Lecture 7: Law of Large Numbers and Central Limit Theorem

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Most remarkable two results in probability theory history

- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
  - Moment Generating Function (MGF)
- Strong Law of Large Numbers

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- $\mathbb{E}[X_i] = \mu$, $\text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

$$S_n = X_1 + X_2 + \ldots + X_n$$

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- First, consider the sample mean, and try to understand how it behaves:

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- We call this law of large numbers.

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- Need to mathematically build up the concept of convergence for the sequence of random variables.

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Convergence in probability

For any $\epsilon > 0$, $\mathbb{P}\Big(|Y_n - a| \geq \epsilon\Big) \xrightarrow{n\to\infty} 0$.

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- Proof requires some knowledge about useful inequalities, which we cover later.

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- The answer is $\frac{1}{2}$

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- Interestingly, it converges to some random variable Z that we know very well.

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- Meaning from scaling perspective.
  - LLN: Scaling S_n by $1/n$, you go to a deterministic world.
  - CLT: Scaling S_n by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each X_i. Very interesting!

Practical Use of CLT

$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

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  - If X_i resembles a normal rv more, smaller n works: symmetry and unimodality[1]

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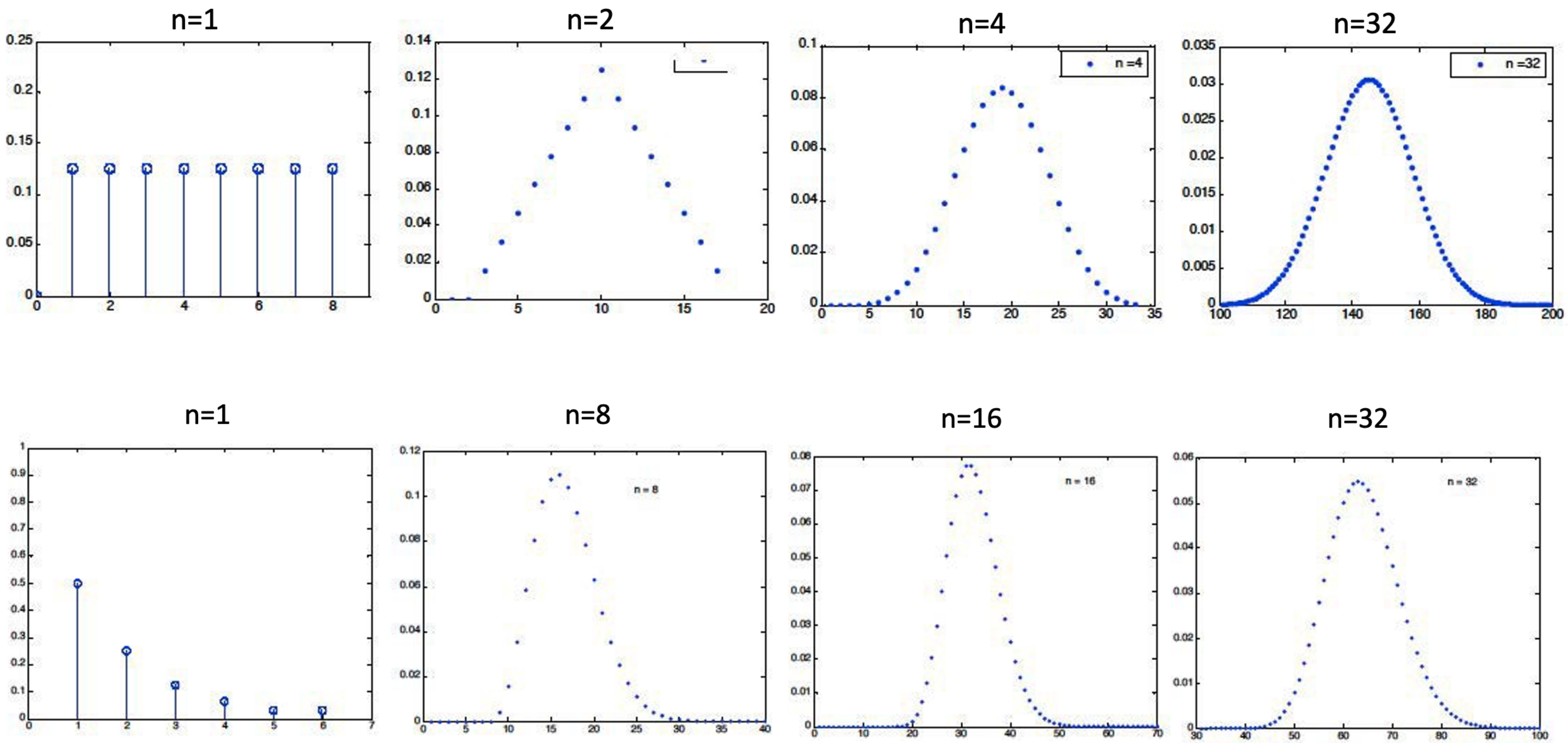
CLT: Examples of n



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Note that we have:

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Markov Inequality

KAIST EE

- (Q) Knowing $\mathbb{E}(X)$, can we say something about the distribution of X?
- Intuition: small $\mathbb{E}(X) \Longrightarrow$ small $\mathbb{P}(X \geq a)$

Markov Inequality

If $X \geq 0$ and $a > 0$, then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.

Proof. For any $a > 0$, define Y_a as:

$$Y_a \triangleq \begin{cases} 0, & \text{if } X < a, \\ a, & \text{if } X \geq a \end{cases}$$

Then, using non-negativity of X, $Y_a \leq X$, which leads to $\mathbb{E}[Y_a] \leq \mathbb{E}[X]$.

Note that we have:

$$\mathbb{E}[Y_a] = a\mathbb{P}(Y_a = a) = a\mathbb{P}(X \geq a).$$

Thus, $a \cdot \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$. $\square$

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- Both bounds are the ones that bound the probability of rare events.

Back to WLLN

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

Weak law of large numbers

M_n converges to μ in probability.

Proof.

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$$\mathbb{P}\Big(|M_n - \mu| \geq \epsilon\Big) \leq \frac{\mathrm{var}(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{n\to\infty} 0$$

Roadmap

- Most remarkable two results in probability theory history

- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
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Ex1) $X \sim \exp(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$

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Ex2) $X \sim N(0,1)$ (homework problem)

$$M(s) = e^{s^2/2}$$

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

Inversion Property

The transform $M_X(s)$ associated with a random variable X uniquely determines the CDF of X, assuming that $M_X(s)$ is finite for all s in some interval $[-a, a]$, where a is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own MGF.

Back to CLT

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$

Proof.

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- If we apply l'hopital's rule twice (please check), we get

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Roadmap

- Most remarkable two results in probability theory history

- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
  - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
  - Moment Generating Function (MGF)
- Strong Law of Large Numbers (Optional)

Questions?

Review Questions

1) What's the practical value of LLN and CLT?
2) Explain LLN and CLT from the scaling perspective.
3) Why are LLN and CLT great?
4) Why do we need different concepts of convergence for random variables?
5) Explain what is convergence in probability.
6) Explain what is convergence in distribution.
7) Why is MGF useful?