

## Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes  
KAIST EE

October 14, 2021

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) Bayes' rule for RVs

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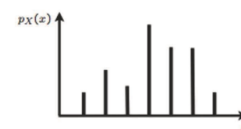
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- Many cases when random variables have “continuous values”, e.g., velocity of a car

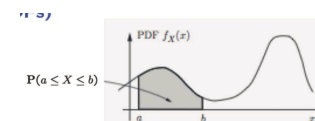
A rv  $X$  is **continuous** if  $\exists$  a function  $f_X$ , called **probability density function (PDF)**, s.t.

$$\mathbb{P}(X \in B) = \int_B f_X(x) dx, \quad \text{every subset } B \in \mathbb{R}$$

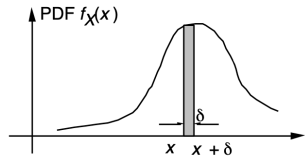
- All of the concepts and methods (expectation, PMFs, and conditioning) for discrete rvs have continuous counterparts



- $\mathbb{P}(a \leq X \leq b) = \sum_{x: a \leq x \leq b} p_X(x)$
- $p_X(x) \geq 0, \sum_x p_X(x) = 1$

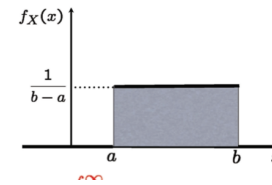
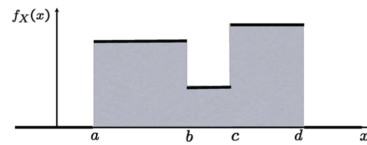
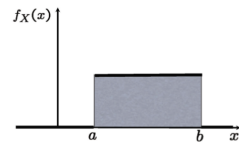


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x) dx = 1$



- $\mathbb{P}(a \leq X \leq a + \delta) \approx f_X(a) \cdot \delta$
- $\mathbb{P}(X = a) = 0$

## Examples



- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{b+a}{2}$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a} \frac{b^3 - a^3}{3} = \frac{a^2 + ab + b^2}{3}$
- $\text{var}[X] = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

L4(1)

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## Roadmap

## Cumulative Distribution Function (CDF)

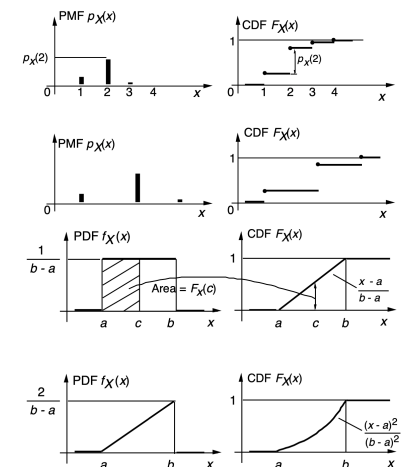
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- Discrete: PMF, Continuous: PDF
- Can we describe all rvs with a single mathematical concept?

$$F_X(x) = \mathbb{P}(X \leq x) =$$

$$\begin{cases} \sum_{k \leq x} p_X(k), & \text{discrete} \\ \int_{-\infty}^x f_X(t) dt, & \text{continuous} \end{cases}$$

- always well defined, because we can always compute the probability for the event  $\{X \leq x\}$
- CCDF (Complementary CDF):  $\mathbb{P}(X > x)$



L4(2)

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L4(2)

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- Non-decreasing
- $F_X(x)$  tends to 1, as  $x \rightarrow \infty$  and  $F_X(x)$  tends to 0, as  $x \rightarrow -\infty$
- If  $X$  is discrete,
  - $F_X(x)$  is a piecewise constant function of  $x$ .
  - $p_X(k) = F_X(k) - F_X(k-1)$
- If  $X$  is continuous
  - $F_X(x)$  is a continuous function of  $x$ .
  - $F_X(x) = \int_{-\infty}^x f_X(t)dt$  and  $f_X(x) = \frac{dF_X}{dx}(x)$

L4(2)

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- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$ , and  $X_i \in \{1, 2, \dots, 10\}$  uniformly at random
- **Question.**  $p_X(x)$ ?
- Approach 1:  $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$ ?

- Approach 2

$$F_X(x) = \mathbb{P}(\max\{X_1, X_2, X_3\} \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x, X_3 \leq x) \\ = \mathbb{P}(X_1 \leq x) \cdot \mathbb{P}(X_2 \leq x) \cdot \mathbb{P}(X_3 \leq x) = \left(\frac{x}{10}\right)^3$$

Thus,

$$p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \dots, 10$$

L4(2)

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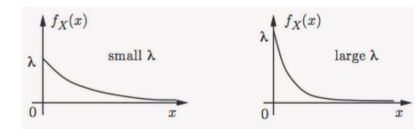
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L4(3)

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- A rv  $X$  is called **exponential with  $\lambda$** , if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



- CDF  $F_X(x) = \int_0^x \lambda e^{-\lambda s} ds = 1 - e^{-\lambda x}$
- CCDF  $\mathbb{P}(X > x) = e^{-\lambda x}$
- **(Check)**  $\mathbb{E}[X] = 1/\lambda$ ,  $\mathbb{E}[X^2] = 2/\lambda^2$ ,  $\text{var}[X] = 1/\lambda^2$

L4(3)

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- $\mathbb{E}(X) = 1/\lambda$ . Use **integration by parts**:  $\int u dv = uv - \int v du$

$$\int_0^{\infty} x \lambda e^{-\lambda x} dx = (-x e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx = 0 + \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

L4(3)

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- $\mathbb{P}(X > x) = e^{-\lambda x}$

- Appropriate for modeling a waiting time until an incident of interest takes place

- $\mathbb{P}(X > x)$ : exponentially decays
- message arriving at a computer, some equipment breaking down, a light bulb burning out, etc

- (Q) What is the discrete rv which models a waiting time? **Geometric**

- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

L4(3)

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## Example

## Geometric vs. Exponential (1)

- A very small meteorite first lands anywhere in Korea



- Time of landing is modeled as an exponential rv with mean 10 days
- The current time is midnight. What is the probability that a meteorite first lands some time between 6 a.m. and 6 p.m. of the first day?

VIDEO PAUSE

- (Solution)

- $\mathbb{E}(X) = 1/\lambda = 10$ . Thus,  $\lambda = \frac{1}{10}$ .

- 6 a.m. from midnight = 1/4 day, 6 p.m. from midnight = 3/4 day

$$\mathbb{P}(1/4 \leq X \leq 3/4) = \mathbb{P}(X \geq 1/4) - \mathbb{P}(X \geq 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

L4(3)

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- Models a system evolution over time: Continuous time vs. Discrete time.

- **Example**. Customer arrivals at my shop
- **Modeling 1**: Every 30 minute I record the number of customers for each 30-min window
- **Modeling 2**: I record the exact time of each customer's arrival
- In modeling 1, every 10 minute? every 1 minute? every 1 sec? every 0.0000001 sec?

- In many cases, continuous case is some type of **limit** of its corresponding discrete case.

- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

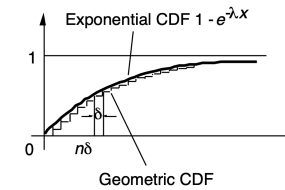
L4(3)

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- 'slot' is one unit time, e.g., 1 hour, 30 mins, 1 min, 10 sec, etc.
- Continuous system = Discrete system with
  - infinitely many slots whose duration is infinitely small.
  - success probability  $p$  over one slot decreases to 0 in the limit
- Given  $X^{exp} \sim \exp(\lambda)$ , let us construct a geometric RV  $X_\delta^{geo}$ 
  - Set the length of a slot to be  $\delta$ , which is a parameter.
  - Set the success probability  $p_\delta$  over a slot to be  $p_\delta = 1 - e^{-\lambda\delta}$  (this looks magical, whose secret will be uncovered soon)
  - $\mathbb{P}(X_\delta^{geo} \leq n) = 1 - (1 - p_\delta)^n = 1 - e^{-\lambda\delta n}$

L4(3)

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- Note that  $\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda x}$ . Then, when  $x = n\delta$ ,  $n = 1, 2, \dots$

$$\mathbb{P}(X^{exp} \leq x) = 1 - e^{-\lambda\delta n} = \mathbb{P}(X_\delta^{geo} \leq n)$$

- If we choose sufficiently small  $\delta$ , the slot length  $\downarrow$  and  $p_\delta \downarrow$

$$\mathbb{P}(X_\delta^{geo} \leq n) \xrightarrow{\delta \rightarrow 0} \mathbb{P}(X^{exp} \leq x), \quad x = n\delta$$

L4(3)

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L4(4)

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- Standard Normal  $\mathcal{N}(0, 1)$

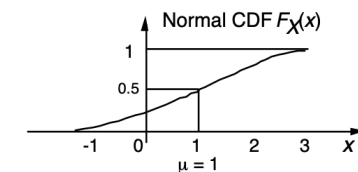
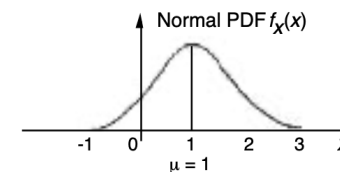
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

- $\mathbb{E}[X] = 0$
- $\text{var}[X] = 1$

- General Normal  $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\text{var}[X] = \sigma^2$



L4(4)

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- PDF's normalization property:  $\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = 1$ 
  - A little bit boring :-). See Problem 14 at pp 189.
- Expectation
  - $f_X(x)$  is symmetric in terms of  $x = \mu$ . Thus, we should have  $\mathbb{E}(X) = \mu$ .
- Variance

$$\begin{aligned} \text{var}(X) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \stackrel{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \left( -ye^{-y^2/2} \right) \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \sigma^2 \end{aligned}$$

$$\int u dv = uv - \int v du: u = y \text{ and } dv = ye^{-y^2/2} \rightarrow du = dy \text{ and } v = -e^{-y^2/2}$$

L4(4)

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- Linear transformation preserves normality (we will verify this in Lecture 5)

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for  $a \neq 0$  and  $b$ ,  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

- Thus, every normal rv can be **standardized**:  
If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Y = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$
- Thus, we can make the **table** which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

L4(4)

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## Example

- Annual snowfall  $X$  is modeled as  $\mathcal{N}(60, 20^2)$ . What is the probability that this year's snowfall is at least 80 inches?
- $Y = \frac{X-60}{20}$ .

$$\begin{aligned} \mathbb{P}(X \geq 80) &= \mathbb{P}(Y \geq \frac{80-60}{20}) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587 \end{aligned}$$

|     | .00   | .01   | .02   | .03   | .04   | .05   | .06   | .07   | .08   | .09   |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |

L4(4)

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## Normal RVs: Why Important?

- Central limit theorem
  - One of the most remarkable findings in the probability theory
  - Sum of **any** random variables  $\approx$  Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

<sup>0</sup>Central limit theorem: 중심극한정리

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Two continuous rvs are **jointly continuous** if a non-negative function  $f_{X,Y}(x, y)$  (called joint PDF) satisfies: for **every subset**  $B$  of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x, y) dx dy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}[(X, Y) \in B] = \iint_{(x,y) \in B} f_{X,Y}(x, y) dx dy$$

Our particular interest:  $B = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$

2. The **marginal** PDFs of  $X$  and  $Y$  are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

3. The **joint CDF** is defined by  $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ , and determines the joint PDF as:

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x, y)$$

4. A function  $g(X, Y)$  of  $X$  and  $Y$  defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$$

- \* Conditional PDF, given an event  $A$

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$   
 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$
- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$   
 $\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x) dx$
- $\int f_{X|A}(x) dx = 1$

- \* Conditional PDF, given  $\{X \in C\}$

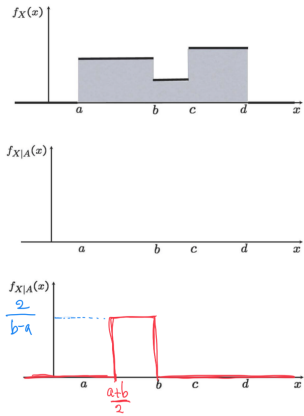
$$f_{X|\{X \in C\}}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | X \in C)$$

$$f_{X|\{X \in C\}}(x) = \begin{cases} 0, & \text{if } x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X \in C)}, & \text{if } x \in C \end{cases}$$

(Q) In the discrete, we consider the event  $\{X = x\}$ , not  $\{X \in B\}$ . Why?

**Notation:**  $A$  is an event, but  $B$  and  $C$  is a subset that includes the possible values which can be taken by the rv  $X$ . Sorry for the confusion, if any.

$$A = \left\{ \frac{a+b}{2} \leq X \leq b \right\}$$



L4(5)

$$\begin{aligned} \mathbb{E}[X] &= \int x f_X(x) dx \\ \mathbb{E}[X|A] &= \int x f_{X|A}(x) dx \\ \mathbb{E}[g(X)] &= \int g(x) f_X(x) dx \\ \mathbb{E}[g(X)|A] &= \int g(x) f_{X|A}(x) dx \end{aligned}$$

$$\mathbb{E}[X|A] = \int_{(a+b)/2}^b x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^b x^2 \frac{2}{b-a} dx =$$

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- **Remember:** Exponential rv is a continuous counterpart of geometric rv.
- Thus, expected to be memoryless. Remember the definition?

**Definition.** A random variable  $X$  is called memoryless if, for any  $n, m \geq 0$ ,

$$\mathbb{P}(X > n+m | X > m) = \mathbb{P}(X > n)$$

- **Proof.** Note that the exponential rv's CCDF  $\mathbb{P}(X > x) = e^{-\lambda x}$ . Then,

$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)} = \frac{e^{-\lambda(n+m)}}{e^{-\lambda m}} = e^{-\lambda n} = \mathbb{P}(X > n)$$

L4(5)

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Partition of  $\Omega$  into  $A_1, A_2, A_3, \dots$

\* Discrete case

#### Total Probability Theorem

$$\begin{aligned} p_X(x) &= \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) \\ &= \sum_i \mathbb{P}(A_i) p_{X|A_i}(x) \end{aligned}$$

#### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

L4(5)

\* Continuous case

#### Total Probability Theorem

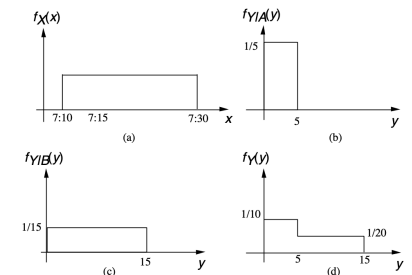
$$f_X(x) = \sum_i \mathbb{P}(A_i) f_{X|A_i}(x)$$

#### Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

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- The train's arrival every quarter hour (0, 15min, 30min, 45min).
- Your arrival  $\sim \mathcal{U}(7:10, 7:30)$  am.
- What is the PDF of waiting time for the first train?
- $X$  : your arrival time,  $Y$  : waiting time.
- The value of  $X$  makes a different waiting time. So, consider two events:  
 $A = \{7:10 \leq X \leq 7:15\}$   
 $B = \{7:15 \leq X \leq 7:30\}$



VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A) f_{Y|A}(y) + \mathbb{P}(B) f_{Y|B}(y)$$

$$f_Y(y) = \frac{1}{4} \frac{1}{5} + \frac{3}{4} \frac{1}{15} = \frac{1}{10}, \quad \text{for } 0 \leq y \leq 5$$

$$f_Y(y) = \frac{1}{4} 0 + \frac{3}{4} \frac{1}{15} = \frac{1}{20}, \quad \text{for } 5 < y \leq 15$$

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- $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- Similarly, for  $f_Y(y) > 0$ ,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Remember: For a fixed event  $A$ ,  $\mathbb{P}(\cdot|A)$  is a legitimate probability law.
- Similarly, For a fixed  $y$ ,  $f_{X|Y}(x|y)$  is a legitimate PDF, since

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y) dx}{f_Y(y)} = 1$$

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- **Multiplication rule.**

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

- **Total prob./exp. theorem.**

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y) dy$$

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbb{E}[X|Y=y] dy$$

- **Independence**

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

(Prob 21 at pp. 191)

- Break a stick of length  $l$  twice
  - first break at  $Y \sim \mathcal{U}[0, l]$
  - second break at  $X \sim \mathcal{U}[0, Y]$

(a) joint PDF  $f_{X,Y}(x,y)$ ?

$$f_Y(y) = \frac{1}{l}, \quad 0 \leq y \leq l$$

$$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 \leq x \leq y$$

Using  $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y)$ ,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{l} \cdot \frac{1}{y}, & 0 \leq x \leq y \leq l, \\ 0, & \text{otherwise} \end{cases}$$

 ${}^0\mathcal{U}[a, b]$ : continuous uniform random variable over the interval  $[a, b]$ 

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(b) marginal PDF  $f_X(x)$ ?

$$f_X(x) = \int f_{X,Y}(x,y) dy = \int_x^l \frac{1}{ly} dy \\ = \frac{1}{l} \ln(l/x), \quad 0 \leq x \leq l$$

(c) Evaluate  $\mathbb{E}(X)$ , using  $f_X(x)$ 

$$\mathbb{E}(X) = \int_0^l x f_X(x) dx = \int_0^l \frac{x}{l} \ln(l/x) dx \\ = \frac{l}{4}$$

(d) Evaluate  $\mathbb{E}(X)$ , using  $X = Y \cdot (X/Y)$ 

If  $Y \perp\!\!\!\perp X/Y$ , it becomes easy, but true?  
Yes, because whatever  $Y$  is, the fraction  $X/Y$  does not depend on it.

$$\mathbb{E}(X) = \mathbb{E}(Y) \mathbb{E}(X/Y) = \frac{l}{2} \cdot \frac{1}{2} = \frac{l}{4}$$

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(e) Evaluate  $\mathbb{E}(X)$ , using TET

$$0\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y) \mathbb{E}[X|Y=y] dy \\ = \int_0^l \frac{1}{l} \mathbb{E}[X|Y=y] dy = \int_0^l \frac{1}{l} \frac{y}{2} dy = \frac{l}{4}$$

- **Message.** There are many ways to reach our goal. Of crucial importance is how to find the best way!

- (1) Continuous Random Variable and PDF (Probability Density Function)
- (2) CDF (Cumulative Distribution Function)
- (3) Exponential RVs
- (4) Gaussian (Normal) RVs
- (5) Continuous RVs: Joint, Conditioning, and Independence
- (6) **Bayes' rule for RVs**

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- $X$ : state/cause/original value  $\rightarrow Y$ : result/resulting action/noisy measurement
- Given:  $\mathbb{P}(X)$  and  $\mathbb{P}(Y|X)$  (cause  $\rightarrow$  result)
- Inference:  $\mathbb{P}(X|Y)$ ?

$$\begin{aligned}
 p_{X,Y}(x,y) &= p_X(x)p_{Y|X}(y|x) \\
 &= p_Y(y)p_{X|Y}(x|y) \\
 p_{X|Y}(x|y) &= \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)} \\
 p_Y(y) &= \sum_{x'} p_X(x')p_{Y|X}(y|x')
 \end{aligned}$$

$$\begin{aligned}
 f_{X,Y}(x,y) &= f_X(x)f_{Y|X}(y|x) \\
 &= f_Y(y)f_{X|Y}(x|y) \\
 f_{X|Y}(x|y) &= \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)} \\
 f_Y(y) &= \int f_X(x')f_{Y|X}(y|x')dx'
 \end{aligned}$$

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- A light bulb  $Y \sim \exp(\lambda)$ . However, there are some quality control problems. So, the parameter  $\lambda$  of  $Y$  is actually a random variable, denoted by  $\Lambda$ , which is  $\Lambda \sim \mathcal{U}[1, 3/2]$ . We test a light bulb and record its lifetime.
- **Question.** What can we say about the underlying parameter  $\lambda$ ? In other words, what is  $f_{\Lambda|Y}(\lambda|y)$ ?
- $f_{\Lambda}(\lambda) = 2$  for  $1 \leq \lambda \leq 3/2$  and  $f_{Y|\Lambda}(y|\lambda) = \text{pdf of } \exp(\lambda)$ . Then, the inference about the parameter given the lifetime of a light bulb is:

$$f_{\Lambda|Y}(\lambda|y) = \frac{f_{\Lambda}(\lambda)f_{Y|\Lambda}(y|\lambda)}{\int_{-\infty}^{\infty} f_{\Lambda}(t)f_{Y|\Lambda}(y|t)dt}$$

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- $X$ : **parameter**  $\rightarrow Y$ : result of **my model**
- Given:  $\mathbb{P}(X)$  and  $\mathbb{P}(Y|X)$  (parameter  $\rightarrow$  model)
- Inference:  $\mathbb{P}(X|Y)$ ? Probabilistic feature of the parameter given the result of the model?

Example.

1. Light bulb's lifetime  $Y \sim \exp(\lambda)$ . Given the lifetime  $y$ , the modified belief about  $\lambda$ ?
2. Romeo and Juliet start dating, but Romeo will be late by a random variable  $Y \sim \mathcal{U}[0, \theta]$ . Given the time of being late  $y$ , the modified belief about  $\theta$ ?

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$K$ : discrete,  $Y$ : continuous

- Inference of  $K$  given  $Y$

$$\begin{aligned}
 p_{K|Y}(k|y) &= \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)} \\
 f_Y(y) &= \sum_{k'} p_K(k')f_{Y|K}(y|k')
 \end{aligned}$$

- $f_{Y|K}(y|k) = f_{Y|A}(y)$ , where  $A = \{K = k\}$

- Inference of  $Y$  given  $K$

$$\begin{aligned}
 f_{Y|K}(y|k) &= \frac{f_Y(y)p_{K|Y}(k|y)}{p_K(k)} \\
 p_K(k) &= \int f_Y(y')p_{K|Y}(k|y')dy'
 \end{aligned}$$

- Wait!  $p_{K|Y}(k|y)$ ? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K = k, Y = y)}{\mathbb{P}(Y = y)} = \frac{0}{0}$$

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- For small  $\delta$  (in other words, taking the limit as  $\delta \rightarrow 0$ ).

Let  $A = \{K = k\}$ .

$$\begin{aligned} p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \leq Y \leq y + \delta) \\ &= \frac{\mathbb{P}(A)\mathbb{P}(y \leq Y \leq y + \delta|A)}{\mathbb{P}(y \leq Y \leq y + \delta)} \\ &\approx \frac{\mathbb{P}(A)f_{Y|A}(y)\delta}{f_Y(y)\delta} \\ &= \frac{\mathbb{P}(A)f_{Y|A}(y)}{f_Y(y)} \end{aligned}$$

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Inference of discrete  $K$  given continuous  $Y$ :

$$p_{K|Y}(k|y) = \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)}, \quad f_Y(y) = \sum_{k'} p_K(k')f_{Y|K}(y|k')$$

- $K$ : -1, +1, original signal, equally likely.  $p_K(1) = 1/2, p_K(-1) = 1/2$ .
- $Y$ : measured signal with Gaussian noise,  $Y = K + W$ ,  $W \sim \mathcal{N}(0, 1)$
- Your received signal = 0.7. What's your guess about the original signal? **+1**
- Your received signal = -0.2. What's your guess about the original signal? **-1**
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

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## Example: Signal Detection (2)

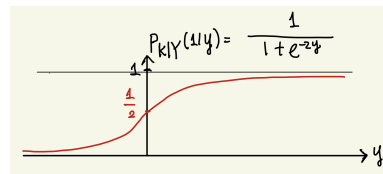
- $Y|\{K = 1\} \sim \mathcal{N}(1, 1)$  and  $Y|\{K = -1\} \sim \mathcal{N}(-1, 1)$ .  
(Remind: linear transformation preserves normality.)

$$\begin{aligned} f_{Y|K}(y|k) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}, \quad k = 1, -1 \\ f_Y(y) &= \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \quad (\text{from TPT}) \end{aligned}$$

- Probability that  $K = 1$ , given  $Y = y$ ? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1 + e^{-2y}}$$

- If  $y > 0$ , the inference probability for  $K = 1$  exceeds  $\frac{1}{2}$ . So, original signal = 1.
- Similarly, compute  $p_{K|Y}(-1|y)$  and then do the inference



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## Example: Buffon's Needle (1)

- A surface is ruled with parallel lines with distance  $d$  from each other.
- We throw a needle of length  $l$  on the surface at random.
- Question.** What is the probability that the needle will intersect one of the lines?

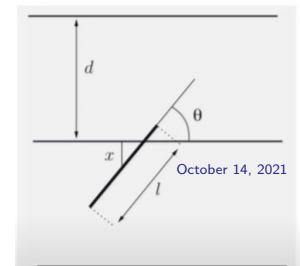
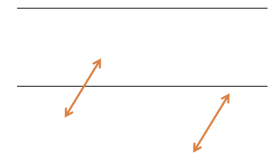
- Assume  $l < d$ . The needle intersects two lines (x).
- When does and does not the needle intersect?
- What are the random quantities to model?

R1. How far is the needle from the nearest parallel line?

**X**: vertical distance from the midpoint of the needle to the nearest of the parallel lines

R2. How does the needle meet the nearest parallel line?

**Θ**: acute angle formed by the axis of the needle and the parallel lines



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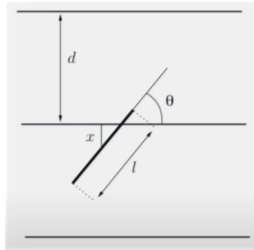
- A random vector  $(X, \Theta)$  with a uniform joint PDF over  $\{(x, \theta) | 0 \leq x \leq d/2, 0 \leq \theta \leq \pi/2\}$  with

$$f_{X,\Theta}(x, \theta) = \begin{cases} 4/(\pi d), & \text{if } x \in [0, d/2] \text{ and } \theta \in [0, \pi/2], \\ 0, & \text{otherwise} \end{cases}$$

- When does the needle intersect a line?  $X \leq \frac{l}{2} \sin \Theta$
- The probability of intersection

$$\begin{aligned} \mathbb{P}(X \leq \frac{l}{2} \sin \Theta) &= \iint_{x \leq (l/2) \sin \theta} f_{X,\Theta}(x, \theta) dx d\theta \\ &= \frac{4}{\pi d} \int_0^{\pi/2} \int_0^{(l/2) \sin \theta} dx d\theta = \frac{4}{\pi d} \int_0^{\pi/2} \frac{l}{2} \sin \theta d\theta = \frac{2l}{\pi d} (-\cos \theta) \Big|_0^{\pi/2} = \frac{2l}{\pi d} \end{aligned}$$

- In history, a method for the experimental evaluation of  $\pi$ .



Questions?

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- 1) What do we mean by "continuous" in continuous random variables?
- 2) Explain PDF and CDF. Why do we need CDF?
- 3) What are joint/marginal/conditional PDFs?
- 4) Explain the relationship between Geometric rvs and Exponential rvs.
- 5) Explain how normality is preserved under linear transformation for Normal (Gaussian) rvs.
- 6) Explain how we can use Bayes' rule for parameter learning.
- 7) Explain the version of Bayes' rule for continuous and mixed random variables.

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