Lecture 2: Conditioning, Bayes' Rule, and Independence

Yi, Yung (이융)

EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Conditional Probability
   - How should I change my belief about event A, if I come to know that event B occurs?

(2) Bayes' Rule and Bayesian Inference
   - prob. of A given that B occurs vs. prob. of B given that A occurs

(3) Independence, Conditional Independence
   - Can I ignore my knowledge about event B, when I consider event A?

Roadmap

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(2) Bayes' Rule and Bayesian Inference

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- Need to build up a new theoretical concept, which we call $\boxed{\text{conditional probability}}$.

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  - Three axioms[1] should be satisfied.

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  - $\mathbb{P}(\Omega|B) = 1$?
  - $\mathbb{P}(B|B) = 1$ from our common sense.
  - True?

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From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference (추론).*

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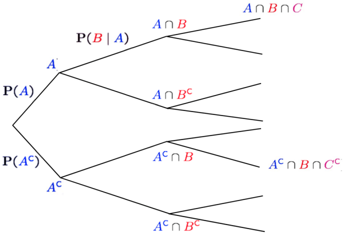
We will study this topic rigorously later in this class (chapter 8).

Multiplication Rule

- $\mathbb{P}(B|A) = \boxed{}$
- $\mathbb{P}(A \cap B) = \boxed{}$
- $\mathbb{P}(A^c \cap B \cap C^c) = \boxed{}$

  $= \boxed{}$



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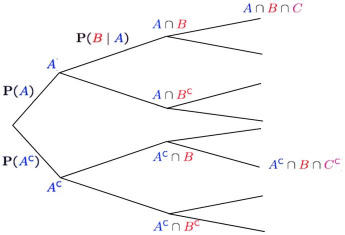

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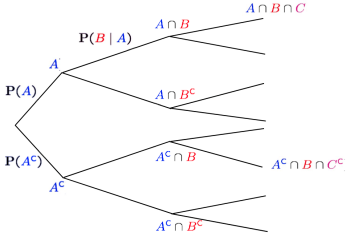

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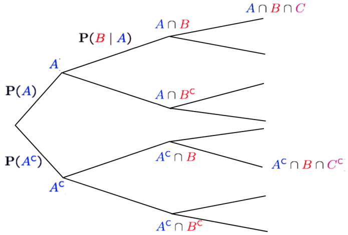

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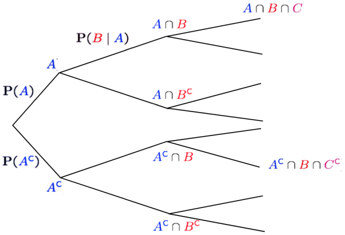

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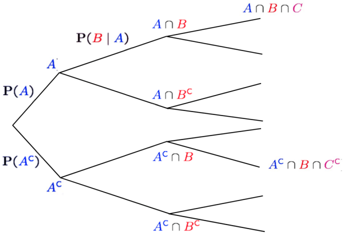

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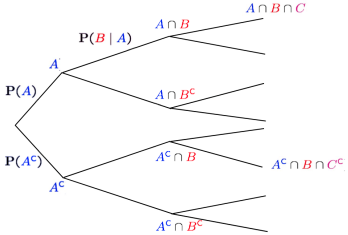

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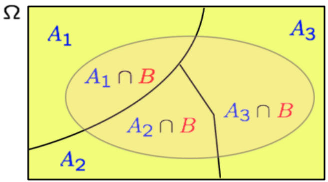
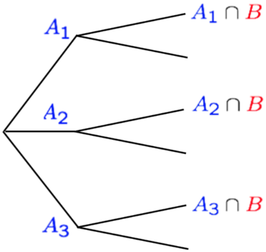
- Partition of Ω into A_1, A_2, A_3





[1]Partition: A_1, A_2, A_3 are mutually exclusive and $\Omega = A_1 \cup A_2 \cup A_3$

Total Probability Theorem

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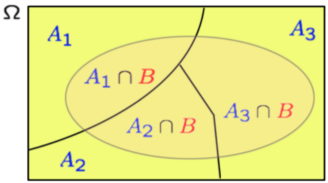

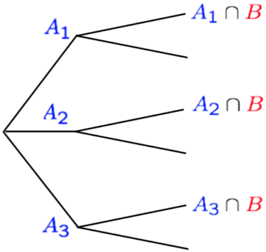

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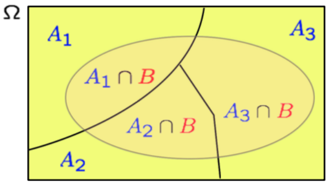

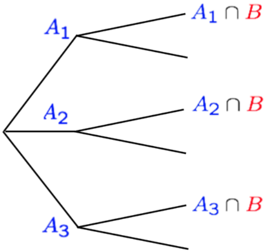

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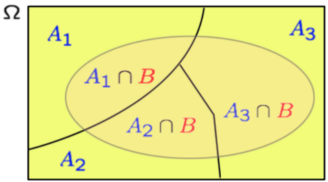

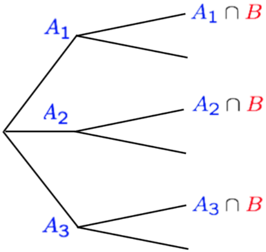

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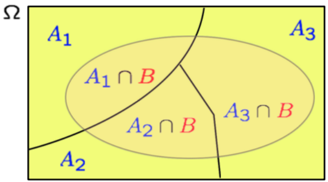
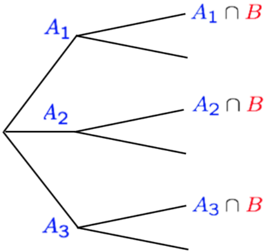
- Weighted average from the point of A_i knowledge.





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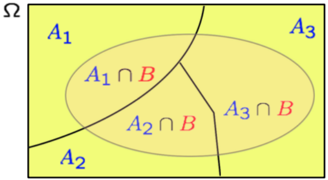

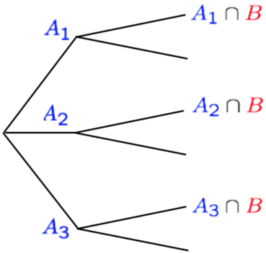

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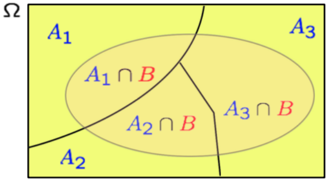
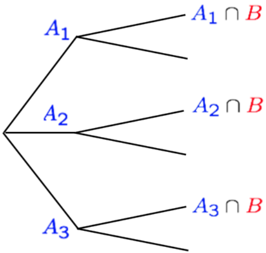


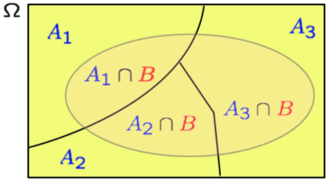


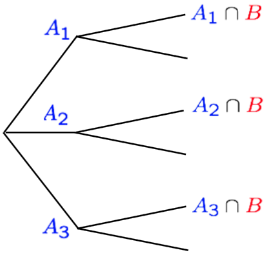
Bayes' Rule

- Partition of Ω into A_1, A_2, A_3
- We know: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(A_i|B)$?
- revised belief about A_i, given B occurs

Bayes' Rule

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\mathbb{P}(B|A_i)}{\sum_j \mathbb{P}(A_j)\mathbb{P}(B|A_j)}$$





Example 1: Airplane-Radar

VIDEO PAUSE

- A : Airplane is flying above
- B : Something on radar screen

$\mathbb{P}(A \cap B) =$

$=$

$\mathbb{P}(B) =$

$=$

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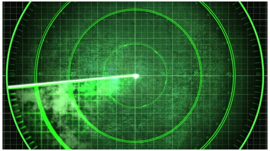

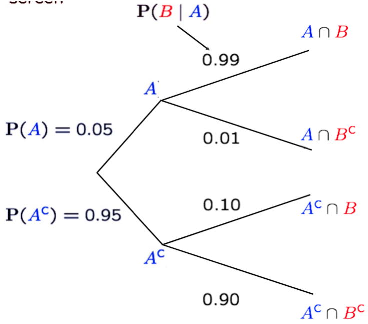

- A : Airplane is flying above
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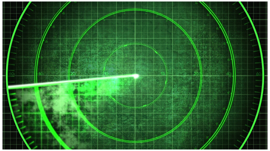
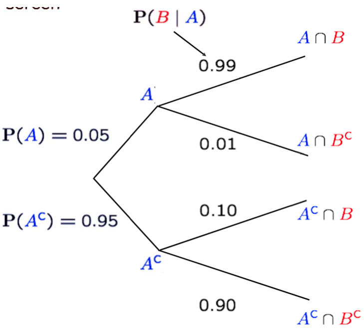
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$
$$=$$

$$\mathbb{P}(B) =$$
$$=$$

$$\mathbb{P}(A|B) = \qquad\qquad =$$





Example 1: Airplane-Radar

- A : Airplane is flying above
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$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$
$$= 0.05 \times 0.99 = 0.0495$$

$$\mathbb{P}(B) =$$
$$=$$

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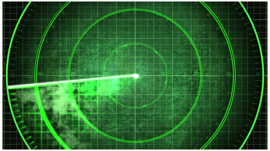

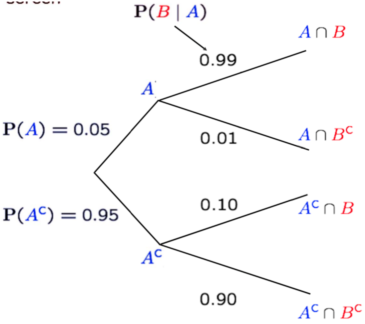

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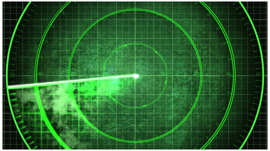

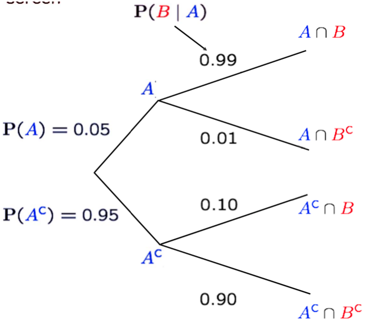

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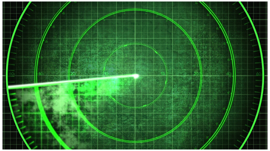

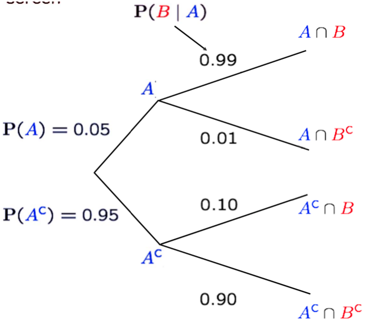

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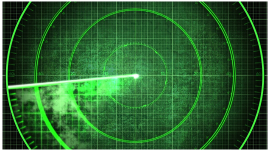

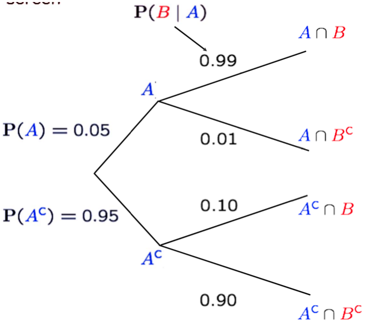

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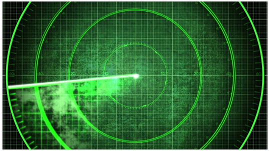

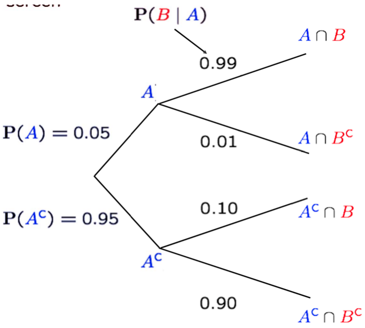

Example 2: Happy/Sad-Shout

- A_1: you are happy, A_2: you are sad
- B: you shout.
- Assume:

$$\mathbb{P}(A_1) = 0.7, \ \mathbb{P}(A_2) = 0.3,$$
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- Calculate $\mathbb{P}(A_1|B)$ and $\mathbb{P}(A_2|B)$.

VIDEO PAUSE

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- Calculate $\mathbb{P}(A_1|B)$ and $\mathbb{P}(A_2|B)$.

$$\begin{aligned}\mathbb{P}(A_1)\mathbb{P}(B|A_1) &= 0.7 \times 0.3 = 0.21\\ \mathbb{P}(A_2)\mathbb{P}(B|A_2) &= 0.3 \times 0.5 = 0.15\\ \mathbb{P}(B) &= 0.21 + 0.15 = 0.36\end{aligned}$$

$$\mathbb{P}(A_1|B) = \frac{0.21}{0.36} \approx 0.583$$
$$\mathbb{P}(A_2|B) = \frac{0.15}{0.36} \approx 0.417$$

Roadmap

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

(3) Independence, Conditional Independence

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.

Why We Care Independence?

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- Event A: I get the grade A in the probability class (my interest).
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- A and B do not seem dependent on each other. So, just forget B!
- Independence makes our analysis and modeling much simpler, because I can remove independent events in the analysis of what I am interested in.

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Independence of A and B, $\boxed{A \perp\!\!\!\perp B}$

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Conditional Independence of A and B given C, $\boxed{A \perp\!\!\!\perp B|C}$

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$$\mathbb{P}(B|A \cap C) = \mathbb{P}(B|C)$$

- Using $\mathbb{P}(A \cap B|C) = \frac{\mathbb{P}[B \cap (A \cap C)]}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C)\mathbb{P}(B|A \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|C)\mathbb{P}(B|C)$,

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$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \times \mathbb{P}(B|C)$$

(Q1) $A \perp\!\!\!\perp B \rightarrow A \perp\!\!\!\perp B | C$?

- Suppose that A and B are independent. If you heard that C occurred, A and B are still independent? VIDEO PAUSE

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  - H_1: 1st toss is a head
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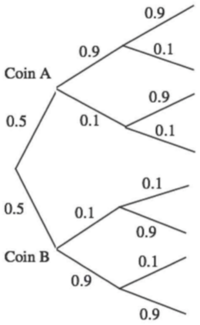

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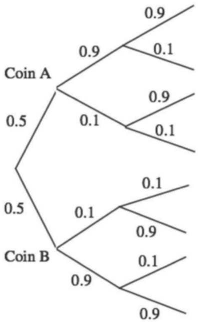

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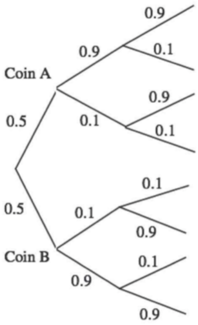
$$\mathbb{P}(H_1 \cap H_2 | B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1 | B)\mathbb{P}(H_2 | B) = 0.9 \times 0.9$$



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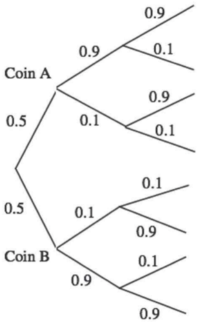

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$$\mathbb{P}(H_1) = \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c)$$
$$= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2}$$
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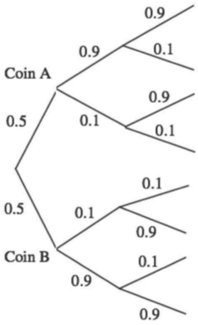

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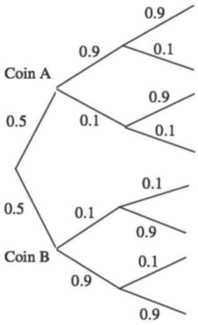
  H_i: i-th toss is head, and B: blue is selected.
- $H_1 \perp\!\!\!\perp H_2 | B$? Yes

$$\mathbb{P}(H_1 \cap H_2 | B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1|B)\mathbb{P}(H_2|B) = 0.9 \times 0.9$$

- $H_1 \perp\!\!\!\perp H_2$? No

$$\begin{aligned}
\mathbb{P}(H_1) &= \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c) \\
&= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2} \\
\mathbb{P}(H_2) &= \mathbb{P}(H_1) \quad \text{(because of symmetry)} \\
\mathbb{P}(H_1 \cap H_2) &= \mathbb{P}(B)\mathbb{P}(H_1 \cap H_2|B) + \mathbb{P}(B^c)\mathbb{P}(H_1 \cap H_2|B^c) \\
&= \frac{1}{2}(0.9 \times 0.9) + \frac{1}{2}(0.1 \times 0.1) \neq \frac{1}{2}
\end{aligned}$$



- Three events: A_1, A_2, A_3. What are the conditions of "their independence"?

Independence of Multiple Events

- Three events: A_1, A_2, A_3. What are the conditions of "their independence"?
- What about this? (Pairwise independence)

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Independence of Multiple Events

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Independence of Multiple Events

The events $A_1, A_2, \ldots, A_n$ ar said to be independent if

$$\mathbb{P}\Big(\bigcap_{i \in S} A_i\Big) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}$$

Questions?

Review Questions

1) What is conditional probability? Why do we need it?
2) Explain the overall framework of Bayesian inference.
3) What is the total probability theorem?
4) What is Bayes' rule? What does it can give us?
5) What's the difference between independence and conditional independence?