# Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

# Roadmap

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- Markov Chain
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

- Assume discrete times $n = 1, 2, \ldots$

- Random process: A sequence of $X_1, X_2, X_3, \cdots$

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  - Markov chain
  - One of the most popular random processes in engineering

# Example: Machine Failure, Repair, and Replacement

- A machine: working or broken down on a given day.
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- What will happen at $(n+1)$-th day depends only on what happens at $n$-th day?

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  Thus, for any $n \geq 0$, we introduce a simple notation $p_{ij}$

  $$p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$$

# Transition Prob. Matrix and State Transition Diagram

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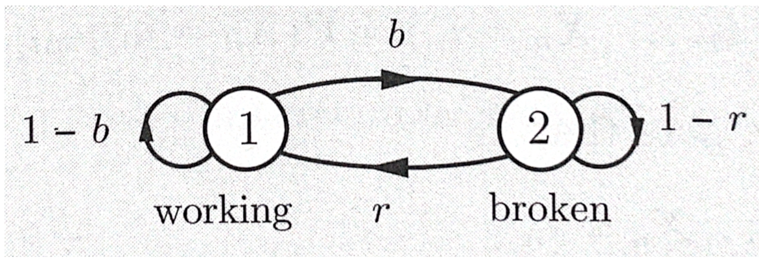
- Transition probability matrix

$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$

- State transition diagram



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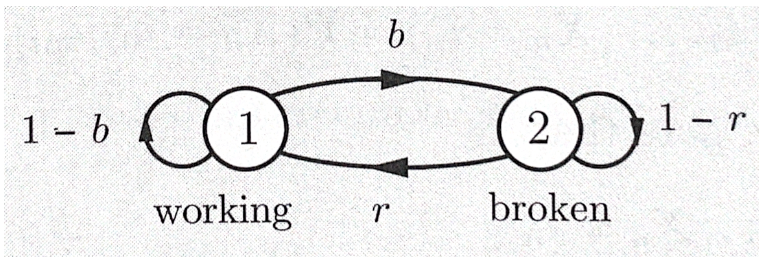


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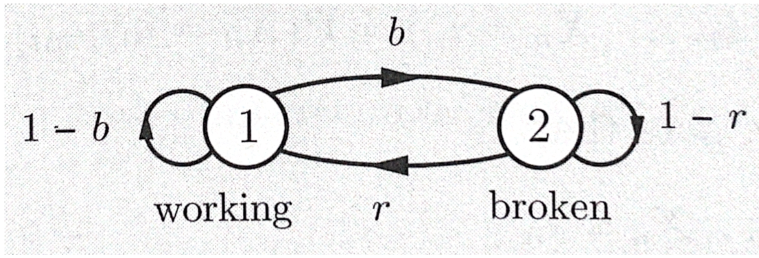


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- $\sum_{j=}^{m} p_{ij} = 1$ (for each row $i$, the column sum $= 1$)

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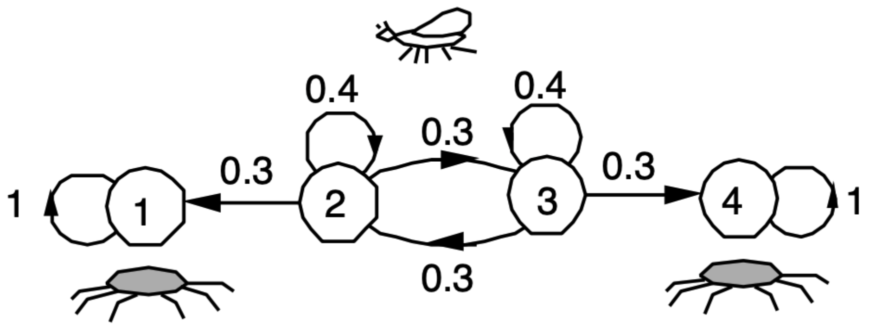


| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
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| 3 | 0 | 0.3 | 0.4 | 0.3 |
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$p_{ij}$

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  $$\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2) p_{22} p_{22} p_{23} p_{34} = \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2$$

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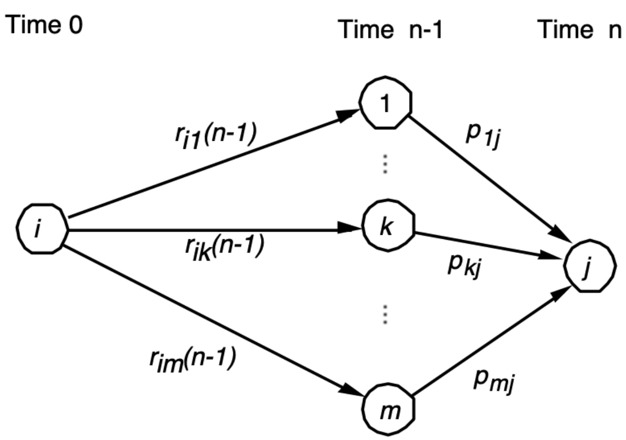

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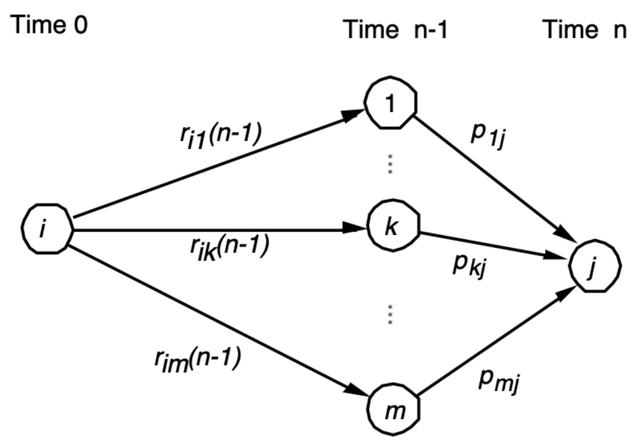

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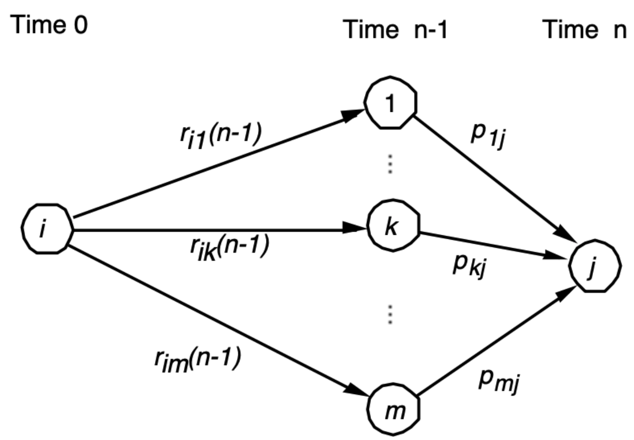
$$= \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$



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# Examples: Different States and Classes

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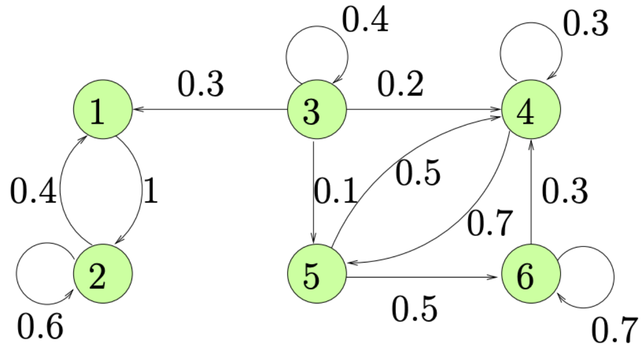

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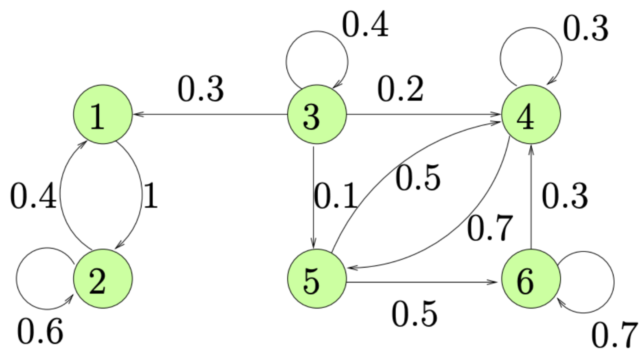

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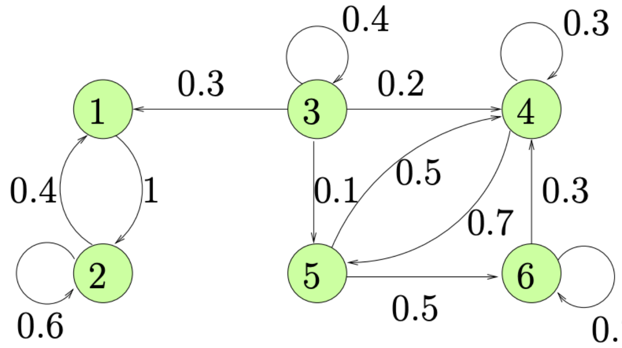

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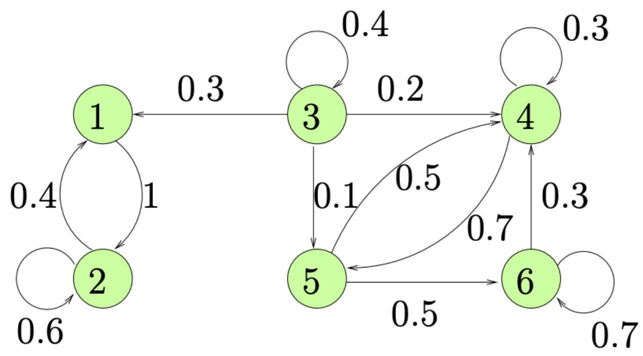

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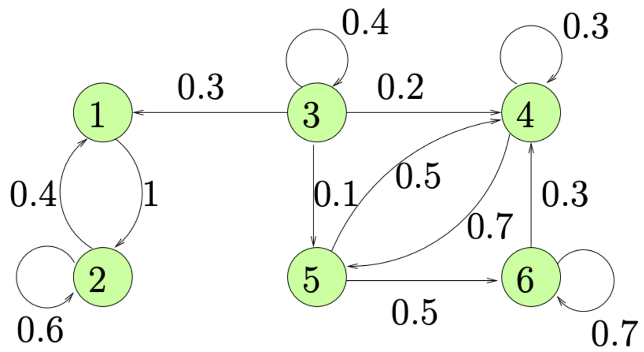
  - Divide into three classes: $\{3\}, \{1, 2\}, \{4, 5, 6\}$



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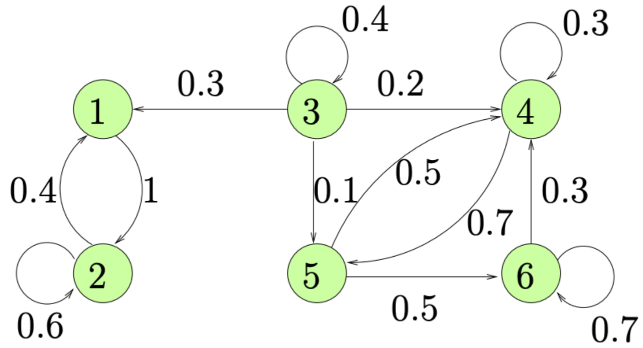

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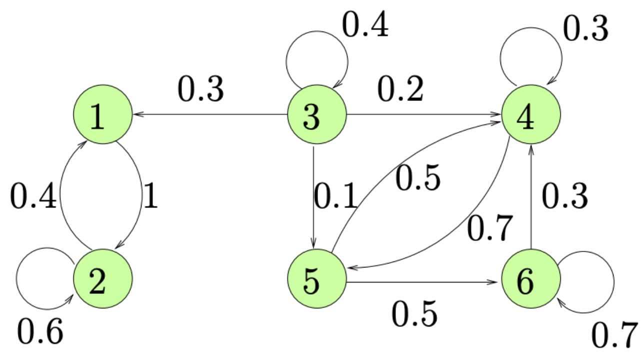

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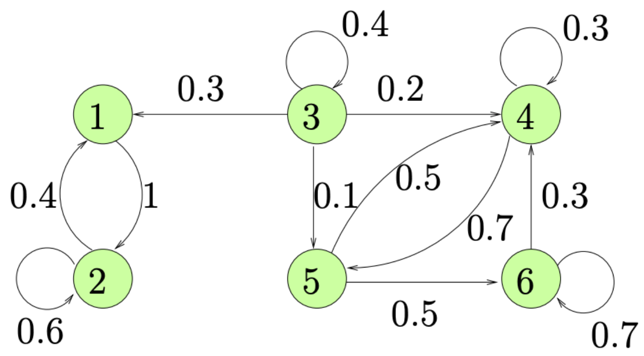


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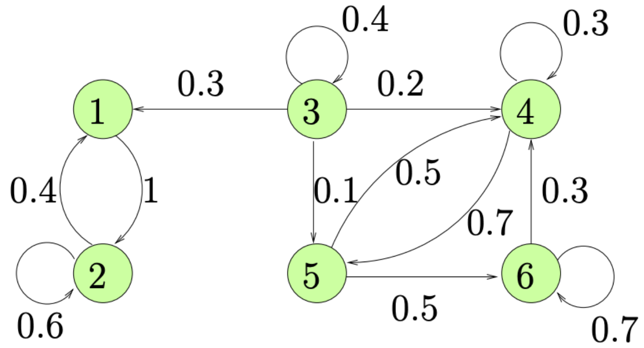


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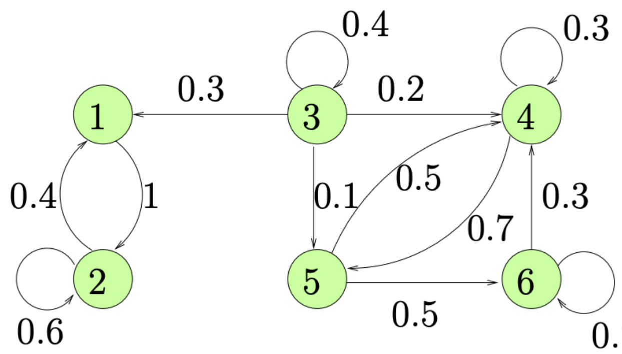


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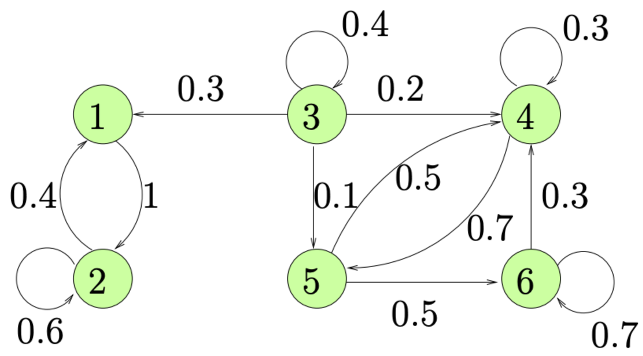


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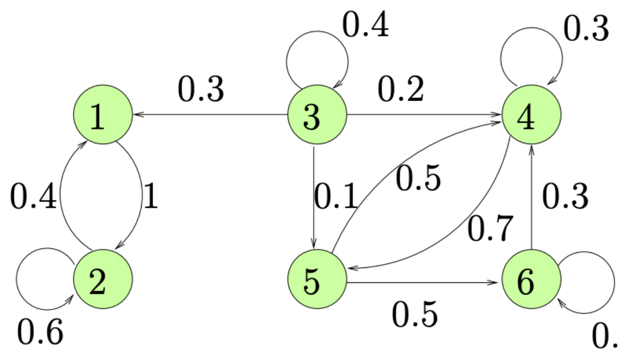


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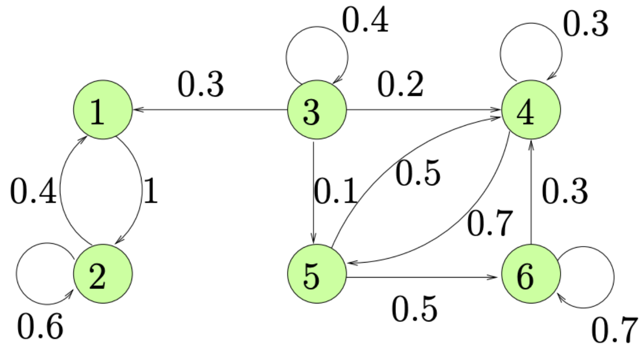

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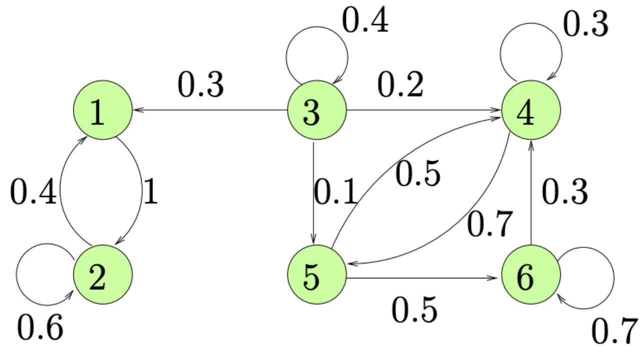

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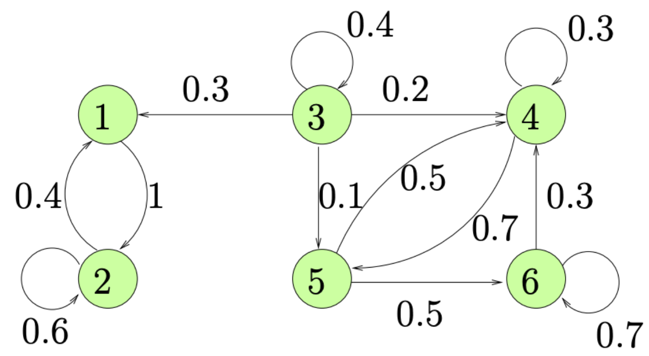

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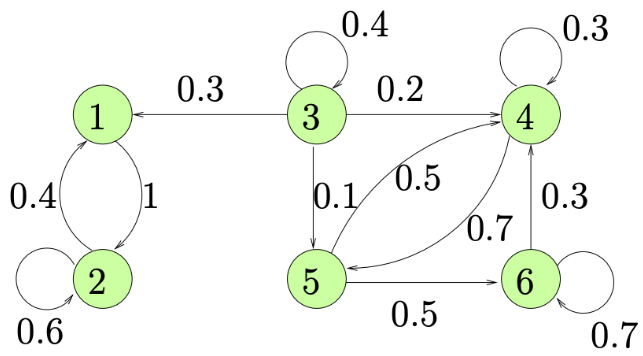

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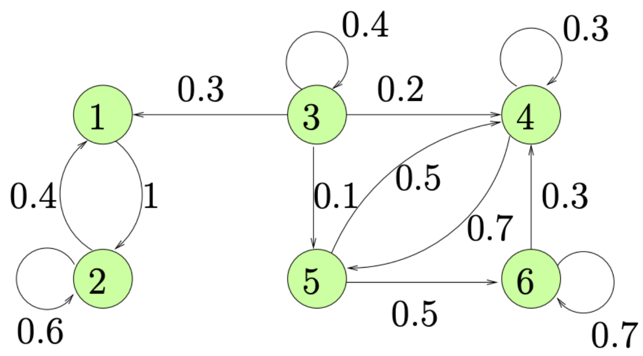

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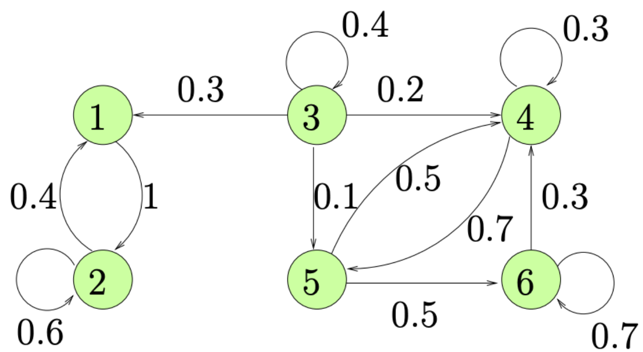


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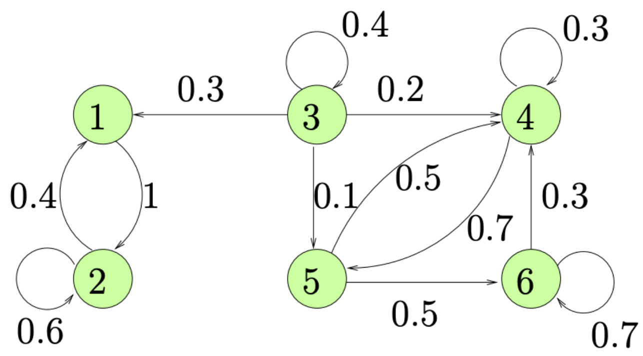


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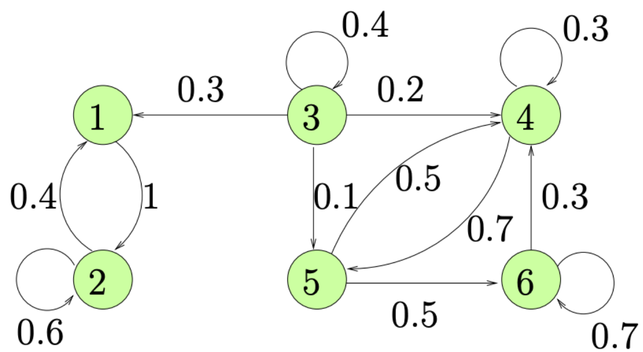


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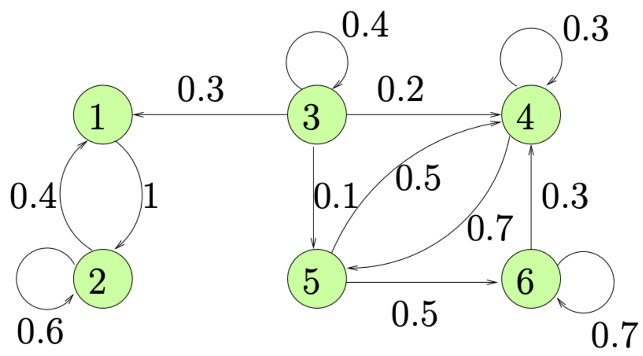


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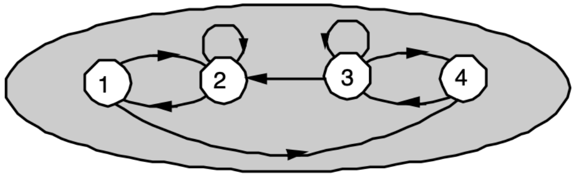
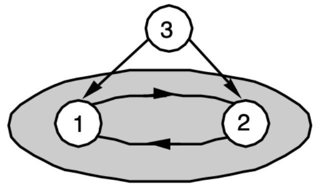
  - A state that is not recurrent is transient.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state $i$, then there is always some probability of returning to $i$. It means that, given enough time, it is certain that it returns to $i$.

# Classification of States (2)

- A set of recurrent states which communicate with each other form a class.




Single class of recurrent states




Single class of recurrent states (1 and 2)
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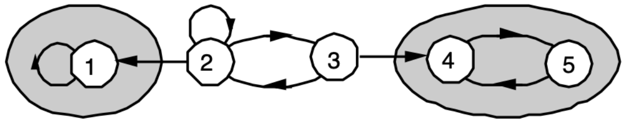


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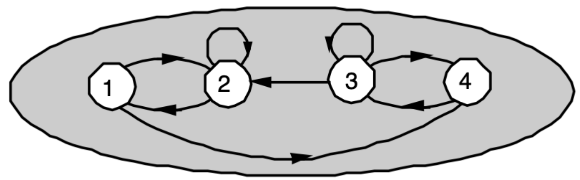

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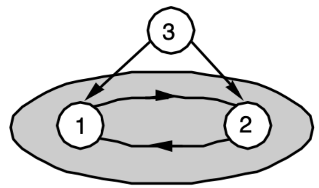

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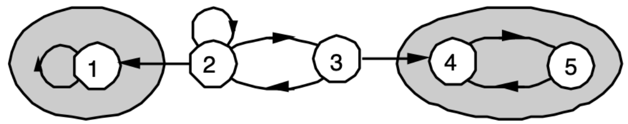

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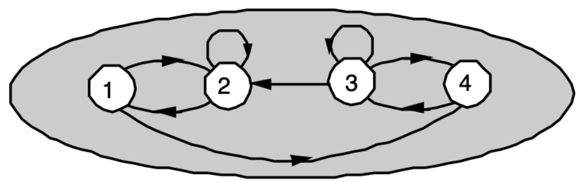


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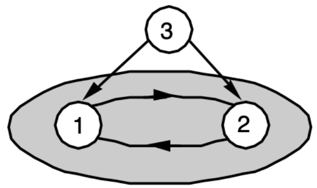


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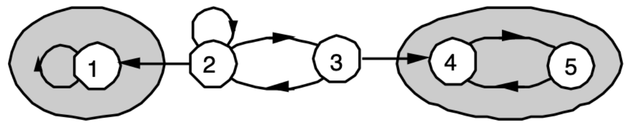


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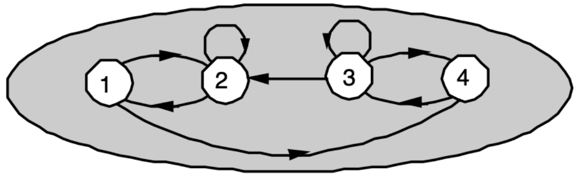


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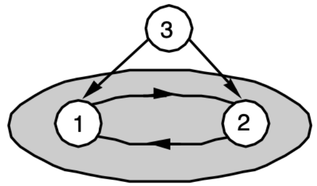


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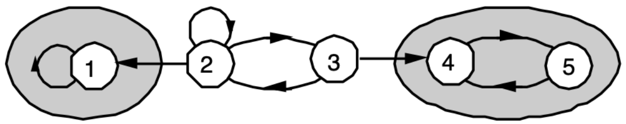


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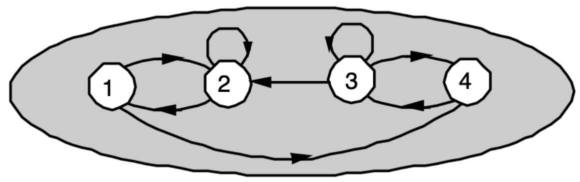


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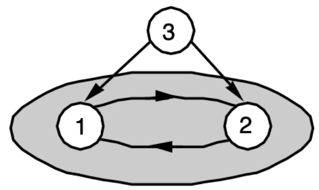


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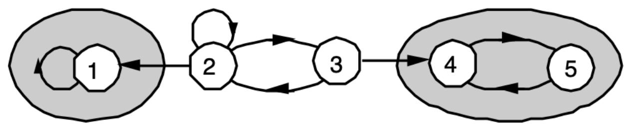


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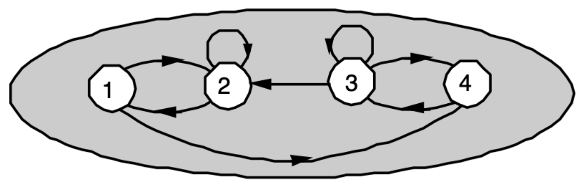
- The MC with only a single recurrent class is said to be irreducible (더이상 분해할 수 없는).

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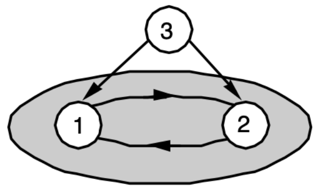


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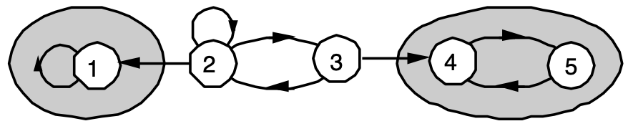


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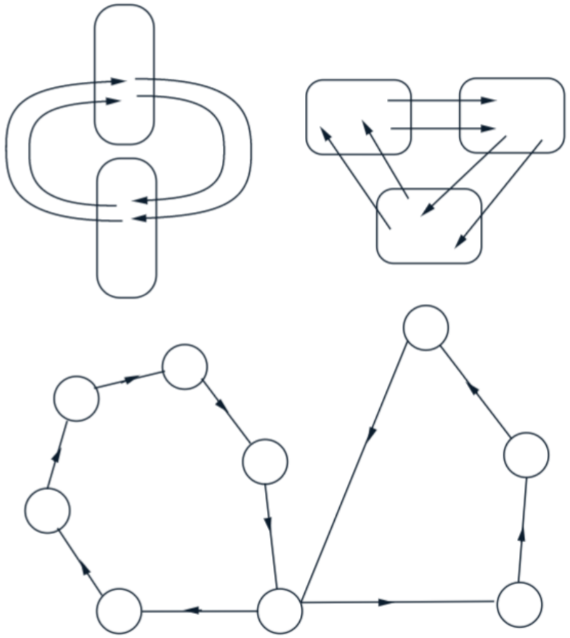
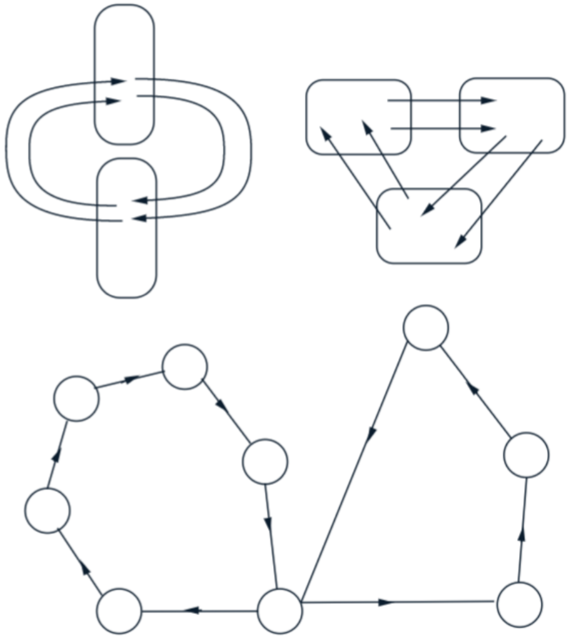
# Periodicity



- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.



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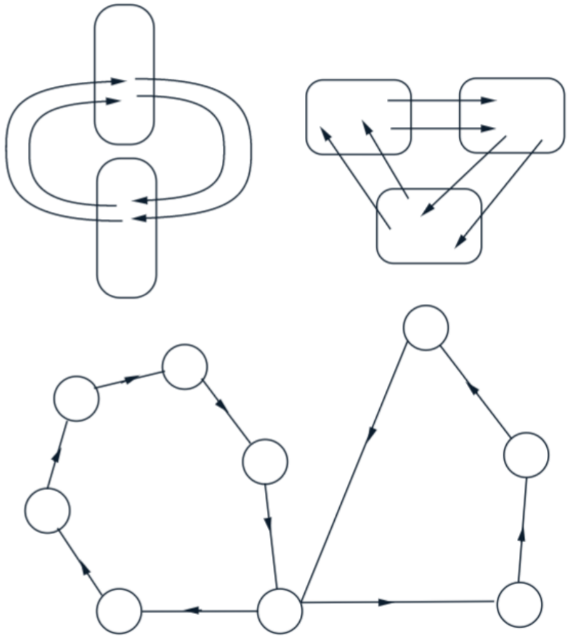
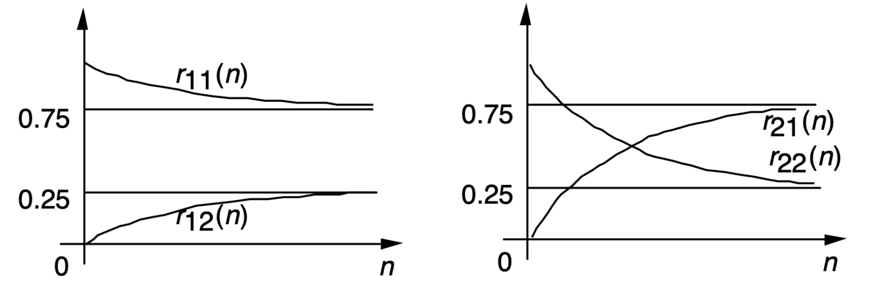
- A recurrent class that is not periodic is said to be aperiodic.



# Roadmap

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- Markov Chain
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors




$n$-step transition probabilities as a function of the number $n$ of transitions

$r_{ij}(1)$

| | UpD | B |
|---|---|---|
| UpD | 0.8 | 0.2 |
| B | 0.6 | 0.4 |

$r_{ij}(2)$

| | |
|---|---|
| .76 | .24 |
| .72 | .28 |

$r_{ij}(3)$

| | |
|---|---|
| .752 | .248 |
| .744 | .256 |

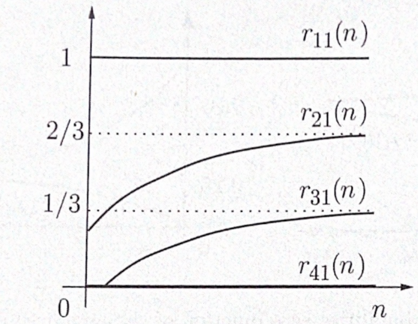
$r_{ij}(4)$

| | |
|---|---|
| .7504 | .2496 |
| .7488 | .2512 |

$r_{ij}(5)$

| | |
|---|---|
| .7501 | .2499 |
| .7498 | .2502 |

Sequence of $n$-step transition probability matrices




$n$-step transition probabilities into state 1

$r_{ij}(1)$

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
| 3 | 0 | 0.3 | 0.4 | 0.3 |
| 4 | 0 | 0 | 0 | 1.0 |

$r_{ij}(2)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .42 | .25 | .24 | .09 |
| .09 | .24 | .25 | .42 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(3)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .50 | .17 | .17 | .16 |
| .16 | .17 | .17 | .50 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(4)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .55 | .12 | .12 | .21 |
| .21 | .12 | .12 | .55 |
| 0 | 0 | 0 | 1.0 |

....

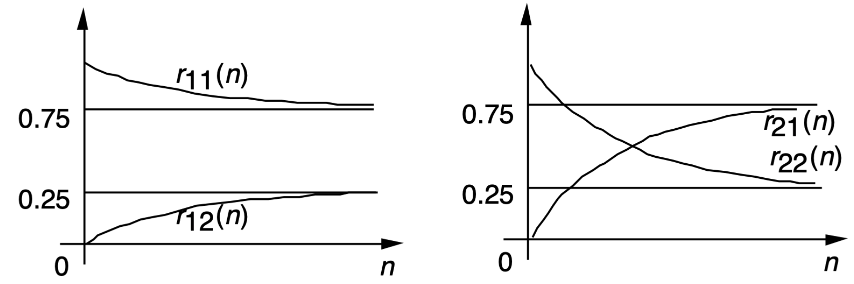
$r_{ij}(\infty)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| 2/3 | 0 | 0 | 1/3 |
| 1/3 | 0 | 0 | 2/3 |
| 0 | 0 | 0 | 1.0 |

Sequence of transition probability matrices

# $n$-step transition prob.: $r_{ij}(n)$ for large $n$

- Convergence irrespective of the starting state

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$r_{ij}(2)$

| | |
|---|---|
| .76 | .24 |
| .72 | .28 |

$r_{ij}(3)$

| | |
|---|---|
| .752 | .248 |
| .744 | .256 |

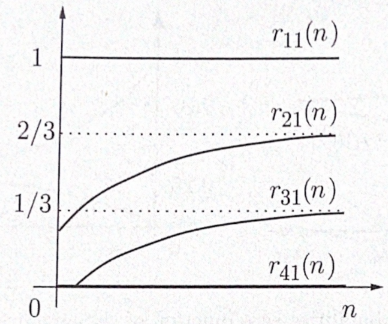
$r_{ij}(4)$

| | |
|---|---|
| .7504 | .2496 |
| .7488 | .2512 |

$r_{ij}(5)$

| | |
|---|---|
| .7501 | .2499 |
| .7498 | .2502 |

Sequence of $n$-step transition probability matrices




$n$-step transition probabilities into state 1

$r_{ij}(1)$

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
| 3 | 0 | 0.3 | 0.4 | 0.3 |
| 4 | 0 | 0 | 0 | 1.0 |

$r_{ij}(2)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .42 | .25 | .24 | .09 |
| .09 | .24 | .25 | .42 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(3)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .50 | .17 | .17 | .16 |
| .16 | .17 | .17 | .50 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(4)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .55 | .12 | .12 | .21 |
| .21 | .12 | .12 | .55 |
| 0 | 0 | 0 | 1.0 |

....

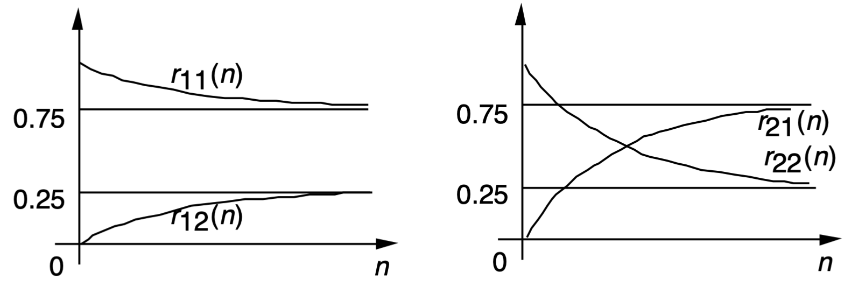
$r_{ij}(\infty)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| 2/3 | 0 | 0 | 1/3 |
| 1/3 | 0 | 0 | 2/3 |
| 0 | 0 | 0 | 1.0 |

Sequence of transition probability matrices

# $n$-step transition prob.: $r_{ij}(n)$ for large $n$

- Convergence irrespective of the starting state




$n$-step transition probabilities as a function of the number $n$ of transitions

$r_{ij}(1)$

| | UpD | B |
|---|---|---|
| UpD | 0.8 | 0.2 |
| B | 0.6 | 0.4 |

$r_{ij}(2)$

| | |
|---|---|
| .76 | .24 |
| .72 | .28 |

$r_{ij}(3)$

| | |
|---|---|
| .752 | .248 |
| .744 | .256 |

$r_{ij}(4)$

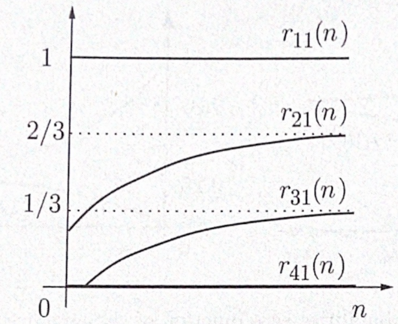
| | |
|---|---|
| .7504 | .2496 |
| .7488 | .2512 |

$r_{ij}(5)$

| | |
|---|---|
| .7501 | .2499 |
| .7498 | .2502 |

Sequence of $n$-step transition probability matrices

- Convergence depending on the starting state

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| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
| 3 | 0 | 0.3 | 0.4 | 0.3 |
| 4 | 0 | 0 | 0 | 1.0 |

$r_{ij}(2)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .42 | .25 | .24 | .09 |
| .09 | .24 | .25 | .42 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(3)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .50 | .17 | .17 | .16 |
| .16 | .17 | .17 | .50 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(4)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .55 | .12 | .12 | .21 |
| .21 | .12 | .12 | .55 |
| 0 | 0 | 0 | 1.0 |

....

$r_{ij}(\infty)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| 2/3 | 0 | 0 | 1/3 |
| 1/3 | 0 | 0 | 2/3 |
| 0 | 0 | 0 | 1.0 |

Sequence of transition probability matrices

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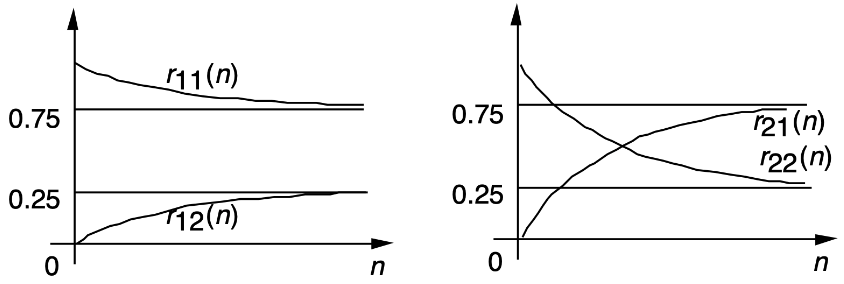


$n$-step transition probabilities as a function of the number $n$ of transitions

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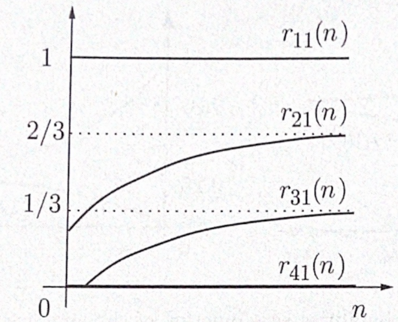


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|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
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$r_{ij}(2)$:

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .42 | .25 | .24 | .09 |
| .09 | .24 | .25 | .42 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(3)$:

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .50 | .17 | .17 | .16 |
| .16 | .17 | .17 | .50 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(4)$:

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .55 | .12 | .12 | .21 |
| .21 | .12 | .12 | .55 |
| 0 | 0 | 0 | 1.0 |

....

$r_{ij}(\infty)$:

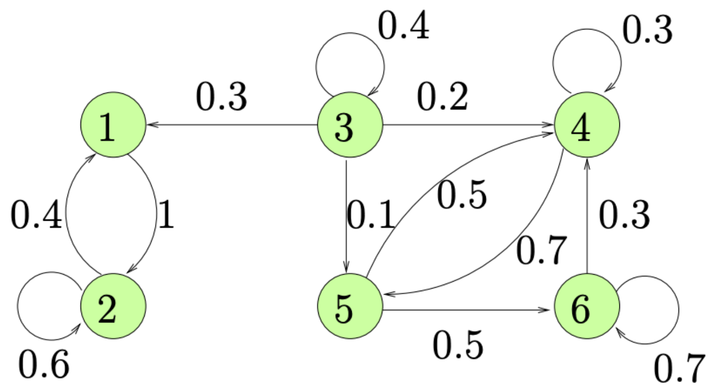
| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| 2/3 | 0 | 0 | 1/3 |
| 1/3 | 0 | 0 | 2/3 |
| 0 | 0 | 0 | 1.0 |

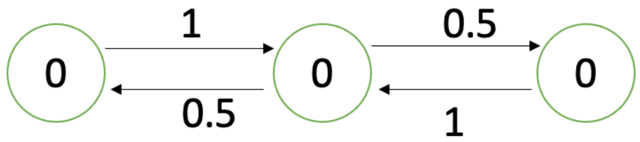
Sequence of transition probability matrices

(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?

# Steady-state behavior

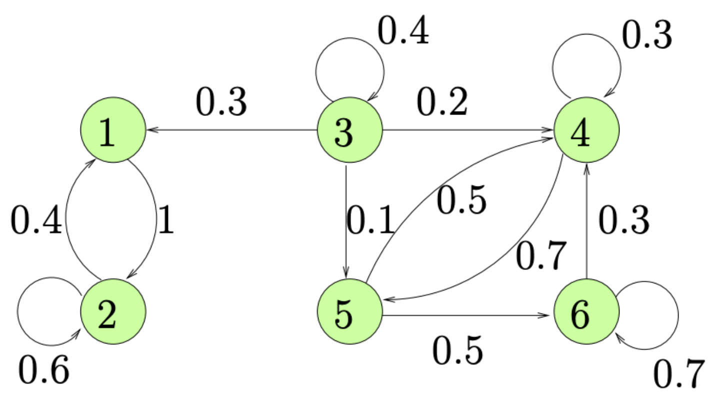
- $r_{ij}(n) \xrightarrow{n \to \infty} \pi_j$, for some $\pi_j \leq 1$?

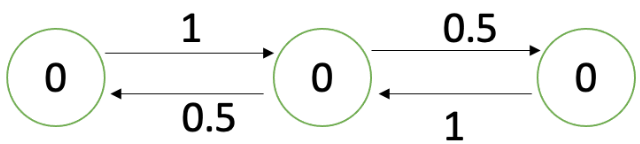


# Steady-state behavior

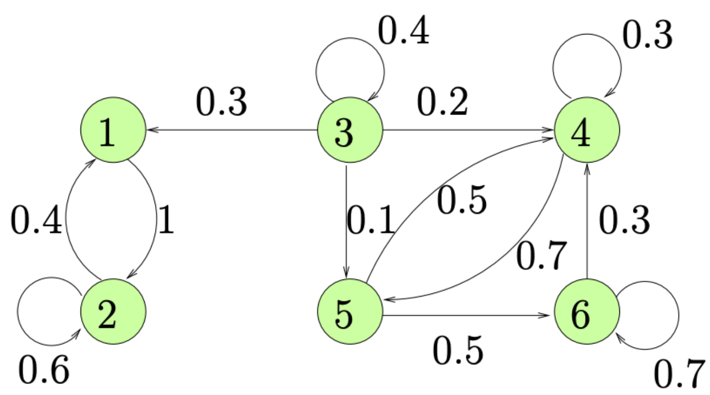
- $r_{ij}(n) \xrightarrow{n \to \infty} \pi_j$, for some $\pi_j \leq 1$?
- Convergence occurs, independent of the starting state, if:

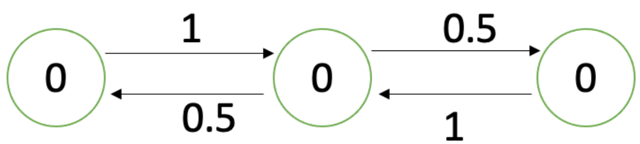


- $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \leq 1$?
- Convergence occurs, independent of the starting state, if:

  **C1.** Only a single recurrent class



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- Convergence occurs, independent of the starting state, if:

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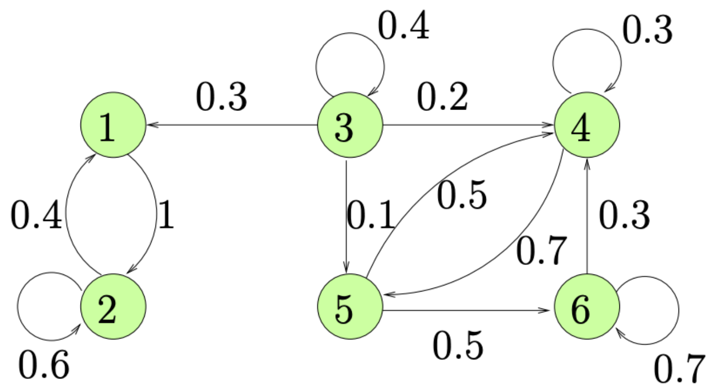
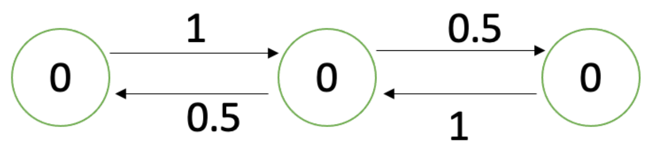
**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.



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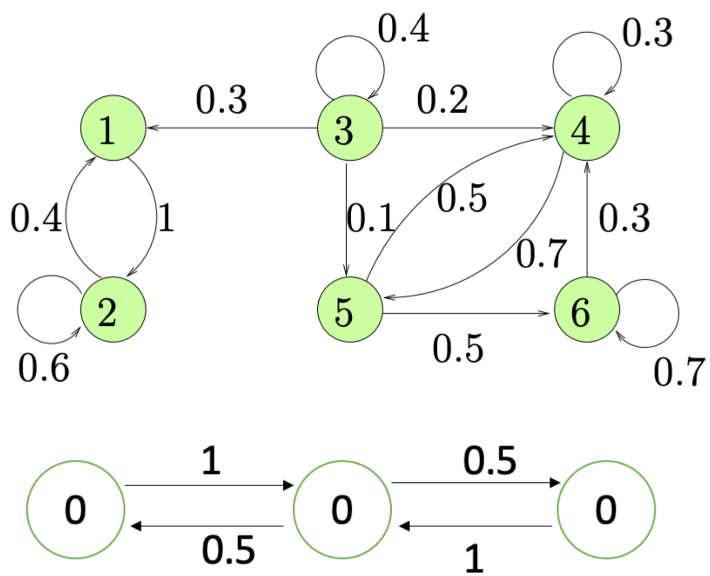

# Steady-state behavior

- $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$?
- Convergence occurs, independent of the starting state, if:
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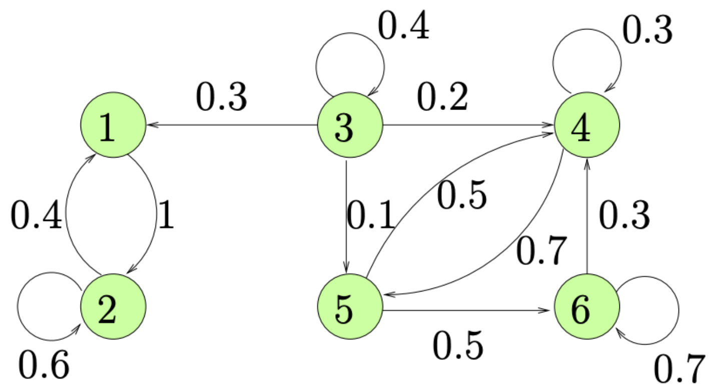
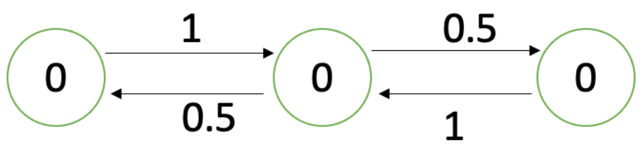
  **C2.** such recurrent class is aperiodic

**C1.** For the case of multiple recurrent classes, one stays at the class including the starting state.

**C2.** Divergent behavior for periodic recurrent classes.



# Balance Equation

- If $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \leq 1$,

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj} \Longrightarrow$$

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- Normalization equation

$$\sum_{i=1}^{m} \pi_i = 1$$

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- Normalization equation

$$\sum_{i=1}^{m} \pi_i = 1$$

- Balance equation + Normalization equation $\Longrightarrow$ Finding the steady-state probabilities $\{\pi_i\}$.

# Example

- A two-state MC with:

$$p_{11} = 0.8, \quad p_{12} = 0.2,$$
$$p_{21} = 0.6, \quad p_{22} = 0.4.$$

- Balance equation:

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$$
$$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$$

- Normalization equation: $\pi_1 + \pi_2 = 1$
- The stationary distribution is: $\pi_1 = 0.25$, $\pi_2 = 0.75$.

# Stationary Distribution

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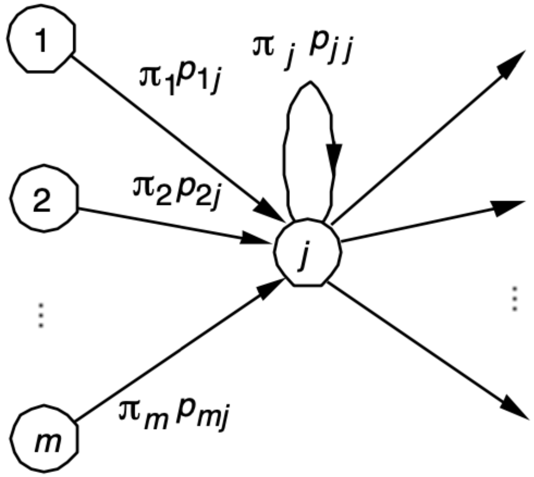
- We say that "the limiting distribution is equal to to the stationary distribution"

# Long-term Frequency Interpretation

- $\pi_j$: the long-term expected fraction of time that the state is equal to $j$.



# Long-term Frequency Interpretation

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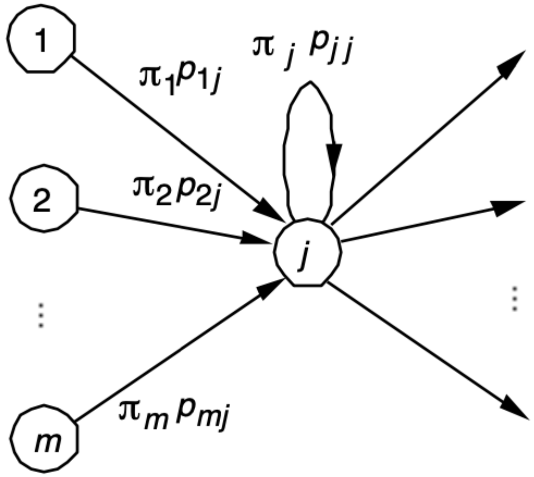
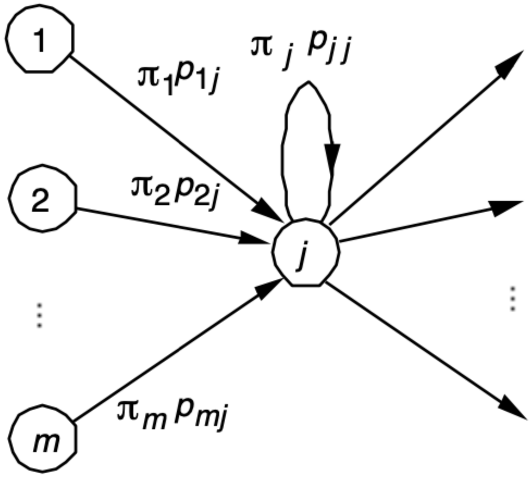


# Long-term Frequency Interpretation

- $\pi_j$: the long-term expected fraction of time that the state is equal to $j$.
- Balance equation: $\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$ means:
  - The expected frequency $\pi_j$ of visits to $j$ is equal to the sum of the expected frequencies $\pi_k p_{kj}$ of transitions that lead to $j$.



# Roadmap

KAIST EE

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- Markov Chain
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

# Absorption Probability

- Definition. A state $k$ is absorbing, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.

[0] The notation $a_i$ should have dependence on $s$, but we omit it for simplicity.

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- Definition. A state $k$ is absorbing, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.
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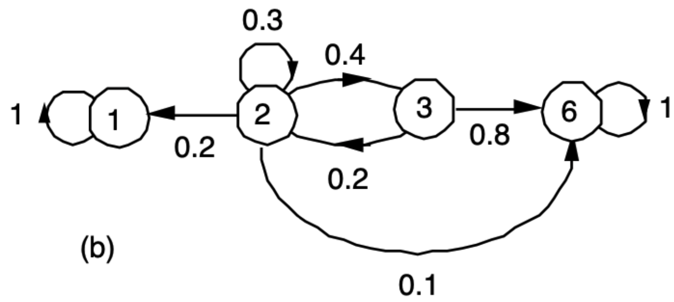
  $a_3 = 0.2a_2 + 0.8a_6$

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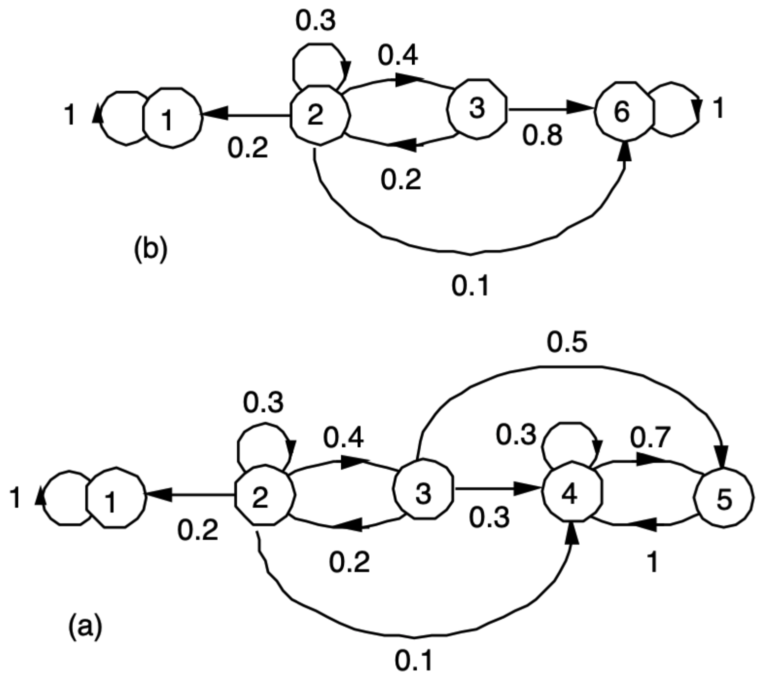


(Q) What if there are some non-absorbing recurrent state?

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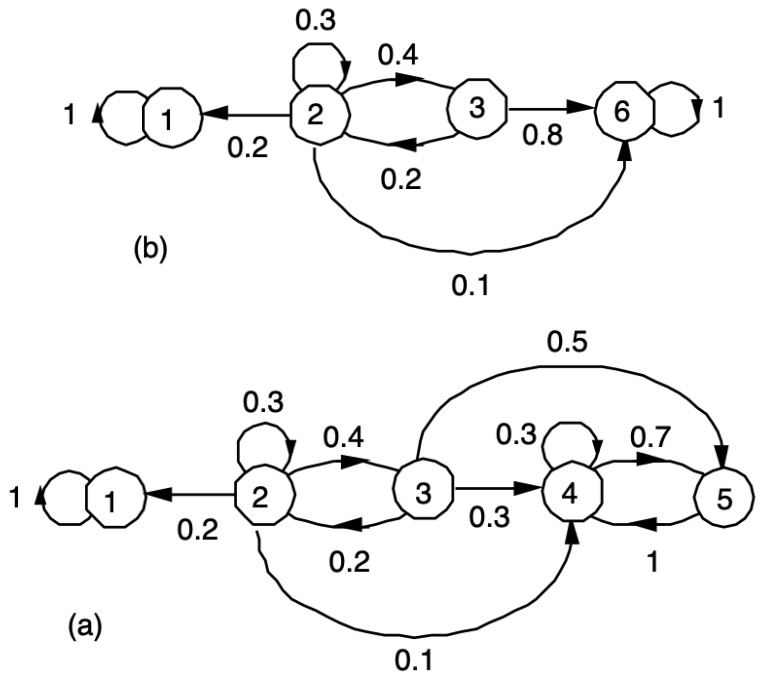


(Q) What if there are some non-absorbing recurrent state?

- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

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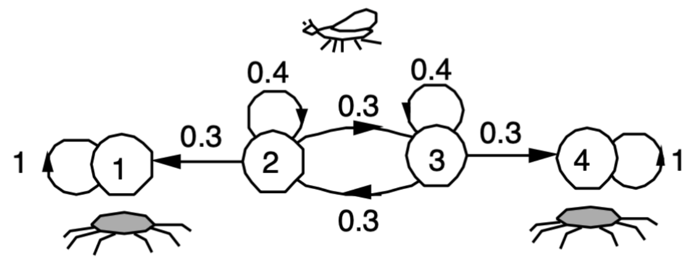
# Expected Time to Any Absorbing State

(Q) Starting from a transient state $i$, expected number of transitions $\mu_i$ until absorption to any absorbing state?



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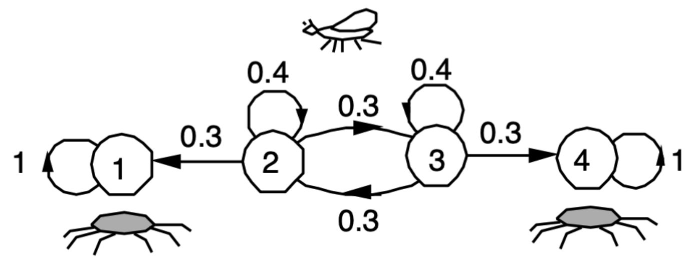


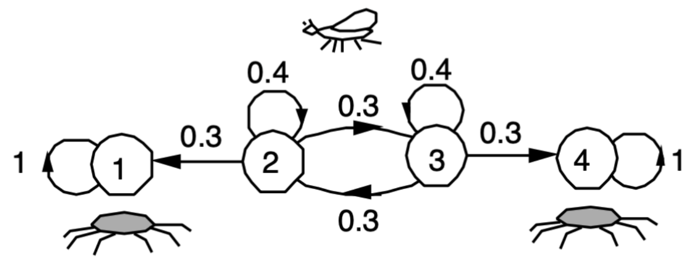

- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad \text{(for recurrent states)}$$

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad \text{(for transient states)}$$

# Expected Time to Any Absorbing State

(Q) Starting from a transient state $i$, expected number of transitions $\mu_i$ until absorption to any absorbing state?




- Spider-fly example

$$\mu_1 = \mu_4 = 0 \quad \text{(for recurrent states)}$$

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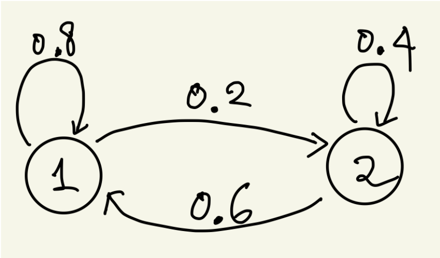
- For generalized description, please see the textbook (pp. 367).

# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

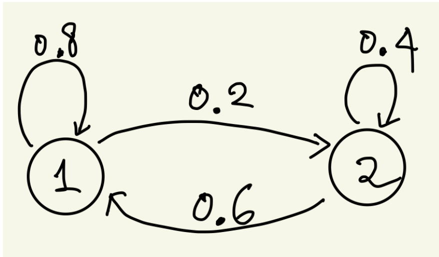



[0]The notation $t_i$ should have the dependence on $s$, but we omit it for simplicity.

# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

(Q) First passage time. Starting from a $i$, expected number of transitions $t_i$ to reach $s$ for the first time?

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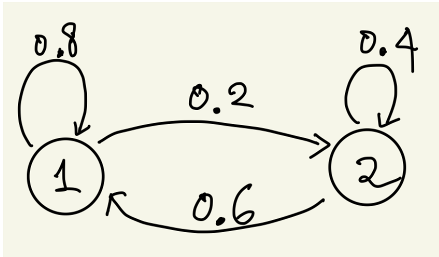



- Mean first passage time from 2 to 1

$$t_1 = 0$$

$$t_2 = 1 + p_{21} t_1 + p_{22} t_2 = 1 + 0.4 t_2 \Longrightarrow t_2 = 5/3$$

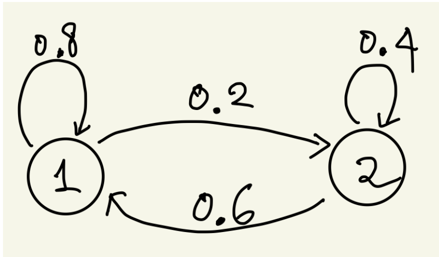
[0]The notation $t_i$ should have the dependence on $s$, but we omit it for simplicity.

# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

(Q) First passage time. Starting from a $i$, expected number of transitions $t_i$ to reach $s$ for the first time?

(Q) First recurrence time. Starting from a $s$, expected number of transitions $t_s^\star$ to reach $s$ for the first time?

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# Expected time to a particular recurrent state $s$

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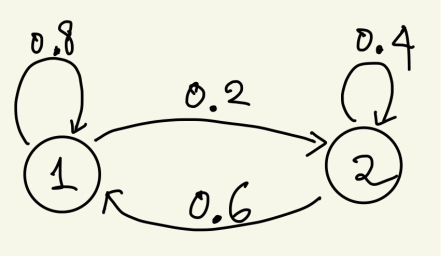



- Mean first passage time from 2 to 1

$$t_1 = 0$$

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2 \Longrightarrow t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^\star = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2\frac{5}{3} = \frac{4}{3}$$

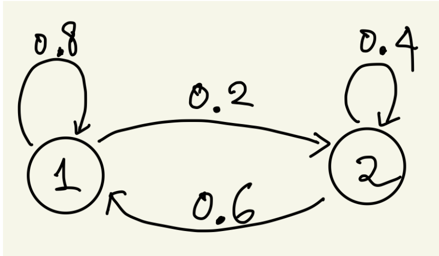
[0] The notation $t_i$ should have the dependence on $s$, but we omit it for simplicity.

# Expected time to a particular recurrent state $s$

- Assume a single recurrent class

(Q) First passage time. Starting from a $i$, expected number of transitions $t_i$ to reach $s$ for the first time?

(Q) First recurrence time. Starting from a $s$, expected number of transitions $t_s^\star$ to reach $s$ for the first time?




- Mean first passage time from 2 to 1

$$t_1 = 0$$

$$t_2 = 1 + p_{21} t_1 + p_{22} t_2 = 1 + 0.4 t_2 \Longrightarrow t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^\star = 1 + p_{11} t_1 + p_{12} t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

- For generalized description, please see the textbook (pp. 368)

[0] The notation $t_i$ should have the dependence on $s$, but we omit it for simplicity.

Questions?

## Review Questions

1) Why do you think Markov chain (MC) is important?
2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
3) What are the limiting distribution and the stationary distribution of MCs?
4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
5) What are recurrent and transient states in MC?