Lecture 8: Random Processes, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

Recap and Markov Chain

- Assume discrete times $n = 1, 2, \dots$

- Random process: A sequence of $X_1, X_2, X_3, \cdots$
- "Simplest" random process
  - Process without memory

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n)$$

  - Bernoulli process
- A random process that is a little more complex than the above?
  - Process that depends only on "yesterday", not the entire history

$$\mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \dots, X_1 = i_1) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$$

  - Markov chain
  - One of the most popular random processes in engineering

Example: Machine Failure, Repair, and Replacement

- A machine: working or broken down on a given day.
  - If working, break down in the next day w.p. b, and continue working w.p. $1 - b$.
  - If broken down, it will be repaired and be working in the next day w.p. r, and continue to be broken down w.p. $1 - r$.
- $X_n \in \{1, 2\}$: status of the machine, 1: working and 2: broken down
- $(X_n)_{n=1}^{\infty}$: A random process satisfying: for any $n \geq 1$,

$$\mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$
$$\mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \quad \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

- What will happen at $(n + 1)$-th day depends only on what happens at n-th day?

Markov Chain: Definition

- Definition. Let $X_1, \ldots, X_n, \ldots$ be a sequence of random variables taking values in some finite space $\mathcal{S} = \{1, 2, \ldots, m\}$, such that for all $i, j \in \mathcal{S}$, $n \geq 0$, the following Markov property is satisfied:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0),$$

- For any fixed n, the future of the process after n is independent of $\{X_1, \ldots, X_n\}$, given X_n (i.e., depends only on X_n)
- The value that X_n can take is called 'state'. Thus, the space $\mathcal{S}$ is called *state space*.
- Time homogeneity. The probability $\mathbb{P}(X_{n+1} = j | X_n = i)$ does NOT depends on n.

  Thus, for any $n \geq 0$, we introduce a simple notation p_{ij}

  $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$

Transition Prob. Matrix and State Transition Diagram

- Transition Probability Matrix. Consider a $m \times m$ matrix $\boldsymbol{P} = [p_{ij}]$, where $p_{ij} \triangleq \mathbb{P}(X_{n+1} = j | X_n = i)$
- Machine example.

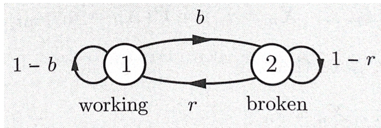
$$p_{11} = \mathbb{P}(X_{n+1} = 1 | X_n = 1) = 1 - b, \qquad p_{12} = \mathbb{P}(X_{n+1} = 2 | X_n = 1) = b$$

$$p_{21} = \mathbb{P}(X_{n+1} = 1 | X_n = 2) = r, \qquad p_{22} = \mathbb{P}(X_{n+1} = 2 | X_n = 2) = 1 - r$$

\- Transition probability matrix

$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}$$

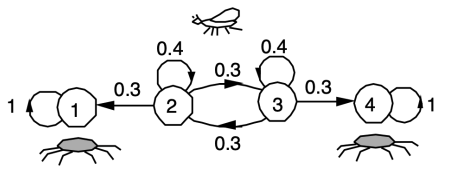
\- State transition diagram



- Both are the complete description of Markov chain.
- $\sum_{j=}^{m} p_{ij} = 1$ (for each row i, the column sum = 1)

Spider-Fly example

- A fly moves along a line in unit increments.
- At each time, it moves one unit (i) left w.p. 0.3, (ii) right w.p. 0.3 and (iii) stays in place w.p. 0.4, independent of the past history of movements.
- Two spiders lurk at positions 1 and 4: if the fly lands there, it is captured by the spider, and the process terminates. Assume that the fly starts in a position between 1 and 4.
- X_n: position of the fly. Please draw the state transition diagram and find the transition probability matrix.



	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

p_{ij}

Probability of a Sample Path

(Q) What is the probability of a sample path in a Markov chain?

$$\mathbb{P}\Big(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\Big)$$

$$= \mathbb{P}\Big(X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\Big) \cdot \mathbb{P}\Big(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\Big)$$

$$= p_{i_{n-1} i_n} \cdot \mathbb{P}\Big(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\Big) = \mathbb{P}(X_0 = i_0) \cdot p_{i_0 i_1} \cdot p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

- Spider-Fly example

  $\mathbb{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) = \mathbb{P}(X_0 = 2) p_{22} p_{22} p_{23} p_{34} = \mathbb{P}(X_0 = 2)(0.4)^2(0.3)^2$

Probability after n Steps

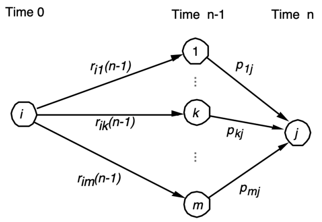
(Q) What is the probability that my state is i, starting from i after n steps?

- n-step transition probability

$$r_{ij}(n) \triangleq \mathbb{P}(X_n = j \mid X_0 = i)$$

- Recursive formula, starting with $r_{ij}(1) = p_{ij}$

$$r_{ij}(n) = \mathbb{P}(X_n = j \mid X_0 = i) =$$
$$\sum_{k=1}^{m} \mathbb{P}(X_{n-1} = k | X_0 = i)\mathbb{P}(X_n = j | X_{n-1} = k, X_0 = i)$$
$$= \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$$

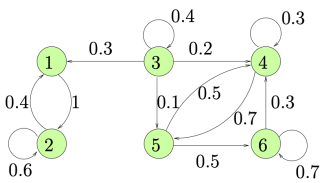


Roadmap

- Basics on Random Process
- Bernoulli Process
- Poisson Process
- Use of Bernoulli and Poisson Processes
- Markov Chain
  - Definition, Transition Probability Matrix, State Transition Diagram
  - Classification of States
  - Steady-state Behaviors and Stationary Distribution
  - Transient Behaviors

Examples: Different States and Classes

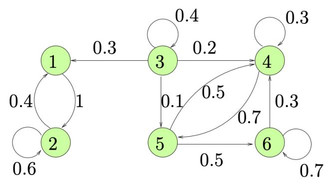
- Classes
  - 3 can only be reached from 3
  - 1 and 2 can reach each other but no other state
  - 4, 5, and 6 all reach each other.
  - Divide into three classes: $\{3\}, \{1, 2\}, \{4, 5, 6\}$
  - Insight 1. Multiple classes may exist.



- Difference between 1 and 3
  - 1: If I start from 1, visit 1 infinite times.
  - 3: If I start from 3, visit 3 only finite times (move to other classes and don't return).
  - Insight 2. Some states are visited infinite times, but some states are not.
- State 2 will share the above properties with 1 (similarly, 4,5, and 6)
- Insigt 3. States in the same class share some properties.

Classification of States (1)

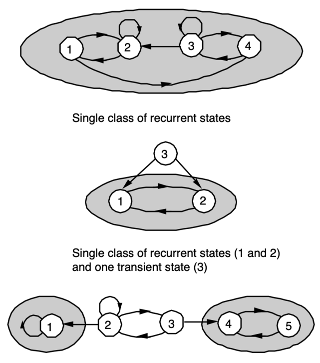
- Definition. State j is accessible from state i, if for some n $r_{ij}(n) > 0$.
  - 6 is accessible from 3, but not the other way around.
- Definition. If i is accessible from j and j is accessible from i, we say that i communicates with j.
  - $1 \leftrightarrow 2$, but 3 does not communicate with 5.



- Definition. Let $A(i) = \{\text{states accessible from } i\}$. State i is recurrent, if $\forall j \in A(i)$, i is also accessible from j. In other words, "I communicate with all of my neighbors!"
  - A state that is not recurrent is transient.
  - 2 is recurrent? Yes. 3 is recurrent? No.
  - If we start from a recurrent state i, then there is always some probability of returning to i. It means that, given enough time, it is certain that it returns to i.

Classification of States (2)

- A set of recurrent states which communicate with each other form a class.
- Markov chain decomposition
  - A MC can be decomposed into one or more recurrent classes, plus possibly some transient states.
  - A recurrent state is accessible from all states in its class, but it not accessible from recurrent states in other classes.
  - A transient state is not accessible from any recurrent state.
  - At least one, possibly more, recurrent states are accessible from a given transient state.
- The MC with only a single recurrent class is said to be irreducible (더이상 분해할 수 없는).



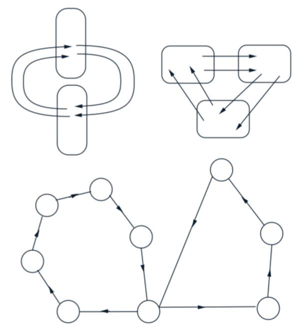
Single class of recurrent states

Single class of recurrent states (1 and 2) and one transient state (3)

Two classes of recurrent states (class of state1 and class of states 4 and 5) and two transient states (2 and 3)

Periodicity

- The states in a recurrent class are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
- A recurrent class that is not periodic is said to be aperiodic.



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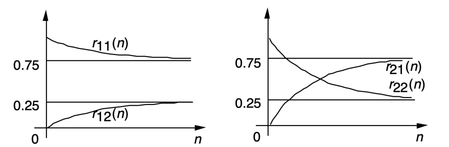
n-step transition prob.: $r_{ij}(n)$ for large n

- Convergence irrespective of the starting state

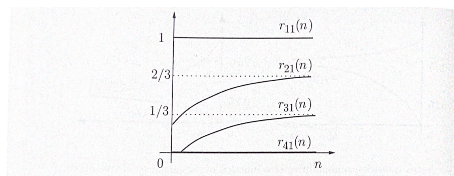


n-step transition probabilities as a function of the number n of transitions

	$r_{ij}(1)$ UpD	$r_{ij}(1)$ B	$r_{ij}(2)$		$r_{ij}(3)$		$r_{ij}(4)$		$r_{ij}(5)$	
UpD	0.8	0.2	.76	.24	.752	.248	.7504	.2496	.7501	.2499
B	0.6	0.4	.72	.28	.744	.256	.7488	.2512	.7498	.2502

Sequence of n-step transition probability matrices

- Convergence depending on the starting state



n-step transition probabilities into state 1

$r_{ij}(1)$:

	1	2	3	4
1	1.0	0	0	0
2	0.3	0.4	0.3	0
3	0	0.3	0.4	0.3
4	0	0	0	1.0

$r_{ij}(2)$:

1.0	0	0	0
.42	.25	.24	.09
.09	.24	.25	.42
0	0	0	1.0

$r_{ij}(3)$:

1.0	0	0	0
.50	.17	.17	.16
.16	.17	.17	.50
0	0	0	1.0

$r_{ij}(4)$:

1.0	0	0	0
.55	.12	.12	.21
.21	.12	.12	.55
0	0	0	1.0

....

$r_{ij}(\infty)$:

1.0	0	0	0
2/3	0	0	1/3
1/3	0	0	2/3
0	0	0	1.0

Sequence of transition probability matrices

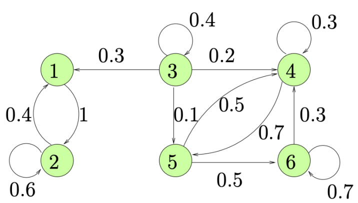
(Q) Under what conditions, convergence occurs? If so, how does it depend on the starting state?

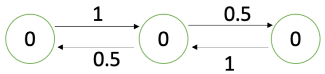
Steady-state behavior

- $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$?
- Convergence occurs, independent of the starting state, if:
  - **C1.** Only a single recurrent class
  - **C2.** such recurrent class is aperiodic

C1. For the case of multiple recurrent classes, one stays at the class including the starting state.

C2. Divergent behavior for periodic recurrent classes.





Balance Equation

- If $r_{ij}(n) \xrightarrow{n\to\infty} \pi_j$, for some $\pi_j \le 1$,

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj} \Longrightarrow \pi_j = \sum_{k=1}^{m} \pi_k p_{kj} \text{ (Balance equation)}$$

- Normalization equation

$$\sum_{i=1}^{m} \pi_i = 1$$

- Balance equation + Normalization equation $\Longrightarrow$ Finding the steady-state probabilities $\{\pi_i\}$.

Example

- A two-state MC with:

$p_{11} = 0.8, \quad p_{12} = 0.2,$

$p_{21} = 0.6, \quad p_{22} = 0.4.$

- Balance equation:

$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}$

$\pi_2 = \pi_2 p_{22} + \pi_1 p_{12}$

- Normalization equation: $\pi_1 + \pi_2 = 1$
- The stationary distribution is: $\pi_1 = 0.25$, $\pi_2 = 0.75$.

Stationary Distribution

- $\{\pi_j\}$ is also called a stationary distribution. Why?
- Distribution, because $\sum_{j=1}^{m} \pi_j = 1$.
- Stationary, because, if you choose the starting state according to $\{\pi_j\}$, then

$$\mathbb{P}(X_0 = j) = \pi_j, \quad j = 1, \dots, m \Longrightarrow \mathbb{P}(X_1 = j) = \sum_{k=1}^{m} \mathbb{P}(X_0 = k)p_{kj} = \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$$

  - Then, $\mathbb{P}(X_n = j) = \pi_j$, for all n and j.
  - If the initial state is chosen according to $\{\pi_j\}$, the state at any future time will have the same distribution (i.e., the distribution does not change over time).
- We say that "the limiting distribution is equal to to the stationary distribution"

Long-term Frequency Interpretation

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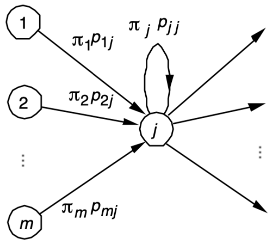
- π_j: the long-term expected fraction of time that the state is equal to j.
- Balance equation: $\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j$ means:
  - The expected frequency π_j of visits to j is equal to the sum of the expected frequencies $\pi_k p_{kj}$ of transitions that lead to j.



Roadmap

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Absorption Probability

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- Definition. A state k is absorbing, if $p_{kk} = 1$, and $p_{kj} = 0$ for all $j \neq k$.
  - states 1 and 6 are absorbing

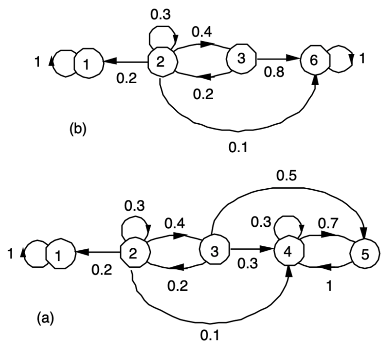
(Q) For a fixed absorbing state s, the probability a_i of reaching s, starting from a transient state i?

- Fix $s = 6$.

  $a_1 = 0, \quad a_6 = 1$

  $a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6$

  $a_3 = 0.2a_2 + 0.8a_6$



(Q) What if there are some non-absorbing recurrent state?

- Convert it into the one only with absorbing recurrent states (from (a) to (b)).

[0] The notation a_i should have dependence on s, but we omit it for simplicity.

Expected Time to Any Absorbing State

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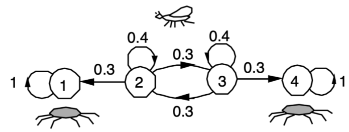
(Q) Starting from a transient state i, expected number of transitions μ_i until absorption to any absorbing state?



- Spider-fly example

  $\mu_1 = \mu_4 = 0 \quad$ (for recurrent states)

  $\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3, \quad \mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3 \quad$ (for transient states)

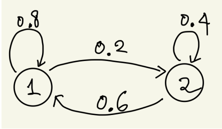
- For generalized description, please see the textbook (pp. 367).

Expected time to a particular recurrent state s

- Assume a single recurrent class

(Q) First passage time. Starting from a i, expected number of transitions t_i to reach s for the first time?

(Q) First recurrence time. Starting from a s, expected number of transitions $t_s^\star$ to reach s for the first time?



- Mean first passage time from 2 to 1

$$t_1 = 0$$
$$t_2 = 1 + p_{21} t_1 + p_{22} t_2 = 1 + 0.4 t_2 \implies t_2 = 5/3$$

- Mean first recurrence time from 1 to 1

$$t_1^\star = 1 + p_{11} t_1 + p_{12} t_2 = 1 + 0 + 0.2 \frac{5}{3} = \frac{4}{3}$$

- For generalized description, please see the textbook (pp. 368)

[0] The notation t_i should have the dependence on s, but we omit it for simplicity.

Questions?

Review Questions

1) Why do you think Markov chain (MC) is important?
2) What is the Markov property and its meaning? What's the key difference of MC from Bernoulli processes?
3) What are the limiting distribution and the stationary distribution of MCs?
4) How are you going to compute the stationary distribution, if you are given a transition probability matrix?
5) What are recurrent and transient states in MC?