Lecture 7: Random Processes, Part I

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

June 12, 2021

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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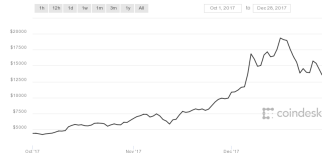
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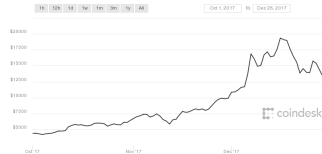
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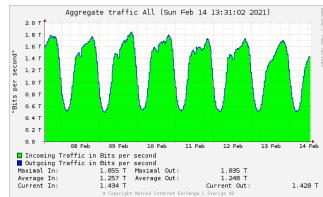
(b) Internet traffic traces

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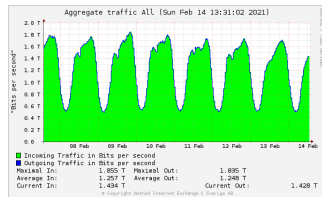 - Sequence of traffic loads in Internet

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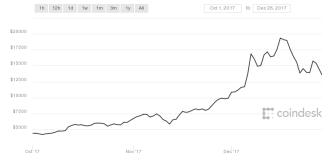 - Sequence of traffic loads in Internet
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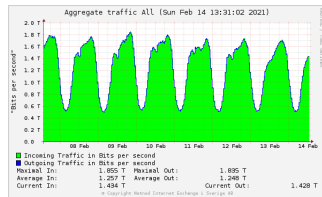
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- The values that X_t (or $X(t)$) can take: discrete or continuous

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Random Processes: Basics (2)

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  - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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Random Processes: Our Interest

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  - Other interesting questions, depending on the target random process

- Types of time and value

(a) continuous time, continuous value

(b) continuous time, discrete value

(c) discrete time, continuous value

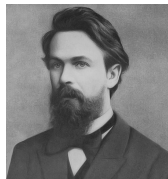(d) discrete time, discrete value
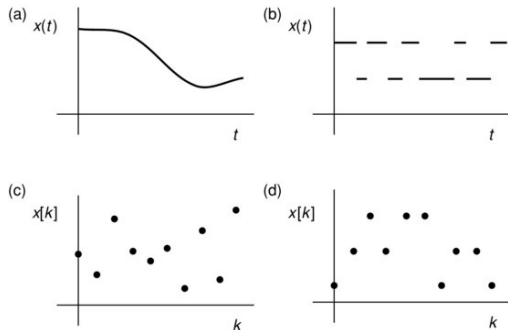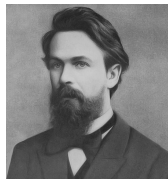
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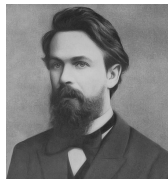- Bernoulli Process (BP)

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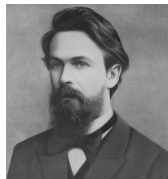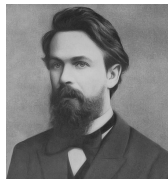- Bernoulli Process (BP)

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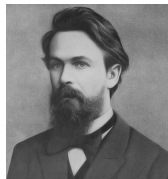- Bernoulli Process (BP)

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- X_i depends on X_{i-1}, but $\perp\!\!\!\perp \{X_{i-2}, X_{i-2}, \ldots, X_1\}$

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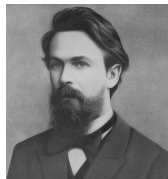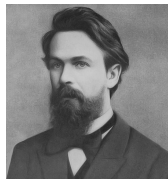- Bernoulli Process (BP)

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- "today" independent of "past"

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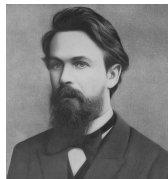

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$$| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 |$$

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- Discrete time, discrete value
- One of the simplest random processes
- A type of "arrival" process

Please pause the video and write down the questions that you want
to ask about Bernoulli process. <mark>VIDEO PAUSE</mark>

Q1.

Q2.

Q3.

Q4.

Q5.

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- This will hold for any n consecutive slots.

(Q2) # of slots T_1 until the first arrival?

(Q1) # of arrivals in the first n slots?

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- $S_n \sim \text{Binomial}(n, p)$

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- This will hold for any n consecutive slots.

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- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots $\implies$ the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6$?

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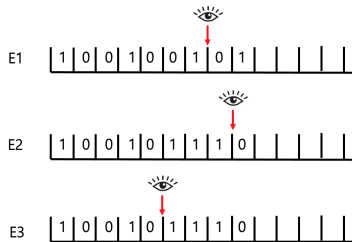(Q4) After time $n = 6$, I start to look at the process $(X_n)_{n=6}^{\infty}$?

- $(X_1, \ldots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^{\infty}$
- Fresh-start after a deterministic time n (doesn't matter what happened until $n = 5$.
- If you watch the on-going Bernoulli process(p) from some time n, you still see the same Bernoulli process(p).

(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

- This means that the time I start to look at the process is a random variable.

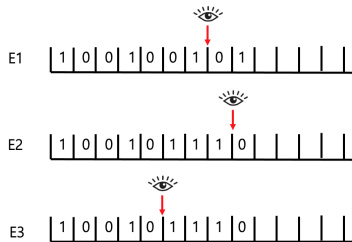(Q5) The process $(X_N, X_{N+1}, X_{N+2}, \ldots)$? Fresh-start even after random N?

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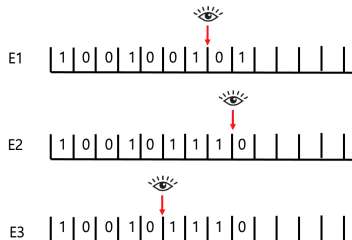
- Examples of N

E1. Time of 3rd arrival

E1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | | | | |

E2 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | | | | |

E3 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | | | | |

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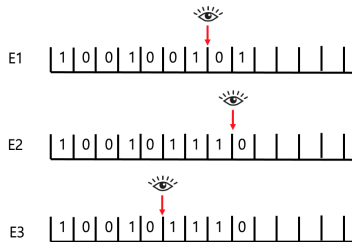**E2.** First time when 3 consecutive arrivals have been observed

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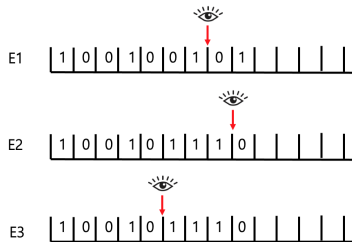**E3.** Time just before 3 consecutive arrivals

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- Difference of N from n
  - The time when I watch the on-going Bernoulli process is random.
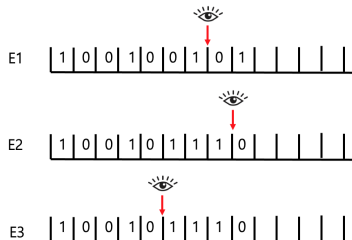 - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω?

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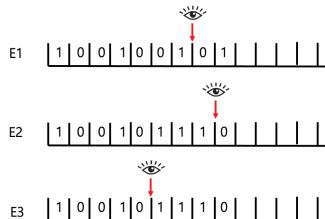**E2.** First time when 3 consecutive arrivals have been observed

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- Do we experience the fresh-start for any N? **E1, E2, and E3?**

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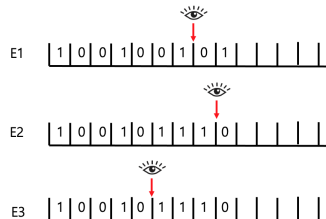**E1.** When I watch the process, N has been already determined. Yes

KAIST EE

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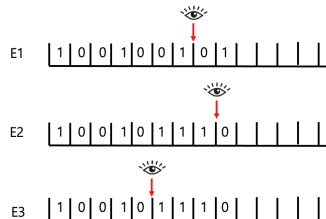
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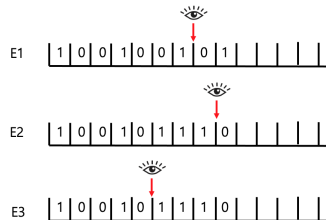**E1.** When I watch the process, N has been already determined. Yes

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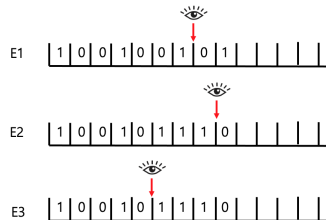
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- The question of $N = n$? can be answered just from the knowledge about $X_1, X_2, \ldots, X_n$? Then, Yes! (see pp. 301 for more formal description)

- In probability theory, a random time N is said to be a stopping time, if the question of "$N = n$?" can be answered only from the present and the past knowledge of $X_1, X_2, \ldots, X_n$.

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- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot
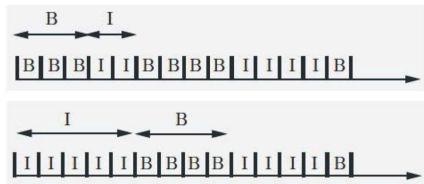
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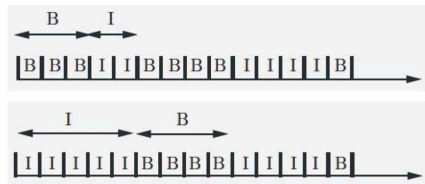
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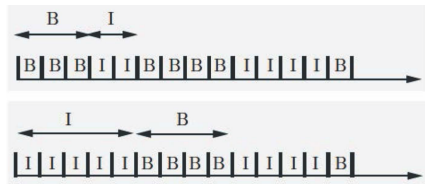- N: time of the first busy slot. N is a stopping time?

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Distribution of Busy Periods (1)

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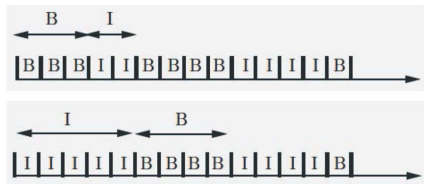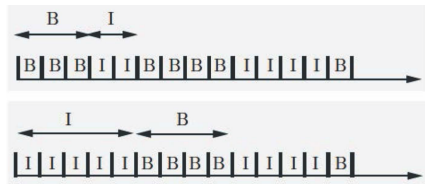- First busy period B_1: starts with the first busy slot and ends just before the first subsequent idle slot

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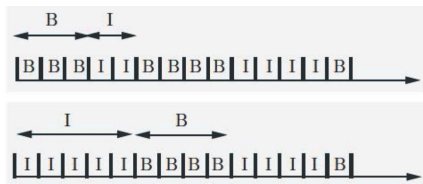
L8(2)

June 12, 2021   18 / 1

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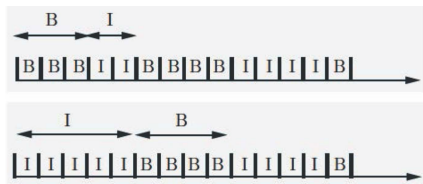
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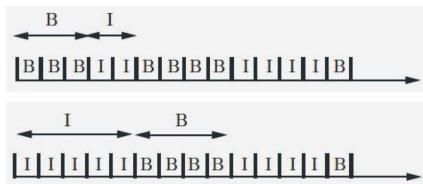
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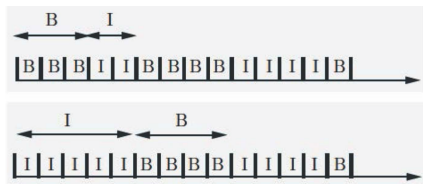
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- $B_3, B_4, \ldots$?

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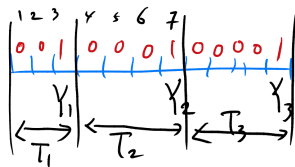
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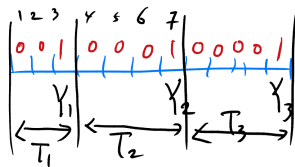
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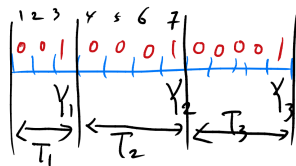
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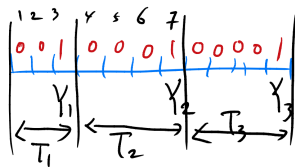
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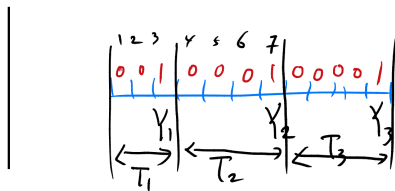
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- We know Y_k's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but its distribution?

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$$= p \times \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \ldots$$

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- Pascal$(1, p)$ = Geom(p)

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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- A random variable $S \sim \text{Bin}(n, p)$: Models the number of successes in a given number n of independent trials with success probability p.

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Design of Continuous Analogue of Bernoulli Process

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- Need a "modeling sense" to make this possible. It's a good practice for engineers!
- VIDEO PAUSE

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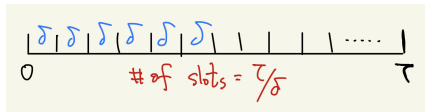- Divide $[0, \tau]$ into slots whose length $= \delta$. Then, $n = \#$ of slots $= \frac{\tau}{\delta}$.
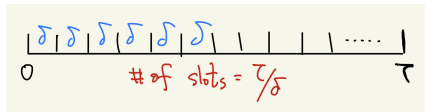
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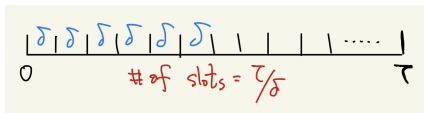 - We know how to handle Bernoulli process with discrete time slots.

- Divide $[0, \tau]$ into slots whose length $= \delta$. Then, $n = \#$ of slots $= \frac{\tau}{\delta}$.

- What's the limit as $\delta \to 0$ (equivalently, $n \to \infty$)

$$\underbrace{|\delta|\delta|\delta|\delta|\delta|\delta|}_{\text{\# of slots} = \tau/\delta} | \quad | \quad | \quad | \quad \cdots \cdots |$$
$$0 \qquad\qquad\qquad\qquad\qquad\qquad \tau$$

- Now, our design idea: during one time slot of length δ,

$\delta \mid \delta \mid \delta \mid \delta \mid \delta \mid \delta \mid \quad \mid \quad \mid \quad \mid \quad \mid \cdots \cdots \mid$

$0 \qquad$ # of slots = τ / δ $\qquad \tau$
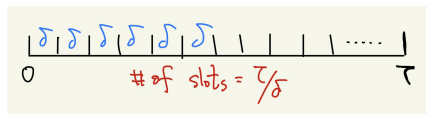
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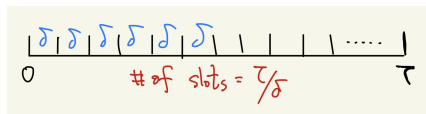

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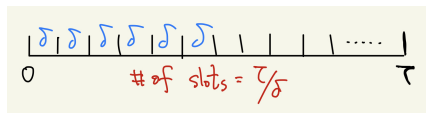$\mathbb{P}(0 \text{ arrival}) = 1 - \mathbb{P}(1 \text{ arrival or } \geq 2 \text{ arrivals})$

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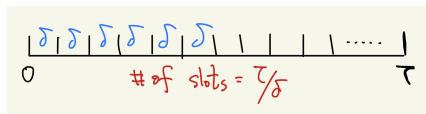$\mathbb{P}(\text{1 arrival}) \propto$ arrival rate and slot length
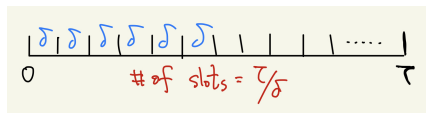
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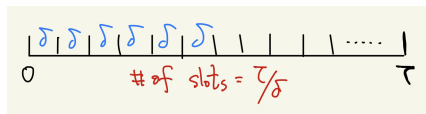|	
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- $o(\delta)$: some function that goes to zero faster than δ.
  - Thus, for very small δ, $o(\delta)$ becomes negligible, compared to δ.
  - Example: $o(\delta) = \delta^{\alpha}$, where any $\alpha > 1$

$$\text{# of slots} = \tau/\delta$$
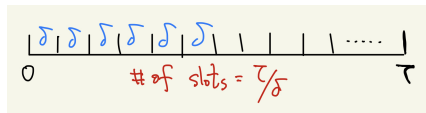
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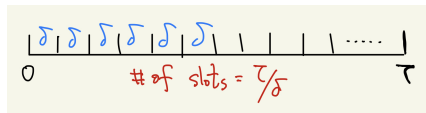
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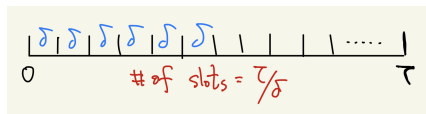
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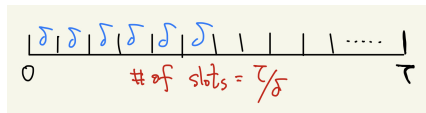
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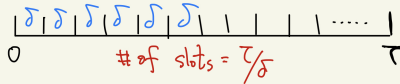
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- This is a continuous twin process of Bernous process, which we call Poisson process.

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

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- (Small interval probability) Let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, which satisfy:

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  $\mathbb{P}(k, \tau) = o_k(\tau)$ for $k = 2, 3, \ldots$, where $\lim_{\tau \to 0} \dfrac{o(\tau)}{\tau} = 0$, $\lim_{\tau \to 0} \dfrac{o_k(\tau)}{\tau} = 0$

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- (Distribution of N_τ) N_τ is the Poisson rv with parameter $\lambda\tau$, i.e., if we let $\mathbb{P}(k, \tau) = \mathbb{P}(N_\tau = k)$, we have:

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
  - Continuous twin of geometric rv in Bernoulli process
  - Memoryless

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour

- Mean and variance of mails received during a day

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- $\mathbb{P}[\text{exactly two msgs during each of the next three hours}]$

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Poisson Process: Example

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  - $\mathbb{P}(1, 1) = \dfrac{(5 \cdot 1)^1 e^{-5 \cdot 1}}{1!} = 5e^{-5}$

- $\mathbb{P}[\text{exactly two msgs during each of the next three hours}]$
  - $\left(\dfrac{5^2 e^{-5}}{2!} \right)^3$

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- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k?

- For a given δ, ⬚ : prob. of k-th arrival over $[y, y + \delta]$.

- For a given δ, $\boxed{\delta \cdot f_{Y_k}(y)}$: prob. of k-th arrival over $[y, y + \delta]$.

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- This is called Erlang rv.

> An Erlang random variable Z with parameter (k, λ) has the following pdf:
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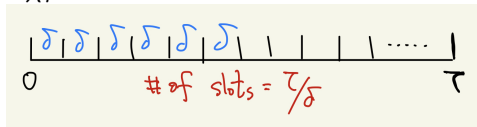
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PMF of # of arrivals		
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(Q6) $\mathbb{E}[\text{number of fish}] = \lambda \cdot 2 + \mathbb{P}(0, 2) \cdot 1$

(1) Introduction of Random Processes

(2) Bernoulli Processes

(3) Poisson Processes: Poisson RV and Basic Idea

(4) Poisson Processes: Definition and Properties

(5) Playing with Bernoulli and Poisson Processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, $T_2, \ldots$, with common parameter p, and let these stand for the interarrival times.

2. Record a success (or arrival) at times T_1, $T_1 + T_2$, $T_1 + T_2 + T_3$, etc.

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- Thus, the answer is p^2.

- Question. How to make software codes of Bernoulli process with p and Poisson process with λ

- Inter-arrival times are very handy.

- Bernoulli process with p: Obtain a sequence of random values following the geometric distribution with parameter p.

- Poisson process with λ: Obtain a sequence of random values following the exponential distribution with parameter λ.

- Bernoulli random variable: Bern(p)

- Bernoulli process: BP(p)

- Poisson random variable: Poisson(λ)

- Poisson process: PP(λ)

- Split BP(p) into two processes. Whenever there is an arrival, keep it w.p. q and discard it w.p. $(1 - q)$.
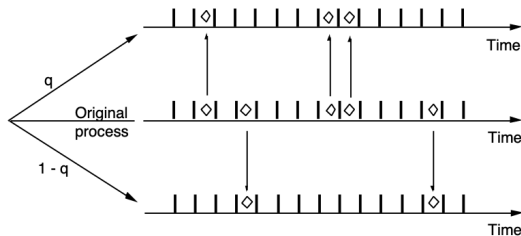
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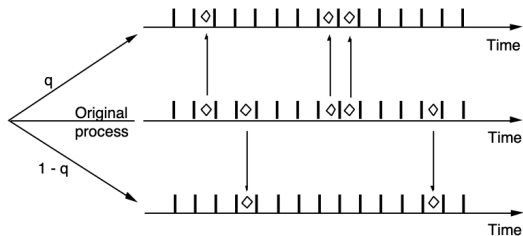
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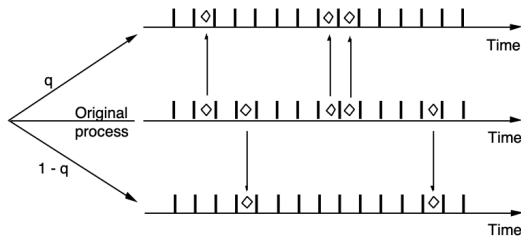
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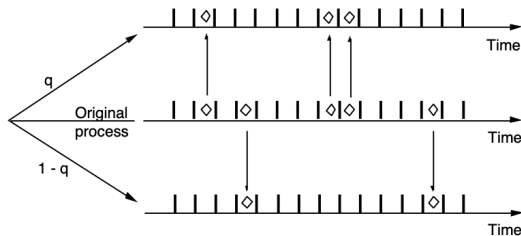
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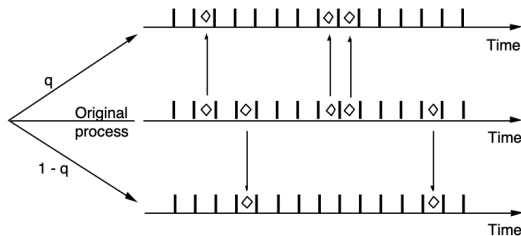
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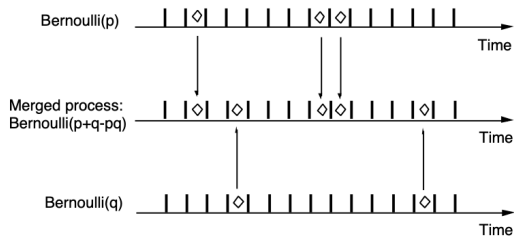
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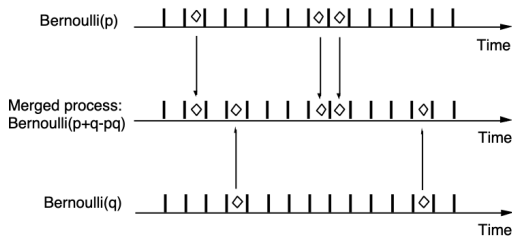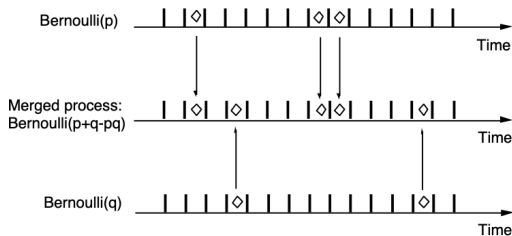
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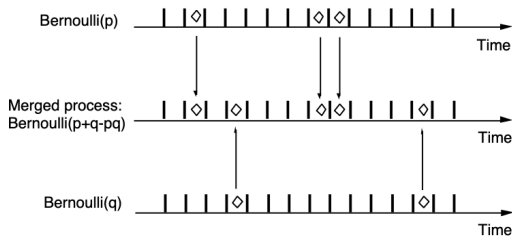
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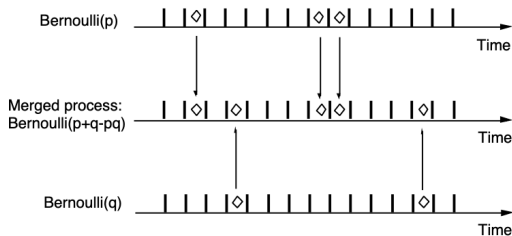
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- Question. What are the split processes?

- Let's focus on the process that we keep

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  - $\mathbb{P}(1 \text{ arrival}) = p\lambda\delta + p \cdot o(\delta) = p\lambda\delta + o(\delta)$
  - $\mathbb{P}(2 \text{ arrivals}) = p \cdot o(\delta) = o(\delta)$
  - $\mathbb{P}(0 \text{ arrival}) = 1 - p\lambda\delta - p \cdot o(\delta) - p \cdot o(\delta) = 1 - p\lambda\delta + o(\delta)$
- $PP(\lambda p)$ and $PP(\lambda(1 - p))$

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  $\mathbb{P}(0 \text{ arrival}) \approx (1 - \lambda_1\delta)(1 - \lambda_2\delta) \approx 1 - (\lambda_1 + \lambda_2)\delta$

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- Merged process: $PP(\lambda_1 + \lambda_2)$

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- $\mathbb{P}(\text{k out of first 10 arrivals are red})$? $\dbinom{10}{k} \left(\dfrac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\dfrac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{10-k}$

1. Competing exponentials

2. Sum of independent Poisson rvs

3. Poisson arrivals during and Exponential interval

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$$\mathbb{E}\Big[T_1 + T_2 + T_3\Big] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

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  - X: number of arrivals of PP(1) over a time interval of length μ
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- Very tedious and not very intuitive.

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- Let L be the number of arrivals from $PP(\lambda)$ until we get the first arrival from $PP(\nu)$.

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \dots$$

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  - $p = 1/100$, $n = 10,000$: small p, but large $n \implies$ Both Poisson and Normal

Questions?

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