Lecture 9: Introduction to Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

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(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

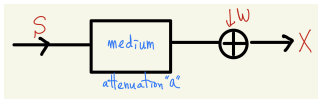KAIST EE

- Examples

# What is Statistical Inference?

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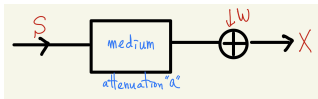
- Process of extracting information about an unknown variable or an unknown model from noisy available data

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2. There exists uncertainty (i.e., noise) as to how well the sample represents the unknown

3. How to obtain samples has impact on inference (e.g., when we need to pay for online surveys)
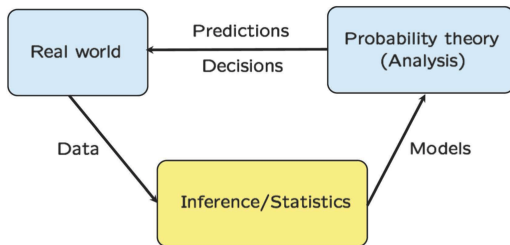
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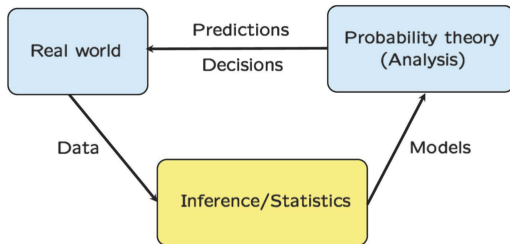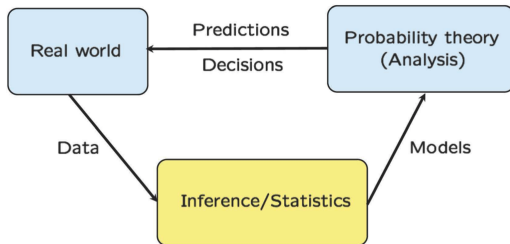
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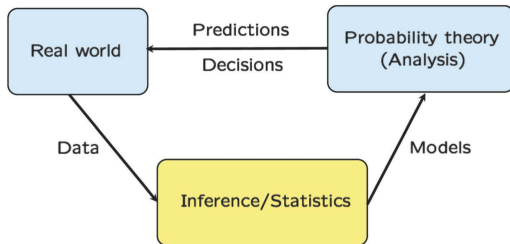 - Analysis is possible, so that predictions and decisions are made.

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- Inference
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- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building
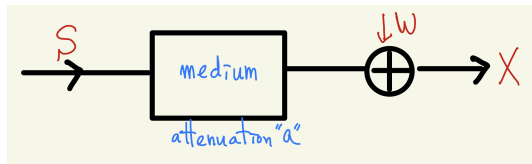
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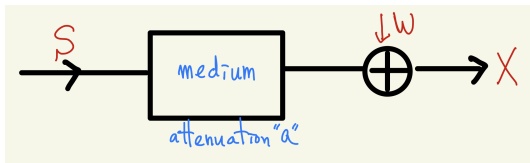
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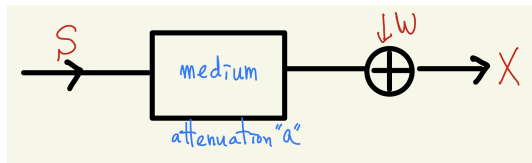
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- Variable estimation
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What to Infer?: Unknown Model vs. Unknown Variable

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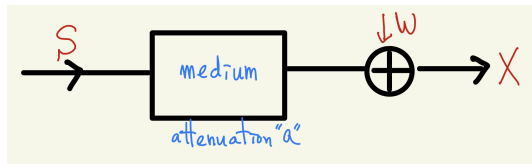
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- Same mathematical structure, because the parameters in models are variables in many cases

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  - Unknown: a few possible ones
  - Goal: small probability of incorrect decision

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  - (Ex) Biased coin with unknown probability of head $\theta \in [0, 1]$. Data of heads and tails. What is θ?

What Kind?: Hypothesis Testing vs. Estimation

- Hypothesis testing
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  - (Ex) Biased coin with unknown probability of head $\theta \in [0, 1]$. Data of heads and tails. What is θ?
  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

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- Distribution of θ (prior) e.g.,

$$\mathbb{P}\left(\theta = \frac{3}{4}\right) = 1/2, \quad \mathbb{P}\left(\theta = \frac{1}{4}\right) = 1/2$$

Different Views: Bayesian vs. Classical (1)

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- Use Bayes' rule and find the posterior:

$$\mathbb{P}\left[\theta = \frac{3}{4}\Big|(HHH)\right] = \frac{27}{28}, \ \mathbb{P}\left[\theta = \frac{1}{4}\Big|(HHH)\right] = \frac{1}{28}$$

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(Note) There are other inference methods, and here we just show examples.

Bayesian approach

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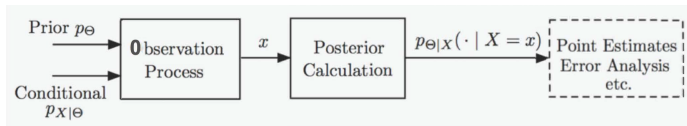
- Fundamental difference about the nature of unknown models or variables

- Random variable or deterministic quantity

- Who is the winner? A century-long debate

- Example of debate: mass of the electron by noisy measurement
  ◦ Classical. while unknown, it is a constant and there is no justification for modeling it as a random variable.

  ◦ Bayesian. Prior distribution reflects our state of knowledge, e.g., some range of candidate values from our previous noisy measurements.

- Particular prior? too arbitrary vs. every statistical procedure's hidden choices

- Pratical issues: Bayesian approach is often computationally intractable (multi-dimensional integrals)

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  - physical quantity or model parameter
  - random variable
  - prior distribution p_Θ and f_Θ

Prior p_Θ → **O**bservation Process ← Conditional $p_{X|\Theta}$ → x → Posterior Calculation → $p_{\Theta|X}(\cdot \mid X = x)$ → Point Estimates Error Analysis etc.
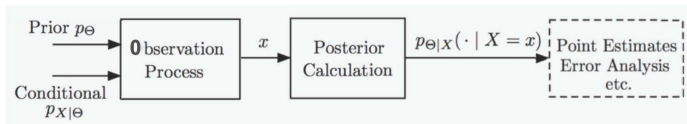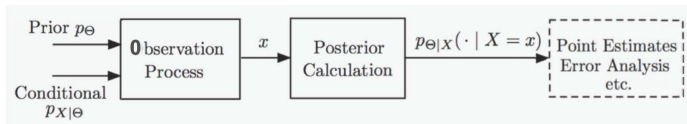
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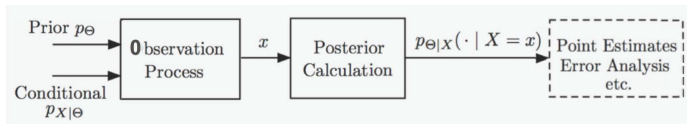
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- That is, the joint distribution of X and Θ ($p_{X,\Theta}(x,\theta)$ and $f_{X,\Theta}(x,\theta)$) is given
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- The posterior distribution is the complete answer of the Bayesian inference.
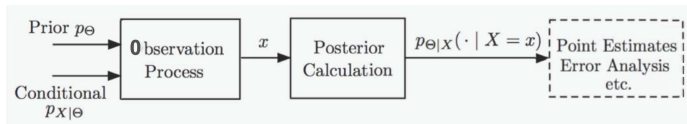
Framework of Bayesian Inference

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- Observations or measurements X
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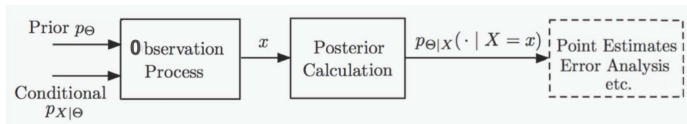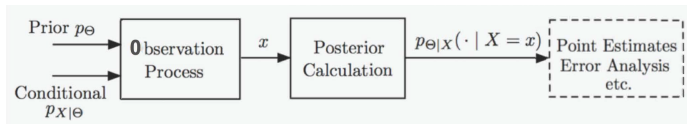 - observation model $p_{X|\Theta}$ and $f_{X|\Theta}$
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- However, one may use it for further processing, depending on what he/she wants, e.g., point estimation.
- Multiple observations and multiple parameters are possible
  - $X = (X_1, \ldots, X_n)$
  - $\Theta = (\Theta_1, \ldots, \Theta_n)$

- Θ: discrete, X: discrete

$$p_{\Theta|X}(\theta|x) = \frac{p_\Theta(\theta)p_{X|\Theta}(x|\theta)}{p_X(x)}$$

$$p_X(x) = \sum_{\theta'} p_\Theta(\theta')p_{X|\Theta}(x|\theta')$$

- Θ: continuous, X: continuous

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- Prior and observation model (likelihood)

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$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} =$$

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$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} = \begin{cases} \frac{1/\theta}{\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta|\log x|}, & x \le \theta \le 1, \\ 0, & \theta < x \text{ or } \theta > 1 \end{cases}$$

○ What happens if we have more observation samples?

- Romeo was late *n* times by $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $X_i \sim \mathcal{U}[0, \theta]$.

- $X_1, \ldots, X_n$ are conditionally independent, given $\Theta = \theta$.

- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.

- Observation: Romeo was late *n* times by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$

- See Example 8.2 at pp. 414 for more detailed treatment.

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.

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- $\{w_1, w_2, \ldots, w_n\}$: a collection of words which suggest "spam".

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$$\mathbb{P}\Big[\Theta = m|(x_1, \ldots, x_n)\Big] = \frac{p_\Theta(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

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- We will look at the prior whose distribution is something called the Beta distribution.

Beta distribution

A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

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- A special case of Beta$(1, 1)$ is $\mathcal{U}[0, 1]$

- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

- In other words, Beta prior $\implies$ Beta posterior (why useful?)

Proof.

(a) First, the posterior pdf is given by:
$$f_{\Theta|X}(\theta|k) = c f_\Theta(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_\Theta(\theta) \theta^k (1-\theta)^{n-k}, \; c \text{ the normalizing constant}$$

(b) Next, for $\text{Beta}(\alpha, \beta)$ prior, $f_\Theta(\theta) = \dfrac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$.

(c) Then, $f_{\Theta|X}(\theta|k) = c \binom{n}{k} f_\Theta(\theta) \theta^k (1-\theta)^{n-k} = \dfrac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}$,
where $d = c \binom{n}{k}$.

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Lemma. Up to recaling, the pdf of the form $e^{-\frac{1}{2}(ax^2 - 2bx + c)}$ is $\mathcal{N}(\frac{b}{a}, \frac{1}{a})$.

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  ◦ Thus, $\sigma^2 = \dfrac{1}{a}$ and $\dfrac{\mu}{\sigma^2} = b \implies \mu = b\sigma^2 = \dfrac{b}{a}$

> Theorem. The product of two Gaussian pdfs $\mathcal{N}(\mu_0, \nu_0)$ and $\mathcal{N}(\mu_1, \nu_1)$ is $\mathcal{N}\left(\dfrac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \dfrac{\nu_0\nu_1}{\nu_0 + \nu_1}\right).$

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Background: Product of Two Gaussian Densities

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$$\implies \mathcal{N}\left(\nu\left(\frac{\mu_0}{\nu_0} + \frac{\mu_1}{\nu_1}\right), \overbrace{\frac{1}{\nu_0^{-1} + \nu_1^{-1}}}^{=\nu}\right) = \mathcal{N}\left(\frac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \frac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$$

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$$f_{X|\Theta}(x|\theta) = c_2 \cdot \exp\left\{-\frac{(\theta - x_1)^2}{2\sigma_1^2}\right\} \cdots \exp\left\{-\frac{(\theta - x_n)^2}{2\sigma_n^2}\right\}$$

$$f_{\Theta|X}(\theta|x) = \frac{f_{\Theta}(\theta) f_{X|\Theta}(x|\theta)}{\int f_{\Theta}(\theta') f_{X|\Theta}(x|\theta') d\theta'}$$

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- Denominator: just a constant, not a function of θ

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   ◦ the prior mean x_0 acts just as another observation.
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- In the example of parameter inference with the Normal prior, with the new observation $x_{n+1} \sim \mathcal{N}(x_{n+1}, \sigma_{n+1}^2)$, the posterior pdf is nothing but the Normal pdf of:

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- Suppose that after $X_1, \ldots, X_n$ are observed, an additional observation X_{n+1} is observed.

- Instead of solving the inference problem from scratch, we can view $f_{\Theta|X_1,\ldots,X_n}$ as our prior, use the new observation to obtain the new posterior $f_{\Theta|X_1,\ldots,X_n,X_{n+1}}$

- In the example of parameter inference with the Normal prior, with the new observation $x_{n+1} \sim \mathcal{N}(x_{n+1}, \sigma_{n+1}^2)$, the posterior pdf is nothing but the Normal pdf of:

$$\text{mean} = \frac{(m/v) + (x_{n+1}/\sigma_{n+1}^2)}{(1/v) + (1/\sigma_{n+1}^2)}, \qquad \text{variance} = \frac{1}{(1/v) + (1/\sigma_{n+1}^2)}$$

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator
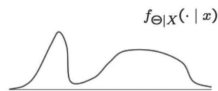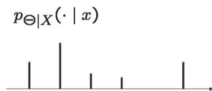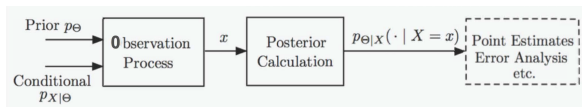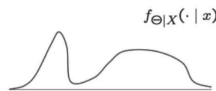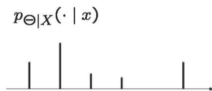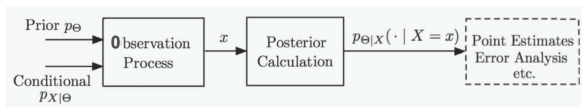
- **Point Estimate**
  - Given observation x, which single value θ are you going to choose as your inference result? People often want just the summary and a simple answer.
  - Very often, θ, our inference target, is by nature a single value, i.e., mass of the electron.

$$p_{\Theta|X}(\cdot \mid x) \qquad\qquad f_{\Theta|X}(\cdot \mid x)$$
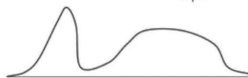
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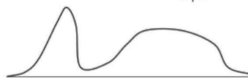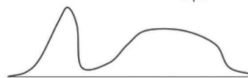
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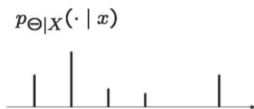
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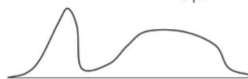
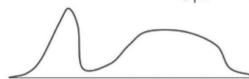**M2.** Choose the mean: Conditional expectation, aka LMS (Least Mean Square)

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- Why MAP and LMS are good? Not mathematically clear yet (We will discuss later)
- Notation: The community uses $\hat{\theta}$ to mean the estiamted value, i.e., hat for estimated value.

- Random observation: X

- Observation instance: x

- Estimate as a mapping from x to a number

$$\hat{\theta} = g(x), \quad \hat{\theta}_{\mathsf{MAP}} = g_{\mathsf{MAP}}(x), \quad \hat{\theta}_{\mathsf{LMS}} = g_{\mathsf{LMS}}(x)$$

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From now on we focus on the MAP estimate, mainly based on the examples that we've discussed in the previous section.

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.
- Observation: Romeo was late by x.
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Slide 16 for more details

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- Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$. $\implies \hat{\theta}_{\mathsf{MAP}} = x$.

Slide 18 for more details

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.
- $\{w_1, w_2, \ldots, w_n\}$: a collection of words which suggest "spam".
- For each i, a Bernoulli $X_i = 1$ if w_i appears and 0 otherwise.
- Assumption: Conditioned on Θ, X_i are independent.
- Posterior PMF

$$\mathbb{P}\Big[\Theta = m | (x_1, \ldots, x_n)\Big] = \frac{p_\Theta(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

Example: Spam Filtering

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|1) > p_\Theta(2) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|2)$$

- Biased coin with probability of head θ
- Unknown θ: modeled by Θ with some prior $f_\Theta(\theta)$
- Observation X: number of heads out of n tosses

  - If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta|\{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
  - $f_{\Theta|X}(\theta|k) \propto \theta^{\alpha+k-1}(1-\theta)^{\beta+n-k-1}$

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$$\hat{\theta}_{\text{MAP}} = \arg \max_\theta \Big[(\alpha + k - 1)\log\theta + (\beta + n - k + 1)\log(1-\theta)\Big] \quad = \frac{\alpha + k - 1}{\alpha + \beta - 2 + n}$$

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- When $\alpha = \beta = 1$ (i.e., $\mathcal{U}[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

Slide 27 for more details

- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\frac{(\theta - m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^n \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^n \frac{1}{\sigma_i^2}}$$

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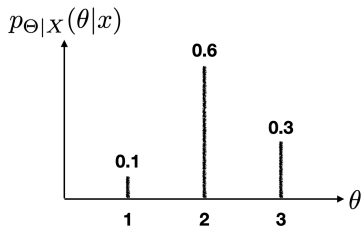
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○ Given $X = x$, θ that minimizes the probability of incorrect decision?

$$= \arg\min_{\hat{\theta}=1,2,3} \mathbb{P}(\hat{\theta} \neq \Theta | X = x)$$

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$p_{\Theta|X}(\theta|x)$

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0.3

0.1

1   2   3 θ

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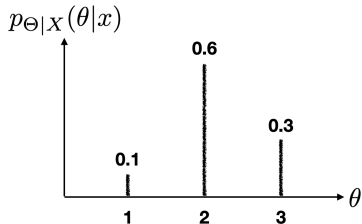$$\hat{\theta}_{MAP} = \arg \min_{\hat{\theta}=1,2,3} \mathbb{P}(\hat{\theta} \neq \Theta | X = x)$$

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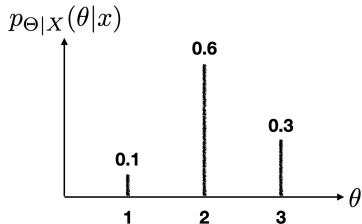
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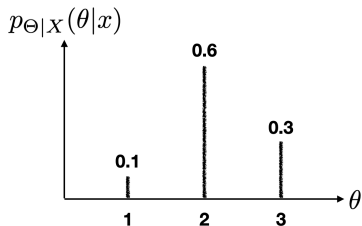

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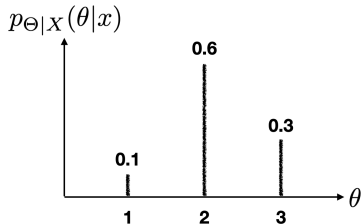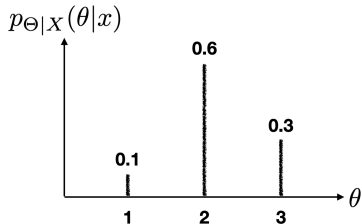

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$$\geq \sum_x \mathbb{P}(\hat{\theta}_{\mathsf{MAP}} \neq \Theta | X = x) p_X(x)$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

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Other Estimate than MAP?

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  ◦ Least Mean Square (LMS) Estimate

- Unknown: θ modeled by Θ with prior $f_\Theta(\cdot)$. Assume $\Theta \sim \mathcal{U}[4, 10]$.

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- MAP estimate
  - Any value $\hat{\theta}_{\text{MAP}} \in [4, 10]$ (why? posterior = prior), not very useful

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Slides 17 and 35 for more details

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
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$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\theta|X = x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta =$

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Example: Romeo and Juliet

Slides 17 and 35 for more details

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$$\mathbb{E}[\Theta|X = k] = \frac{k + \alpha}{k + \alpha + n - k + \beta} = \frac{k + \alpha}{\alpha + \beta + n}$$

Example: Biased Coin with Beta Prior

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◦ Fact. If $\Theta \sim \text{Beta}(\alpha, \beta)$,

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- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\dfrac{(\theta-m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

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$$f_{\Theta,X}(\theta,x) = f_\Theta(\theta)f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}, & \text{if } 4 \le \theta \le 10, \ \theta - 1 \le x \le \theta + 1, \\ 0, & \text{otherwise} \end{cases}$$

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$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\Theta | X = x]$: midpoint of the corresponding vertical section
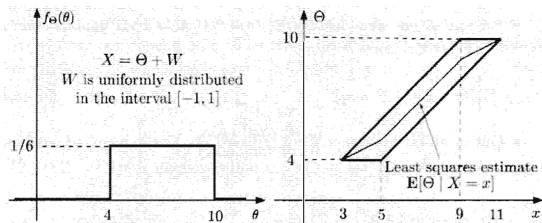
- What is conditional MSE? $\mathbb{E}\left[(\Theta - \mathbb{E}[\Theta|X=x])^2|X=x\right]$

- Given $X = 3$, it's the variance of $\mathcal{U}[4,4] = 0$

- Given $X = 5$, it's the variance of $\mathcal{U}[4,6] = (6-4)^2/12 = 1/3$

- The rising pattern between $X = 3$ and $X = 5$ is quadratic. This is because the expectation increases linearly, where the variance increases in a quadratic manner.
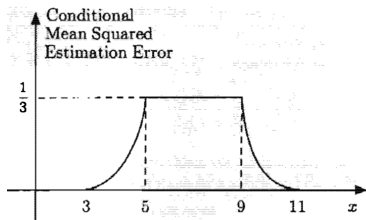
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- Any alternative to LMS estimator?

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- Linear models are always the first choice for a simple design in engineering.

LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big),$$

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◦ If $\rho > 0$:
- Baseline ($\mathbb{E}[\Theta]$) + correction term
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○ If $\rho = 0$ (uncorrelated):
- Just baseline ($\mathbb{E}[\Theta]$)
- $\hat{\Theta}_L = \mathbb{E}[\Theta]$
- No use of data X

- MSE $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$?

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$$\mathbb{E}\left[(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X)^2\right] = \mathrm{var}(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X)$$

$$= \mathrm{var}(\Theta) + \left(\rho\frac{\sigma_\Theta}{\sigma_X}\right)^2 \mathrm{var}(X) - 2\left(\rho\frac{\sigma_\Theta}{\sigma_X}\right)\mathrm{cov}(\Theta, X)$$

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\left(X - \mathbb{E}(X)\right)$$

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  - Assume $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$ (for simplicity). Then, MSE $= \mathbb{E}\left[(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X)^2\right]$
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- Uncertainty about Θ after observation decreases by the factor of $1 - \rho^2$

- What happens if $|\rho| = 1$ or $\rho = 0$?

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$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) \tag{1}$$

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Linear LMS (LLMS) Estimator: Proof

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$$\mathbb{E}\Big[(Y - b)^2\Big], \quad Y = \Theta - aX$$

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Slide pp. 43

$$\text{ERR}(a, b) = \mathbb{E}[(Y - \mathbb{E}[Y])^2] = \text{var}(Y)$$
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- Using $\rho = \frac{\text{cov}(\Theta, X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.

- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.

- Random observation: X

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- Question. What is the LLMS estimator $\hat{\Theta}_{\mathsf{L}}$?

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$$\text{var}[X] = \mathbb{E}[\text{var}[X|\Theta]] + \text{var}[\mathbb{E}[X|\Theta]]$$
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$$\text{cov}(\Theta, X) = 1/6 - 1/2 \cdot 1/4 = 1/24$$

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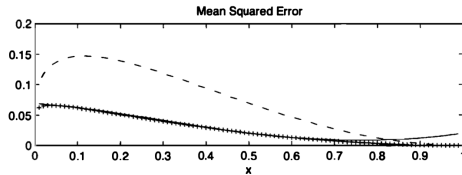- Using $\mathbb{E}[\Theta] = 1/2$ and $\mathbb{E}[\Theta^2] = 1/3$,

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Example: Biased Coin with Uniform Prior

- Biased coin with probability of head θ
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- $\hat{\Theta}_L = \hat{\Theta}_{\mathrm{LMS}}$! Intuitive?

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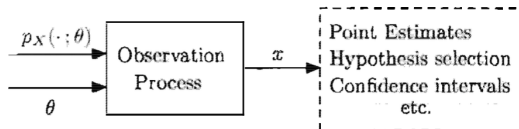$$\text{cov}(\Theta, X) = \mathbb{E}[\Theta X] - \mathbb{E}[\Theta]\mathbb{E}[X] = \mathbb{E}[\Theta X] - n/4$$

$$\mathbb{E}[\Theta X] = \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]]$$
$$= \mathbb{E}[n\Theta^2] = n/3$$

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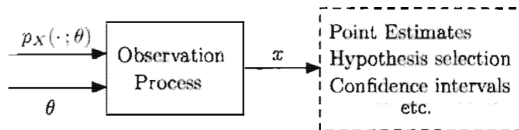
$$\hat{\Theta}_L = \frac{1}{2} + \frac{n/12}{n(n + 2)/12}(X - \frac{n}{2}) = \frac{X + 1}{n + 2}$$

- $\hat{\Theta}_{MAP} = \frac{X}{n}$
- $\hat{\Theta}_{LMS} = \frac{X+1}{n+2}$
- $\hat{\Theta}_L = \hat{\Theta}_{LMS}$! Intuitive?
- Yes, because the LMS esitmator was linear.

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

$p_X(\cdot\,; \theta)$ → Observation Process → x → Point Estimates / Hypothesis selection / Confidence intervals / etc.
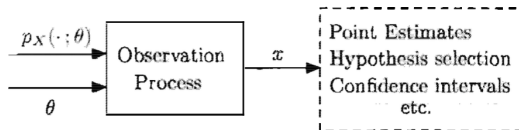
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- Observations or measurements X

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  - deterministic (not random) quantity (thus, no prior distribution)

  - No prior, No posterior probabilities

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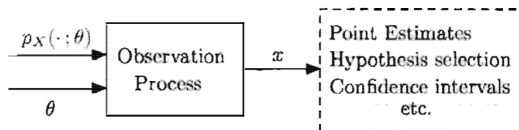 - Notation: $p_X(x; \theta)$ and $f_X(x; \theta)$, θ-parameterized distribution of observations

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- Choosing one among multiple probabilistic models
  - Each θ corresponds to a probabilistic model

- Problem types
  - Estimation: θ: prob. of head?
  - Hypothesis testing: $\theta = 1/2$ or $\theta = 1/4$?
  - Significance testing: $\theta = 1/2$ or not?

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- Key inference methods
  - ML (Maximum Likelihood) estimation

  - Linear regression

  - Likelihood ratio test

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- Just a taste in this course.

Maximum Likelihood Estimation

- Random observation $x = (x_1, x_2, \ldots, x_n)$ of $X = (X_1, X_2, \ldots, X_n)$
  - Assume a scalar θ and a vector of multiple observations in this lecture.

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    - The probability that the observed value x arises when the parameter is θ.

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- Very often, X_is are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \ldots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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- MAP in the Bayesian inference

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ML vs. MAP

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- Thus, when Θ is **uniform** (complete ignorance of Θ) in MAP, **MAP == ML**

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.

- Unknown: θ modeled by a rv $\Theta \sim U[0, 1]$.

- MAP: $\hat{\theta}_{\mathsf{MAP}} = x$

- LMS: $\hat{\theta}_{\mathsf{LMS}} = (1 - x)/|\log x|$

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Example: Estimation of Parameter of Exponential rv

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- Reminder. $X \sim \exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \mathbb{E}[X] = 1/\lambda$$

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  $$\arg \max_\theta f_X(x; \theta) = \arg \max_\theta \prod_{i=1}^n \theta e^{-\theta x_i} = \arg \max_\theta \left(n \log \theta - \theta \sum_{i=1}^n x_i \right)$$

Questions?

1) What is statistical inference?

2) Draw the building blocks of Bayesian inference and explain how it works.

3) What are MAP and LMS estimators and their underlying philosophies?

4) What is LLMS estimator and why is it useful?

5) Compare the classical and Bayesian inference.

6) What is the ML estimator and how is it related to the MAP estimator?