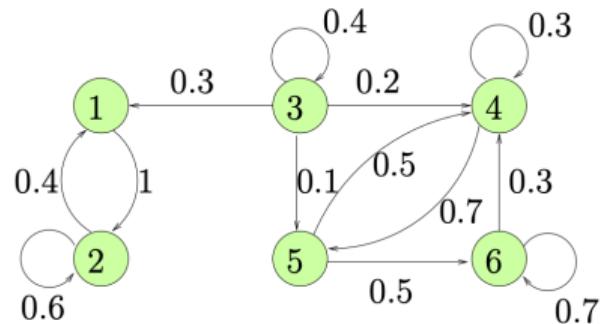
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C1. For the case of multiple recurrent classes, one stays at the class including the starting state.



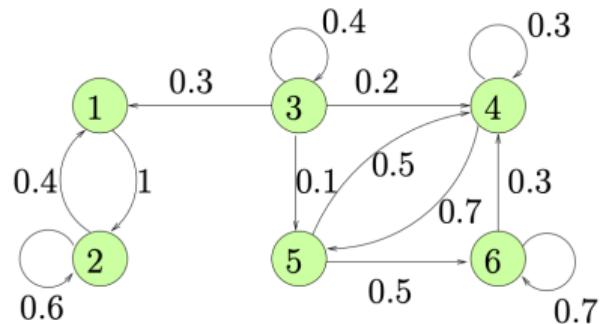
(a) multiple recurrent classes

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 - Convergence occurs, **independent of** the starting state, if:
 - C1.** Only a **single recurrent class**
 - C2.** such recurrent class is **aperiodic**
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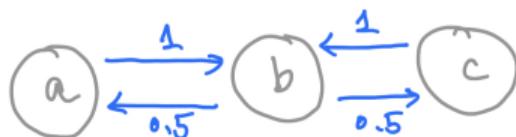


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 - C2.** such recurrent class is **aperiodic**
- C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.
- C2.** Divergent behavior for periodic recurrent classes.



(a) multiple recurrent classes



(b) single recurrent, but periodic class

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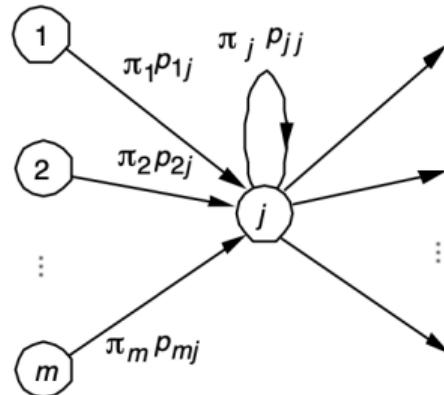
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- $\sum_{i=1}^m \pi_i = 1$: (Normalization equation)
- Balance eqn. + Normalization eqn. \implies Finding the steady-state probabilities $\{\pi_i\}$.
 - Solving linear equations

- Probability: often interpreted as the **relative frequencies** out of many independent trials
- $\pi_j = \lim_{n \rightarrow \infty} \frac{v_{ij}(n)}{n}$, where $v_{ij}(n)$ is the expected number of visits to state j up to the first n transitions
- In other words, π_j : long-term **expected fraction of time** that the MC is at the state j .
- $\pi_j p_{jk}$: the long-term expected **fraction of transitions** that move the state **from j to k** .

- Balance equation: $\sum_{k=1}^m \pi_k p_{kj} = \pi_j$
 - The expected frequency of visits to j = The sum of the expected frequencies of transitions that lead to j .



- A two-state MC with: $\begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$
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$$\pi_1 =$$

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- We say that "the limiting distribution (steady-state distribution) is equal to the stationary distribution"

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Example 2

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VIDEO PAUSE

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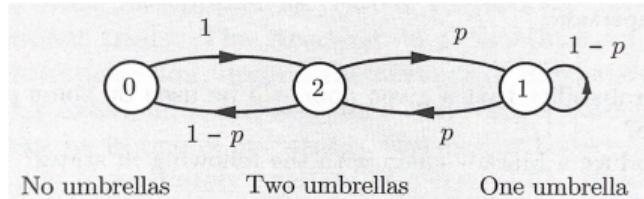
VIDEO PAUSE

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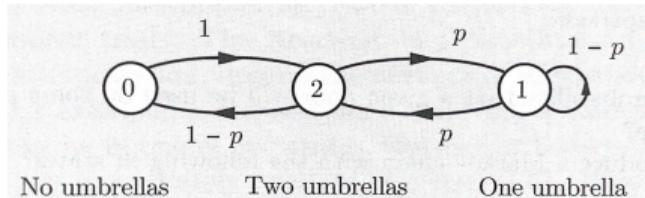


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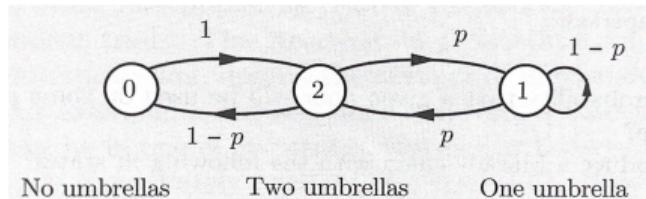
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VIDEO PAUSE

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- Single recurrent class and aperiodic
- Balance and normalization equation

$$\pi_0 = \quad , \quad \pi_1 =$$

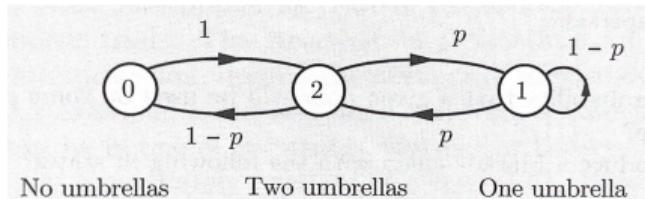
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Example 2

- An absent-minded professor: two umbrellas from home to office and back.
- If it rains and an umbrella is available, she takes it. If it is not raining, she always forgets to take an umbrella.
- Suppose that it rains w.p. $0 < p < 1$ each time when she commutes, independent of other times.
- (Q) What is the steady-state probability that she gets wet during a commute?
- (Hint) If you think that this can be modeled by a MC, think about what should be chosen as states. What is changing over time?

VIDEO PAUSE

- state $i \in \{0, 1, 2\}$: i umbrellas available in her location.
- Transition diagram



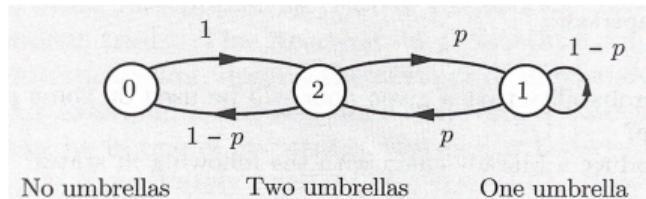
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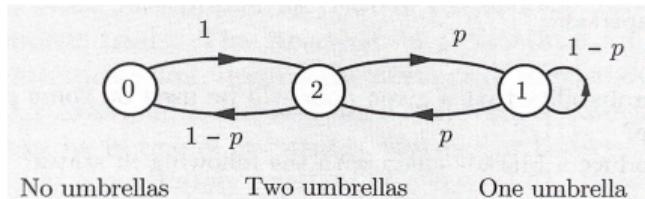
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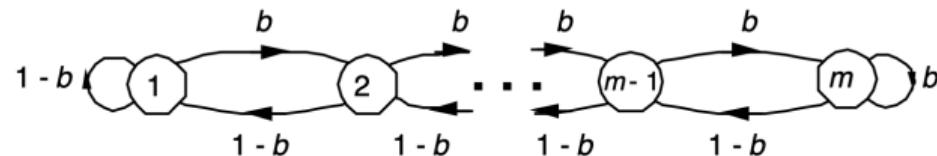
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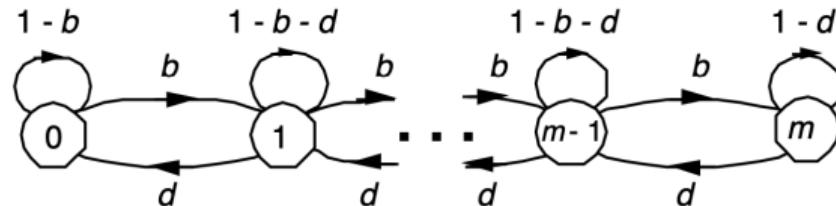
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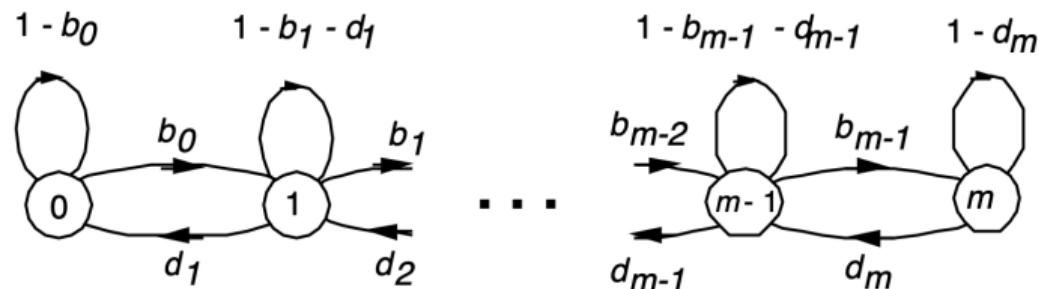
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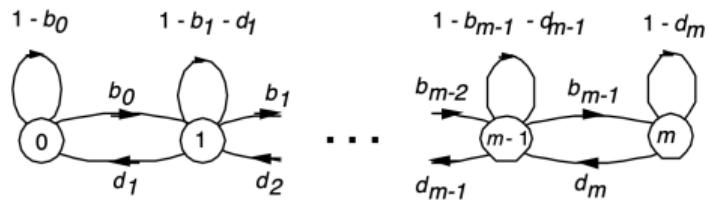
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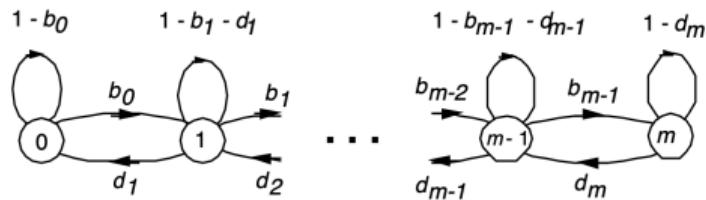
$$\begin{aligned} b_i &= \mathbb{P}(X_{n+1} = i + 1 | X_n = i), && \text{birth probability at state } i \\ d_i &= \mathbb{P}(X_{n+1} = i - 1 | X_n = i), && \text{death probability at state } i \end{aligned}$$



- State transition diagram



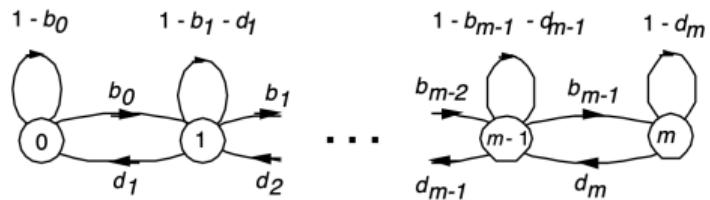
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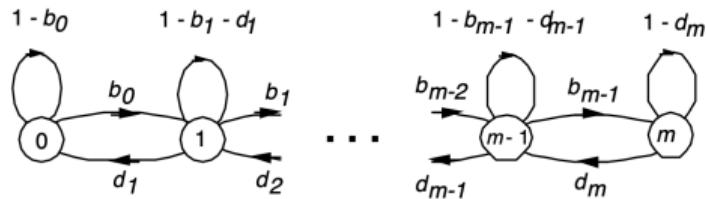
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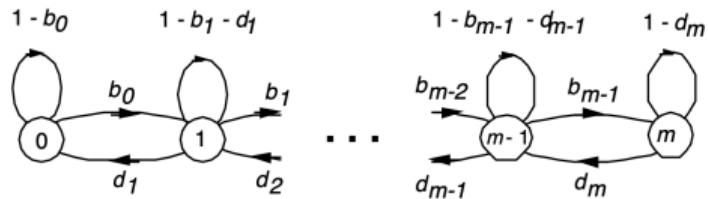
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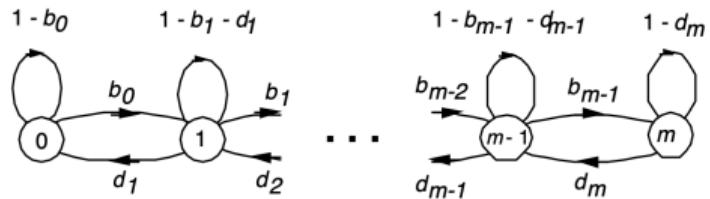
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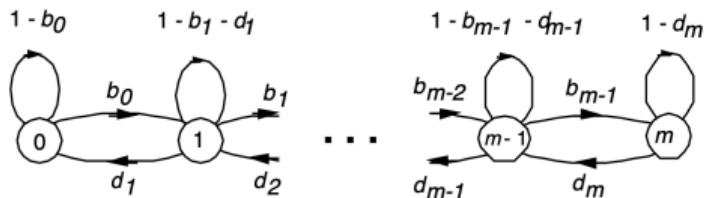
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Birth-Death Process (2)

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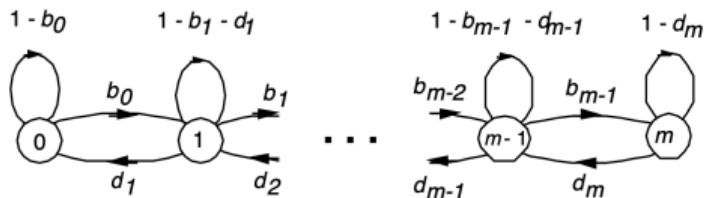
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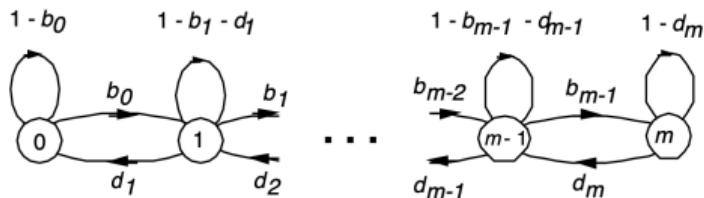
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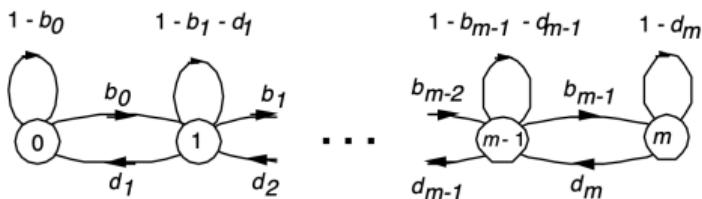
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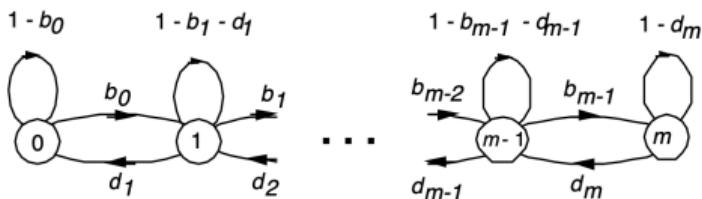
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- Examples 3 and 4 are the special cases of birth-death process. So, please compute the steady-state probabilities for both examples as your homeworks.

Markov Chain

- (1) Definition, Transition Probability Matrix, State Transition Diagram
- (2) n -step Transition Probability
- (3) Classification of States
- (4) Steady-state Behaviors and Stationary Distribution
- (5) Transient Behaviors

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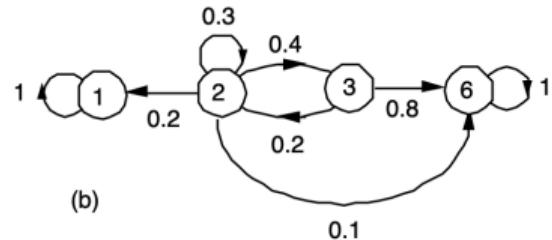
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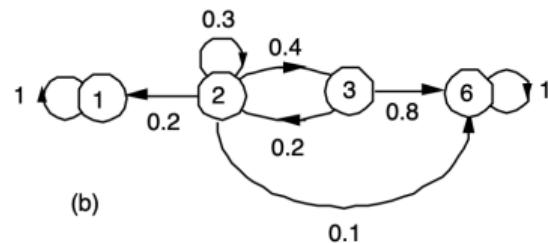
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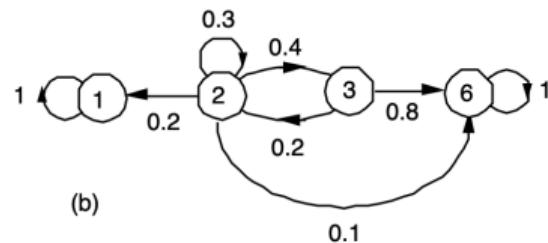
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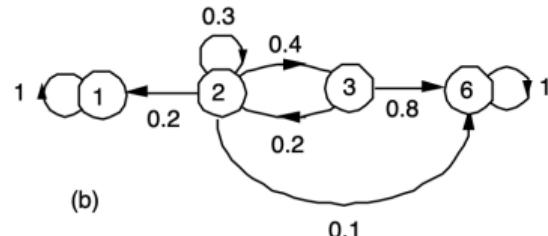
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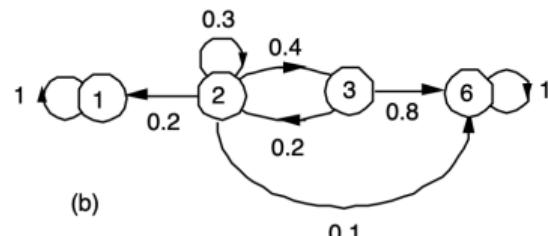
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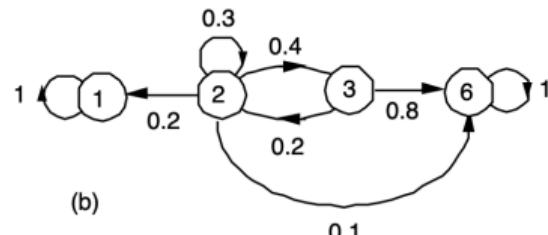


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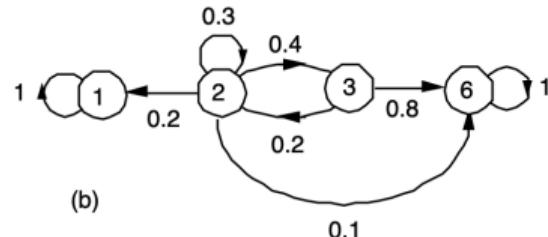
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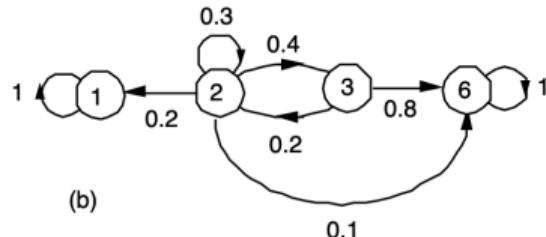
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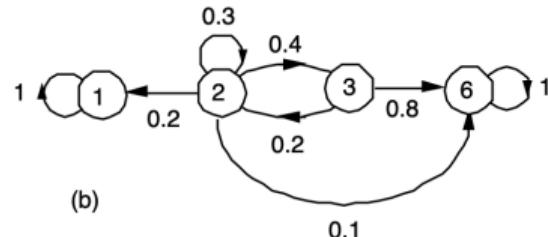
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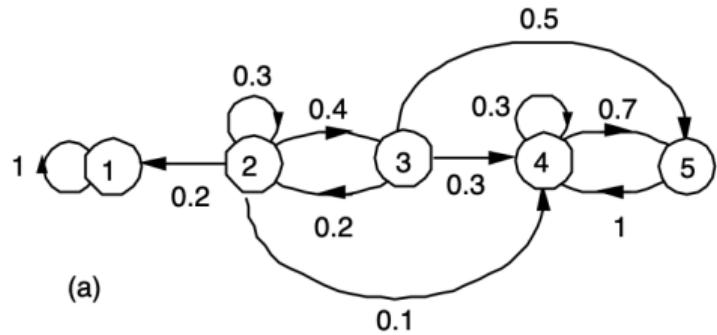
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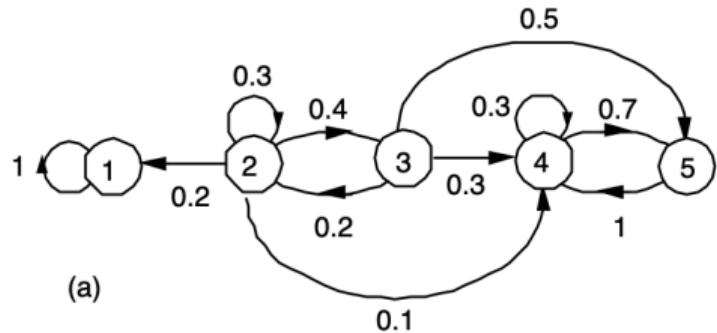


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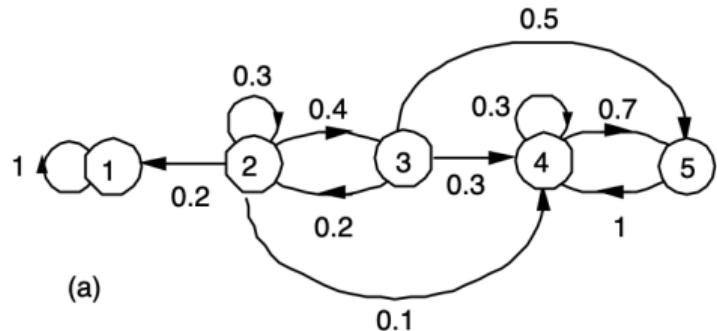
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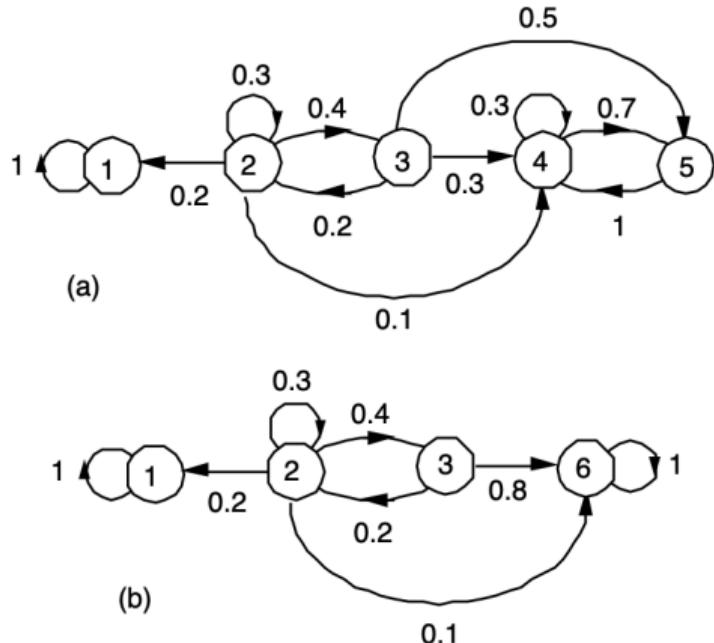
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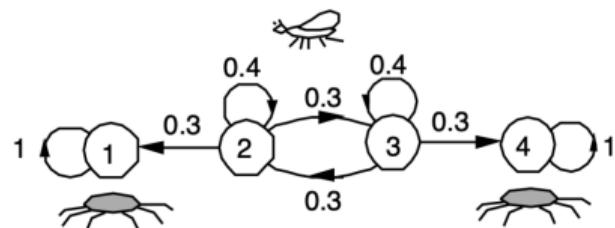
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- Thus, convert it into the one only with absorbing recurrent states ((a) \rightarrow (b)).

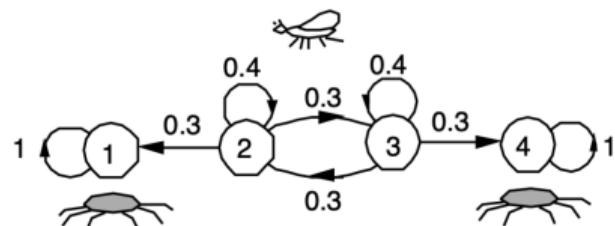


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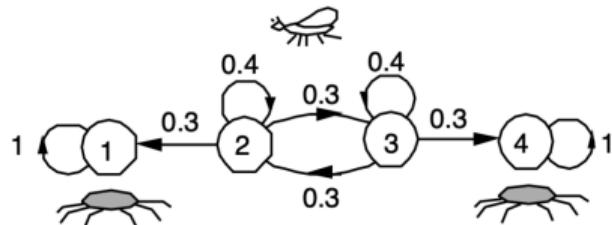


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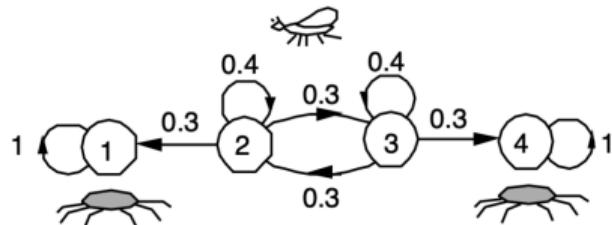


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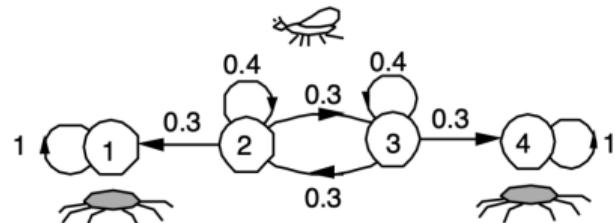


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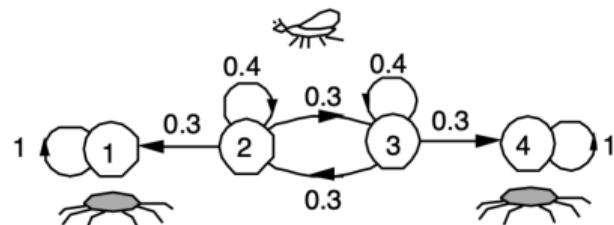


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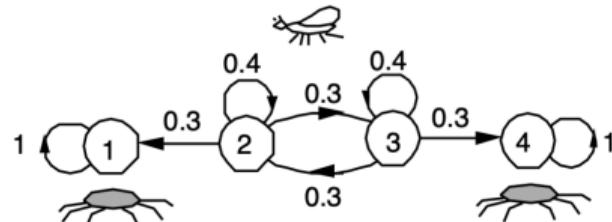
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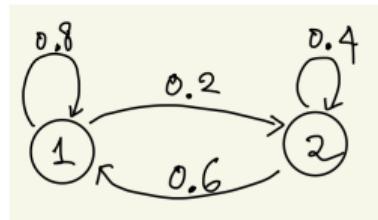
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- Again, for general MCs, convert them into the one with only recurrent states that are absorbing

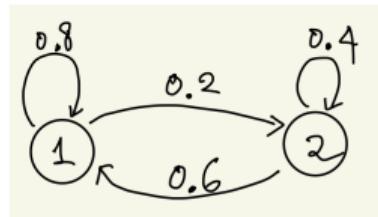


- Assume a single recurrent class for simplicity



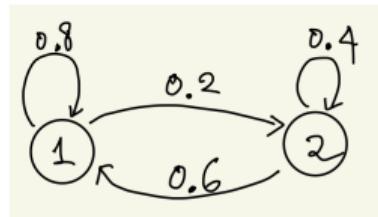
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$$t_1 = 0$$

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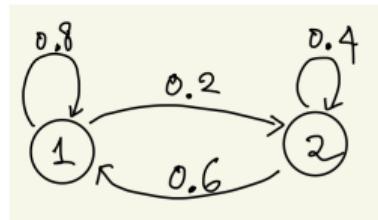
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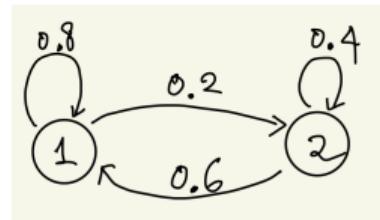
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- Mean first recurrence time from 1 to 1

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2\frac{5}{3} = \frac{4}{3}$$



Questions?

- 1) What is MC? Explain its definition and also its relation to BP and PP. Why do you think Markov chain (MC) is important?
- 2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
- 3) What are two representation tools to completely describe a given MC?
- 4) How are you going to compute the steady-state probability, if you are given a transition probability matrix? What is the meaning of the balance equation?
- 5) What is n -step transition probability? What is its relation to \mathbf{P}^n , where \mathbf{P} is the state transition matrix? What is the limiting distribution?

- 6) What are recurrent and transient states in MC? How can we talk about periodicity of a given MC?
- 7) What are the limiting distribution (steady-state distribution) and the stationary distribution of MCs? When are they equal? What is the steady-state convergence theorem and its meaning?
- 8) What is birth-death process? What is the local balance equation?
- 9) What are the main questions to investigate the transient behavior of MCs?