Lecture 5: Random Variable, Part III

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

Derived Distribution: $Y = g(X)$

- Given the PDF of X, What is the PDF of $Y = g(X)$?

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- What are easy or difficult cases?

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- Examples: $Y = X$, $Y = X + 1$, $Y = X^2$, etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x : g(x) = y} p_X(x) \end{aligned}$$

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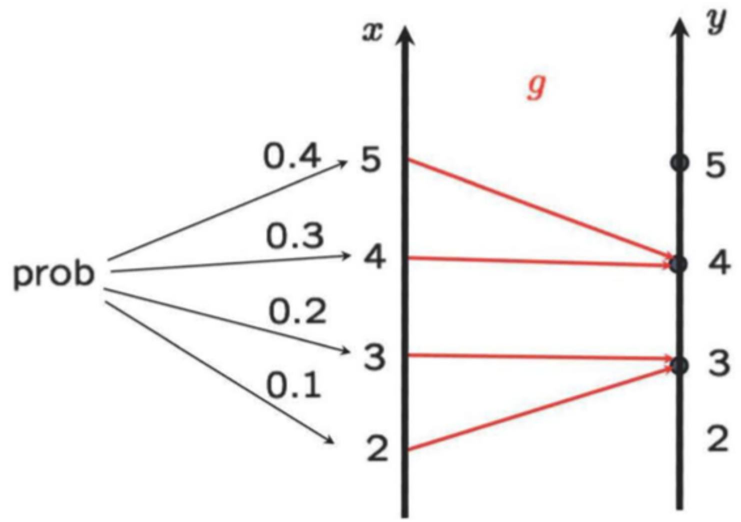

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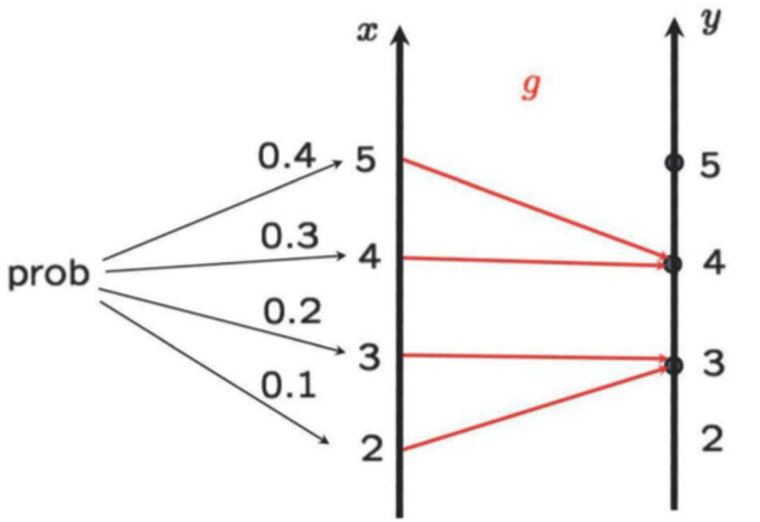
$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b,\ a \neq 0$

If $a > 0$,

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Linear: $Y = aX + b$, $a \neq 0$

If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \dfrac{y-b}{a}) = F_X(\dfrac{y-b}{a})$

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Special case. X is normal. Then, Y is also normal, i.e., $Y \sim N(a\mu + b, a^2\sigma^2)$

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Step 1. Find the CDF of Y:

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Step 2. Differentiate: $f_Y(y) = \frac{dF_Y}{dy}(y)$

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Ex1. $X \sim uniform[0,1]$. $Y = \sqrt{X}$.

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Ex2. $X \sim \textit{uniform}[0,2]$. $Y = X^3$.

$$F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \frac{1}{2}y^{1/3}$$

$$f_Y(y) = \frac{1}{6}y^{-2/3}, \quad 0 \le y \le 8$$

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Ex3. X with $f_X(x)$. $Y = X^2$.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$
$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0$$

Functions of multiple rvs: $Y = g(X, Y)$ (1)

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f_Z(z) = \begin{cases} 2z, & \text{if } 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
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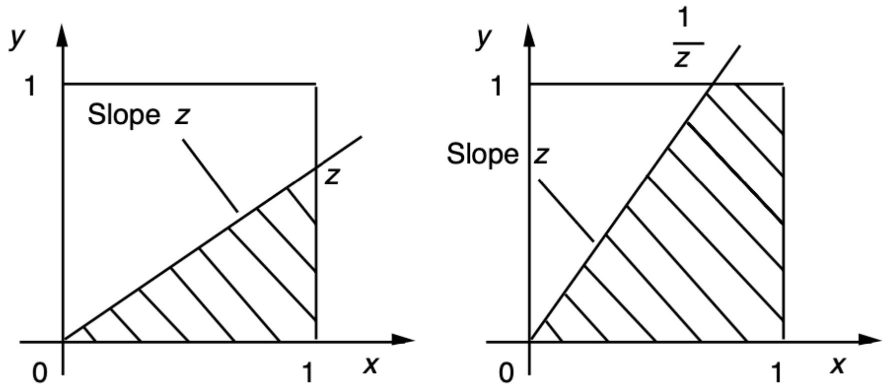

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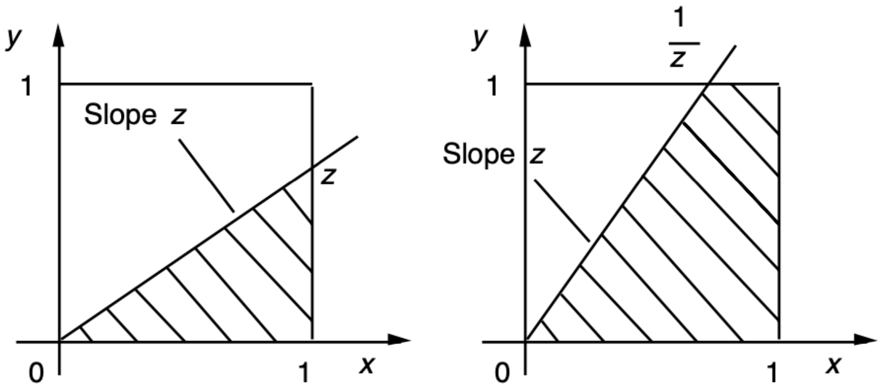

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$$f_Z(z) = \begin{cases} 1/2, & 0 \leq z \leq 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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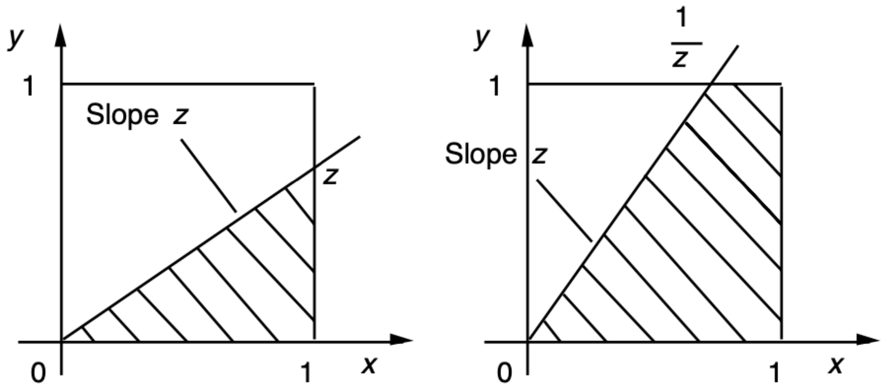

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&= \sum_x \mathbb{P}(X = x, Y = z - x) \\
&= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
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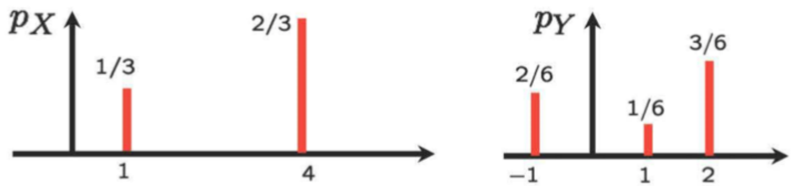
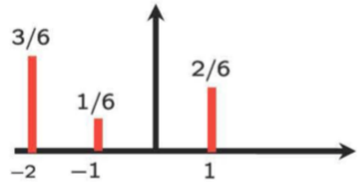
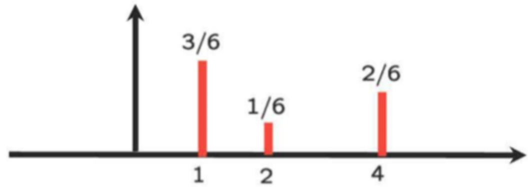
- Interpretation (for a given z)

(i) Flip (horizontally) $p_Y(y)$ ($p_Y(-x)$)

(ii) Put it underneath $p_X(x)$ ($p_Y(-x+z)$)







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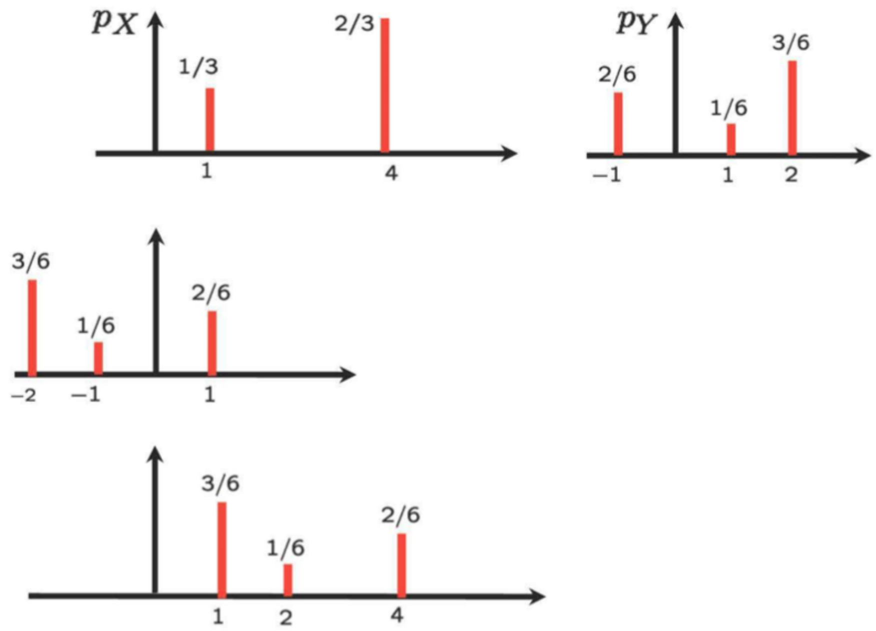

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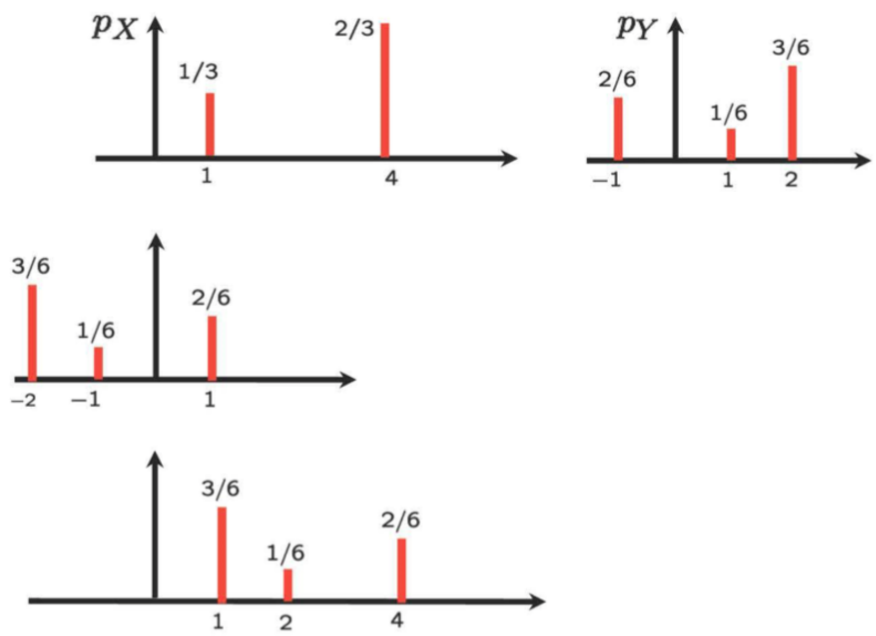

$Y = X + Y$, $X \perp\!\!\!\perp Y$: Continuous

- Same logic as the discrete case

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Sum of two independent normal rvs

$X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_x, \sigma_x^2)$
Then, $X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

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- Why normal rvs are used to model the sum of random noises.

- (Extension) The sum of finitely many independent normals is also normal.

Roadmap

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

OK. Let's Design!

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- Dependent: Positive (If $X \uparrow$, $Y \uparrow$) or Negative (If $X \uparrow$, $Y \downarrow$)

OK. Let's Design!

- Simple case: $\mathbb{E}[X] = \mu_x = 0$ and $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If $X \uparrow$, $Y \uparrow$) or Negative (If $X \uparrow$, $Y \downarrow$)
- What about $\mathbb{E}[XY]$? Seems good.

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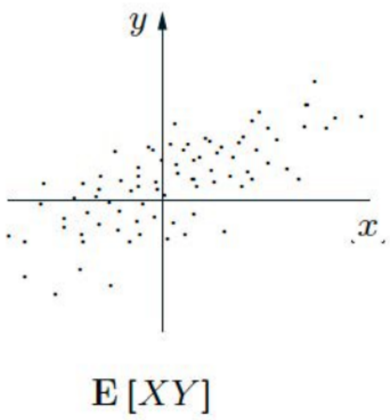

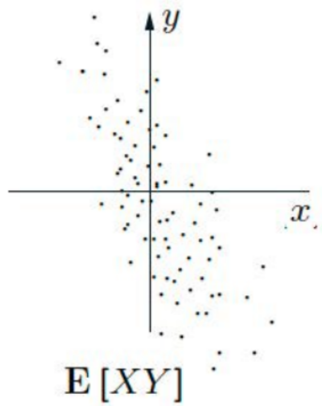

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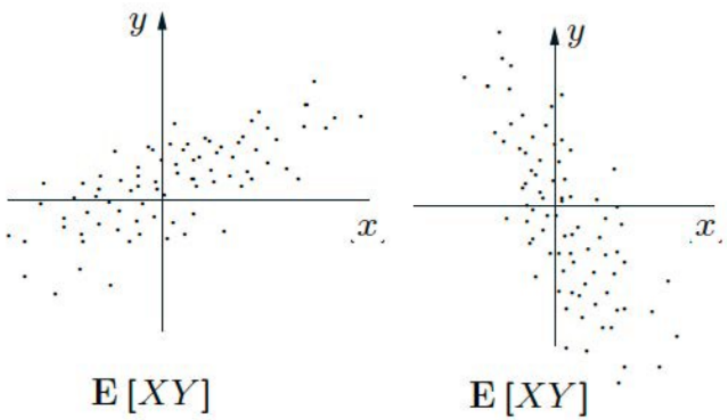

(Q) What about $\mathbb{E}[X + Y]$?

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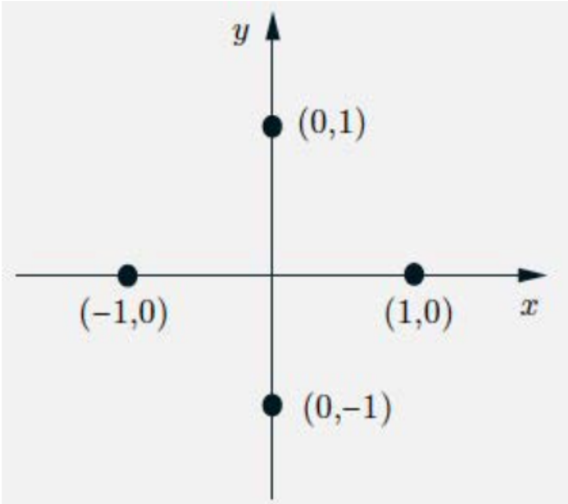

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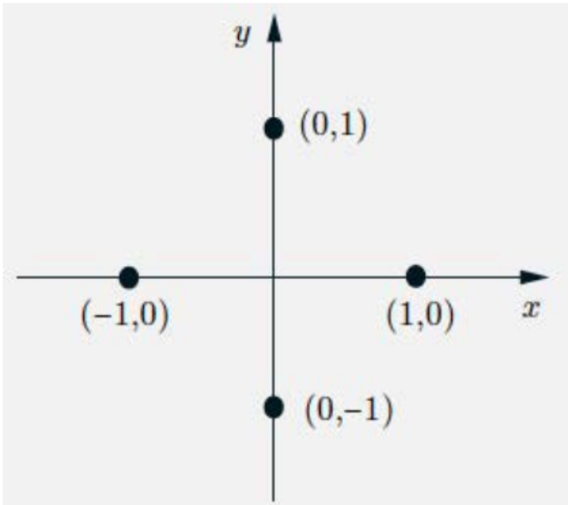

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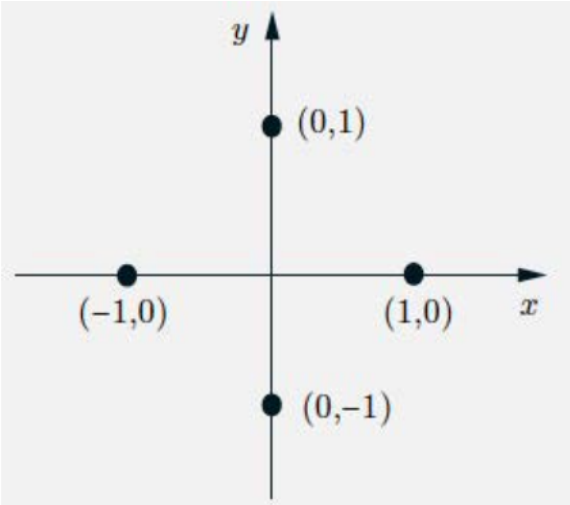
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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Example: The hat problem in Lecture 3. Remember?

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- (Q) var$[X]$
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$.
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&= \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\
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- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$ (linear relation, VERY related)

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

A Special Random Variable

- Consider a rv Y, such that

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

Conditional Expectation $\mathbb{E}[X|Y]$

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A random variable $g(Y) = \boxed{\phantom{\mathbb{E}[X|Y]}}$, called $\boxed{\phantom{\text{the conditional expectation of } X \text{ given } Y}}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

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- Thus, having a distribution, expectation, variance, all the things that a random variable has
- Often confusing because of the notation

Expectation of $\mathbb{E}[X|Y]$

Expectation of Conditional Expectation

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y = y]p_Y(y) \\ &= \mathbb{E}[X]\end{aligned}$$

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- Forecasts on sales: calculating expected value, given any available information

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- A function of Y
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has

Expectation and Variance of $\mathbb{E}[X|Y]$ and $\mathrm{var}[X|Y]$

	$\mathbb{E}[X\|Y]$	$\mathrm{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\mathrm{var}(X\|Y)\right]$
Variance	$\mathrm{var}\left[\mathbb{E}(X\|Y)\right]$	$\mathrm{var}\left[\mathrm{var}(X\|Y)\right]$

Law of Total Variance

KAIST EE

Law of total variance

$$\mathrm{var}[X] =$$

Proof.

$$(1)$$

$$(2)$$

Law of Total Variance

Law of total variance

$$\text{var}[X] = \mathbb{E}\Big[\text{var}(X|Y)\Big] + \text{var}[\mathbb{E}(X|Y)]$$

Proof.

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$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \mathrm{var}[X]$$

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Questions?

Review Questions

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
2) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?
3) What are covariance and correlation coefficient? Why do we need them?
4) Please explain the concepts of conditional expectation and conditional variance.