

## Lecture 2: Conditioning, Bayes' Rule, and Independence

Yi, Yung (이웅)

EE210: Probability and Introductory Random Processes  
KAIST EE

April 17, 2021

- (1) Conditional Probability
  - How should I change my belief about event  $A$ , if I come to know that event  $B$  occurs?
- (2) Bayes' Rule and Bayesian Inference
  - prob. of  $A$  given that  $B$  occurs vs. prob. of  $B$  given that  $A$  occurs
- (3) Independence, Conditional Independence
  - Can I ignore my knowledge about event  $B$ , when I consider event  $A$ ?

April 17, 2021 1 / 22

April 17, 2021 2 / 22

- (1) **Conditional Probability**
- (2) Bayes' Rule and Bayesian Inference
- (3) Independence, Conditional Independence

- Pick a person  $a$  at random
  - event  $A$ :  $a$ 's age  $\leq 20$
  - event  $B$ :  $a$  is married
- (Q1) What is the probability of  $A$ ?
- (Q2) What is the probability of  $A$ , if I know that that  $B$  is true?
- Clearly, the above two should be different. I will assign **lower** probability for (Q2).
- **Question**: How should I change my belief, given some additional information?
- Need to build up a new theoretical concept, which we call **conditional probability**.

- First, let's choose the notation. "Probability of  $A$ , given  $B$  occurs". What do you recommend?

$$\mathbb{P}(A)(B), \mathbb{P}_B(A), \mathbb{P}^B(A), (B)\mathbb{P}(A), \dots$$

- People's choice is ...  $\mathbb{P}(A | B)$
- From now on, given  $B$ ,  $\mathbb{P}(\cdot | B)$  should be a new **probability law**.
  - Three axioms<sup>1</sup> should be satisfied.

<sup>1</sup>Non-negativity, Normalization, Countable Additivity  
L2(1)

- Second, let's define  $\mathbb{P}(A|B)$ . What would it be a good definition?
- Probability of  $A$  given  $B \rightarrow$  both  $A$  and  $B$  occur. Then, what about this?

$$\mathbb{P}(A | B) \triangleq \mathbb{P}(A \cap B)$$

- Is it good or bad? Why good? Why bad?
- Reasons why it is bad:
  - $\mathbb{P}(\cdot | B)$  should be a new probability law (thus, three axioms)
  - $\mathbb{P}(\Omega | B) = 1$ ?
  - $\mathbb{P}(B | B) = 1$  from our common sense.
  - True?

- How to fix this? **Normalization**

$$\mathbb{P}(A | B) \triangleq \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \text{ for } \mathbb{P}(B) > 0.$$

- Note that this is a **definition**, not a **theorem**.
- So, it's not about right or wrong. It's about how happy we are about this definition.
- All properties of the law  $\mathbb{P}(\cdot)$  is applied to the conditional law  $\mathbb{P}(\cdot | B)$ .
  - Non-negativity**.  $\mathbb{P}(A|B)$  for any event  $A$ ?
  - Finite additivity** and thus **countable additivity**. For any two disjoint  $A$  and  $C$ ,

$$\mathbb{P}(A \cup C | B) = \frac{\mathbb{P}[(A \cup C) \cap B]}{\mathbb{P}(B)} = \frac{\mathbb{P}[(A \cap B) \cup (C \cap B)]}{\mathbb{P}(B)} = \mathbb{P}(A | B) + \mathbb{P}(C | B)$$

- (1) Conditional Probability
- (2) **Bayes' Rule and Bayesian Inference**
- (3) Independence, Conditional Independence

From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference* (추론).

- $A_1$ : Happy ( $:-$ ),  $A_2$ : Sad ( $:-$ )
- $B$ : Shout
- Assume that somebody gives you the following information:  
 $\mathbb{P}(A_1)$ ,  $\mathbb{P}(A_2)$ ,  $\mathbb{P}(B|A_1)$ ,  $\mathbb{P}(B|A_2)$ .
- **Question:**  $\mathbb{P}(A_1|B)$  and  $\mathbb{P}(A_2|B)$ ?

- $A_i$ : state/cause/original value
- $B$ : result/resulting action/noisy measurement
- In reality,  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$  (cause  $\rightarrow$  result) can be given from my model
- Inference:  $\mathbb{P}(\text{cause} | \text{result})$ ?

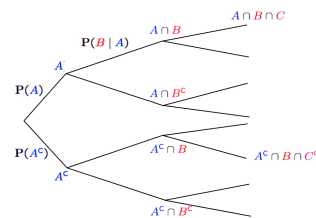
We will study this topic rigorously later in this class (chapter 8).

## Multiplication Rule

- $\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$
- $\mathbb{P}(A^c \cap B \cap C^c) = \mathbb{P}(A^c \cap B) \cdot \mathbb{P}(C^c|A^c \cap B)$   
 $= \mathbb{P}(A^c) \cdot \mathbb{P}(B|A^c) \cdot \mathbb{P}(C^c|A^c \cap B)$

Generally,

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, A_2, \dots, A_{n-1})$$



VIDEO PAUSE

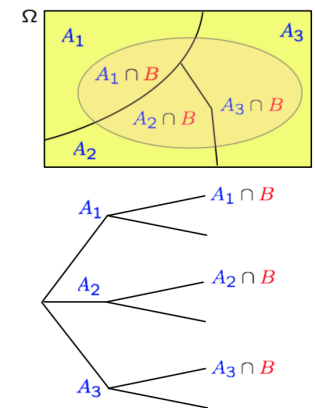
## Total Probability Theorem

- **Partition** of  $\Omega$  into  $A_1, A_2, A_3$
- We know:  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$
- **What is  $\mathbb{P}(B)$ ?** (probability of result)

### Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

- $\mathbb{P}(A_i \cap B) = \mathbb{P}(A_i)\mathbb{P}(B|A_i)$
- Weighted average from the point of  $A_i$  knowledge.

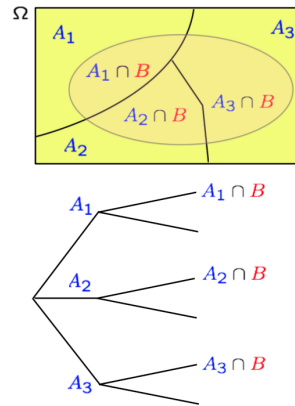


<sup>1</sup>Partition:  $A_1, A_2, A_3$  are mutually exclusive and  $\Omega = A_1 \cup A_2 \cup A_3$

- Partition of  $\Omega$  into  $A_1, A_2, A_3$
- We know:  $\mathbb{P}(A_i)$  and  $\mathbb{P}(B|A_i)$
- What is  $\mathbb{P}(A_i|B)$ ?
- revised belief about  $A_i$ , given  $B$  occurs

## Bayes' Rule

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j)\mathbb{P}(B|A_j)}$$



L2(2)

April 17, 2021 13 / 22

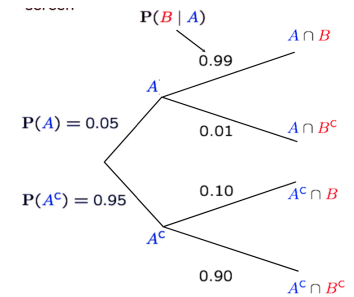
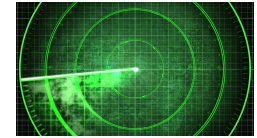
## VIDEO PAUSE

- $A$  : Airplane is flying above
- $B$  : Something on radar screen

$$\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\mathbb{P}(B|A) \\ &= 0.05 \times 0.99 = 0.0495 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B) \\ &= 0.05 \times 0.99 + 0.95 \times 0.1 = 0.1445 \end{aligned}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.0495}{0.1445} \approx 0.34$$



L2(2)

April 17, 2021 14 / 22

- $A_1$ : you are happy,  $A_2$ : you are sad
- $B$ : you shout.
- Assume:
  - $\mathbb{P}(A_1) = 0.7, \mathbb{P}(A_2) = 0.3,$
  - $\mathbb{P}(B|A_1) = 0.3, \mathbb{P}(B|A_2) = 0.5.$

- Calculate  $\mathbb{P}(A_1|B)$  and  $\mathbb{P}(A_2|B)$ .

## VIDEO PAUSE

$$\begin{aligned} \mathbb{P}(A_1)\mathbb{P}(B|A_1) &= 0.7 \times 0.3 = 0.21 \\ \mathbb{P}(A_2)\mathbb{P}(B|A_2) &= 0.3 \times 0.5 = 0.15 \\ \mathbb{P}(B) &= 0.21 + 0.15 = 0.36 \end{aligned}$$

$$\begin{aligned} \mathbb{P}(A_1|B) &= \frac{0.21}{0.36} \approx 0.583 \\ \mathbb{P}(A_2|B) &= \frac{0.15}{0.36} \approx 0.417 \end{aligned}$$

- (1) Conditional Probability
- (2) Bayes' Rule and Bayesian Inference
- (3) Independence, Conditional Independence

L2(2)

April 17, 2021 15 / 22

L2(3)

April 17, 2021 16 / 22

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.

- Event  $A$ : I get the grade A in the probability class (my interest).
- Event  $B$ : My friend is rich.
- $A$  and  $B$  do not seem dependent on each other. So, just forget  $B$ !
- Independence makes our analysis and modeling **much simpler**, because I can remove independent events in the analysis of what I am interested in.

L2(3)

April 17, 2021 17 / 22

L2(3)

April 17, 2021 18 / 22

## Independence

- Occurrence of  $A$  provides **no new information** about  $B$ . Thus, knowledge about  $A$  does **NOT** change my belief about  $B$ .

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

- Using  $\mathbb{P}(B|A) = \mathbb{P}(B \cap A) / \mathbb{P}(A)$ ,

Independence of  $A$  and  $B$ ,  $A \perp\!\!\!\perp B$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$$

- The above definition show the **symmetry** of independence more clearly.
- **Q1.**  $A$  and  $B$  disjoint  $\implies A \perp\!\!\!\perp B$ ?  
No. Actually, really dependent, because if you know that  $A$  occurred, then, we know that  $B$  did not occur.
- **Q2.** If  $A \perp\!\!\!\perp B$ , then  $A \perp\!\!\!\perp B^c$ ? Yes.

L2(3)

April 17, 2021 19 / 22

## Conditional Independence

- Remember: for a probability law  $\mathbb{P}(\cdot)$ , given some event  $C$ ,  $\mathbb{P}(\cdot|C)$  is a new probability law.
- Thus, we can talk about independence under  $\mathbb{P}(\cdot|C)$ .
- Given that  $C$  occurs, occurrence of  $A$  provides no new information about  $B$ .

$$\mathbb{P}(B|A \cap C) = \mathbb{P}(B|C)$$

- Using  $\mathbb{P}(A \cap B|C) = \frac{\mathbb{P}(B \cap (A \cap C))}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C) \mathbb{P}(B|A \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|C) \mathbb{P}(B|C)$ ,

Conditional Independence of  $A$  and  $B$  given  $C$ ,  $A \perp\!\!\!\perp B|C$

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \times \mathbb{P}(B|C)$$

L2(3)

April 17, 2021 20 / 22

### (Q1) $A \perp\!\!\!\perp B \rightarrow A \perp\!\!\!\perp B|C$ ?

- Suppose that  $A$  and  $B$  are independent. If you heard that  $C$  occurred,  $A$  and  $B$  are still independent?

VIDEO PAUSE

- Two independent coin tosses
  - $H_1$ : 1st toss is a head
  - $H_2$ : 2nd toss is a head
  - $D$ : two tosses have different results.
- $\mathbb{P}(H_1|D) = 1/2, \mathbb{P}(H_2|D) = 1/2$
- $\mathbb{P}(H_1 \cap H_2|D) = 0,$
- No.

L2(3)

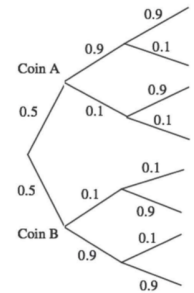
April 17, 2021 21 / 22

### (Q2) $A \perp\!\!\!\perp B|C \rightarrow A \perp\!\!\!\perp B$ ?

- Two coins: Blue and Red. Choose one uniformly at random, and proceed with two independent tosses.
- $\mathbb{P}(\text{head of blue}) = 0.9$  and  $\mathbb{P}(\text{head of red}) = 0.1$   
 $H_i$ :  $i$ -th toss is head, and  $B$ : blue is selected.
- $H_1 \perp\!\!\!\perp H_2|B$ ? Yes

$$\mathbb{P}(H_1 \cap H_2|B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1|B)\mathbb{P}(H_2|B) = 0.9 \times 0.9$$

- $H_1 \perp\!\!\!\perp H_2$ ? No  
$$\begin{aligned} \mathbb{P}(H_1) &= \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c) \\ &= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2} \\ \mathbb{P}(H_2) &= \mathbb{P}(H_1) \quad (\text{because of symmetry}) \\ \mathbb{P}(H_1 \cap H_2) &= \mathbb{P}(B)\mathbb{P}(H_1 \cap H_2|B) + \mathbb{P}(B^c)\mathbb{P}(H_1 \cap H_2|B^c) \\ &= \frac{1}{2}(0.9 \times 0.9) + \frac{1}{2}(0.1 \times 0.1) \neq \frac{1}{2} \end{aligned}$$



L2(3)

April 17, 2021 22 / 22

## Independence of Multiple Events

- Three events:  $A_1, A_2, A_3$ . What are the conditions of “their independence”?
- What about this? (Pairwise independence)  
 $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3), \mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$
- What about  $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$ ?
- We need both.

### Independence of Multiple Events

The events  $A_1, A_2, \dots, A_n$  are said to be independent if

$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \dots, n\}$$

## Questions?

L2(3)

April 17, 2021 23 / 22

L2(3)

April 17, 2021 24 / 22

- 1) What is conditional probability? Why do we need it?
- 2) Explain the overall framework of Bayesian inference.
- 3) What is the total probability theorem?
- 4) What is Bayes' rule? What does it can give us?
- 5) What's the difference between independence and conditional independence?