Lecture 5: Random Variable, Part III

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs.

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

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- Examples: $Y = X$, $Y = X + 1$, $Y = X^2$, etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear: $Y = aX + b$

- Take all values of x such that $g(x) = y$, i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x:g(x)=y} p_X(x) \end{aligned}$$

Discrete Case

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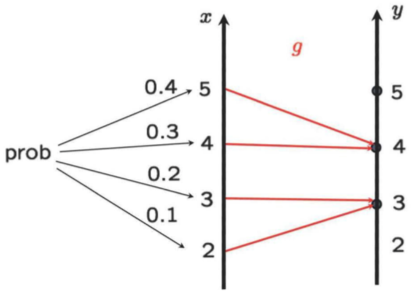

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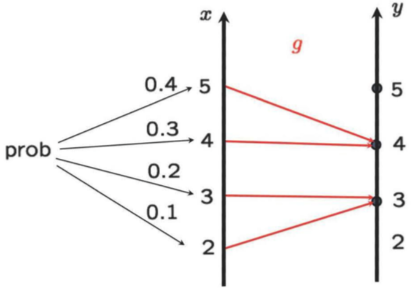
$$p_Y(y) = \mathbb{P}(g(X) = y) = \sum_{x: g(x)=y} p_X(x)$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$
$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



Linear: $Y = aX + b$, $a \neq 0$, X: Continuous

If $a > 0$,

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If $a > 0$, $\quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

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$$\rightarrow f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

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$$\text{If } a < 0, \quad F_Y(y) = \mathbb{P}(aX + b \le y) = \mathbb{P}(X \ge \frac{y-b}{a}) = 1 - F_X(\frac{y-b}{a})$$

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Therefore,

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Linear: $Y = aX + b$, when X is exponential

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if} \quad x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if} \quad (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If $b = 0$ and $a > 0$, Y is exponential with parameter $\frac{\lambda}{a}$, but generally not.

Linear: $Y = aX + b$, when X is normal

- Remember? Linear transformation preserves normality. Time to prove.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

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If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Proof.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

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- Proof.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\}$$

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$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y)$$

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Ex1. $Y = X^2$.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \geq 0$$

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Ex2. $X \sim \mathcal{U}[0,1]$. $Y = \sqrt{X}$.

$$F_Y(y) = \mathbb{P}(\sqrt{X} \le y) = \mathbb{P}(X \le y^2) = y^2$$
$$f_Y(y) = 2y, \quad 0 \le y \le 1$$

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Ex3. $X \sim \mathcal{U}[0,2]$. $Y = X^3$.

$$F_Y(y) = \mathbb{P}(X^3 \le y) = \mathbb{P}(X \le \sqrt[3]{y}) = \frac{1}{2} y^{1/3}$$
$$f_Y(y) = \frac{1}{6} y^{-2/3}, \quad 0 \le y \le 8$$

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When $Y = g(X)$ is monotonic, a general formula can be drawn (see the textbook at pp 207)

Functions of multiple rvs: $Z = g(X, Y)$ (1)

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\begin{aligned}
F_Z(z) &= \mathbb{P}(\max(X, Y) \le z) = \mathbb{P}(X \le z, Y \le z) \\
&= \mathbb{P}(X \le z)\mathbb{P}(Y \le z) = z^2 \qquad\qquad \text{(from } X \perp\!\!\!\perp Y\text{)}
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f_Z(z) = \begin{cases} 2z, & \text{if } 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
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Ex2. $X, Y \sim \mathcal{U}[0, 1]$, and $X \perp\!\!\!\perp Y$. $Z = Y/X$.

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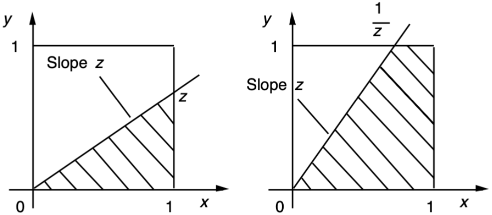
$F_Z(z) = \mathbb{P}(Y/X \le z)$

- Depending on the value of z, two cases need to be considered separately.



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Ex2. $X, Y \sim \mathcal{U}[0,1]$, and $X \perp\!\!\!\perp Y$. $Z = Y/X$.

$$F_Z(z) = \mathbb{P}(Y/X \leq z)$$

$$= \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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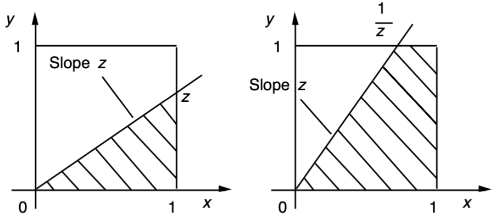

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$$= \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} 1/2, & 0 \leq z \leq 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

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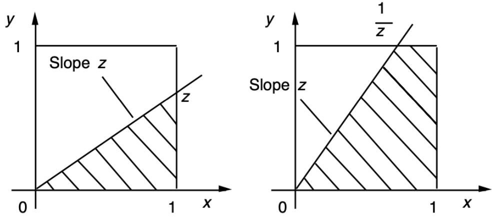

Functions of multiple rvs: $Z = g(X, Y)$ (2)

Basically, follow two step approach: (i) CDF and (ii) differentiate.

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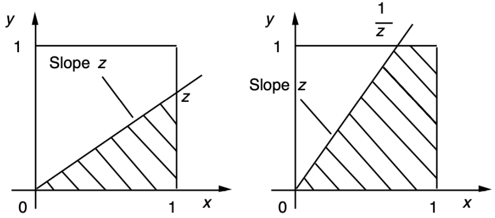

(Note) Sometimes, the problem is tricky, which requires careful case-by-case handing. :-)

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- Assume that $X, Y \in \mathbb{Z}$

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&= \sum_{\{(x,y):x+y=z\}} \mathbb{P}(X = x, Y = y) \\
&= \sum_x \mathbb{P}(X = x, Y = z - x) \\
&= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
&= \sum_x p_X(x)p_Y(z - x)
\end{aligned}
$$

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&= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
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\end{aligned}
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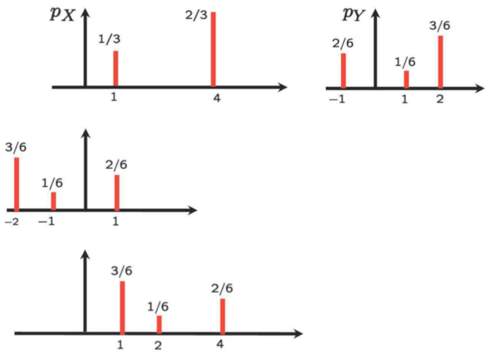
- Interpretation (for a given z)
(i) Flip (horizontally) $p_Y(y)$ ($p_Y(-x)$)
(ii) Put it underneath $p_X(x)$ ($p_Y(-x+z)$)



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- $p_Z(z)$ is called $\boxed{\qquad\qquad}$ of the PMFs of X and Y.

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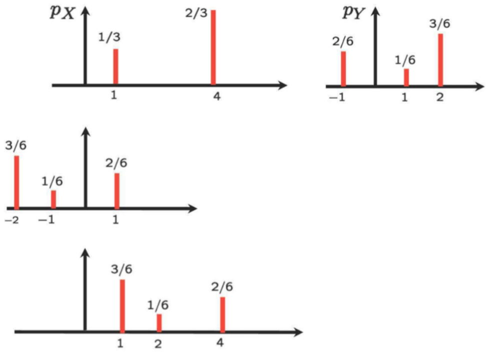

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- Assume that $X, Y \in \mathbb{Z}$

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&= \sum_{\{(x,y):x+y=z\}} \mathbb{P}(X = x, Y = y) \\
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&= \sum_x \mathbb{P}(X = x)\mathbb{P}(Y = z - x) \\
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- $p_Z(z)$ is called $\boxed{\text{convolution}}$ of the PMFs of X and Y.

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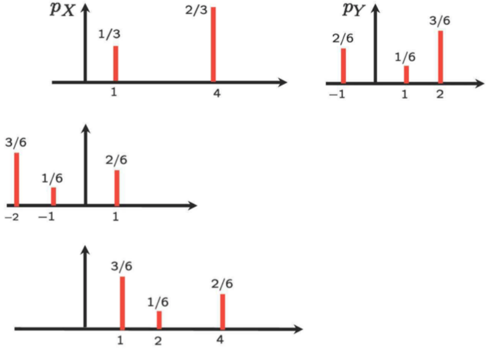

- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

- Example. $X, Y \sim \mathcal{U}[0,1]$ and $X \perp\!\!\!\perp Y$. What is the PDF of $Z = X + Y$?

-

$Y = X + Y$, $X \perp\!\!\!\perp Y$, Normal (1)

- Very special, but useful case
  - X and Y are normal.

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Sum of two independent normal rvs

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- Why normal rvs are used to model the sum of random noises.
- Extension. The sum of finitely many independent normals is also normal.

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx
\end{aligned}
$$

- The details of integration is a little bit tedious, but note where we use the independence condition.

$$
f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right\}
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Dependence Degree: Motivating Example

- covariance의 필요성을 이야기해주는 example을 찾아서 먼저 이야기를 해준다.

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- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - A lot of metrics in our professional sports leagues (baseball, basketball, etc)
  - Cybermetrics in MLB (Major League Baseball): `http://m.mlb.com/glossary/advanced-stats`

OK. Let's Design!

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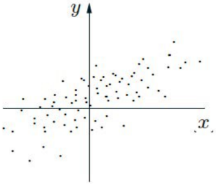

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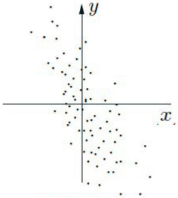

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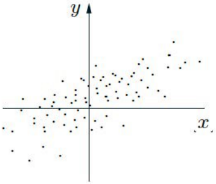

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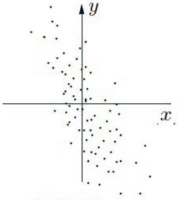

$\mathbf{E}[XY]$

(Q) What about $\mathbb{E}[X+Y]$?

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$$\text{cov}(X, Y) = \mathbb{E}\Big[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])\Big]$$

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Example: $\mathrm{cov}(X, Y) = 0$, but not independent

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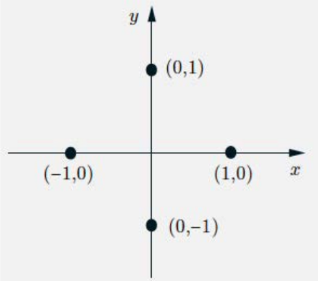

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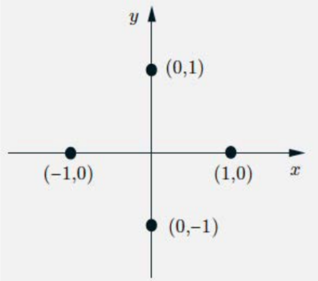

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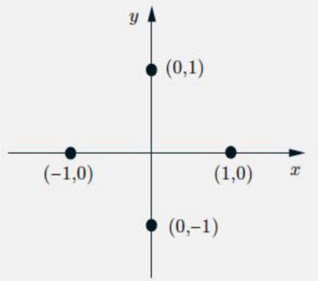
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 0$. So, $\text{cov}(X, Y) = 0$
- Are they independent? No, because if $X = 1$, then we should have $Y = 0$.



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Example: The hat problem in Lecture 3. Remember?

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- (Q) var$[X]$
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$.
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&= \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\
&= \frac{1}{n}\frac{1}{n-1} - \frac{1}{n^2} = \frac{1}{n^2(n-1)}
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\end{aligned}$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

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- $-1 \le \rho \le 1$
- $|\rho| = 1 \implies X - \mu_X = c(Y - \mu_Y)$ (linear relation, VERY related)

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A Special Random Variable

- Consider a rv Y, such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

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- If $h(y) = y^2$, then a new rv $h(Y)$ is:

$$h(Y) = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 4, & \text{w.p. } 1/2 \end{cases}$$

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- What about? $X_{exp}(Y)$, $\mathbb{E}[X_Y]$, $\mathbb{E}_X[Y]$?

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- A function of Y

Conditional Expectation $\mathbb{E}[X|Y]$

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Conditional Expectation

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- Often confusing because of the notation.

Expectation of Conditional Expectation

$\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] = \mathbb{E}[X]$,   Law of iterated expectations

Proof.

$$\begin{aligned}\mathbb{E}\Big[\mathbb{E}[X|Y]\Big] &= \sum_y \mathbb{E}[X|Y = y] p_Y(y) \\ &= \mathbb{E}[X]\end{aligned}$$

Examples and Meaning

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  Revised forecast $\neq \mathbb{E}[X]$
- Law of iterated expectations
  $\mathbb{E}[\text{revised forecast}] = \text{original one}$

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

(2) Derived distribution of $Z = X + Y$

(3) Covariance: Degree of dependence between two rvs

(4) Correlation coefficient

(5) Conditional expectation and law of iterative expectations

(6) Conditional variance and law of total variance

(7) Random number of sum of random variables

Conditional Variance $\text{var}[X|Y]$

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

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Conditional Variance

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A random variable $g(Y) = \boxed{\text{var}[X|Y]}$ and called $\boxed{\text{conditional variance of } X \text{ given } Y}$, takes the value $g(y) = \text{var}[X|Y = y]$, if Y happens to take the value y.

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	$\mathbb{E}[X\|Y]$	$\text{var}[X\|Y]$
Expectation	$\mathbb{E}\left[\mathbb{E}(X\|Y)\right]$	$\mathbb{E}\left[\text{var}(X\|Y)\right]$
Variance	$\text{var}\left[\mathbb{E}(X\|Y)\right]$	$\text{var}\left[\text{var}(X\|Y)\right]$

Law of total variance

$$\text{var}[X] =$$

Proof.

(1)

(2)

Law of Total Variance

Law of total variance

$$\mathsf{var}[X] = \mathbb{E}\Big[\mathsf{var}(X|Y)\Big] + \mathsf{var}[\mathbb{E}(X|Y)]$$

Proof.

$$(1)$$

$$(2)$$

Law of total variance

$$\mathrm{var}[X] = \mathbb{E}\Big[\mathrm{var}(X|Y)\Big] + \mathrm{var}[\mathbb{E}(X|Y)]$$

Proof.

$$\mathrm{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

$$\tag{1}$$

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$$\mathrm{var}[X] = \mathbb{E}\Big[\mathrm{var}(X|Y)\Big] + \mathrm{var}[\mathbb{E}(X|Y)]$$

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$$(1) + (2) = \mathbb{E}[X^2] + (\mathbb{E}[X])^2 = \text{var}[X]$$

Roadmap

(1) Derived distribution of $Y = g(X)$ or $Z = g(X, Y)$

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Sum of a random number of rvs

- N : number of stores visited (random)

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Questions?

1) What are the key steps to get the derived distributions of $Y = g(X)$ or $Z = g(X, Y)$?
2) How can we compute the distribution of $Z + X + Y$ when X and Y are independent?
3) What are covariance and correlation coefficient? Why do we need them?
4) Please explain the concepts of conditional expectation and conditional variance.