Lecture 6: Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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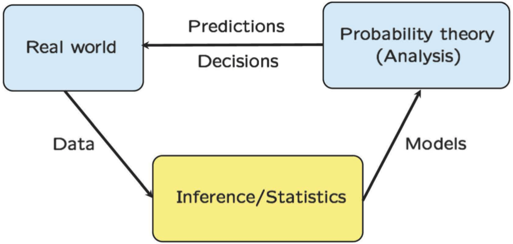
Inference: Big Picture



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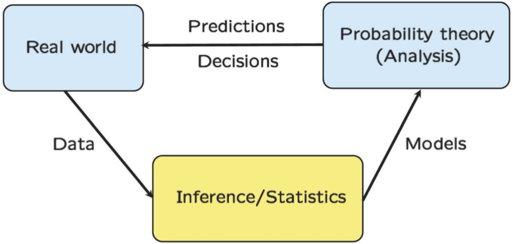

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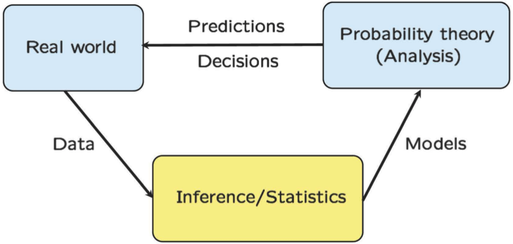

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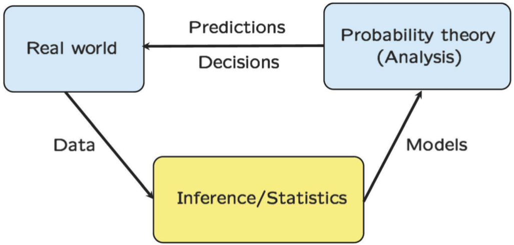
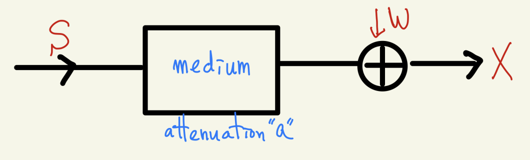
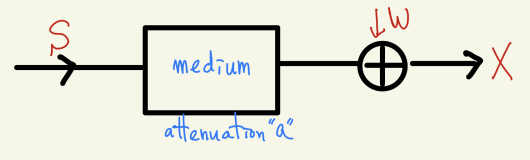


- Inference
  - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building

What to Infer?: Unknown Model vs. Unknown Variable



- $X = aS + W$



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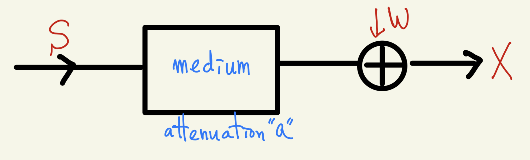

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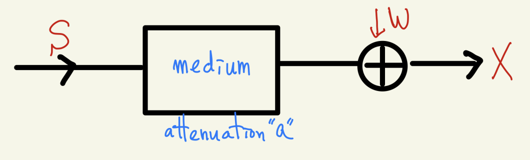

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  - know the original signal S, observe X
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- Variable estimation
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- Same mathematical structure, because the parameters in models are variables in many cases

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  - (Ex) Biased coin with unknown probability of head $\theta \in [0, 1]$. Data of heads and tails. What is θ?
  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

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- Use Bayes' rule and find the posterior:

$$\mathbb{P}\left[\theta = \frac{3}{4} \middle| (HHH)\right] = \frac{27}{28}, \ \mathbb{P}\left[\theta = \frac{1}{4} \middle| (HHH)\right] = \frac{1}{28}$$

Inference with Different Views: Bayesian vs. Classical (1)

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- Find the probability of (H, H, H), if $\theta = \frac{1}{4}$ or $\frac{3}{4}$ (likelihood)

$$\mathbb{P}\left[(HHH)|\theta = \frac{3}{4}\right] = \left(\frac{3}{4}\right)^3$$

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- *Classical approach* (Chapter 9)

Inference with Different Views: Bayesian vs. Classical (2)

Bayesian approach

Classical approach

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- Unknown: random variable with some distribution (prior)

Classical approach

- Unknown: deterministic value

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Classical approach

- Unknown: deterministic value
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Inference with Different Views: Bayesian vs. Classical (2)

Bayesian approach

- Unknown: random variable with some distribution (prior)
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- Observed data x gives: posterior distribution $p_{\Theta|X}(\theta|x)$

Classical approach

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- Who is the winner? A century-long debate (see p. 409 for discussion)

Inference with Different Views: Bayesian vs. Classical (2)

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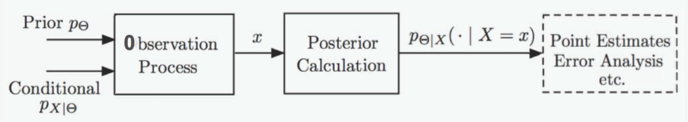
Framework of Bayesian Inference



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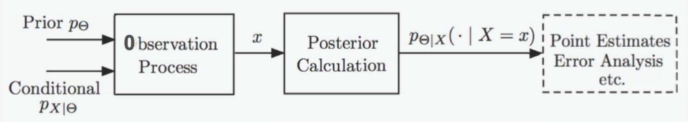

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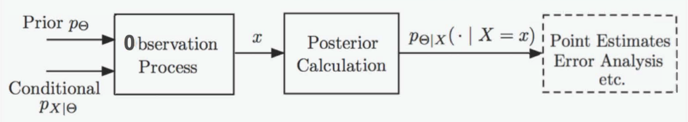

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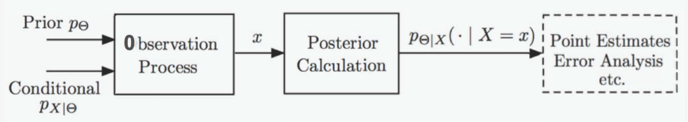

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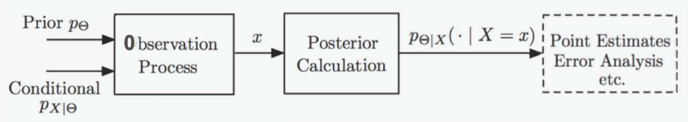

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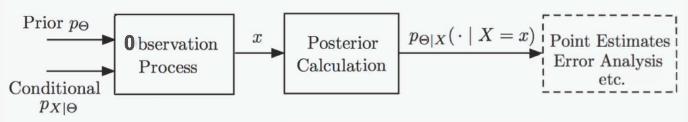

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- Find the posterior distribution $p_{X|\Theta}$ and $f_{X|\Theta}$.
  - Use Bayes' rule
- Using the posterior distribution, apply one of the methods of choosing the final $\hat{\theta}$ for estimation and hypothesis testing.

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Point Estimation

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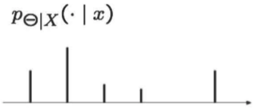

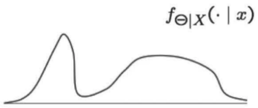

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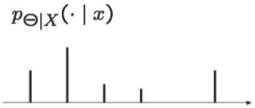

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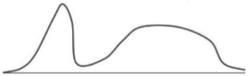

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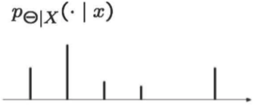

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- Why MAP and LMS are good? Not mathematically clear yet (later)

Estimator as a function

- Random observation: X
- Observation instance: x
- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\text{MAP}} = g_{\text{MAP}}(x), \quad \hat{\theta}_{\text{LMS}} = g_{\text{LMS}}(x)$$

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  - Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$.
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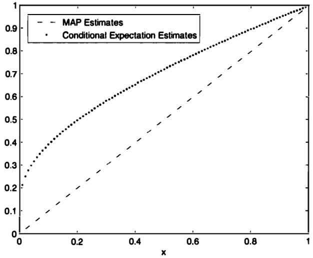

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A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

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- When $\alpha = \beta = 1$ (i.e., $U[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1)\prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_\Theta(2)\prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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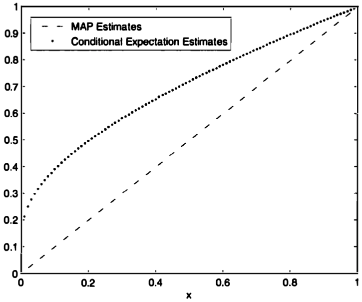

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- Using the above fact,

$$\mathbb{E}[\Theta | X = k] = \frac{k+\alpha}{k+\alpha+n-k+\beta} = \frac{k+\alpha}{\alpha+\beta+n}$$

- Remind. If $\Theta \sim Beta(\alpha, \beta)$, then $\Theta | \{X = k\} \sim Beta(k + \alpha, n - k + \beta)$
- Fact. If $\Theta \sim Beta(\alpha, \beta)$,

$$\mathbb{E}[\Theta] = \frac{1}{B(\alpha, \beta)} \int_0^1 \theta \theta^{\alpha - 1} (1 - \theta)^{\beta - 1} d\theta = \frac{B(\alpha + 1, \beta)}{B(\alpha, \beta)} = \frac{\alpha}{\alpha + \beta}$$

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- For $\alpha = \beta = 1$ ($\Theta = Uniform[0, 1]$),

$$\mathbb{E}[\Theta | X = k] = \frac{k + 1}{n + 2}$$

Example: Signal Recovery from Noisy Measurement (1)

- Unknown: $\Theta \sim \mathit{Uniform}[4, 10]$
- Observe Θ with random error W as X. $W \sim \mathit{Uniform}[-1, 1]$

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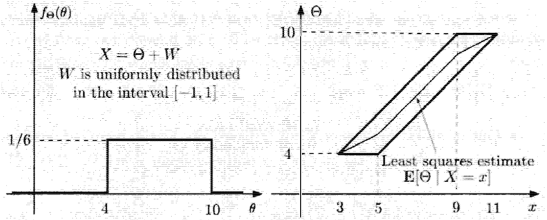

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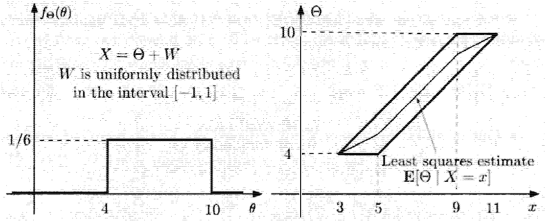

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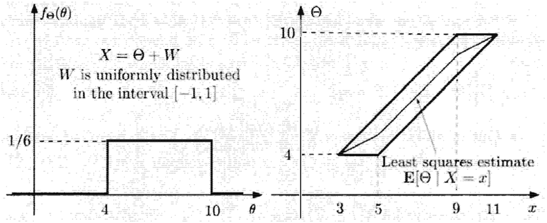
- $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$ = midpoint of the corresponding vertical section



Example: Signal Recovery from Noisy Measurement (2)

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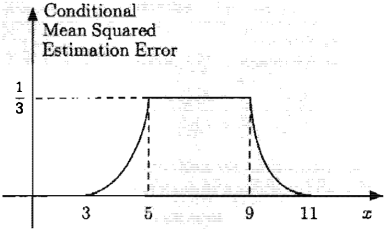
- Conditional MSE

$$\mathbb{E}\Big[(\Theta - \mathbb{E}[\Theta|X = x])^2 | X = x\Big]$$



Hardness of LMS Estimation

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{f_X(x)}$$

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  - AlexNet in image recognition: 61M parameters (though not a Bayesian inference)
- Any alternative to LMS estimator?

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

Linear LMS (LLMS) Estimator: Approach

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KAIST EE

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- Linear models are always the first choice for a simple design in engineering.

Linear LMS (LLMS) Estimator: Solution First

LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)$$

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- If $\rho > 0$:
  - Baseline ($\mathbb{E}[\Theta]$) + correction term
  - If $X > \mathbb{E}[X] \implies \hat{\Theta}_L > \mathbb{E}[\Theta]$
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- If $\rho = 0$ (uncorrelated):
  - Just baseline ($\mathbb{E}[\Theta]$)
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data X

Linear LMS (LLMS) Estimator: Proof

$$
\begin{aligned}
\hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\mathrm{cov}(\Theta, X)}{\mathrm{var}(X)}\Big(X - \mathbb{E}(X)\Big) & (1) \\
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\end{aligned}
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(3)

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- (3) is minimized when $a = \frac{\text{cov}(\Theta,X)}{\text{var}[X]}$. Then,

$$\hat{\Theta}_L = aX + b = aX + \mathbb{E}(\Theta) - a\mathbb{E}(X)$$

$$= (1)$$

- Using $\rho = \frac{\text{cov}(\Theta,X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

Example: Romeo and Juliet

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.
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- $\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta]\mathbb{E}[X]$

$$
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$$
\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24
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- LLMS estimator is:

$$\begin{aligned} \hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) \\ &= \frac{1}{2} + \frac{1/24}{7/144}(X - \frac{1}{4}) = \frac{6}{7}X + \frac{2}{7} \end{aligned}$$

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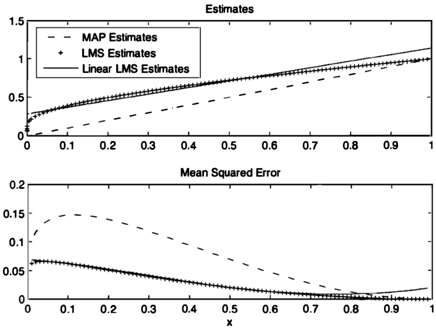

Example: Biased Coin with Uniform Prior

- Biased coin with probability of head θ
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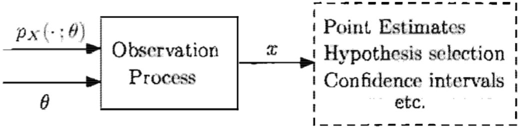
Yes, because the LMS esitmator was linear.

KAIST EE

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator

- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

Framework of Classical Inference (1)



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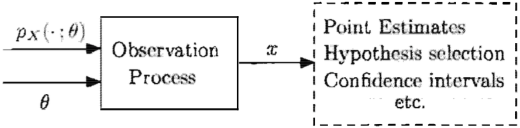

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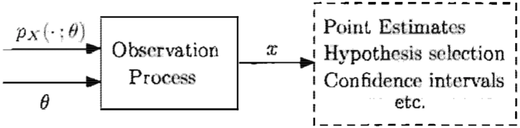

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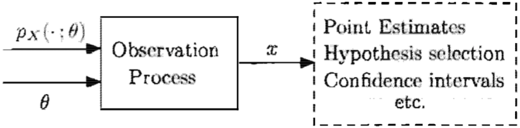

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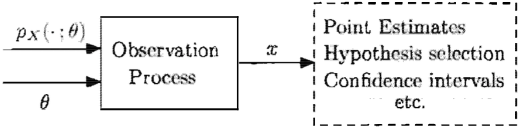

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- Choosing one among multiple probabilistic models
  - Each θ corresponds to a probabilistic model

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- Problem types
  - Estimation
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  - Significance testing

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- Just a taste in this course due to time constraint.

Maximum Likelihood Estimation

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- Very often, X_i are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \ldots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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- ML and MAP: How are they related?

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- When Θ is uniform (complete ignorance of Θ), MAP == ML

Example: Romeo and Juliet

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.
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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \mathbb{E}[X] = 1/\lambda$$

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$$\arg\max_{\theta} f_X(x; \theta) = \arg\max_{\theta} \prod_{i=1}^{n} \theta e^{-\theta x_i} = \arg\max_{\theta} \left(n \log \theta - \theta \sum_{i=1}^{n} x_i \right)$$

Questions?

1) What is statistical inference?
2) Draw the building blocks of Bayesian inference and explain how it works.
3) What are MAP and LMS estimators and their underlying philosophies?
4) What is LLMS estimator and why is it useful?
5) Compare the classical and Bayesian inference.
6) What is the ML estimator and how is it related to the MAP estimator?