Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

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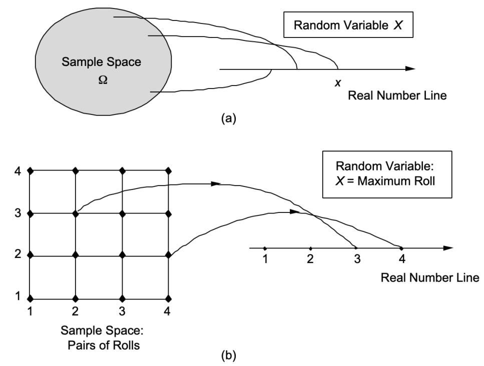
Random Variable: Idea

- In reality, many outcomes are numerical, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

Random Variable: More Formally

- Mathematically, a random variable X is a $\boxed{\text{function}}$ which maps from Ω to $\mathbb{R}$.
- Notation. Random variable X, numerical value x.
- Different random variables can be defined on the same sample space.
- For a fixed value x, we can associate an $\boxed{\text{event}}$ that a random variable X has the value x, i.e., $\boxed{\{\omega \in \Omega \mid X(\omega) = x\}}$
- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,
$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}\Big(\{\omega \in \Omega \mid X(\omega) = x\}\Big)$$
- For a discrete random variable X, we call $p_X(x)$ $\boxed{\text{probability mass function}}$ (PMF).

[1] Finite or countably infinite.

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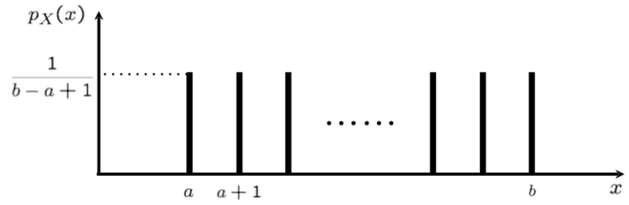
Bernoulli X with parameter $p \in [0, 1]$

- Only binary values
$$X = \begin{cases} 0, & \text{w.p.} \quad 1-p, \\ 1, & \text{w.p.} \quad p \end{cases}$$
  In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.
- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:
$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \le b$
- Choose a number out of $\Omega = \{a, a+1, \ldots, b\}$ uniformly at random.
- $p_X(i) = \frac{1}{b-a+1}$, $i \in \Omega$

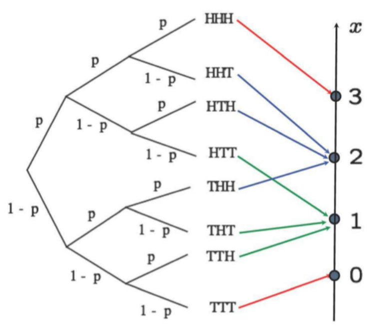


- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p

- Models the number of successes in a given number of independent trials
- n independent trials, where one trial has the success probability p.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



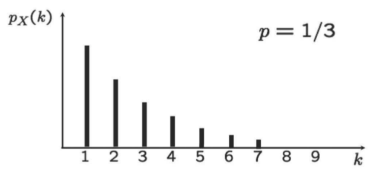
[1] $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, which we read '$n$ choose k'.

Geometric X with parameter p

- Infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the first success.

$$p_X(k) = (1-p)^{k-1} p$$

- Models waiting times until something happens.



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Expectation/Mean

- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

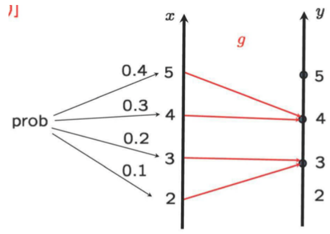
Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.
- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.
- For a constant c, $\mathbb{E}[c] = c$.

Expectation of a function of a RV

- For a rv X, $Y = g(X)$ is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- Compute $\mathbb{E}[Y]$ for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2)$$
$$= 2.8 + 0.9 = 3.7$$

Linearity of Expectation

$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Variance

- Measures how much the $\boxed{\text{spread}}$ of a PMF is.
- What about $\mathbb{E}[X - \mu]$, where $\mu = \mathbb{E}[X]$? Zero
- Then, what about $\mathbb{E}[(X - \mu)^2]$?

Variance, Standard Deviation

$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$

$\sigma_X = \sqrt{\text{var}[X]}$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

  $\text{var}[X] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$
  $= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$

- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$

  $\text{var}[Y] = \mathbb{E}[(X + b)^2] - (\mathbb{E}[X + b])^2$

- $Y = aX$, $\text{var}[Y] = a^2\text{var}[X]$

  $\text{var}[Y] = \mathbb{E}[a^2 X^2] - (a\mathbb{E}[X])^2$

Example: Variance of a Bernoulli rv (p)

$$\mu = \mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p$$
$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1 - p) = p$$
$$\text{var}[X] = \mathbb{E}[X^2] - \mu^2 = p - p^2$$
$$= p(1 - p)$$

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Joint PMF

- Joint PMF. For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$p_{X,Y}(x,y) \triangleq \mathbb{P}\Big(\{X = x\} \cap \{Y = y\}\Big)$$

- $\sum_x \sum_y p_{X,Y}(x,y) = 1$
- Marginal PMF.

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

Example. VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) = 2/20$

$p_X(4) = 2/20 + 1/20 = 3/20$

$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$

Functions of Multiple RVs

- Consider a rv $Z = g(X,Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

$$p_Z(z) = \mathbb{P}(g(X,Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x,y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

Linearity of Expectation for Multiple RVs

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

(easy to prove, using the definition.)

- $\mathbb{E}[X_1 \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- Example. Mean of a binomial rv Y with (n, p)
- Y: number of successes in n Bernoulli trials with p
- $Y = X_1 + \ldots X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

Message. When some rv X is write as a linear combination of other rvs, it is often easy to handle X.

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Conditional PMF: Conditioning on an event

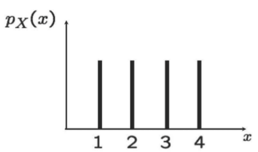
Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A.

- $p_X(x) \triangleq \mathbb{P}(X = x)$
- $\mathbb{E}[X] = \sum_x x p_X(x)$
- $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

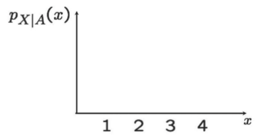
Example: Conditional PMF

$A = \{X \geq 2\}$



$$\mathbb{E}[X] = \frac{1}{4}(1+2+3+4) = 2.5$$

$$\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{4}(1+2^2+3^2+4^2) - 2.5^2$$



$$\mathbb{E}[X|A] = \frac{1}{3}(2+3+4) = 3$$

$$\text{var}[X|A] = \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 = \frac{1}{3}(2^2+3^2+4^2) - 3^2 = 2/3$$

Conditional PMF: Conditioning on a RV

What do we mean by "conditioning on a rv"? Consider $A = \{Y = y\}$ for a rv Y.

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$

- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
- $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
- $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
- $\text{var}[X|Y = y] \triangleq \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

Conditional PMF

- Conditional PMF

$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

- $\sum_x p_{X|Y}(x|y) = 1$
- Multiplication rule.

$$p_{X,Y}(x,y) = p_Y(y)p_{X|Y}(x|y) = p_X(x)p_{Y|X}(y|x)$$

- $p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X|Y}(2|2) = \frac{1}{1+3+1}$

$p_{X|Y}(3|2) = \frac{3}{1+3+1}$

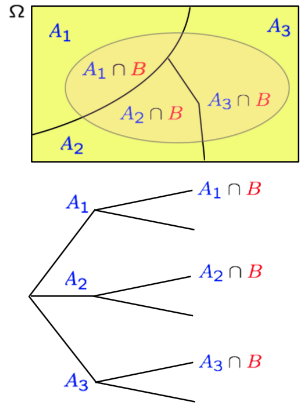
$\mathbb{E}[X|Y=3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$

Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$? (probability of result)

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

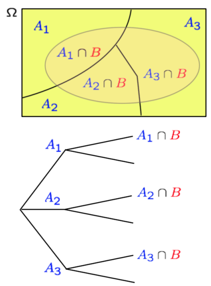


Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$



Total Expectation Theorem for $\{A_i\}$

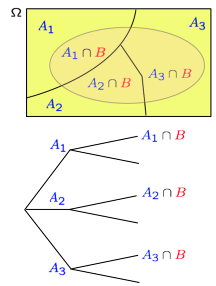
- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$



Total Expectation Theorem for $\{Y = y\}$

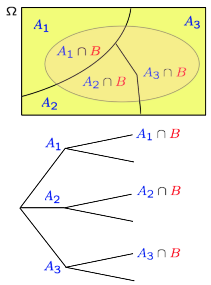
- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y] = \sum_y p_Y(y)\mathbb{E}[X|Y = y]$$



Example 1: Total Expectation Theorem

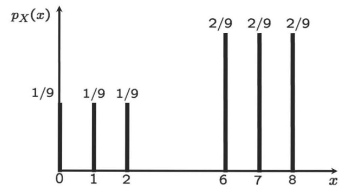
- Using the definition of expectation,

$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8) = \frac{3+12+14+16}{9} = 5$$

- Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\}, \ A_2 = \{X \in \{6,7,8\}\}$$

$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i] = 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



Background: Memoryless Property

- Some random variable often does not have memory.
- Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n+m | X > m) = \mathbb{P}(X > n)$$

- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

- Suppose that X is the time of waiting for a bus and X is memoryless. At the bus stop, I have waited for the bus for 10 mins. Then, the time until the bus arrival does not depend on how much I have waited for a bus. No memory.

Background: Memoryless Property of Geometric RVs

- Theorem. Any geometric random variable is memoryless.
- Remind. Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1-p)^{k-1}p, \quad \mathbb{P}(X > k) = 1 - \sum_{k'=1}^{k}(1-p)^{k'-1}p = (1-p)^k$$

- Proof.

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)} = \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)$$

- Meaning. Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

Example 2: Mean of Geometric rv

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- Write softwares over and over, and each time w.p. p of working correctly (independent from prev. programs).
- X: number of trials until the program works correctly.
- (Q) mean of X
- X is geometric
- Direct computation is boring.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

- Total expectation theorem and memorylessness helps a lot.

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$\begin{aligned}\mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\ &= 1 + \mathbb{P}(A_1)\mathbb{E}[X-1|X=1] \\ &\quad + \mathbb{P}(A_2)\mathbb{E}[X-1|X>1] \\ &= 1 + (1-p)\mathbb{E}[X]\end{aligned}$$

$$\mathbb{E}[X] = 1 + (1-p)\tfrac{1}{p} = 1/p.$$

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Independence, Conditional Independence

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- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$

$$\mathbb{P}(\{X = x\} \cap B|C) = \mathbb{P}(X = x|C) \cdot \mathbb{P}(B|C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}|Z = z) = \mathbb{P}(X = x|Z = z) \cdot \mathbb{P}(Y = y|Z = z), \quad \text{for all } x, y$$

$$p_{X,Y|Z}(x, y) = p_{X|Z}(x) \cdot p_{Y|Z}(y)$$

Example

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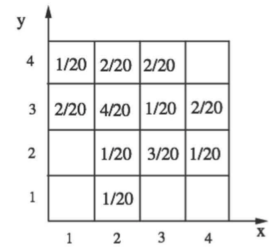
- $X \perp\!\!\!\perp Y$?

$p_{X,Y}(1,1) = 0, \quad p_X(1) = 3/20$

$p_Y(1) = 1/20.$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?

- Yes.



$Y = 4$ (1/3)	1/9	2/9
$Y = 3$ (2/3)	2/9	4/9
	$X = 1$ (1/3)	$X = 2(2/3)$

Expectation and Variance

- Always true.
  $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Generally, $\mathbb{E}[g(X,Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$
- However, if $X \perp\!\!\!\perp Y$,
  $$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$
  $$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[g(Y)]$$
- Proof.
  $$\mathbb{E}[g(X)h(Y)] = \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y) = \sum_x x p_X(x) \sum_y y p_Y(y)$$

- Always true.
  $\text{var}[aX] = a^2\text{var}[X]$, $\text{var}[X+a] = \text{var}[X]$
- Generally, $\text{var}[X+Y] \neq \text{var}[X] + \text{var}[Y]$
- However, if $X \perp\!\!\!\perp Y$,
  $\text{var}[X+Y] = \text{var}[X] + \text{var}[Y]$
- Practice.
  - $X = Y \implies \text{var}[X+Y] = 4\text{var}[X]$
  - $X = -Y \implies \text{var}[X+Y] = 0$
  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X-3Y] = \text{var}[X] + 9\text{var}[Y]$

$\text{var}[X+Y] \neq \text{var}[X] + \text{var}[Y]$

- Why not generally true?
  $$\begin{aligned}\text{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\ &= \mathbb{E}[X^2 + Y^2 + 2XY] - ((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) \\ &= \text{var}[X] + \text{var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\end{aligned}$$
- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

Example: The hat problem (1)

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise.
  $$X = \sum_{i=1}^{n} X_i.$$
- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (symmetry)

Example: The hat problem (2)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1.$
- Key step 2. Are X_is are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \to X_2 = 1$, and $X_1 = 0 \to X_2 = 0$. Thus, dependent.
  $$\begin{aligned}\text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \mathbb{E}\Big[\sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j\Big] - (\mathbb{E}[X])^2\end{aligned}$$
  $$\mathbb{E}[X_i^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$
  $$\mathbb{E}[X_iX_j] = \mathbb{E}[X_1X_2] = 1 \times \mathbb{P}(X_1X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$
- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFS?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?