

## Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes  
KAIST EE

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### Roadmap

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables

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- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
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- (5) Conditioning for random variables
- (6) Independence for random variables

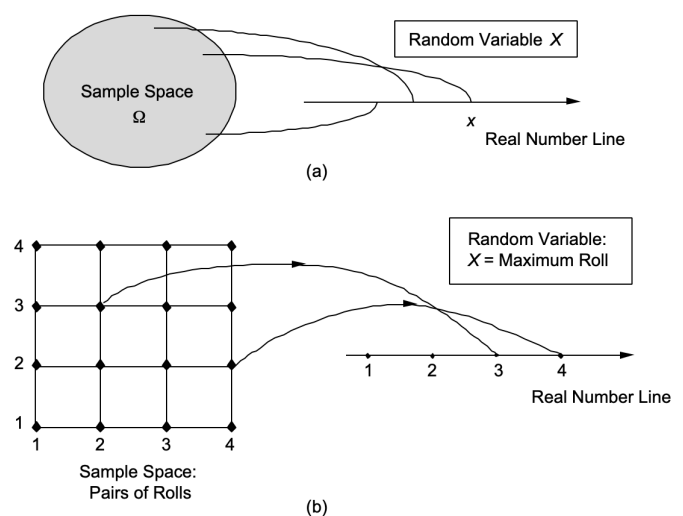
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## Random Variable: Idea

- In reality, many outcomes are **numerical**, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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- Mathematically, a random variable  $X$  is a **function** which maps from  $\Omega$  to  $\mathbb{R}$ .
- Notation.** Random variable  $X$ , numerical value  $x$ .
- Different random variables can be defined on the same sample space.
- For a fixed value  $x$ , we can associate an **event** that a random variable  $X$  has the value  $x$ , i.e.,  $\{\omega \in \Omega \mid X(\omega) = x\}$
- Assume that values  $x$  are discrete<sup>1</sup> such as  $1, 2, 3, \dots$ .  
For notational convenience,
 
$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$
- For a discrete random variable  $X$ , we call  $p_X(x)$  **probability mass function** (PMF).

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<sup>1</sup>Finite or countably infinite.

## Example

- Rolls a dice,  $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable  $X = 1$  for even numbers and  $X = 0$  for odd numbers
- Event  $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$ , but simply  $A_1 = \{X = 1\}$
- Event  $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$ , but simply  $A_0 = \{X = 0\}$
- Remember that the random variable  $X$  is a **function** from  $\Omega$  to  $\mathbb{R}$

- (1) Random variable: Idea and formal definition
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Bernoulli  $X$  with parameter  $p \in [0, 1]$ 

- Only **binary** values

$$X = \begin{cases} 0, & \text{w.p. } 1 - p, \\ 1, & \text{w.p. } p \end{cases}$$

In other words,  $p_X(0) = 1 - p$  and  $p_X(1) = p$  from our PMF notation.

- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an **indicator rv** of an event  $A$ . Define a rv  $\mathbf{1}_A$  as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

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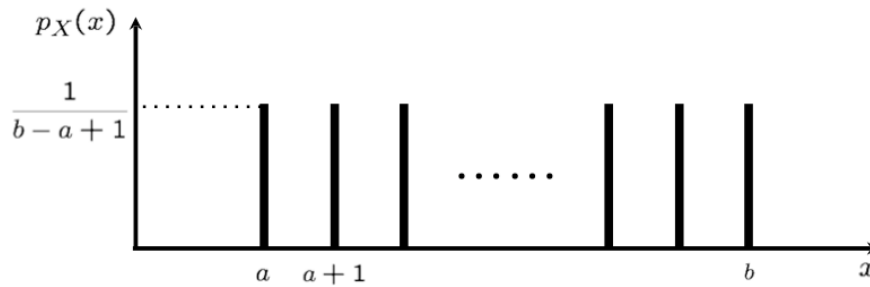
<sup>1</sup>w.p.: with probability

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- integers  $a, b$ , where  $a \leq b$
- Choose a number out of  $\Omega = \{a, a + 1, \dots, b\}$  uniformly at random.
- $p_X(i) = \frac{1}{b-a+1}$ ,  $i \in \Omega$



- Models complete **ignorance** (I don't know anything about  $X$ )

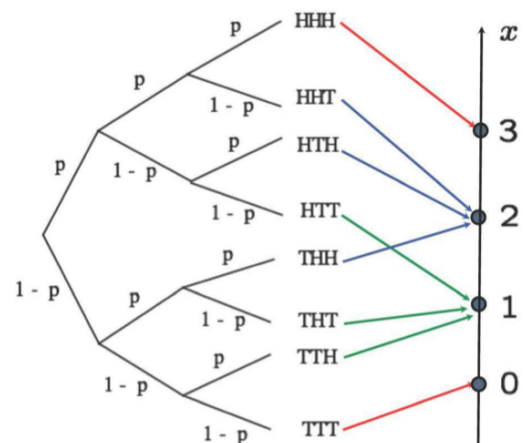
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- Models the number of **successes** in a given number of **independent** trials
- $n$  independent trials, where one trial has the success probability  $p$ .

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



<sup>1</sup> $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ , which we read ' $n$  choose  $k$ '.

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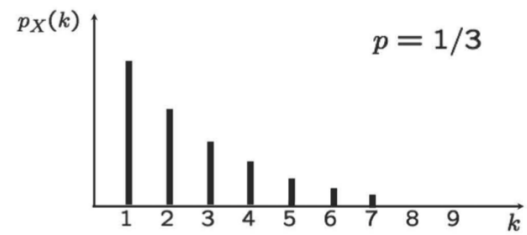
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- Infinitely many independent Bernoulli trials, where each trial has success probability  $p$
- Random variable: number of trials until the **first success**.

$$p_X(k) = (1 - p)^{k-1}p$$

- Models **waiting** times until something happens.



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- Average

## Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$ : relative frequency of value  $x$  (trials with  $x$ /total trials)
- **Example.** Bernoulli rv with  $p$

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p = p_X(1)$$

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## Properties of Expectation

Not very surprising. Easy to prove using the definition.

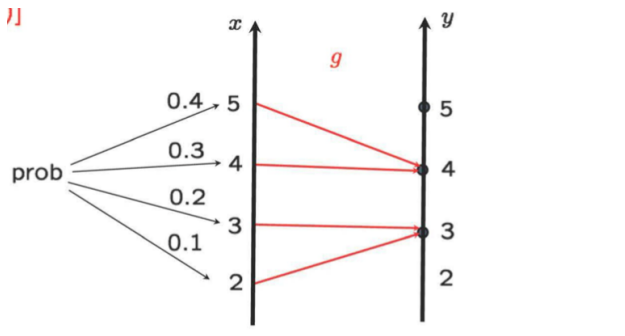
- If  $X \geq 0$ ,  $\mathbb{E}[X] \geq 0$ .
- If  $a \leq X \leq b$ ,  $a \leq \mathbb{E}[X] \leq b$ .
- For a constant  $c$ ,  $\mathbb{E}[c] = c$ .

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- For a rv  $X$ ,  $Y = g(X)$  is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$
- Compute  $\mathbb{E}[Y]$  for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2) = 2.8 + 0.9 = 3.7$$

## Linearity of Expectation

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

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# Variance

- Measures how much the **spread** of a PMF is.
- What about  $\mathbb{E}[X - \mu]$ , where  $\mu = \mathbb{E}[X]$ ? Zero
- Then, what about  $\mathbb{E}[(X - \mu)^2]$ ?

## Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

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- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\ &= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2\end{aligned}$$

- $Y = X + b, \text{var}[Y] = \text{var}[X]$

$$\text{var}[Y] = \mathbb{E}[(X + b)^2] - (\mathbb{E}[X + b])^2$$

- $Y = aX, \text{var}[Y] = a^2\text{var}[X]$

$$\text{var}[Y] = \mathbb{E}[a^2X^2] - (a\mathbb{E}[X])^2$$

Example: Variance of a Bernoulli rv ( $p$ )

$$\mu = \mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p$$

$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1 - p) = p$$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\ &= p(1 - p)\end{aligned}$$

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- Joint PMF.** For two random variables  $X, Y$ , consider two events  $\{X = x\}$  and  $\{Y = y\}$ , and

$$p_{X,Y}(x,y) \triangleq \mathbb{P}(\{X=x\} \cap \{Y=y\})$$

- $\sum_x \sum_y p_{X,Y}(x,y) = 1$

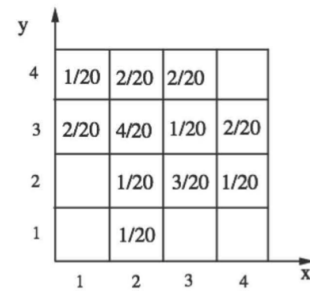
- Marginal PMF.

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

$$p_Y(y) = \sum_x p_{X,Y}(x, y)$$

Example.

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$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

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- Consider a rv  $Z = g(X, Y)$ . (Ex)  $X + Y, X^2 + Y^2$ . Then, PMF of  $Z$  is:

$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X, Y}(x, y)$$

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- Remember:  $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

(easy to prove, using the definition.)

- $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- **Example.** Mean of a binomial rv  $Y$  with  $(n, p)$

- $Y$ : number of successes in  $n$  Bernoulli trials with  $p$

- $Y = X_1 + \dots + X_n$ , where  $X_i$  is a Bernoulli rv.

- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

**Message.** When some rv  $X$  is written as a linear combination of other rvs,  $X$  becomes easy to handle.

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Remember two probability laws:  $\mathbb{P}(\cdot)$  and  $\mathbb{P}(\cdot|A)$  for an event  $A$ .

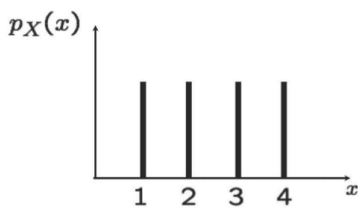
- $p_X(x) \triangleq \mathbb{P}(X = x)$
  - $\mathbb{E}[X] = \sum_x x p_X(x)$
  - $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
  - $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
  - $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
  - $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
  - $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
  - (Note)  $p_{X|A}(x)$ ,  $\mathbb{E}[X|A]$ ,  $\mathbb{E}[g(X)|A]$ , and  $\text{var}[X|A]$  are all just notations!

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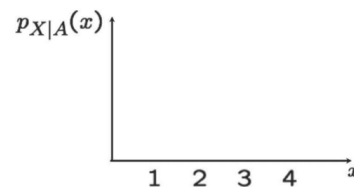
## Example: Conditional PMF

$$A = \{X \geq 2\}$$



$$\mathbb{E}[X] = \frac{1}{4} (1 + 2 + 3 + 4) = 2.5$$

$$\begin{aligned} \text{var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{4} (1 + 2^2 + 3^2 + 4^2) - 2.5^2 \end{aligned}$$



$$\mathbb{E}[X|A] = \frac{1}{3} (2 + 3 + 4) = 3$$

$$\begin{aligned} \text{var}[X|A] &= \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 \\ &= \frac{1}{3} (2^2 + 3^2 + 4^2) - 3^2 = 2/3 \end{aligned}$$

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What do we mean by “conditioning on a rv”? Consider  $A = \{Y = y\}$  for a rv  $Y$ .

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$

- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
- $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
- $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
- $\text{var}[X|Y = y] \triangleq \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

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## Conditional PMF

- Conditional PMF

$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

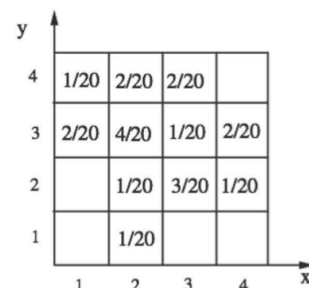
for  $y$  such that  $p_Y(y) > 0$ .

- $\sum_x p_{X|Y}(x|y) = 1$
- Multiplication rule

$$\begin{aligned} p_{X,Y}(x,y) &= p_Y(y) p_{X|Y}(x|y) \\ &= p_X(x) p_{Y|X}(y|x) \end{aligned}$$

- $p_{X,Y,Z}(x,y,z) = p_X(x) p_{Y|X}(y|x) p_{Z|X,Y}(z|x,y)$

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$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

$$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$$

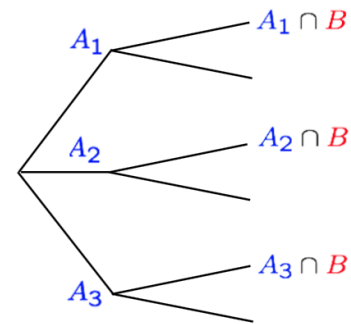
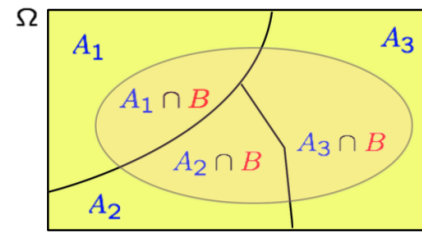
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- Partition of  $\Omega$  into  $A_1, A_2, A_3$
- Known:  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$
- What is  $\mathbb{P}(B)$ ?

## Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i) \mathbb{P}(B|A_i)$$



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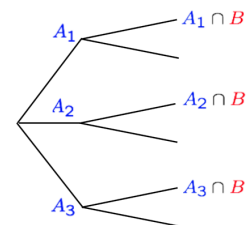
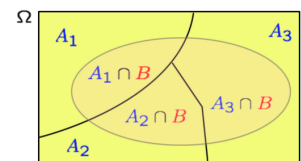
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Total Probability Theorem:  $B = \{X = x\}$ 

- Partition of  $\Omega$  into  $A_1, A_2, A_3$

## Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$



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## Total Expectation Theorem for $\{A_i\}$

- Partition of  $\Omega$  into  $A_1, A_2, A_3$

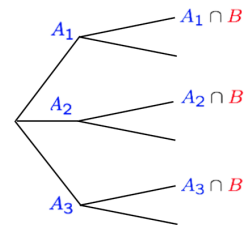
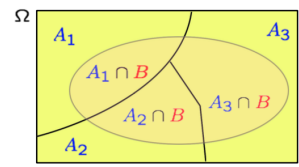
### Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

- Weighted average of expectations from  $A_i$ 's perspective



## Total Expectation Theorem for $\{Y = y\}$

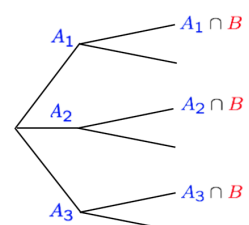
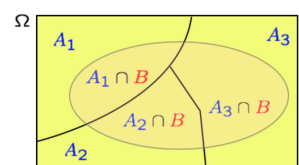
- Partition of  $\Omega$  into  $A_1, A_2, A_3$

### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y) \mathbb{E}[X | Y = y] = \sum_y p_Y(y) \mathbb{E}[X | Y = y]$$



- **Question.** What is  $\mathbb{E}(X)$ ?

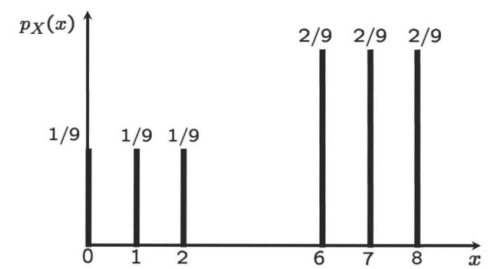
(1) Just using the definition of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{9}(0 + 1 + 2) + \frac{2}{9}(6 + 7 + 8) \\ &= \frac{3 + 12 + 14 + 16}{9} = 5\end{aligned}$$

(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0, 1, 2\}\}, A_2 = \{X \in \{6, 7, 8\}\}$$

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1,2} \mathbb{P}(A_i) \mathbb{E}[X|A_i] \\ &= 1/3 \cdot 1 + 2/3 \cdot 7 = 5\end{aligned}$$



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- Write softwares over and over, and each time w.p.  $p$  of working correctly (independent from previous programs).
- $X$ : number of trials until the program works correctly.
- **(Q)**  $\mathbb{E}(X)$ ?
- $X$  is a geometric rv
- Direct computation is boring.

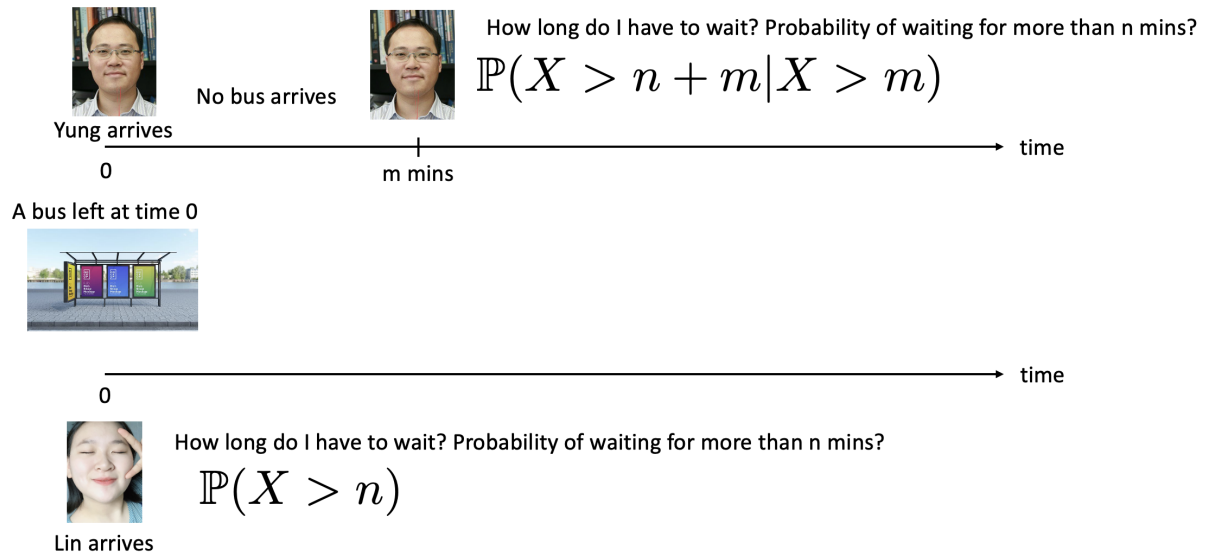
$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p + 2(1-p)p + 3(1-p)^2p + \dots$$

- Total expectation theorem and a notion of **memorylessness** helps a lot.

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- Some random variable often does not have **memory**.
- Definition.** A random variable  $X$  is called **memoryless** if, for any  $n, m \geq 0$ ,  

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$
- Meaning.** Conditioned on  $X > m$ ,  $X - m$ 's distribution is the same as the original  $X$ .  

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

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- **Theorem.** Any **geometric** random variable is **memoryless**.
- **Remind.** Geometric rv  $X$  with parameter  $p$

$$\mathbb{P}(X = k) = (1 - p)^{k-1}p, \quad \mathbb{P}(X > k) = \sum_{i=k+1}^{\infty} (1 - p)^{i-1}p = (1 - p)^k$$

- **Proof.**

$$\begin{aligned} \mathbb{P}(X > n + m | X > m) &= \frac{\mathbb{P}(X > n + m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > m)} \\ &= \frac{(1 - p)^{n+m}}{(1 - p)^m} = (1 - p)^n = \mathbb{P}(X > n) \end{aligned}$$

- **Meaning.** Conditioned on  $X > m$ ,  $X - m$  is geometric with the same parameter.

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- $A_1 = \{X = 1\}$  (first try is success),  $A_2 = \{X > 1\}$  (first try is failure).

$$\begin{aligned} \mathbb{E}[X] &= 1 + \mathbb{E}[X - 1] \\ &= 1 + \mathbb{P}(A_1)\mathbb{E}[X - 1 | X = 1] + \mathbb{P}(A_2)\mathbb{E}[X - 1 | X > 1] && \text{(from TET)} \\ &= 1 + (1 - p)\mathbb{E}[X] && \text{(from memorylessness)} \end{aligned}$$

- Thus,  $\mathbb{E}[X] = \frac{1}{p}$

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## Independence, Conditional Independence

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$

$$\mathbb{P}(\{X = x\} \cap B | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(B | C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

$$\mathbb{P}(\{X = x\} \cap \{Y = y\} | Z = z) = \mathbb{P}(X = x | Z = z) \cdot \mathbb{P}(Y = y | Z = z), \quad \text{for all } x, y$$

$$p_{X,Y|Z}(x, y) = p_{X|Z}(x) \cdot p_{Y|Z}(y)$$

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- $X \perp\!\!\!\perp Y$ ?

$$p_{X,Y}(1,1) = 0$$

$$p_X(1) = 3/20$$

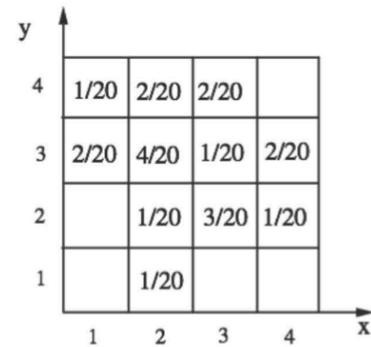
$$p_Y(1) = 1/20$$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$ ?

VIDEO PAUSE

$Y = 4$ (1/3)	1/9	2/9
$Y = 3$ (2/3)	2/9	4/9
	$X = 1$ (1/3)	$X = 2$ (2/3)

- Yes.



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## Expectation and Variance

- Always true.  
 $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Generally,  $\mathbb{E}[g(X, Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$
- However, if  $X \perp\!\!\!\perp Y$ ,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

- **Proof.**

$$\begin{aligned} \mathbb{E}[g(X)h(Y)] &= \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y) \\ &= \sum_x g(x)p_X(x) \sum_y h(y)p_Y(y) \end{aligned}$$

- Always true.  
 $\text{var}[aX] = a^2\text{var}[X]$ ,  $\text{var}[X + a] = \text{var}[X]$
- Generally,  $\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$  (next slide)
- However, if  $X \perp\!\!\!\perp Y$ ,  
 $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y]$
- **Practice.**
  - $X = Y \implies \text{var}[X + Y] = 4\text{var}[X]$
  - $X = -Y \implies \text{var}[X + Y] = 0$
  - $X \perp\!\!\!\perp Y \implies \text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

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$$\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$$

- Why not generally true?

$$\begin{aligned}\text{var}[X + Y] &= \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 \\ &= \mathbb{E}[X^2 + Y^2 + 2XY] - ((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) \\ &= \text{var}[X] + \text{var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\end{aligned}$$

- $X \perp\!\!\!\perp Y$  is a sufficient condition for  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study **covariance**.

## Example: The hat problem (1)

- $n$  people throw their hats in a box and then pick one at random
- $X$ : number of people with their own hat
- $\mathbb{E}[X]$ ?  $\text{var}[X]$ ?
- All permutations are equally likely as  $1/n!$ . Thus, this equals to picking one hat at a time.
- Key step 1.** Define a rv  $X_i = 1$  if  $i$  selects its own hat and 0 otherwise.

$$X = \sum_{i=1}^n X_i.$$

- $\{X_i\}, i = 1, 2, \dots, n$ : identically distributed (from symmetry)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1$ .
- **Key step 2.** Are  $X_i$ s independent? If yes, easy to get  $\text{var}(X)$ .
- Assume  $n = 2$ . Then,  $X_1 = 1 \rightarrow X_2 = 1$ , and  $X_1 = 0 \rightarrow X_2 = 0$ . Thus, **dependent**.

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\left[\sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j\right] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_1^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

## Questions?

- 1) What is Random Variable? Why is it useful?
- 2) What is PMF (Probability Mass Function)?
- 3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
- 4) What are joint and marginal PMFs?
- 5) Describe and explain the total probability/expectation theorem for random variables?
- 6) When is it useful to use total probability/expectation theorem?
- 7) What is conditional independence?