

Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

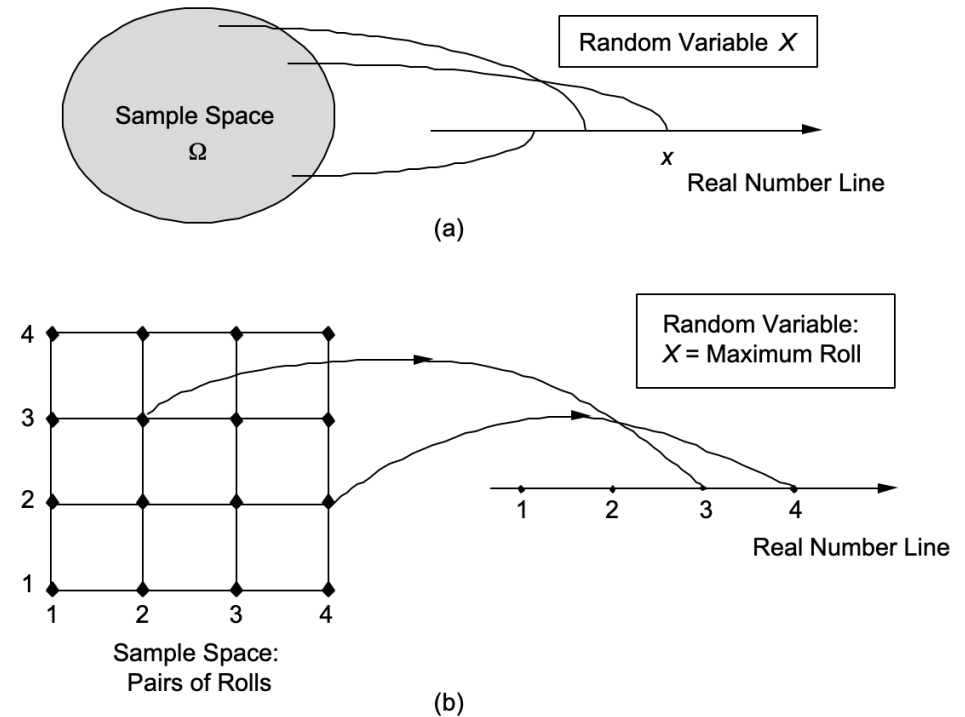
August 26, 2021

- (1) Random variable: Idea and formal definition
- (2) Popular discrete random variables
- (3) Summarizing random variables: Expectation and Variance
- (4) (Functions of) multiple random variables
- (5) Conditioning for random variables
- (6) Independence for random variables

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Random Variable: Idea

- In reality, many outcomes are **numerical**, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

- Mathematically, a random variable X is a **function** which maps from Ω to \mathbb{R} .
- Notation.** Random variable X , numerical value x .
- Different random variables can be defined on the same sample space.
- For a fixed value x , we can associate an **event** that a random variable X has the value x , i.e., $\{\omega \in \Omega \mid X(\omega) = x\}$
- Assume that values x are discrete¹ such as $1, 2, 3, \dots$.
For notational convenience,
$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$
- For a discrete random variable X , we call $p_X(x)$ **probability mass function** (PMF).

¹Finite or countably infinite.

- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$
- Remember that the random variable X is a **function** from Ω to \mathbb{R}

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- Only **binary** values

$$X = \begin{cases} 0, & \text{w.p. } 1 - p, \\ 1, & \text{w.p. } p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

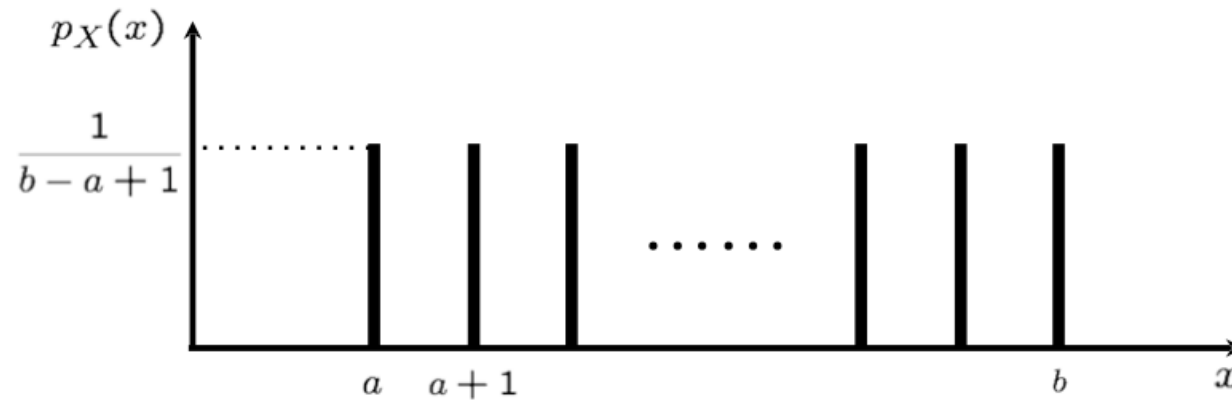
- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an **indicator rv** of an event A . Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

¹w.p.: with probability

Uniform X with parameter a, b

- integers a, b , where $a \leq b$
- Choose a number out of $\Omega = \{a, a + 1, \dots, b\}$ uniformly at random.
- $p_X(i) = \frac{1}{b-a+1}$, $i \in \Omega$

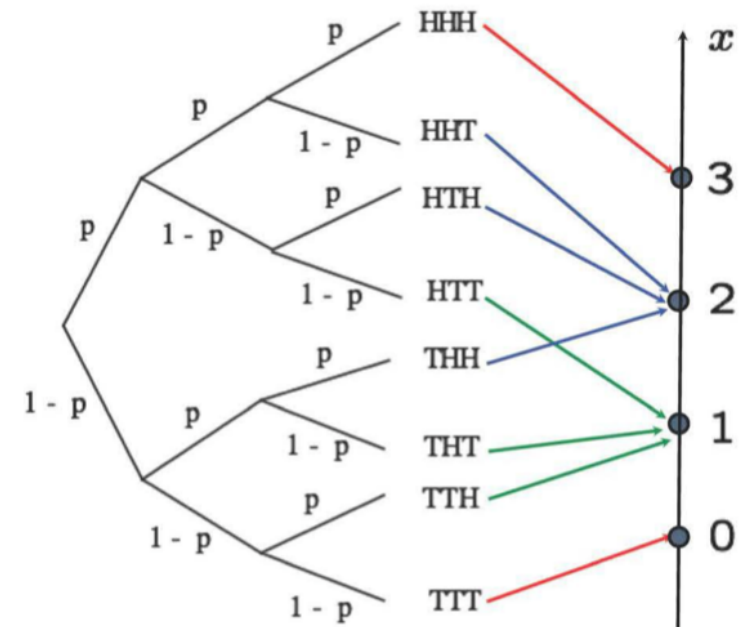


- Models complete **ignorance** (I don't know anything about X)

Binomial X with parameter n, p

- Models the number of **successes** in a given number of **independent** trials
- n independent trials, where one trial has the success probability p .

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



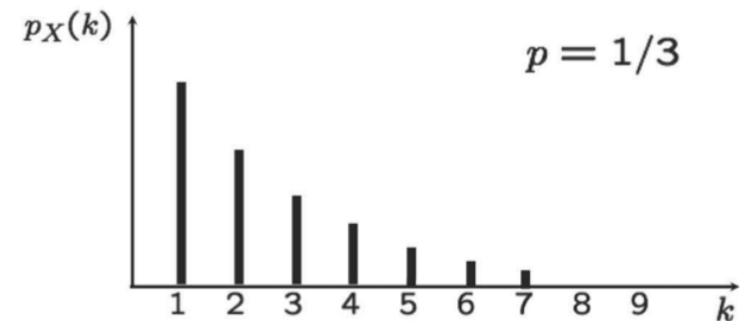
¹ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, which we read ' n choose k '.
L3(2)

Geometric X with parameter p

- Infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the **first success**.

$$p_X(k) = (1 - p)^{k-1} p$$

- Models **waiting** times until something happens.



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- Average

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x /total trials)
- **Example.** Bernoulli rv with p

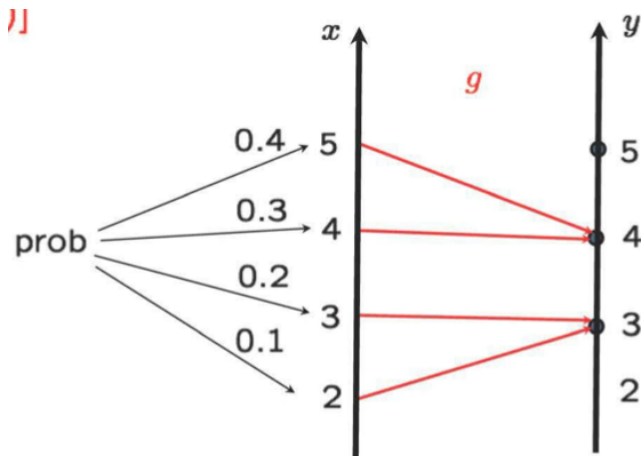
$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p = p_X(1)$$

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.
- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.
- For a constant c , $\mathbb{E}[c] = c$.

Expectation of a function of a RV

- For a rv X , $Y = g(X)$ is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$
- Compute $\mathbb{E}[Y]$ for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2) = 2.8 + 0.9 = 3.7$$

Linearity of Expectation

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

- Measures how much the spread of a PMF is.
- What about $\mathbb{E}[X - \mu]$, where $\mu = \mathbb{E}[X]$? Zero
- Then, what about $\mathbb{E}[(X - \mu)^2]$?

Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\ &= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2\end{aligned}$$

- $Y = X + b, \text{var}[Y] = \text{var}[X]$

$$\text{var}[Y] = \mathbb{E}[(X + b)^2] - (\mathbb{E}[X + b])^2$$

- $Y = aX, \text{var}[Y] = a^2\text{var}[X]$

$$\text{var}[Y] = \mathbb{E}[a^2X^2] - (a\mathbb{E}[X])^2$$

Example: Variance of a Bernoulli rv (p)

$$\mu = \mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p$$

$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1 - p) = p$$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\ &= p(1 - p)\end{aligned}$$

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- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y, X^2 + Y^2$. Then, PMF of Z is:

$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y): g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

- Similarly,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

(easy to prove, using the definition.)

- $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- **Example.** Mean of a binomial rv Y with (n, p)

- Y : number of successes in n Bernoulli trials with p

- $Y = X_1 + \dots + X_n$, where X_i is a Bernoulli rv.

- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

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Conditional PMF: Conditioning on an event

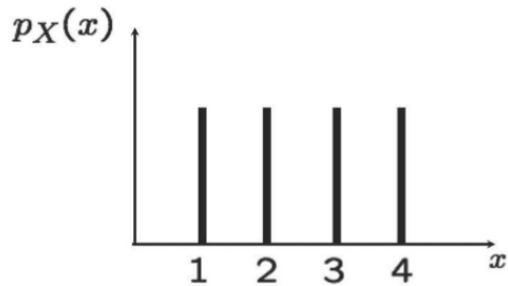
Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A .

- $p_X(x) \triangleq \mathbb{P}(X = x)$
- $\mathbb{E}[X] = \sum_x x p_X(x)$
- $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

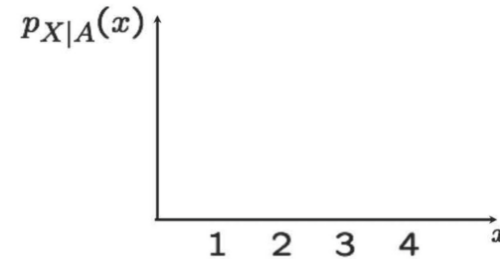
Example: Conditional PMF

$$A = \{X \geq 2\}$$



$$\mathbb{E}[X] = \frac{1}{4}(1 + 2 + 3 + 4) = 2.5$$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{4}(1 + 2^2 + 3^2 + 4^2) - 2.5^2\end{aligned}$$



$$\mathbb{E}[X|A] = \frac{1}{3}(2 + 3 + 4) = 3$$

$$\begin{aligned}\text{var}[X|A] &= \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 \\ &= \frac{1}{3}(2^2 + 3^2 + 4^2) - 3^2 = 2/3\end{aligned}$$

What do we mean by “conditioning on a rv”? Consider $A = \{Y = y\}$ for a rv Y .

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$

- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
- $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
- $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
- $\text{var}[X|Y = y] \triangleq \mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

VIDEO PAUSE

- Conditional PMF

$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

- $\sum_x p_{X|Y}(x|y) = 1$

- Multiplication rule

$$\begin{aligned} p_{X,Y}(x, y) &= p_Y(y)p_{X|Y}(x|y) \\ &= p_X(x)p_{Y|X}(y|x) \end{aligned}$$

- $p_{X,Y,Z}(x, y, z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x, y)$

y				
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		
	1	2	3	4
	x			

$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

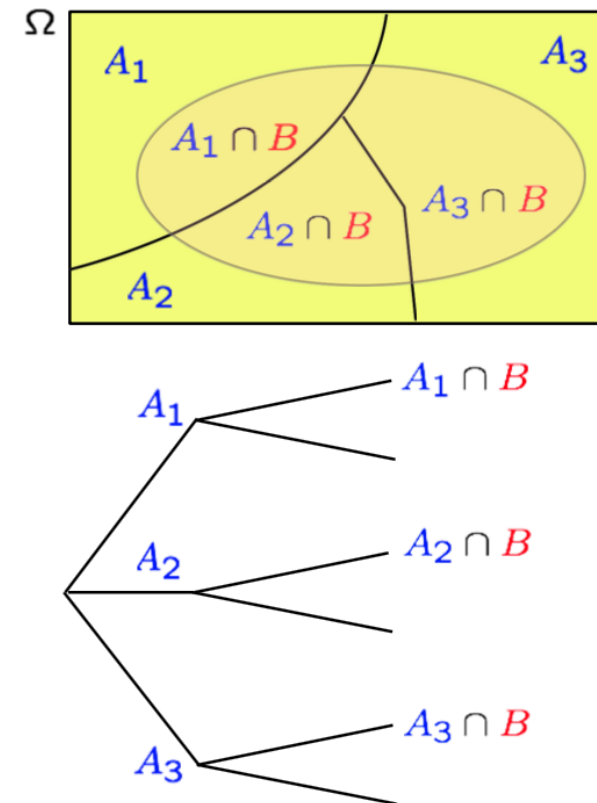
$$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$$

Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i) \mathbb{P}(B|A_i)$$

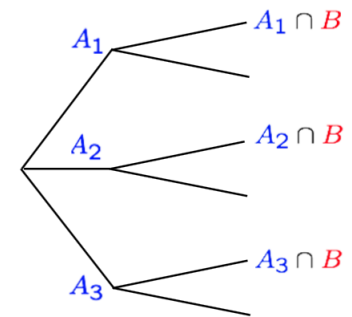
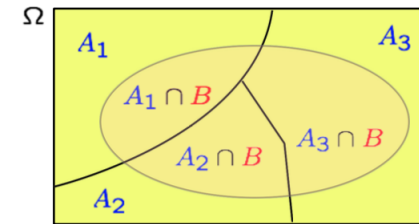


Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$



Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

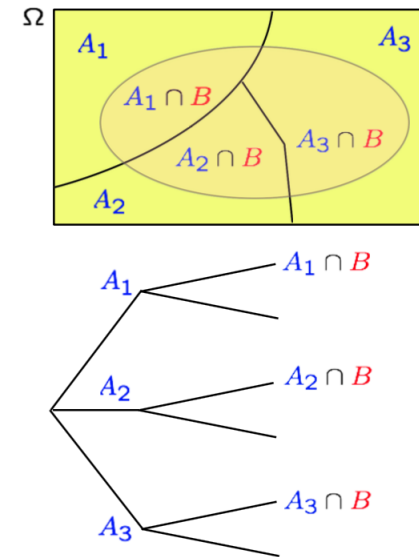
Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i) \mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X | A_i]$$

- Weighted average of expectations from A_i 's perspective



Total Expectation Theorem for $\{Y = y\}$

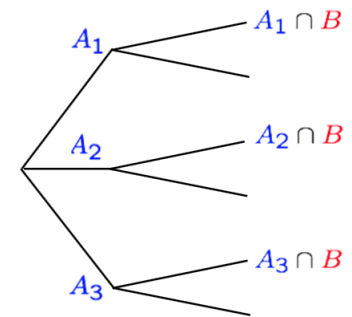
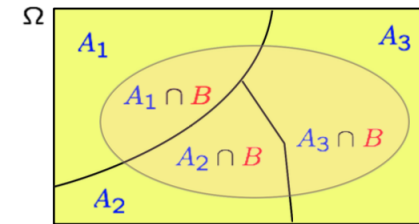
- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i) \mathbb{E}[X|A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y) \mathbb{E}[X|Y = y] = \sum_y p_Y(y) \mathbb{E}[X|Y = y]$$



Example 1: Total Expectation Theorem

- **Question.** What is $\mathbb{E}(X)$?

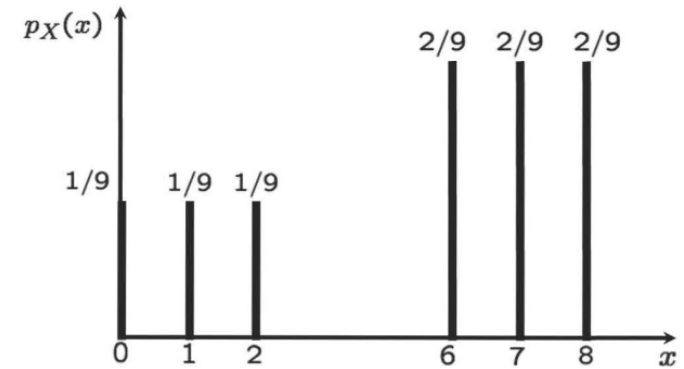
(1) Just using the definition of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{9}(0 + 1 + 2) + \frac{2}{9}(6 + 7 + 8) \\ &= \frac{3 + 12 + 14 + 16}{9} = 5\end{aligned}$$

(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0, 1, 2\}\}, \quad A_2 = \{X \in \{6, 7, 8\}\}$$

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1,2} \mathbb{P}(A_i) \mathbb{E}[X|A_i] \\ &= 1/3 \cdot 1 + 2/3 \cdot 7 = 5\end{aligned}$$

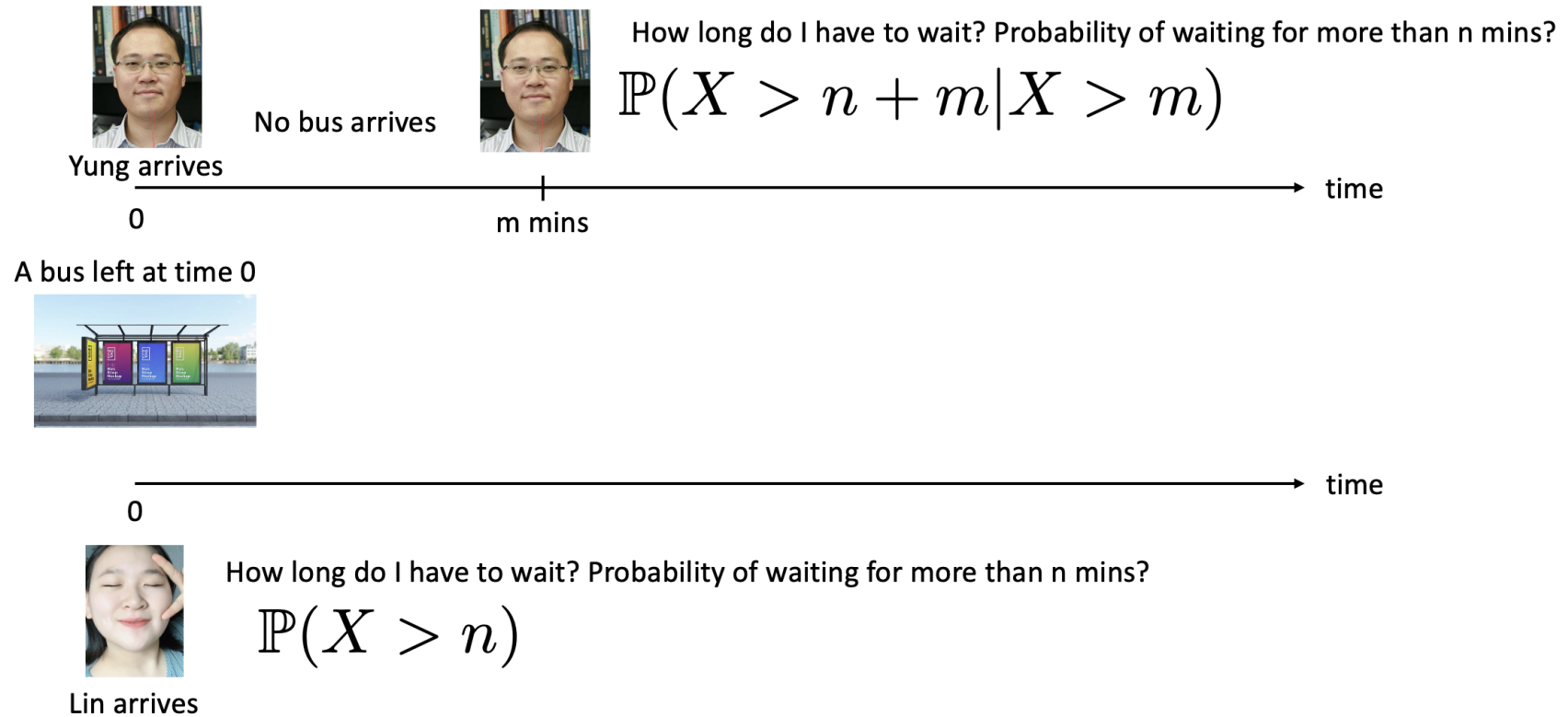


- Write softwares over and over, and each time w.p. p of working correctly (independent from previous programs).
- X : number of trials until the program works correctly.
- (Q) $\mathbb{E}(X)$?
- X is a geometric rv
- Direct computation is boring.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p + 2(1-p)p + 3(1-p)^2p + \dots$$

- Total expectation theorem and a notion of **memorylessness** helps a lot.

Memoryless Property: Motivating Example



- Some random variable often does not have **memory**.
- **Definition.** A random variable X is called **memoryless** if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n + m | X > m) = \mathbb{P}(X > n)$$

- **Meaning.** Conditioned on $X > m$, $X - m$'s distribution is the same as the original X .

$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

- **Theorem.** Any **geometric** random variable is **memoryless**.
- **Remind.** Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1 - p)^{k-1}p, \quad \mathbb{P}(X > k) = \sum_{i=k+1}^{\infty} (1 - p)^{i-1}p = (1 - p)^k$$

- **Proof.**

$$\begin{aligned} \mathbb{P}(X > n + m | X > m) &= \frac{\mathbb{P}(X > n + m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n + m)}{\mathbb{P}(X > m)} \\ &= \frac{(1 - p)^{n+m}}{(1 - p)^m} = (1 - p)^n = \mathbb{P}(X > n) \end{aligned}$$

- **Meaning.** Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$\mathbb{E}[X] = 1 + \mathbb{E}[X - 1]$$

$$= 1 + \mathbb{P}(A_1)\mathbb{E}[X - 1|X = 1] + \mathbb{P}(A_2)\mathbb{E}[X - 1|X > 1] \quad (\text{from TET})$$

$$= 1 + (1 - p)\mathbb{E}[X] \quad (\text{from memorylessness})$$

- Thus, $\mathbb{E}[X] = \frac{1}{p}$

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- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$

$$\mathbb{P}(\{X = x\} \cap B | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(B | C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

$$\mathbb{P}(\{X = x\} \cap \{Y = y\} | Z = z) = \mathbb{P}(X = x | Z = z) \cdot \mathbb{P}(Y = y | Z = z), \quad \text{for all } x, y$$

$$p_{X,Y|Z}(x, y) = p_{X|Z}(x) \cdot p_{Y|Z}(y)$$

Example

- $X \perp\!\!\!\perp Y$?

$$p_{X,Y}(1,1) = 0$$

$$p_X(1) = 3/20$$

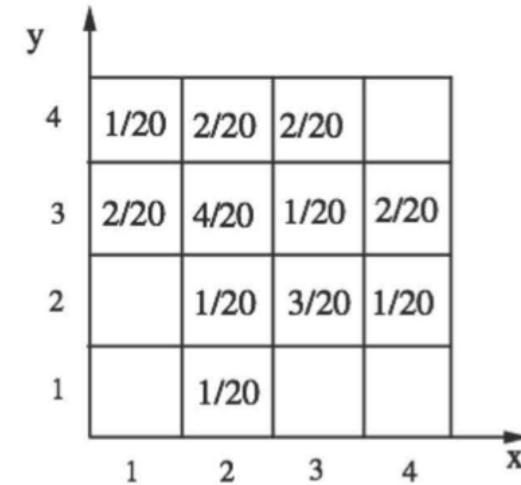
$$p_Y(1) = 1/20$$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?

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$Y = 4$ (1/3)	1/9	2/9
$Y = 3$ (2/3)	2/9	4/9
	$X = 1$ (1/3)	$X = 2$ (2/3)

- Yes.



- Always true.

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

- Generally, $\mathbb{E}[g(X, Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$

- However, if $X \perp\!\!\!\perp Y$,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

- **Proof.**

$$\begin{aligned}\mathbb{E}[g(X)h(Y)] &= \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y) \\ &= \sum_x g(x)p_X(x) \sum_y h(y)p_Y(y)\end{aligned}$$

- Always true.

$$\text{var}[aX] = a^2\text{var}[X], \text{var}[X + a] = \text{var}[X]$$

- Generally, $\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$
(next slide)

- However, if $X \perp\!\!\!\perp Y$,

$$\text{var}[X + Y] = \text{var}[X] + \text{var}[Y]$$

- **Practice.**

- $X = Y \implies \text{var}[X + Y] = 4\text{var}[X]$
- $X = -Y \implies \text{var}[X + Y] = 0$
- $X \perp\!\!\!\perp Y \implies \text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

$$\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$$

- Why not generally true?

$$\begin{aligned}\text{var}[X + Y] &= \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 \\ &= \mathbb{E}[X^2 + Y^2 + 2XY] - ((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]) \\ &= \text{var}[X] + \text{var}[Y] + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\end{aligned}$$

- $X \perp\!\!\!\perp Y$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study **covariance**.

Example: The hat problem (1)

- n people throw their hats in a box and then pick one at random
- X : number of people with their own hat
- $\mathbb{E}[X]$? $\text{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- **Key step 1.** Define a rv $X_i = 1$ if i selects its own hat and 0 otherwise.

$$X = \sum_{i=1}^n X_i.$$

- $\{X_i\}, i = 1, 2, \dots, n$: identically distributed (from symmetry)

Example: The hat problem (2)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1$.
- **Key step 2.** Are X_i s are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \rightarrow X_2 = 1$, and $X_1 = 0 \rightarrow X_2 = 0$. Thus, **dependent**.

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}\left[\sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j\right] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_1^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1|X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

- 1) What is Random Variable? Why is it useful?
- 2) What is PMF (Probability Mass Function)?
- 3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
- 4) What are joint and marginal PMFs?
- 5) Describe and explain the total probability/expectation theorem for random variables?
- 6) When is it useful to use total probability/expectation theorem?
- 7) What is conditional independence?