Lecture 8: Random Processes, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- A lot of applications in engineering systems

- Basics on Random Process

- Bernoulli Process

- Poisson Process

- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split

- Markov Chain

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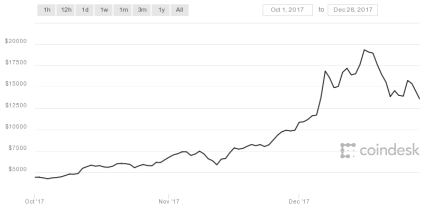
Things that evolve in time

- Many probabilistic experiments that evolve in time



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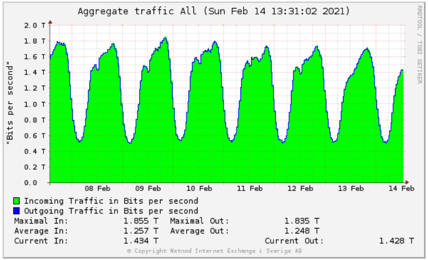
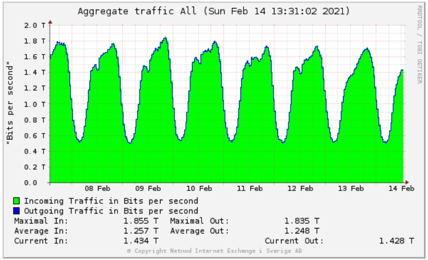


- Many probabilistic experiments that evolve in time
  - Sequence of daily prices of a stock
  - Sequence of scores in football
  - Sequence of failure times of a machine
  - Sequence of traffic loads in Internet
- Random process is a mathematical model for it.



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- For a fixed time t, X_t is a random variable.
- The values that X_t can take: discrete or continuous

4 Types of Random Processes

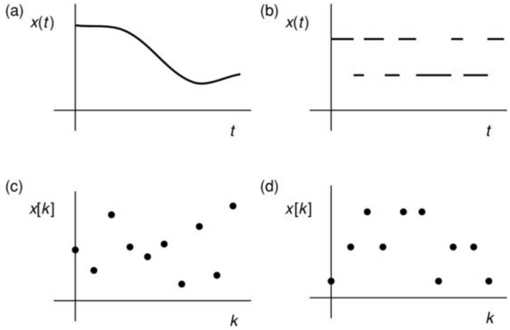
- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

(d) discrete time, discrete value



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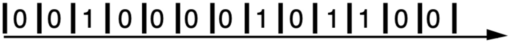

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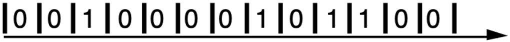

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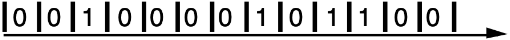

- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

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- Next: Key questions and answers about Bernoulli process

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Number of arrivals and Time until the first arrival

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- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

Memoryless and Fresh-start after Deterministic n

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- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

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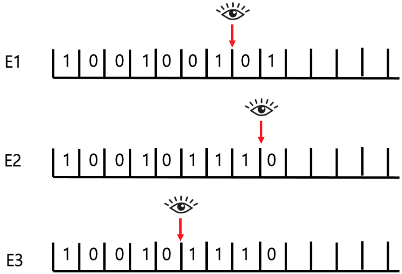

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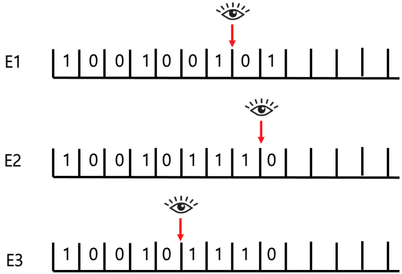

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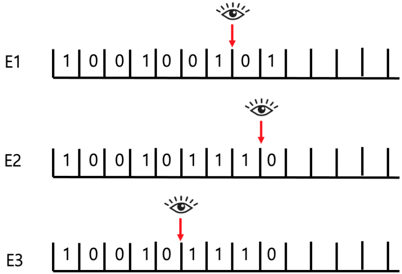

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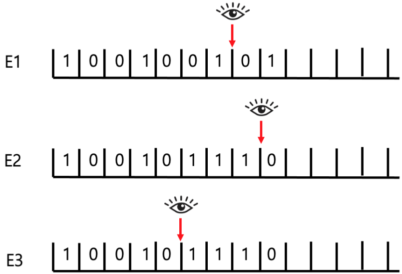

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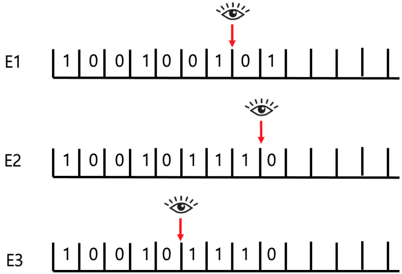

- Difference of N from n
  - The time when I watch the on-going Bernoulli process is random.

Fresh-start after Random N (2)

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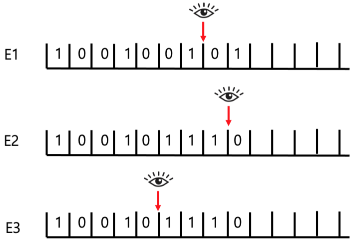

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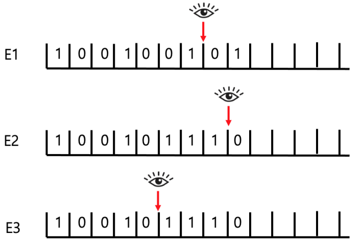

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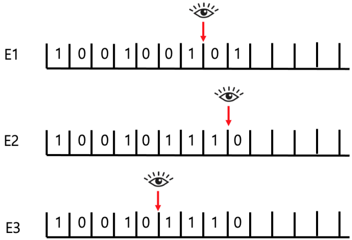

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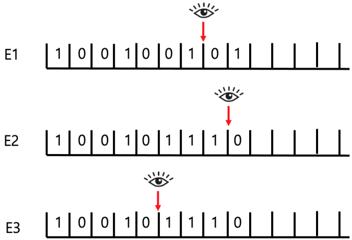

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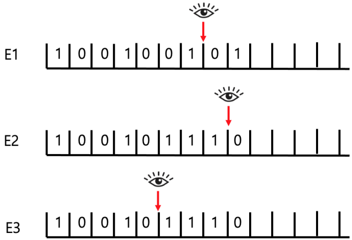

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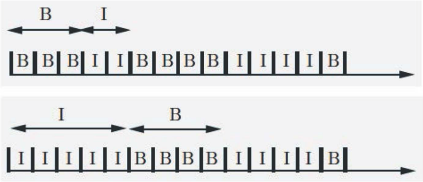
- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \dots, X_n$? Then, Yes! (see pp. 301 for more formal description)

- Regard an arrival as business of a server

Distribution of Busy Periods

- Regard an arrival as business of a server
- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot



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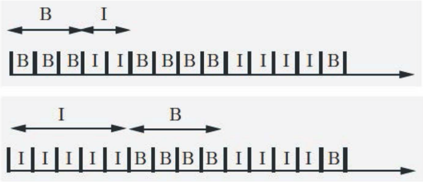

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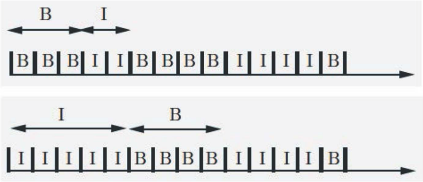

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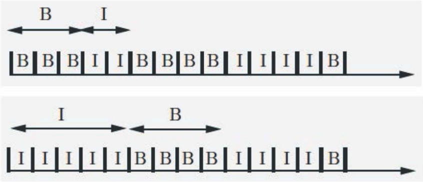

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- B_1 is geometric with parameter $(1-p)$

- Regard an arrival as business of a server
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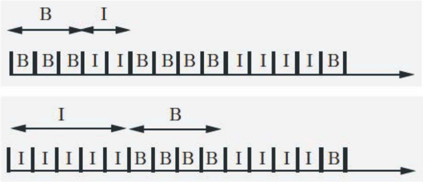

- (Q6) Distribution of B_1?
- N: time of the first busy slot. Fresh-start after N.
- B_1 is geometric with parameter $(1 - p)$
- Question: What about the second busy period B_2? B_3, B_4?

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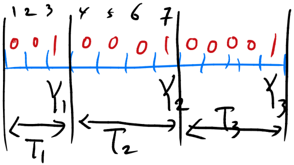

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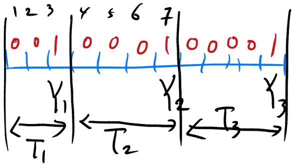

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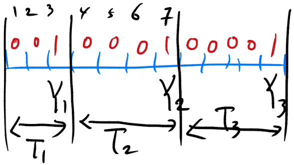

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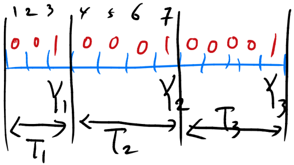

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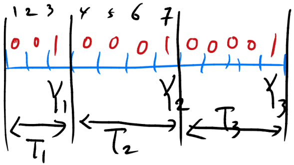

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- A lot of applications in engineering systems
- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split
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- Need a "modeling sense" to make this possible. It's a good practice for engineers!

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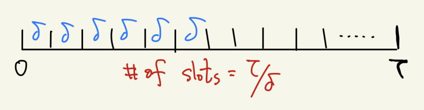

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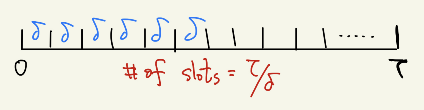

- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)

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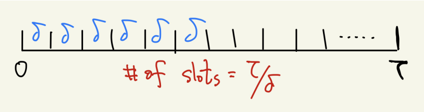

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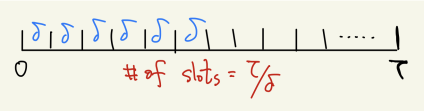

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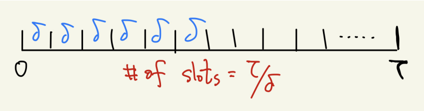

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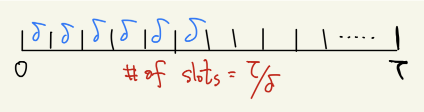

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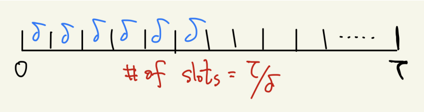

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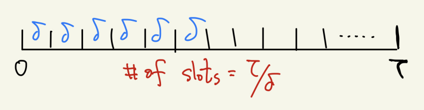

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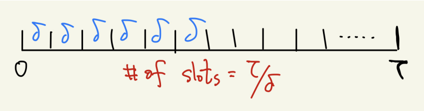

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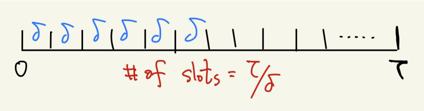

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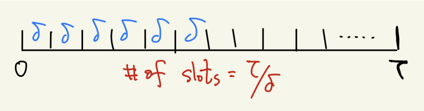

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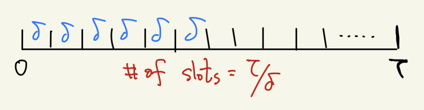
- $o(\delta)$: some function that goes to zero faster than δ goes to zero.
  - Thus, for very small δ, $o(\delta)$ becomes negligible.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$



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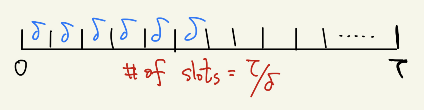

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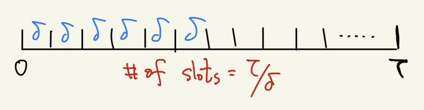

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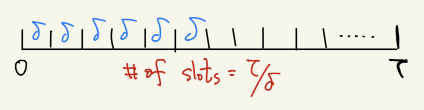

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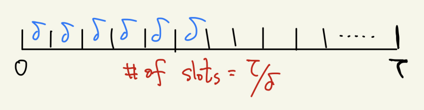

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- # of arrivals over $[0, \tau]$, $\sim$ *Poisson*$(\lambda\tau)$
- This is a continuous twin process of Bernous process, which we call Poisson process.

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  - Thus, N_s can be a random variable over any interval of length s.
- (Small interval probability) The probabilities $\mathbb{P}(k, s)$ satisfy:

$$\mathbb{P}(0, s) = 1 - \lambda\tau + o(s)$$

$$\mathbb{P}(1, s) = \lambda s + o_1(s)$$

$$\mathbb{P}(k, s) = o_k(s) \quad \text{for } k = 2, 3, \ldots,$$

where

$$\lim_{s \to 0} \frac{o(s)}{s} = 0, \quad \lim_{s \to 0} \frac{o_k(s)}{s} = 0$$

Poisson Process: $\mathbb{P}(k, \tau)$, N_τ, and T

- (Q1) Number of arrivals of any interval with length $\tau \sim Poisson(\lambda\tau)$, i.e.,

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$$f_T(t) = \frac{dF_T(t)}{dt} = \lambda e^{-\lambda t}, \quad t \ge 0.$$

- (Q1) Number of arrivals of any interval with length $\tau \sim \mathit{Poisson}(\lambda\tau)$, i.e.,

$$\mathbb{P}(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \dots$$

- $\mathbb{E}[N_\tau] = \lambda\tau$ and $\mathrm{var}[N_\tau] = \lambda\tau$
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- $T \sim \mathit{expo}(\lambda)$. Thus $\mathbb{E}[T] = 1/\lambda$ and $\mathrm{var}[T] = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

Poisson Process: Example

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages per hour
- Mean and variance of mails received during a day
- $\mathbb{P}$[one new message in the next hour]
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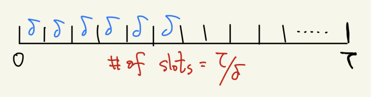
- Time of first arrival: geometric / exponential
- Time of k-th arrivals: Pascal / Erlang

Poisson Process vs. Bernoulli Process

- $n = \tau/\delta$, $p = \lambda\delta$, $np = \lambda\tau$



	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time CDF	Exponential	Geometric
Arrival Rate	λ/unit time	p/per trial

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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

Roadmap

- A lot of applications in engineering systems

- Basics on Random Process

- Bernoulli Process

- Poisson Process

- Use of Bernoulli and Poisson Processes
  - Coding of both processes
  - Merge and Split

- Markov Chain

- Inter-arrival times facilitates coding of both processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.
2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

Alternative Description of the Poisson Process

1. Start with a sequence of independent exponential random variables $T_1, T_2, \ldots$, with common parameter λ, and let these stand for the interarrival times.
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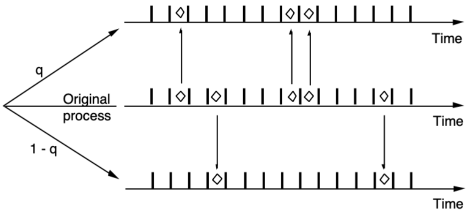
- Split $Bernoulli(p)$ into two processes with biased coin of head probability q



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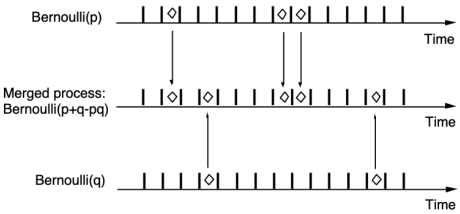

Split and Merge: Bernoulli Process

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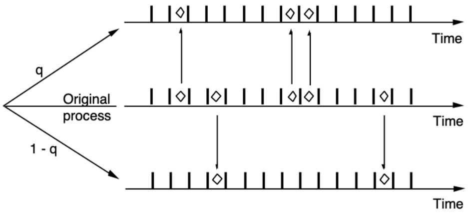

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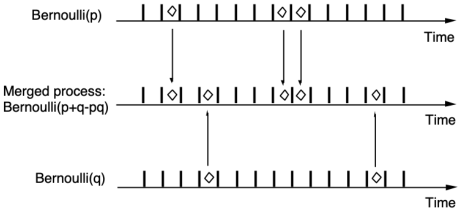

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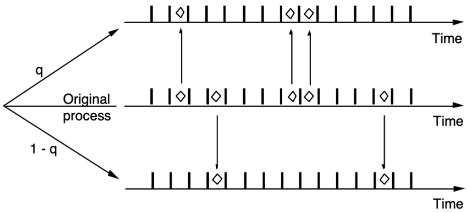
- Split processes: also Bernoulli processes
- $Bernoulli(pq)$ and $Bernoulli(p(1-q))$



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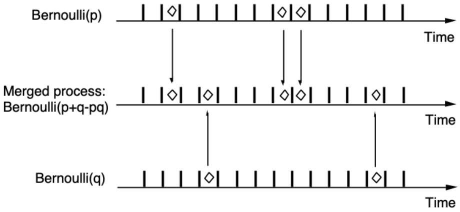

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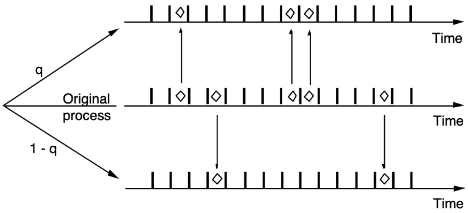
- Split decisions are independent of arrivals
- Split processes: also Bernoulli processes
- $Bernoulli(pq)$ and $Bernoulli(p(1-q))$



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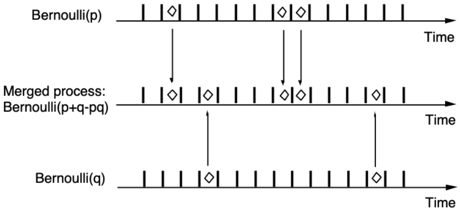

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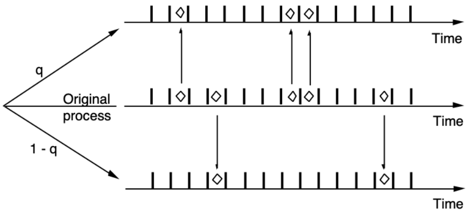

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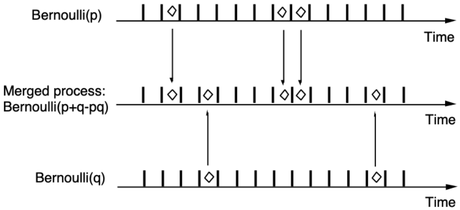

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- $T_1 \sim exp(3\lambda)$, $T_2 \sim exp(2\lambda)$, $T_3 \sim exp(\lambda)$

$$\mathbb{E}\left[T_1 + T_2 + T_3\right] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

Questions?

Review Questions

1)