# Lecture 6: Law of Large Numbers and Central Limit Theorem

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EE210: Probability and Introductory Random Processes
KAIST EE

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# Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

- Two most remarkable findings in probability theory

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- $X_1, X_2, \ldots, X_n$: i.i.d (independent and identically distributed) random variables
- $\mathbb{E}[X_i] = \mu$, $\text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

$$\boxed{S_n = X_1 + X_2 + \ldots + X_n}$$

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- Easy case: Sum of normal rvs = a normal rv, however, generally very challenging.
- Possible apporach. Take a certain scaling with respect to $n$ that corresponds to a new glass, and investigate the system for large $n$

- Consider the sample mean, and try to understand how $S_n$ behaves:

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- We call this law of large numbers (LLN).

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- Need to build up the new concept of convergence for the sequence of rvs.

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- A special case: when $Y = a$ for some constant $a$: $Y_n \xrightarrow{\text{in prob.}} a$
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# Example 1: Convergence in Probability

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- A sequence of iid rvs $X_n \sim \mathcal{U}[0,1]$, and let

$$Y_n = \min\{X_1, X_2, \ldots, X_n\}$$

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- Proof. For any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(|Y_n - 0| \geq \epsilon) &= \mathbb{P}(X_1 \geq \epsilon, \ldots, X_n \geq \epsilon) = \mathbb{P}(X_1 \geq \epsilon) \times \cdots \times \mathbb{P}(X_n \geq \epsilon) \\ &= (1-\epsilon)^n \xrightarrow{n \to \infty} 0\end{aligned}$$

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  $\mathbb{P}(|Y_n - 0| \geq \epsilon) = \mathbb{P}(Y \geq n\epsilon) = e^{-n\epsilon} \xrightarrow{n\to\infty} 0$

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- For any $\epsilon > 0$,

$$\mathbb{P}(|Y_n| \geq \epsilon) = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

- Thus, $Y_n$ converges to 0 in probability.

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

- Roughly, $M_n$ concetrates around $\mu$

# Weak Law of Large Numbers (WLLN)

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- The proof requires some knowledge about useful inequalities, which we will cover later.

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- For example, assume that a large number of identically distributed noises come to you. Then, you can roughly approximate it as ($n \times$ average noise)
- Provides an interpretation of expectations (as well as probabilities) in terms of a long sequence of identical independent experiments. For example, what is the probability of head of a coin? Toss 1000 times, and count the number of heads.

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- The answer is $\frac{1}{2}$

- Reshaping the equation:

$$\frac{\sqrt{n}}{\sigma} \times (M_n - \mu) =$$

# Central Limit Theorem: Start with Scaling (2)

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- Some deterministic number just like WLLG?

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$$\frac{\sqrt{n}}{\sigma} \times (M_n - \mu) = \sqrt{n}\left(\frac{S_n - n\mu}{\sigma n}\right) = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

- Let $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$. Then, $\mathbb{E}[Z_n] = 0$ and $\text{var}(Z_n) = 1$.
- $Z_n$ is well-centered with a variance irrespective of $n$.
- We expect that $Z_n$ converges to something meaningful as $n \to \infty$, but what?
- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.

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- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.
  - Need a new concept of convergence: "convergence in distribution"

- Consider a sequence of rvs $(Y_n)_{n=1,2,\ldots}$ and a rv $Y$.

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- Another type of convergence of rvs
- Comparison with convergence in probability?
  - Convergence in probability $\implies$ Convergence in distribution, but the reverse is not true.
  - The proof is beyond what this class covers, but it will be interesting to find an example that shows convergence in distribution, which is not convergence in probability.

# Example: in Distribution, but not in Probability

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$$\begin{aligned}\mathbb{P}(|X_n - X| \geq \epsilon) &= \mathbb{P}(|X_n - 1 + X_n| \geq \epsilon) = \mathbb{P}(|2X_n - 1| \geq \epsilon) \\ &= \mathbb{P}(1 \geq \epsilon) \qquad (\text{because } |2X_n - 1| = 1)\end{aligned}$$

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- We can find $\epsilon$ small enough so that the above does not go to zero.

- $S_n = X_1 + X_2 + \cdots + X_n, \qquad Z_n = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$

**Central Limit Theorem**

$Z_n$ convergens to $Z$ in distribution, where $Z \sim \mathcal{N}(0, 1)$.

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**Central Limit Theorem**

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- Very surprising!
- Irrespecitive of the distribution of $X_i$, $Z$ is normal.

- For simplicity, assume that $\mathbb{E}(X_i) = 0$ and $\mathrm{var}(X_i) = 1$, $i = 1, 2, \ldots, n$

# LLG vs. CLT: Different Scaling Glasses

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> Scaling $S_n$ by $1/n$, you go to a deterministic world.

- Central Limit Theorem

> Scaling $S_n$ by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each $X_i$.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \qquad \mathbb{P}(Z_n \le z) \xrightarrow{n\to\infty} \mathbb{P}(Z \le z),\ Z \sim \mathcal{N}(0,1)$$

[1]Only unique mode. A single maximum or minimum.

## Practical Use of CLT

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  - If $X_i$ resembles a normal rv more, smaller $n$ works: symmetry and unimodality[1]

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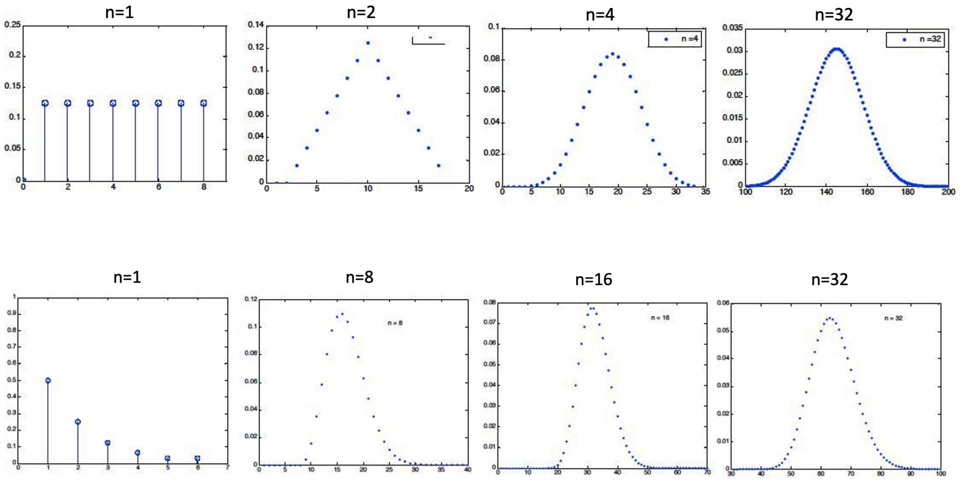
# CLT: Examples of Required $n$



# Examples of CLT (1)

$\mathbb{P}(S_n \leq a) \approx b$: Given two parameters, find the third

- Package weights $X_i$: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$ )

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- Load container with $n = 100$ packages

$$\mathbb{P}(S_{100} \geq 210) = \mathbb{P}\left[\frac{S_{100} - 100 \cdot 2}{2\sqrt{100}} \geq \frac{210 - 200}{20}\right] = \mathbb{P}(Z_{100} \geq 0.5)$$

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$$\approx \mathbb{P}(Z \geq 0.5) = 1 - \mathbb{P}(Z \leq 0.5) = 1 - \Phi(0.5)$$

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# Examples of CLT (2)

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- The value of $a$ such that $\Phi(\frac{a-200}{20}) = 0.95$? $\frac{a-200}{20} = 1.645$ and $a = 232.9$

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- The value of $n$ such that $\frac{210-2n}{2\sqrt{n}} = 1.645$? $n = 89$

# Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

# Markov Inequality

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If $X \geq 0$ and $a > 0$, then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.

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Proof. For any $a > 0$, define $Y_a$ as:

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Thus, $a \cdot \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$. $\square$

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- $X \sim \exp(1)$. Then, $\mathbb{E}[X] = 1/\lambda = 1$ and $\text{var}[X] = 1/\lambda^2 = 1$.

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- Both bounds are the ones that bound the probability of rare events.

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

**Weak law of large numbers**

$M_n$ converges to $\mu$ in probability.

Proof. For any given $\epsilon > 0$,

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We ask the same question, and try to answer it, using WLLN or CLT.

See how the answers becomes different.

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$$\mathbb{P}(|M_n - p| \geq \epsilon) = \qquad\qquad =$$

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- Compare: 50,000 from LLN vs. 9604 from CLT

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

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- If the context is clear, we omit $X$ and use just $M(s)$.

Ex1) Let $p_X(x)$ is given as:

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2 \\ 1/6, & \text{if } x = 3 \\ 1/3, & \text{if } x = 5 \end{cases}$$

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$$M(s) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{sy} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}+sy} dy$$

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KAIST EE

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- Question. MGF of $\mathcal{N}(\mu, \sigma^2)$?

# Useful Properties of MGF

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

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- Thus, $\text{var}(X) = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

### Inversion Property

The transform $M_X(s)$ associated with a random variable $X$ uniquely determines the CDF of $X$, assuming that $M_X(s)$ is finite for all $s$ in some interval $[-a, a]$, where $a$ is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own unique MGF.

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- Given the following MGF of rv $X$, what is the distribution of $X$?

$$M(s) = \frac{1}{4}e^{-s} + \frac{1}{2} + \frac{1}{8}e^{4s} + \frac{1}{8}e^{5s}$$

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- We can see that

$$p_X(-1) = \frac{1}{4},\ p_X(0) = \frac{1}{2},\ p_X(4) = \frac{1}{8},\ p_X(5) = \frac{1}{8}$$

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- $X$ is a geometric rv with parameter $p$

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- Proof.

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$$= \qquad\qquad =$$

# Back to CLT Proof (1)

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$
- $Z_n = \dfrac{S_n}{\sqrt{n}} = \dfrac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$
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- For simplicity, let $M(\cdot) = M_{X_1}(\cdot)$

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$\square$

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- If we apply l'hopital's rule twice (please check), we get

$$\lim_{y\to 0} \frac{\log M(ys)}{y^2} = \frac{s^2}{2}$$

$\square$

Questions?

# Review Questions

1) What's the practical value of LLN and CLT?
2) Explain LLN and CLT from the scaling perspective.
3) Why are LLN and CLT great?
4) Why do we need different concepts of convergence for random variables?
5) Explain what is convergence in probability.
6) Explain what is convergence in distribution.
7) Why is MGF useful?