Lecture 1: Probabilistic Model

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Probabilistic Model
  - Mathematical description of uncertain situations

(2) Sample Space, Event, Probability Law
  - Elements of probability theory

(3) Probability Axioms
  - 3 axioms for the completeness of a theory

Roadmap

(1) Probabilistic Model

(2) Sample Space, Event, Probability Law

(3) Probability Axioms

What Do We Want?

Modeling: Understand reality with a simple (mathematical) model

- Experiment
- Flip two coins

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  - Assign a number to each outcome or a set of outcomes
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  - Assign a number to each outcome or a set of outcomes
  - Mathematical description of an uncertain situation
- Which model is good or bad?

Probabilistic Model

Goal: Build up a probabilistic model. Hmm... How?

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Elements of Probabilistic Model

1. All outcomes of my interest: $\boxed{}$

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Question: What are the conditions of Ω and $\mathbb{P}(\cdot)$ under which their induced probability model becomes "legitimate"?

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3. (a) Just figuring out prob. of H or T.
   $\Longrightarrow \Omega = \{H, T\}$

   (b) The impact of the weather (rain or no rain) on the coin's behavior.

   $$\Longrightarrow \Omega = \{(H, R), (T, R), (H, NR), (T, NR)\},$$

   R(Rain), NR(No Rain).

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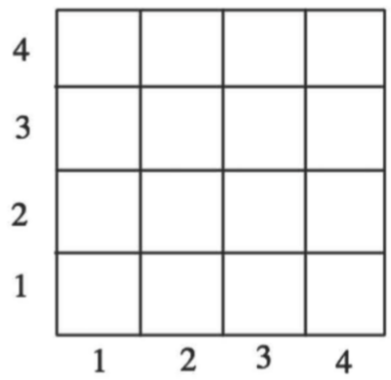
- *Discrete case:* Two rolls of a tetrahedral die

  - $\Omega = \{(1,1),(1,2),\ldots,(4,4)\}$



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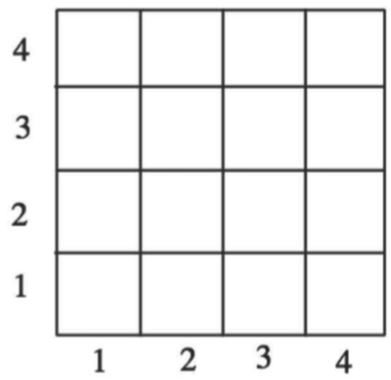
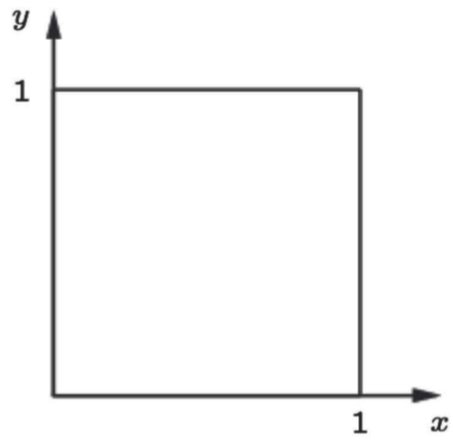


◦ *Continuous case:* Dropping a needle in a plain

- $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$



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- Roll a dice. What is the probability of odd numbers?

  $\mathbb{P}(\{1, 3, 5\})$, where $\{1, 3, 5\} \subset \Omega$ is an event.

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  - If $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$
  - many others

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- Note that coming up with the above axioms is far from trivial.

Examples

A1: Nonnegativity, A2: Normalization, A3: Finite additivity

Prove the following properties using the axioms: VIDEO PAUSE

1. For any event A, $\mathbb{P}(A) \leq 1$

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   $$\mathbb{P}(B) \overset{\text{A3}}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A) \overset{\text{A1}}{\geq} \mathbb{P}(A)$$

Probability Calculation Steps

1. Specify the sample space
2. Specify a probability law
   - from my earlier belief, from data, from expert's opinion
3. Identify an event of interest
4. Calculate

Toss a (biased) coin

1. $\Omega = \{H, T\}$
2. $\mathbb{P}(\{H\}) = 1/4$, $\mathbb{P}(\{T\}) = 3/4$,
3. probability of head or tail
4. $1/4$, $3/4$

Discrete but infinite sample space

- $\Omega = \{1, 2, 3, \dots\}$, $\mathbb{P}(\{n\}) = \frac{1}{2^n}$, $n = 1, 2, \dots$

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$$\begin{aligned} \mathbb{P}(\text{even}) &= \mathbb{P}(\{2, 4, 6, \ldots\}) \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = 1/3 \end{aligned}$$

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- Is the above right? If not, why?
  - Wrong: Finite additivity axiom does not allow this.

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Probability Axioms: Version 2

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A2. Normalization: $\mathbb{P}(\Omega) = 1$

A3. $\boxed{\text{Countable additivity:}}$ If $A_1, A_2, A_3, \ldots$ is an infite sequence of disjoint events, then $\mathbb{P}(A_1 \cup A_2 \cup \cdots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots$.

Interpretation of Probability Theory

- A narrow view: A branch of math
  - axioms $\rightarrow$ theorems
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- Beliefs: $\mathbb{P}(\text{He is reelected}) = 0.7$

Anyway, we believe that probabilistic reasoning is very helpful to understand the world with many uncertain situations.

Questions?

You build up the very basics of a probabilistic model.

What else do we need to build up?

Review Questions

1) Please explain what a probabilistic model is and why we need it.
2) What is the mathematical definition of event?
3) What are the key elements of the probabilistic model?
4) Please list up the probability axioms and explain them.
5) Why do we need countable additivity in the probability axioms?