Lecture 2: Conditioning, Bayes' Rule, and Independence

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Conditional Probability
  - How should I change my belief about event A, if I come to know that event B occurs?

(2) Bayes' Rule and Bayesian Inference
  - prob. of A given that B occurs vs. prob. of B given that A occurs

(3) Independence, Conditional Independence
  - Can I ignore my knowledge about event B, when I consider event A?

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

(3) Independence, Conditional Independence

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- From now on, given B, $\mathbb{P}(\cdot|B)$ should be a new **probability law.**
  - Three axioms[1] should be satisfied.

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  - $\mathbb{P}(\Omega|B) = 1$?
  - $\mathbb{P}(B|B) = 1$ from our common sense.
  - True?

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$$\mathbb{P}(A \cup C| B) = \frac{\mathbb{P}\Big[(A \cup C) \cap B\Big]}{\mathbb{P}(B)} = \frac{\mathbb{P}\Big[(A \cap B) \cup (C \cap B)\Big]}{\mathbb{P}(B)} = \mathbb{P}(A| B) + \mathbb{P}(C| B)$$

Roadmap

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

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From now on, using the theory of probability and conditional probability constructed so far, we will develop interesting properties and theorems which are very useful to answer some exciting questions.

That is *Bayes' Rule* to make some *inference (추론).*

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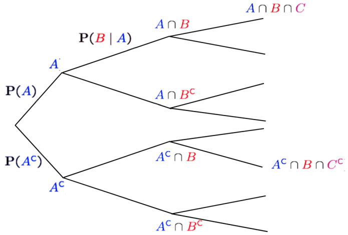
We will study this topic rigorously later in this class (chapter 8).

Multiplication Rule

- $\mathbb{P}(B|A) = \boxed{}$
- $\mathbb{P}(A \cap B) = \boxed{}$
- $\mathbb{P}(A^c \cap B \cap C^c) = \boxed{}$

  $= \boxed{}$



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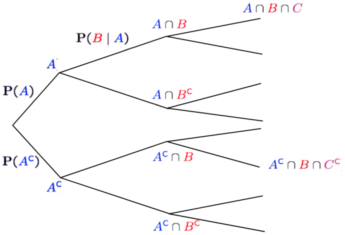

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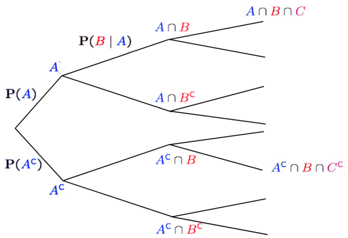

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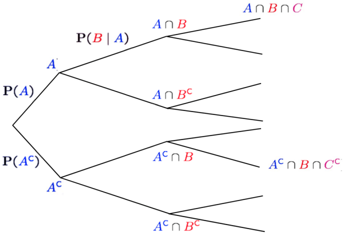

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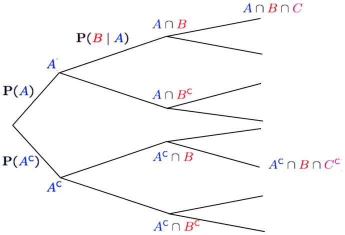

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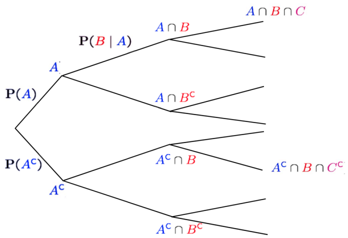

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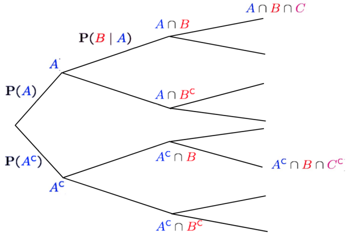

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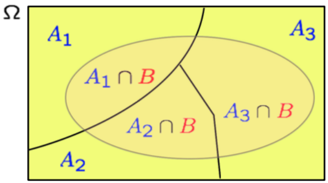
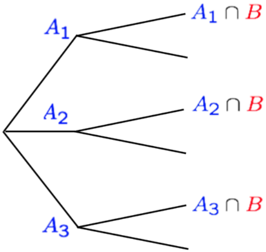
$$\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1, A_2) \cdots \mathbb{P}(A_n|A_1, A_2, \ldots, A_{n-1})$$

Total Probability Theorem

- Partition of Ω into A_1, A_2, A_3





[1] Partition: A_1, A_2, A_3 are mutually exclusive and $\Omega = A_1 \cup A_2 \cup A_3$

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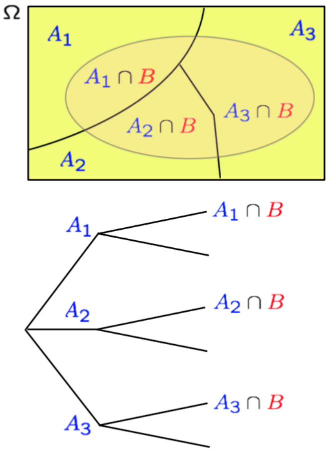

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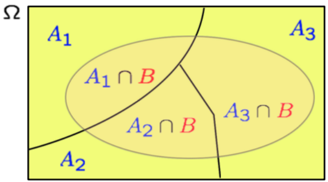

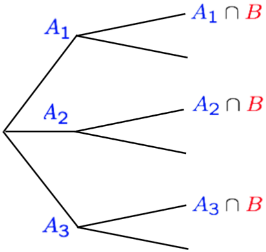

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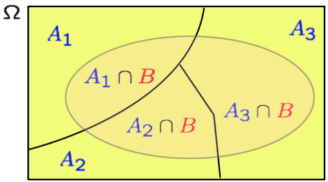

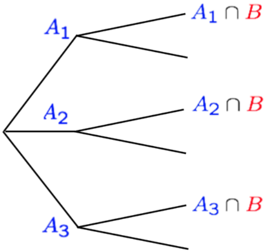

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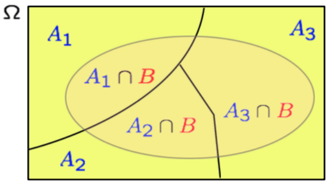
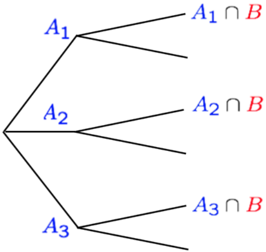
- $\mathbb{P}(A_i \cap B) = \mathbb{P}(A_i)\mathbb{P}(B|A_i)$
- Weighted average from the point of A_i knowledge.





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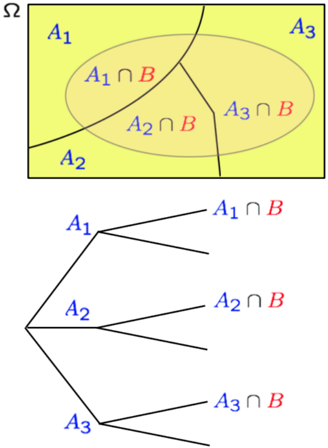

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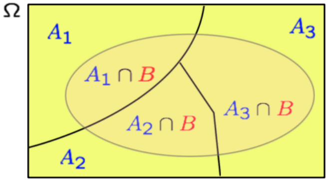

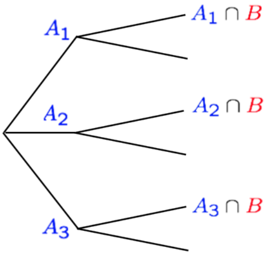

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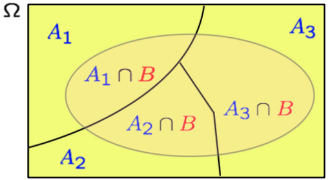

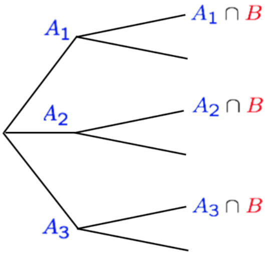

Example 1: Airplane-Radar

VIDEO PAUSE

- A : Airplane is flying above
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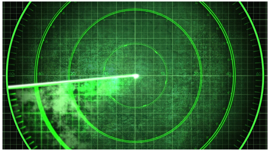

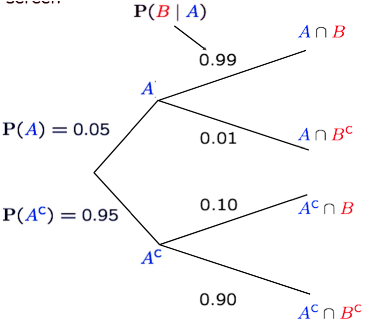

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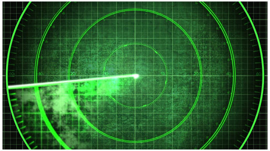

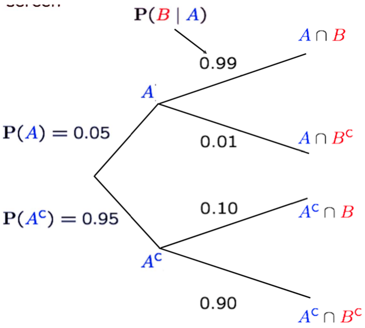

Example 1: Airplane-Radar

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$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$
$$= 0.05 \times 0.99 = 0.0495$$

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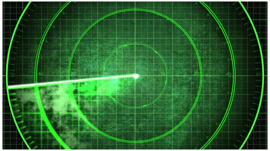

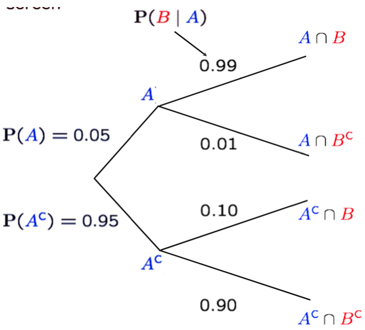

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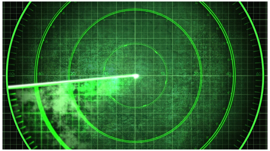

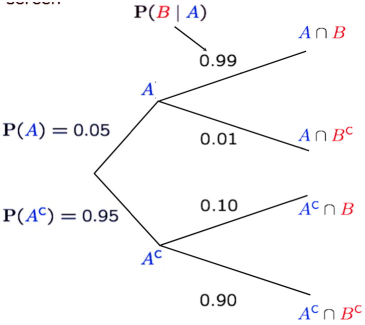

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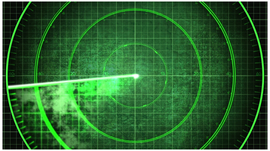

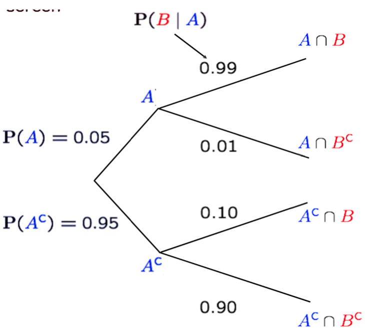

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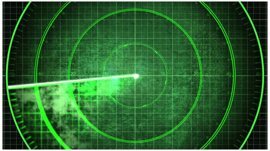

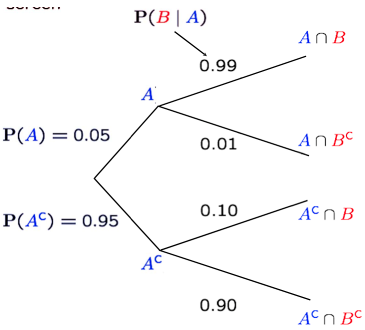

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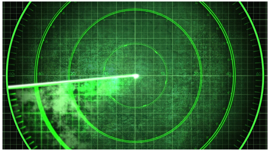

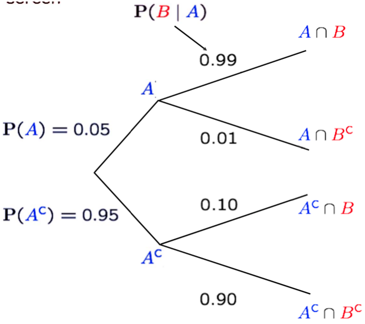

Example 2: Happy/Sad-Shout

- A_1: you are happy, A_2: you are sad
- B: you shout.
- Assume:

$$\mathbb{P}(A_1) = 0.7,\ \mathbb{P}(A_2) = 0.3,$$
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- Calculate $\mathbb{P}(A_1|B)$ and $\mathbb{P}(A_2|B)$.

VIDEO PAUSE

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- Calculate $\mathbb{P}(A_1|B)$ and $\mathbb{P}(A_2|B)$.

$$\begin{aligned}\mathbb{P}(A_1)\mathbb{P}(B|A_1) &= 0.7 \times 0.3 = 0.21\\ \mathbb{P}(A_2)\mathbb{P}(B|A_2) &= 0.3 \times 0.5 = 0.15\\ \mathbb{P}(B) &= 0.21 + 0.15 = 0.36\end{aligned}$$

$$\mathbb{P}(A_1|B) = \frac{0.21}{0.36} \approx 0.583$$
$$\mathbb{P}(A_2|B) = \frac{0.15}{0.36} \approx 0.417$$

Roadmap

(1) Conditional Probability

(2) Bayes' Rule and Bayesian Inference

(3) Independence, Conditional Independence

Bayesian inference was really fun.

Now, let's develop a new concept from conditioning.

That is *Independence*.

Why We Care Independence?

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- Independence makes our analysis and modeling much simpler, because I can remove independent events in the analysis of what I am interested in.

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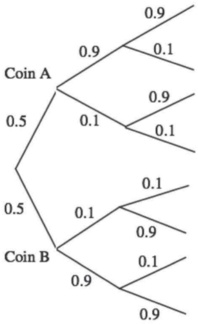

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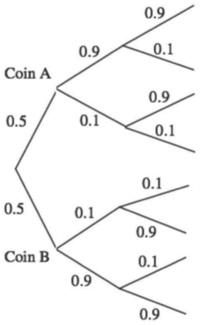

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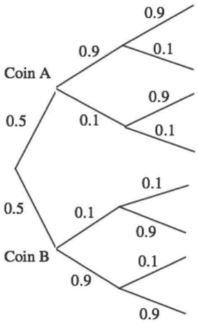

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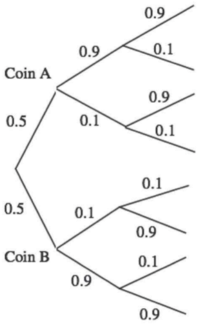

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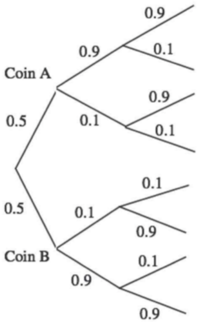

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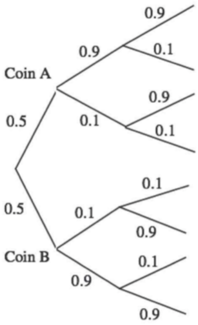
$$\mathbb{P}(H_1 \cap H_2 | B) = 0.9 \times 0.9, \quad \mathbb{P}(H_1|B)\mathbb{P}(H_2|B) = 0.9 \times 0.9$$

- $H_1 \perp\!\!\!\perp H_2$? No

$$\begin{aligned} \mathbb{P}(H_1) &= \mathbb{P}(B)\mathbb{P}(H_1|B) + \mathbb{P}(B^c)\mathbb{P}(H_1|B^c) \\ &= \frac{1}{2}0.9 + \frac{1}{2}0.1 = \frac{1}{2} \\ \mathbb{P}(H_2) &= \mathbb{P}(H_1) \quad \text{(because of symmetry)} \\ \mathbb{P}(H_1 \cap H_2) &= \mathbb{P}(B)\mathbb{P}(H_1 \cap H_2|B) + \mathbb{P}(B^c)\mathbb{P}(H_1 \cap H_2|B^c) \\ &= \frac{1}{2}(0.9 \times 0.9) + \frac{1}{2}(0.1 \times 0.1) \neq \frac{1}{2} \end{aligned}$$



Independence of Multiple Events

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KAIST EE

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- What about this? (Pairwise independence)

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  $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2),\ \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_3),\ \mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2)\mathbb{P}(A_3)$
- What about $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$?
- We need both.

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Independence of Multiple Events

The events $A_1, A_2, \ldots, A_n$ ar said to be independent if

$$\mathbb{P}\Big(\bigcap_{i \in S} A_i\Big) = \prod_{i \in S} \mathbb{P}(A_i), \quad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}$$

Questions?

1) What is conditional probability? Why do we need it?
2) Explain the overall framework of Bayesian inference.
3) What is the total probability theorem?
4) What is Bayes' rule? What does it can give us?
5) What's the difference between independence and conditional independence?