

Lecture 6: Statistical Inference

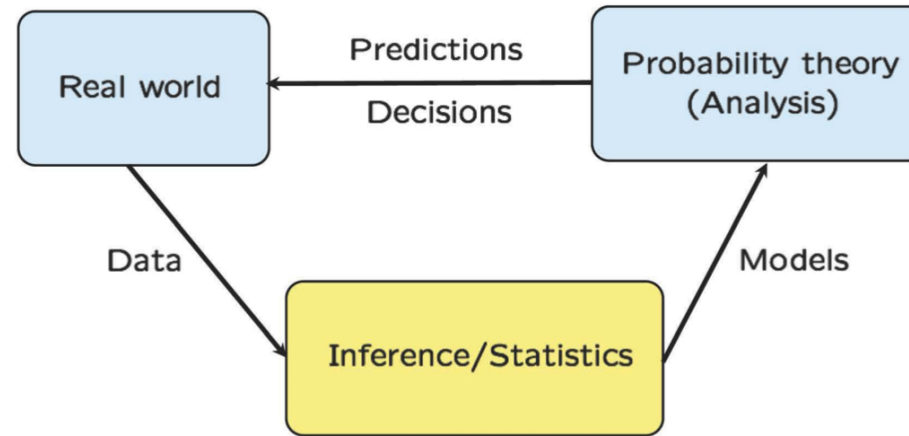
Yi, Yung (이윤)

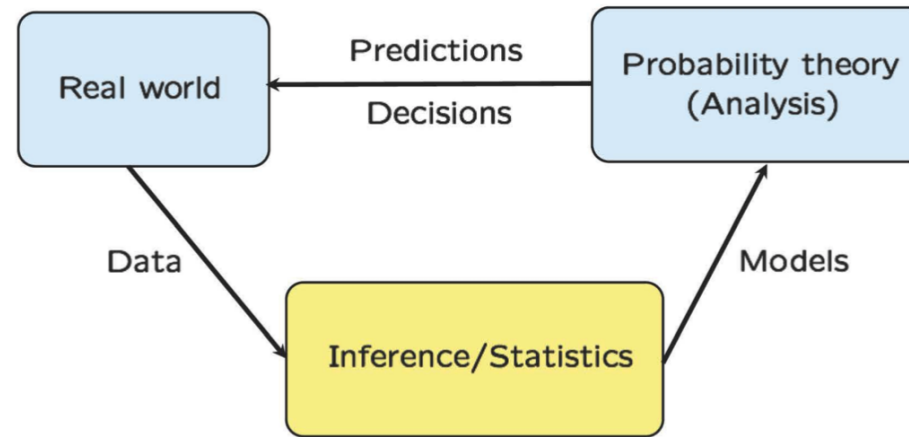
EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

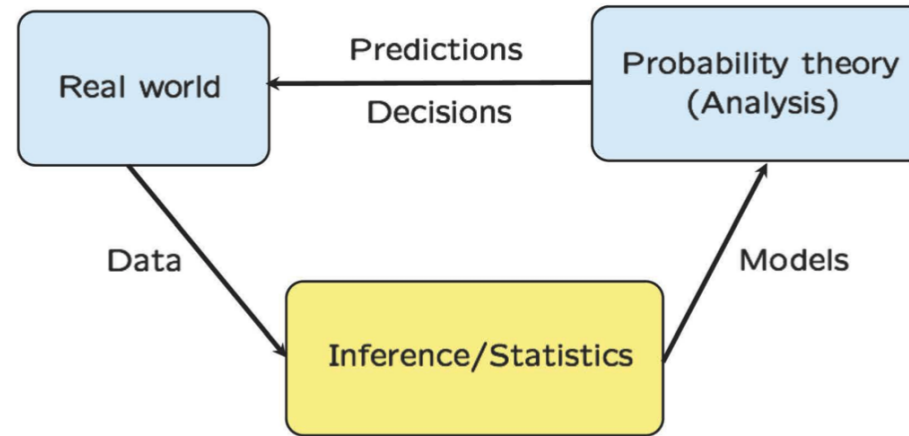
- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
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- LLMS (Linear LMS) Estimator
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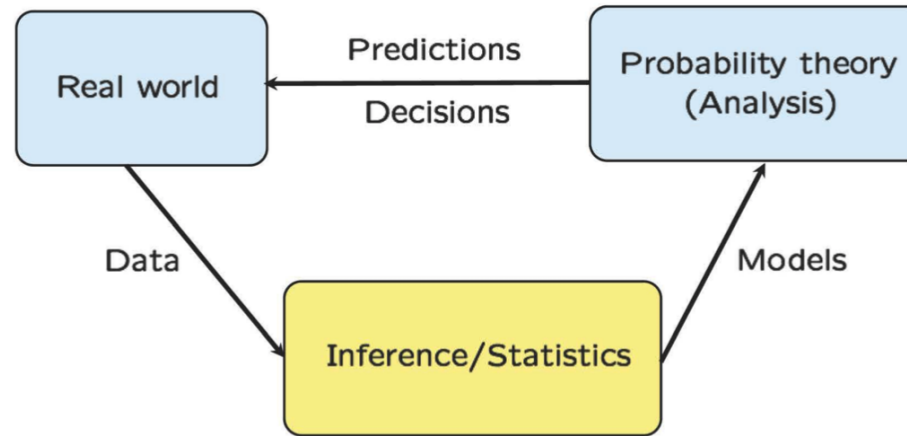




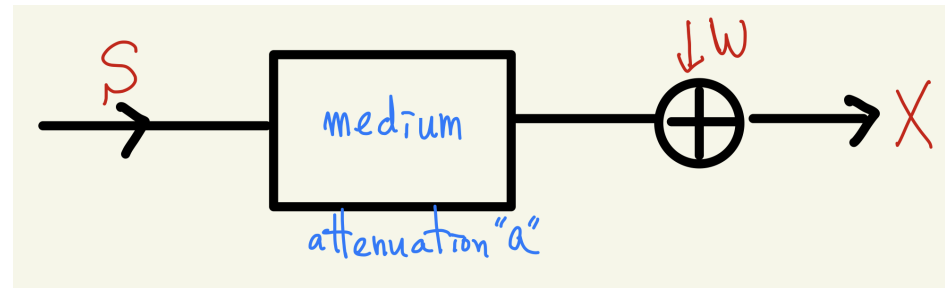
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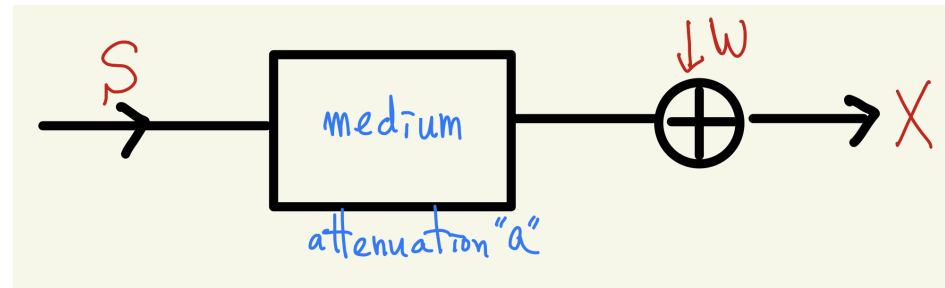
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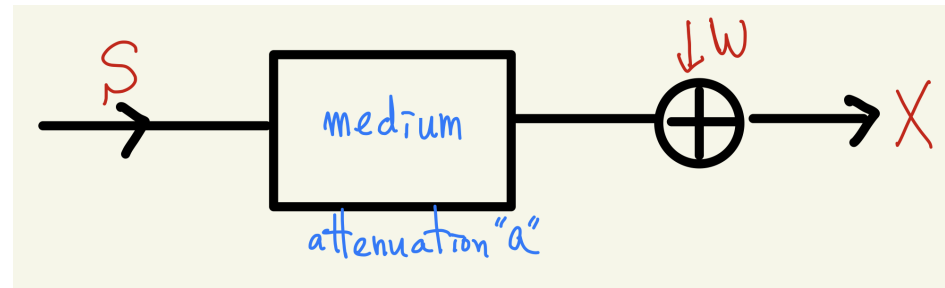
- Inference
 - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
 - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
 - Connecting big data and big model building



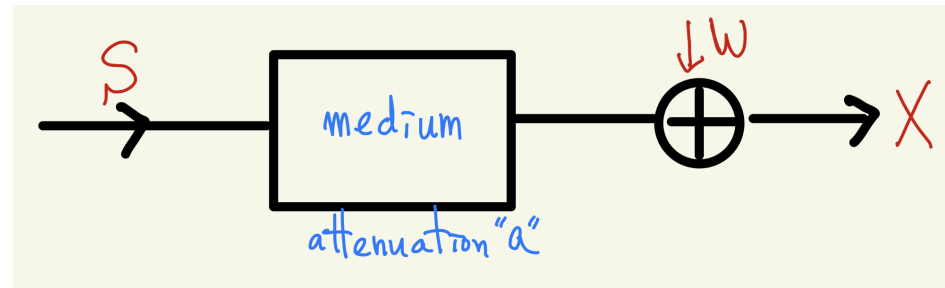
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- Same mathematical structure, because the parameters in models are variables in many cases

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 - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

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- Use Bayes' rule and find the **posterior**:

$$\mathbb{P}\left[\theta = \frac{3}{4} \middle| (HHH)\right] = \frac{27}{28}, \quad \mathbb{P}\left[\theta = \frac{1}{4} \middle| (HHH)\right] = \frac{1}{28}$$

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- **Classical approach** (Chapter 9)

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- Who is the winner? A century-long debate (see p. 409 for discussion)

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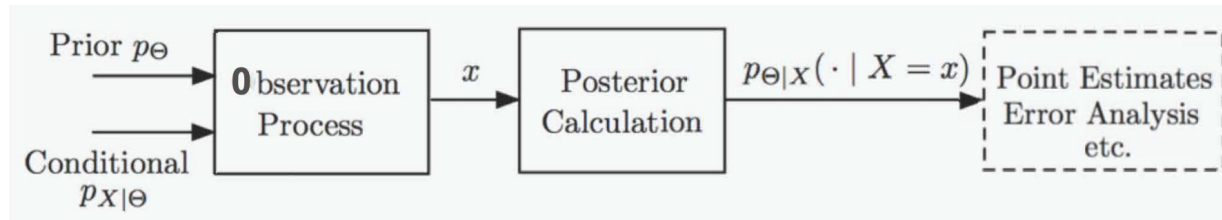
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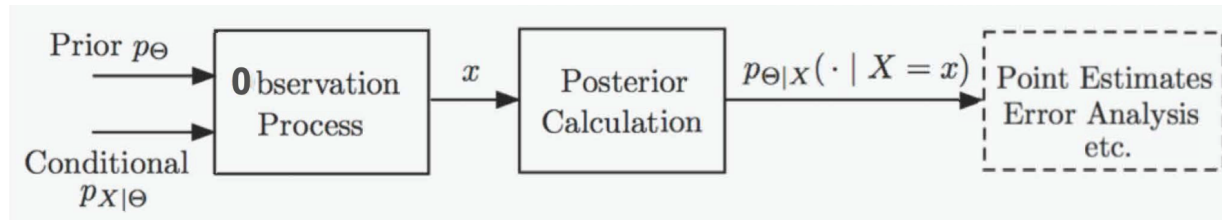
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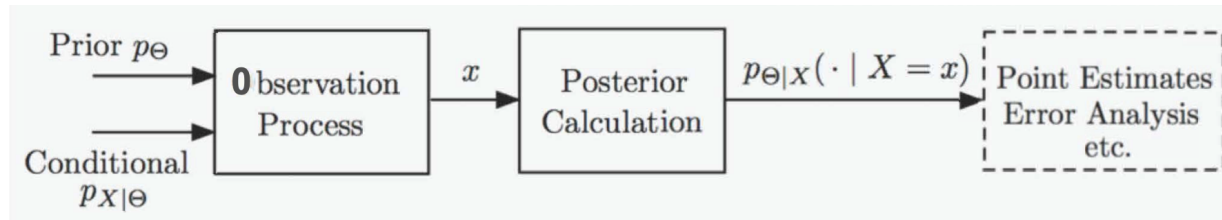
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Framework of Bayesian Inference

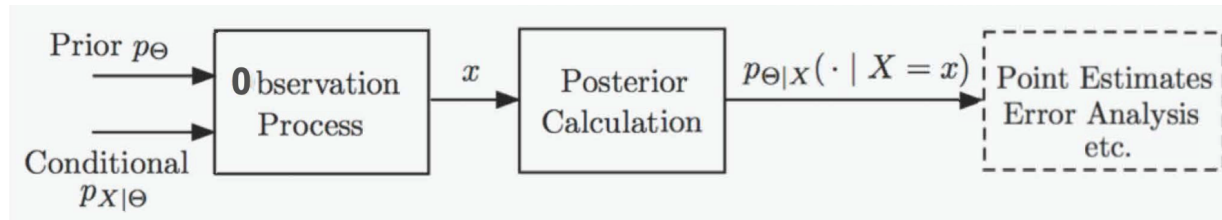




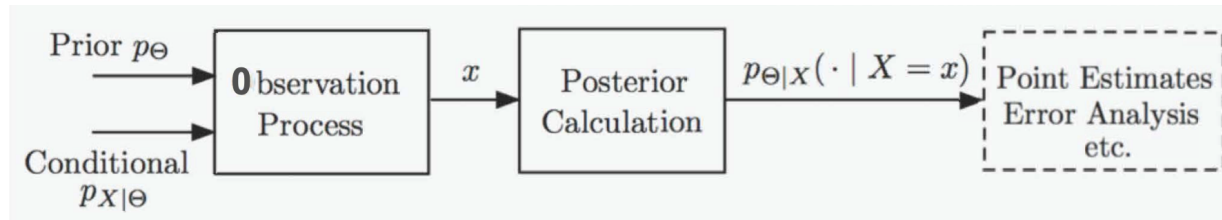
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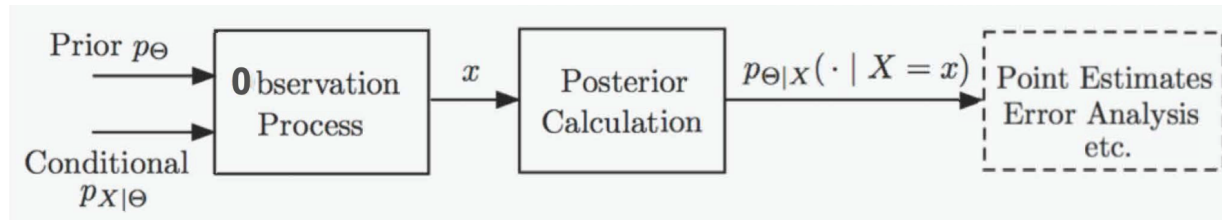


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 - Using the posterior distribution, apply one of the methods of choosing the final $\hat{\theta}$ for estimation and hypothesis testing.

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- Why MAP and LMS are good? Not mathematically clear yet (later)

- Random observation: X
- Observation instance: x
- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\text{MAP}} = g_{\text{MAP}}(x), \quad \hat{\theta}_{\text{LMS}} = g_{\text{LMS}}(x)$$

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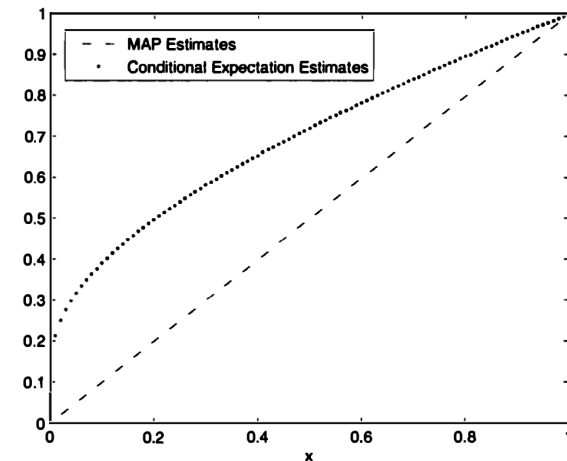
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- If $\Theta \sim \text{Beta}(\alpha, \beta)$, what is $\hat{\theta}_{\text{MAP}}$?

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$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1 - \theta)^{n-k}, \text{ } c \text{ the normalizing constant}$$

- If $\Theta \sim \text{Beta}(\alpha, \beta)$, what is $\hat{\theta}_{\text{MAP}}$?
- What is $\text{Beta}(\alpha, \beta)$?

Example 2: Biased Coin with Beta Prior (2)

Beta distribution

A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $B(\alpha, \beta)$, called Beta function, is a normalizing constant, given by

$$B(\alpha, \beta) = \int_0^1 \theta^{\alpha-1} (1 - \theta)^{\beta-1} d\theta = \frac{(\alpha - 1)! (\beta - 1)!}{(\alpha + \beta - 1)!}$$

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- A special case of $Beta(1, 1)$ is $Uniform[0, 1]$

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- When $\alpha = \beta = 1$ (i.e., $U[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_{\Theta}(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_{\Theta}(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

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Thus, **Claim 1** holds. We now take the expectation of the above equations, the law of iterated expectations leads to **Claim 2**.

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
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- ML (Maximum Likelihood) Estimator

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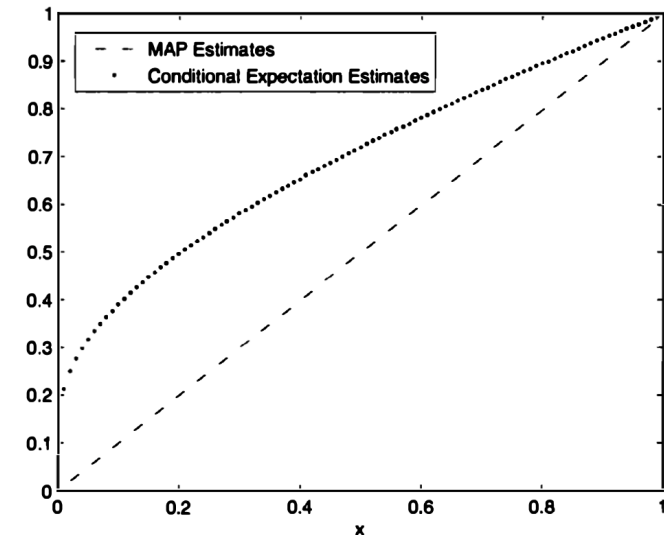
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- For $\alpha = \beta = 1$ ($\Theta = \text{Uniform}[0, 1]$),

$$\mathbb{E}[\Theta|X = k] = \frac{k + 1}{n + 2}$$

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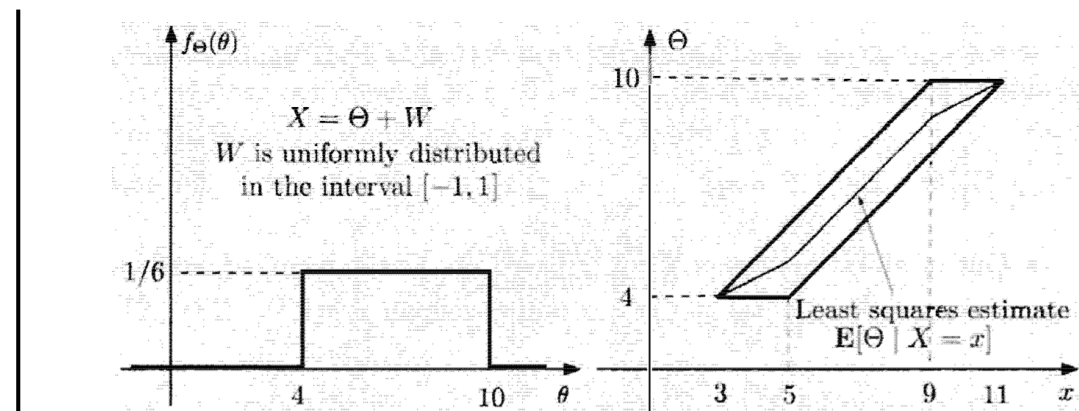


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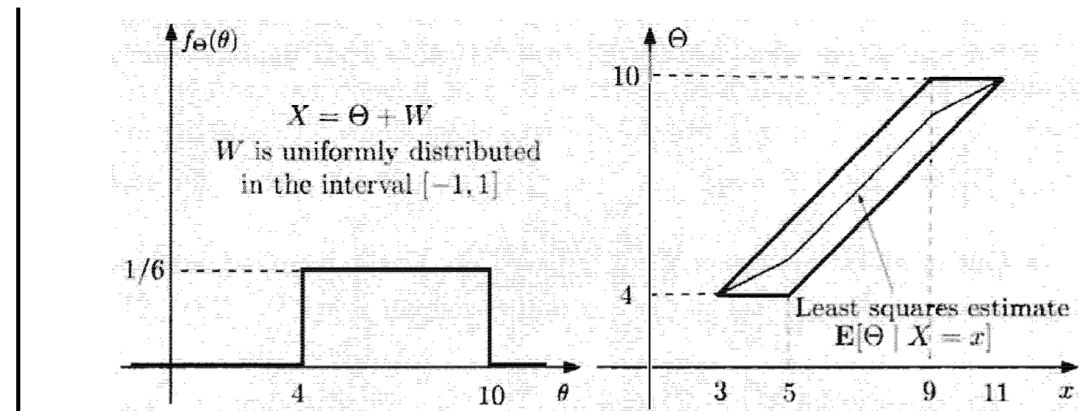
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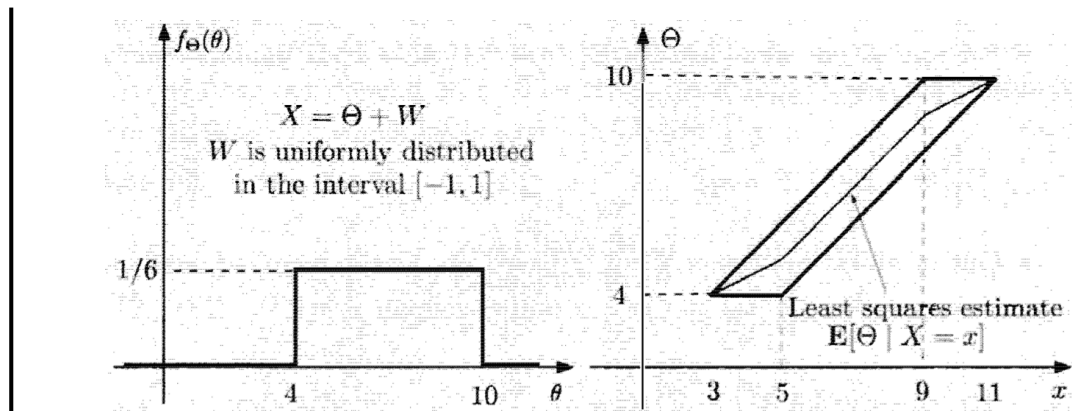
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- $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$ = midpoint of the corresponding vertical section



Example: Signal Recovery from Noisy Measurement (2)

- Unknown: $\Theta \sim \text{Uniform}[4, 10]$
- Observe Θ with random error W as X . $W \sim \text{Uniform}[-1, 1]$

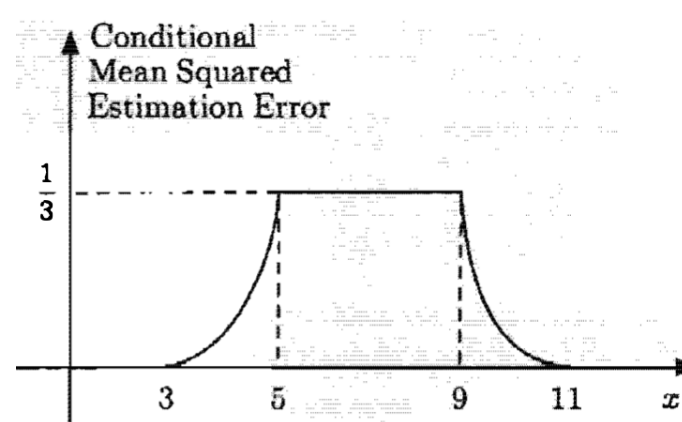
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- Conditional MSE

$$\mathbb{E}[(\Theta - \mathbb{E}[\Theta|X = x])^2 | X = x]$$



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- Any alternative to LMS estimator?

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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- Linear models are always the first choice for a simple design in engineering.



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$$a = \frac{\rho \sigma_{\Theta} \sigma_X}{\sigma_X^2} = \frac{\rho \sigma_{\Theta}}{\sigma_X}$$

- Then, we have (2).

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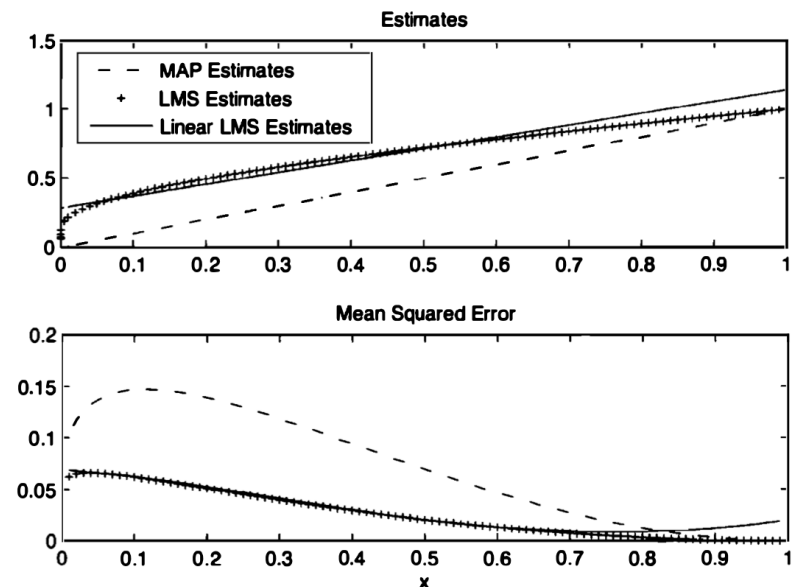
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- Biased coin with probability of head θ
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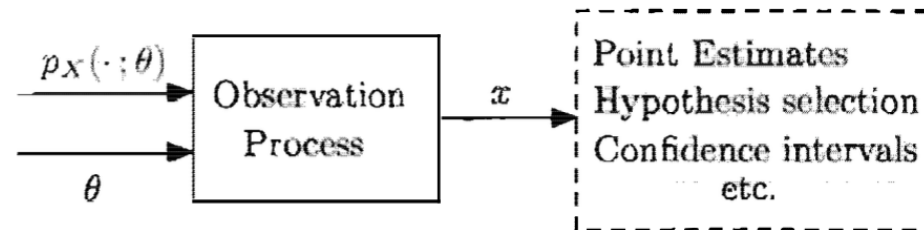
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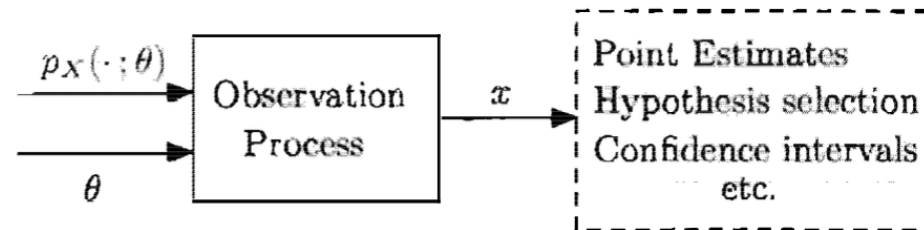
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Yes, because the LMS estimator was linear.

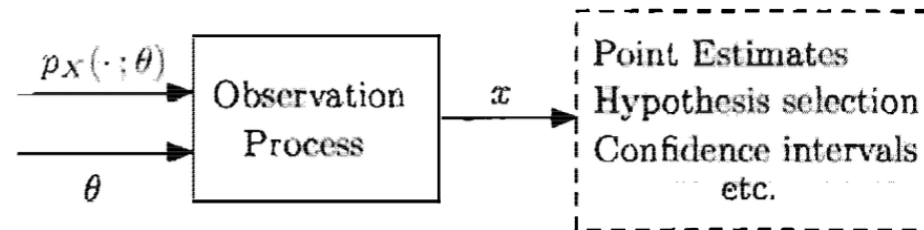
- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator



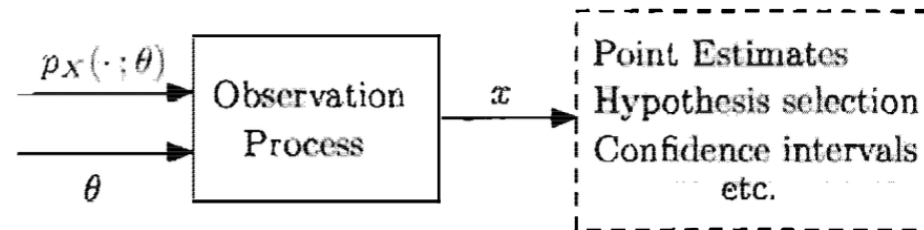
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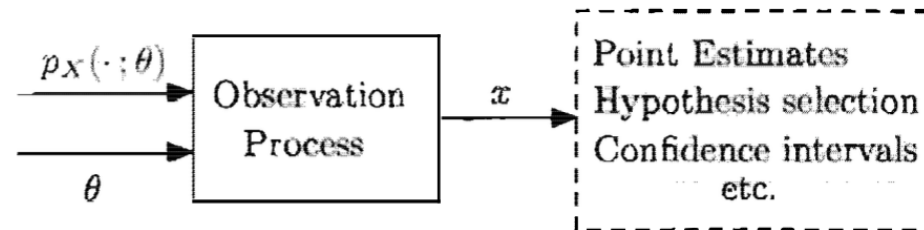
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- Choosing one among multiple probabilistic models
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 - Estimation
 - Hypothesis testing
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- Very often, X_i are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \dots, x_n; \theta) = \log \prod_{i=1}^n p_{X_i}(x_i; \theta) = \sum_{i=1}^n \log p_{X_i}(x_i; \theta)$$

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$$\arg \max_{\theta} f_X(x; \theta) = \arg \max_{\theta} \prod_{i=1}^n \theta e^{-\theta x_i} = \arg \max_{\theta} \left(n \log \theta - \theta \sum_{i=1}^n x_i \right)$$

Questions?

- 1) What is statistical inference?
- 2) Draw the building blocks of Bayesian inference and explain how it works.
- 3) What are MAP and LMS estimators and their underlying philosophies?
- 4) What is LLMS estimator and why is it useful?
- 5) Compare the classical and Bayesian inference.
- 6) What is the ML estimator and how is it related to the MAP estimator?