

## Lecture 6: Statistical Inference

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EE210: Probability and Introductory Random Processes  
KAIST EE

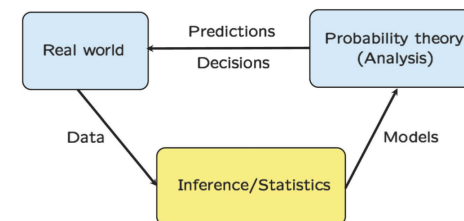
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- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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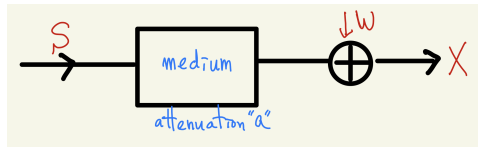
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- Inference
  - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building

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- $X = aS + W$
- Modeling building
  - know the original signal  $S$ , observe  $X$
  - infer the model parameter  $a$
- Variable estimation
  - know  $a$ , observe  $X$
  - infer the original signal  $S$
- Same mathematical structure, because the parameters in models are variables in many cases

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- Hypothesis testing
  - Unknown: a few possible ones
  - Goal: small probability of incorrect decision
  - (Ex) Something detected on the radar. Is it a bird or an airplane?
- Estimation
  - Unknown: a value included in an infinite, typically continuous set
  - Goal: Finding the value close to the true value
  - (Ex) Biased coin with unknown probability of head  $\theta \in [0, 1]$ . Data of heads and tails. What is  $\theta$ ?
  - (Note) If you have the candidate values of  $\theta = \{1/4, 1/2, 3/4\}$ , then it's a hypothesis testing problem

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- Biased coin with parameter  $\theta$  (probability of head). Assume that  $\theta \in \{1/4, 3/4\}$ .
- Throw the coin 3 times and get  $(H, H, H)$ . Goal: infer  $\theta$ , 1/4 or 3/4?
- Distribution of  $\theta$  (prior) e.g.,
 
$$\mathbb{P}(\theta = \frac{3}{4}) = 1/2, \quad \mathbb{P}(\theta = \frac{1}{4}) = 1/2$$
- Use Bayes' rule and find the posterior:
 
$$\mathbb{P}[\theta = \frac{3}{4} | (HHH)] = \frac{27}{28}, \quad \mathbb{P}[\theta = \frac{1}{4} | (HHH)] = \frac{1}{28}$$
- Choose  $\theta$  with larger posterior probability.
- Bayesian approach (Chapter 8)
- Find the probability of  $(H, H, H)$ , if  $\theta = \frac{1}{4}$  or  $\frac{3}{4}$  (likelihood)
 
$$\mathbb{P}[(HHH) | \theta = \frac{3}{4}] = \left(\frac{3}{4}\right)^3$$

$$\mathbb{P}[(HHH) | \theta = \frac{1}{4}] = \left(\frac{1}{4}\right)^3$$
- Choose  $\theta$  with a larger likelihood.
- Classical approach (Chapter 9)

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## Bayesian approach

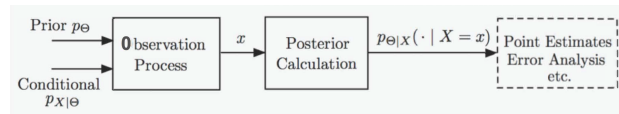
- Unknown: random variable with some distribution (prior)
- Unknown model as chosen randomly from a given model class
- Observed data  $x$  gives: posterior distribution  $p_{\Theta|X}(\theta|x)$
- Choose  $\theta$  with larger posterior probability (other methods exist)

## Classical approach

- Unknown: deterministic value
- Unknown model as one of multiple probabilistic models
- Observed data  $x$  gives: likelihood  $p(X; \theta)$
- Choose  $\theta$  with larger likelihood (other methods exist)

- Who is the winner? A century-long debate (see p. 409 for discussion)

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- Unknown  $\Theta$ 
    - physical quantity or model parameter
    - random variable
    - prior distribution  $p_\Theta$  and  $f_\Theta$
  - Observations or measurements  $X$ 
    - observation model  $p_{X|\Theta}$  and  $f_{X|\Theta}$
  - That is, the joint distribution of  $X$  and  $\Theta$ ,  $p_{X,\Theta}$  and  $f_{X,\Theta}$ , is given
- Find the posterior distribution  $p_{X|\Theta}$  and  $f_{X|\Theta}$ .
    - Use Bayes' rule
  - Using the posterior distribution, apply one of the methods of choosing the final  $\hat{\theta}$  for estimation and hypothesis testing.

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- Given observation  $x$ , which  $\theta$  are you going to choose?

M1. Choose the largest: Maximum a posteriori probability (MAP) rule

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} p_{\Theta|X}(\theta|x), \quad \hat{\theta}_{\text{MAP}} = \arg \max_{\theta} f_{\Theta|X}(\theta|x)$$

M2. Choose the mean: Conditional expectation, aka LMS (Least Mean Square)

$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta|X=x]$$

- Why MAP and LMS are good? Not mathematically clear yet (later)

- Random observation:  $X$

- Observation instance:  $x$

- Estimate as a mapping from  $x$  to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\text{MAP}} = g_{\text{MAP}}(x), \quad \hat{\theta}_{\text{LMS}} = g_{\text{LMS}}(x)$$

- Estimator as a mapping from  $X$  to a random variable

$$\hat{\Theta} = g(X), \quad \hat{\Theta}_{\text{MAP}} = g_{\text{MAP}}(X), \quad \hat{\Theta}_{\text{LMS}} = g_{\text{LMS}}(X)$$

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## Example 1: Romeo and Juliet

- Romeo and Juliet start dating.
  - Romeo: late by  $X \sim U[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim U[0, 1]$ .

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'}$$

$$= \frac{1/\theta}{\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta|\log x|}, \quad x \leq \theta \leq 1,$$

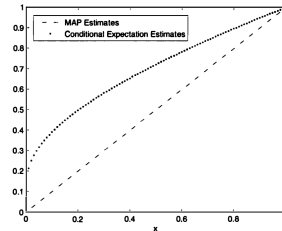
and  $f_{\Theta|X}(\theta|x) = 0$ ,  $\theta < x$  or  $\theta > 1$ .

- MAP rule
  - Given  $x$ ,  $f_{\Theta|X}(\theta|x)$  is decreasing in  $\theta$  over  $[x, 1]$ .
  - $\hat{\theta}_{\text{MAP}} = x$ .

- Conditional expectation estimator

$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\theta|X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta$$

$$= (1-x)/|\log x|$$



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## Example 2: Biased Coin with Beta Prior (1)

- Biased coin with probability of head  $\theta$
- Unknown  $\theta$ : modeled by  $\Theta$  with some prior  $f_{\Theta}(\theta)$
- Observation  $X$ : number of heads out of  $n$  tosses
- Posterior PDF

$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k}, \quad c \text{ the normalizing constant}$$

- If  $\Theta \sim \text{Beta}(\alpha, \beta)$ , what is  $\hat{\theta}_{\text{MAP}}$ ?
- What is  $\text{Beta}(\alpha, \beta)$ ?

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## Example 2: Biased Coin with Beta Prior (2)

### Beta distribution

A continuous rv  $\Theta$  follows a beta distribution with integer parameters  $\alpha, \beta > 0$ , if

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $B(\alpha, \beta)$ , called Beta function, is a normalizing constant, given by

$$B(\alpha, \beta) = \int_0^1 \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

- A special case of  $\text{Beta}(1, 1)$  is  $\text{Uniform}[0, 1]$

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## Example 2: Biased Coin with Beta Prior (3)

- If  $\Theta \sim \text{Beta}(\alpha, \beta)$ , then  $\Theta|X=k \sim \text{Beta}(k+\alpha, n-k+\beta)$ 
  - Very useful: Beta prior  $\implies$  Beta posterior

- Proof.** For  $\text{Beta}(\alpha, \beta)$  prior,

$$f_{\Theta}(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$f_{\Theta|X}(\theta|k) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k} = \frac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}$$

where  $d = c \binom{n}{k}$ .

- Taking the logarithm,

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} [(\alpha+k-1) \log \theta + (\beta+n-k+1) \log(1-\theta)] = \frac{\alpha+k-1}{\alpha+\beta-2+n}$$

- When  $\alpha = \beta = 1$  (i.e.,  $U[0, 1]$  prior),  $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- E-mail: spam (1) or legitimate (2),  $\Theta \in \{1, 2\}$ , with prior  $p_{\Theta}(1)$  and  $p_{\Theta}(2)$ .
- $\{w_1, w_2, \dots, w_n\}$ : a collection of words which suggest "spam".
- For each  $i$ , a Bernoulli  $X_i = 1$  if  $w_i$  appears and 0 otherwise.
- Observation model  $p_{X_i|\Theta(x_i|1)}$  and  $p_{X_i|\Theta(x_i|2)}$  are known. Conditioned on  $\Theta$ ,  $X_i$  are independent.
- Posterior PMF

$$\mathbb{P}(\Theta = m | (x_1, \dots, x_n)) = \frac{p_{\Theta}(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_{\Theta}(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_{\Theta}(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_{\Theta}(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

- MAP estimate is intuitive, but we need more mathematical support.
- Claim 1.** For a given  $x$ , the MAP rule minimizes the probability of an incorrect decision.
- Claim 2.** The MAP rule minimizes the overall probability of an incorrect decision, averaged over  $x$ .
- Proof.** Let  $I$  and  $I_{map}$  be the indicator rv, representing the correct decision by any general estimator and the MAP, respectively.

$$\mathbb{E}[I|X=x] = \mathbb{P}[g(X) = \Theta|X=x] \leq \mathbb{P}[g_{map}(X) = \Theta|X=x] = \mathbb{E}[I_{map}|X=x]$$

Thus, **Claim 1** holds. We now take the expectation of the above equations, the law of iterated expectations leads to **Claim 2**.

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- Unknown:  $\theta$  modeled by  $\Theta$  with prior  $f_{\Theta}(\cdot)$ . Assume  $\Theta \sim \text{Uniform}[4, 10]$ .
- No observations available
- MAP estimate
  - Any value  $\hat{\theta}_{map} \in [4, 10]$  (why? posterior = prior), not very useful
- What is your other choice?
  - Expectation:  $\hat{\theta} = \mathbb{E}[\Theta] = 7$
  - looks reasonable, but why?
- Because it minimizes mean squared error (MSE)

$$\min_{\hat{\theta}} \mathbb{E}[(\Theta - \hat{\theta})^2] = \min_{\hat{\theta}} \left( \text{var}(\Theta - \hat{\theta}) + (\mathbb{E}[\Theta - \hat{\theta}])^2 \right) = \min_{\hat{\theta}} \left( \text{var}(\Theta) + (\mathbb{E}[\Theta - \hat{\theta}])^2 \right)$$

- minimized when  $\hat{\theta} = \mathbb{E}[\Theta]$ .

- Unknown:  $\theta$  modeled by  $\Theta$  with prior  $f_{\Theta}(\cdot)$ .
- Observation  $X = x$  with model  $f_{X|\Theta}(x|\theta)$
- Minimizing conditional mean squared error

$$\min_{\hat{\theta}} \mathbb{E}[(\Theta - \hat{\theta})^2 | X = x]$$

- minimized when  $\hat{\theta} = \mathbb{E}[\Theta | X = x]$ .
- LMS estimator  $\hat{\Theta} = \mathbb{E}[\Theta | X]$
- Performance (MSE: Mean Squared Error)
  - When  $X = x$ ,  $\mathbb{E}[(\Theta - \mathbb{E}[\Theta | X = x])^2 | X = x] = \text{var}(\Theta | X = x)$
  - Averaged over  $X$ :  $\mathbb{E}[(\Theta - \mathbb{E}[\Theta | X])^2] = \mathbb{E}[\text{var}(\Theta | X = x)]$

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- Romeo and Juliet start dating.
  - Romeo: late by  $X \sim U[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim U[0, 1]$ .

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

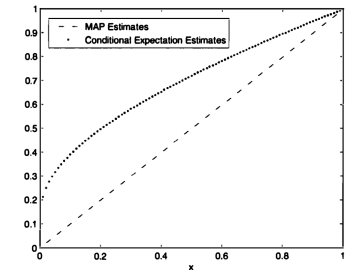
$$f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_{\Theta|X}(\theta|x) &= \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'} \\ &= \frac{1/\theta}{\int_x^1 \frac{1}{\theta'}d\theta'} = \frac{1}{\theta|\log x|}, \quad x \leq \theta \leq 1, \end{aligned}$$

and  $f_{\Theta|X}(\theta|x) = 0$ ,  $\theta < x$  or  $\theta > 1$ .

- MAP rule
  - $\hat{\theta}_{\text{MAP}} = x$ .
- LMS estimator

$$\begin{aligned} \hat{\theta}_{\text{LMS}} &= \mathbb{E}[\theta | X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta \\ &= (1-x)/|\log x| \end{aligned}$$



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- Remind.** If  $\Theta \sim \text{Beta}(\alpha, \beta)$ , then  $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

- Fact.** If  $\Theta \sim \text{Beta}(\alpha, \beta)$ ,

$$\mathbb{E}[\Theta] = \frac{1}{B(\alpha, \beta)} \int_0^1 \theta \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\alpha}{\alpha + \beta}$$

- Using the above fact,

$$\mathbb{E}[\Theta | X = k] = \frac{k + \alpha}{k + \alpha + n - k + \beta} = \frac{k + \alpha}{\alpha + \beta + n}$$

- For  $\alpha = \beta = 1$  ( $\Theta = \text{Uniform}[0, 1]$ ),

$$\mathbb{E}[\Theta | X = k] = \frac{k+1}{n+2}$$

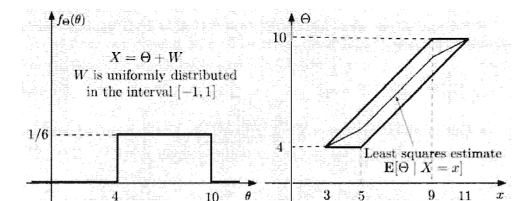
- Unknown:  $\Theta \sim \text{Uniform}[4, 10]$
- Observe  $\Theta$  with random error  $W$  as  $X$ .  $W \sim \text{Uniform}[-1, 1]$

$$X = \Theta + W$$

- Given  $\Theta = \theta$ ,  $X = \theta + W \sim \text{Uniform}[\theta - 1, \theta + 1]$ .

$$f_{\Theta, X}(\theta, x) = f_{\Theta}(\theta)f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}, & \text{if } 4 \leq \theta \leq 10, \theta - 1 \leq x \leq \theta + 1, \\ 0, & \text{otherwise} \end{cases}$$

-  $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$  = midpoint of the corresponding vertical section



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- Unknown:  $\Theta \sim \text{Uniform}[4, 10]$
- Observe  $\Theta$  with random error  $W$  as  $X$ .  $W \sim \text{Uniform}[-1, 1]$

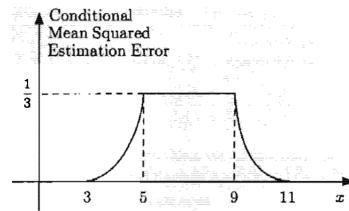
$$X = \Theta + W$$

- Given  $\Theta = \theta$ ,  $X = \theta + W \sim \text{Uniform}[\theta - 1, \theta + 1]$ .

$$f_{\Theta, X}(\theta, x) = f_{\Theta}(\theta)f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}, & \text{if } 4 \leq \theta \leq 10, \theta - 1 \leq x \leq \theta + 1, \\ 0, & \text{otherwise} \end{cases}$$

- Conditional MSE

$$\mathbb{E}[(\Theta - \mathbb{E}[\Theta|X=x])^2 | X=x]$$



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$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{f_X(x)}$$

$$f_X(x) = \int f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'$$

- Observation model  $f_{X|\Theta}(x|\theta)$  may not be always available
- Finding the posterior distribution is hard for multi-dimensional  $\Theta$
- $\Theta$  is very often high-dimensional, especially in the era of big data and deep learning
  - AlexNet in image recognition: 61M parameters (though not a Bayesian inference)
- Any alternative to LMS estimator?

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- Give up optimality, but choose a simple, but good one.
- General estimators  $\hat{\Theta} = g(X)$ , LMS estimator  $\hat{\Theta}_{LMS} = \mathbb{E}[\Theta|X]$
- We consider a restricted class of  $g(X)$ :  $\hat{\Theta} = aX + b$ .
- Our goal is:

$$\min_{a,b} \mathbb{E}[(\Theta - aX - b)^2]$$

- Linear models are always the first choice for a simple design in engineering.

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## LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) = \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X} (X - \mathbb{E}(X))$$

- No distributions on  $\Theta$  and  $X$ : only means, variances, and covariances
- MSE  $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$ ? Assume  $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$ .  $\mathbb{E}[(\Theta - \rho \frac{\sigma_\Theta}{\sigma_X} X)^2] = (1 - \rho^2) \text{var}[\Theta]$ 
  - Uncertainty about  $\Theta$  **decreases** by the factor of  $1 - \rho^2$
  - What happens if  $|\rho| = 1$  or  $\rho = 0$ ?
- If  $\rho > 0$ :
  - Baseline ( $\mathbb{E}[\Theta]$ ) + correction term
  - If  $X > \mathbb{E}[X] \Rightarrow \hat{\Theta}_L > \mathbb{E}[\Theta]$
  - If  $X < \mathbb{E}[X] \Rightarrow \hat{\Theta}_L < \mathbb{E}[\Theta]$
- If  $\rho = 0$  (uncorrelated):
  - Just baseline ( $\mathbb{E}[\Theta]$ )
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data  $X$

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$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) \quad (1)$$

$$= \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X} (X - \mathbb{E}(X)) \quad (2)$$

$$\min_{a,b} \text{ERR}(a, b) = \min_{a,b} \mathbb{E}[(\Theta - aX - b)^2]$$

- Assume  $a$  was found.

$$\mathbb{E}[(Y - b)^2], \quad Y = \Theta - aX$$

- Minimized when  $b = \mathbb{E}(Y) = \mathbb{E}(\Theta) - a\mathbb{E}(X)$ .

$$\begin{aligned} \text{ERR}(a, b) &= \mathbb{E}[(Y - \mathbb{E}(Y))^2] = \text{var}(Y) \\ &= \text{var}[\Theta] + a^2 \text{var}[X] - 2a \text{cov}(\Theta, X) \end{aligned} \quad (3)$$

- (3) is minimized when  $a = \frac{\text{cov}(\Theta, X)}{\text{var}[X]}$ . Then,

$$\begin{aligned} \hat{\Theta}_L &= aX + b = aX + \mathbb{E}(\Theta) - a\mathbb{E}(X) \\ &= (1) \end{aligned}$$

- Using  $\rho = \frac{\text{cov}(\Theta, X)}{\sigma_\Theta \sigma_X}$ , we get:

$$a = \frac{\rho \sigma_\Theta \sigma_X}{\sigma_X^2} = \frac{\rho \sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

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## Example: Romeo and Juliet

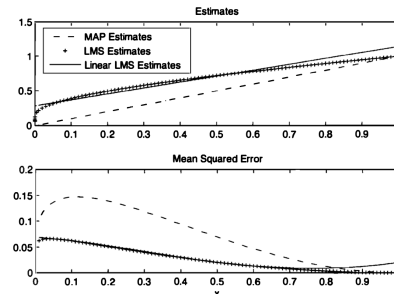
- Romeo and Juliet start dating. Romeo: late by  $X \sim U[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim U[0, 1]$ .
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[\Theta/2] = 1/4$
- Using  $\mathbb{E}[\Theta] = 1/2$  and  $\mathbb{E}[\Theta^2] = 1/3$ ,
 
$$\begin{aligned} \text{var}[X] &= \mathbb{E}[\text{var}[X|\Theta]] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \frac{1}{12} \mathbb{E}[\Theta^2] + \frac{1}{4} \text{var}[\Theta] = \frac{7}{144} \end{aligned}$$
- $\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta] \mathbb{E}[X]$ 

$$\begin{aligned} \mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6 \end{aligned}$$

$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

- LLMS estimator is:

$$\begin{aligned} \hat{\Theta}_L &= \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) \\ &= \frac{1}{2} + \frac{1/24}{7/144} (X - \frac{1}{4}) = \frac{6}{7} X + \frac{2}{7} \end{aligned}$$



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## Example: Biased Coin with Uniform Prior

- Biased coin with probability of head  $\theta$
- Unknown  $\Theta \sim \text{uniform}[0, 1]$ ,
  - $\mathbb{E}[\Theta] = 1/2$ ,  $\text{var}[X] = 1/12$
- $n$  tosses,  $X$ : number of heads.
- $p_{X|\Theta}(k|\theta)$ :  $\text{Binomial}(n, \theta)$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\Theta]] = \mathbb{E}[n\Theta] = n/2$

$$\begin{aligned} \text{var}(X) &= \mathbb{E}[\text{var}(X|\Theta)] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[n\Theta(1 - \Theta)] + \text{var}[n\Theta] \\ &= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12} = \frac{n(n+2)}{12} \end{aligned}$$

$$\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta] \mathbb{E}[X] = \mathbb{E}[\Theta X] - n/4$$

$$\begin{aligned} \mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[n\Theta^2] = n/3 \end{aligned}$$

$$\text{cov}(\Theta, X) = \frac{n}{3} - \frac{n}{4} = \frac{12}{n}$$

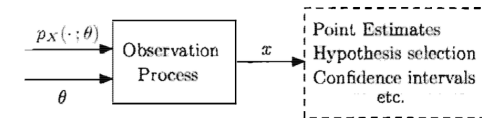
$$\hat{\Theta}_L = \frac{1}{2} + \frac{n/12}{n(n+2)/12} (X - \frac{n}{2}) = \frac{X+1}{n+2}$$

- What was the LMS estimator?  $\frac{X+1}{n+2}$   
 - Same! Intuitive?  
 Yes, because the LMS estimator was linear.

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- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator



- Unknown  $\theta$ 
  - **deterministic (not random)** quantity (thus, no prior distribution)
  - No prior, No posterior probabilities
- Observations or measurements  $X$ 
  - Random observation  $X$ 's distribution just depends on  $\theta$
  - Notation:  $p_X(x; \theta)$  and  $f_X(x; \theta)$ ,  $\theta$ -parameterized distribution of observations
- Choosing one among multiple probabilistic models
  - Each  $\theta$  corresponds to a probabilistic model

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- Problem types
  - **Estimation**
  - Hypothesis testing
  - Significance testing
- Key inference methods
  - **ML (Maximum Likelihood) estimation**
  - Linear regression
  - Likelihood ratio test
  - Significant testing
- Just a taste in this course due to time constraint.

- Random observation  $x = (x_1, x_2, \dots, x_n)$  of  $X = (X_1, X_2, \dots, X_n)$ 
  - Assume a scalar  $\theta$  and a vector of observation in this lecture.
- **Likelihood  $p_X(x_1, x_2, \dots, x_n; \theta)$** 
  - $p_X(x_1, x_2, \dots, x_n; \theta)$ 
    - NOT the probability that the unknown parameter is equal to  $\theta$ .
    - but, the probability that **the observed value  $x$  arises when the parameter is  $\theta$** .
  - ML (Maximum Likelihood) estimation

$$\hat{\theta}_{ml} = \arg \max_{\theta} p_X(x_1, x_2, \dots, x_n; \theta)$$

- Very often,  $X_i$  are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \dots, x_n; \theta) = \log \prod_{i=1}^n p_{X_i}(x_i; \theta) = \sum_{i=1}^n \log p_{X_i}(x_i; \theta)$$

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- ML and MAP: How are they related?

- MAP in the Bayesian inference

$$\hat{\theta}_{map} = \arg \max_{\theta} p_{\Theta|X}(\theta|x) = \arg \max_{\theta} \frac{p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{p_X(x)} = \frac{1}{p_X(x)} \arg \max_{\theta} p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)$$

- ML in the classical inference

$$\hat{\theta}_{ml} = \arg \max_{\theta} p_X(x; \theta)$$

- $p_{X|\Theta}(x|\theta)$  in the Bayesian setting corresponds to  $p_X(x; \theta)$  in the classical setting.
- When  $\Theta$  is **uniform** (complete ignorance of  $\Theta$ ), MAP == ML

- Romeo and Juliet start dating. Romeo: late by  $X \sim U[0, \theta]$ .

- Unknown:  $\theta$  modeled by a rv  $\Theta \sim U[0, 1]$ .

- MAP:  $\hat{\theta}_{MAP} = x$

- LMS:  $\hat{\theta}_{LMS} = (1 - x)/|\log x|$

- LLMS:  $\hat{\theta}_L = \frac{6}{7}x + \frac{2}{7}$

- ML:  $\hat{\theta}_{ML} = \hat{\theta}_{MAP} = x$

- $n$  identical, independent exponential rvs,  $X_1, X_2, \dots, X_n$  with parameter  $\theta$ .

- Observation  $x_1, x_2, \dots, x_n$

- What is the ML estimate of  $\theta$ ?

- **Reminder.**  $X \sim \exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \mathbb{E}[X] = 1/\lambda$$

- Any guess?  $\hat{\theta}_{ML} = \frac{n}{x_1 + x_2 + \dots + x_n}$

$$\arg \max_{\theta} f_X(x; \theta) = \arg \max_{\theta} \prod_{i=1}^n \theta e^{-\theta x_i} = \arg \max_{\theta} \left( n \log \theta - \theta \sum_{i=1}^n x_i \right)$$

Questions?

- 1) What is statistical inference?
- 2) Draw the building blocks of Bayesian inference and explain how it works.
- 3) What are MAP and LMS estimators and their underlying philosophies?
- 4) What is LLMS estimator and why is it useful?
- 5) Compare the classical and Bayesian inference.
- 6) What is the ML estimator and how is it related to the MAP estimator?