Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Outline

- Continuous Random Variable
- PDF (Probability Density Function)
- CDF (Cumulative Distribution Function)
- Exponential and Normal Distribution
- Joint PDF, Conditional PDF
- Bayes' rule for continous and even mixed cases

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

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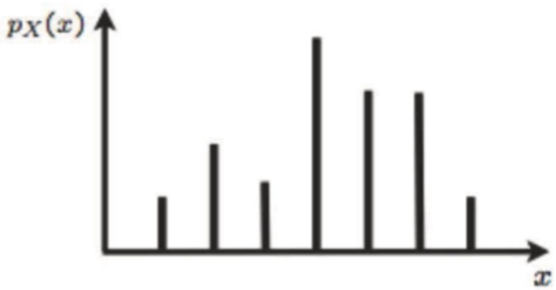

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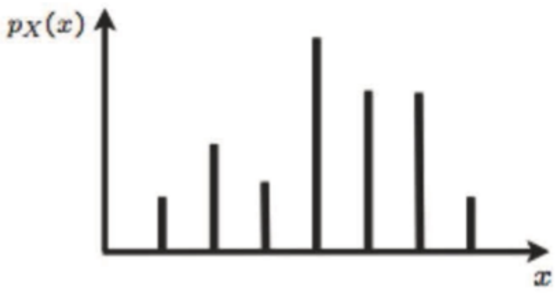

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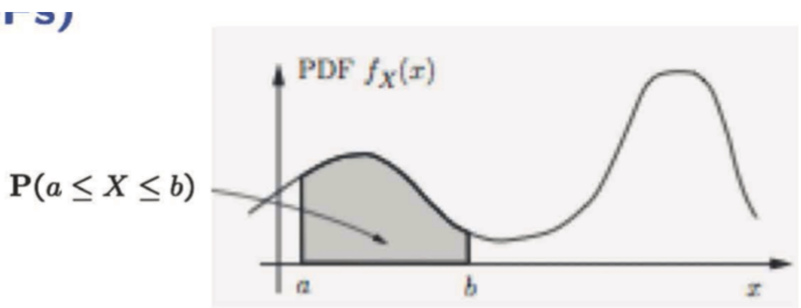
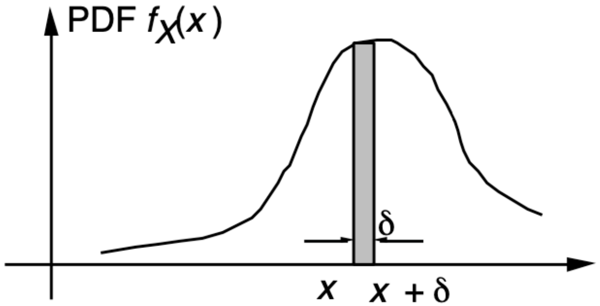


- $\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$
- $f_X(x) \ge 0, \ \int_{-\infty}^{\infty} f_X(x)dx = 1$

PDF and Examples



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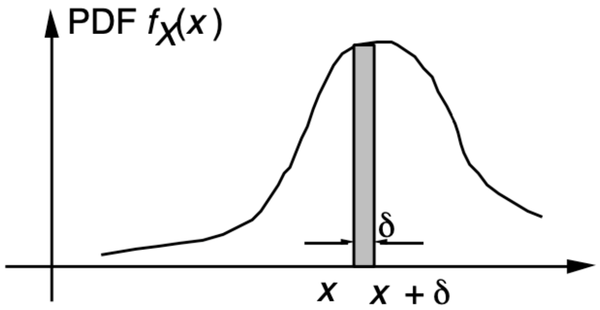

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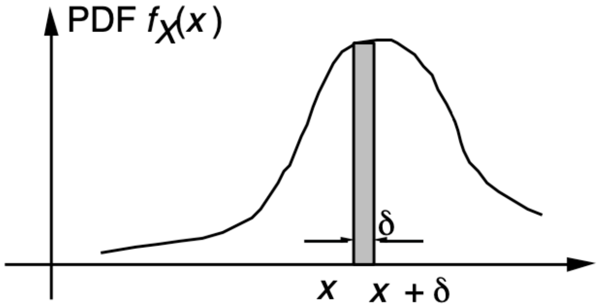

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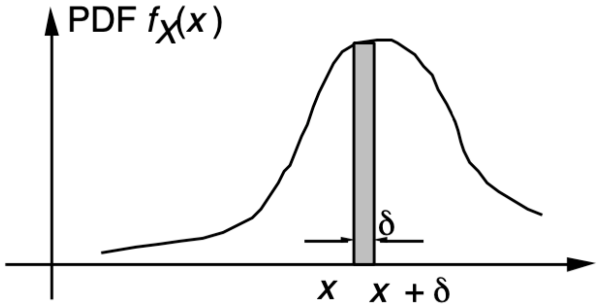

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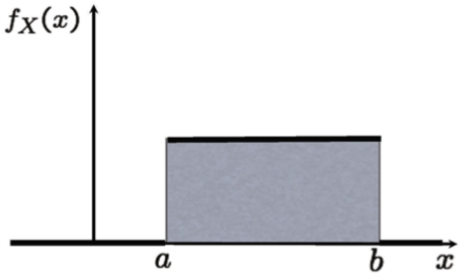
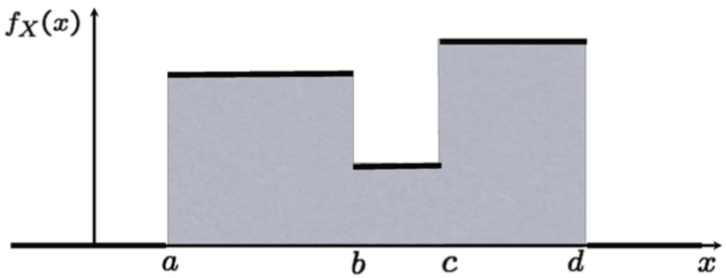
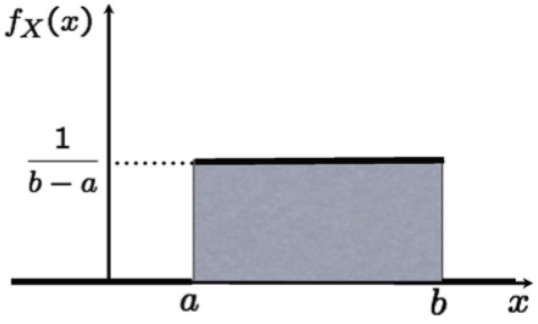




Expectation and Variance



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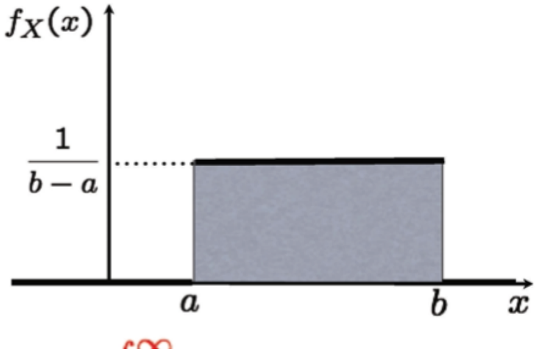

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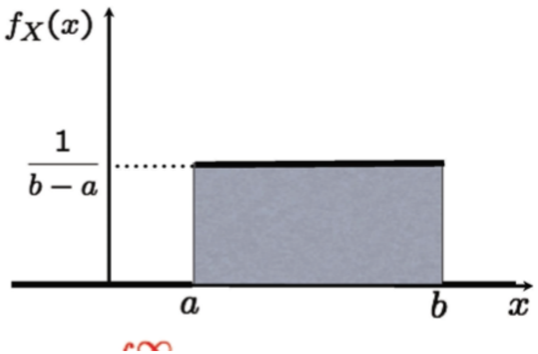

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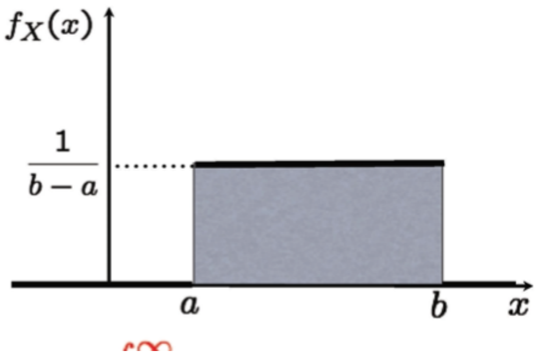

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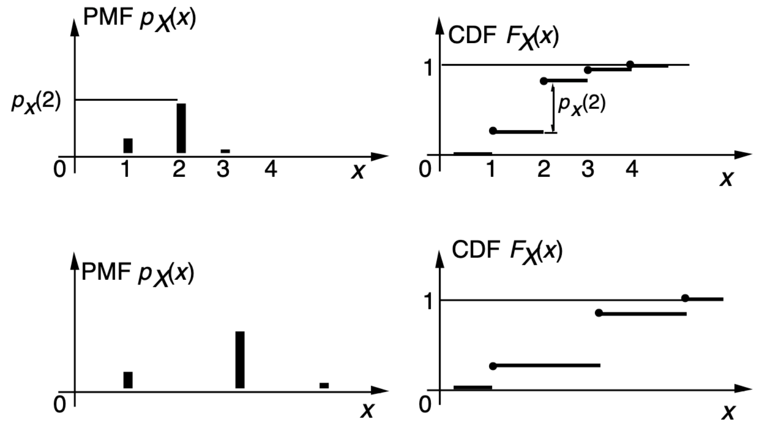

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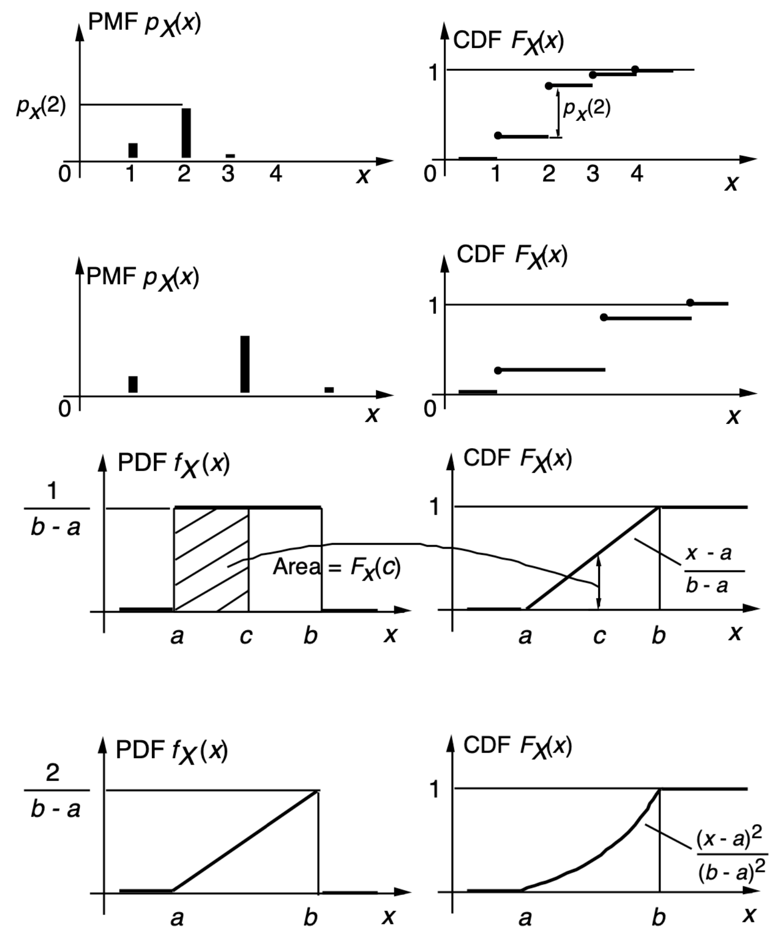

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Now, let's look at famous continuous random variables popularly used in our life.

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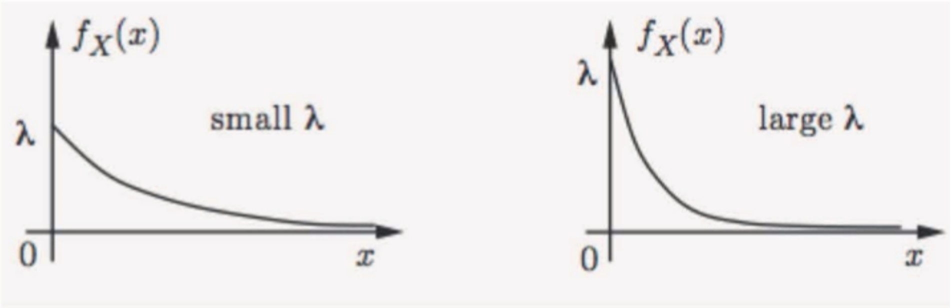

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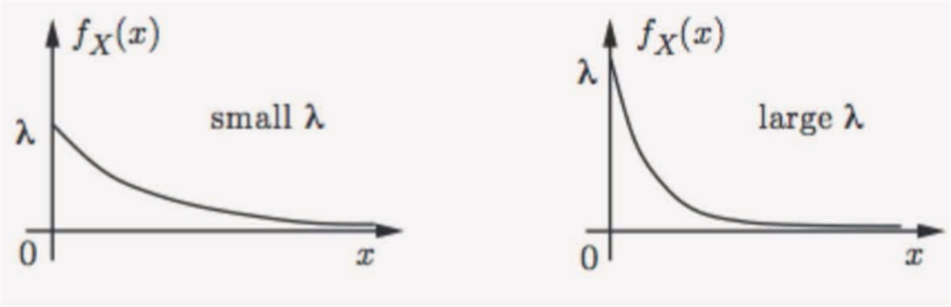

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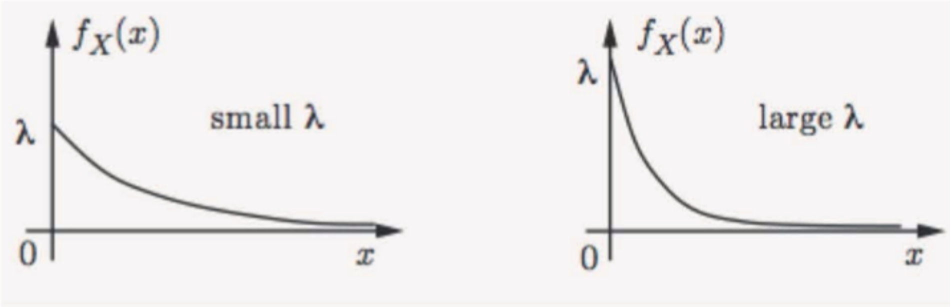

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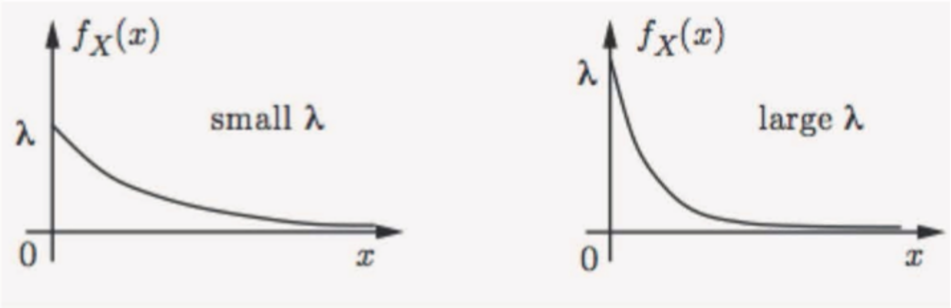

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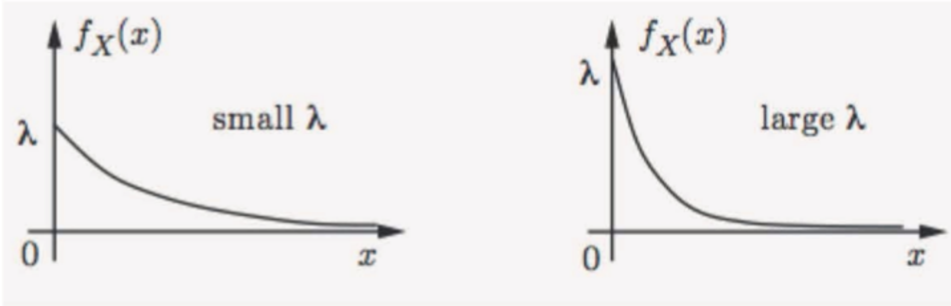
- (Q) What is the discrete rv which models a waiting time?



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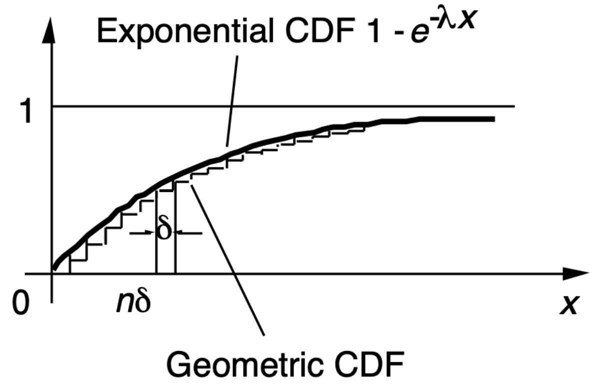

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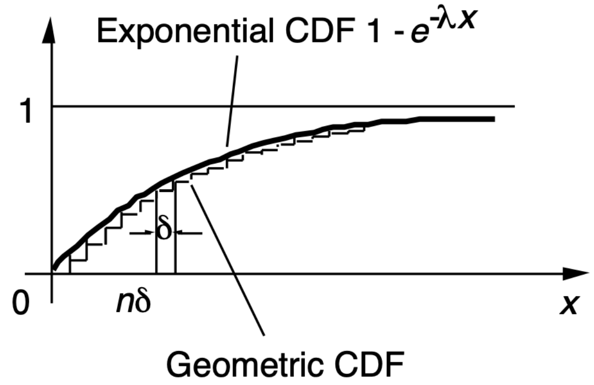

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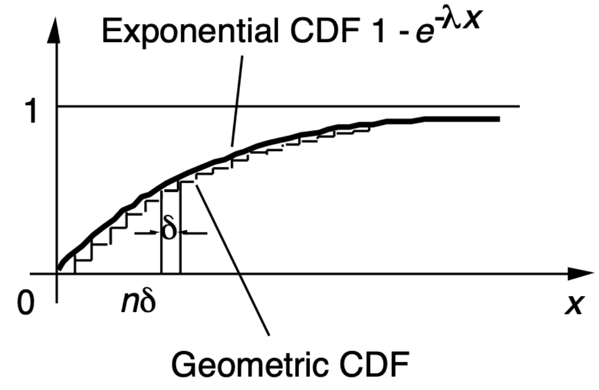

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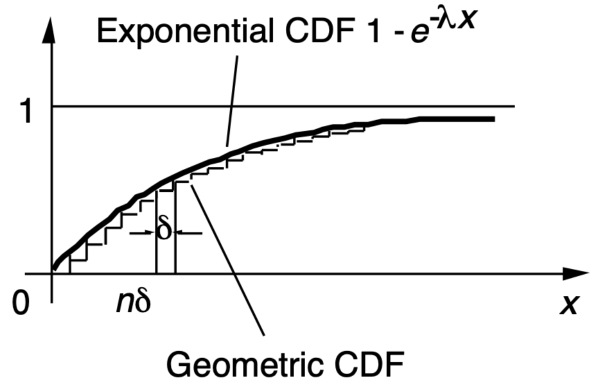

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- As $n \to \infty$, the slot length $\delta \to 0$ thus $p_n \to 0$
- The CDF values of exponential and n-th geometric rvs become equal whenever $x = \delta, 2\delta, 3\delta, \ldots,$ i.e.,

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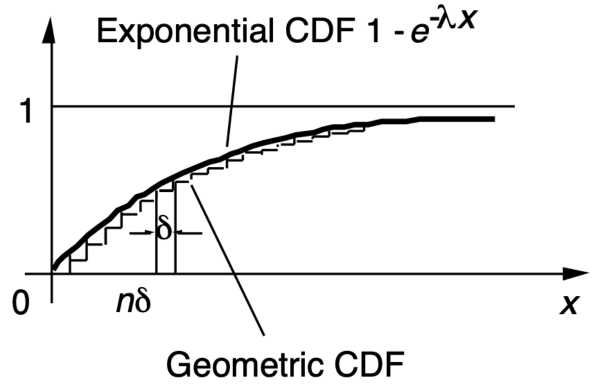

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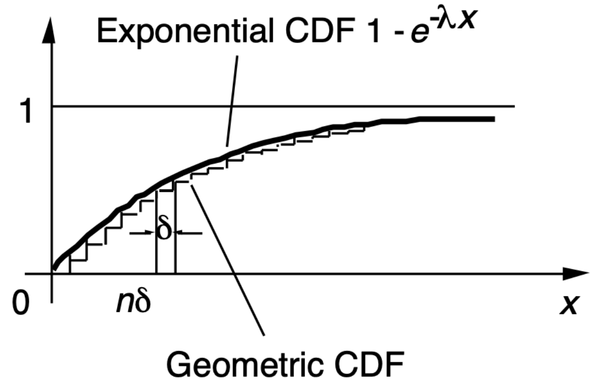

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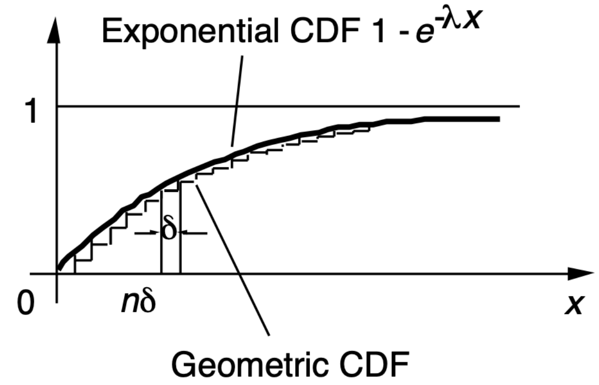

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- As n grows, $F^{n}_{geo}(n)$ approaches $F_{exp}(n\delta)$.

Normal (also called Gaussian) Random Variable

Why important?

- Central limit theorem (중심극한정리)
  - One of the most remarkable findings in the probability theory

- Convenient analytical properties

- Modeling aggregate noise with many small, independent noise terms

Normal: PDF, Expectation, Variance

- Standard Normal $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

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- $\mathbb{E}[X] = 0$
- $\mathrm{var}[X] = 1$

- General Normal $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

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- $\mathrm{var}[X] = \sigma^2$

Normal: PDF, Expectation, Variance

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- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$

Need to check:

- a legitimate PDF or not

- expectation/variance

Normal: Useful Property

Normal: Useful Property

- Linear transformation preserves normality

Linear transformation of Normal

If $X \sim Norm(\mu, \sigma^2)$, then for $a \neq 0$ and b $Y = aX + b \sim Norm(a\mu + b, a^2\sigma^2)$.

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- Thus, every normal rv can be $\boxed{}$:

  If $X \sim Norm(\mu, \sigma^2)$, then $\boxed{} \sim Norm(0, 1)$

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- Thus, every normal rv can be $\boxed{\text{standardized}}$:

  If $X \sim Norm(\mu, \sigma^2)$, then $\boxed{Y = \frac{X-\mu}{\sigma}} \sim Norm(0, 1)$

- Thus, we can make the table which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt$$

Example

- Annual snowfall X is modeled as $Norm(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

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1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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Example

- Annual snowfall X is modeled as $Norm(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?
- $Y = \frac{X-60}{20}$.

$$
\begin{aligned}
\mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80-60}{20}) \\
&= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\
&= 1 - 0.8413 = 0.1587
\end{aligned}
$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

** Continuous counterparts are intuitively understandable. So, we will be quick at reviewing them.

Continuous: Joint PDF and CDF (1)

Jointly Continuous

Two continuous rvs are $\boxed{}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for $\boxed{\text{every}}$ subset B of the two dimensional plane,

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

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1. The joint PDF is used to calculate probabilities

$$\mathbb{P}((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

   Our particular interest: $B = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$

Continuous: Joint PDF and CDF (2)

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2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

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3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

Continuous: Joint PDF and CDF (2)

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4. A function $g(X,Y)$ of X and Y defines a new random variable, and

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$$

Continuous: Conditional PDF given an event

* Conditional PDF, given an event

* Conditional PDF, given $X \in B$

Continuous: Conditional PDF given an event

* Conditional PDF, given an event

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

* Conditional PDF, given $X \in B$

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  $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

- $\mathbb{P}(X \in B) = \int_B f_X(x)dx$
  $\mathbb{P}(X \in B|A) = \int_B f_{X|A}(x)dx$

  Note: A is an event, but B is a subset that includes the possible values which can be taken by the rv X.

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Continuous: Conditional PDF given an event

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- $\int f_{X|A}(x) = 1$

* Conditional PDF, given $X \in B$

Continuous: Conditional PDF given an event

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* Conditional PDF, given $X \in B$

$$\mathbb{P}(x \leq X \leq x + \delta | X \in B) \approx f_{X|X \in B}(x) \cdot \delta$$

$$f_{X|X \in B}(x) = \begin{cases} 0, & \text{if} \quad x \notin B \\ \frac{f_X(x)}{\mathbb{P}(B)}, & \text{if} \quad x \in B \end{cases}$$

Continuous: Conditional PDF given an event

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- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
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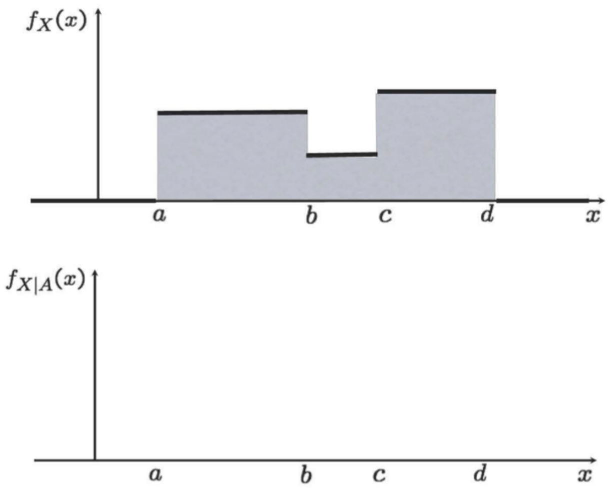
(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

Continuous: Conditional Expectation

$A = \{\frac{a+b}{2} \leq X \leq b\}$



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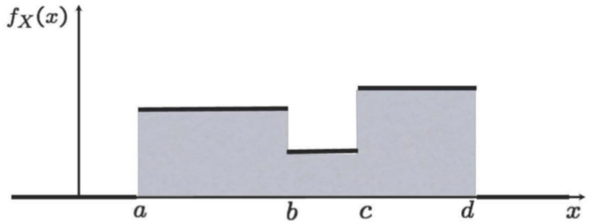

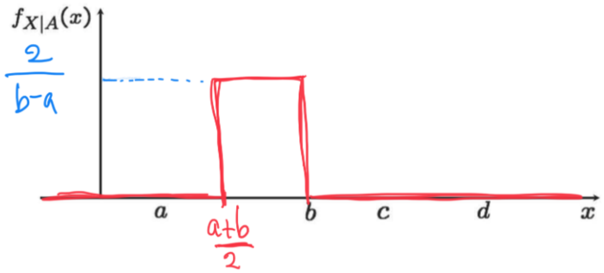

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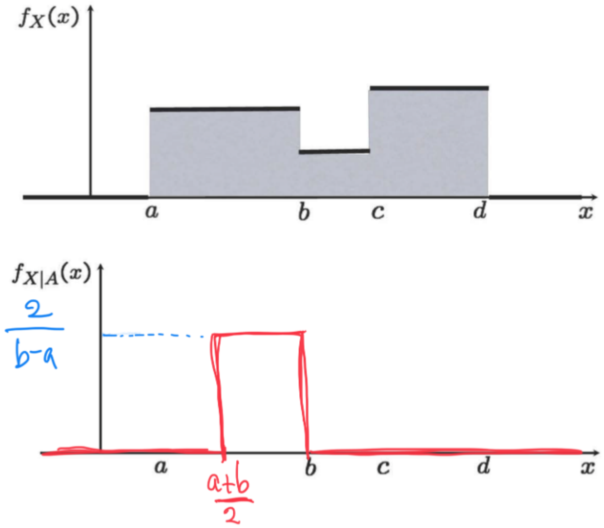
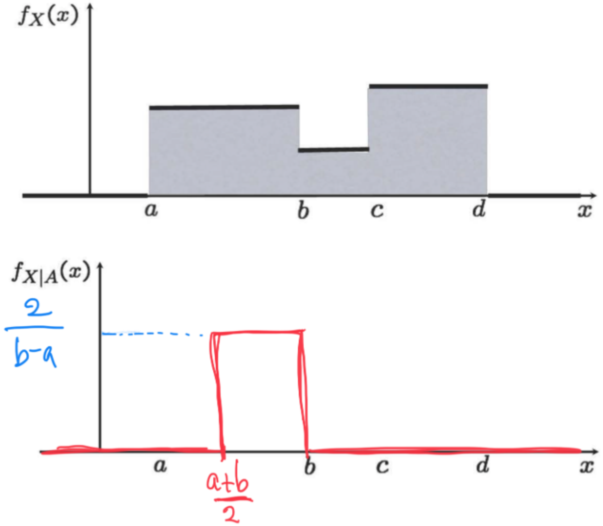


- $\mathbb{E}[X] = \int x f_X(x) dx$
  $\mathbb{E}[X|A] = \int x f_{X|A}(x) dx$

Continuous: Conditional Expectation

$A = \{\frac{a+b}{2} \leq X \leq b\}$



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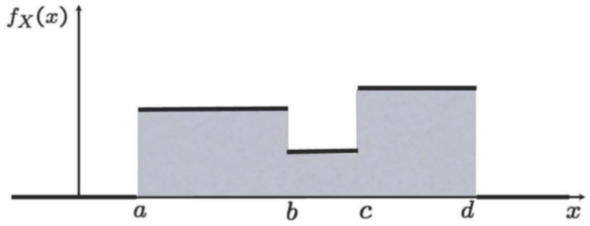

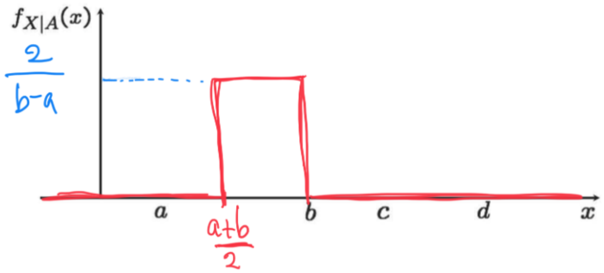

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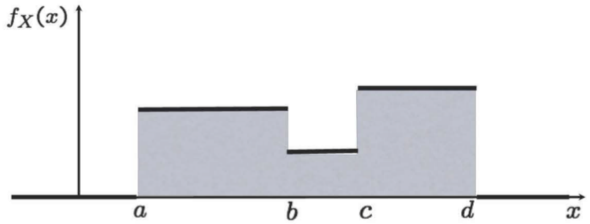

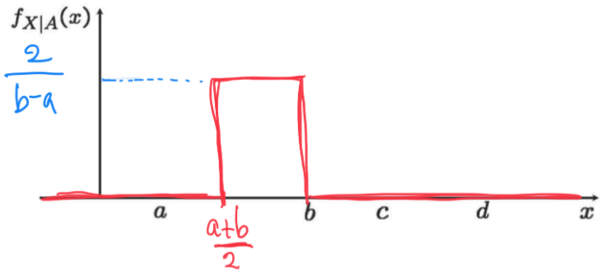

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$$\mathbb{E}[X|A] = \int_{(a+b)/2}^{b} x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

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Total Probability/Expectation Theorem

Partition of Ω into $A_1, A_2, A_3, \ldots$

* Discrete case

* Continuous case

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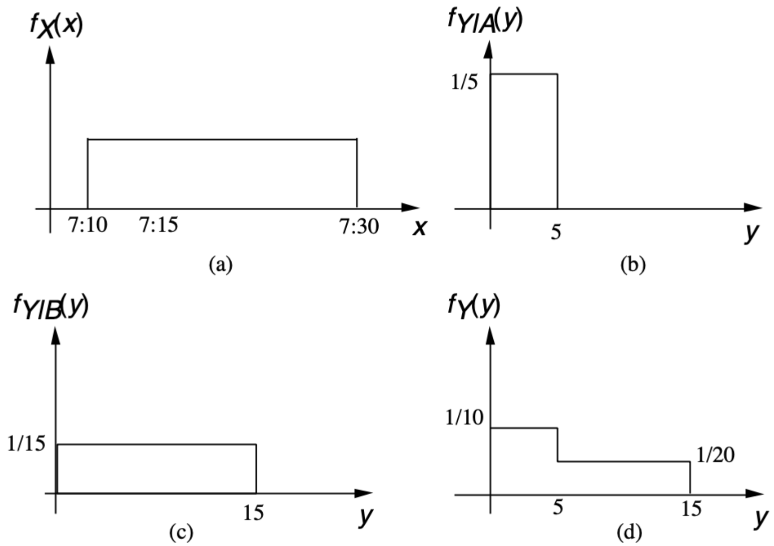

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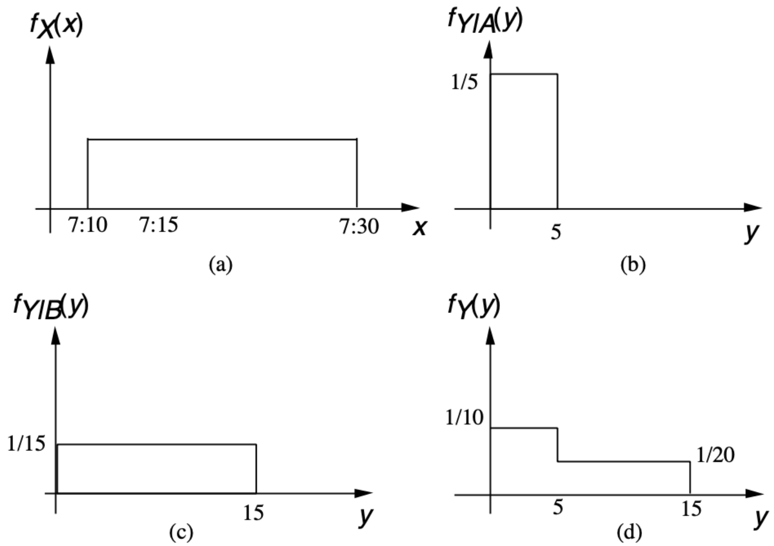

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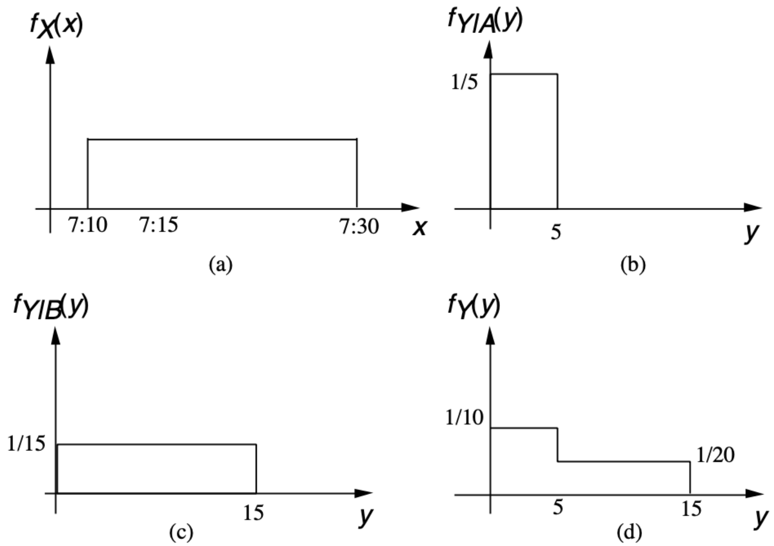

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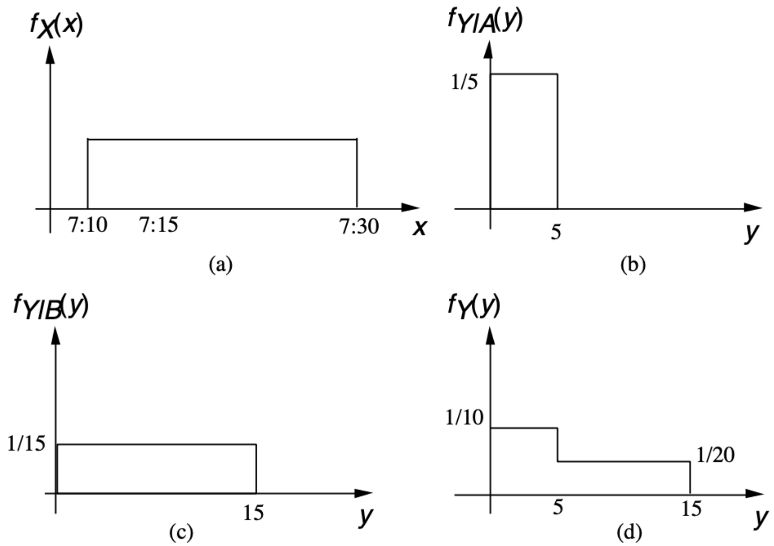

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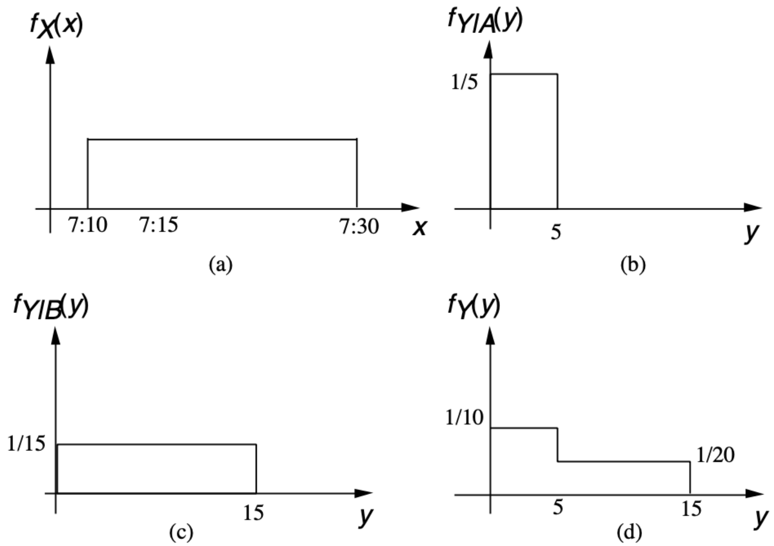

$$f_Y(y) = \mathbb{P}(A)f_{Y|A}(y) + \mathbb{P}(B)f_{Y|B}(y)$$

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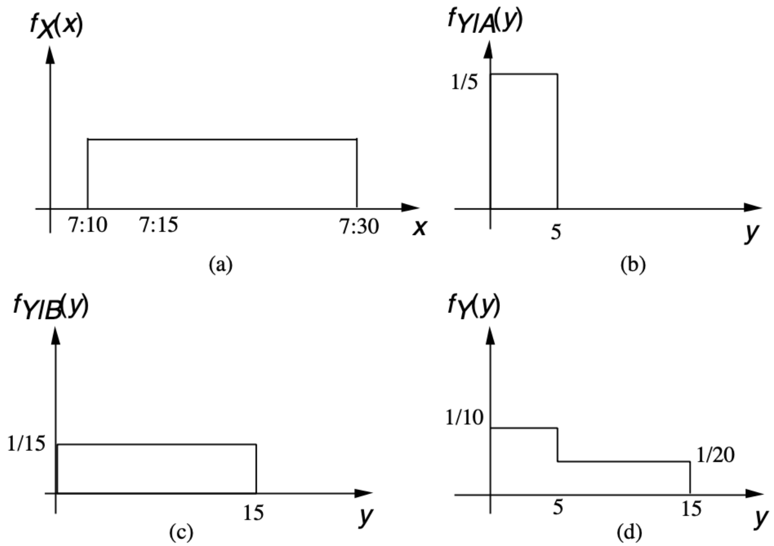

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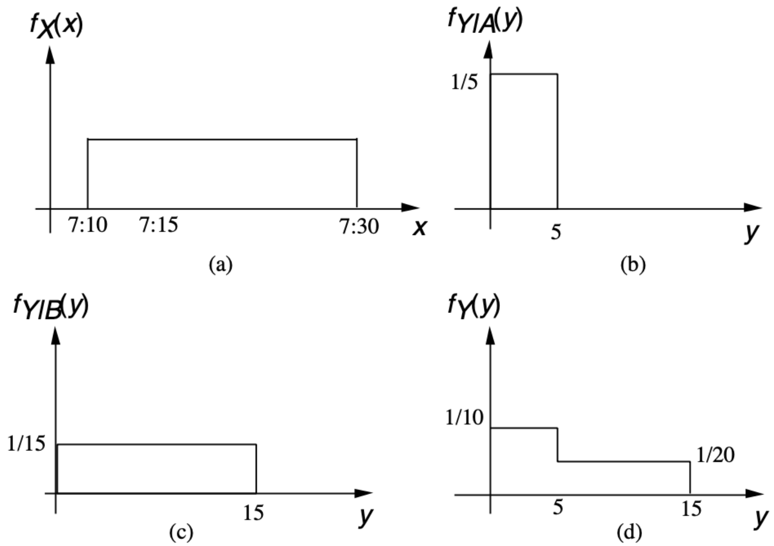

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- Independence.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

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$$\mathbb{E}[Y] = \int_0^l \frac{1}{l}\mathbb{E}[Y|X = x]dx$$
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- $f_X(x)$ and $f_{Y|X}(y|x)$ seems easy to compute. Thus,

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x) = \frac{1}{l} \cdot \frac{1}{x}$$

You can do many other things with the joint PDF.

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

Bayes Rule for Continuous

- X: state/cause/original value $\to$ Y: result/resulting action/noisy measurement
- Model: $\mathbb{P}(X)$ (prior) and $\mathbb{P}(Y|X)$ (cause $\to$ result)
- Inference: $\mathbb{P}(X|Y)$?

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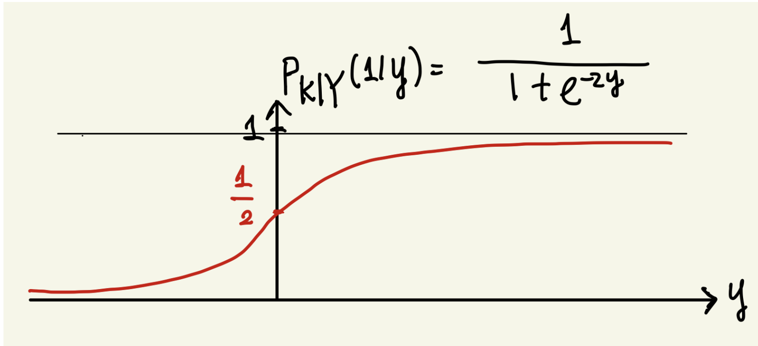

Questions?

Review Questions

1) What is PDF and CDF?
2) Why do we need CDF?
3) What are joint/marginal/conditional PDFs?
4) Explain memorylessness of exponential random variables.
5) Explain the version of Bayes' rule for continuous and mixed random variables.