Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

August 31, 2021

Roadmap

(1) Probabilistic Model
- Mathematical description of uncertain situations

(2) Sample Space, Event, Probability Law
- Elements of probability theory

(3) Probability Axioms
- 3 axioms for the completeness of a theory

Roadmap

(1) Probabilistic Model

(2) Sample Space, Event, Probability Law

(3) Probability Axioms

What Do We Want?

Modeling: Understand reality with a simple (mathematical) model

- Experiment — Flip two coins
- Observation: a random outcome — for example, (H, H)
- All outcomes — $\{(H, H), (H, T), (T, H), (T, T)\}$

- Our goal: Build up a $\boxed{\text{probabilistic model}}$ for an experiment with random outcomes
- Probabilistic model?
  - Assign a number to each outcome or a set of outcomes
  - Mathematical description of an uncertain situation
- Which model is good or bad?

Probabilistic Model

Goal: Build up a probabilistic model. Hmm... How?

The first thing: What are the *elements* of a probabilistic model?

Elements of Probabilistic Model

1. All outcomes of my interest: $\boxed{\text{Sample Space } \Omega}$
2. Assigned numbers to each outcome of Ω: $\boxed{\text{Probability Law } \mathbb{P}(\cdot)}$

Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

Roadmap

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1. Sample Space Ω

The set of all outcomes of $\boxed{\text{my interest}}$

(1) Mutually exclusive

(2) Collectively exhaustive

(3) At the $\boxed{\text{right granularity}}$
(not too concrete, not too abstract)

1. Toss a coin. What about this?
   $\Omega = \{H, T, HT\}$
2. Toss a coin. What about this? $\Omega = \{H\}$
3. (a) Just figuring out prob. of H or T.
   $\Longrightarrow \Omega = \{H, T\}$

   (b) The impact of the weather (rain or no rain) on the coin's behavior.

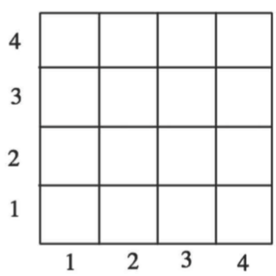
   $$\Longrightarrow \Omega = \{(H, R), (T, R), (H, NR), (T, NR)\},$$

   R(Rain), NR(No Rain).

Examples: Sample Space Ω

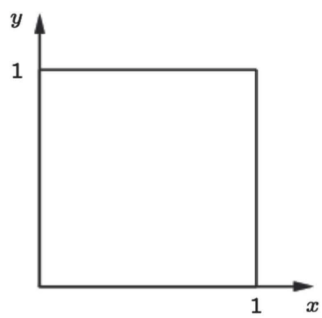
- *Discrete case:* Two rolls of a tetrahedral die

  - $\Omega = \{(1,1), (1,2), \ldots, (4,4)\}$



- *Continuous case:* Dropping a needle in a plain

  - $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x, y \le 1\}$



2. Probability Law

- Assign numbers to what? Each outcome?
- What is the probability of dropping a needle at $(0.5, 0.5)$ over the 1×1 plane?
- Assign numbers to each $\boxed{\text{subset}}$ of Ω
- a subset of Ω: $\boxed{\text{an event}}$
- $\mathbb{P}(A)$: Probability of an event A.
  - This is where probability meets set theory.
- Roll a dice. What is the probability of odd numbers?

  $\mathbb{P}(\{1,3,5\})$, where $\{1,3,5\} \subset \Omega$ is an event.

Roadmap

(1) Probabilistic Model

(2) Sample Space, Event, Probability Law

(3) Probability Axioms

How should we construct $\mathbb{P}(\cdot)$?

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- Need to construct $\mathbb{P}(\cdot)$ that naturally satisfies the intention of a probability theory designer just like you. What about the followings as starting points?
  - $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
  - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
  - For two disjoint[1] events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
  - $\mathbb{P}(\Omega) = 1$ (Why not $\mathbb{P}(\Omega) = 10$?)
  - $\mathbb{P}(\emptyset) = 0$
  - If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$
  - many others

[1] Their intersection is empty.

Probability Axioms

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- Surprisingly, we need just the following three rules (called axioms):

Probability Axioms: Version 1

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. (Finite) additivity: For two disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

- No other things are necessary, and we can prove all other things from the above axioms.
- Note that coming up with the above axioms is far from trivial.

Examples

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A1: Nonnegativity, A2: Normalization, A3: Finite additivity

Prove the following properties using the axioms: VIDEO PAUSE

1. For any event A, $\mathbb{P}(A) \leq 1$

   $$1 \overset{\text{A2}}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(A^c) \Longrightarrow \mathbb{P}(A) = 1 - \mathbb{P}(A^c) \overset{\text{A1}}{\leq} 1$$

2. $\mathbb{P}(\emptyset) = 0$

   $$\mathbb{P}(\Omega \cup \emptyset) \overset{\text{A3}}{=} \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) \overset{\text{A2}}{=} 1 + \mathbb{P}(\emptyset) \overset{\text{from 1.}}{\Longrightarrow} \mathbb{P}(\emptyset) = 0$$

3. If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$

   $$\mathbb{P}(B) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \overset{\text{A1}}{\geq} \mathbb{P}(A)$$

Probability Calculation Steps

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1. Specify the sample space
2. Specify a probability law
   - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4$, $\mathbb{P}(\{T\}) = 3/4$,
3. probability of head or tail
4. $1/4$, $3/4$

Discrete but infinite sample space

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- $\Omega = \{1, 2, 3, \ldots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \ldots$
- Is the above probability law legitimate? seems OK

$$\mathbb{P}(\Omega) = \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{1/2}{1 - 1/2} = 1$$

- $\mathbb{P}(\text{even numbers})$?

$$\begin{aligned}\mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \ldots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = 1/3\end{aligned}$$

- Is the above right? If not, why?
  - Wrong: Finite additivity axiom does not allow this.

Probability Axioms Version 1 2

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Probability Axioms: Version 1 2

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. (Finite) additivity: For two disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

A4. $\boxed{\text{Countable additivity:}}$ If $A_1, A_2, A_3, \ldots$ is an infite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$.

Interpretation of Probability Theory

- A narrow view: A branch of math
  - axioms $\rightarrow$ theorems
  - Mathematicians work very hard to find the smallest set of necessary axioms (just like atoms in physics)
- Frequencies: $\mathbb{P}(H) = 1/2$
  - Understanding an uncertain situation: fractions of successes out of many experiments
- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$

Anyway, we believe that probabilistic reasoning is very helpful to understand the world with many uncertain situations.

Questions?

You build up the very basics of a probabilistic model.

What else do we need to build up?

Review Questions

1) Explain what a probabilistic model is and why we need it.
2) What is the mathematical definition of event?
3) What are the key elements of the probabilistic model?
4) List up the probability axioms and explain them. Are you going to choose the same axioms to build up the probability theory?
5) Why do we need countable additivity in the probability axioms?