

Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

April 27, 2021

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

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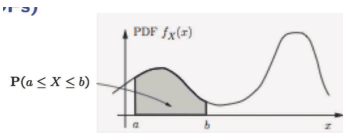
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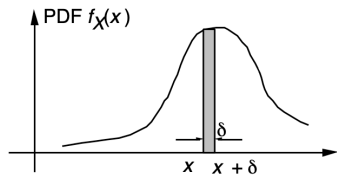
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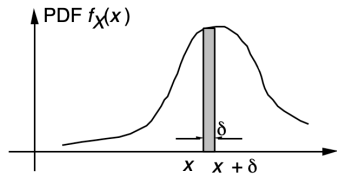


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x) dx = 1$



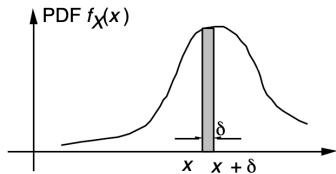
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Examples



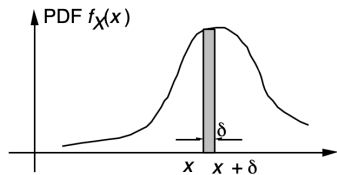
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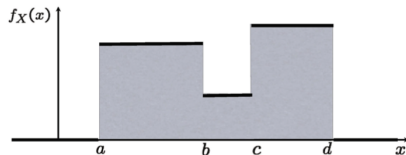
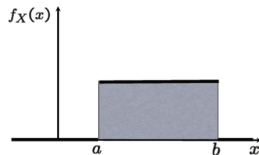
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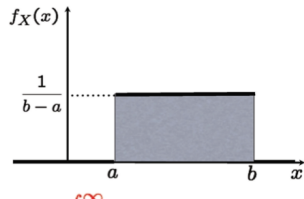
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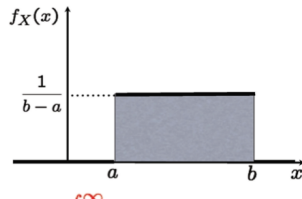
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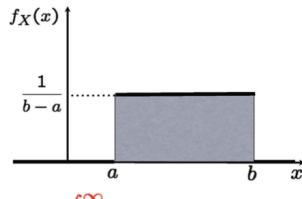




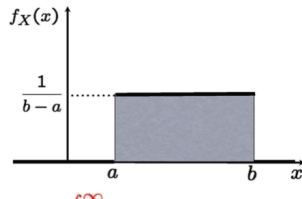
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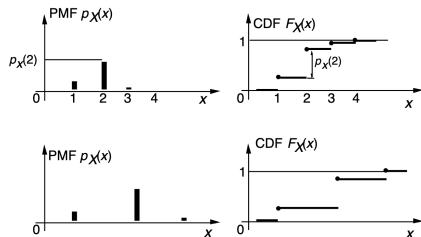
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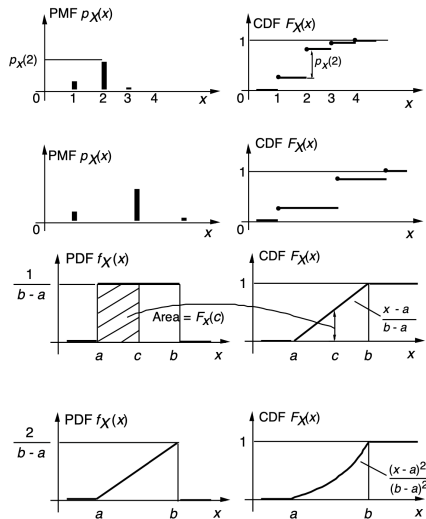


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- If X is continuous
 - $F_X(x)$ is a continuous function of x .
 - $F_X(x) = \int_{-\infty}^x f_X(t)dt$ and $f_X(x) = \frac{dF_X}{dx}(x)$

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$, and $X_i \in \{1, 2, \dots, 10\}$ uniformly at random
- **Question.** $p_X(x)$?
- Approach 1: $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$?
- Approach 2

$$\begin{aligned} F_X(x) &= \mathbb{P}(\max\{X_1, X_2, X_3\} \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x) \\ &= \mathbb{P}(X_1 \leq x) \cdot \mathbb{P}(X_2 \leq x) \cdot \mathbb{P}(X_3 \leq x) = \left(\frac{x}{10}\right)^3 \end{aligned}$$

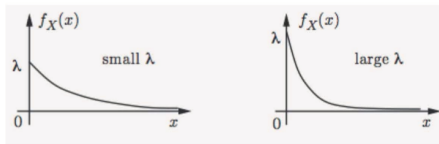
Thus,

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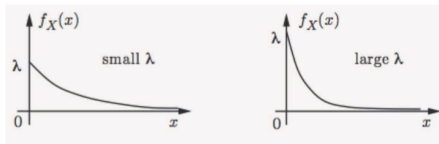
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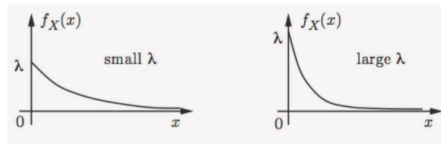
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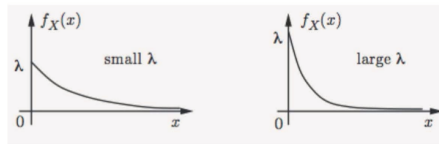
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- (Check) $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$, $\text{var}[X] = 1/\lambda^2$

- $\mathbb{E}(X) = 1/\lambda$. Use **integration by parts**: $\int u dv = uv - \int v du$

$$\int_0^{\infty} x \lambda e^{-\lambda x} dx = (-x e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx = 0 + \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

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- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

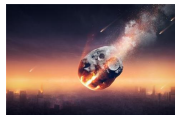
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VIDEO PAUSE

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- (Solution)
 - $\mathbb{E}(X) = 1/\lambda = 10$. Thus, $\lambda = \frac{1}{10}$.
 - 6 a.m. from midnight = 1/4 day, 6 p.m. from midnight = 3/4 day

$$\mathbb{P}(1/4 \leq X \leq 3/4) = \mathbb{P}(X \geq 1/4) - \mathbb{P}(X \geq 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

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- In many cases, continuous case is some type of **limit** of its corresponding discrete case.
- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

- 'slot' is one unit time, e.g., 1 hour, 30 mins, 1 min, 10 sec, etc.

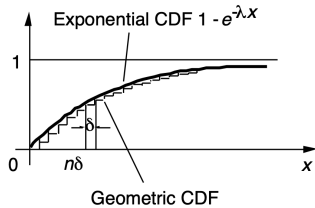
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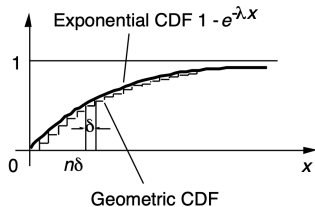
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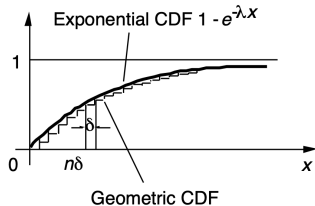
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 - $\mathbb{P}(X_{\delta}^{geo} \leq n) = 1 - (1 - p_{\delta})^n = 1 - e^{-\lambda\delta n}$





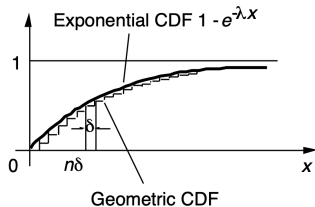
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- If we choose sufficiently small δ , the slot length \downarrow and $p_{\delta} \downarrow$

$$\mathbb{P}(X_{\delta}^{geo} \leq n) \xrightarrow{\delta \rightarrow 0} \mathbb{P}(X^{exp} \leq x), \quad x = n\delta$$

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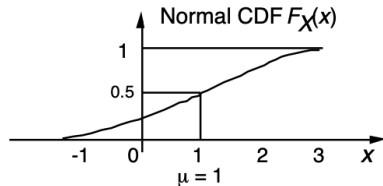
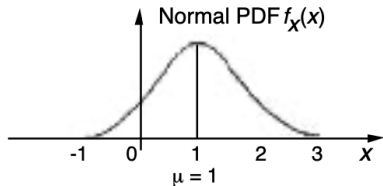
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- $\text{var}[X] = 1$

- General Normal $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$



- PDF's normalization property: $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

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- Expectation
 - $f_X(x)$ is symmetric in terms of $x = \mu$. Thus, we should have $\mathbb{E}(X) = \mu$.
- Variance

$$\begin{aligned} \text{var}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \stackrel{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} (-ye^{-y^2/2}) \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sigma^2 \end{aligned}$$

$$\int u dv = uv - \int v du: u = y \text{ and } dv = ye^{-y^2/2} \rightarrow du = dy \text{ and } v = -e^{-y^2/2}$$

- Linear transformation preserves normality (we will verify this in Lecture 5)

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b , $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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- Thus, every normal rv can be :

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\sim \mathcal{N}(0, 1)$

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If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b , $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be **standardized**:

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

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If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b , $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be **standardized**:

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

- Thus, we can make the **table** which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?
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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
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2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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- $Y = \frac{X-60}{20}$.

$$\begin{aligned}\mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80 - 60}{20}) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587\end{aligned}$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

- Central limit theorem
 - One of the most remarkable findings in the probability theory
 - Sum of **any** random variables \approx Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

⁰Central limit theorem: 중심극한정리

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

Two continuous rvs are if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for **every subset** B of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy,$$

Two continuous rvs are **jointly continuous** if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for **every subset** B of the two dimensional plane,

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Two continuous rvs are **jointly continuous** if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for **every subset** B of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}[(X, Y) \in B] = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy$$

Our particular interest: $B = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$

2. The **marginal** PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

2. The **marginal** PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The **joint CDF** is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

2. The **marginal** PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The **joint CDF** is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

4. A function $g(X, Y)$ of X and Y defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

* Conditional PDF, given an event A

* Conditional PDF, given $\{X \in C\}$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X . Sorry for the confusion, if any.

* Conditional PDF, given an event A

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

* Conditional PDF, given $\{X \in C\}$

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 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$
- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$

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- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$
- $\int f_{X|A}(x) dx = 1$

* Conditional PDF, given $\{X \in C\}$

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- $\int f_{X|A}(x) dx = 1$

* Conditional PDF, given $\{X \in C\}$

$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$$

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X . Sorry for the confusion, if any.

* Conditional PDF, given an event A

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$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

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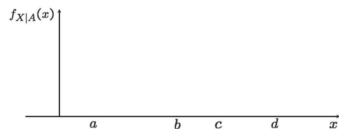
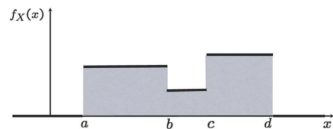
(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X . Sorry for the confusion, if any.

- $\mathbb{E}[X] = \int x f_X(x) dx$
 $\mathbb{E}[X|A] = \int x f_{X|A}(x) dx$

- $\mathbb{E}[X] = \int x f_X(x) dx$
 $\mathbb{E}[X|A] = \int x f_{X|A}(x) dx$
- $\mathbb{E}[g(X)] = \int g(x) f_X(x) dx$
 $\mathbb{E}[g(X)|A] = \int g(x) f_{X|A}(x) dx$

$$A = \left\{ \frac{a+b}{2} \leq X \leq b \right\}$$

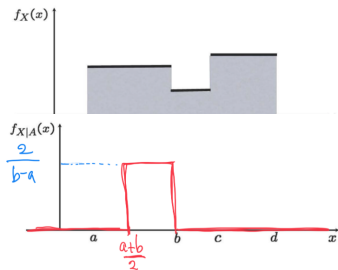


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Partition of Ω into A_1, A_2, A_3, \dots

* Discrete case

* Continuous case

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Total Probability Theorem

$$\begin{aligned} p_X(x) &= \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) \\ &= \sum_i \mathbb{P}(A_i) p_{X|A_i}(x) \end{aligned}$$

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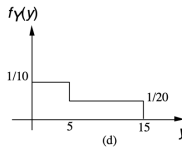
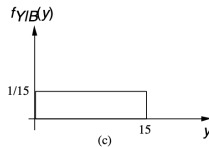
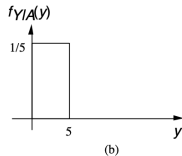
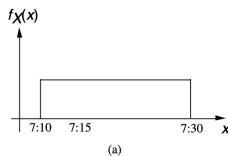
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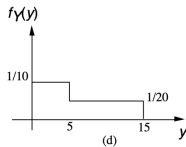
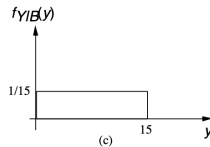
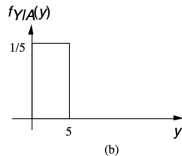
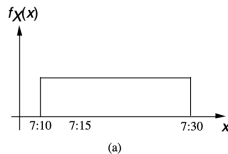
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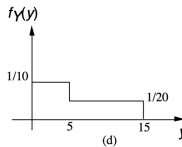
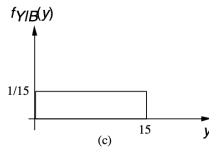
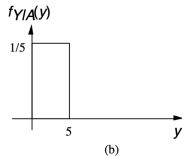
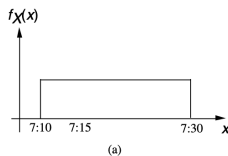
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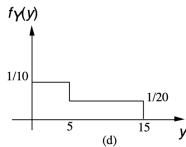
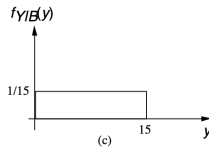
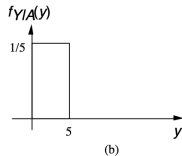
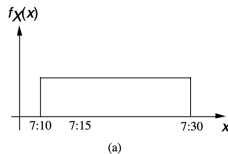
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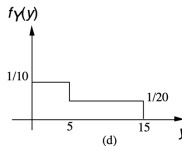
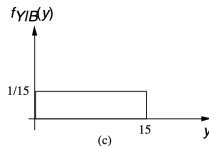
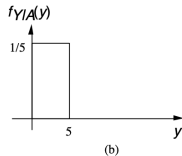
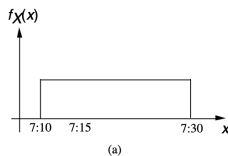
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VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A)f_{Y|A}(y) + \mathbb{P}(B)f_{Y|B}(y)$$

$$\text{for } 0 \leq y \leq 5$$

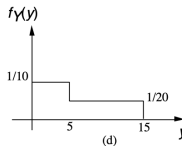
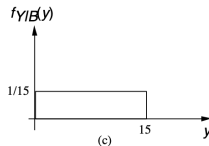
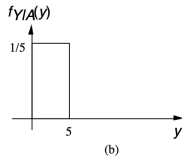
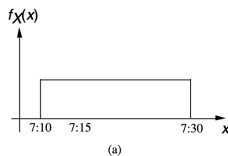
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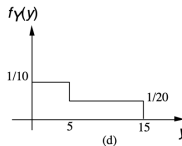
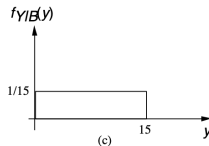
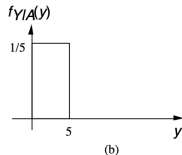
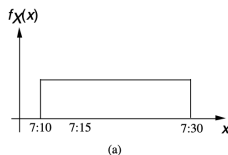
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- **Independence**

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

(Prob 21 at pp. 191)

- Break a stick of length l twice
 - first break at $Y \sim \mathcal{U}[0, l]$
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${}^0\mathcal{U}[a, b]$: continuous uniform random variable over the interval $[a, b]$

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(a) joint PDF $f_{X,Y}(x, y)$?

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Using $f_{X,Y}(x, y) = f_Y(y)f_{X|Y}(x|y)$,

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(b) marginal PDF $f_X(x)$?

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x, y) dy = \int_x^l \frac{1}{ly} dy \\ &= \frac{1}{l} \ln(l/x), \quad 0 \leq x \leq l \end{aligned}$$

⁰ $\mathcal{U}[a, b]$: continuous uniform random variable over the interval $[a, b]$

(c) Evaluate $\mathbb{E}(X)$, using $f_X(x)$

(d) Evaluate $\mathbb{E}(X)$, using $X = Y \cdot (X/Y)$

If $Y \perp\!\!\!\perp X/Y$, it becomes easy, but true?

(e) Evaluate $\mathbb{E}(X)$, using TET

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- **Message.** There are many ways to reach our goal. Of crucial importance is how to find the best way!

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

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- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (cause \rightarrow result)
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1. Light bulb's lifetime $Y \sim \exp(\lambda)$. Given the lifetime **y**, the modified belief about **λ** ?
2. Romeo and Juliet start dating, but Romeo will be late by a random variable $Y \sim \mathcal{U}[0, \theta]$. Given the time of being late **y**, the modified belief about **θ** ?

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- Wait! $p_{K|Y}(k|y)$? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

- For small δ (in other words, taking the limit as $\delta \rightarrow 0$).

Let $A = \{K = k\}$.

$$\begin{aligned} p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \leq Y \leq y + \delta) \\ &= \frac{\mathbb{P}(A)\mathbb{P}(y \leq Y \leq y + \delta|A)}{\mathbb{P}(y \leq Y \leq y + \delta)} \\ &\approx \frac{\mathbb{P}(A)f_{Y|A}(y)\delta}{f_Y(y)\delta} \\ &= \frac{\mathbb{P}(A)f_{Y|A}(y)}{f_Y(y)} \end{aligned}$$

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- Your received signal = 0.7. What's your guess about the original signal? +1
- Your received signal = -0.2. What's your guess about the original signal? -1
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

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- Probability that $K = 1$, given $Y = y$? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$

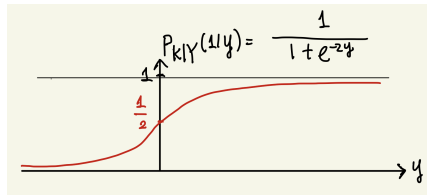
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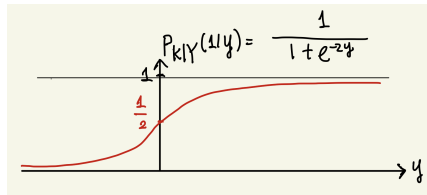
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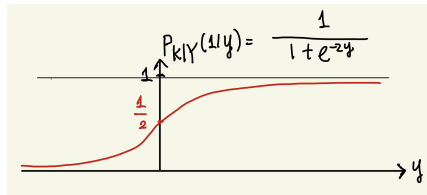
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- If $y > 0$, the inference probability for $K = 1$ exceeds $\frac{1}{2}$. So, original signal = 1.
- Similarly, compute $p_{K|Y}(-1|y)$ and then do the inference



Questions?

- 1) What is PDF and CDF?
- 2) Why do we need CDF?
- 3) What are joint/marginal/conditional PDFs?
- 4) Explain memorylessness of exponential random variables.
- 5) Explain the version of Bayes' rule for continuous and mixed random variables.