Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

Derived Distribution: $Y = g(X)$

- Given the PDF of X, What is the PDF of $Y = g(X)$?

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- What are easy or difficult cases?

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- Examples: $Y = X$, $Y = X + 1$, $Y = X^2$, etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

Discrete Case

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x:g(x)=y} p_X(x) \end{aligned}$$

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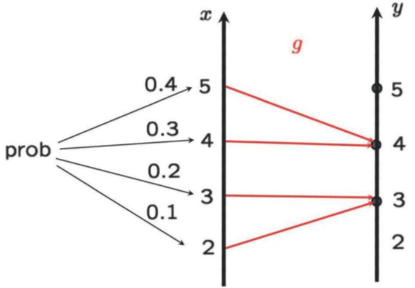

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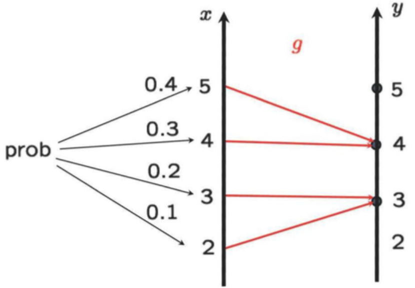
$$p_Y(y) = \mathbb{P}(g(X) = y) = \sum_{x:g(x)=y} p_X(x)$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$

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If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

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Special case. X is normal. Then, Y is also normal, i.e., $Y \sim N(a\mu + b, a^2\sigma^2)$

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Step 1. Find the CDF of Y:

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Ex1. $X \sim uniform[0,1]$. $Y = \sqrt{X}$.

$$F_Y(y) = \mathbb{P}(\sqrt{X} \leq y) = \mathbb{P}(X \leq y^2) = y^2$$

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Ex2. $X \sim uniform[0,2]$. $Y = X^3$.

$$F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \frac{1}{2}y^{1/3}$$

$$f_Y(y) = \frac{1}{6}y^{-2/3}, \quad 0 \le y \le 8$$

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Ex3. X with $f_X(x)$. $Y = X^2$.

$$F_Y(y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \ge 0$$

Functions of multiple rvs: $Y = g(X, Y)$ (1)

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f_Z(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}
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Functions of multiple rvs: $Y = g(X, Y)$ (2)

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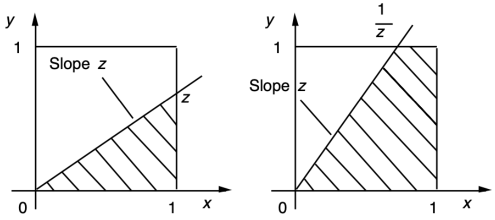
- Depending on the value of z, two cases need to be considered separately.



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$$F_Z(z) = \mathbb{P}(Y/X \leq z)$$
$$= \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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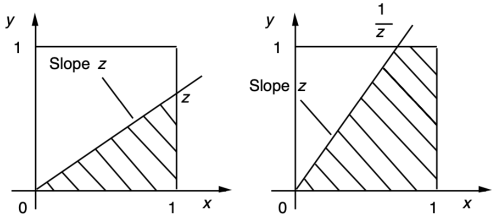

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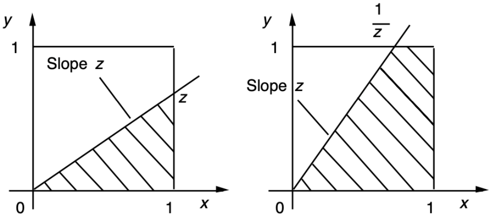

- A very basic case with many applications
- Assume that $X, Y \in \mathbb{Z}$

Functions of multiple rvs: $Z = X + Y$, $X \perp\!\!\!\perp Y$

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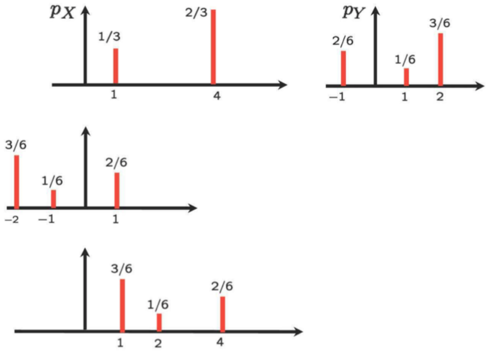
- Interpretation (for a given z)

(i) Flip (horizontally) $p_Y(y)$ ($p_Y(-x)$)

(ii) Put it underneath $p_X(x)$ ($p_Y(-x+z)$)



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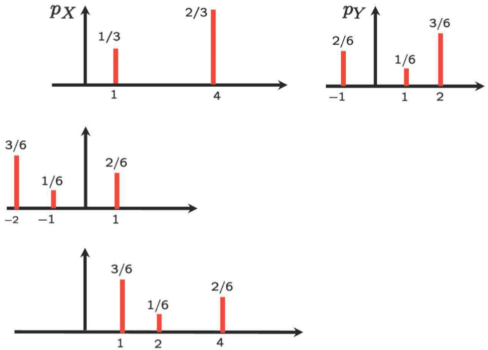

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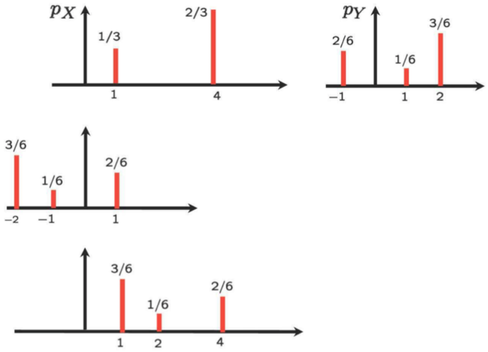

$Y = X + Y$, $X \perp\!\!\!\perp Y$: Continuous

- Same logic as the discrete case

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  - X and Y are normal.

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$X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_x, \sigma_x^2)$
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- Why normal rvs are used to model the sum of random noises.
- (Extension) The sum of finitely many independent normals is also normal.

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
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- Bayes' rule for random variables
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- Goal: Given two rvs X and Y, assign some number that quantifies the degree of their dependence

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

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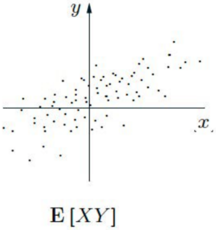
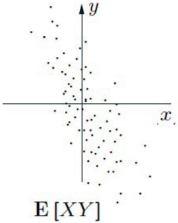
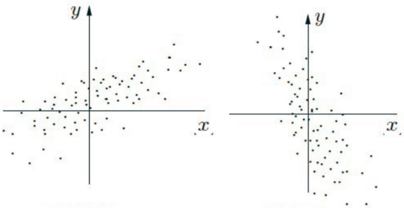
- What about $\mathbb{E}[XY]$? Seems good.
    - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$ when $X \perp\!\!\!\perp Y$
    - More data points (thus increases) when $xy > 0$ (both positive or negative)





OK. Let's Design!

- Simple case: $\mathbb{E}[X] = \mu_X = 0$ and $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If $X \uparrow$, $Y \uparrow$) or Negative (If $X \uparrow$, $Y \downarrow$)
- What about $\mathbb{E}[XY]$? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$ when $X \perp\!\!\!\perp Y$
  - More data points (thus increases) when $xy > 0$ (both positive or negative)



$\mathbf{E}[XY]$ $\mathbf{E}[XY]$

(Q) What about $\mathbb{E}[X + Y]$?

What If $\mu_X \neq 0, \mu_Y \neq 0$?

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Example: $\mathrm{cov}(X, Y) = 0$, but not independent

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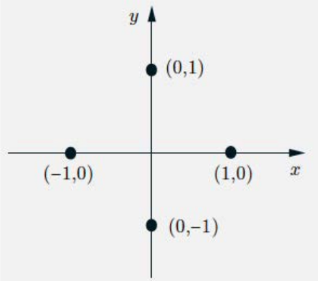

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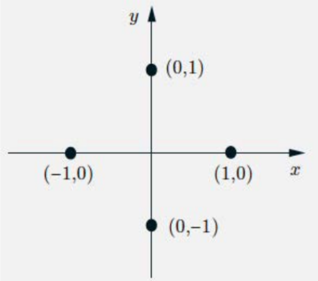

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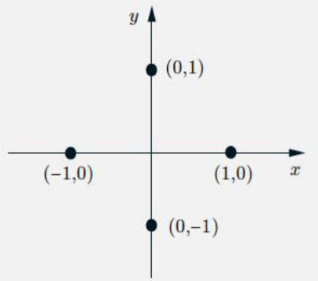
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- (Q) $\text{var}[X]$
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KAIST EE

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- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$ (linear relation, VERY related)

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- (Derived) Distribution of $Y = g(X)$ or $Z = g(X, Y)$
- Quantifying the degree of dependence between two rvs.
- Conditional expectation/variance
- (Random) Sum of random variables

A Special Random Variable

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

Conditional Expectation $\mathbb{E}[X|Y]$

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[redacted]

A random variable $g(Y) = \boxed{\qquad\qquad}$, called $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$, takes the value $g(y) = \mathbb{E}[X|Y = y]$, if Y happens to take the value y.

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- Thus, having a distribution, expectation, variance, all the things that a random variable has
- Often confusing because of the notation

Expectation of $\mathbb{E}[X|Y]$

[redacted]

$$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

Proof.

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y = y]p_Y(y)\\ &= \mathbb{E}[X]\end{aligned}$$

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- X : February sales
- Forecast in the beg. of the year: $\mathbb{E}[X]$
- End of Jan. new information $Y = y$ (Jan. sales)
  Revised forecast: $\mathbb{E}[X|Y=y]$
  Revised forecast $\neq \mathbb{E}[X]$

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- $\mathbb{E}[X|Y]=Y/2$
- $\mathbb{E}[X]=\mathbb{E}[\mathbb{E}[X|Y]]=\mathbb{E}[Y/2]=\frac{1}{2}\frac{l}{2}=l/4$

- Forecasts on sales: calculating expected value, given any available information
- X : February sales
- Forecast in the beg. of the year: $\mathbb{E}[X]$
- End of Jan. new information $Y=y$ (Jan. sales)
  Revised forecast: $\mathbb{E}[X|Y=y]$
  Revised forecast $\neq \mathbb{E}[X]$
- Law of iterated expectations
  $\mathbb{E}$[revised forecast] $=$ original one

Conditional Variance var$[X|Y]$

$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Conditional Variance $\mathrm{var}[X|Y]$

$$\mathrm{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\mathrm{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\text{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

[redacted]

A random variable $g(Y) =$ $\boxed{\qquad\qquad}$ and called $\boxed{\qquad\qquad\qquad\qquad\qquad\qquad}$, takes the value $g(y) = \text{var}[X|Y = y]$, if Y happens to take the value y.

Conditional Variance $\mathrm{var}[X|Y]$

$$\mathrm{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$\mathrm{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2|Y = y]$$

[redacted]

A random variable $g(Y) = \boxed{\mathrm{var}[X|Y]}$ and called $\boxed{\text{conditional variance of } X \text{ given } Y}$, takes the value $g(y) = \mathrm{var}[X|Y = y]$, if Y happens to take the value y.

Conditional Variance $\mathrm{var}[X|Y]$

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[redacted]

A random variable $g(Y) = \boxed{\mathrm{var}[X|Y]}$ and called $\boxed{\text{conditional variance of } X \text{ given } Y}$, takes the value $g(y) = \mathrm{var}[X|Y = y]$, if Y happens to take the value y.

- A function of Y
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has