

## Lecture 9: Introduction to Statistical Inference

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EE210: Probability and Introductory Random Processes  
KAIST EE

June 2, 2021

- (1) Overview on Statistical Inference
- (2) Bayesian Inference: Framework
- (3) Examples
- (4) MAP (Maximum A Posteriori) Estimator
- (5) LMS (Least Mean Squares) Estimator
- (6) LLMS (Linear LMS) Estimator
- (7) Classical Inference: ML Estimator

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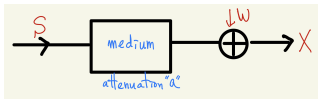
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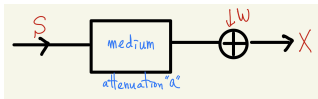
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- Process of extracting information about an **unknown variable** or an **unknown model** from **noisy available data**

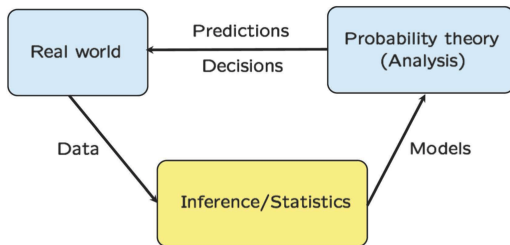




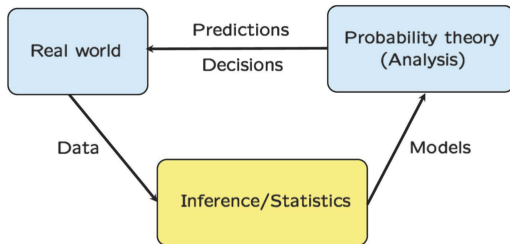
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3. How to obtain samples has impact on inference (e.g., when we need to pay for online surveys)

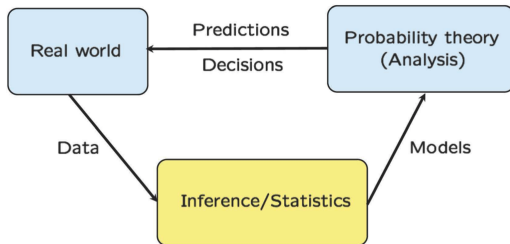


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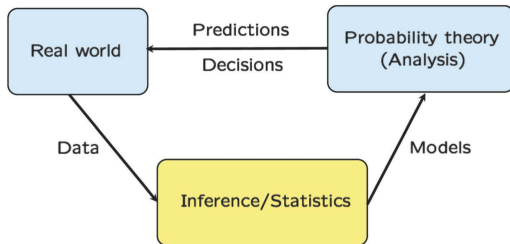
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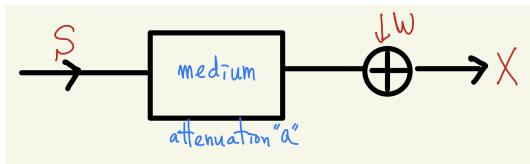
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  - Analysis is possible, so that predictions and decisions are made.



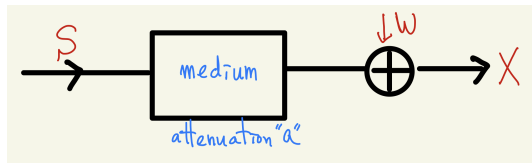
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- Inference
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- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building

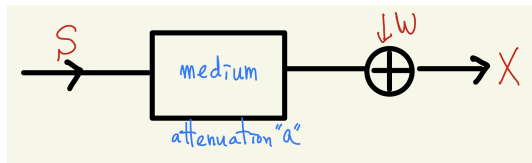




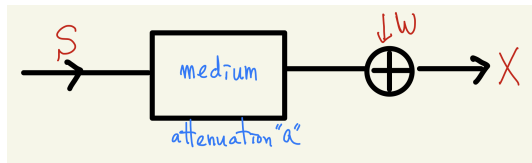
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- Same mathematical structure, because the parameters in models are variables in many cases



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  - (Ex) Biased coin with unknown probability of head  $\theta \in [0, 1]$ . Data of heads and tails. What is  $\theta$ ?
  - (Note) If you have the candidate values of  $\theta = \{1/4, 1/2, 3/4\}$ , then it's a hypothesis testing problem

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(Note) There are other inference methods, and here we just show examples.

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Classical approach

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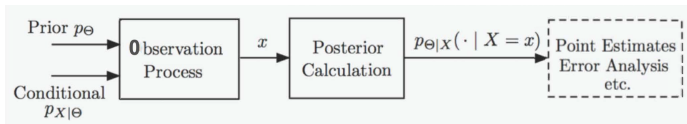
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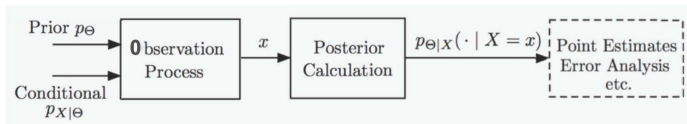
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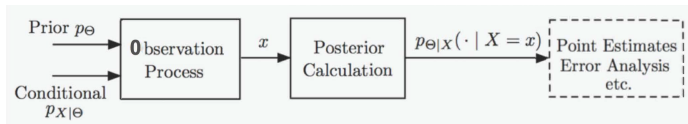
- Fundamental difference about the nature of unknown models or variables
- Random variable or deterministic quantity
- Who is the winner? A century-long debate
- Example of debate: mass of the electron by noisy measurement
  - **Classical.** while unknown, it is a constant and there is no justification for modeling it as a random variable.
  - **Bayesian.** Prior distribution reflects our state of knowledge, e.g., some range of candidate values from our previous noisy measurements.
- Particular prior? too arbitrary vs. every statistical procedure's hidden choices
- Practical issues: Bayesian approach is often computationally intractable (multi-dimensional integrals)

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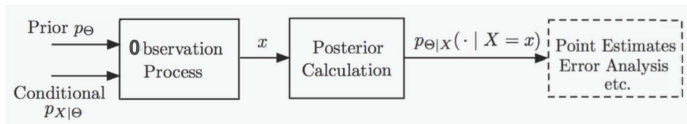




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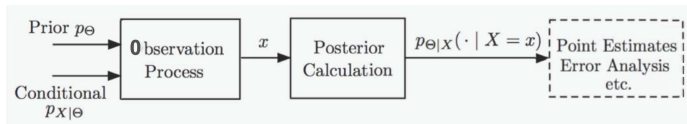


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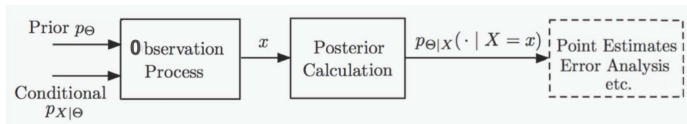


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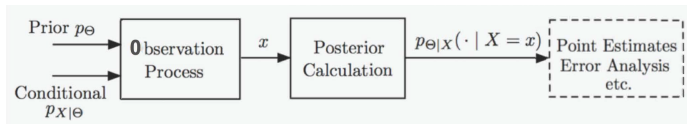




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  - However, one may use it for further processing, depending on what he/she wants, e.g., point estimation.



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- **Multiple** observations and **multiple** parameters are possible
  - $X = (X_1, \dots, X_n)$
  - $\Theta = (\Theta_1, \dots, \Theta_n)$

- $\Theta$ : discrete,  $X$ : discrete

$$p_{\Theta|X}(\theta|x) = \frac{p_{\Theta}(\theta)p_{X|\Theta}(x|\theta)}{p_X(x)}$$

$$p_X(x) = \sum_{\theta'} p_{\Theta}(\theta')p_{X|\Theta}(x|\theta')$$

- $\Theta$ : continuous,  $X$ : continuous

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- Prior

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- Prior and observation model (likelihood)

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

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- Posterior

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'} =$$

- Romeo and Juliet start dating, where Romeo is late by  $X \sim \mathcal{U}[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim \mathcal{U}[0, 1]$ .
- Observation: Romeo was late by  $x$ .
- Prior and observation model (likelihood)

$$f_{\Theta}(\theta) = \begin{cases} 1, & 0 \leq \theta \leq 1 \\ 0, & \text{otherwise} \end{cases}, \quad f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

- Posterior

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int_0^1 f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'} = \begin{cases} \frac{1/\theta}{\int_x^1 \frac{1}{\theta'}d\theta'} = \frac{1}{\theta|\log x|}, & x \leq \theta \leq 1, \\ 0, & \theta < x \text{ or } \theta > 1 \end{cases}$$

- What happens if we have more observation samples?
  - Romeo was late *n times* by  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ ,  $X_i \sim \mathcal{U}[0, \theta]$ .
  - $X_1, \dots, X_n$  are conditionally independent, given  $\Theta = \theta$ .
  - Unknown:  $\theta$  modeled by a rv  $\Theta \sim \mathcal{U}[0, 1]$ .
  - Observation: Romeo was late *n times* by  $\mathbf{x} = (x_1, x_2, \dots, x_n)$
  - See Example 8.2 at pp. 414 for more detailed treatment.

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- Observation model  $p_{X_i|\Theta(x_i|1)}$  and  $p_{X_i|\Theta(x_i|2)}$  are known.
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- Posterior PMF

$$\mathbb{P}[\Theta = m | (x_1, \dots, x_n)] = \frac{p_{\Theta}(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_{\Theta}(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

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- **Question.** Suppose that you have freedom to choose the form of the prior distribution. What prior will you choose? Requirement of “good” priors?
- We will look at the prior whose distribution is something called the **Beta distribution**.



## Beta distribution

A continuous rv  $\Theta$  follows a beta distribution with integer parameters  $\alpha, \beta > 0$ , if

$$f_{\Theta}(\theta) = \begin{cases} \theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise,} \end{cases}$$



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- See <https://youtu.be/8yaRt24qA1M> for the integration in the Beta function formula.
- A special case of Beta(1, 1) is  $\mathcal{U}[0, 1]$

- If  $\Theta \sim \text{Beta}(\alpha, \beta)$ , then  $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
- In other words, Beta prior  $\implies$  Beta posterior (why useful?)

Proof.

(a) First, the posterior pdf is given by:

$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1 - \theta)^{n-k}, \text{ } c \text{ the normalizing constant}$$

(b) Next, for  $\text{Beta}(\alpha, \beta)$  prior,  $f_{\Theta}(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}$ .

(c) Then,  $f_{\Theta|X}(\theta|k) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1 - \theta)^{n-k} = \frac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1} (1 - \theta)^{\beta+n-k-1}$ ,  
where  $d = c \binom{n}{k}$ .

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 $X = \theta + W$ , where  $W \sim \mathcal{N}(0, \sigma^2)$
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**Lemma.** Up to recaling, the pdf of the form  $e^{-\frac{1}{2}(ax^2-2bx+c)}$  is  $\mathcal{N}(\frac{b}{a}, \frac{1}{a})$ .



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  - $-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2) = -\frac{1}{2}(ax^2 - 2bx + c)$
  - Thus,  $\sigma^2 = \frac{1}{a}$  and  $\frac{\mu}{\sigma^2} = b \implies \mu = b\sigma^2 = \frac{b}{a}$

**Theorem.** The product of two Gaussian pdfs  $\mathcal{N}(\mu_0, \nu_0)$  and  $\mathcal{N}(\mu_1, \nu_1)$  is  $\mathcal{N}\left(\frac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \frac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$ .

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**Theorem.** The product of  $n + 1$  Gaussian pdfs  $\mathcal{N}(\mu_0, \nu_0)$ ,  $\mathcal{N}(\mu_1, \nu_1), \dots$ ,  $\mathcal{N}(\mu_n, \nu_n)$ , is  $\mathcal{N}(\mu, \nu)$ , where

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$$X_i = \theta + W_i, \quad W_i \sim \mathcal{N}(0, \sigma_i^2), \quad i = 1, \dots, n$$

- $\Theta, W_1, \dots, W_n$  are independent and let  $X = (X_1, \dots, X_n)$ ,  $x = (x_1, \dots, x_n)$ .
- Our interest. The **posterior pdf**  $f_{\Theta|X}(\theta|x)$ .

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- Observation model.** Noting that  $X_1, X_2, \dots, X_n$  are independent,

$$f_{X|\Theta}(x|\theta) = c_2 \cdot \exp \left\{ -\frac{(\theta - x_1)^2}{2\sigma_1^2} \right\} \cdots \exp \left\{ -\frac{(\theta - x_n)^2}{2\sigma_n^2} \right\}$$

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'}$$

- Numerator:**  $f_{\Theta}(\theta)f_{X|\Theta}(x|\theta) = c_1 c_2 \cdot \exp \left\{ - \sum_{i=0}^n \frac{(x_i - \theta)^2}{2\sigma_i^2} \right\}$ , which can be reexpressed as the following, using the **product of  $n + 1$  Gaussians**:

$$c_1 c_2 \cdot \exp \left\{ - \sum_{i=0}^n \frac{(x_i - \theta)^2}{2\sigma_i^2} \right\} = d \cdot \exp \left\{ - \frac{(\theta - m)^2}{2v} \right\},$$

$$\text{where } m = \frac{\sum_{i=0}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}, \quad v = \frac{1}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}$$

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- Denominator:** just a constant, not a function of  $\theta$

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{\int f_{\Theta}(\theta')f_{X|\Theta}(x|\theta')d\theta'}$$

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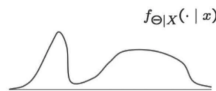
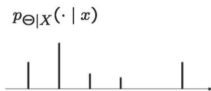
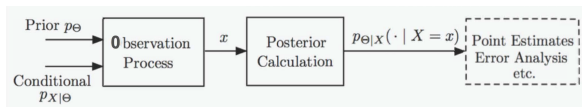
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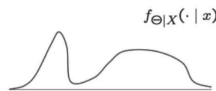
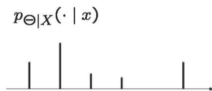
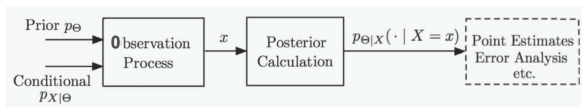
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$$\text{mean} = \frac{(m/v) + (x_{n+1}/\sigma_{n+1}^2)}{(1/v) + (1/\sigma_{n+1}^2)}, \quad \text{variance} = \frac{1}{(1/v) + (1/\sigma_{n+1}^2)}$$

- (1) Overview on Statistical Inference
- (2) Bayesian Inference: Framework
- (3) Examples
- (4) MAP (Maximum A Posteriori) Estimator
- (5) LMS (Least Mean Squares) Estimator
- (6) LLMS (Linear LMS) Estimator
- (7) Classical Inference: ML Estimator





- **Point Estimate**

- Given observation  $x$ , which **single** value  $\theta$  are you going to choose as your inference result? People often want just the summary and a simple answer.
- Very often,  $\theta$ , our inference target, is by nature a single value, i.e., mass of the electron.



$$p_{\Theta|X}(\cdot | x)$$



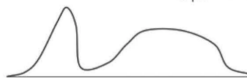
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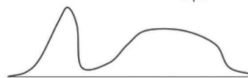
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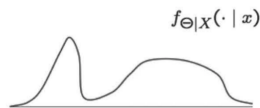
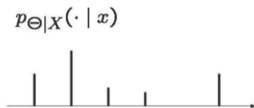


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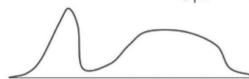


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- Notation: The community uses  $\hat{\theta}$  to mean the estimated value, i.e., **hat** for estimated value.

- Random observation:  $X$
- Observation instance:  $x$
- **Estimate** as a mapping from  $x$  to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\text{MAP}} = g_{\text{MAP}}(x), \quad \hat{\theta}_{\text{LMS}} = g_{\text{LMS}}(x)$$



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From now on we focus on the MAP estimate, mainly based on the examples that we've discussed in the previous section.

Slide 16 for more details

- Romeo and Juliet start dating, where Romeo is late by  $X \sim \mathcal{U}[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim \mathcal{U}[0, 1]$ .
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- Given  $x$ ,  $f_{\Theta|X}(\theta|x)$  is decreasing in  $\theta$  over  $[x, 1]$ .  $\implies \hat{\theta}_{\text{MAP}} = x$ .

Slide 18 for more details

- E-mail: **spam** (1) or **legitimate** (2),  $\Theta \in \{1, 2\}$ , with prior  $p_{\Theta}(1)$  and  $p_{\Theta}(2)$ .
- $\{w_1, w_2, \dots, w_n\}$ : a collection of words which suggest “spam”.
- For each  $i$ , a Bernoulli  $X_i = 1$  if  $w_i$  appears and 0 otherwise.
- Assumption: Conditioned on  $\Theta$ ,  $X_i$  are independent.
- Posterior PMF

$$\mathbb{P}[\Theta = m | (x_1, \dots, x_n)] = \frac{p_{\Theta}(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_{\Theta}(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$



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- MAP rule for this hypothesis testing problem. Decided that the message is **spam** if

$$p_{\Theta}(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_{\Theta}(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

Slide 21 for more details

- Biased coin with probability of head  $\theta$
- Unknown  $\theta$ : modeled by  $\Theta$  with some prior  $f_{\Theta}(\theta)$
- Observation  $X$ : number of heads out of  $n$  tosses

- If  $\Theta \sim \text{Beta}(\alpha, \beta)$ , then  $\Theta| \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
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- When  $\alpha = \beta = 1$  (i.e.,  $\mathcal{U}[0, 1]$  prior),  $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

Slide 27 for more details

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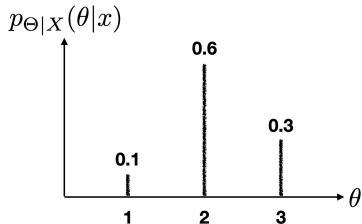
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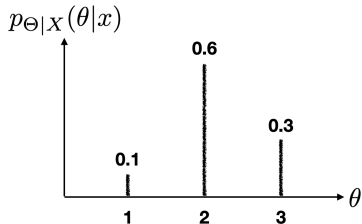


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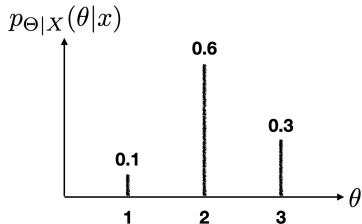


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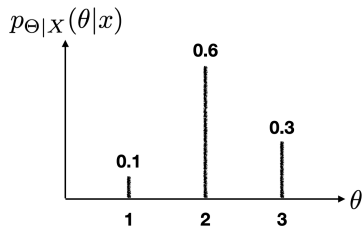


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$$= \arg \min_{\hat{\theta}=1,2,3} \mathbb{P}(\hat{\theta} \neq \Theta | X = x)$$

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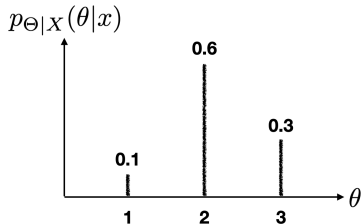


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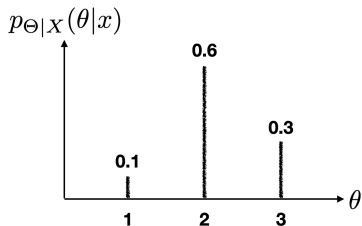
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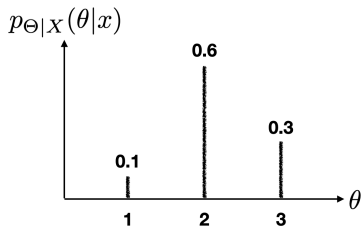
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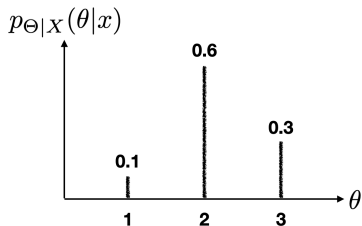
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- MAP estimate is intuitive, but we need more mathematical evidence for its performance guarantee. We would trust its quality if it is **optimal in some sense**.



- MAP:  $\hat{\theta}_{\text{MAP}} = 2$

- Given  $X = x$ ,  $\theta$  that minimizes the probability of incorrect decision?

$$\hat{\theta}_{\text{MAP}} = \arg \min_{\hat{\theta}=1,2,3} \mathbb{P}(\hat{\theta} \neq \Theta | X = x)$$

- Average probability of incorrect decision

$$\begin{aligned} \mathbb{P}(\hat{\Theta} \neq \Theta) &= \sum_x \mathbb{P}(\hat{\Theta} \neq \Theta | X = x) p_X(x) \\ &= \sum_x \mathbb{P}(\hat{\theta} \neq \Theta | X = x) p_X(x) \\ &\geq \sum_x \mathbb{P}(\hat{\theta}_{\text{MAP}} \neq \Theta | X = x) p_X(x) \end{aligned}$$

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Thus, **Claim 1** holds. We now take the expectation of the above equations, the law of iterated expectations leads to **Claim 2**.

- (1) Overview on Statistical Inference
- (2) Bayesian Inference: Framework
- (3) Examples
- (4) MAP (Maximum A Posteriori) Estimator
- (5) **LMS (Least Mean Squares) Estimator**
- (6) LLMS (Linear LMS) Estimator
- (7) Classical Inference: ML Estimator



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- Least Mean Square (LMS) Estimate



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$$\min_{\hat{\theta}} \mathbb{E}[(\Theta - \hat{\theta})^2] = \min_{\hat{\theta}} \left( \text{var}(\Theta - \hat{\theta}) + \left( \mathbb{E}[\Theta - \hat{\theta}] \right)^2 \right) = \min_{\hat{\theta}} \left( \text{var}(\Theta) + \left( \mathbb{E}[\Theta - \hat{\theta}] \right)^2 \right)$$

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- minimized when  **$\hat{\theta} = \mathbb{E}[\Theta]$** .

## What's the Form?: LMS Estimator (2)

- Unknown:  $\theta$  modeled by  $\Theta$  with prior  $f_{\Theta}(\cdot)$ .
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Slides 17 and 35 for more details

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Slides 17 and 35 for more details

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$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\theta|X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta =$$

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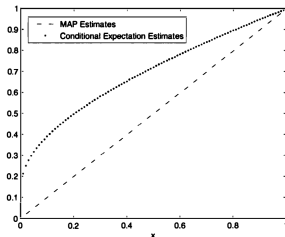
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Slides 21 and 37 for more details

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- Observation  $X$ : number of heads out of  $n$  tosses
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- For  $\alpha = \beta = 1$  ( $\mathcal{U}[0, 1]$  prior):  $\mathbb{E}[\Theta|X = k] = \frac{k + 1}{n + 2}$

Slides 27 and 38 for more details

- The posterior pdf  $f_{\Theta|X}(\theta|x) = a \cdot \exp \left\{ -\frac{(\theta - m)^2}{2v} \right\}$ , where

$$m = \frac{\sum_{i=0}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}, \quad v = \frac{1}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}$$

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Slides 27 and 38 for more details

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- What is the LMS estimate?

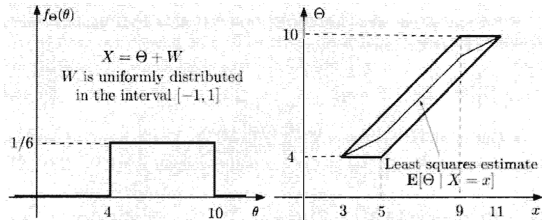
$$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta|X = x] = m$$

- Send signal  $\theta$  with the uniform noise  $W \sim \mathcal{U}[-1, 1]$ . Observe  $X$
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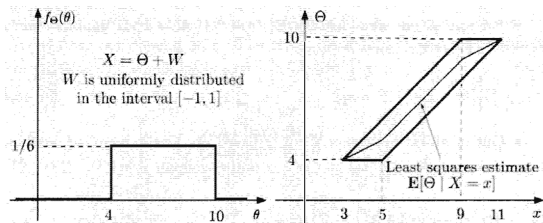


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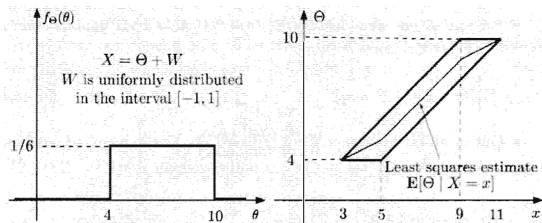
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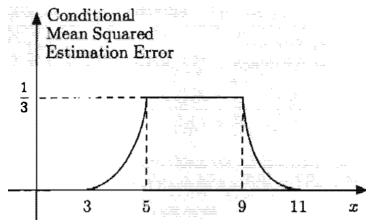
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$\hat{\theta}_{\text{LMS}} = \mathbb{E}[\Theta | X = x]$ : **midpoint** of the corresponding vertical section



- What is conditional MSE?  $\mathbb{E}[(\Theta - \mathbb{E}[\Theta|X = x])^2|X = x]$
- Given  $X = 3$ , it's the variance of  $\mathcal{U}[4, 4] = 0$
- Given  $X = 5$ , it's the variance of  $\mathcal{U}[4, 6] = (6 - 4)^2/12 = 1/3$
- The rising pattern between  $X = 3$  and  $X = 5$  is quadratic. This is because the expectation increases linearly, where the variance increases in a quadratic manner.



$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta)f_{X|\Theta}(x|\theta)}{f_X(x)}$$

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- Finding the posterior distribution is hard for multi-dimensional  $\Theta$
- $\Theta$  is very often high-dimensional, especially in the era of big data and deep learning
  - AlexNet in image recognition: 61M parameters
  - GPT-3 in natural language processing: 175B parameters

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- Observation model  $f_{X|\Theta}(x|\theta)$  may not be always available
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- $\Theta$  is very often high-dimensional, especially in the era of big data and deep learning
  - AlexNet in image recognition: 61M parameters
  - GPT-3 in natural language processing: 175B parameters
- Any alternative to LMS estimator?

- (1) Overview on Statistical Inference
- (2) Bayesian Inference: Framework
- (3) Examples
- (4) MAP (Maximum A Posteriori) Estimator
- (5) LMS (Least Mean Squares) Estimator
- (6) LLMS (Linear LMS) Estimator
- (7) Classical Inference: ML Estimator



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- Linear models are always the first choice for a simple design in engineering.

## LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X)) = \mathbb{E}(\Theta) + \rho \frac{\sigma_{\Theta}}{\sigma_X} (X - \mathbb{E}(X)),$$

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  - If  $X > \mathbb{E}[X] \Rightarrow \hat{\Theta}_L > \mathbb{E}[\Theta]$
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- If  $\rho = 0$  (uncorrelated):
  - Just baseline ( $\mathbb{E}[\Theta]$ )
  - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
  - No use of data  $X$



- MSE  $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$ ?

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  - Assume  $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$  (for simplicity). Then,  $\text{MSE} = \mathbb{E}\left[(\Theta - \rho \frac{\sigma_{\Theta}}{\sigma_X} X)^2\right]$

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- Uncertainty about  $\Theta$  after observation **decreases** by the factor of  $1 - \rho^2$
- What happens if  $|\rho| = 1$  or  $\rho = 0$ ?

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Slide pp. 43

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Slide pp. 43

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- Using  $\rho = \frac{\text{cov}(\Theta, X)}{\sigma_{\Theta}\sigma_X}$ , we get:

$$a = \frac{\rho\sigma_{\Theta}\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_{\Theta}}{\sigma_X}$$

- Then, we have (2).



Slides 17, 35, and 45 for more details

- Romeo and Juliet start dating, where Romeo is late by  $X \sim \mathcal{U}[0, \theta]$ .
- Unknown:  $\theta$  modeled by a rv  $\Theta \sim \mathcal{U}[0, 1]$ .
- Random observation:  $X$
- $\hat{\Theta}_{\text{MAP}} = X$ , and  $\hat{\Theta}_{\text{LMS}} = (1 - X)/| \log X$ .

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- $\hat{\Theta}_{\text{MAP}} = X$ , and  $\hat{\Theta}_{\text{LMS}} = (1 - X)/| \log X$ .
- **Question.** What is the LLMS estimator  $\hat{\Theta}_L$ ?

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)} (X - \mathbb{E}(X))$$

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- Using  $\mathbb{E}[\Theta] = 1/2$  and  $\mathbb{E}[\Theta^2] = 1/3$ ,

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[\text{var}[X|\Theta]] + \text{var}[\mathbb{E}[X|\Theta]] \\ &= \frac{1}{12}\mathbb{E}[\Theta^2] + \frac{1}{4}\text{var}[\Theta] = \frac{7}{144}\end{aligned}$$

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- $\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta]\mathbb{E}[X]$

$$\begin{aligned} \mathbb{E}[\Theta X] &= \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]] \\ &= \mathbb{E}[\Theta^2/2] = 1/6 \end{aligned}$$

$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

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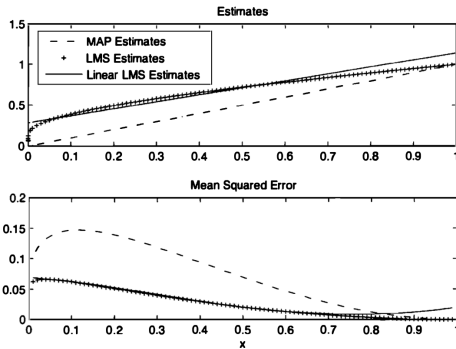
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- Biased coin with probability of head  $\theta$
- Unknown  $\Theta \sim \mathcal{U}[0, 1]$ ,
  - $\mathbb{E}[\Theta] = 1/2$ ,  $\text{var}[\Theta] = 1/12$
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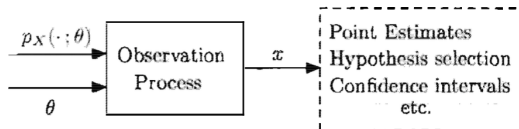
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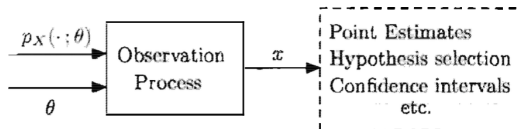
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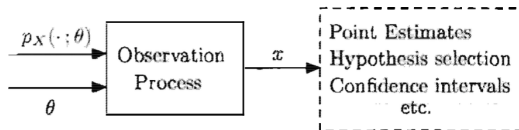
- (1) Overview on Statistical Inference
- (2) Bayesian Inference: Framework
- (3) Examples
- (4) MAP (Maximum A Posteriori) Estimator
- (5) LMS (Least Mean Squares) Estimator
- (6) LLMS (Linear LMS) Estimator
- (7) Classical Inference: ML Estimator



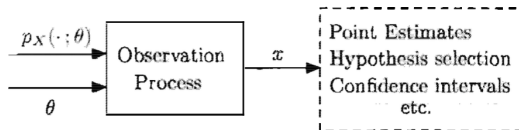
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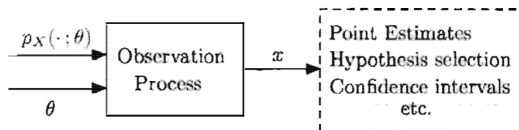
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  - Estimation
  - Hypothesis testing
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- Very often,  $X_i$ s are independent. Then, ML equals to maximizing the log-likelihood:

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- Thus, when  $\Theta$  is **uniform** (complete ignorance of  $\Theta$ ) in MAP, **MAP == ML**

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Questions?

- 1) What is statistical inference?
- 2) Draw the building blocks of Bayesian inference and explain how it works.
- 3) What are MAP and LMS estimators and their underlying philosophies?
- 4) What is LLMS estimator and why is it useful?
- 5) Compare the classical and Bayesian inference.
- 6) What is the ML estimator and how is it related to the MAP estimator?