

## Lecture 4: Random Variable, Part II

Yi, Yung (이웅)

EE210: Probability and Introductory Random Processes  
KAIST EE

June 2, 2021

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

June 2, 2021 1 / 45

June 2, 2021 2 / 45

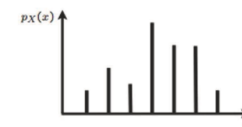
- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

- Many cases when random variables have "continuous values", e.g., velocity of a car

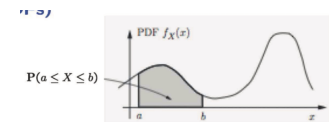
A rv  $X$  is **continuous** if  $\exists$  a function  $f_X$ , called **probability density function (PDF)**, s.t.

$$\mathbb{P}(X \in B) = \int_B f_X(x) dx, \quad \text{every subset } B \in \mathbb{R}$$

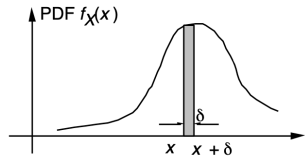
- All of the concepts and methods (expectation, PMFs, and conditioning) for discrete rvs have continuous counterparts



- $\mathbb{P}(a \leq X \leq b) = \sum_{x: a \leq x \leq b} p_X(x)$
- $p_X(x) \geq 0, \sum_x p_X(x) = 1$

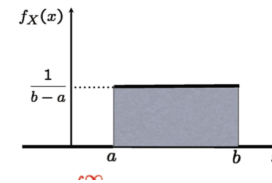
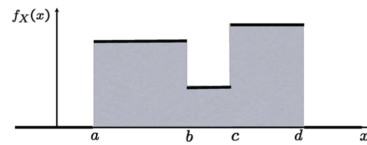
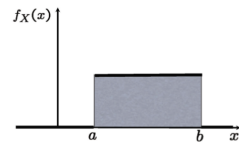


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x) dx = 1$



- $\mathbb{P}(a \leq X \leq a + \delta) \approx f_X(a) \cdot \delta$
- $\mathbb{P}(X = a) = 0$

## Examples



- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{b+a}{2}$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a} \frac{b^3 - a^3}{3} = \frac{a^2 + ab + b^2}{3}$
- $\text{var}[X] = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

L4(1)

June 2, 2021 5 / 45

L4(1)

June 2, 2021 6 / 45

## Roadmap

## Cumulative Distribution Function (CDF)

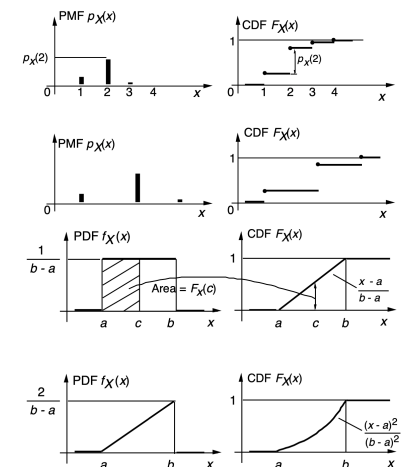
- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

- Discrete: PMF, Continuous: PDF
- Can we describe all rvs with a single mathematical concept?

$$F_X(x) = \mathbb{P}(X \leq x) =$$

$$\begin{cases} \sum_{k \leq x} p_X(k), & \text{discrete} \\ \int_{-\infty}^x f_X(t) dt, & \text{continuous} \end{cases}$$

- always well defined, because we can always compute the probability for the event  $\{X \leq x\}$
- CCDF (Complementary CDF):  $\mathbb{P}(X > x)$



L4(2)

June 2, 2021 7 / 45

L4(2)

June 2, 2021 8 / 45

- Non-decreasing
- $F_X(x)$  tends to 1, as  $x \rightarrow \infty$  and  $F_X(x)$  tends to 0, as  $x \rightarrow -\infty$
- If  $X$  is discrete,
  - $F_X(x)$  is a piecewise constant function of  $x$ .
  - $p_X(k) = F_X(k) - F_X(k-1)$
- If  $X$  is continuous
  - $F_X(x)$  is a continuous function of  $x$ .
  - $F_X(x) = \int_{-\infty}^x f_X(t)dt$  and  $f_X(x) = \frac{dF_X}{dx}(x)$

L4(2)

June 2, 2021 9 / 45

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$ , and  $X_i \in \{1, 2, \dots, 10\}$  uniformly at random
- **Question.**  $p_X(x)$ ?
- Approach 1:  $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$ ?

- Approach 2

$$F_X(x) = \mathbb{P}(\max\{X_1, X_2, X_3\} \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x) \\ = \mathbb{P}(X_1 \leq x) \cdot \mathbb{P}(X_2 \leq x) \cdot \mathbb{P}(X_3 \leq x) = \left(\frac{x}{10}\right)^3$$

Thus,

$$p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \dots, 10$$

L4(2)

June 2, 2021 10 / 45

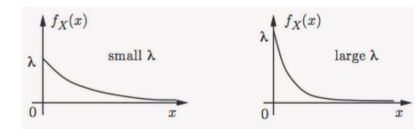
- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) **Exponential RVs**
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

L4(3)

June 2, 2021 11 / 45

- A rv  $X$  is called **exponential with  $\lambda$** , if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



- CDF  $F_X(x) = \int_0^x \lambda e^{-\lambda s} ds = 1 - e^{-\lambda x}$
- CCDF  $\mathbb{P}(X > x) = e^{-\lambda x}$
- **(Check)**  $\mathbb{E}[X] = 1/\lambda$ ,  $\mathbb{E}[X^2] = 2/\lambda^2$ ,  $\text{var}[X] = 1/\lambda^2$

L4(3)

June 2, 2021 12 / 45

- $\mathbb{E}(X) = 1/\lambda$ . Use **integration by parts**:  $\int u dv = uv - \int v du$   

$$\int_0^\infty x \lambda e^{-\lambda x} dx = (-xe^{-\lambda x}) \Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \frac{1}{\lambda}$$
- $\mathbb{E}(X^2)$   

$$\int_0^\infty x^2 \lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x}) \Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx = 0 + \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda^2}$$
- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

L4(3)

June 2, 2021 13 / 45

- $\mathbb{P}(X > x) = e^{-\lambda x}$
- Appropriate for modeling a waiting time until an incident of interest takes place
  - $\mathbb{P}(X > x)$ : exponentially decays
  - message arriving at a computer, some equipment breaking down, a light bulb burning out, etc
- (Q) What is the discrete rv which models a waiting time? **Geometric**
- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

L4(3)

June 2, 2021 14 / 45

- A very small meteorite first lands anywhere in Korea
- Time of landing is modeled as an exponential rv with mean 10 days
- The current time is midnight. What is the probability that a meteorite first lands some time between 6 a.m. and 6 p.m. of the first day?
- (Solution)
  - $\mathbb{E}(X) = 1/\lambda = 10$ . Thus,  $\lambda = \frac{1}{10}$ .
  - 6 a.m. from midnight = 1/4 day, 6 p.m. from midnight = 3/4 day
$$\mathbb{P}(1/4 \leq X \leq 3/4) = \mathbb{P}(X \geq 1/4) - \mathbb{P}(X \geq 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$



VIDEO PAUSE

L4(3)

June 2, 2021 15 / 45

- Models a system evolution over time: Continuous time vs. Discrete time.
  - **Example**. Customer arrivals at my shop
  - **Modeling 1**: Every 30 minute I record the number of customers for each 30-min window
  - **Modeling 2**: I record the exact time of each customer's arrival
  - In modeling 1, every 10 minute? every 1 minute? every 1 sec? every 0.0000001 sec?
- In many cases, continuous case is some type of **limit** of its corresponding discrete case.
- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

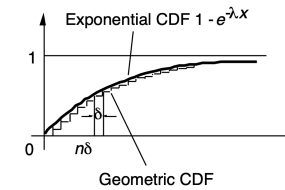
L4(3)

June 2, 2021 16 / 45

- 'slot' is one unit time, e.g., 1 hour, 30 mins, 1 min, 10 sec, etc.
- Continuous system = Discrete system with
  - infinitely many slots whose duration is infinitely small.
  - success probability  $p$  over one slot decreases to 0 in the limit
- Given  $X^{exp} \sim \exp(\lambda)$ , let us construct a geometric RV  $X_\delta^{geo}$ 
  - Set the length of a slot to be  $\delta$ , which is a parameter.
  - Set the success probability  $p_\delta$  over a slot to be  $p_\delta = 1 - e^{-\lambda\delta}$  (this looks magical, whose secret will be uncovered soon)
  - $\mathbb{P}(X_\delta^{geo} \leq n) = 1 - (1 - p_\delta)^n = 1 - e^{-\lambda\delta n}$

L4(3)

June 2, 2021 17 / 45



- Note that  $\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda x}$ . Then, when  $x = n\delta$ ,  $n = 1, 2, \dots$ 

$$\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda\delta n} = \mathbb{P}(X_\delta^{geo} \leq n)$$

- If we choose sufficiently small  $\delta$ , the slot length  $\downarrow$  and  $p_\delta \downarrow$

$$\mathbb{P}(X_\delta^{geo} \leq n) \xrightarrow{\delta \rightarrow 0} \mathbb{P}(X^{exp} \leq x), \quad x = n\delta$$

L4(3)

June 2, 2021 18 / 45

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

L4(4)

June 2, 2021 19 / 45

- Standard Normal  $\mathcal{N}(0, 1)$

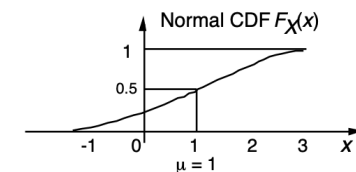
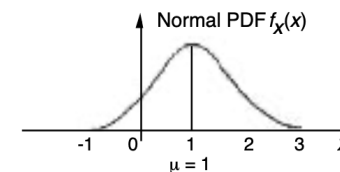
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

- General Normal  $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$



L4(4)

June 2, 2021 20 / 45

- PDF's normalization property:  $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

◦ A little bit boring :-). See Problem 14 at pp 189.

- Expectation

◦  $f_X(x)$  is symmetric in terms of  $x = \mu$ . Thus, we should have  $\mathbb{E}(X) = \mu$ .

- Variance

$$\begin{aligned} \text{var}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \stackrel{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \left( -ye^{-y^2/2} \right) \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sigma^2 \end{aligned}$$

$$\int u dv = uv - \int v du: u = y \text{ and } dv = ye^{-y^2/2} \rightarrow du = dy \text{ and } v = -e^{-y^2/2}$$

L4(4)

June 2, 2021 21 / 45

- Linear transformation preserves normality (we will verify this in Lecture 5)

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for  $a \neq 0$  and  $b$ ,  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

- Thus, every normal rv can be **standardized**:

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Y = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

- Thus, we can make the **table** which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

L4(4)

June 2, 2021 22 / 45

## Example

- Annual snowfall  $X$  is modeled as  $\mathcal{N}(60, 20^2)$ . What is the probability that this year's snowfall is at least 80 inches?

$$Y = \frac{X-60}{20}$$

$$\begin{aligned} \mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80-60}{20}) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587 \end{aligned}$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

L4(4)

June 2, 2021 23 / 45

## Normal RVs: Why Important?

- Central limit theorem
  - One of the most remarkable findings in the probability theory
  - Sum of **any** random variables  $\approx$  Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

<sup>0</sup>Central limit theorem: 중심극한정리

L4(4)

June 2, 2021 24 / 45

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

Two continuous rvs are **jointly continuous** if a non-negative function  $f_{X,Y}(x,y)$  (called joint PDF) satisfies: for **every subset**  $B$  of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}[(X, Y) \in B] = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy$$

Our particular interest:  $B = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$

2. The **marginal** PDFs of  $X$  and  $Y$  are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

3. The **joint CDF** is defined by  $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$ , and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$$

4. A function  $g(X, Y)$  of  $X$  and  $Y$  defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$$

- \* Conditional PDF, given an event  $A$

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$   
 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$
- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$   
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$
- $\int f_{X|A}(x) dx = 1$

**Notation:**  $A$  is an event, but  $B$  and  $C$  is a subset that includes the possible values which can be taken by the rv  $X$ . Sorry for the confusion, if any.

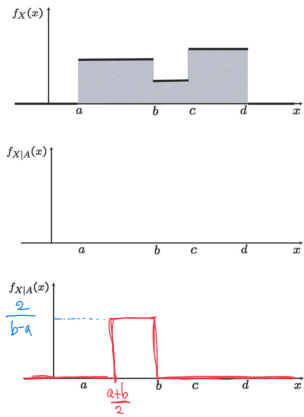
- \* Conditional PDF, given  $\{X \in C\}$

$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$$

**(Q)** In the discrete, we consider the event  $\{X = x\}$ , not  $\{X \in B\}$ . Why?

$$A = \left\{ \frac{a+b}{2} \leq X \leq b \right\}$$



L4(5)

- $\mathbb{E}[X] = \int x f_X(x) dx$   
 $\mathbb{E}[X|A] = \int x f_{X|A}(x) dx$
- $\mathbb{E}[g(X)] = \int g(x) f_X(x) dx$   
 $\mathbb{E}[g(X)|A] = \int g(x) f_{X|A}(x) dx$

$$\mathbb{E}[X|A] = \int_{(a+b)/2}^b x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^b x^2 \frac{2}{b-a} dx =$$

June 2, 2021 29 / 45

- **Remember:** Exponential rv is a continuous counterpart of geometric rv.
- Thus, expected to be memoryless. Remember the definition?

**Definition.** A random variable  $X$  is called memoryless if, for any  $n, m \geq 0$ ,

$$\mathbb{P}(X > n+m | X > m) = \mathbb{P}(X > n)$$

- **Proof.** Note that the exponential rv's CCDF  $\mathbb{P}(X > x) = e^{-\lambda x}$ . Then,

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)} = \frac{e^{-\lambda(n+m)}}{e^{-\lambda m}} = e^{-\lambda n} = \mathbb{P}(X > n)$$

L4(5)

June 2, 2021 30 / 45

Partition of  $\Omega$  into  $A_1, A_2, A_3, \dots$

\* Discrete case

#### Total Probability Theorem

$$\begin{aligned} p_X(x) &= \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) \\ &= \sum_i \mathbb{P}(A_i) p_{X|A_i}(x) \end{aligned}$$

#### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

\* Continuous case

#### Total Probability Theorem

$$f_X(x) = \sum_i \mathbb{P}(A_i) f_{X|A_i}(x)$$

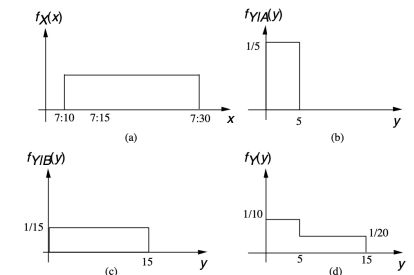
#### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

L4(5)

June 2, 2021 31 / 45

- The train's arrival every quarter hour (0, 15min, 30min, 45min).
- Your arrival  $\sim \mathcal{U}(7:10, 7:30)$  am.
- What is the PDF of waiting time for the first train?
- $X$  : your arrival time,  $Y$  : waiting time.
- The value of  $X$  makes a different waiting time. So, consider two events:  
 $A = \{7:10 \leq X \leq 7:15\}$   
 $B = \{7:15 \leq X \leq 7:30\}$



VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A) f_{Y|A}(y) + \mathbb{P}(B) f_{Y|B}(y)$$

$$f_Y(y) = \frac{1}{4} \frac{1}{5} + \frac{3}{4} \frac{1}{15} = \frac{1}{10}, \quad \text{for } 0 \leq y \leq 5$$

$$f_Y(y) = \frac{1}{4} 0 + \frac{3}{4} \frac{1}{15} = \frac{1}{20}, \quad \text{for } 5 < y \leq 15$$

L4(5)

June 2, 2021 32 / 45



- $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- Similarly, for  $f_Y(y) > 0$ ,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Remember: For a fixed event  $A$ ,  $\mathbb{P}(\cdot|A)$  is a legitimate probability law.
- Similarly, For a fixed  $y$ ,  $f_{X|Y}(x|y)$  is a legitimate PDF, since

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y) dx}{f_Y(y)} = 1$$

L4(5)

June 2, 2021 33 / 45

- **Multiplication rule.**

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

- **Total prob./exp. theorem.**

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy$$

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbb{E}[X|Y=y] dy$$

- **Independence**

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

(Prob 21 at pp. 191)

- Break a stick of length  $l$  twice
  - first break at  $Y \sim \mathcal{U}[0, l]$
  - second break at  $X \sim \mathcal{U}[0, Y]$

(a) joint PDF  $f_{X,Y}(x,y)$ ?

$$f_Y(y) = \frac{1}{l}, \quad 0 \leq y \leq l$$

$$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 \leq x \leq y$$

Using  $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y)$ ,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{l} \cdot \frac{1}{y}, & 0 \leq x \leq y \leq l, \\ 0, & \text{otherwise} \end{cases}$$

 ${}^0\mathcal{U}[a, b]$ : continuous uniform random variable over the interval  $[a, b]$ 

L4(5)

June 2, 2021 34 / 45

(b) marginal PDF  $f_X(x)$ ?

$$f_X(x) = \int f_{X,Y}(x,y) dy = \int_x^l \frac{1}{ly} dy \\ = \frac{1}{l} \ln(l/x), \quad 0 \leq x \leq l$$

(c) Evaluate  $\mathbb{E}(X)$ , using  $f_X(x)$ 

$$\mathbb{E}(X) = \int_0^l x f_X(x) dx = \int_0^l \frac{x}{l} \ln(l/x) dx \\ = \frac{l}{4}$$

(d) Evaluate  $\mathbb{E}(X)$ , using  $X = Y \cdot (X/Y)$ 

If  $Y \perp\!\!\!\perp X/Y$ , it becomes easy, but true?  
Yes, because whatever  $Y$  is, the fraction  $X/Y$  does not depend on it.

$$\mathbb{E}(X) = \mathbb{E}(Y) \mathbb{E}(X/Y) = \frac{l}{2} \cdot \frac{1}{2} = \frac{l}{4}$$

L4(5)

June 2, 2021 35 / 45

(e) Evaluate  $\mathbb{E}(X)$ , using TET

$$0\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbb{E}[X|Y=y] dy \\ = \int_0^l \frac{1}{l} \mathbb{E}[X|Y=y] dy = \int_0^l \frac{1}{l} \frac{y}{2} dy = \frac{l}{4}$$

- **Message.** There are many ways to reach our goal. Of crucial importance is how to find the best way!

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) **Bayes' rule for RVs**

L4(6)

June 2, 2021 36 / 45

- $X$ : state/cause/original value  $\rightarrow Y$ : result/resulting action/noisy measurement
- Given:  $\mathbb{P}(X)$  and  $\mathbb{P}(Y|X)$  (cause  $\rightarrow$  result)
- Inference:  $\mathbb{P}(X|Y)$ ?

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x)p_{Y|X}(y|x) \\ &= p_Y(y)p_{X|Y}(x|y) \\ p_{X|Y}(x|y) &= \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)} \\ p_Y(y) &= \sum_{x'} p_X(x')p_{Y|X}(y|x') \end{aligned}$$

L4(6)

June 2, 2021 37 / 45

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_{Y|X}(y|x) \\ &= f_Y(y)f_{X|Y}(x|y) \\ f_{X|Y}(x|y) &= \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)} \\ f_Y(y) &= \int f_X(x')f_{Y|X}(y|x')dx' \end{aligned}$$

- A light bulb  $Y \sim \exp(\lambda)$ . However, there are some quality control problems. So, the parameter  $\lambda$  of  $Y$  is actually a random variable, denoted by  $\Lambda$ , which is  $\Lambda \sim \mathcal{U}[1, 3/2]$ . We test a light bulb and record its lifetime.
- **Question.** What can we say about the underlying parameter  $\lambda$ ? In other words, what is  $f_{\Lambda|Y}(\lambda|y)$ ?
- $f_{\Lambda}(\lambda) = 2$  for  $1 \leq \lambda \leq 3/2$  and  $f_{Y|\Lambda}(y|\lambda) = \text{pdf of } \exp(\lambda)$ . Then, the inference about the parameter given the lifetime of a light bulb is:

$$f_{\Lambda|Y}(\lambda|y) = \frac{f_{\Lambda}(\lambda)f_{Y|\Lambda}(y|\lambda)}{\int_{-\infty}^{\infty} f_{\Lambda}(t)f_{Y|\Lambda}(y|t)dt}$$

L4(6)

June 2, 2021 38 / 45

- $X$ : **parameter**  $\rightarrow Y$ : result of **my model**
- Given:  $\mathbb{P}(X)$  and  $\mathbb{P}(Y|X)$  (parameter  $\rightarrow$  model)
- Inference:  $\mathbb{P}(X|Y)$ ? Probabilistic feature of the parameter given the result of the model?

### Example.

1. Light bulb's lifetime  $Y \sim \exp(\lambda)$ . Given the lifetime  $y$ , the modified belief about  $\lambda$ ?
2. Romeo and Juliet start dating, but Romeo will be late by a random variable  $Y \sim \mathcal{U}[0, \theta]$ . Given the time of being late  $y$ , the modified belief about  $\theta$ ?

L4(6)

June 2, 2021 39 / 45

$K$ : discrete,  $Y$ : continuous

- Inference of  $K$  given  $Y$

$$\begin{aligned} p_{K|Y}(k|y) &= \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)} \\ f_Y(y) &= \sum_{k'} p_K(k')f_{Y|K}(y|k') \end{aligned}$$

- $f_{Y|K}(y|k) = f_{Y|A}(y)$ , where  $A = \{K = k\}$

- Inference of  $Y$  given  $K$

$$\begin{aligned} f_{Y|K}(y|k) &= \frac{f_Y(y)p_{K|Y}(k|y)}{p_K(k)} \\ p_K(k) &= \int f_Y(y')p_{K|Y}(k|y')dy' \end{aligned}$$

- Wait!  $p_{K|Y}(k|y)$ ? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

L4(6)

June 2, 2021 40 / 45

- For small  $\delta$  (in other words, taking the limit as  $\delta \rightarrow 0$ ).

Let  $A = \{K = k\}$ .

$$\begin{aligned} p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \leq Y \leq y + \delta) \\ &= \frac{\mathbb{P}(A)\mathbb{P}(y \leq Y \leq y + \delta|A)}{\mathbb{P}(y \leq Y \leq y + \delta)} \\ &\approx \frac{\mathbb{P}(A)f_{Y|A}(y)\delta}{f_Y(y)\delta} \\ &= \frac{\mathbb{P}(A)f_{Y|A}(y)}{f_Y(y)} \end{aligned}$$

L4(6)

June 2, 2021 41 / 45

Inference of discrete  $K$  given continuous  $Y$ :

$$p_{K|Y}(k|y) = \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)}, \quad f_Y(y) = \sum_{k'} p_K(k')f_{Y|K}(y|k')$$

- $K$ : -1, +1, original signal, equally likely.  $p_K(1) = 1/2, p_K(-1) = 1/2$ .
- $Y$ : measured signal with Gaussian noise,  $Y = K + W$ ,  $W \sim \mathcal{N}(0, 1)$
- Your received signal = 0.7. What's your guess about the original signal? **+1**
- Your received signal = -0.2. What's your guess about the original signal? **-1**
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

L4(6)

June 2, 2021 42 / 45

## Example: Signal Detection (2)

- $Y|\{K = 1\} \sim \mathcal{N}(1, 1)$  and  $Y|\{K = -1\} \sim \mathcal{N}(-1, 1)$ .  
(Remind: linear transformation preserves normality.)

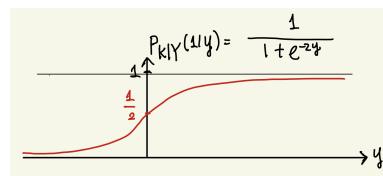
$$f_{Y|K}(y|k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}, \quad k = 1, -1$$

$$f_Y(y) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \quad (\text{from TPT})$$

- Probability that  $K = 1$ , given  $Y = y$ ? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$

- If  $y > 0$ , the inference probability for  $K = 1$  exceeds  $\frac{1}{2}$ . So, original signal = 1.
- Similarly, compute  $p_{K|Y}(-1|y)$  and then do the inference



## Questions?

L4(6)

June 2, 2021 43 / 45

L4(6)

June 2, 2021 44 / 45

- 1) What is PDF and CDF?
- 2) Why do we need CDF?
- 3) What are joint/marginal/conditional PDFs?
- 4) Explain memorylessness of exponential random variables.
- 5) Explain the version of Bayes' rule for continuous and mixed random variables.