

## Lecture 5: Random Variable, Part III

Yi, Yung (이웅)

EE210: Probability and Introductory Random Processes  
KAIST EE

June 2, 2021

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### Roadmap

- (1) Derived distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- (2) Derived distribution of  $Z = X + Y$
- (3) Covariance: Degree of dependence between two rvs.
- (4) Correlation coefficient
- (5) Conditional expectation and law of iterative expectations
- (6) Conditional variance and law of total variance
- (7) Random number of sum of random variables

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L5(1)

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Derived Distribution:  $Y = g(X)$ 

- Given the PDF of  $X$ , What is the PDF of  $Y = g(X)$ ?
- Wait! Didn't we cover this topic? No. We covered just  $\mathbb{E}[g(X)]$ .
- Examples:  $Y = X$ ,  $Y = X + 1$ ,  $Y = X^2$ , etc.
- What are easy or difficult cases?
- Easy cases
  - Discrete
  - Linear:  $Y = aX + b$

L5(1)

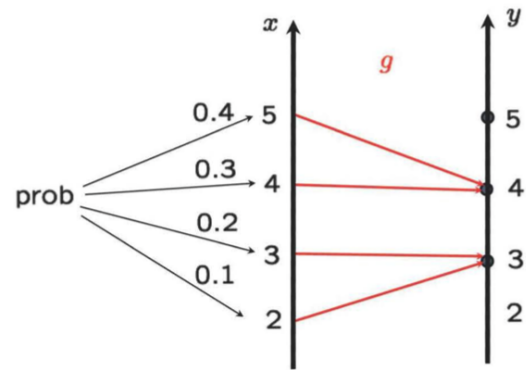
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- Take all values of  $x$  such that  $g(x) = y$ , i.e.,

$$\begin{aligned} p_Y(y) &= \mathbb{P}(g(X) = y) \\ &= \sum_{x: g(x)=y} p_X(x) \end{aligned}$$

$$p_Y(3) = p_X(2) + p_X(3) = 0.1 + 0.2 = 0.3$$

$$p_Y(4) = p_X(4) + p_X(5) = 0.3 + 0.4 = 0.7$$



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Linear:  $Y = aX + b$ ,  $a \neq 0$ ,  $X$ : Continuous

$$\text{If } a > 0, \quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\rightarrow f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\text{If } a < 0, \quad F_Y(y) = \mathbb{P}(aX + b \leq y) = \mathbb{P}\left(X \geq \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\rightarrow f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Therefore,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, & \text{if } (y-b)/a \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- If  $b = 0$  and  $a > 0$ ,  $Y$  is exponential with parameter  $\frac{\lambda}{a}$ , but generally not.

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- Remember? Linear transformation preserves normality. Time to prove.

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for  $a \neq 0$  and  $b$ ,  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ .

- **Proof.**

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2\right\} \\ &= \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\left\{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right\} \end{aligned}$$

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Step 1. Find the CDF of  $Y$ :

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y)$$

Step 2. Differentiate:  $f_Y(y) = \frac{dF_Y}{dy}(y)$

Ex1.  $Y = X^2$ .

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \\ &\quad \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}), \quad y \geq 0 \end{aligned}$$

Ex2.  $X \sim \mathcal{U}[0, 1]$ .  $Y = \sqrt{X}$ .

$$\begin{aligned} F_Y(y) &= \mathbb{P}(\sqrt{X} \leq y) = \mathbb{P}(X \leq y^2) = y^2 \\ f_Y(y) &= 2y, \quad 0 \leq y \leq 1 \end{aligned}$$

Ex3.  $X \sim \mathcal{U}[0, 2]$ .  $Y = X^3$ .

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X^3 \leq y) = \mathbb{P}(X \leq \sqrt[3]{y}) = \frac{1}{2} y^{1/3} \\ f_Y(y) &= \frac{1}{6} y^{-2/3}, \quad 0 \leq y \leq 8 \end{aligned}$$

When  $Y = g(X)$  is monotonic, a **general formula** can be drawn (see the textbook at pp 207)

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Basically, follow two-step approach: (i) CDF and (ii) differentiate.

Ex1.  $X, Y \sim \mathcal{U}[0, 1]$ , and  $X \perp\!\!\!\perp Y$ .  $Z = \max(X, Y)$ .

\*  $\mathbb{P}(X \leq z) = \mathbb{P}(Y \leq z) = z, \quad z \in [0, 1]$ .

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\max(X, Y) \leq z) = \mathbb{P}(X \leq z, Y \leq z) \\ &= \mathbb{P}(X \leq z) \mathbb{P}(Y \leq z) = z^2 \quad (\text{from } X \perp\!\!\!\perp Y) \end{aligned}$$

$$f_Z(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

L5(1)

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Basically, follow two step approach: (i) CDF and (ii) differentiate.

Ex2.  $X, Y \sim \mathcal{U}[0, 1]$ , and  $X \perp\!\!\!\perp Y$ .  $Z = Y/X$ .

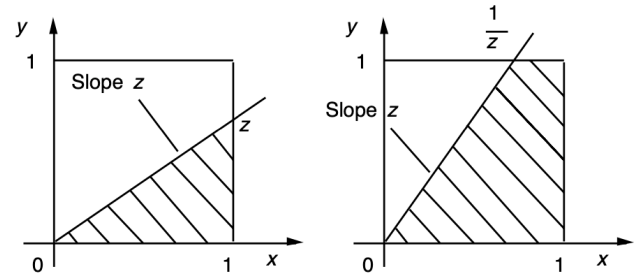
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$$F_Z(z) = \mathbb{P}(Y/X \leq z)$$

$$= \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/2z, & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} 1/2, & 0 \leq z \leq 1 \\ 1/(2z^2), & z > 1 \\ 0, & \text{otherwise} \end{cases}$$

- Depending on the value of  $z$ , two cases need to be considered separately.



(Note) Sometimes, the problem is tricky, which requires careful case-by-case handling. :-)

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## Roadmap

- (1) Derived distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- (2) **Derived distribution of  $Z = X + Y$**
- (3) Covariance: Degree of dependence between two rvs.
- (4) Correlation coefficient
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L5(2)

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- Sum of two independent rvs
- A very basic case with many applications
- Assume that  $X, Y \in \mathbb{Z}$

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_{\{(x,y): x+y=z\}} \mathbb{P}(X = x, Y = y) = \sum_x \mathbb{P}(X = x, Y = z - x)$$

$$= \sum_x \mathbb{P}(X = x) \mathbb{P}(Y = z - x) = \sum_x p_X(x) p_Y(z - x)$$

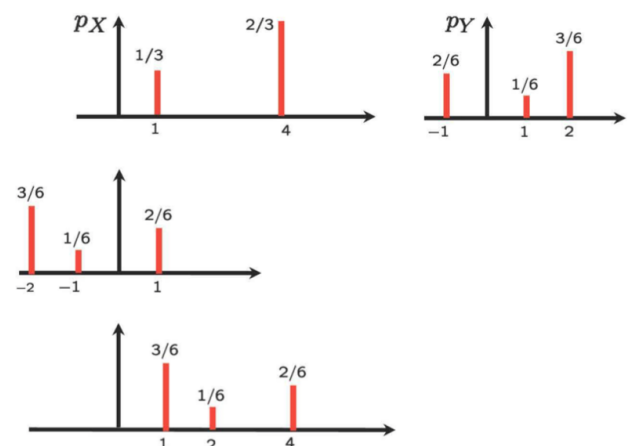
- $p_Z(z)$  is called **convolution** of the PMFs of  $X$  and  $Y$ .

L5(2)

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- Convolution:  $p_Z(z) = \sum_x p_X(x) p_Y(z - x)$
- Interpretation for a given  $z$ :
  - Flip (horizontally) the PMF of  $Y$  ( $p_Y(-x)$ )
  - Put it underneath the PMF of  $X$
  - Right-shift the flipped PMF by  $z$  ( $p_Y(-x + z)$ )

Example.  $z = 3$



L5(2)

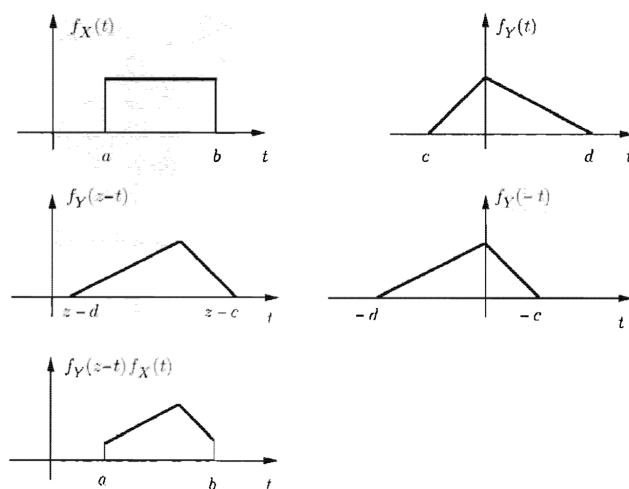
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- Same logic as the discrete case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

- Youtube animation for convolution:  
<https://www.youtube.com/watch?v=C1N55M1VD2o>

For a fixed  $z$ ,



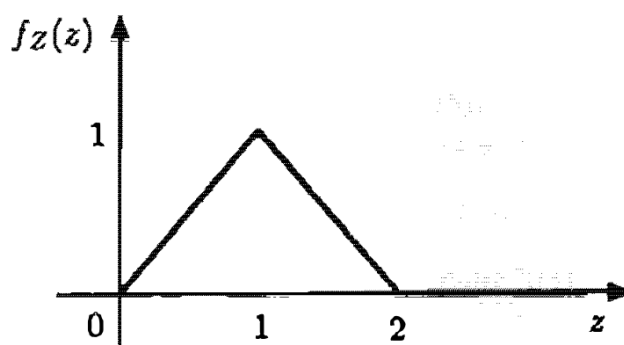
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## Example

- Example.**  $X, Y \sim \mathcal{U}[0, 1]$  and  $X \perp\!\!\!\perp Y$ . What is the PDF of  $Z = X + Y$ ? Draw the PDF of  $Z$ .



L5(2)

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<https://www.youtube.com/watch?v=MQm6ZP1F6ms>

L5(2)

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## $Y = X + Y, X \perp Y, \text{Normal (1)}$

- Very special, but useful case
  - $X$  and  $Y$  are **normal**.

### Sum of two independent normal rvs

$X \sim \mathcal{N}(\mu_x, \sigma_x^2)$  and  $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$  Then,  $X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

- Why normal rvs are used to model the **sum of random noises**.
- **Extension**. The sum of **finitely many** independent normals is also normal.

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$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{(z-x-\mu_y)^2}{2\sigma_y^2}\right\} dx \end{aligned}$$

- The details of integration is a little bit tedious. :-)

$$f_Z(z) = \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} \exp\left\{-\frac{(z - \mu_x - \mu_y)^2}{2(\sigma_x^2 + \sigma_y^2)}\right\}$$

L5(2)

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## Roadmap

- (1) Derived distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- (2) Derived distribution of  $Z = X + Y$
- (3) **Covariance: Degree of dependence between two rvs**
- (4) Correlation coefficient
- (5) Conditional expectation and law of iterative expectations
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- (7) Random number of sum of random variables

L5(3)

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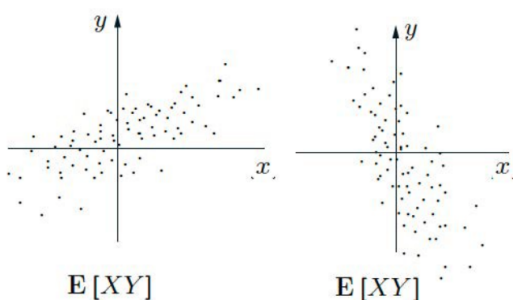
- Goal: Given two rvs  $X$  and  $Y$ , assign some number that quantifies the degree of their dependence.
  - feeling/weather, university ranking/annual salary,
- Requirements
  - R1.** Increases (resp. decreases) as they become more (resp. less) dependent. 0 when they are independent.
  - R2.** Shows the 'direction' of dependence by  $+$  and  $-$
  - R3.** Always bounded by some numbers (i.e., dimensionless metric). For example,  $[-1, 1]$
- Good engineers: Good at making good metrics
  - Metric of how our society is economically polarized
  - Cybermetrics in MLB (Major League Baseball):  
<http://m.mlb.com/glossary/advanced-stats>

L5(3)

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## OK. Let's Design!

- Simple case:  $\mathbb{E}[X] = \mu_X = 0$  and  $\mathbb{E}[Y] = \mu_Y = 0$
- Dependent: Positive (If  $X \uparrow$ ,  $Y \uparrow$ ) or Negative (If  $X \uparrow$ ,  $Y \downarrow$ )
- What about  $\mathbb{E}[XY]$ ? Seems good.
  - $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0$  when  $X \perp\!\!\!\perp Y$
  - More data points (thus increases) when  $xy > 0$  (both positive or negative)
  - $|\mathbb{E}[XY]|$  also quantifies the **amount of spread**.



(Q) What about  $\mathbb{E}[X + Y]$ ?

- When they are positively dependent, but have negative values?

L5(3)

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- **Solution:** Centering.  $X \rightarrow X - \mu_X$  and  $Y \rightarrow Y - \mu_Y$

### Covariance

$$\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])]$$

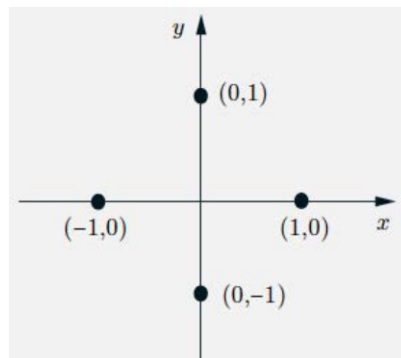
- After some algebra,  $\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $X \perp\!\!\!\perp Y \implies \text{cov}(X, Y) = 0$
- $\text{cov}(X, Y) = 0 \implies X \perp\!\!\!\perp Y$ ? NO.
- When  $\text{cov}(X, Y) = 0$ , we say that  $X$  and  $Y$  are **uncorrelated**.

L5(3)

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### Example: $\text{cov}(X, Y) = 0$ , but not independent

- $p_{X,Y}(1, 0) = p_{X,Y}(0, 1) = p_{X,Y}(-1, 0) = p_{X,Y}(0, -1) = 1/4$ .
- $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ , and  $\mathbb{E}[XY] = 0$ . So,  $\text{cov}(X, Y) = 0$
- Are they independent? No, because if  $X = 1$ , then we should have  $Y = 0$ .



L5(3)

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$$\text{cov}(X, X) = \text{var}(X)$$

$$\text{cov}(aX + b, Y) = \mathbb{E}[(aX + b)Y] - \mathbb{E}[aX + b]\mathbb{E}[Y] = a \cdot \text{cov}(X, Y)$$

$$\text{cov}(X, Y + Z) = \mathbb{E}[X(Y + Z)] - \mathbb{E}[X]\mathbb{E}[Y + Z] = \text{cov}(X, Y) + \text{cov}(X, Z)$$

$$\text{var}[X + Y] = \mathbb{E}[(X + Y)^2] - (\mathbb{E}[X + Y])^2 = \text{var}[X] + \text{var}[Y] + 2\text{cov}(X, Y)$$

$$\text{var}\left[\sum X_i\right] = \sum \text{var}[X_i] + \sum_{i \neq j} \text{cov}(X_i, X_j)$$

L5(3)

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## Example: The hat problem in Lecture 3. Remember?

- $n$  people throw their hats in a box and then pick one at random
- $X$ : number of people with their own hat
- (Q)  $\text{var}[X]$
- Key step 1. Define a rv  $X_i = 1$  if  $i$  selects own hat and 0 otherwise. Then,  $X = \sum_{i=1}^n X_i$ .
- Key step 2. Are  $X_i$ s are independent?
- $X_i \sim \text{Bern}(1/n)$ . Thus,  $\mathbb{E}[X_i] = 1/n$  and  $\text{var}[X_i] = \frac{1}{n}(1 - \frac{1}{n})$

- For  $i \neq j$ ,

$$\begin{aligned} \text{cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\ &= \mathbb{P}(X_i = 1 \text{ and } X_j = 1) - \frac{1}{n^2} \\ &= \mathbb{P}(X_i = 1)\mathbb{P}(X_j = 1 | X_i = 1) - \frac{1}{n^2} \\ &= \frac{1}{n} \frac{1}{n-1} - \frac{1}{n^2} = \frac{1}{n^2(n-1)} \end{aligned}$$

$$\begin{aligned} \text{var}[X] &= \text{var}\left[\sum X_i\right] \\ &= \sum \text{var}[X_i] + \sum_{i \neq j} \text{cov}(X_i, X_j) \\ &= n \frac{1}{n} \left(1 - \frac{1}{n}\right) + n(n-1) \frac{1}{n^2(n-1)} = 1 \end{aligned}$$

L5(3)

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L5(4)

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## Bounding the metric: Correlation Coefficient

- Reqs. **R1** and **R2** are satisfied.
- **R3.** Always bounded by some numbers (dimensionless metric)
- How? **Normalization**, but by what?

### Correlation Coefficient

$$\rho(X, Y) = \mathbb{E} \left[ \frac{(X - \mu_X)}{\sigma_X} \cdot \frac{(Y - \mu_Y)}{\sigma_Y} \right] = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

- **Theorem.**
  1.  $-1 \leq \rho \leq 1$  (proof at the next slide)
  2.  $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$  for some constant  $c$  ( $c > 0$  when  $\rho = 1$  and  $c < 0$  when  $\rho = -1$ ). In other words, linear relation, meaning VERY related.

L5(4)

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- **Cauchy-Schwarz inequality.** For any rvs  $X$  and  $Y$ ,  $(\mathbb{E}(XY))^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$

- **Proof of  $-1 \leq \rho \leq 1$ :**

Let  $\tilde{X} = X - \mathbb{E}(X)$  and  $\tilde{Y} = Y - \mathbb{E}(Y)$ . Then,  $(\rho(X, Y))^2 = \frac{(\mathbb{E}[\tilde{X}\tilde{Y}])^2}{\mathbb{E}(\tilde{X}^2)\mathbb{E}(\tilde{Y}^2)} \leq 1$

- **Proof of CSI:** For any constant  $a$ ,

$$0 \leq \mathbb{E}[(X - aY)^2] = \mathbb{E}[X^2 - 2aXY + a^2Y^2] = \mathbb{E}(X^2) - 2a\mathbb{E}(XY) + a^2\mathbb{E}(Y^2)$$

Now, choose  $a = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$ . Then,

$$\mathbb{E}(X^2) - 2\frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\mathbb{E}(XY) + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2}\mathbb{E}(Y^2) = \mathbb{E}(X^2) - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}(Y^2)} \geq 0$$

L5(4)

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2.  $|\rho| = 1 \Leftrightarrow X - \mu_X = c(Y - \mu_Y)$ 

( $\Rightarrow$ ) Suppose that  $|\rho| = 1$ . In the proof of CSI,

$$\mathbb{E}\left[\left(\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y}\right)^2\right] = \mathbb{E}(\tilde{X}^2) - \frac{(\mathbb{E}[\tilde{X}\tilde{Y}])^2}{\mathbb{E}(\tilde{Y}^2)} = \mathbb{E}(\tilde{X}^2)(1 - \rho^2) = 0$$

$$\tilde{X} - \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y} = 0 \Leftrightarrow \tilde{X} = \frac{\mathbb{E}(\tilde{X}\tilde{Y})}{\mathbb{E}(\tilde{Y}^2)}\tilde{Y} = \rho\sqrt{\frac{\mathbb{E}(\tilde{X}^2)}{\mathbb{E}(\tilde{Y}^2)}}\tilde{Y}$$

( $\Leftarrow$ ) If  $\tilde{Y} = c\tilde{X}$ , then

$$\rho(X, Y) = \frac{\mathbb{E}(\tilde{X}c\tilde{X})}{\sqrt{\mathbb{E}[\tilde{X}^2]\mathbb{E}[(c\tilde{X})^2]}} = \frac{c}{|c|}$$

L5(4)

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L5(5)

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## A Special Random Variable

- Consider a rv  $Y$ , such that

$$Y = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 2, & \text{w.p. } 1/2 \end{cases}$$

- If  $h(y) = y^2$ , then a new rv  $h(Y)$  is:

$$h(Y) = \begin{cases} 0, & \text{w.p. } 1/4 \\ 1, & \text{w.p. } 1/4 \\ 4, & \text{w.p. } 1/2 \end{cases}$$

- Consider other rv  $X$ , which, we assume, has:

$$g(y) = \mathbb{E}[X|Y = y] = \begin{cases} 3, & \text{if } y = 0 \\ 8, & \text{if } y = 1 \\ 9, & \text{if } y = 2 \end{cases}$$

- Then, a rv  $g(Y)$  is:

$$g(Y) = \begin{cases} 3, & \text{w.p. } 1/4 \\ 8, & \text{w.p. } 1/4 \\ 9, & \text{w.p. } 1/2 \end{cases}$$

- The rv  $g(Y)$  looks special, so let's give a fancy notation to it.

- What about?  $X_{\text{exp}}(Y)$ ,  $\mathbb{E}[X_Y]$ ,  $\mathbb{E}_X[Y]$ ?

L5(5)

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## Conditional Expectation

A random variable  $g(Y) = \mathbb{E}[X|Y]$ , called **conditional expectation of  $X$  given  $Y$** , takes the value  $g(y) = \mathbb{E}[X|Y = y]$ , if  $Y$  happens to take the value  $y$ .

- A function of  $Y$
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has.
- Often confusing because of the notation.

L5(5)

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## Expectation of Conditional Expectation

$$\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X], \quad \text{Law of iterated expectations}$$

**Proof.**

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X|Y]] &= \sum_y \mathbb{E}[X|Y = y] p_Y(y) \\ &= \mathbb{E}[X] \end{aligned}$$

L5(5)

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- Stick of length  $l$
- Uniformly break at point  $Y$ , and break what is left uniformly at point  $X$ .
- $\mathbb{E}[X|Y = y] = y/2$
- $\mathbb{E}[X|Y] = Y/2$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y/2] = \frac{1}{2} \frac{l}{2} = l/4$

- Forecasts on sales: calculating expected value, given any available information
- $X$  : February sales
- Forecast in the beg. of the year:  $\mathbb{E}[X]$
- End of Jan. new information  $Y = y$  (Jan. sales)  
Revised forecast:  $\mathbb{E}[X|Y = y]$   
Revised forecast  $\neq \mathbb{E}[X]$
- Law of iterated expectations  
 $\mathbb{E}[\text{revised forecast}] = \text{original one}$

L5(5)

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## Example: Averaging Quiz Scores by Section

- A class:  $n$  students, student  $i$ 's quiz score:  $x_i$
- Average quiz score:  $m = \frac{1}{n} \sum_{i=1}^n x_i$
- Students: partitioned into sections  $A_1, \dots, A_k$  and  $n_s$ : number of students in section  $s$
- average score in section  $s =$   
 $m_s = \frac{1}{n_s} \sum_{i \in A_s} x_i$
- whole average: (i) taking the average  $m_s$  of each section and (ii) forming a weighted average

$$\sum_{s=1}^k \frac{n_s}{n} m_s = \sum_{s=1}^k \frac{n_s}{n} \frac{1}{n_s} \sum_{i \in A_s} x_i = \frac{1}{n} \sum_{i=1}^n x_i = m$$

- Understanding from  $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$
- $X$ : score of a randomly chosen student,  $Y$ : section of a student ( $\in \{1, \dots, k\}$ )

$$\begin{aligned} m &= \mathbb{E}(X) = \mathbb{E}[\mathbb{E}[X|Y]] \\ &= \sum_{s=1}^k \mathbb{E}(X|Y = s) \mathbb{P}(Y = s) \\ &= \sum_{s=1}^k \left( \frac{1}{n_s} \sum_{i \in A_s} x_i \right) \frac{n_s}{n} = \sum_{s=1}^k m_s \frac{n_s}{n} \end{aligned}$$

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- (1) Derived distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- (2) Derived distribution of  $Z = X + Y$
- (3) Covariance: Degree of dependence between two rvs
- (4) Correlation coefficient
- (5) Conditional expectation and law of iterative expectations
- (6) **Conditional variance and law of total variance**
- (7) Random number of sum of random variables

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## Conditional Variance $\text{var}[X|Y]$

$$\text{var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$g(y) = \text{var}[X|Y = y] = \mathbb{E}[(X - \mathbb{E}[X|Y = y])^2 | Y = y]$$

$$g(Y) = \text{var}[X|Y] = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

### Conditional Variance

A random variable  $g(Y) = \boxed{\text{var}[X|Y]}$  and called **conditional variance of  $X$  given  $Y$** , takes the value  $g(y) = \text{var}[X|Y = y]$ , if  $Y$  happens to take the value  $y$ .

- A function of  $Y$
- A random variable
- Thus, having a distribution, expectation, variance, all the things that a random variable has

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	$\mathbb{E}[X Y]$	$\text{var}[X Y]$
Expectation	$\mathbb{E}[\mathbb{E}(X Y)]$	$\mathbb{E}[\text{var}(X Y)]$
Variance	$\text{var}[\mathbb{E}(X Y)]$	$\text{var}[\text{var}(X Y)]$

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## Law of Total Variance

### Law of total variance (LTV)

$$\text{var}[X] = \mathbb{E}[\text{var}(X|Y)] + \text{var}[\mathbb{E}(X|Y)]$$

**Proof.**

$$\text{var}(X|Y) = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$$

$$\mathbb{E}[\text{var}(X|Y)] = \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \quad (1)$$

$$\text{var}[\mathbb{E}(X|Y)] = \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}(X|Y)])^2 = \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2 \quad (2)$$

$$(1) + (2) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \text{var}[X]$$

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- Same setting as that in page 36
- $X$ : score of a randomly chosen student,  $Y$ : section of a student ( $\in \{1, \dots, k\}$ )
- Let's intuitively understand:  $\text{var}[X] = \mathbb{E}[\text{var}(X|Y)] + \text{var}[\mathbb{E}(X|Y)]$
- $\mathbb{E}[\text{var}(X|Y)] = \sum_{k=1}^s \mathbb{P}(Y = s) \text{var}(X|Y = s) = \sum_{k=1}^s \frac{n_s}{n} \text{var}(X|Y = s)$ 
  - Weighted average of the section variances
  - **average score variability within individual sections**
- $\text{var}[\mathbb{E}(X|Y)]$ : variability of the average of the different sections
  - $\mathbb{E}(X|Y = s)$ : average score in section  $s$
  - **variability between sections**

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- Stick of length  $l$
- Uniformly break at point  $Y$ , and break what is left uniformly at point  $X$ .
- **Question.**  $\text{var}(X)$ ?
- LTV:  $\text{var}[X] = \mathbb{E}[\text{var}(X|Y)] + \text{var}[\mathbb{E}(X|Y)]$
- **Fact.** If a rv  $X \sim \mathcal{U}[0, \theta]$ , then  $\text{var}(X) = \frac{\theta^2}{12}$
- Since  $X \sim \mathcal{U}[0, Y]$ ,  $\text{var}(X|Y) = \frac{Y^2}{12} \rightarrow \mathbb{E}[\text{var}[X|Y]] = \frac{1}{12} \int_0^l \frac{1}{l} y^2 dy = \frac{l^2}{36}$
- $\mathbb{E}(X|Y) = Y/2 \rightarrow \text{var}(\mathbb{E}[X|Y]) = \frac{1}{4} \text{var}[Y] = \frac{1}{4} \frac{l^2}{12} = \frac{l^2}{48}$
- $\text{var}(X) = \frac{l^2}{36} + \frac{l^2}{48} = \frac{7l^2}{144}$

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- (1) Derived distribution of  $Y = g(X)$  or  $Z = g(X, Y)$
- (2) Derived distribution of  $Z = X + Y$
- (3) Covariance: Degree of dependence between two rvs
- (4) Correlation coefficient
- (5) Conditional expectation and law of iterative expectations
- (6) Conditional variance and law of total variance
- (7) Random number of sum of random variables

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## Sum of a random number of rvs

- $N$  : number of stores visited (random)
- $X_i$ : money spent in store  $i$ , independent of other  $X_j$  and  $N$ ,  $X_i$ s are identically distributed with  $\mathbb{E}[X_i] = \mu$
- $Y = X_1 + X_2 + \dots X_N$ . What are  $\mathbb{E}[Y]$  and  $\text{var}[Y]$ ?
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N\mathbb{E}[X_i]] = \mathbb{E}[N]\mathbb{E}[X_i] = \mu\mathbb{E}[N]$
- $\text{var}[Y] = \mathbb{E}[\text{var}(Y|N)] + \text{var}[\mathbb{E}(Y|N)] = \mathbb{E}[N\text{var}[X_i] + \mu^2\text{var}[N]$   
 $\text{var}(\mathbb{E}[Y|N]) = \text{var}(N\mu) = \mu^2\text{var}[N]$   
 $\text{var}[Y|N] = N\text{var}[X_i]$   
 $\mathbb{E}[\text{var}(Y|N)] = \mathbb{E}[N\text{var}[X_i]] = \mathbb{E}[N]\text{var}[X_i]$

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## Questions?

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## Review Questions

- 1) What are the key steps to get the derived distributions of  $Y = g(X)$  or  $Z = g(X, Y)$ ?
- 2) How can we compute the distribution of  $Z = X + Y$  when  $X$  and  $Y$  are independent?
- 3) What are covariance and correlation coefficient? Why do we need them?
- 4) Please explain the concepts of conditional expectation and conditional variance.

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