Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

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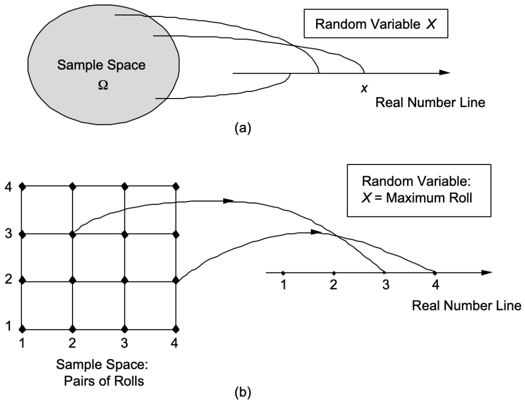
Random Variable: Idea

- In reality, many outcomes are [blank box], e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



(b) Two rolls of tetrahedral dice

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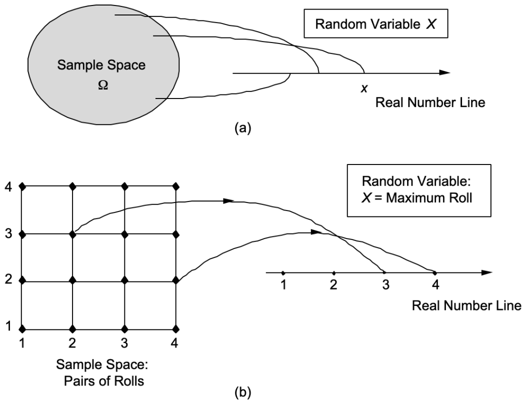

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Random Variable: More Formally

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$$

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- Rolls a dice, $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Define a random variable $X = 1$ for even numbers and $X = 0$ for odd numbers
- Event $A_1 = \{\omega \in \Omega \mid X(\omega) = 1\} = \{2, 4, 6\} \subset \Omega$, but simply $A_1 = \{X = 1\}$
- Event $A_0 = \{\omega \in \Omega \mid X(\omega) = 0\} = \{1, 3, 5\} \subset \Omega$, but simply $A_0 = \{X = 0\}$

- Remember that the random variable X is a **function** from Ω to $\mathbb{R}$

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Bernoulli X with parameter $p \in [0, 1]$

- Only binary values

[1] w.p.: with probability

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$$X = \begin{cases} 0, & \text{w.p.} \quad 1-p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail

- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv $\mathbf{1}_A$ as:

$$\mathbf{1}_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[1] w.p.: with probability

Uniform X with parameter a, b

- integers a, b, where $a \leq b$

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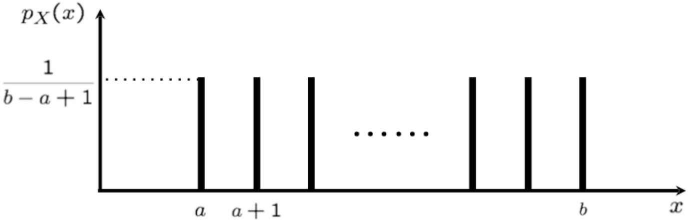

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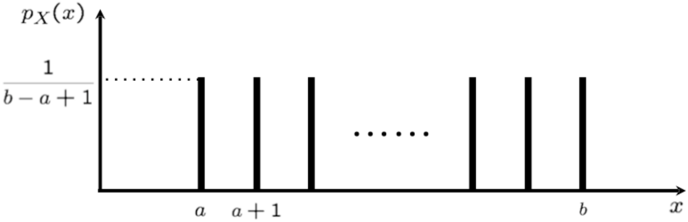

- Models complete ignorance (I don't know anything about X)

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- Models the number of successes in a given number of independent trials

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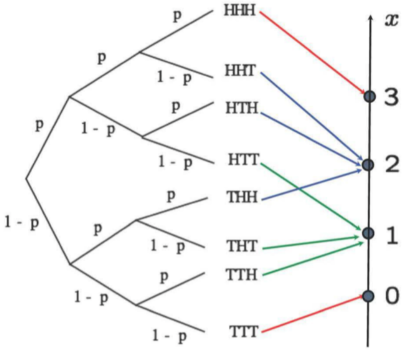

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L3(2)

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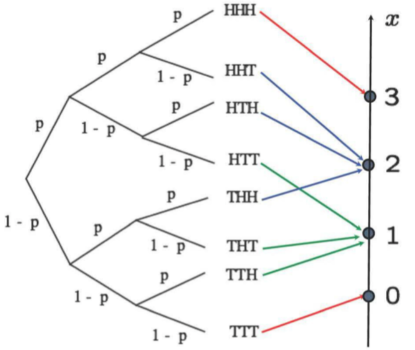
$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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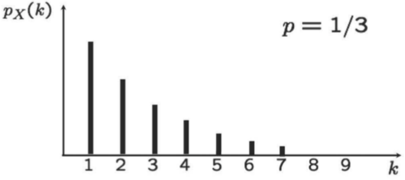
$$p_X(k) = (1-p)^{k-1} p$$



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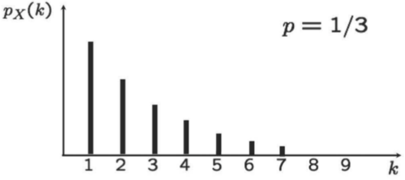
- Models waiting times until something happens.



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Expectation/Mean

- Average

Definition

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- Example. Bernoulli rv with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

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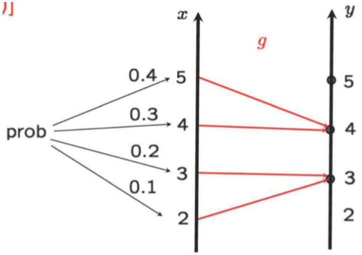

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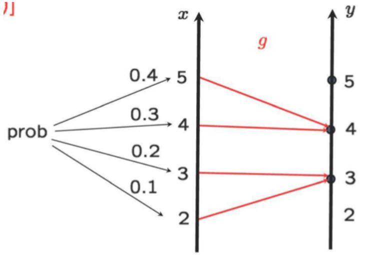

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$$= 2.8 + 0.9 = 3.7$$

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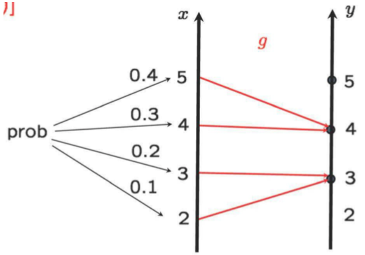

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Linearity of Expectation

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Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$

$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$
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Example: Variance of a Bernoulli rv (p)

Variance: Useful Property

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Example: Variance of a Bernoulli rv (p)

$$\begin{aligned}
\mu = \mathbb{E}[X] &= 1 \times p + 0 \times (1-p) = p \\
\mathbb{E}[X^2] &= 1 \times p + 0 \times (1-p) = p \\
\text{var}[X] &= \mathbb{E}[X^2] - \mu^2 = p - p^2 \\
&= p(1-p)
\end{aligned}$$

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Joint PMF

- For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$\mathbb{P}(\{X = x\} \cap \{Y = y\})$$

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- $\boxed{\text{Joint PMF.}}$ For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

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- $\boxed{}$

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

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Example.

VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X,Y}(1,3) =$

$p_X(4) =$

$\mathbb{P}(X = Y) =$

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- **Marginal PMF.**

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2		1/20	3/20	1/20
1		1/20		

$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Linearity of Expectation for Multiple RVs

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Message. When some rv X is written as a linear combination of other rvs, X becomes easy to handle.

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A.

Conditional PMF: Conditioning on an event

Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$ for an event A.

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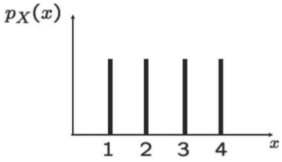
- (Note) $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

$A = \{X \geq 2\}$



$\mathbb{E}[X] =$

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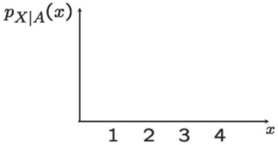

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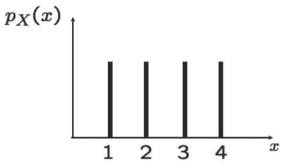

$$\mathbb{E}[X] = \frac{1}{4}(1+2+3+4) = 2.5$$

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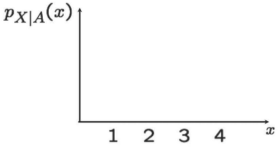

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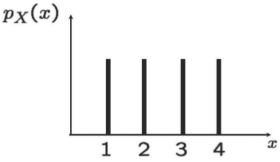

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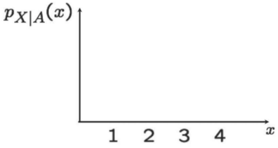

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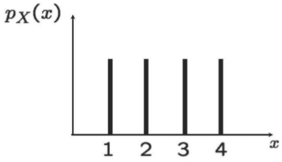

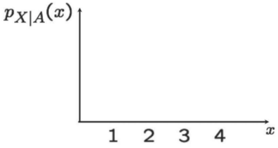

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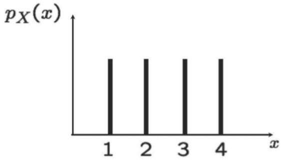

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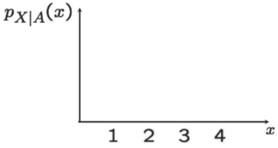

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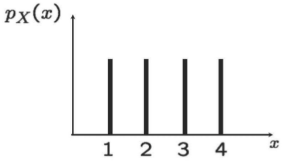

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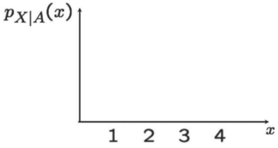

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What do we mean by "conditioning on a rv"? Consider $A = \{Y = y\}$ for a rv Y.

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- Conditional PMF

- Multiplication rule

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$

- Conditional PMF

$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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KAIST EE

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VIDEO PAUSE

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$p_{X|Y}(2|2) =$

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$\mathbb{E}[X|Y = 3] =$

KAIST EE

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$$p_{X|Y}(2|2) = \frac{1}{1+3+1}$$

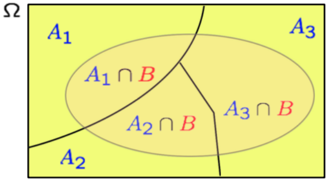
$$p_{X|Y}(3|2) = \frac{3}{1+3+1}$$

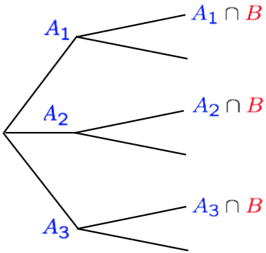
$$\mathbb{E}[X|Y = 3] = 1(2/9)+2(4/9)+3(1/9)+4(2/9)$$

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$?

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$



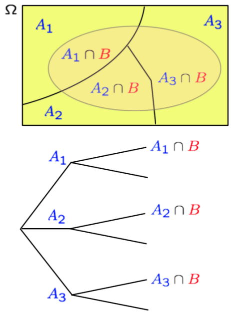


Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x | A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$



Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

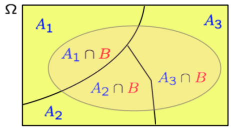
Total Probability Theorem

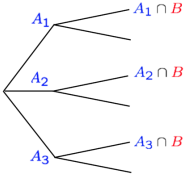
$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

- Weighted average of expectations from A_i's perspective





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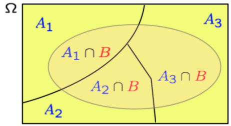
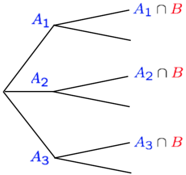




Total Expectation Theorem for $\{Y = y\}$

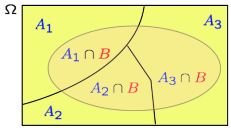
- Partition of Ω into A_1, A_2, A_3

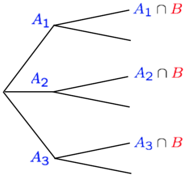
Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y] = \sum_y p_Y(y)\mathbb{E}[X|Y = y]$$

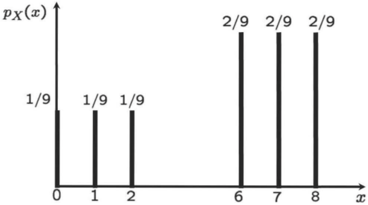




- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$\mathbb{E}[X] =$

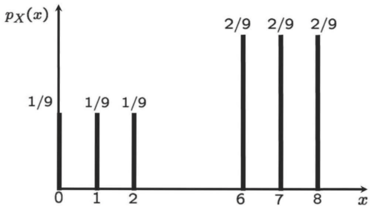


Example 1: Total Expectation Theorem

- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8) \\ &= \frac{3+12+14+16}{9} = 5\end{aligned}$$



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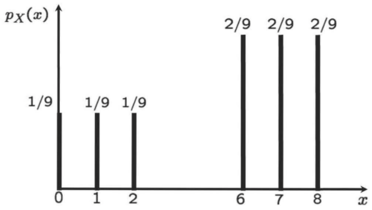
(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$



Example 1: Total Expectation Theorem

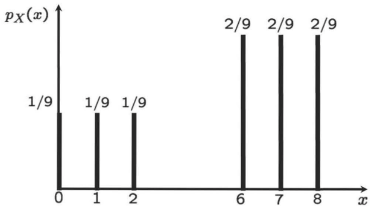
- Question. What is $\mathbb{E}(X)$?

(1) Just using the definition of expectation,

$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$
$$= \frac{3+12+14+16}{9} = 5$$

(2) Let's use TET, for which consider

$$A_1 = \{X \in \{0,1,2\}\},\ A_2 = \{X \in \{6,7,8\}\}$$
$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 5$$



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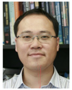
- Total expectation theorem and a notion of memorylessness helps a lot.

Memoryless Property: Motivating Example

No bus arrives



How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n + m | X > m)$$

Yung arrives

time

0

m mins

A bus left at time 0

time

0

How long do I have to wait? Probability of waiting for more than n mins?

$$\mathbb{P}(X > n)$$

Lin arrives

Background: Memoryless Property

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- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.
$$\mathbb{P}(X - m > n | X > m) = \mathbb{P}(X > n)$$

Background: Memoryless Property of Geometric RVs

- Theorem. Any **geometric** random variable is memoryless.

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- Remind. Geometric rv X with parameter p

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$$\mathbb{P}(X > n+m | X > m) = \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} = \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)}$$
$$= \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)$$

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Back to Example 2: Mean of Geometric rv

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

  $$\begin{aligned} \mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\ &= && \text{(from TET)} \\ &= && \text{(from memorylessness)} \end{aligned}$$

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- Thus, $\mathbb{E}[X] = \dfrac{1}{p}$

Roadmap

(1) Random variable: Idea and formal definition

(2) Popular discrete random variables

(3) Summarizing random variables: Expectation and Variance

(4) (Functions of) multiple random variables

(5) Conditioning for random variables

(6) Independence for random variables

Independence, Conditional Independence

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

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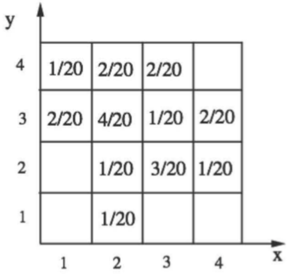
Example

- $X \perp\!\!\!\perp Y$?

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?



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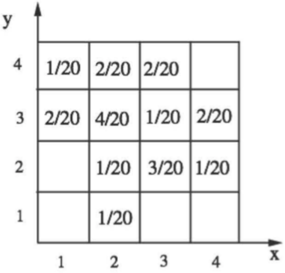

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VIDEO PAUSE

$Y = 4$		
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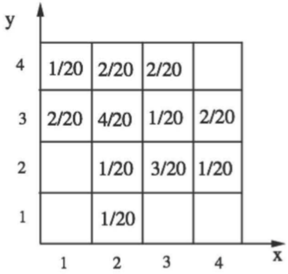

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$Y = 3$ $(2/3)$	$2/9$	$4/9$
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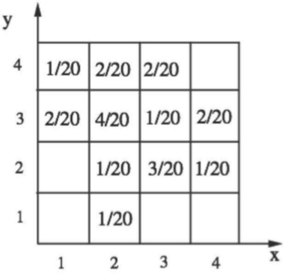

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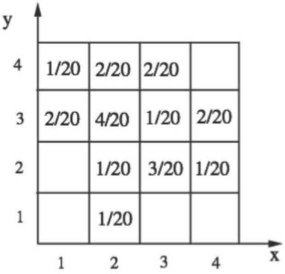

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  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

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- Why not generally true?

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$$\begin{aligned}
\mathrm{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}[X^2 + Y^2 + 2XY] - \left((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right) \\
&= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)
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&= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)
\end{aligned}$$

- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (from symmetry)

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- $\text{var}(X) = 2 - 1 = 1$

Questions?

Review Questions

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFs?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?