Lecture 6: Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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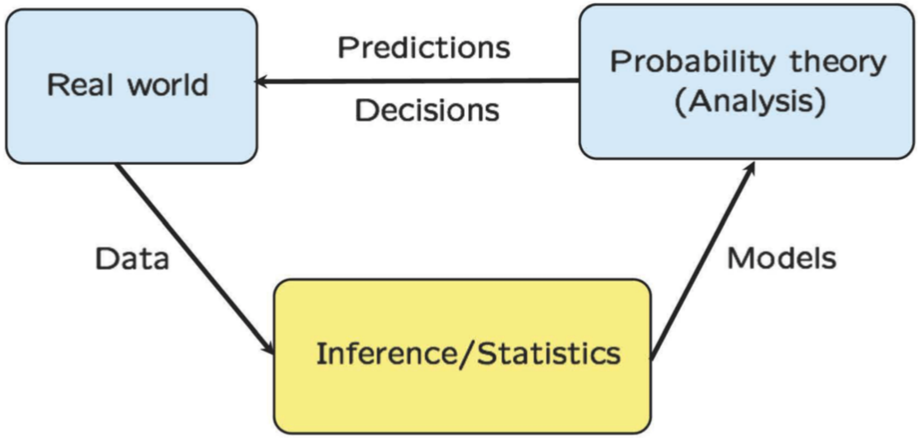
Inference: Big Picture



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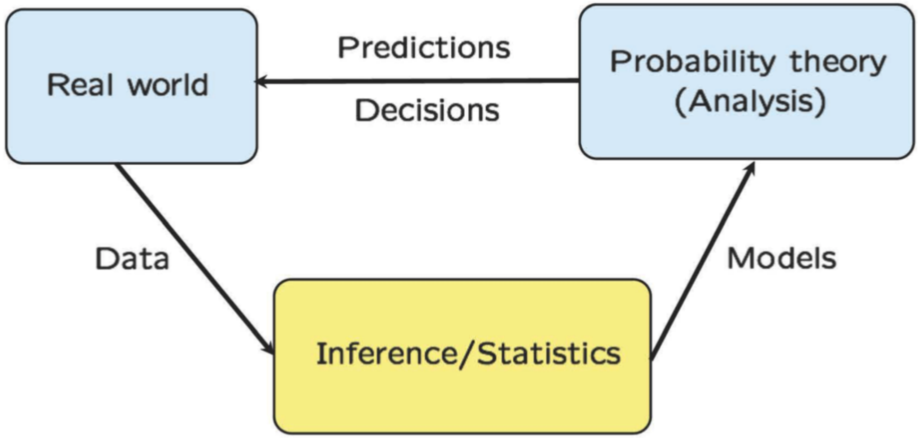

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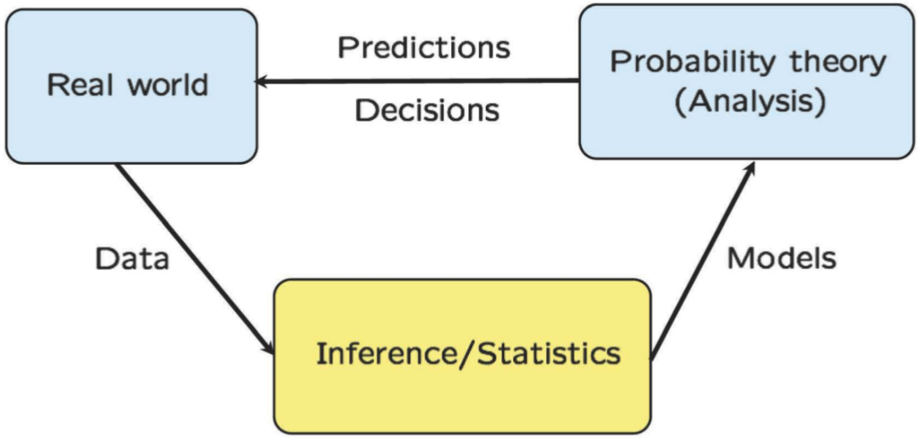

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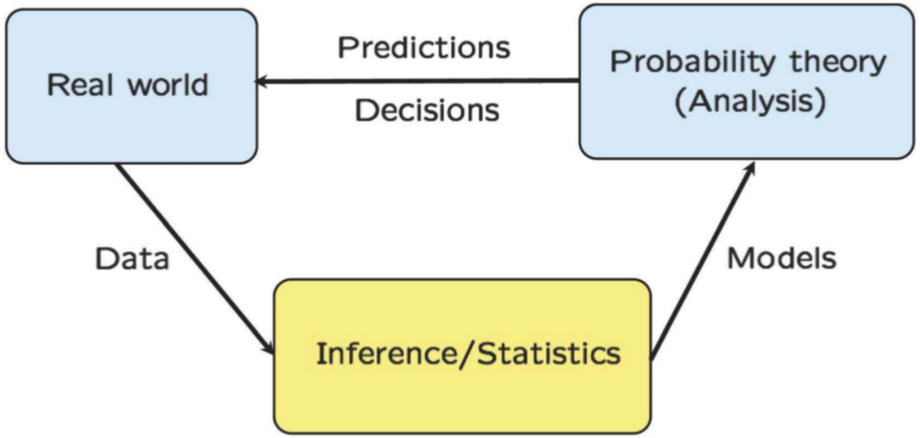
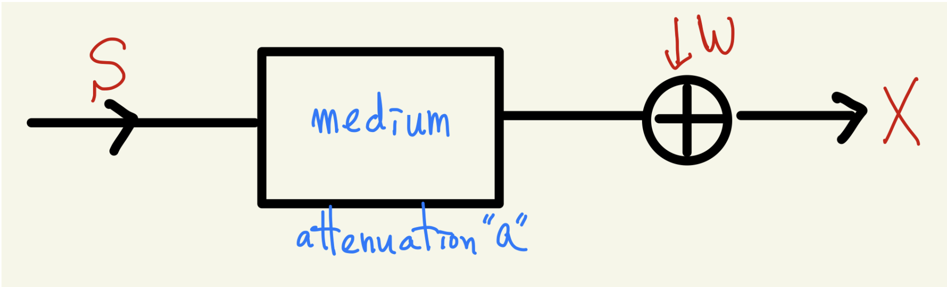
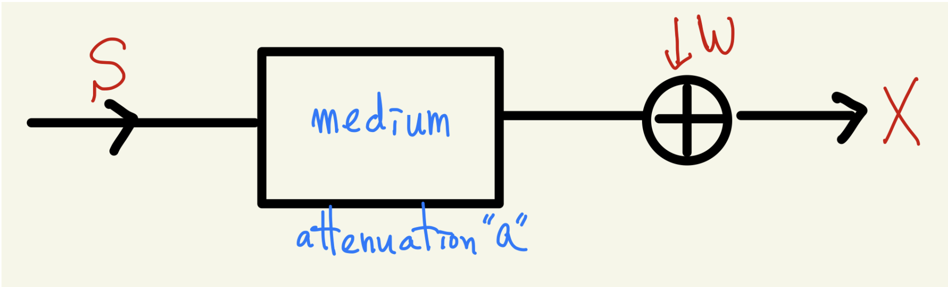


- Inference
  - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building



- $X = aS + W$

What to Infer?: Unknown Model vs. Unknown Variable



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- Modeling building
  - know the original signal S, observe X
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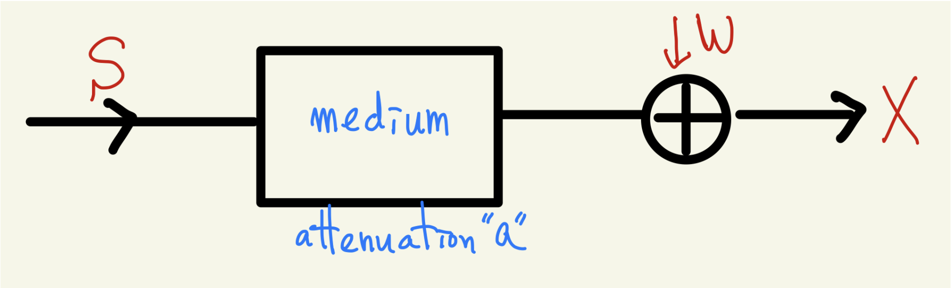

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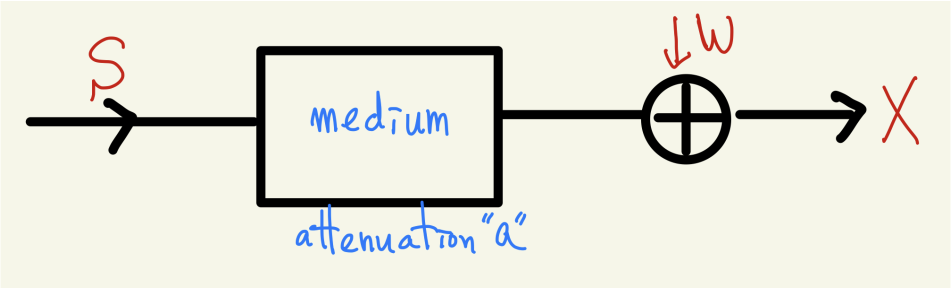

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- Modeling building
  - know the original signal S, observe X
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- Variable estimation
  - know a, observe X
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- Same mathematical structure, because the parameters in models are variables in many cases

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  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

Inference with Different Views: Bayesian vs. Classical (1)

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- *Classical approach* (Chapter 9)

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Bayesian approach

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Classical approach

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- Who is the winner? A century-long debate (see p. 409 for discussion)

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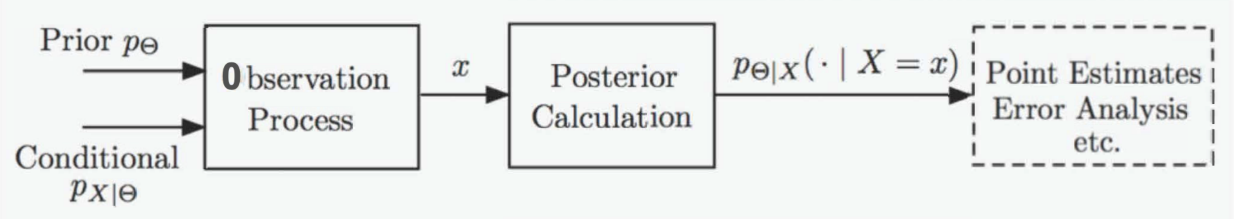

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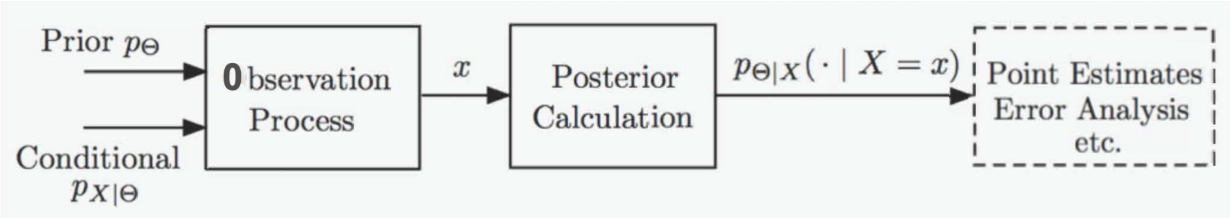

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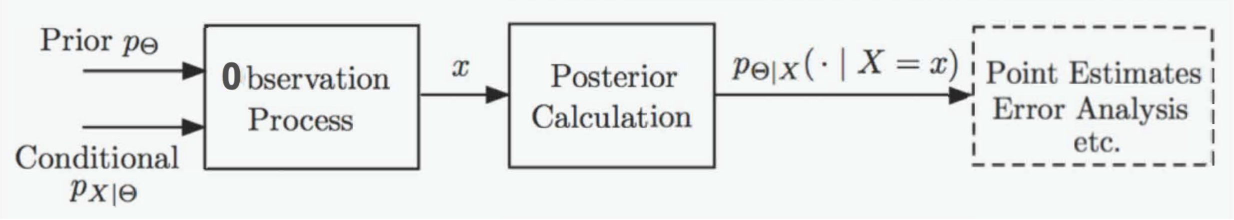

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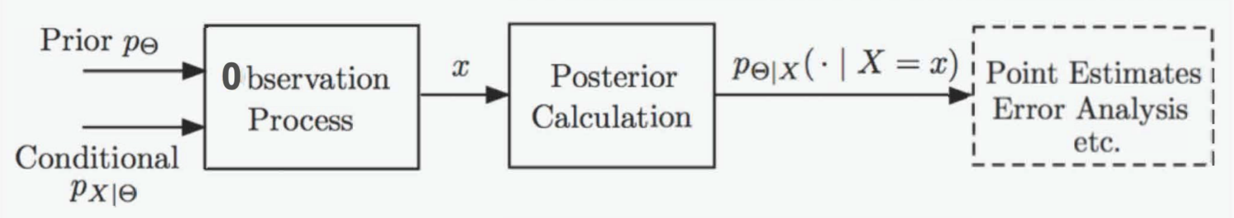

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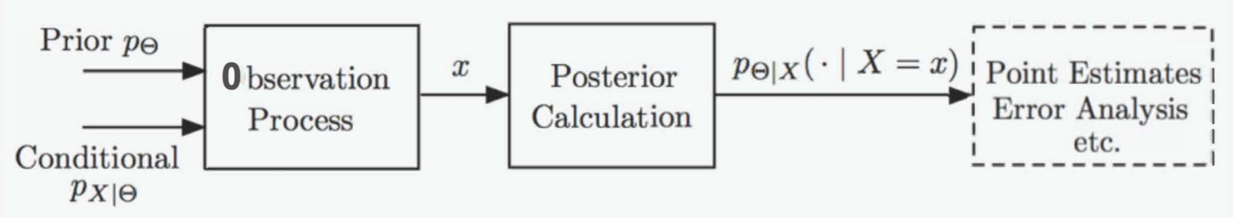

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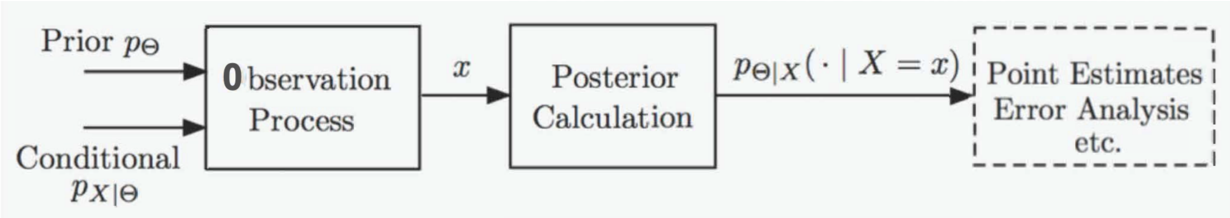

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- Find the posterior distribution $p_{X|\Theta}$ and $f_{X|\Theta}$.
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- Using the posterior distribution, apply one of the methods of choosing the final $\hat{\theta}$ for estimation and hypothesis testing.

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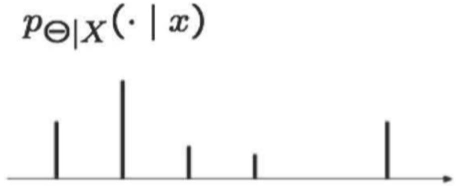

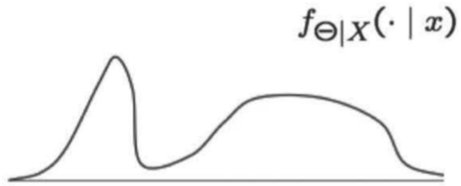

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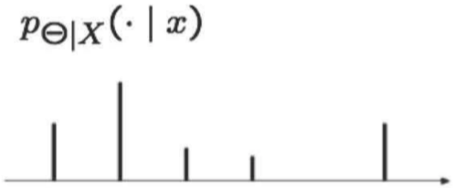
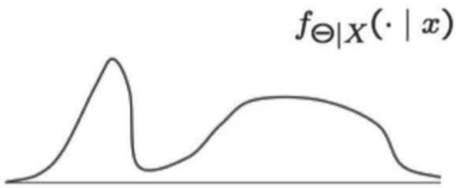
Point Estimation





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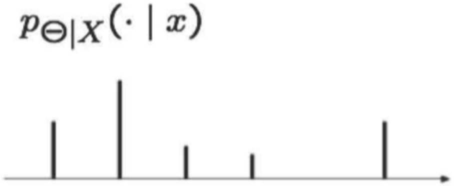

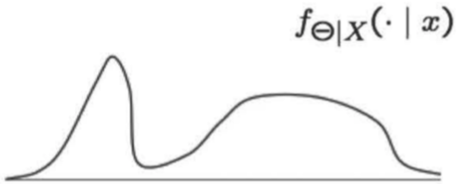

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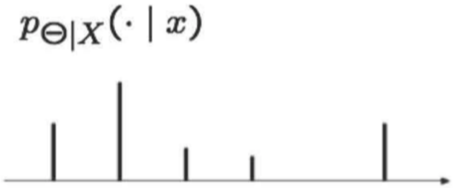

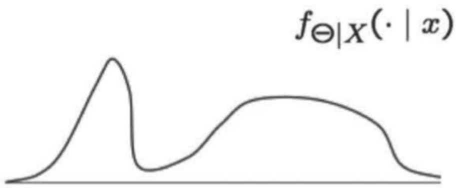

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- Why MAP and LMS are good? Not mathematically clear yet (later)

Estimator as a function

- Random observation: X
- Observation instance: x
- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\mathsf{MAP}} = g_{\mathsf{MAP}}(x), \quad \hat{\theta}_{\mathsf{LMS}} = g_{\mathsf{LMS}}(x)$$

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- MAP rule
  - Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$.
  - $\hat{\theta}_{\text{MAP}} = x$.

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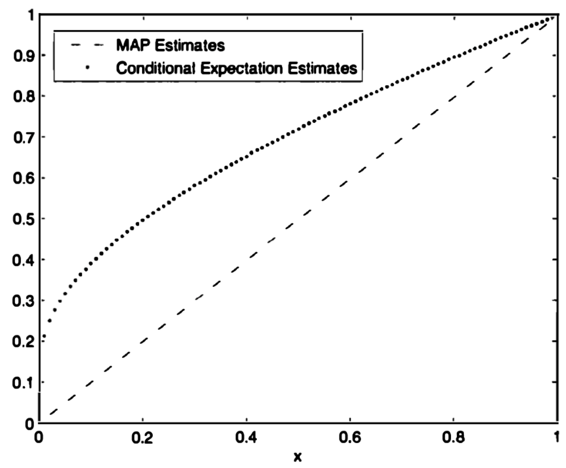

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A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

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where $B(\alpha, \beta)$, called Beta function, is a normalizing constant, given by

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- When $\alpha = \beta = 1$ (i.e., $U[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_\Theta(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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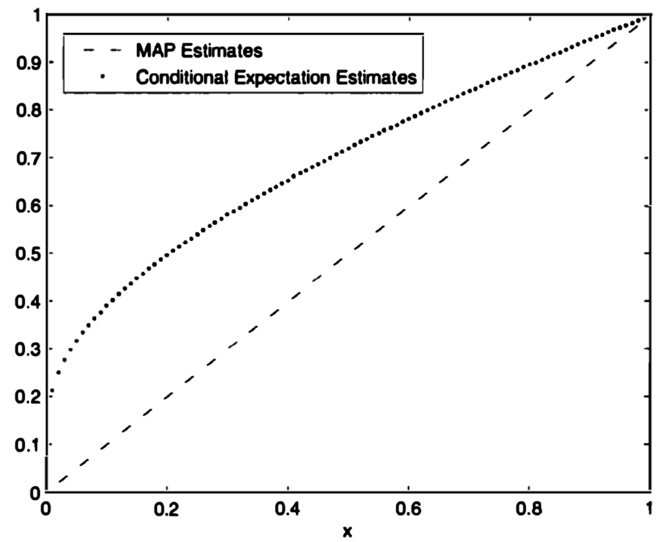

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- For $\alpha = \beta = 1$ ($\Theta = Uniform[0, 1]$),

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Example: Signal Recovery from Noisy Measurement (1)

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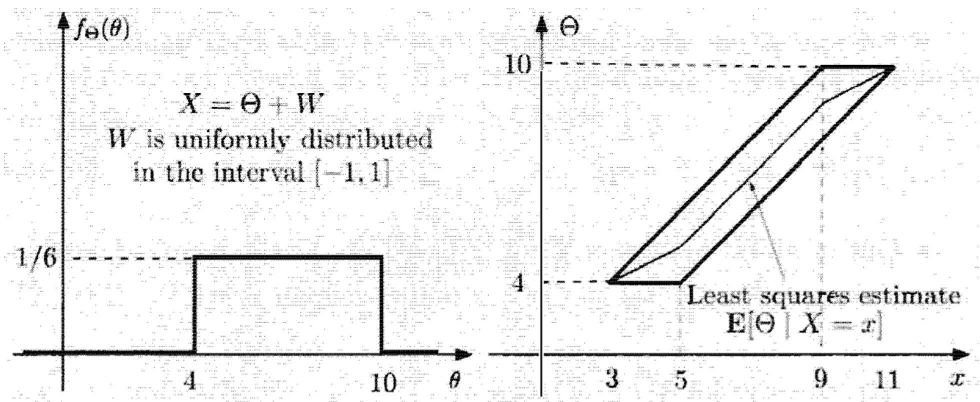

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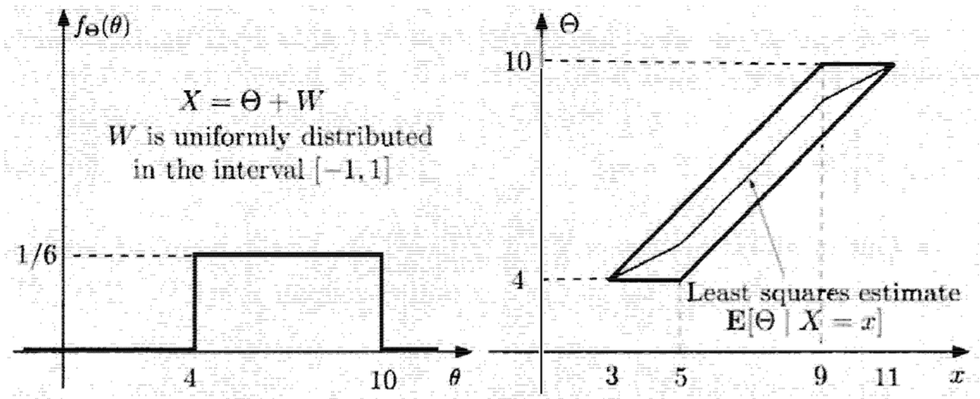

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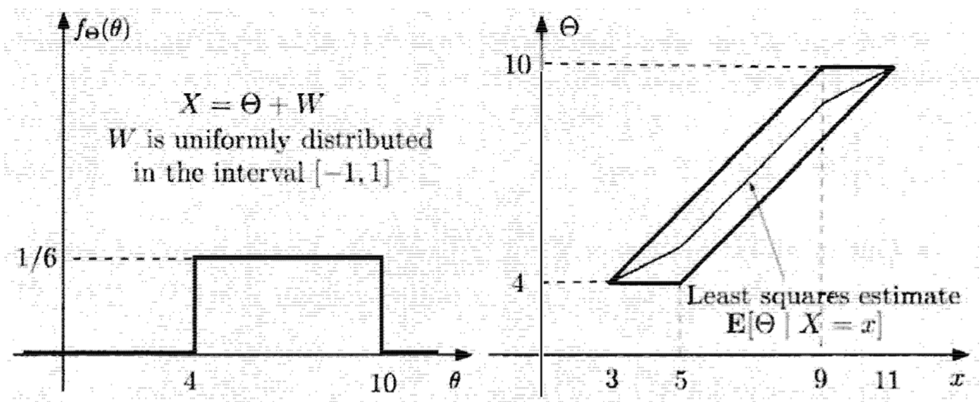
- $\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta|X = x] =$ midpoint of the corresponding vertical section



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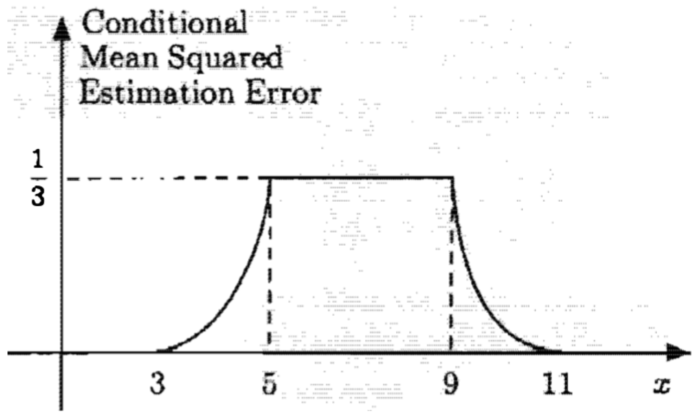

Hardness of LMS Estimation

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- Any alternative to LMS estimator?

Roadmap

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

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- Linear models are always the first choice for a simple design in engineering.

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- If $\rho = 0$ (uncorrelated):
  - Just baseline ($\mathbb{E}[\Theta]$)
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data X

Linear LMS (LLMS) Estimator: Proof

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(3)

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$$= (1)$$

- Using $\rho = \frac{\text{cov}(\Theta,X)}{\sigma_\Theta\sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

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- LLMS estimator is:

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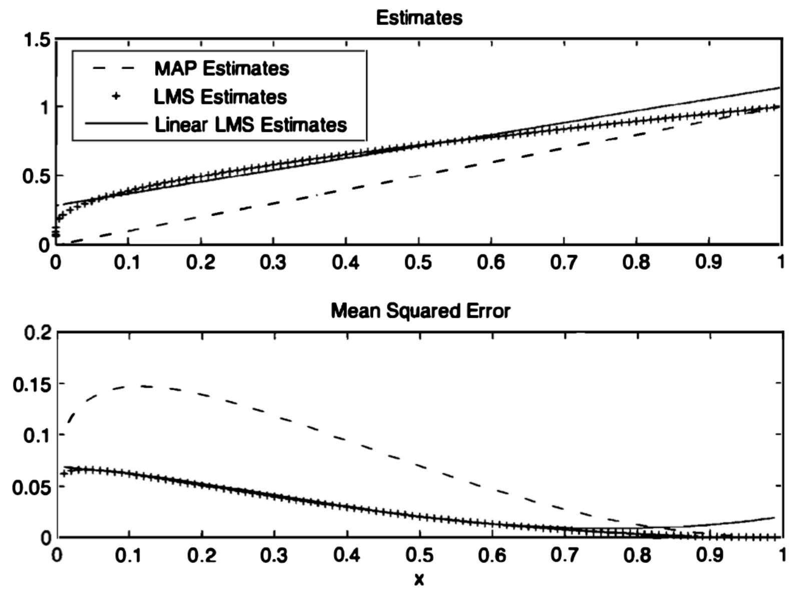

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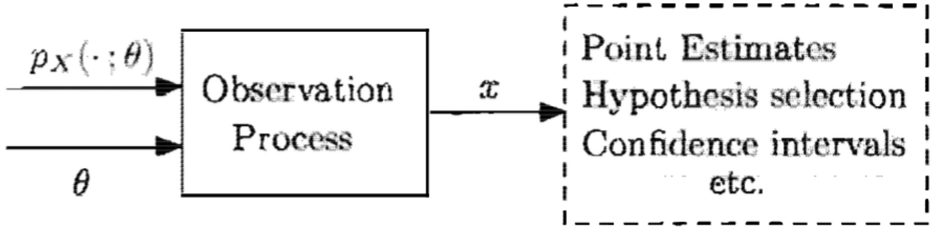
Yes, because the LMS esitmator was linear.

- Basics on Statistic Inference
- Framework of Bayesian Inference
- MAP (Maximum A Posteriori) Estimator
- LMS (Least Mean Squares) Estimator
- LLMS (Linear LMS) Estimator
- Framework of Classical Inference
- ML (Maximum Likelihood) Estimator

Framework of Classical Inference (1)



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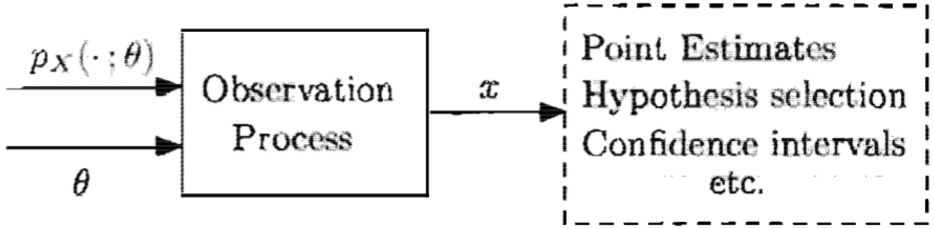

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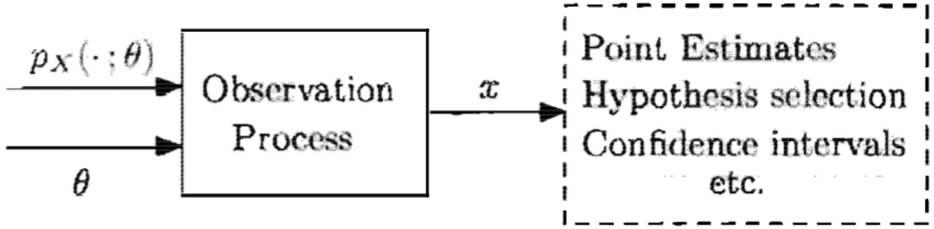

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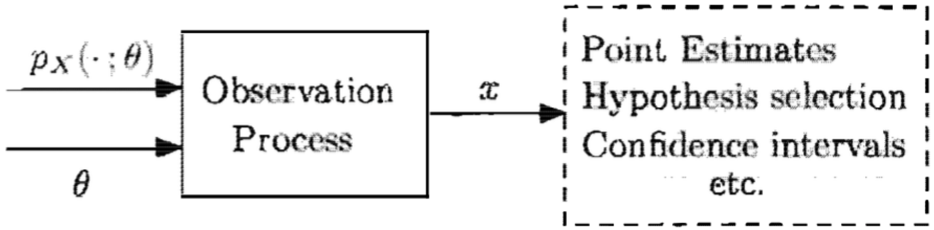

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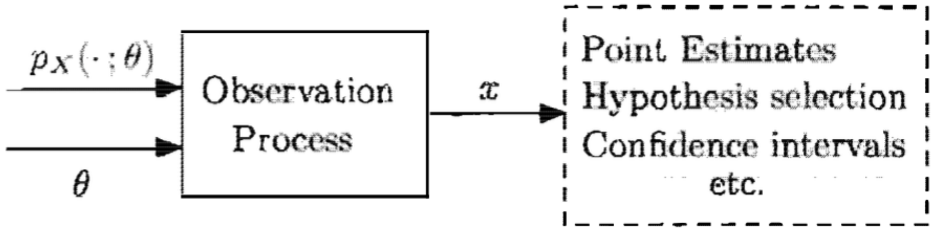

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- Very often, X_i are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \ldots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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- When Θ is uniform (complete ignorance of Θ), MAP == ML

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Questions?

Review Questions

1) What is statistical inference?
2) Draw the building blocks of Bayesian inference and explain how it works.
3) What are MAP and LMS estimators and their underlying philosophies?
4) What is LLMS estimator and why is it useful?
5) Compare the classical and Bayesian inference.
6) What is the ML estimator and how is it related to the MAP estimator?