Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

September 23, 2021

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Roadmap

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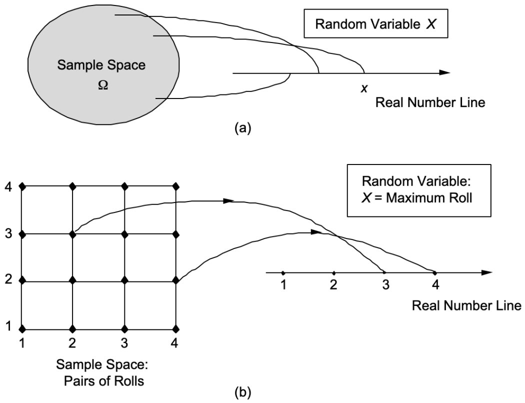
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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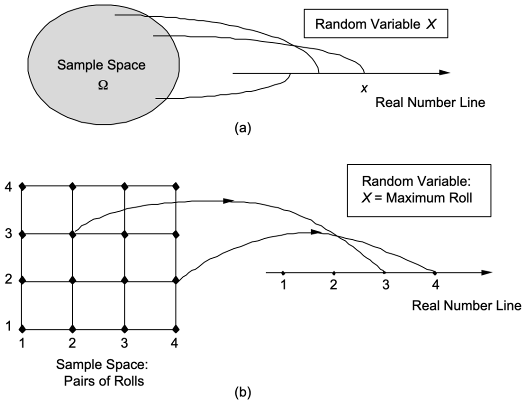

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Random Variable: More Formally

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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Example

- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$

- Remember that the random variable X is a **function** from Ω to $\mathbb{R}$

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Bernoulli X with parameter $p \in [0, 1]$

- Only binary values

[1] w.p.: with probability

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$$X = \begin{cases} 0, & \text{w.p.} \quad 1 - p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an indicator rv of an event A. Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs}, \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$

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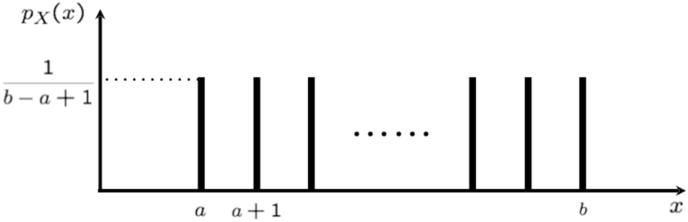

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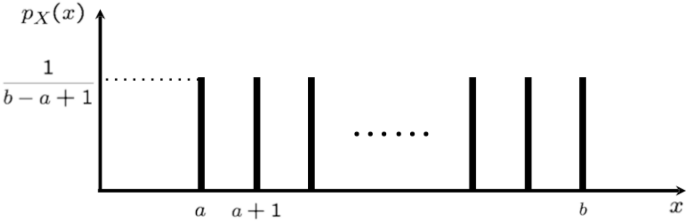

- Models complete ignorance (I don't know anything about X)

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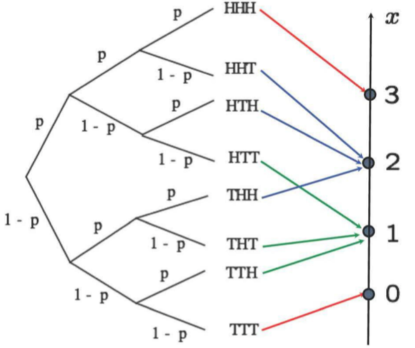

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L3(2)

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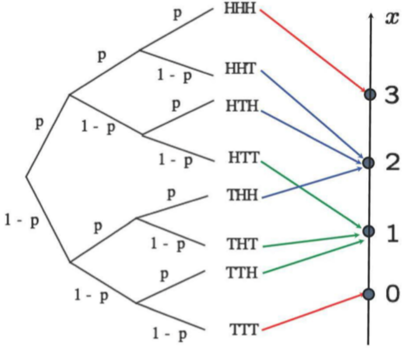
$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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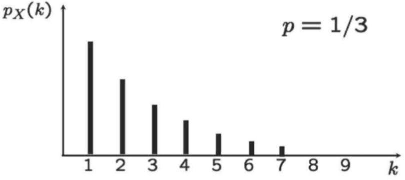
$$p_X(k) = (1-p)^{k-1}p$$



- Infinitely many independent Bernoulli trials, where each trial has success probability p
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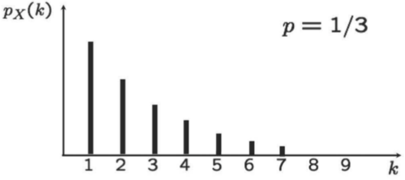
- Models waiting times until something happens.



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- Average

Definition

$$\mathbb{E}[X] = \sum_{x} x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)

Expectation/Mean

- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

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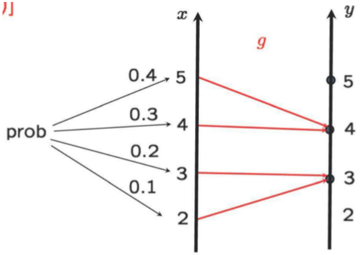

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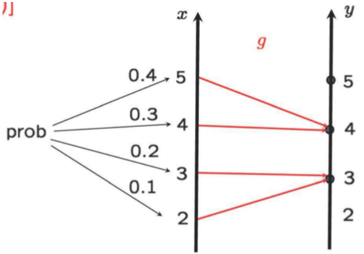

$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2) = 2.8 + 0.9 = 3.7$$

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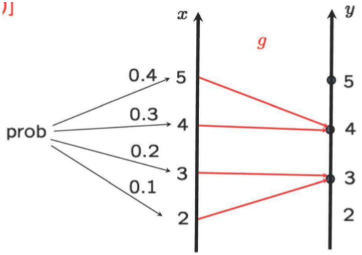

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Linearity of Expectation

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

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Example: Variance of a Bernoulli rv (p)

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Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}
\mu = \mathbb{E}[X] &= 1 \times p + 0 \times (1-p) = p \\
\mathbb{E}[X^2] &= 1 \times p + 0 \times (1-p) = p \\
\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\
&= p(1-p)
\end{aligned}$$

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- For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$\mathbb{P}(\{X = x\} \cap \{Y = y\})$$

- $\boxed{\text{Joint PMF.}}$ For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

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- $\boxed{\phantom{\text{Marginal PMF.}}}$

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

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- **Joint PMF.** For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

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Example. VIDEO PAUSE

y					
4	1/20	2/20	2/20		
3	2/20	4/20	1/20	2/20	
2		1/20	3/20	1/20	
1		1/20			
	1	2	3	4	x

$p_{X,Y}(1,3) =$

$p_X(4) =$

$\mathbb{P}(X = Y) =$

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y \ x	1	2	3	4
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3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

Functions of Multiple RVs

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

Linearity of Expectation for Multiple RVs

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Linearity of Expectation for Multiple RVs

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

(easy to prove, using the definition.)

Linearity of Expectation for Multiple RVs

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Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Conditional PMF: Conditioning on an event

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KAIST EE

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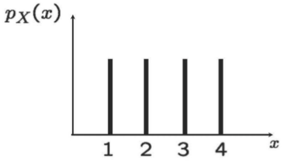
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$A = \{X \geq 2\}$



$\mathbb{E}[X] =$

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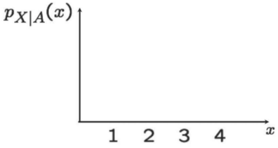

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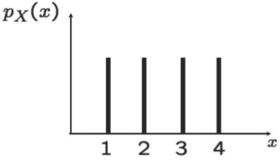

$\mathbb{E}[X] = \frac{1}{4}(1 + 2 + 3 + 4) = 2.5$

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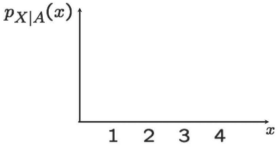

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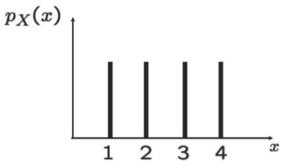

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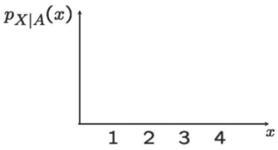

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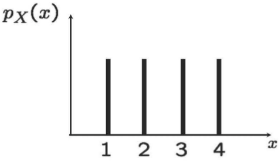

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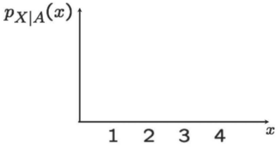

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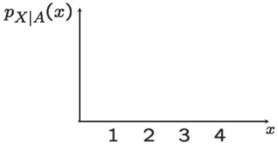

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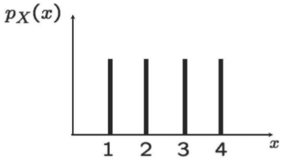

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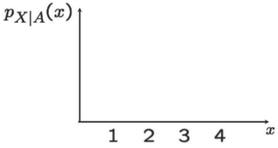

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- Conditional PMF

- Multiplication rule

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$

Conditional PMF

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$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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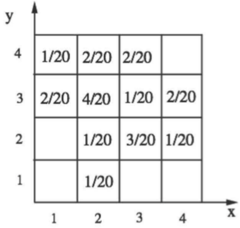
VIDEO PAUSE



$p_{X|Y}(2|2) =$

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$\mathbb{E}[X|Y = 3] =$

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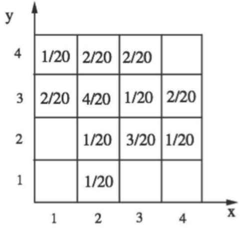


$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

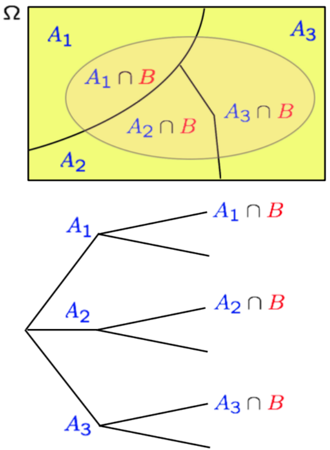
$$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$$

Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

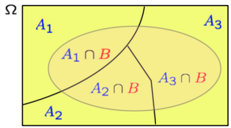


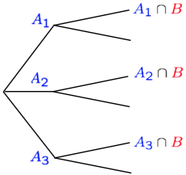
Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$





Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

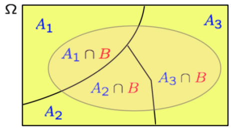
Total Probability Theorem

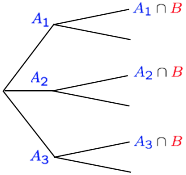
$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

- Weighted average of expectations from A_i's perspective





Total Expectation Theorem for $\{Y = y\}$

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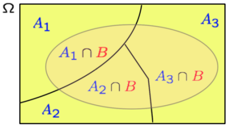

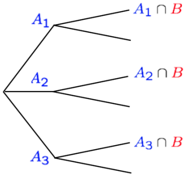

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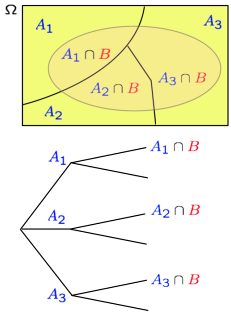
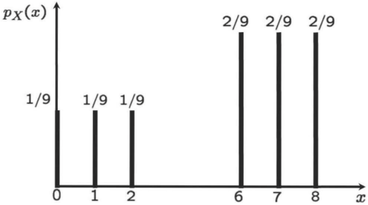


- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$$\mathbb{E}[X] =$$



Example 1: Total Expectation Theorem

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(1) Just using the definition of expectation,

$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$
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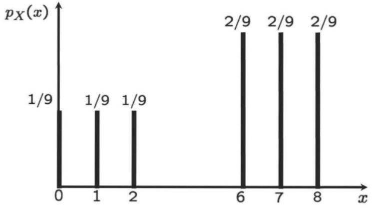

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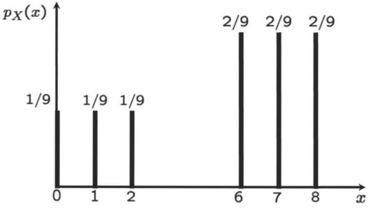
(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$



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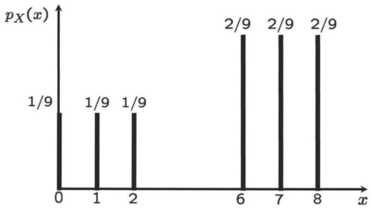
$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$

$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



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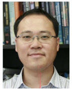
- Total expectation theorem and a notion of memorylessness helps a lot.

Memoryless Property: Motivating Example

Yung arrives

No bus arrives



How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n + m | X > m)$$

time

0

m mins

A bus left at time 0

time

0

Lin arrives

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n)$$

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- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

Background: Memoryless Property of Geometric RVs

- Theorem. Any **geometric** random variable is memoryless.

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- Remind. Geometric rv X with parameter p

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- **Proof.**

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)}$$
$$= \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)$$

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Back to Example 2: Mean of Geometric rv

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

  $$\begin{aligned} \mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\ &= && \text{(from TET)} \\ &= && \text{(from memorylessness)} \end{aligned}$$

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- Thus, $\mathbb{E}[X] = \dfrac{1}{p}$

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Independence, Conditional Independence

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- Two events

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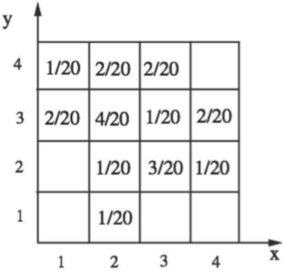
Example

- $X \perp\!\!\!\perp Y$?

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?



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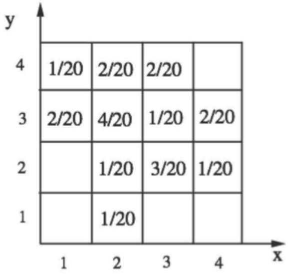

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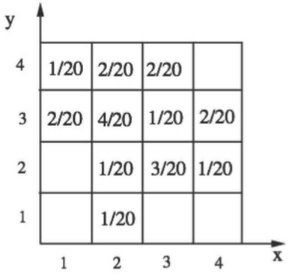
VIDEO PAUSE

$Y = 4$		
$Y = 3$		
	$X = 1$	$X = 2$



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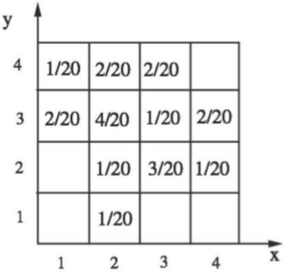

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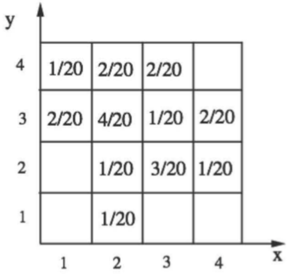
\- Yes.



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KAIST EE

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  - $X \perp\!\!\!\perp Y \implies$
    $\mathrm{var}[X-3Y] = \mathrm{var}[X] + 9\mathrm{var}[Y]$

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$$
\begin{aligned}
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&= \mathbb{E}[X^2 + Y^2 + 2XY] - \left((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right) \\
&= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)
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- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

Example: The hat problem (1)

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- X: number of people with their own hat
- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.

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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (from symmetry)

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Questions?

1) What is a random variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Geometric rvs. When are they useful and what are their PMFs?
4) Explain the memoryless property.
5) What are joint and marginal PMFs?
6) Describe and explain the total probability/expectation theorem for random variables? When is it useful to use total probability/expectation theorem?
7) Explain the definition and the meaning of expectation and variance. Why do we need them?
8) What is the difference between independence/conditional independence for events and those for random variables?