Lecture 6: Law of Large Numbers and Central Limit Theorem

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EE210: Probability and Introductory Random Processes
KAIST EE

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Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

◦ Two most remarkable findings in probability theory

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- $X_1, X_2, \ldots, X_n$: i.i.d (independent and identically distributed) random variables
- $\mathbb{E}[X_i] = \mu$, $\text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

$$S_n = X_1 + X_2 + \ldots + X_n$$

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- Figure out the distribution of S_n. Very challenging. Even just for $Z = X + Y$, finding the distribution, for example, requires the complex convolution.

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- Easy case: Sum of normal rvs = a normal rv, however, generally very challenging.
- Possible appoach. Take a certain scaling with respect to n that corresponds to a new glass, and investigate the system for large n

Sample Mean

- Consider the sample mean, and try to understand how S_n behaves:

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- We call this law of large numbers (LLN).

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- Need to build up the new concept of convergence for the sequence of rvs.

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  - For any $\epsilon > 0$ and for any $\delta > 0$, there exists $N = N(\delta)$, such that for all $n \geq N$, $\mathbb{P}(|Y_n - Y| \geq \epsilon) \leq \delta$.
  - For any $\epsilon > 0$, $\mathbb{P}\left(\{|Y_n - Y| \geq \epsilon\}\right) \xrightarrow{n \to \infty} 0$.

Convergence in Probability (2)

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- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - Y| \geq \epsilon\}) \xrightarrow{n\to\infty} 0.$
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- A special case: when $Y = a$ for some constant a: $Y_n \xrightarrow{\text{in prob.}} a$
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- Proof. For any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(|Y_n - 0| \geq \epsilon) &= \mathbb{P}(X_1 \geq \epsilon, \ldots, X_n \geq \epsilon) = \mathbb{P}(X_1 \geq \epsilon) \times \cdots \times \mathbb{P}(X_n \geq \epsilon) \\ &= (1-\epsilon)^n \xrightarrow{n\to\infty} 0\end{aligned}$$

Example 2: Convergence in Probability

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- For any $\epsilon > 0$, $\mathbb{P}(\{|Y_n - a| \geq \epsilon\}) \xrightarrow{n\to\infty} 0.$

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- a sequence of rvs $Y_n = Y/n$ (note that these are dependent)

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Example 3: Convergence in Probability

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- For any $\epsilon > 0$,

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- Thus, Y_n converges to 0 in probability.

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- The proof requires some knowledge about useful inequalities, which we will cover later.

WLLN: Why Useful?

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- For example, assume that a large number of identically distributed noises come to you. Then, you can roughly approximate it as $(n \times \text{average noise})$
- Provides an interpretation of expectations (as well as probabilities) in terms of a long sequence of identical independent experiments. For example, what is the probability of head of a coin? Toss 1000 times, and count the number of heads.

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- The answer is $\frac{1}{2}$

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- Some deterministic number just like WLLG?

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- We expect that Z_n converges to something meaningful as $n \to \infty$, but what?
- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.

Central Limit Theorem: Start with Scaling (2)

- Reshaping the equation:

$$\frac{\sqrt{n}}{\sigma} \times (M_n - \mu) = \sqrt{n}\left(\frac{S_n - n\mu}{\sigma n}\right) = \frac{S_n - n\mu}{\sigma\sqrt{n}}.$$

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- Some deterministic number just like WLLG?
- Interestingly, it converges to some well-known random variable.
  - Need a new concept of convergence: "convergence in distribution"

Convergence in Distribution

- Consider a sequence of rvs $(Y_n)_{n=1,2,\ldots}$ and a rv Y.

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- Another type of convergence of rvs
- Comparison with convergence in probability?
  - Convergence in probability $\implies$ Convergence in distribution, but the reverse is not true.
  - The proof is beyond what this class covers, but it will be interesting to find an example that shows convergence in distribution, which is not convergence in probability.

Example: in Distribution, but not in Probability

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- We can find ϵ small enough so that the above does not go to zero.

- $S_n = X_1 + X_2 + \cdots + X_n, \qquad Z_n = \dfrac{S_n - n\mu}{\sigma\sqrt{n}}$

Central Limit Theorem

Z_n convergens to Z in distribution, where $Z \sim \mathcal{N}(0, 1)$.

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Central Limit Theorem

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- Very surprising!
- Irrespecitive of the distribution of X_i, Z is normal.

- For simplicity, assume that $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$, $i = 1, 2, \ldots, n$

LLG vs. CLT: Different Scaling Glasses

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> Scaling S_n by $1/n$, you go to a deterministic world.

- Central Limit Theorem

> Scaling S_n by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each X_i.

Practical Use of CLT

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \qquad \mathbb{P}(Z_n \le z) \xrightarrow{n\to\infty} \mathbb{P}(Z \le z),\ Z \sim \mathcal{N}(0,1)$$

[1]Only unique mode. A single maximum or minimum.

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- Can approximate S_n with a normal rv $\sim (n\mu, n\sigma^2)$
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- How large should n be?
  - A moderate n (20 or 30) usually works, which is the power of CLT.
  - If X_i resembles a normal rv more, smaller n works: symmetry and unimodality[1]

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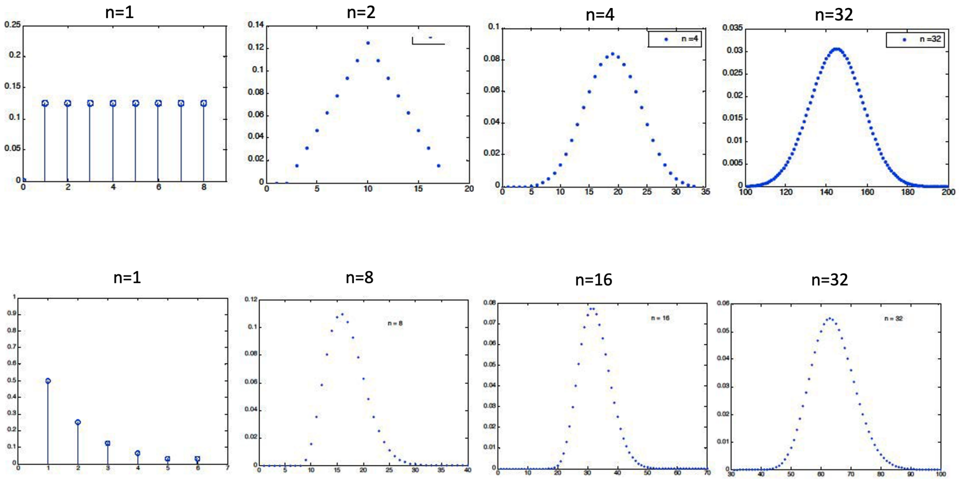
CLT: Examples of Required n



$\mathbb{P}(S_n \leq a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)

Examples of CLT (1)

$\mathbb{P}(S_n \leq a) \approx b$: Given two parameters, find the third

- Package weights X_i: iid exponential $\lambda = 1/2$ ($\mu = 1/\lambda = 2$ and $\sigma^2 = 1/\lambda^2 = 4$)
- Load container with $n = 100$ packages

$$\mathbb{P}(S_{100} \geq 210) = \mathbb{P}\left[\frac{S_{100} - 100 \cdot 2}{2\sqrt{100}} \geq \frac{210 - 200}{20}\right] = \mathbb{P}(Z_{100} \geq 0.5)$$

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$$\mathbb{P}(S_{100} \geq a) = \mathbb{P}\left[\frac{S_{100} - 100 \cdot 2}{2\sqrt{100}} \geq \frac{a - 200}{20}\right] = \mathbb{P}(Z_{100} \geq \frac{a-200}{20})$$

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- The value of a such that $\Phi(\frac{a-200}{20}) = 0.95$? $\frac{a-200}{20} = 1.645$ and $a = 232.9$

Examples of CLT (3)

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- The value of n such that $\frac{210-2n}{2\sqrt{n}} = 1.645$? $n = 89$

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

- (Q) Knowing $\mathbb{E}(X)$, can we say something about the distribution of X?

Markov Inequality

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Thus, $a \cdot \mathbb{P}(X \geq a) \leq \mathbb{E}[X]$. $\square$

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- Both bounds are the ones that bound the probability of rare events.

$$M_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \ldots X_n}{n}$$

Weak law of large numbers

M_n converges to μ in probability.

Proof. For any given $\epsilon > 0$,

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M_n converges to μ in probability.

Proof. For any given $\epsilon > 0$,

$$\mathbb{P}\left(|M_n - \mu| \geq \epsilon\right) \leq \frac{\text{var}(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \xrightarrow{n\to\infty} 0$$

Comparison: WLLN vs. CLT

We ask the same question, and try to answer it, using WLLN or CLT.

See how the answers becomes different.

Example: Polling using WLLN

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- Compare: 50,000 from LLN vs. 9604 from CLT

Roadmap

(1) Weak Law of Large Numbers: Result and Meaning

(2) Central Limit Theorem: Result and Meaning

(3) Weak Law of Large Numbers: Proof
- Inequalities: Markov and Chebyshev

(4) Central Limit Theorem: Proof
- Moment Generating Function (MGF)

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- If the context is clear, we omit X and use just $M(s)$.

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- Question. MGF of $\mathcal{N}(\mu, \sigma^2)$?

Useful Properties of MGF

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

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- Thus, $\mathsf{var}(X) = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

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The transform $M_X(s)$ associated with a random variable X uniquely determines the CDF of X, assuming that $M_X(s)$ is finite for all s in some interval $[-a, a]$, where a is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own unique MGF.

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- We can see that

$$p_X(-1) = \frac{1}{4},\ p_X(0) = \frac{1}{2},\ p_X(4) = \frac{1}{8},\ p_X(5) = \frac{1}{8}$$

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- X is a geometric rv with parameter p

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$$\begin{aligned}\mathbb{E}\left[e^{sS_n/\sqrt{n}}\right] &= \mathbb{E}\left[e^{sX_1/\sqrt{n}}\right] \times \cdots \times \mathbb{E}\left[e^{sX_n/\sqrt{n}}\right] \\ &= \left(\mathbb{E}\left[e^{sX_1/\sqrt{n}}\right]\right)^n = \left(M_{X_1}\left(\frac{s}{\sqrt{n}}\right)\right)^n\end{aligned}$$

Back to CLT Proof (1)

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$
- $Z_n = \frac{S_n}{\sqrt{n}} = \frac{X_1 + X_2 + \ldots X_n}{\sqrt{n}}$
- We will show: MGF of Z_n converges to MFG of $\mathcal{N}(0, 1)$ (using inversion property)
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$$\begin{aligned}\mathbb{E}\left[e^{sS_n/\sqrt{n}}\right] &= \mathbb{E}\left[e^{sX_1/\sqrt{n}}\right] \times \cdots \times \mathbb{E}\left[e^{sX_n/\sqrt{n}}\right] \\ &= \left(\mathbb{E}\left[e^{sX_1/\sqrt{n}}\right]\right)^n = \left(M_{X_1}\left(\frac{s}{\sqrt{n}}\right)\right)^n\end{aligned}$$

- For simplicity, let $M(\cdot) = M_{X_1}(\cdot)$

Back to CLT Proof (2)

- $M(0) = 1,\ M'(0) = 0,\ M''(0) = 1$

$\square$

Back to CLT Proof (2)

- $M(0) = 1,\ M'(0) = 0,\ M''(0) = 1$
- $\left(M\left(\frac{s}{\sqrt{n}}\right)\right)^n \xrightarrow{n\to\infty}$ what???

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- For convenience, do the change of variable $y = \frac{1}{\sqrt{n}}$. Then, $\displaystyle\lim_{y\to 0} \frac{\log M(ys)}{y^2}$
- If we apply l'hopital's rule twice (please check), we get

$$\lim_{y\to 0} \frac{\log M(ys)}{y^2} = \frac{s^2}{2}$$

□

Questions?

Review Questions

1) What's the practical value of LLN and CLT?
2) Explain LLN and CLT from the scaling perspective.
3) Why are LLN and CLT great?
4) Why do we need different concepts of convergence for random variables?
5) Explain what is convergence in probability.
6) Explain what is convergence in distribution.
7) Why is MGF useful?