

Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

September 23, 2021

- (1) Probabilistic Model
 - Mathematical description of uncertain situations
- (2) Sample Space, Event, Probability Law
 - Elements of probability theory
- (3) Probability Axioms
 - 3 axioms for the completeness of a theory

- (1) Probabilistic Model
- (2) Sample Space, Event, Probability Law
- (3) Probability Axioms

Modeling: Understand reality with a simple (mathematical) model

- Experiment
 - Flip two coins
- Observation: a random outcome
 - for example, (H, H)
- All outcomes
 - $\{(H, H), (H, T), (T, H), (T, T)\}$

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- **Our goal:** Build up a **probabilistic model** for an experiment with random outcomes
 - **Probabilistic model?**
 - Assign a number to each outcome or a set of outcomes
 - Mathematical description of an uncertain situation
 - Which model is good or bad?

Goal: Build up a probabilistic model. Hmm... How?

The first thing: What are the *elements* of a probabilistic model?

Elements of Probabilistic Model

1. All outcomes of my interest: Sample Space Ω
2. Assigned numbers to each outcome of Ω : Probability Law $\mathbb{P}(\cdot)$

Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

- (1) Probabilistic Model
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1. Sample Space Ω

The set of all outcomes of my interest

- (1) Mutually exclusive
- (2) Collectively exhaustive
- (3) At the right granularity
(not too concrete, not too abstract)

- 1. Toss a coin. What about this?
 $\Omega = \{H, T, HT\}$
- 2. Toss a coin. What about this? $\Omega = \{H\}$
- 3. (a) Just figuring out prob. of H or T.
 $\implies \Omega = \{H, T\}$

(b) The impact of the weather (rain or no rain) on the coin's behavior.

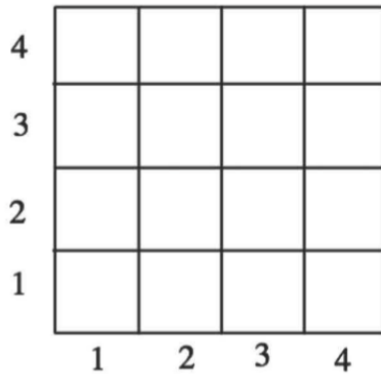
 $\implies \Omega = \{(H, R), (T, R), (H, NR), (T, NR)\},$

R(Rain), NR(No Rain).

Examples: Sample Space Ω

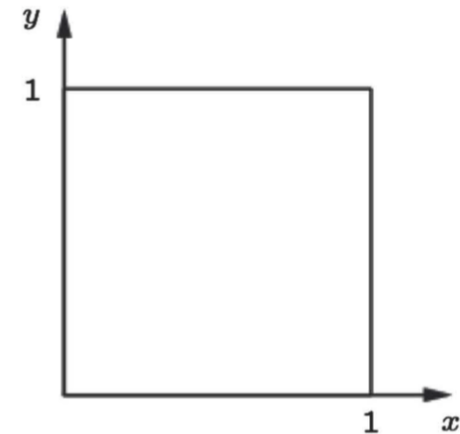
- *Discrete case:* Two rolls of a tetrahedral die

- $\Omega = \{(1, 1), (1, 2), \dots, (4, 4)\}$



- *Continuous case:* Dropping a needle in a plain

- $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$



- Assign numbers to what? Each outcome?
- What is the probability of dropping a needle at $(0.5, 0.5)$ over the 1×1 plane?
- Assign numbers to each subset of Ω
- a subset of Ω : an event
- $\mathbb{P}(A)$: Probability of an event A .
 - This is where probability meets set theory.
- Roll a dice. What is the probability of odd numbers?

$\mathbb{P}(\{1, 3, 5\})$, where $\{1, 3, 5\} \subset \Omega$ is an event.

- (1) Probabilistic Model
- (2) Sample Space, Event, Probability Law
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How should we construct $\mathbb{P}(\cdot)$?

- Need to construct $\mathbb{P}(\cdot)$ that naturally satisfies the intention of a probability theory designer just like you. What about the followings as starting points?
 - $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$
 - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
 - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
 - For two disjoint¹ events A and B , $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$
 - $\mathbb{P}(\Omega) = 1$ (Why not $\mathbb{P}(\Omega) = 10$?)
 - $\mathbb{P}(\emptyset) = 0$
 - If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$
 - many others

¹Their intersection is empty.

- Surprisingly, we need just the following three rules (called **axioms**):

Probability Axioms: Version 1

A1. **Nonnegativity**: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. **Normalization**: $\mathbb{P}(\Omega) = 1$

A3. **(Finite) additivity**: For two disjoint events A and B , $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

- No other things are necessary, and we can prove all other things from the above axioms.
- Note that coming up with the above axioms is far from trivial.

A1: Nonnegativity, A2: Normalization, A3: Finite additivity

Prove the following properties using the axioms:

VIDEO PAUSE

1. For any event A , $\mathbb{P}(A) \leq 1$

$$1 \stackrel{A2}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) \stackrel{A3}{=} \mathbb{P}(A) + \mathbb{P}(A^c) \implies \mathbb{P}(A) = 1 - \mathbb{P}(A^c) \stackrel{A1}{\leq} 1$$

2. $\mathbb{P}(\emptyset) = 0$

$$\mathbb{P}(\Omega \cup \emptyset) \stackrel{A3}{=} \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) \stackrel{A2}{=} 1 + \mathbb{P}(\emptyset) \xrightarrow{\text{from 1.}} \mathbb{P}(\emptyset) = 0$$

3. If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$

$$\mathbb{P}(B) \stackrel{A3}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \stackrel{A1}{\geq} \mathbb{P}(A)$$

1. Specify the sample space
2. Specify a probability law
 - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4, \mathbb{P}(\{T\}) = 3/4,$
3. probability of head or tail
4. $1/4, 3/4$

- $\Omega = \{1, 2, 3, \dots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \dots$
- Is the above probability law legitimate? seems OK

$$\mathbb{P}(\Omega) = \frac{1}{2} + \frac{1}{2^2} + \dots = \frac{1/2}{1 - 1/2} = 1$$

- $\mathbb{P}(\text{even numbers})?$

$$\begin{aligned}\mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \dots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots = 1/3\end{aligned}$$

- Is the above right? If not, why?
 - Wrong: **Finite** additivity axiom does not allow this.

Probability Axioms: Version 1 2

A1. Nonnegativity: $\mathbb{P}(A) \geq 0$ for any event $A \subset \Omega$

A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. (Finite) additivity: For two disjoint events A and B , $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

A4. Countable additivity: If A_1, A_2, A_3, \dots is an infinite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$.

- A narrow view: A branch of math
 - axioms \rightarrow theorems
 - Mathematicians work very hard to find the smallest set of necessary axioms (just like atoms in physics)
- Frequencies: $\mathbb{P}(H) = 1/2$
 - Understanding an uncertain situation: fractions of successes out of many experiments
- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$

Anyway, we believe that probabilistic reasoning is very helpful to understand the world with many uncertain situations.

Questions?

You build up the very basics of a probabilistic model.

What else do we need to build up?

- 1) Explain what a probabilistic model is and why we need it.
- 2) What is the mathematical definition of event?
- 3) What are the key elements of the probabilistic model?
- 4) List up the probability axioms and explain them. Are you going to choose the same axioms to build up the probability theory?
- 5) Why do we need countable additivity in the probability axioms?