Lecture 7: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

November 21, 2022

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

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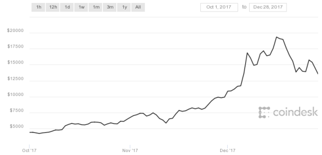
- Many probabilistic experiments that evolve in time



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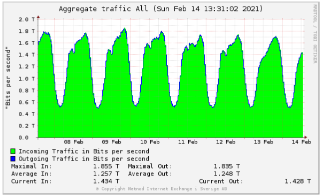

(b) Internet traffic traces

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  - Sequence of scores in football
  - Sequence of failure times of a machine
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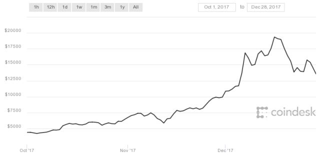

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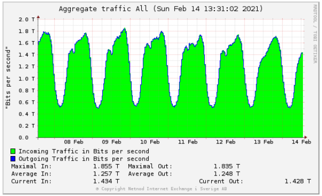

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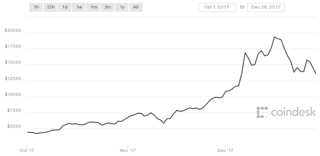
- Random process is a mathematical model for it.



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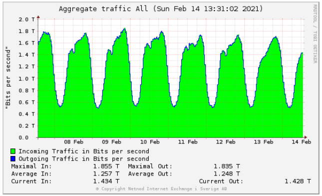

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- The values that X_t (or $X(t)$) can take: discrete or continuous

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  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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  - Other interesting questions, depending on the target random process

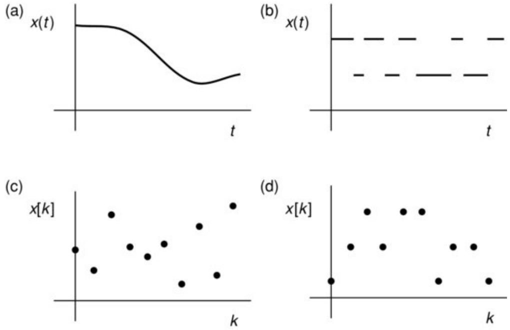
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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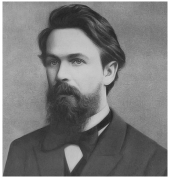

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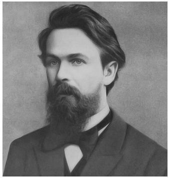

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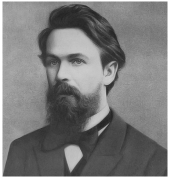

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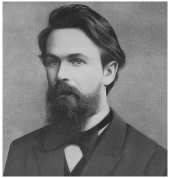

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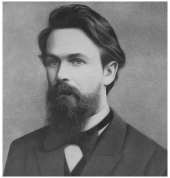

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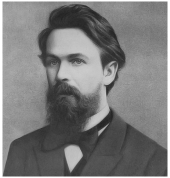

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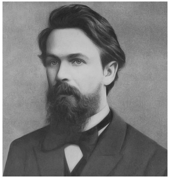

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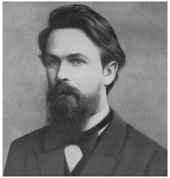

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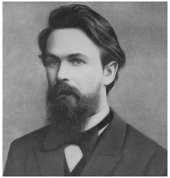

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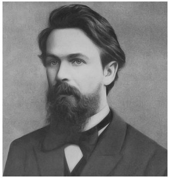

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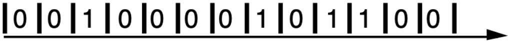

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- Discrete time, discrete value

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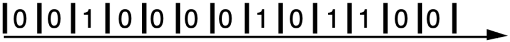
  - We call index 1, 2, ... time slots (or simply slots)



- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Please pause the video and write down the questions that you want to ask about Bernoulli process. VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

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Number of arrivals and Time until the first arrival

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- T_1 is geometric? Memoryless
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- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots $\Longrightarrow$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

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- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

Fresh-start after Random time N (1)

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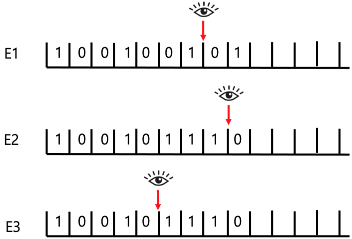

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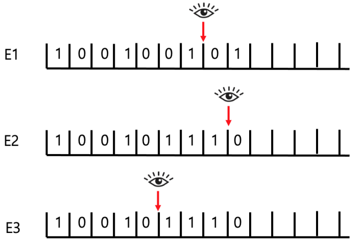

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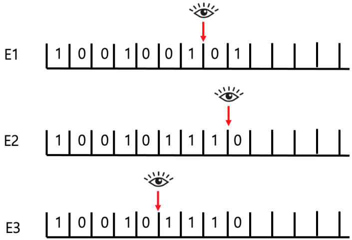

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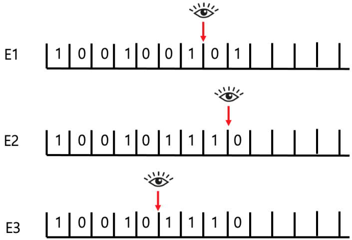

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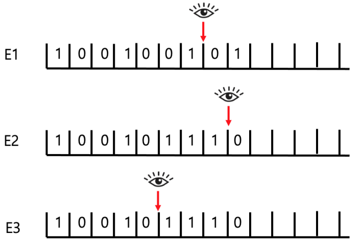

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  - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω?

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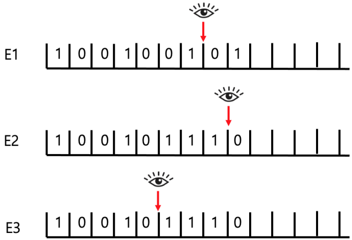

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- Do we experience the fresh-start for any N? **E1, E2, and E3?**

Fresh-start after Random N (2)

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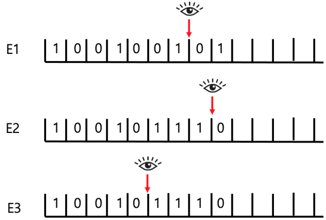

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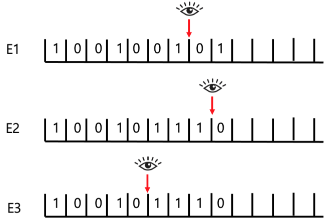

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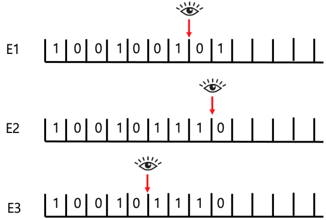

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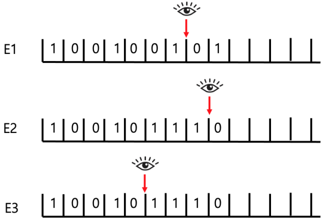

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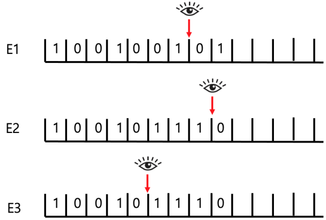

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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

Stopping Time

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  - Yes. Time when 10 consecutive arrivals have been observed
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- Regard an arrival as a server being busy (just for our easy understanding)

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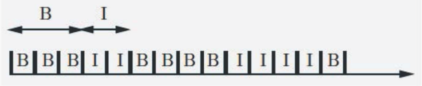
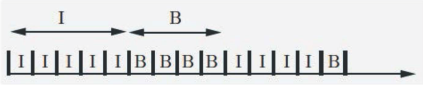
- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot





Distribution of Busy Periods (1)

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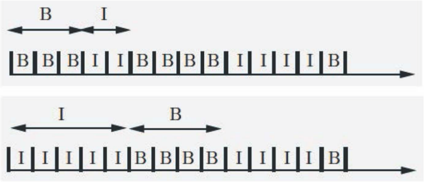

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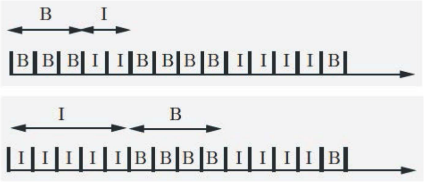

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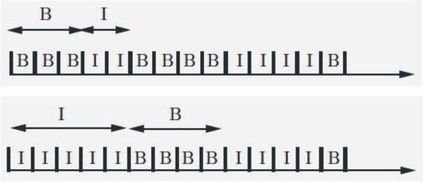

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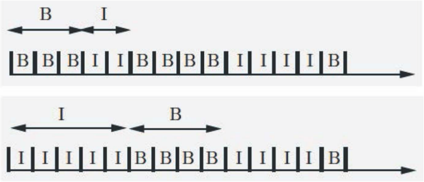

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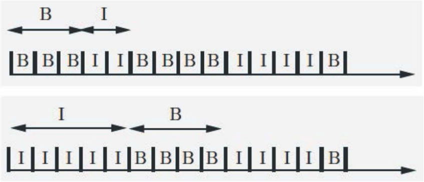
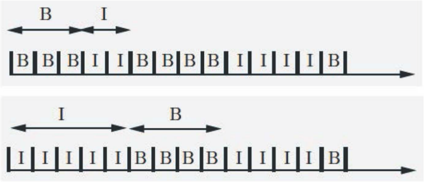


- Question. What about the second busy period B_2?

Distribution of Busy Periods (2)



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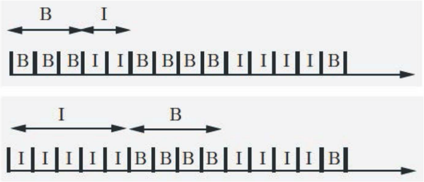
Distribution of Busy Periods (2)



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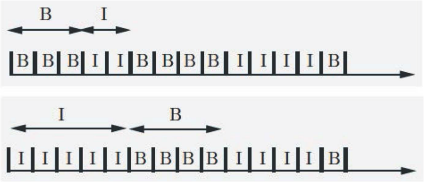

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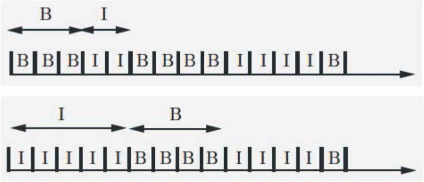

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- $B_3, B_4, \ldots$?

- Time of the first arrival $Y_1 \sim \text{Geom}(p)$

Time of k-th arrival

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VIDEO PAUSE

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VIDEO PAUSE

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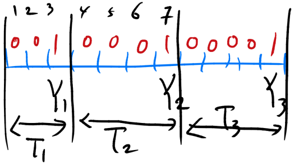

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VIDEO PAUSE

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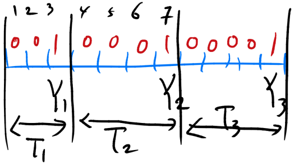

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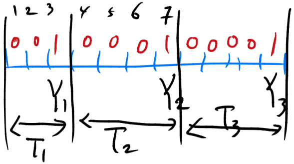

- After each T_k, the fresh-start occurs.

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VIDEO PAUSE

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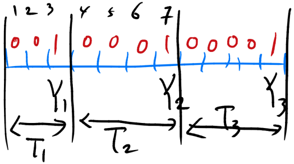

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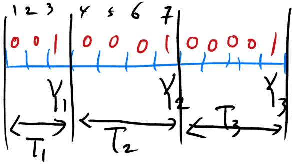

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Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

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- $n < \frac{-\ln 0.999}{0.0001} = 10.005$

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

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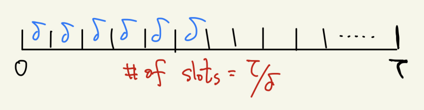

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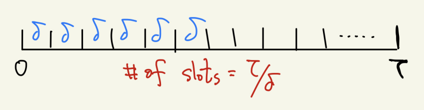
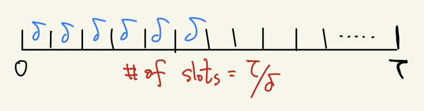


- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)



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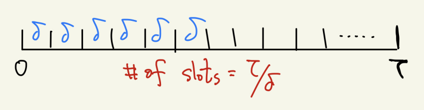

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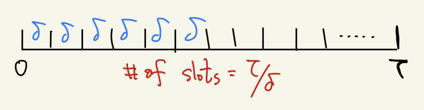

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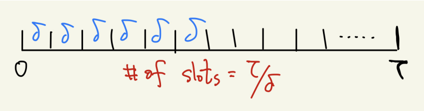

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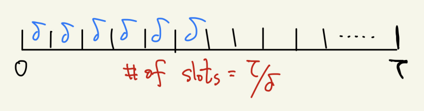

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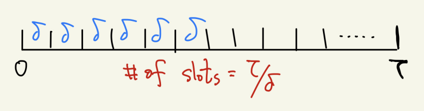

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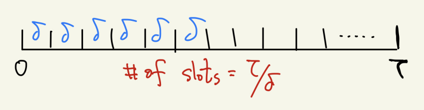

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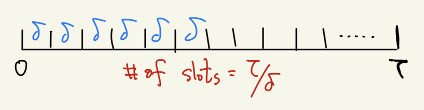

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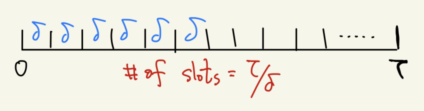

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- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

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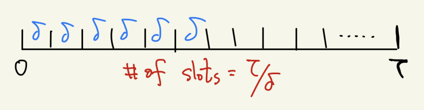

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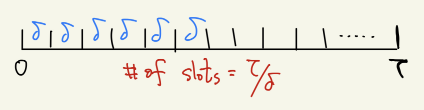

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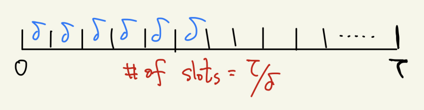

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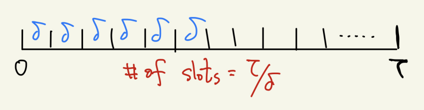

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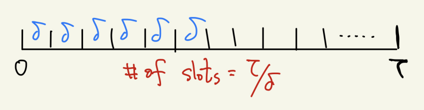
Key Design Idea to Develop a Continuous Twin (3)



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- This is a continuous twin process of Bernous process, which we call Poisson process.

Roadmap

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

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- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

$$\mathbb{P}(0, \tau) = 1 - \lambda\tau + o(\tau)$$

$$\mathbb{P}(1, \tau) = \lambda\tau + o_1(\tau)$$

$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \ldots, \quad \text{where} \quad \lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0$$

Poisson Process: Definition (2)

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  - N_τ becomes the number of arrivals over any interval of length τ.
- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

$$\mathbb{P}(k, \tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
- $\mathbb{P}$[one new message in the next hour]
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- $Y_k = T_1 + T_2 + \cdots + T_k$ is sum of i.i.d. exponential rvs.
- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

PDF of Y_k

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- This is called Erlang rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

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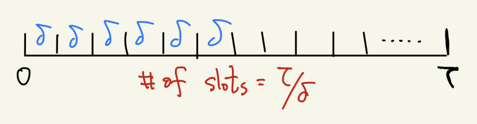
Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



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time of arrival	Discrete	Continuous
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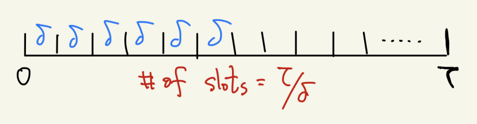

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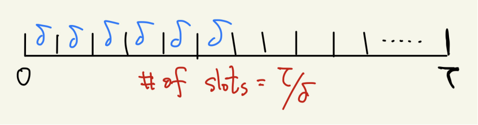

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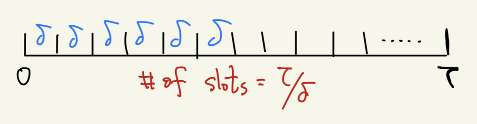

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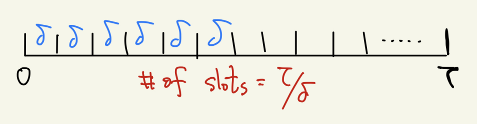

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$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

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(Q3) $\mathbb{P}(\text{Catch at least two fish})$

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(Q4) $\mathbb{E}[\text{future fi. time}|\text{already fished for 3h}]$

Fresh-start. So,
$\mathbb{E}[exp(\lambda)] = 1/\lambda = 1/0.6$

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Example: Poisson Fishing (Problem 10, page 329)

- Catching fish: Poisson process $\lambda = 0.6/\text{hour}$.
- Fish for 2 hours. Within 2 hours, if he catches at least one fish, then he stops at 2 hours. Otherwise, he fishes until one fish is caught.

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\begin{aligned}
2 + \mathbb{E}[F-2] &= 2 + \mathbb{P}(F=2)\cdot 0 + \\
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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0,2)\cdot 1$

Description via Inter-arrival Times

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
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- Thus, the answer is p^2.

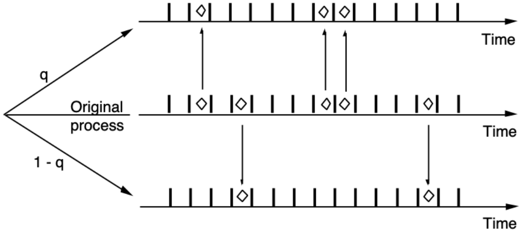
Coding of Random Arrivals

- Question. How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.
- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

Notations In the Rest of These Slides

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1-q)$.



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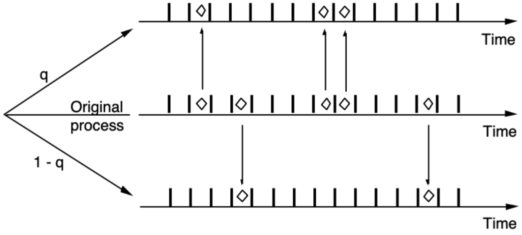

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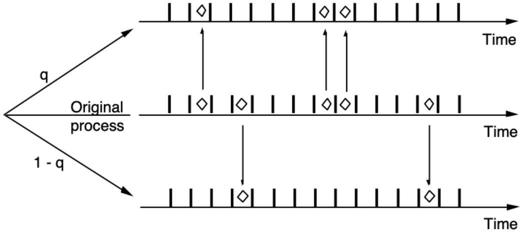

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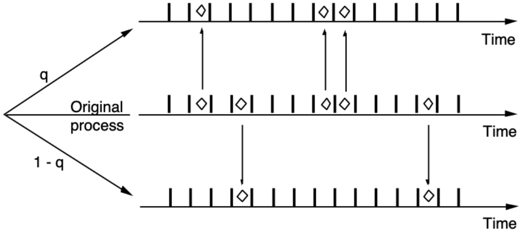

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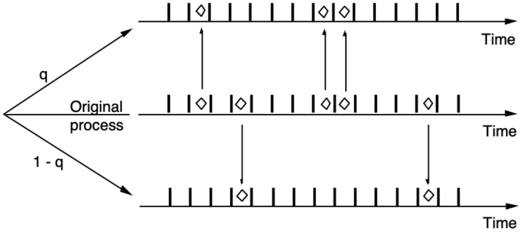
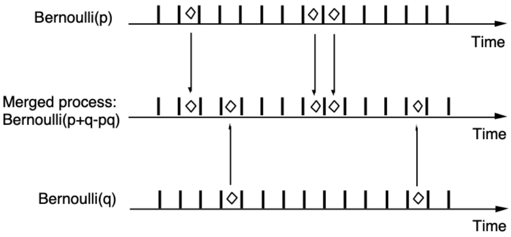
- Are they independent? No.



- Merge BP(p) and BP(q) into one process.



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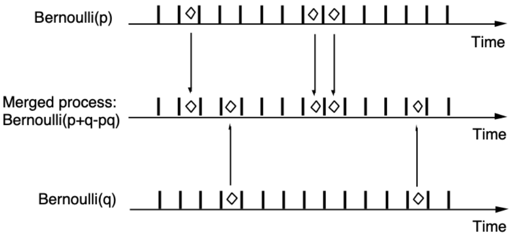

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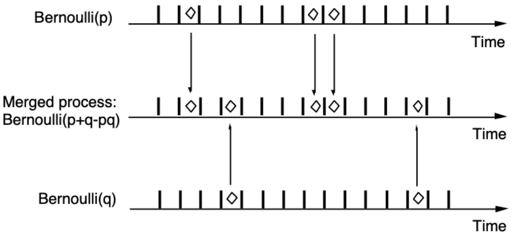

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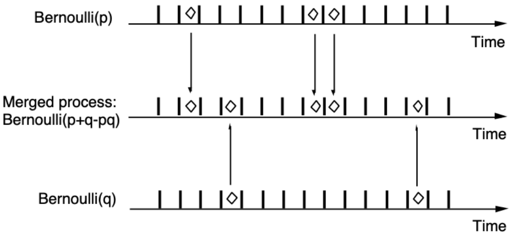

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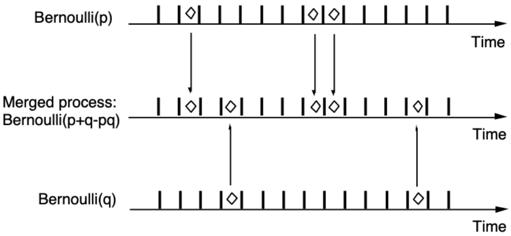
- $\mathbb{P}(\text{arrival from proc. 1} \mid \text{arrival in the merged proc.}) = \frac{p}{p+q-pq}$



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- Split a Poisson process PP(λ) into two processes by keeping each arrival w.p. p and discarding it w.p. $(1 - p)$

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  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
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  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- PP(λp) and PP($\lambda(1-p)$)

Merge: Poisson Process (1)

- Merge from PP(λ_1) and PP(λ_2)

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$\mathbb{P}(0 \text{ arrival}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1-(\lambda_1+\lambda_2)\delta$

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KAIST EE

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1. Competing exponentials

2. Sum of independent Poisson rvs

3. Poisson arrivals during and Exponential interval

Competing Exponential (1)

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- Approach 1
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$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Sum of Independent Poisson rvs

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  - Y: number of arrivals of PP(1) over a time interval of length ν
  - Two intervals do not overlap and located in a consecutive manner $\implies X \perp\!\!\!\perp Y$
- Distribution of $X + Y$: the number of arrivals of PP(1) over a time interval of length $\mu + \nu$
- Thus, $X + Y \sim \text{Poisson}(\mu + \nu)$

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- Problem 24, pp. 335
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- Very tedious and not very intuitive.

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- Let L be the number of arrivals from PP(λ) until we get the first arrival from PP(ν).

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \ldots$$

Approximation of Binomial: Poisson vs. Normal

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  - $p = 1/100$, $n = 10,000$: small p, but large n $\Longrightarrow$ Both Poisson and Normal

KAIST EE

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

(6) Random Incidence Phenomenon

Example: Survey of Utilization of Town Buses

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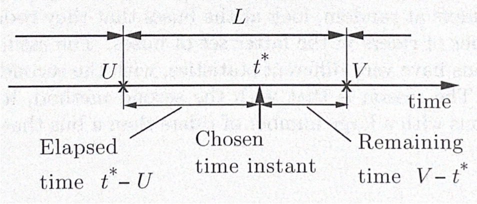

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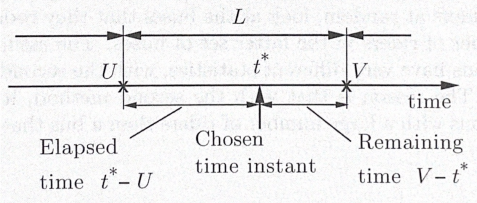

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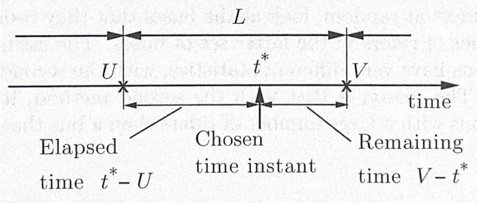

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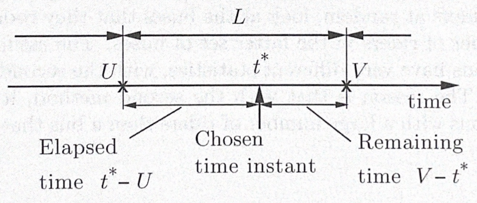
- Question. What is the distribution of L?

VIDEO PAUSE

Random Incidence Phenomemon (2)



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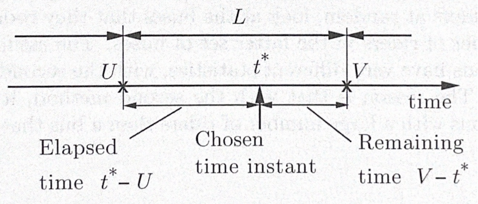

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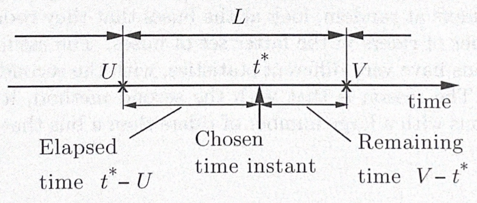

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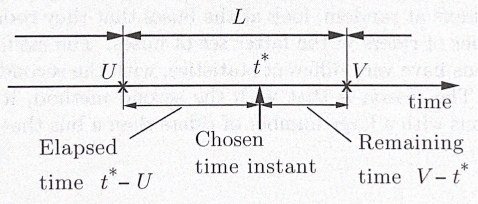

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- One might superficially argue that $L \sim \text{Exp}(\lambda)$, but it is NOT.

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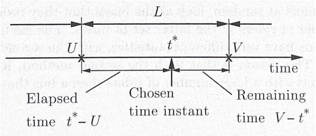

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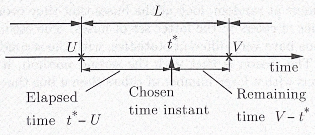

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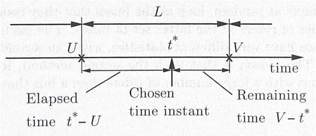

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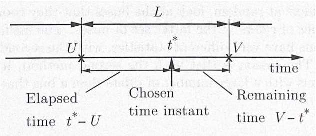

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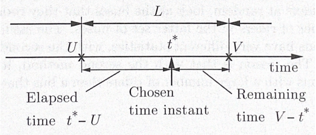

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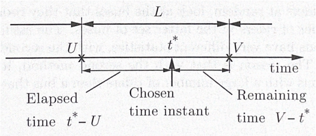

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page 37

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$$f_L(l) = \lambda^2 \cdot l \cdot e^{-\lambda l}, \quad l \geq 0$$

- Mean $= 2/\lambda$
- Why not $\text{Exp}(\lambda)$? An observer who arrives at an arbitrary time is more likely to fall in a large rather than a small interarrival interval.

Back to Survey of Utilization of Town Buses

- Two Approaches
  M1. (1) select a few buses at random, and (2) calculate the average number of riders in the selected bus

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- (i) M1 $=$ M2? (ii) M1 $>$ M2? (iii) M1 $<$ M2?
- Answer: M1 $<$ M2
- More likely to select a bus with a large number of riders than a bus that is near-empty.

Questions?

Review Questions

1) Explain what is random process. Why do we need such a concept?
2) Explain Bernoulli processes. When do we use Bernoulli processes?
3) Explain Poisson processes. When do we use Poisson processes?
4) What are the relations between those two processces? What features do they share?
5) In both processces, ho do we compute (i) number of arrivals over a given interval of time, (ii) time until the first arrival, (iii) time until k-th arrival?
6) What is Pascal rvs and Erlang rvs?
7) What is the "stopping time" and how is it related to fresh-restart?
8) See many examples on how merging and splitting of BP and PP can be used for intuitive soloving of many problems.