

Lecture 7: Random Processes, Part I

Yi, Yung (이웅)

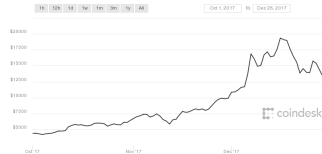
EE210: Probability and Introductory Random Processes
KAIST EE

June 2, 2021

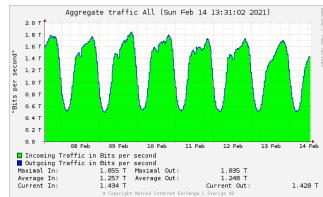
- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
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- (5) Playing with Bernoulli and Poisson Processes

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- Many probabilistic experiments that evolve in time

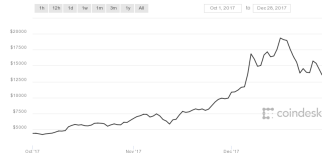


(a) Prices of a cryptocurrency

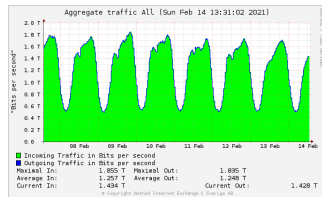


(b) Internet traffic traces

- Many probabilistic experiments that **evolve in time**
 - Sequence of daily prices of a stock
 - Sequence of scores in football
 - Sequence of failure times of a machine
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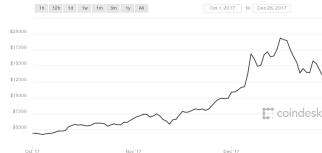


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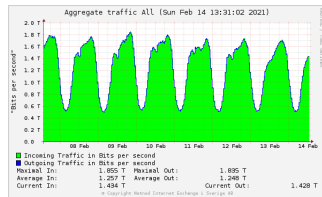


(b) Internet traffic traces

- Many probabilistic experiments that **evolve in time**
 - Sequence of daily prices of a stock
 - Sequence of scores in football
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- Random process is a mathematical model for it.



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- For a **fixed** time t , X_t (or $X(t)$) is a random variable.
- The values that X_t (or $X(t)$) can take: discrete or continuous

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 - $X(3.7, \omega_1) = 3409$, $X(2, \omega_2) = 5000$, $X(7.8, \omega_3) = 2800$, etc.

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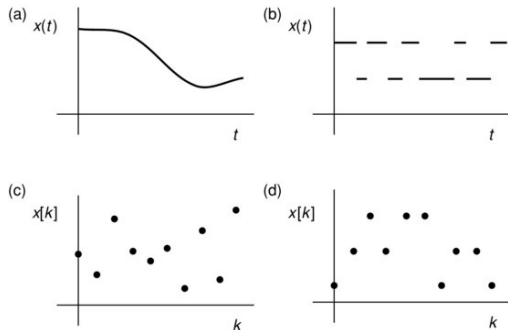
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 - Other interesting questions, depending on the target random process

- Types of time and value

- (a) continuous time, continuous value
- (b) continuous time, discrete value
- (c) discrete time, continuous value
- (d) discrete time, discrete value



- The simplest RP
- discrete time

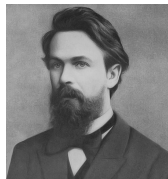
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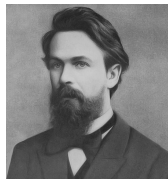
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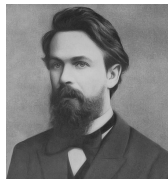
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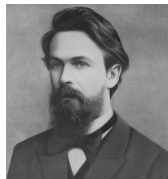


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- $[X(s)]_{s=0}^t \perp\!\!\!\perp [X(s)]_{s=t}^{t+a}$

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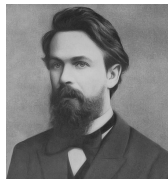


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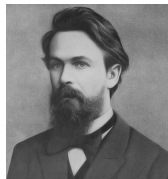
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- One-step more general than BP/PP
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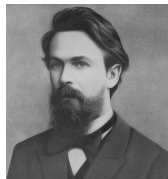
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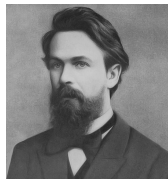
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- “today” independent of “past”

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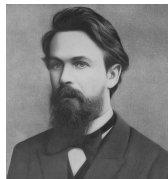
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- Discrete time, discrete value
- One of the simplest random processes
- A type of “arrival” process

Please pause the video and write down the questions that you want to ask about Bernoulli process.

VIDEO PAUSE

Q1.

Q2.

Q3.

Q4.

Q5.

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- This will hold for any n consecutive slots.

(Q2) # of slots T_1 until the first arrival?

(Q1) # of arrivals in the first n slots?

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- Still, geometric.
- But, more than that, as we will see. Independence across time slots leads to many useful properties, allowing the quick solution of many problems.

Independence across slots \implies the fresh-start anytime when I look at the process?

(Q3) $U = X_1 + X_2 \perp\!\!\!\perp V = X_5 + X_6?$

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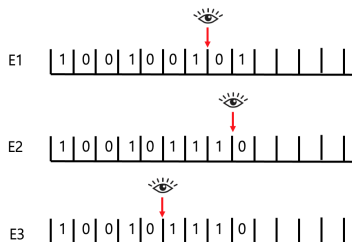
- $(X_1, \dots, X_5) \perp\!\!\!\perp (X_n)_{n=6}^\infty$
- **Fresh-start** after a deterministic time n (doesn't matter what happened until $n = 5$).
- If you watch the on-going Bernoulli process(p) from some time n , you still see the same Bernoulli process(p).

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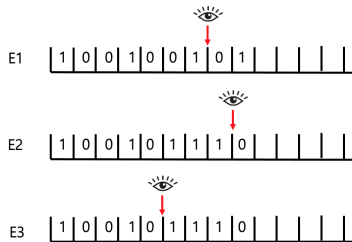
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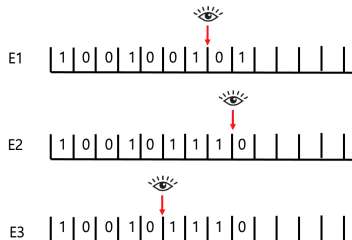


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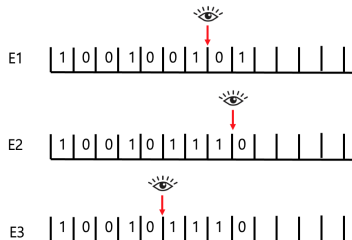
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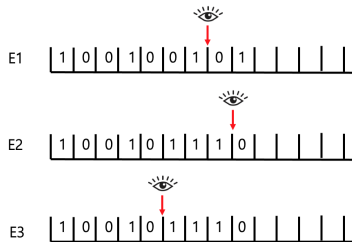
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- Difference of N from n
 - The time when I watch the on-going Bernoulli process is **random**.
 - N is a random variable, i.e., $N : \Omega \mapsto \mathbb{R}$. What is Ω ?

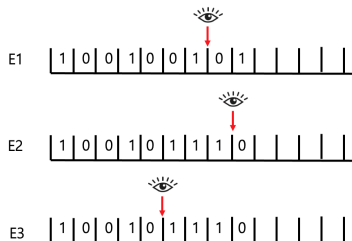
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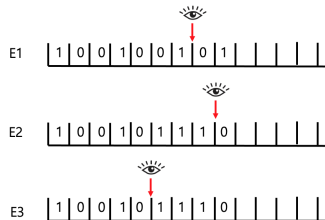
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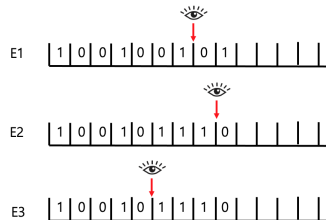


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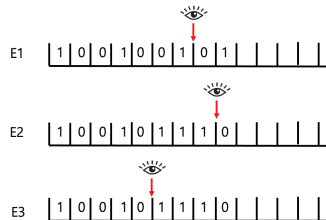
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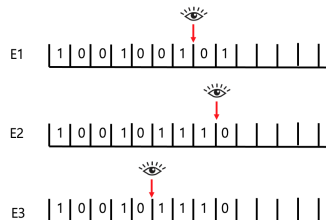
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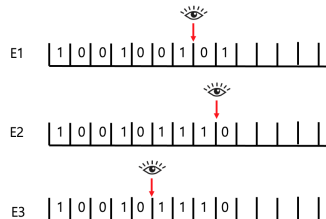
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- The question of $N = n$? can be answered just from the knowledge about X_1, X_2, \dots, X_n ? Then, Yes! (see pp. 301 for more formal description)

- In probability theory, a random time N is said to be a **stopping time**, if the question of “ $N = n$?” can be answered only from the present and the past knowledge of X_1, X_2, \dots, X_n .

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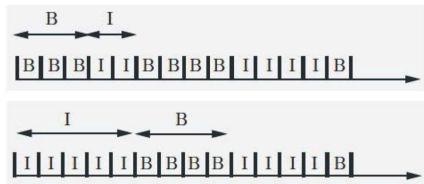
VIDEO PAUSE

 - **Yes.** Time when 10 consecutive arrivals have been observed

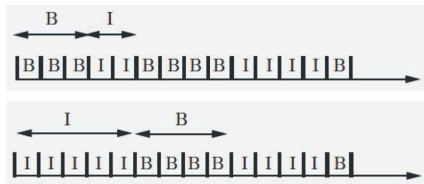
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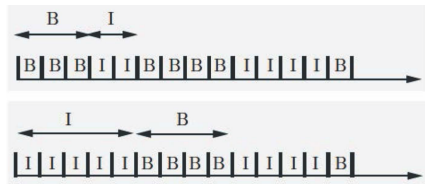


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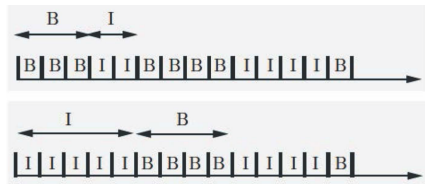
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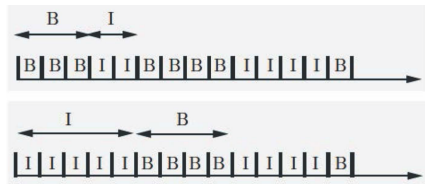
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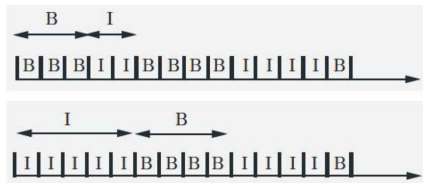


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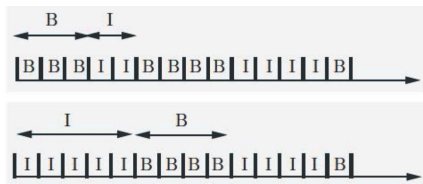
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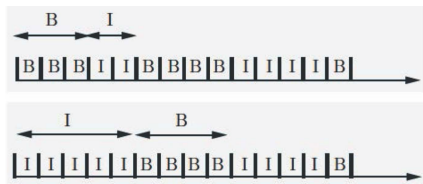
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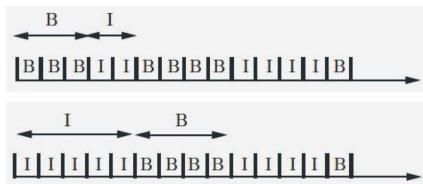
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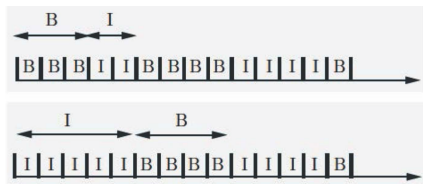
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- B_3, B_4, \dots ?

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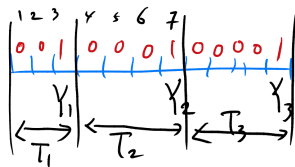
VIDEO PAUSE

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VIDEO PAUSE

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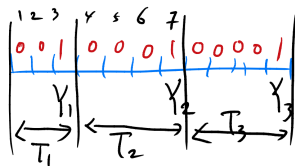


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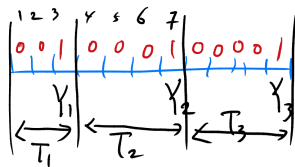


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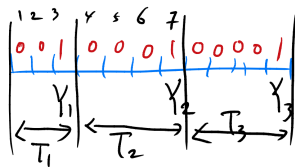
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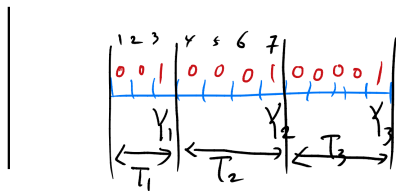
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- We know Y_k 's expectation and variance: $\mathbb{E}[Y_k] = \frac{k}{p}$, $\text{var}[Y_k] = \frac{k(1-p)}{p^2}$, but **its distribution?**

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- $\text{Pascal}(1, p) = \text{Geom}(p)$

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes

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$$p_S(k) = \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

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- Need a “modeling sense” to make this possible. It’s a good practice for engineers!
- **VIDEO PAUSE**

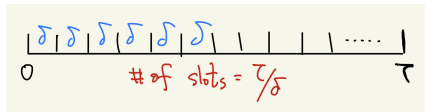
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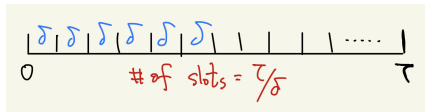
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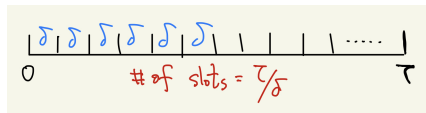
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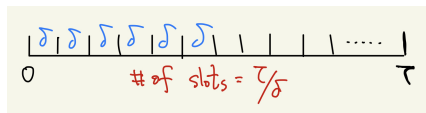


- What's the limit as $\delta \rightarrow 0$ (equivalently, $n \rightarrow \infty$)



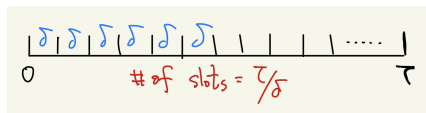
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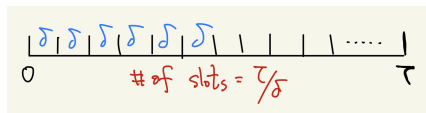
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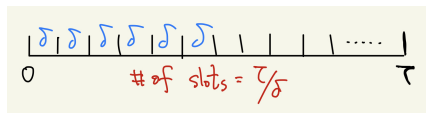
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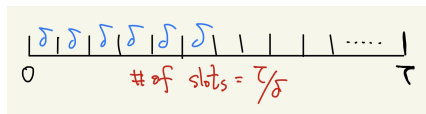
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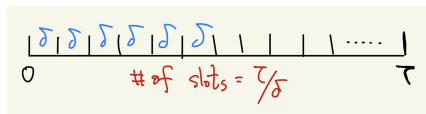
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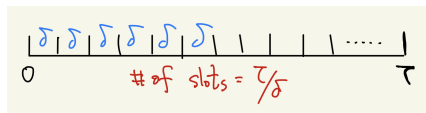
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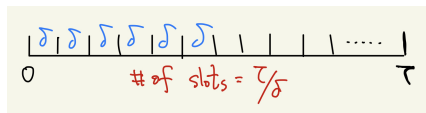
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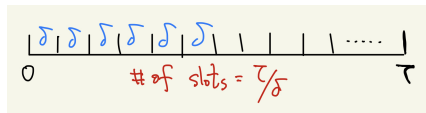
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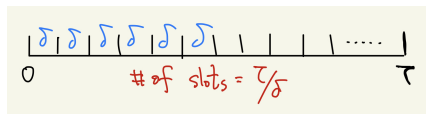
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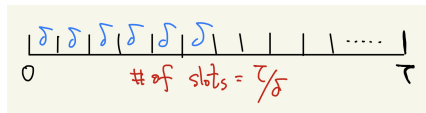
- $o(\delta)$: some function that goes to zero faster than δ .
 - Thus, for very small δ , $o(\delta)$ becomes negligible, compared to δ .
 - Example: $o(\delta) = \delta^\alpha$, where any $\alpha > 1$



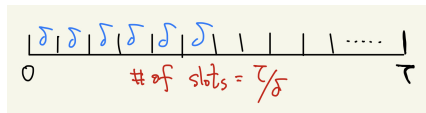
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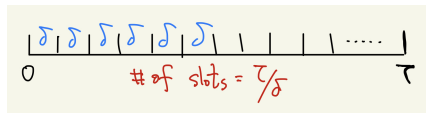
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- This is a continuous twin process of Bernous process, which we call **Poisson process**.

- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
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$$\mathbb{P}(k, \tau) = o_k(\tau) \quad \text{for } k = 2, 3, \dots, \quad \text{where} \quad \lim_{\tau \rightarrow 0} \frac{o(\tau)}{\tau} = 0, \quad \lim_{\tau \rightarrow 0} \frac{o_k(\tau)}{\tau} = 0$$

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- $T \sim \text{Exp}(\lambda)$. Thus, $\mathbb{E}(T) = 1/\lambda$ and $\text{var}(T) = 1/\lambda^2$
 - Continuous twin of geometric rv in Bernoulli process
 - Memoryless

- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
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- Receive emails according to a Poisson process at rate $\lambda = 5$ messages/hour
- Mean and variance of mails received during a day
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- $\mathbb{E}[Y_k] = k/\lambda$ and $\text{var}[Y_k] = k/\lambda^2$, but what is the distribution of Y_k ?

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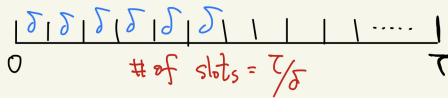
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- This is called **Erlang** rv.

An Erlang random variable Z with parameter (k, λ) has the following pdf:

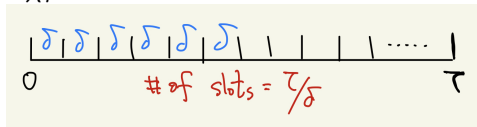
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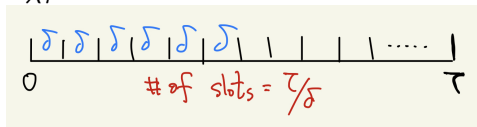
	Bernoulli process	Poisson process
time of arrival	Discrete	Continuous
PMF of # of arrivals		
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Time of k -th arrival		
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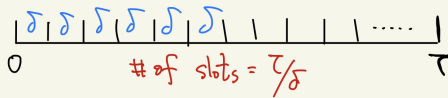
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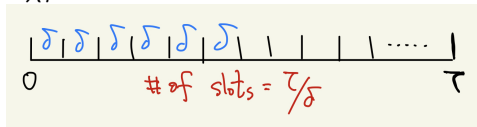
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- (1) Introduction of Random Processes
- (2) Bernoulli Processes
- (3) Poisson Processes: Poisson RV and Basic Idea
- (4) Poisson Processes: Definition and Properties
- (5) Playing with Bernoulli and Poisson Processes

Alternative Description of the Bernoulli Process

1. Start with a sequence of independent geometric random variables T_1, T_2, \dots , with common parameter p , and let these stand for the interarrival times.
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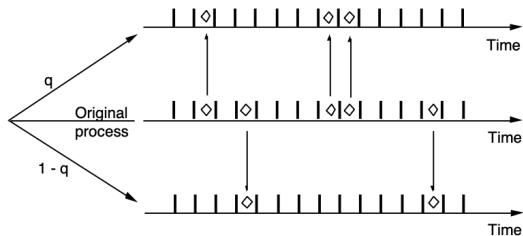
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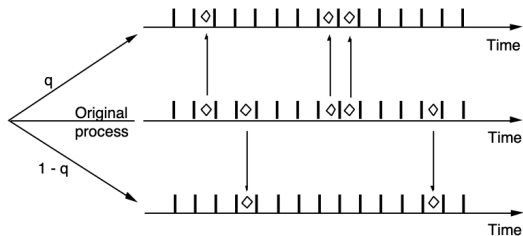
- **Question.** How to make software codes of Bernoulli process with p and Poisson process with λ
- Inter-arrival times are very handy.
- **Bernoulli process with p :** Obtain a sequence of random values following the **geometric distribution** with parameter p .
- **Poisson process with λ :** Obtain a sequence of random values following the **exponential distribution** with parameter λ .

- Bernoulli random variable: $\text{Bern}(p)$
- Bernoulli process: $\text{BP}(p)$
- Poisson random variable: $\text{Poisson}(\lambda)$
- Poisson process: $\text{PP}(\lambda)$

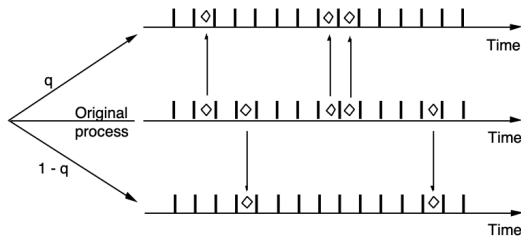
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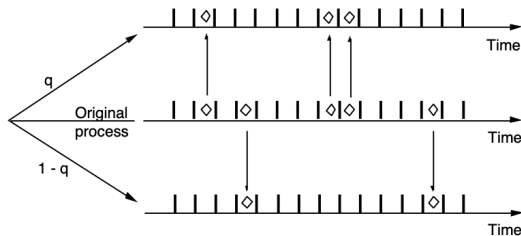
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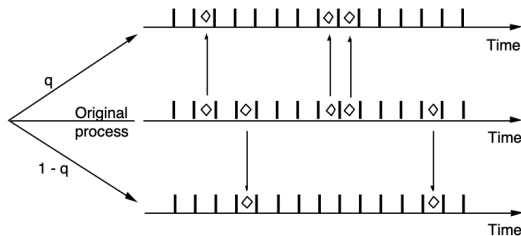
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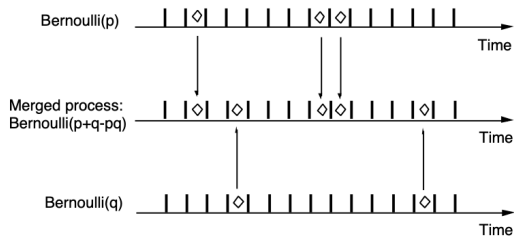
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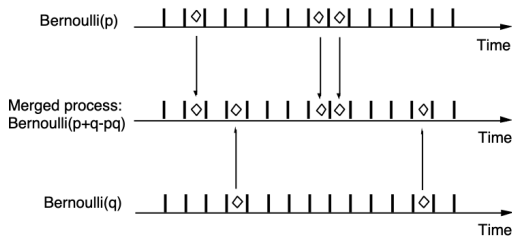
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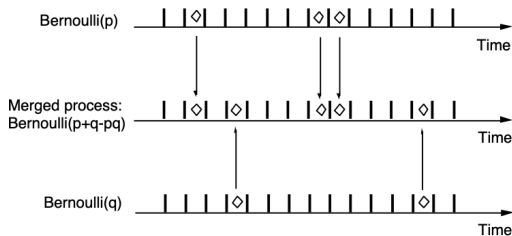
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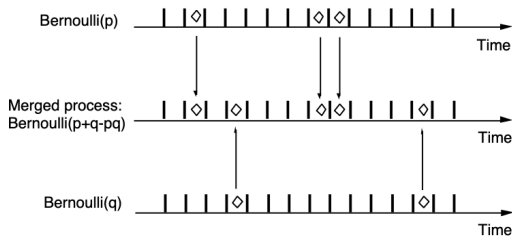
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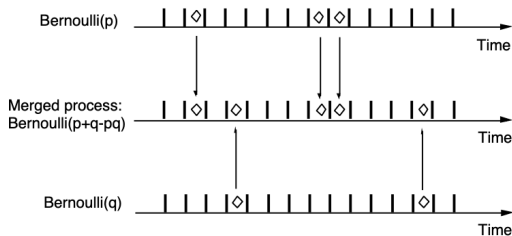
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1. Competing exponentials
2. Sum of independent Poisson rvs
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- Let L be the number of arrivals from $PP(\lambda)$ until we get the first arrival from $PP(\nu)$.

$$p_L(l) = p_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right) \left(\frac{\lambda}{\lambda+\nu}\right)^l, \quad l = 0, 1, \dots$$

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 - $p = 1/100, n = 10,000$: small p , but large $n \implies$ **Both Poisson and Normal**

Questions?

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