

# Lecture 7: Law of Large Numbers and Central Limit Theorem

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EE210: Probability and Introductory Random Processes  
KAIST EE

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- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
  - Moment Generating Function (MGF)
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- $\mathbb{E}[X_i] = \mu, \text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

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- Take a certain **scaling** with respect to  $n$  that corresponds to a **new glass**, and investigate the system for large  $n$
- First, consider the sample mean, and try to understand how it behaves:

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- We call this **law of large numbers**.

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- $M_n$  is a random variable, which is a function from  $\Omega$  to  $\mathbb{R}$ .
- Need to mathematically build up the concept of convergence for the sequence of random variables.



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## Convergence in probability

For any  $\epsilon > 0$ ,  $\mathbb{P}(|Y_n - a| \geq \epsilon) \xrightarrow{n \rightarrow \infty} 0$ .

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- Proof requires some knowledge about useful inequalities, which we cover later.

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# Central Limit Theorem: Start with Scaling (1)

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- The answer is  $\frac{1}{2}$

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- Interestingly, it converges to some **random variable  $Z$**  that we know very well.





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For every  $z$ ,

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- **Meaning from scaling perspective.**
  - LLN: Scaling  $S_n$  by  $1/n$ , you go to a deterministic world.
  - CLT: Scaling  $S_n$  by  $1/\sqrt{n}$ , you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each  $X_i$ . Very interesting!



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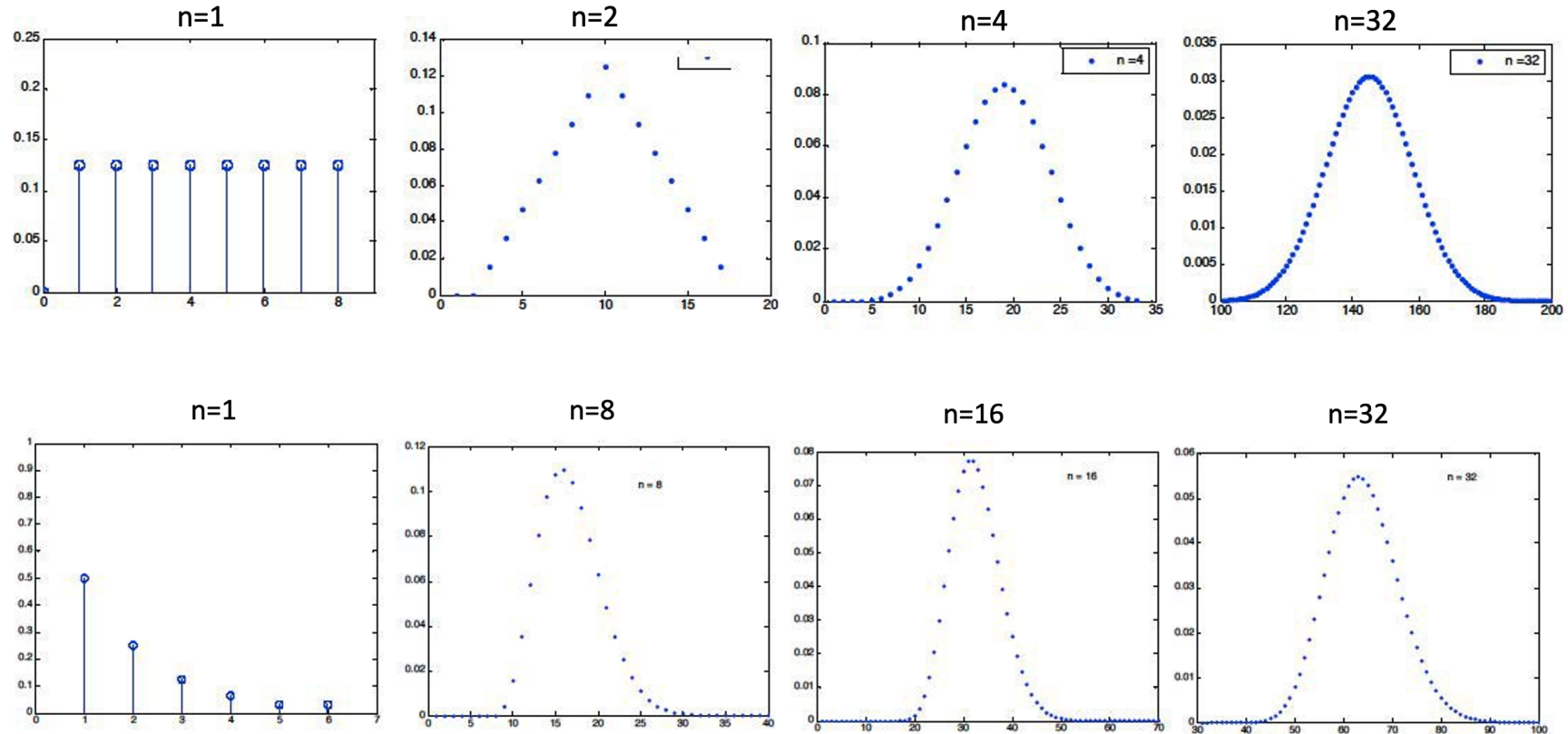
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- How large should  $n$  be?
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  - If  $X_i$  resembles a normal rv more, smaller  $n$  works: symmetry and unimodality<sup>1</sup>

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# CLT: Examples of $n$





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**Proof.** For any  $a > 0$ , define  $Y_a$  as:

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Then, using non-negativity of  $X$ ,  $Y_a \leq X$ , which leads to  $\mathbb{E}[Y_a] \leq \mathbb{E}[X]$ .



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If  $X \geq 0$  and  $a > 0$ , then  $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$ .

**Proof.** For any  $a > 0$ , define  $Y_a$  as:

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Thus,  $a \cdot \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$ . □





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## Example

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- Both bounds are the ones that bound the probability of rare events.

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

### Weak law of large numbers

$M_n$  converges to  $\mu$  in probability.

Proof.

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- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
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Ex1)  $X \sim \exp(\lambda)$ ,  $f_X(x) = \lambda e^{-\lambda x}$ ,  $x \geq 0$

$$\begin{aligned} M(s) &= \lambda \int_0^{\infty} e^{sx} e^{-\lambda x} dx \\ &= \lambda \left. \frac{e^{(s-\lambda)x}}{s-\lambda} \right|_0^{\infty} \quad (\text{if } s < \lambda) \\ &= \frac{\lambda}{\lambda - s} \end{aligned}$$

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Ex2)  $X \sim N(0, 1)$  (homework problem)

$$M(s) = e^{s^2/2}$$

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

## Inversion Property

The transform  $M_X(s)$  associated with a random variable  $X$  uniquely determines the CDF of  $X$ , assuming that  $M_X(s)$  is finite for all  $s$  in some interval  $[-a, a]$ , where  $a$  is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own MGF.

## Back to CLT

- Without loss of generality, assume  $\mathbb{E}(X_i) = 0$  and  $\text{var}(X_i) = 1$

Proof.

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- If we apply l'hospital's rule twice (please check), we get

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- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
  - Moment Generating Function (MGF)
- Strong Law of Large Numbers (Optional)

Questions?

- 1) What's the practical value of LLN and CLT?
- 2) Explain LLN and CLT from the scaling perspective.
- 3) Why are LLN and CLT great?
- 4) Why do we need different concepts of convergence for random variables?
- 5) Explain what is convergence in probability.
- 6) Explain what is convergence in distribution.
- 7) Why is MGF useful?