Lecture 6: Law of Large Numbers and Central Limit Theorem

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Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

- Two most remarkable findings in probability theory

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Our interest: Sum of Random Variables

- Example 1. n students who decides their presence, depending on their feeling. Each student is happy or sad at random, and only happy students will show their presence. How many students will show their presence?
- Example 2. I am hearing some sound. There are n noisy sources from outside.

- $X_1, X_2, \ldots, X_n$: i.i.d (independent and identically distributed) random variables
- $\mathbb{E}[X_i] = \mu$, var$[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

$$\boxed{S_n = X_1 + X_2 + \ldots + X_n}$$

What About Distribution of S_n?

$$S_n = X_1 + X_2 + \ldots + X_n$$

- Figure out the distribution of S_n. Very challenging. Even just for $Z = X + Y$, finding the distribution, for example, requires the complex convolution.

  $p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x p_X(x) p_Y(z - x)$
- Easy case: Sum of normal rvs = a normal rv, however, generally very challenging.
- Possible appoach. Take a certain scaling with respect to n that corresponds to a new glass, and investigate the system for large n

Sample Mean

- Consider the sample mean, and try to understand how S_n behaves:

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

- $\mathbb{E}(M_n) = \mu$, $\text{var}(M_n) = \sigma^2/n$
- For large n, the variance $\text{var}(M_n)$ decays. We expect that, for large n, M_n looses its randomness and concentrates around μ.
- We call this law of large numbers (LLN).

Let's Establish Mathematically

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

- What about this? What's wrong?

$$M_n \xrightarrow{n\to\infty} \mu$$

- Ordinary convergence for the sequence of real numbers: $a_n \to L$

  For every $\epsilon > 0$, there exists $N = N(\epsilon)$, such that for every $n \geq N$, $|a_n - L| \leq \epsilon$.

  `https://www.youtube.com/watch?v=4nBmsRA6eVw`
- However, M_n is a random variable, which is a function from Ω to $\mathbb{R}$.
- Need to build up the new concept of convergence for the sequence of rvs.

Convergence in Probability (1)

- What we want: a sequence of rvs $(Y_n)_{n=1,2,\ldots}$ converges to a rv Y in some sense
- For any given $\epsilon > 0$, consider the sequence of events $A_n = \{|Y_n - Y| \geq \epsilon\}$, and compute its sequence of probabilities $a_n = \mathbb{P}(A_n) = \mathbb{P}(|Y_n - Y| \geq \epsilon)$.
- Now, $\{a_n\}$ are just the real numbers, and show that $a_n \to 0$ as $n \to \infty$.
- To show that $a_n \to 0$ as $n \to \infty$, which is just the ordinary convergence, we show:
  - For any $\delta > 0$, there exists $N = N(\delta)$, such that for all $n \geq N$, $|a_n - 0| \leq \delta$
- Convergence in probability: $Y_n \xrightarrow{\text{in prob.}} Y$
  - For any $\epsilon > 0$ and for any $\delta > 0$, there exists $N = N(\delta)$, such that for all $n \geq N$, $\mathbb{P}(|Y_n - Y| \geq \epsilon) \leq \delta$.
  - For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - Y| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

Convergence in Probability (2)

- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - Y| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.
- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

- A special case: when $Y = a$ for some constant a: $Y_n \xrightarrow{\text{in prob.}} a$
- https://youtu.be/Ajar_6MAOLw?t=248

Example 1: Convergence in Probability

- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

- A sequence of iid rvs $X_n \sim \mathcal{U}[0,1]$, and let
$$Y_n = \min\{X_1, X_2, \ldots, X_n\}$$
- Our intuition: Y_n converges to 0, as $n \to 0$. Why?
- Proof. For any $\epsilon > 0$,
$$\mathbb{P}(|Y_n - 0| \geq \epsilon) = \mathbb{P}(X_1 \geq \epsilon, \ldots, X_n \geq \epsilon) = \mathbb{P}(X_1 \geq \epsilon) \times \cdots \times \mathbb{P}(X_n \geq \epsilon)$$
$$= (1-\epsilon)^n \xrightarrow{n\to\infty} 0$$

Example 2: Convergence in Probability

- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

- Y: exponential rv with the parameter $\lambda = 1$ (Remind: $\mathbb{P}(Y > y) = e^{-\lambda y}$)
- a sequence of rvs $Y_n = Y/n$ (note that these are dependent)
- Our intuition: Y_n converges to 0
- Proof. For any $\epsilon > 0$,
$$\mathbb{P}(|Y_n - 0| \geq \epsilon) = \mathbb{P}(Y \geq n\epsilon) = e^{-n\epsilon} \xrightarrow{n\to\infty} 0$$

Example 3: Convergence in Probability

- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0$.

- Consider a sequence of rvs Y_n with the following distribution:
$$\mathbb{P}(Y_n = y) = \begin{cases} 1 - \frac{1}{n}, & \text{for } y = 0 \\ \frac{1}{n}, & \text{for } y = n^2 \\ 0, & \text{otherwise} \end{cases}$$
- For any $\epsilon > 0$,
$$\mathbb{P}(|Y_n| \geq \epsilon) = \frac{1}{n} \xrightarrow{n\to\infty} 0$$
- Thus, Y_n converges to 0 in probability.

Weak Law of Large Numbers (WLLN)

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \dots X_n}{n}$$

- Roughly, M_n concetrates around μ

Weak law of large numbers

M_n converges to μ in probability, i.e., $M_n \xrightarrow{\text{in prob.}} \mu$

- Why "Weak"? There exists a stronger version, which we call "strong" law of large numbers. We will not cover the strong law of large numbers in this class.
- The proof requires some knowledge about useful inequalities, which we will cover later.

WLLN: Why Useful?

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \dots X_n}{n}$$

- If we take the scaling of S_n by $1/n$, it behaves like a deterministic number. This significantly simplifies how we understand the world.
- For example, assume that a large number of identically distributed noises come to you. Then, you can roughly approximate it as ($n \times$ average noise)
- Provides an interpretation of expectations (as well as probabilities) in terms of a long sequence of identical independent experiments. For example, what is the probability of head of a coin? Toss 1000 times, and count the number of heads.

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Central Limit Theorem: Start with Scaling (1)

- Loosely speaking, WLLG says:
  $(M_n - \mu) \xrightarrow{n\to\infty} 0$
- However, we don't know how $M_n - \mu$ converges to 0. For example, what's the speed of convergence?
- Question. What should be "something"? Something should blow up for large n.

$$\boxed{\text{(something)}} \times (M_n - \mu) \xrightarrow{n\to\infty} \text{meaningful thing}$$

$$\boxed{n^{\alpha}} \times (M_n - \mu) \xrightarrow{n\to\infty} \text{meaningful thing}$$

- What's α for our magic?
- The answer is $\frac{1}{2}$

Central Limit Theorem: Start with Scaling (2)

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- Reshaping the equation:

$$\frac{\sqrt{n}}{\sigma} \times (M_n - \mu) = \sqrt{n}\left(\frac{S_n - n\mu}{\sigma n}\right) = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

- Let $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$. Then, $\mathbb{E}[Z_n] = 0$ and $\mathrm{var}(Z_n) = 1$.
- Z_n is well-centered with a variance irrespective of n.
- We expect that Z_n converges to something meaningful as $n \to \infty$, but what?
- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.
  - Need a new concept of convergence: "convergence in distribution"

Convergence in Distribution

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- Consider a sequence of rvs $(Y_n)_{n=1,2,\ldots}$ and a rv Y.

Convergence in Distribution: $Y_n \xrightarrow{\text{in dist.}} Y$

For every y,

$$\mathbb{P}(Y_n \le y) \xrightarrow{n\to\infty} \mathbb{P}(Y \le y)$$

- Another type of convergence of rvs
- Comparison with convergence in probability?
  - Convergence in probability $\implies$ Convergence in distribution, but the reverse is not true.
  - The proof is beyond what this class covers, but it will be interesting to find an example that shows convergence in distribution, which is not convergence in probability.

Example: in Distribution, but not in Probability

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- $X_n \sim \text{Bernoulli}(1/2)$, for all $n \ge 1$.
- $X = 1 - X_n$.
- Note that $X \sim \text{Bernoulli}(1/2)$. It means that the distributions of X_n and X are equal. It is trivial that X_n converges to X in distribution.
- What about convergence in probability?

$$\begin{aligned}\mathbb{P}(|X_n - X| \ge \epsilon) &= \mathbb{P}(|X_n - 1 + X_n| \ge \epsilon) = \mathbb{P}(|2X_n - 1| \ge \epsilon) \\ &= \mathbb{P}(1 \ge \epsilon) \qquad (\text{because } |2X_n - 1| = 1)\end{aligned}$$

- We can find ϵ small enough so that the above does not go to zero.

Central Limit Theorem: Formalism

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- $S_n = X_1 + X_2 + \cdots + X_n, \qquad Z_n = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$

Central Limit Theorem

Z_n convergens to Z in distribution, where $Z \sim \mathcal{N}(0,1)$.

- Very surprising!
- Irrespective of the distribution of X_i, Z is normal.

LLG vs. CLT: Different Scaling Glasses

- For simplicity, assume that $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$, $i = 1, 2, \ldots, n$
- Law of Large Numbers

> Scaling S_n by $1/n$, you go to a deterministic world.

- Central Limit Theorem

> Scaling S_n by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each X_i.

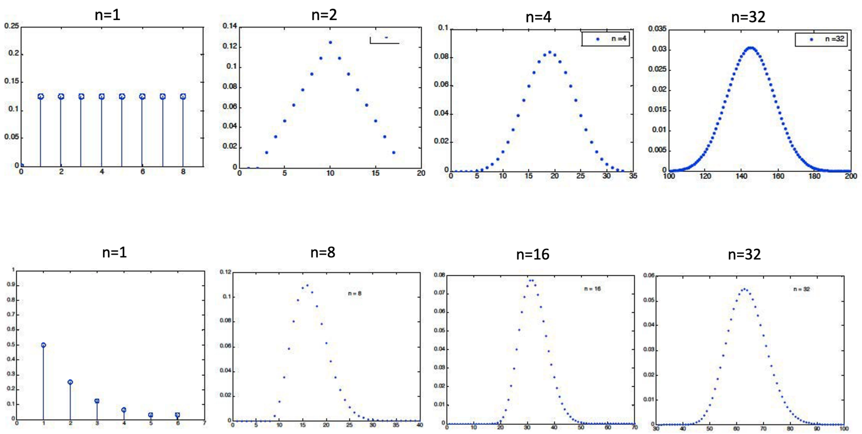
Practical Use of CLT

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \qquad \mathbb{P}(Z_n \le z) \xrightarrow{n\to\infty} \mathbb{P}(Z \le z), \ Z \sim \mathcal{N}(0,1)$$

- Can approximate Z_n with a standard normal rv
- Can approximate S_n with a normal rv $\sim (n\mu, n\sigma^2)$
  - $S_n = n\mu + Z_n\sigma\sqrt{n}$
- How large should n be?
  - A moderate n (20 or 30) usually works, which is the power of CLT.
  - If X_i resembles a normal rv more, smaller n works: symmetry and unimodality[1]

[1] Only unique mode. A single maximum or minimum.

CLT: Examples of Required n



Examples of CLT (1)

> $\mathbb{P}(S_n \le a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)
- Load container with $n = 100$ packages

$$\mathbb{P}(S_{100} \ge 210) = \mathbb{P}\Big[\frac{S_{100} - 100\cdot 2}{2\sqrt{100}} \ge \frac{210-200}{20}\Big] = \mathbb{P}(Z_{100} \ge 0.5)$$

$$\approx \mathbb{P}(Z \ge 0.5) = 1 - \mathbb{P}(Z \le 0.5) = 1 - \Phi(0.5)$$

Examples of CLT (2)

$\mathbb{P}(S_n \le a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)
- $n = 100$ packages, and choose the "capacity" a, so that $\mathbb{P}(S_n \ge a) \approx 0.05$

$$\mathbb{P}(S_{100} \ge a) = \mathbb{P}\Big[\frac{S_{100} - 100 \cdot 2}{2\sqrt{100}} \ge \frac{a-200}{20}\Big] = \mathbb{P}(Z_{100} \ge \frac{a-200}{20})$$
$$\approx \mathbb{P}(Z \ge \frac{a-200}{20}) = 1 - \mathbb{P}(Z \le \frac{a-200}{20}) = 1 - \Phi(\frac{a-200}{20}) = 0.05$$

- The value of a such that $\Phi(\frac{a-200}{20}) = 0.95$? $\frac{a-200}{20} = 1.645$ and $a = 232.9$

Examples of CLT (3)

$\mathbb{P}(S_n \le a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)
- How large n, so that $\mathbb{P}(S_n \ge 210) \approx 0.05$?

$$\mathbb{P}(S_n \ge 210) = \mathbb{P}\Big[\frac{S_n - 2n}{2\sqrt{n}} \ge \frac{210-2n}{2\sqrt{n}}\Big] \approx 1 - \Phi(\frac{210-2n}{2\sqrt{n}}) = 0.05$$

- The value of n such that $\frac{210-2n}{2\sqrt{n}} = 1.645$? $n = 89$

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Markov Inequality

- (Q) Knowing $\mathbb{E}(X)$, can we say something about the distribution of X?
- Intuition: small $\mathbb{E}(X) \Longrightarrow$ small $\mathbb{P}(X \ge a)$

Markov Inequality

If $X \ge 0$ and $a > 0$, then $\mathbb{P}(X \ge a) \le \frac{\mathbb{E}(X)}{a}$.

Proof. For any $a > 0$, define Y_a as:

$$Y_a \triangleq \begin{cases} 0, & \text{if } X < a, \\ a, & \text{if } X \ge a \end{cases}$$

Then, using non-negativity of X, $Y_a \le X$, which leads to $\mathbb{E}[Y_a] \le \mathbb{E}[X]$.

Note that we have:

$\mathbb{E}[Y_a] = a\mathbb{P}(Y_a = a) = a\mathbb{P}(X \ge a)$.

Thus, $a \cdot \mathbb{P}(X \ge a) \le \mathbb{E}[X]$. $\square$

Chebyshev Inequality

- (Q) Knowing both $\mathbb{E}(X)$ and $\text{var}(X)$, can we say something about the distribution of X?
- Intuition: small $\text{var}(X) \implies X$ is unlikely to be too far away from its mean.
- $\mathbb{E}(X) = \mu$, $\text{var}(X) = \sigma^2$.

Chebyshev Inequality

$$\mathbb{P}(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}, \quad \text{for all } c > 0$$

- Proof.

$$\mathbb{P}(|X-\mu| \geq c) = \mathbb{P}\left((X-\mu)^2 \geq c^2\right) \leq \frac{\mathbb{E}\left[(X-\mu)^2\right]}{c^2} = \frac{\text{var}(X)}{c^2}$$

Example: MI vs. CI

- $X \sim \exp(1)$. Then, $\mathbb{E}[X] = 1/\lambda = 1$ and $\text{var}[X] = 1/\lambda^2 = 1$.
- Exact CCDF: $\mathbb{P}(X \geq a) = e^{-a}$

- Markov inequality

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} = \frac{1}{a}$$

- Chebyshev inequality

$$\mathbb{P}(X \geq a) = \mathbb{P}(X - 1 \geq a - 1)$$
$$\leq \mathbb{P}(|X-1| \geq a-1) \leq \frac{1}{(a-1)^2}$$

- For reasonably large a, CI provides much better bound.
- Knowing the variance helps
- Both bounds are the ones that bound the probability of rare events.

Back to WLLN Proof

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

Weak law of large numbers

M_n converges to μ in probability.

Proof. For any given $\epsilon > 0$,

$$\mathbb{P}(|M_n - \mu| \geq \epsilon) \leq \frac{\text{var}(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{n\to\infty} 0$$

Comparison: WLLN vs. CLT

We ask the same question, and try to answer it, using WLLN or CLT.

See how the answers becomes different.

Example: Polling using WLLN

- p: fraction of voters who support "Yung".
- Interview n randomly selected voters and record the result in $M_n = \frac{X_1+\ldots+X_n}{n}$ which is an estimate of p, where the Bernoulli rv $X_i = 1$ if i-th interviewee answers "yes", and 0 otherwise.
- $\mathbb{P}(|M_n - p| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \leq \frac{1}{4n\epsilon^2}$ (because $p(1-p) \leq 1/4$)
- Question. What is n so that the probability that our estimate is incorrect by more than 0.1 is no larger than 0.25?
  - $\epsilon = 0.1$ and $\frac{1}{4n\epsilon^2} \leq 0.25 \implies n \geq 100$
- Question. What is n so that the probability that our estimate is incorrect by more than 0.01 is no larger than 0.05?
  - $\epsilon = 0.01$ and $\frac{1}{4n\epsilon^2} \leq 0.05 \implies n \geq 50000$

Example: Polling using CLT

$$\mathbb{P}(|M_n - p| \geq \epsilon) = \mathbb{P}\left[\left|\frac{S_n - np}{n}\right| \geq \epsilon\right] = \mathbb{P}\left[\left|\frac{S_n - np}{\sigma\sqrt{n}}\right| \geq \frac{\epsilon\sqrt{n}}{\sigma}\right]$$
$$\leq \mathbb{P}\left[\left|\frac{S_n - np}{\sigma\sqrt{n}}\right| \geq 2\epsilon\sqrt{n}\right] = 2\Big(1 - \Phi(2\epsilon\sqrt{n})\Big) \text{ (because } \sigma = \sqrt{p(1-p)} \leq 1/2)$$

- Question. What is n so that the probability that our estimate is incorrect by more than 0.01 is no larger than 0.05?
  - $\epsilon = 0.01$ and $2\Big(1 - \Phi(2\epsilon\sqrt{n})\Big) = 0.05$, i.e., $\Phi(2\epsilon\sqrt{n}) = 0.975 \implies$ $2 \times 0.01 \times \sqrt{n} = 1.96$ and thus $n = 9604$
- Compare: 50,000 from LLN vs. 9604 from CLT

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Moment Generating Function (MGF)

- For a rv X, it is a kind of transform
- The moment generating function (MGF) $M_X(s)$ of a rv X is a function of a scalar parameter s, defined by:

$$M_X(s) = \mathbb{E}[e^{sX}]$$

$$M(s) = \sum_x e^{sx} p_X(x) \quad \text{(discrete)}$$

$$M(s) = \int e^{sx} f_X(x) dx \quad \text{(continuous)}$$

- If the context is clear, we omit X and use just $M(s)$.

Examples

Ex1) Let $p_X(x)$ is given as:

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2 \\ 1/6, & \text{if } x = 3 \\ 1/3, & \text{if } x = 5 \end{cases}$$

$$M(s) = \mathbb{E}(e^{sX}) = \tfrac{1}{2}e^{2s} + \tfrac{1}{6}e^{3s} + \tfrac{1}{3}e^{5s}$$

Ex2) $X \sim \exp(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$

$$M(s) = \lambda \int_0^\infty e^{sx} e^{-\lambda x} dx$$
$$= \lambda \frac{e^{(s-\lambda)x}}{s-\lambda}\bigg|_0^\infty \quad (\text{if } s < \lambda) = \frac{\lambda}{\lambda - s}$$

Ex3) Let a rv $Y = aX + b$.

$$M_Y(s) = \mathbb{E}(e^{sY}) = \mathbb{E}(e^{s(aX+b)})$$
$$= e^{sb}\mathbb{E}(e^{saX}) = e^{sb}M_X(sa)$$

Ex4) $X \sim \mathcal{N}(0,1)$

$$M(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{sy} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}+sy} dy$$
$$= e^{\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y-s)^2} dy$$
$$= e^{s^2/2} \text{ (because it is the pdf of } \mathcal{N}(s,1))$$

- Question. MGF of $\mathcal{N}(\mu, \sigma^2)$?

Useful Properties of MGF

1. $M'(0) = \mathbb{E}[X]$

$$\frac{d}{ds}M(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{sx} f_X(x)dx = \int_{-\infty}^{\infty} \frac{d}{ds} e^{sx} f_X(x)dx = \int_{-\infty}^{\infty} x e^{sx} f_X(x)dx$$
$$= \frac{d}{ds}M(s)\bigg|_{s=0} = \mathbb{E}[X]$$

2. Similarly, $M''(0) = \mathbb{E}[X^2]$
3. $\frac{d^n}{ds^n}M(s)\Big|_{s=0} = \mathbb{E}[X^n]$
4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

Example

- Exponential rv with parameter λ. We know that $\mathbb{E}(X) = 1/\lambda$ and $\text{var}(X) = 1/\lambda^2$, which we will compute using the MGF.
- Remind: $M(s) = \frac{\lambda}{\lambda - s}$
- The first and the second moments are:

$$M'(s) = \frac{\lambda}{(\lambda - s)^2} \quad \rightarrow \quad \mathbb{E}(X) = M'(0) = 1/\lambda$$
$$M''(s) = \frac{2\lambda}{(\lambda - s)^3} \quad \rightarrow \quad \mathbb{E}(X^2) = M''(0) = 2/\lambda^2$$

- Thus, $\text{var}(X) = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

Inversion Property

Inversion Property

The transform $M_X(s)$ associated with a random variable X uniquely determines the CDF of X, assuming that $M_X(s)$ is finite for all s in some interval $[-a, a]$, where a is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own unique MGF.

Example 1

- Given the following MGF of rv X, what is the distribution of X?
  $M(s) = \frac{1}{4}e^{-s} + \frac{1}{2} + \frac{1}{8}e^{4s} + \frac{1}{8}e^{5s}$
- Note that $M(s) = \sum_x e^{sx} p_X(x)$
- We can see that
  $$p_X(-1) = \frac{1}{4},\ p_X(0) = \frac{1}{2},\ p_X(4) = \frac{1}{8},\ p_X(5) = \frac{1}{8}$$

Example 2

- Given the following MGF of rv X, what is the distribution of X?
  $M(s) = \frac{pe^s}{1-(1-p)e^s}$
- Note that $M(s) = \sum_x e^{sx} p_X(x)$
- $M(s)$ can be reexpressed as the following geometric sum: when $(1-p)e^s < 1$,
  $M(s) = pe^s(1 + (1-p)e^s + (1-p)^2e^{2s} + (1-p)^3e^{3s} + \cdots)$
- $p_X(k)$: coefficient of the term e^{ks}, which means:
  $p_X(1) = p,\ p_X(2) = p(1-p),\ p_X(3) = p(1-p)^2,\ p_X(4) = p(1-p)^3, \ldots$
- X is a geometric rv with parameter p

Back to CLT Proof (1)

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$
- $Z_n = \frac{S_n}{\sqrt{n}} = \frac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$
- We will show: MGF of Z_n converges to MFG of $\mathcal{N}(0,1)$ (using inversion property)
- Proof.
  $$\mathbb{E}\left[e^{sS_n/\sqrt{n}}\right] = \mathbb{E}\left[e^{sX_1/\sqrt{n}}\right] \times \cdots \times \mathbb{E}\left[e^{sX_n/\sqrt{n}}\right]$$
  $$= \left(\mathbb{E}\left[e^{sX_1/\sqrt{n}}\right]\right)^n = \left(M_{X_1}\left(\frac{s}{\sqrt{n}}\right)\right)^n$$
- For simplicity, let $M(\cdot) = M_{X_1}(\cdot)$

Back to CLT Proof (2)

- $M(0) = 1,\ M'(0) = 0,\ M''(0) = 1$
- $\left(M\left(\frac{s}{\sqrt{n}}\right)\right)^n \xrightarrow{n\to\infty}$ what???
- Taking log, $n \log M\left(\frac{s}{\sqrt{n}}\right) \xrightarrow{n\to\infty}$ what???
- For convenience, do the change of variable $y = \frac{1}{\sqrt{n}}$. Then, $\lim_{y\to 0} \frac{\log M(ys)}{y^2}$
- If we apply l'hopital's rule twice (please check), we get
  $\lim_{y\to 0} \frac{\log M(ys)}{y^2} = \frac{s^2}{2}$

$\square$

Questions?

1) What's the practical value of LLN and CLT?
2) Explain LLN and CLT from the scaling perspective.
3) Why are LLN and CLT great?
4) Why do we need different concepts of convergence for random variables?
5) Explain what is convergence in probability.
6) Explain what is convergence in distribution.
7) Why is MGF useful?