Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

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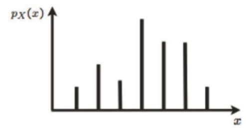
Continuous RV and Probability Density Function

- Many cases when random variables have "continuous values", e.g., velocity of a car

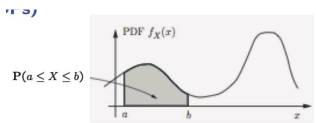
A rv X is continuous if $\exists$ a function f_X, called $\boxed{\text{probability density function (PDF)}}$, s.t.

$$\mathbb{P}(X \in B) = \int_B f_X(x)dx, \qquad \text{every subset } B \in \mathbb{R}$$

- All of the concepts and methods (expectation, PMFs, and conditioning) for discrete rvs have continuous counterparts

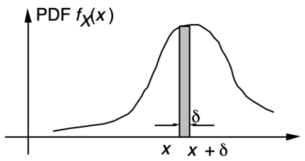


- $\mathbb{P}(a \le X \le b) = \sum_{x:a\le x\le b} p_X(x)$
- $p_X(x) \ge 0$, $\sum_x p_X(x) = 1$



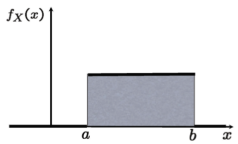
- $\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$
- $f_X(x) \ge 0$, $\int_{-\infty}^{\infty} f_X(x)dx = 1$

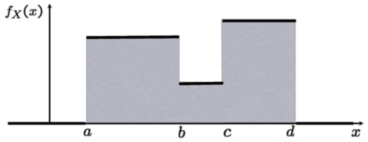
PDF and Examples



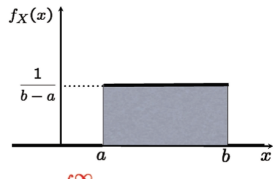
- $\mathbb{P}(a \le X \le a+\delta) \approx \boxed{f_X(a)\cdot\delta}$
- $\mathbb{P}(X = a) = 0$

Examples





Expectation and Variance



- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{b+a}{2}$
- $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{b-a}\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3}$
- $\text{var}[X] = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$

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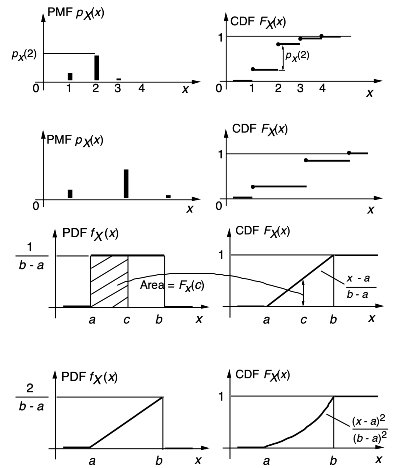
Cumulative Distribution Function (CDF)

- Discrete: PMF, Continuous: PDF
- Can we describe all rvs with a single mathematical concept?

$$F_X(x) = \mathbb{P}(X \le x) = \begin{cases} \sum_{k \le x} p_X(k), & \text{discrete} \\ \int_{-\infty}^{x} f_X(t)dt, & \text{continuous} \end{cases}$$

- always well defined, because we can always compute the probability for the event $\{X \le x\}$
- CCDF (Complementary CDF): $\mathbb{P}(X > x)$



CDF Properties

- Non-decreasing
- $F_X(x)$ tends to 1, as $x \to \infty$ and $F_X(x)$ tends to 0, as $x \to -\infty$
- If X is discrete,
  - $F_X(x)$ is a piecewise constant function of x.
  - $p_X(k) = F_X(k) - F_X(k-1)$
- If X is continuous
  - $F_X(x)$ is a continuous function of x.
  - $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ and $f_X(x) = \frac{dF_X}{dx}(x)$

Example: Maximum of Random Variables

- Take a test three times, and your final score will be the maximum of test scores
- $X = \max\{X_1, X_2, X_3\}$, and $X_i \in \{1, 2, \cdots, 10\}$ uniformly at random
- Question. $p_X(x)$?
- Approach 1: $\mathbb{P}(\max\{X_1, X_2, X_3\} = x)$?
- Approach 2

$$F_X(x) = \mathbb{P}(\max\{X_1, X_2, X_3\} \le x) = \mathbb{P}(X_1 \le x, X_2 \le x, X_3 \le x)$$
$$= \mathbb{P}(X_1 \le x) \cdot \mathbb{P}(X_2 \le x) \cdot \mathbb{P}(X_3 \le x) = \left(\frac{x}{10}\right)^3$$

Thus,

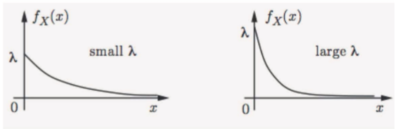
$$p_X(x) = \left(\frac{x}{10}\right)^3 - \left(\frac{x-1}{10}\right)^3, \quad x = 1, 2, \cdots, 10$$

Roadmap

Exponential RV with parameter $\lambda > 0$

- A rv X is called exponential with λ, if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$



- CDF $F_X(x) = \int_0^x \lambda e^{-\lambda s} ds = 1 - e^{-\lambda x}$
- CCDF $\mathbb{P}(X > x) = e^{-\lambda x}$
- (Check) $\mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$, $\mathrm{var}[X] = 1/\lambda^2$

Exponential RV: Mean and Variance

- $\mathbb{E}(X) = 1/\lambda$. Use integration by parts: $\int u dv = uv - \int v du$

$$\int_0^\infty x\lambda e^{-\lambda x} dx = (-xe^{-\lambda x})\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}$$

- $\mathbb{E}(X^2)$

$$\int_0^\infty x^2\lambda e^{-\lambda x} dx = (-x^2 e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx = 0 + \frac{2}{\lambda}\mathbb{E}(X) = \frac{2}{\lambda^2}$$

- $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{\lambda^2}$

Exponential RV: Model of Continuous Waiting Time

- $\mathbb{P}(X > x) = e^{-\lambda x}$
- Appropriate for modeling a waiting time until an incident of interest takes place
  - $\mathbb{P}(X > x)$: exponentially decays
  - message arriving at a computer, some equipment breaking down, a light bulb burning out, etc
- (Q) What is the discrete rv which models a waiting time? Geometric
- What is the relationship between exponential rv and geometric rv? We will see this relationship soon, but let's look at an example first.

Example

- A very small meteorite first lands anywhere in Korea

- Time of landing is modeled as an exponential rv with mean 10 days
- The current time is midnight. What is the probability that a meteorite first lands some time between 6 a.m. and 6 p.m. of the first day? VIDEO PAUSE
- (Solution)
  - $\mathbb{E}(X) = 1/\lambda = 10$. Thus, $\lambda = \frac{1}{10}$.
  - 6 a.m. from midnight $= 1/4$ day, 6 p.m. from midnight $= 3/4$ day

$$\mathbb{P}(1/4 \le X \le 3/4) = \mathbb{P}(X \ge 1/4) - \mathbb{P}(X \ge 3/4) = e^{-1/40} - e^{-3/40} = 0.0476$$

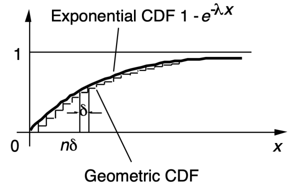
Geometric vs. Exponential (1)

- Models a system evolution over time: Continuous time vs. Discrete time.
  - Example. Customer arrivals at my shop
  - Modeling 1: Every 30 minute I record the number of customers for each 30-min window
  - Modeling 2: I record the exact time of each customer's arrival
  - In modeling 1, every 10 minute? every 1 minute? every 1 sec? every 0.0000001 sec?
- In many cases, continuous case is some type of limit of its corresponding discrete case.
- Can we mathematically describe how geometric and exponential rvs meet each other in the limit?

Geometric vs. Exponential (2)

- 'slot' is one unit time, e.g., 1 hour, 30 mins, 1 min, 10 sec, etc.
- Continuous system = Discrete system with
  - infinitely many slots whose duration is infinitely small.
  - success probability p over one slot decreases to 0 in the limit
- Given $X^{exp} \sim \exp(\lambda)$, let us construct a geometric RV X_δ^{geo}
  - Set the length of a slot to be δ, which is a parameter.
  - Set the success probability p_δ over a slot to be $p_\delta = 1 - e^{-\lambda\delta}$ (this looks magical, whose secrete will be uncovered soon)
  - $\mathbb{P}(X_\delta^{geo} \le n) = 1 - (1 - p_\delta)^n = 1 - e^{-\lambda\delta n}$

Geometric vs. Exponential (3)



- Note that $\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda x}$. Then, when $x = n\delta$, $n = 1, 2, \ldots$

$$\mathbb{P}(X^{exp} \le x) = 1 - e^{-\lambda\delta n} = \mathbb{P}(X_\delta^{geo} \le n)$$

- If we choose sufficiently small δ, the slot length $\downarrow$ and $p_\delta \downarrow$

$$\boxed{\mathbb{P}(X_\delta^{geo} \le n) \xrightarrow{\delta \to 0} \mathbb{P}(X^{exp} \le x),\ x = n\delta}$$

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Normal: PDF, Expectation, Variance

- Standard Normal $\mathcal{N}(0, 1)$

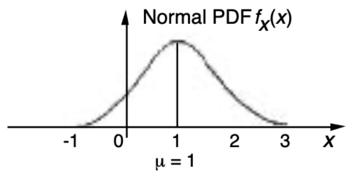
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

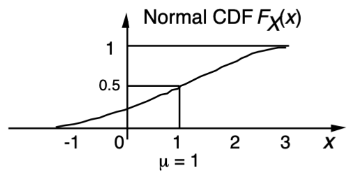
- $\mathbb{E}[X] = 0$
- $\mathrm{var}[X] = 1$

- General Normal $\mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- $\mathbb{E}[X] = \mu$
- $\mathrm{var}[X] = \sigma^2$





Check: PDF, Expectation, Variance

- PDF's normalization property: $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = 1$
  - A little bit boring :-). See Problem 14 at pp 189.
- Expectation
  - $f_X(x)$ is symmetric in terms of $x = \mu$. Thus, we should have $\mathbb{E}(X) = \mu$.
- Variance

$$\mathrm{var}(X) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}dx \overset{y=\frac{x-\mu}{\sigma}}{=} \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{-y^2/2}dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left(-ye^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy = \sigma^2$$

$\int udv = uv - \int vdu$: $u = y$ and $dv = ye^{-y^2/2} \rightarrow du = dy$ and $v = -e^{-y^2/2}$

Normality: Preserved under Linear Transformation

- Linear transformation preserves normality (we will verify this in Lecture 5)

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for $a \neq 0$ and b, $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

- Thus, every normal rv can be standardized:
  If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\boxed{Y = \frac{X-\mu}{\sigma}} \sim \mathcal{N}(0, 1)$
- Thus, we can make the table which records the following CDF values:
  $\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-t^2/2}dt$

Example

- Annual snowfall X is modeled as $\mathcal{N}(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?
- $Y = \frac{X-60}{20}$.

$$\mathbb{P}(X \geq 80) = \mathbb{P}(Y \geq \frac{80-60}{20})$$
$$= \mathbb{P}(Y \geq 1) = 1 - \Phi(1)$$
$$= 1 - 0.8413 = 0.1587$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Normal RVs: Why Important?

- Central limit theorem
  - One of the most remarkable findings in the probability theory
  - Sum of any random variables ≈ Normal random variable
- Modeling aggregate noise with many small, independent noise terms
- Convenient analytical properties, allowing closed forms in many cases
- Highly popular in communication and machine learning areas

[0] Central limit theorem: 중심극한정리

Continuous: Joint PDF and CDF (1)

Two continuous rvs are $\boxed{\text{jointly continuous}}$ if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for every subset B of the two dimensional plane,

$$\mathbb{P}((X,Y)\in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy,$$

1. The joint PDF is used to calculate probabilities

$$\mathbb{P}\Big[(X,Y)\in B\Big] = \iint_{(x,y)\in B} f_{X,Y}(x,y)dxdy$$

Our particular interest: $B = \{(x,y) \mid a \le x \le b, c \le y \le d\}$

Continuous: Joint PDF and CDF (2)

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

3. The joint CDF is defined by $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$, and determines the joint PDF as:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x,y)$$

4. A function $g(X,Y)$ of X and Y defines a new random variable, and

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$$

Continuous: Conditional PDF given an event

* Conditional PDF, given an event A

- $f_X(x)\cdot\delta \approx \mathbb{P}(x \le X \le x+\delta)$
  $f_{X|A}(x)\cdot\delta \approx \mathbb{P}(x \le X \le x+\delta | A)$
- $\mathbb{P}(X\in B) = \int_B f_X(x)dx$
  $\mathbb{P}(X\in B|A) = \int_B f_{X|A}(x)dx$
- $\int f_{X|A}(x)dx = 1$

* Conditional PDF, given $\{X \in C\}$

$$f_{X|\{X\in C\}}(x)\cdot\delta \approx \mathbb{P}(x \le X \le x+\delta | X\in C)$$

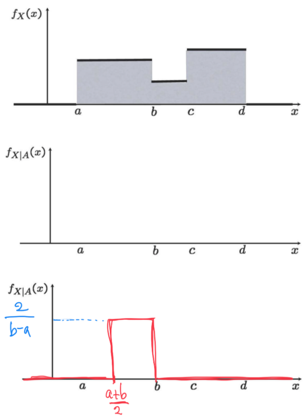
$$f_{X|\{X\in C\}}(x) = \begin{cases} 0, & \text{if} \quad x \notin C \\ \frac{f_X(x)}{\mathbb{P}(X\in C)}, & \text{if} \quad x \in C \end{cases}$$

(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

Notation: A is an event, but B and C is a subset that includes the possible values which can be taken by the rv X. Sorry for the confusion, if any.

Continuous: Conditional Expectation

$A = \{\frac{a+b}{2} \le X \le b\}$



- $\mathbb{E}[X] = \int x f_X(x) dx$
  $\mathbb{E}[X|A] = \int x f_{X|A}(x) dx$
- $\mathbb{E}[g(X)] = \int g(x) f_X(x) dx$
  $\mathbb{E}[g(X)|A] = \int g(x) f_{X|A}(x) dx$

$$\mathbb{E}[X|A] = \int_{(a+b)/2}^{b} x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^{b} x^2 \frac{2}{b-a} dx =$$

Exponential RV: Memoryless

- Remember: Exponential rv is a continuous counterpart of geometric rv.
- Thus, expected to be memoryless. Remember the definition?

  Definition. A random variable X is called memoryless if, for any $n, m \ge 0$,
  $\boxed{\mathbb{P}(X > n+m | X > m) = \mathbb{P}(X > n)}$
- Proof. Note that the exponential rv's CCDF $\mathbb{P}(X > x) = e^{-\lambda x}$. Then,

$$\mathbb{P}(X > n+m|X > m) = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)} = \frac{e^{-\lambda(n+m)}}{e^{-\lambda m}} = e^{-\lambda n} = \mathbb{P}(X > n)$$

Total Probability/Expectation Theorem

Partition of Ω into $A_1, A_2, A_3, \ldots$

* Discrete case

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

* Continuous case

Total Probability Theorem

$$f_X(x) = \sum_i \mathbb{P}(A_i) f_{X|A_i}(x)$$

Total Expectation Theorem

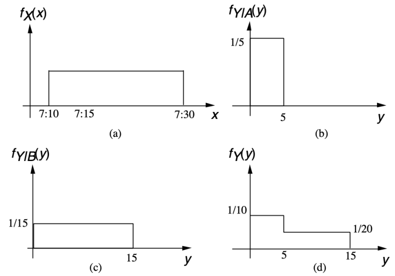
$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Example: Train Arrival

- The train's arrival every quarter hour (0, 15min, 30min, 45min).
- Your arrival $\sim \mathcal{U}$(7:10, 7:30) am.
- What is the PDF of waiting time for the first train?
- X : your arrival time, Y : waiting time.
- The value of X makes a different waiting time. So, consider two events:

  $A = \{7{:}10 \le X \le 7{:}15\}$

  $B = \{7{:}15 \le X \le 7{:}30\}$



VIDEO PAUSE

$$f_Y(y) = \mathbb{P}(A) f_{Y|A}(y) + \mathbb{P}(B) f_{Y|B}(y)$$

$$f_Y(y) = \frac{1}{4}\frac{1}{5} + \frac{3}{4}\frac{1}{15} = \frac{1}{10}, \quad \text{for } 0 \le y \le 5$$

$$f_Y(y) = \frac{1}{4}0 + \frac{3}{4}\frac{1}{15} = \frac{1}{20}, \quad \text{for } 5 < y \le 15$$

Continuous: Conditional PDF given a RV

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- $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- Similarly, for $f_Y(y) > 0$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Remember: For a fixed event A, $\mathbb{P}(\cdot|A)$ is a legitimate probability law.
- Similarly, For a fixed y, $f_{X|Y}(x|y)$ is a legitimate PDF, since

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx}{f_Y(y)} = 1$$

- Multiplication rule.

$$f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$$

- Total prob./exp. theorem.

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y)dy$$
$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y)\mathbb{E}[X|Y=y]dy$$

- Independence

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \quad \text{for all } x \text{ and } y$$

Example: Stick-breaking (1)

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(Prob 21 at pp. 191)

- Break a stick of length l twice
  - first break at $Y \sim \mathcal{U}[0,l]$
  - second break at $X \sim \mathcal{U}[0,Y]$

(a) joint PDF $f_{X,Y}(x,y)$?

$$f_Y(y) = \frac{1}{l}, \quad 0 \le y \le 1$$
$$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 \le x \le y$$

Using $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y)$,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{l}\cdot\frac{1}{y}, & 0 \le x \le y \le l, \\ 0, & \text{otherwise} \end{cases}$$

(b) marginal PDF $f_X(x)$?

$$f_X(x) = \int f_{X,Y}(x,y)dy = \int_x^l \frac{1}{ly}dy$$
$$= \frac{1}{l}\ln(l/x), \quad 0 \le x \le l$$

[0] $\mathcal{U}[a,b]$: continuous uniform random variable over the interval $[a,b]$

Example: Stick-breaking (2)

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(c) Evaluate $\mathbb{E}(X)$, using $f_X(x)$

$$\mathbb{E}(X) = \int_0^l x f_X(x)dx = \int_0^l \frac{x}{l}\ln(l/x)dx$$
$$= \frac{l}{4}$$

(d) Evaluate $\mathbb{E}(X)$, using $X = Y \cdot (X/Y)$

If $Y \perp\!\!\!\perp X/Y$, it becomes easy, but true? Yes, because whatever Y is, the fraction X/Y does not depend on it.

$$\mathbb{E}(X) = \mathbb{E}(Y)\mathbb{E}(X/Y) = \frac{l}{2}\cdot\frac{1}{2} = \frac{l}{4}$$

(e) Evaluate $\mathbb{E}(X)$, using TET

$$0\mathbb{E}[X] = \int_{-\infty}^{\infty} f_Y(y)\mathbb{E}[X|Y=y]dy$$
$$= \int_0^l \frac{1}{l}\mathbb{E}[X|Y=y]dy = \int_0^l \frac{1}{l}\frac{y}{2}dy = \frac{l}{4}$$

- Message. There are many ways to rearch our goal. Of crucial importance is how to find the best way!

Roadmap

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(1) Continuous Random Variable and PDF (Probability Density Function)

(2) CDF (Cumulative Distribution Function)

(3) Exponential RVs

(4) Gaussian (Normal) RVs

(5) Continuous RVs: Joint, Conditioning, and Independence

(6) Bayes' rule for RVs

Bayes Rule for Continuous

- X: state/cause/original value $\to$ Y: result/resulting action/noisy measurement
- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (cause $\to$ result)
- Inference: $\mathbb{P}(X|Y)$?

$$p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y|x) = p_Y(y)p_{X|Y}(x|y)$$
$$p_{X|Y}(x|y) = \frac{p_X(x)p_{Y|X}(y|x)}{p_Y(y)}$$
$$p_Y(y) = \sum_{x'} p_X(x')p_{Y|X}(y|x')$$

$$f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x) = f_Y(y)f_{X|Y}(x|y)$$
$$f_{X|Y}(x|y) = \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$
$$f_Y(y) = \int f_X(x')f_{Y|X}(y|x')dx'$$

Example

- A light bulb $Y \sim \exp(\lambda)$. However, there are some quality control problems. So, the parameter λ of Y is actually a random variable, denoted by Λ, which is $\Lambda \sim \mathcal{U}[1, 3/2]$. We test a light bulb and record its lifetime.
- Question. What can we say about the underlying paramter λ? In other words, what is $f_{\Lambda|Y}(\lambda|y)$?
- $f_\Lambda(\lambda) = 2$ for $1 \le \lambda \le 3/2$ and $f_{Y|\Lambda}(y|\lambda) =$ pdf of $\exp(\lambda)$. Then, the inference about the parameter given the lifetime of a light bulb is:

$$f_{\Lambda|Y}(\lambda|y) = \frac{f_\Lambda(\lambda)f_{Y|\Lambda}(y|\lambda)}{\int_{-\infty}^{\infty} f_\Lambda(t)f_{Y|\Lambda}(y|t)dt}$$

Using Bayes Rule for Parameter Learning

- X: **parameter** $\to$ Y: result of **my model**
- Given: $\mathbb{P}(X)$ and $\mathbb{P}(Y|X)$ (parameter $\to$ model)
- Inference: $\mathbb{P}(X|Y)$? Probabilistic feature of the parameter given the result of the model?

Example.

1. Light bulb's lifetime $Y \sim \exp(\lambda)$. Given the lifetime y, the modified belief about λ?
2. Romeo and Juliet start dating, but Romeo will be late by a random variable $Y \sim \mathcal{U}[0, \theta]$. Given the time of being late y, the modified belief about θ?

Bayes Rule for Mixed Case

K: discrete, Y: continuous

- Inference of K given Y

$$p_{K|Y}(k|y) = \frac{p_K(k)f_{Y|K}(y|k)}{f_Y(y)}$$
$$f_Y(y) = \sum_{k'} p_K(k')f_{Y|K}(y|k')$$

- $f_{Y|K}(y|k) = f_{Y|A}(y)$, where $A = \{K = k\}$

- Inference of Y given K

$$f_{Y|K}(y|k) = \frac{f_Y(y)p_{K|Y}(k|y)}{p_K(k)}$$
$$p_K(k) = \int f_Y(y')p_{K|Y}(k|y')dy'$$

- Wait! $p_{K|Y}(k|y)$? Well-defined?

$$p_{K|Y}(k|y) = \frac{\mathbb{P}(K=k, Y=y)}{\mathbb{P}(Y=y)} = \frac{0}{0}$$

$p_{K|Y}(k|y)$?

- For small δ (in other words, taking the limit as $\delta \to 0$).

  Let $A = \{K = k\}$.

$$
\begin{aligned}
p_{K|Y}(k|y) &\approx \mathbb{P}(A|y \le Y \le y+\delta) \\
&= \frac{\mathbb{P}(A)\mathbb{P}(y \le Y \le y+\delta|A)}{\mathbb{P}(y \le Y \le y+\delta)} \\
&\approx \frac{\mathbb{P}(A) f_{Y|A}(y)\delta}{f_Y(y)\delta} \\
&= \frac{\mathbb{P}(A) f_{Y|A}(y)}{f_Y(y)}
\end{aligned}
$$

Example: Signal Detection (1)

Inference of discrete K given continuous Y:

$$p_{K|Y}(k|y) = \frac{p_K(k) f_{Y|K}(y|k)}{f_Y(y)}, \quad f_Y(y) = \sum_{k'} p_K(k') f_{Y|K}(y|k')$$

- K: -1, +1, original signal, equally likely. $p_K(1) = 1/2, p_K(-1) = 1/2$.
- Y: measured signal with Gaussian noise, $Y = K + W$, $W \sim \mathcal{N}(0,1)$
- Your received signal = 0.7. What's your guess about the original signal? +1
- Your received signal = -0.2. What's your guess about the original signal? -1
- Your intuition: If positive received signal, +1. If negative received signal, -1. How can we mathematically verify this?

Example: Signal Detection (2)

- $Y|\{K=1\} \sim \mathcal{N}(1,1)$ and $Y|\{K=-1\} \sim \mathcal{N}(-1,1)$. (Remind: linear transformation preserves normality.)

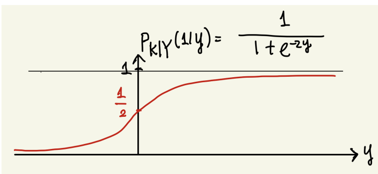
$$f_{Y|K}(y|k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-k)^2}, \quad k = 1, -1$$

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y+1)^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-1)^2} \qquad \text{(from TPT)}$$

- Probability that $K=1$, given $Y=y$? After some algebra,

$$p_{K|Y}(1|y) = \frac{1}{1+e^{-2y}}$$

  - If $y > 0$, the inference probability for $K=1$ exceeds $\frac{1}{2}$. So, original signal = 1.
  - Similarly, compute $p_{K|Y}(-1|y)$ and then do the inference



Example: Buffon's Needle (1)

- A surface is ruled with parallel lines with distance d from each other.
- We throw a needle of length l on the surface at random.
- Question. What is the probability that the needle will intersect one of the lines?

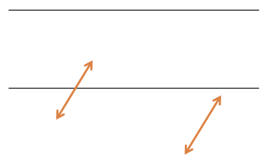
  - Assume $l < d$. The needle intersects two lines (x).
  - When does and does not the needle intersect?
  - What are the random quantities to model?

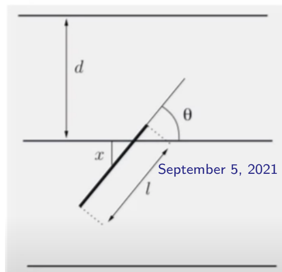
  R1. How far is the needle from the nearest parallel line?

  X: vertical distance from the midpoint of the needle to the nearest of the parallel lines

  R2. How does the needle meet the nearest parallel line?

  Θ: acute angle formed by the axis of the needle and the parallel lines





Example: Buffon's Needle (2)

- A random vector (X, Θ) with a uniform joint PDF over $\{(x,\theta) | 0 \le x \le d/2,\ 0 \le \theta \le \pi/2\}$ with

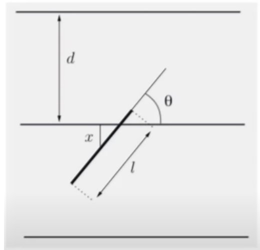
$$f_{X,\Theta}(x,\theta) = \begin{cases} 4/(\pi d), & \text{if } x \in [0, d/2] \text{ and } \theta \in [0, \pi/2], \\ 0, & \text{otherwise} \end{cases}$$

- When does the needle intersect a line? $X \le \frac{l}{2} \sin \Theta$
- The probability of intersection

$$\mathbb{P}(X \le \frac{l}{2}\sin\Theta) = \iint_{x \le (l/2)\sin\theta} f_{X,\Theta}(x,\theta)\ dx\ d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(l/2)\sin\theta} dx d\theta = \frac{4}{\pi d}\int_0^{\pi/2} \frac{l}{2}\sin\theta d\theta = \frac{2l}{\pi d}(-\cos\theta)\Big|_0^{\pi/2} = \frac{2l}{\pi d}$$

- In history, a method for the experimental evaluation of π.



Questions?

Review Questions

1) What do we mean by "continuous" in continuous random variables?
2) Explain PDF and CDF. Why do we need CDF?
3) What are joint/marginal/conditional PDFs?
4) Explain the relationship between Geometric rvs and Exponential rvs.
5) Explain how normality is preserved under linear transformaion for Normal (Gaussian) rvs.
6) Explain how we can use Bayes' rule for parameter learning.
7) Explain the version of Bayes' rule for continuous and mixed random variables.