

Lecture 4: Random Variable, Part II

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EE210: Probability and Introductory Random Processes
KAIST EE

MONTH DAY, 2021

- Continuous Random Variable
- PDF (Probability Density Function)
- CDF (Cumulative Distribution Function)
- Exponential and Normal Distribution
- Joint PDF, Conditional PDF
- Bayes' rule for continuous and even mixed cases

- Famous discrete random variables used in the community
 - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
 - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables



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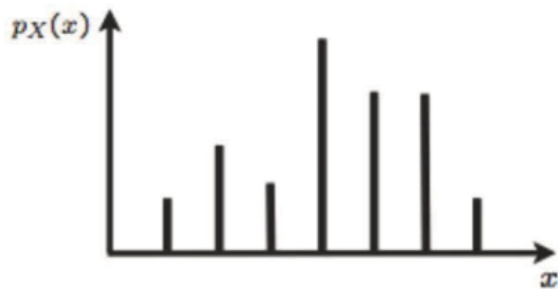
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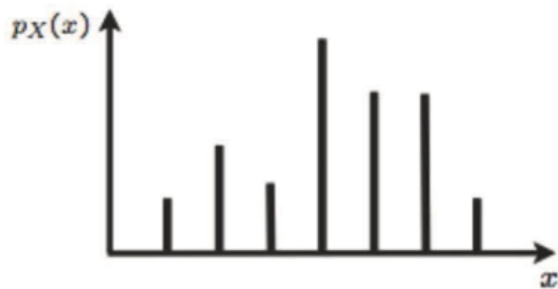
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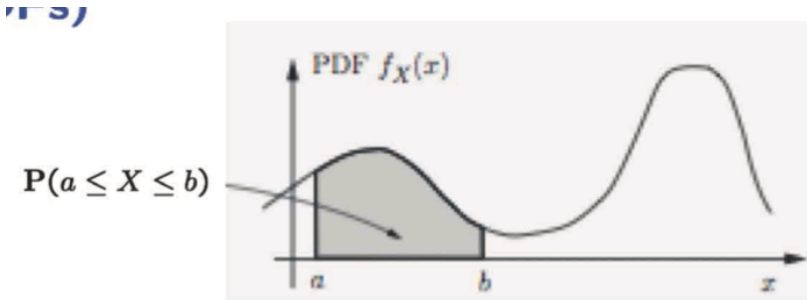
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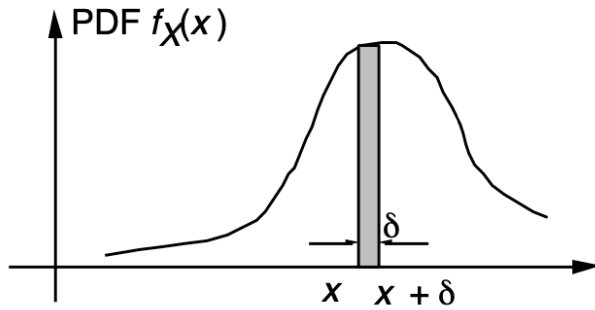
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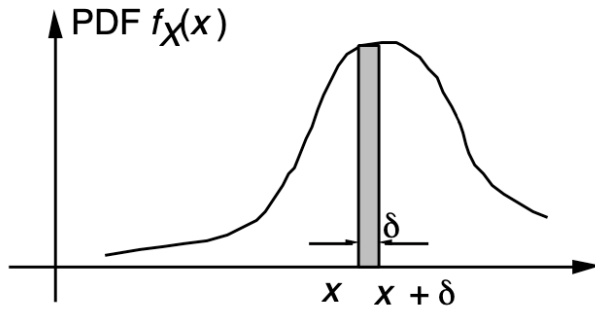


- $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$
- $f_X(x) \geq 0, \int_{-\infty}^{\infty} f_X(x) dx = 1$



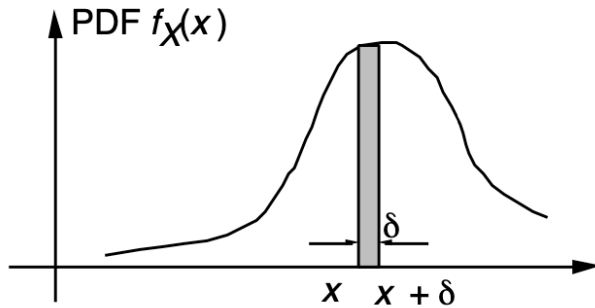
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Examples



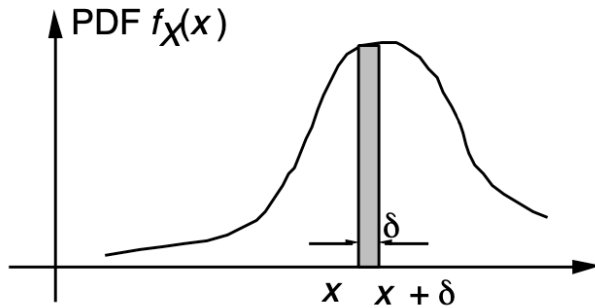
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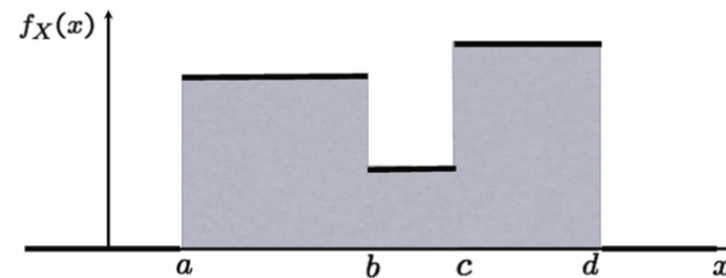
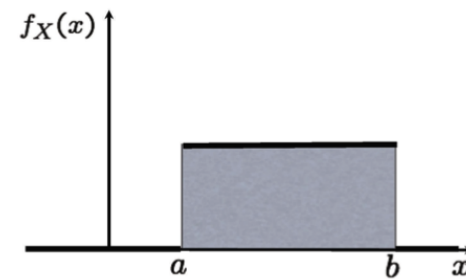
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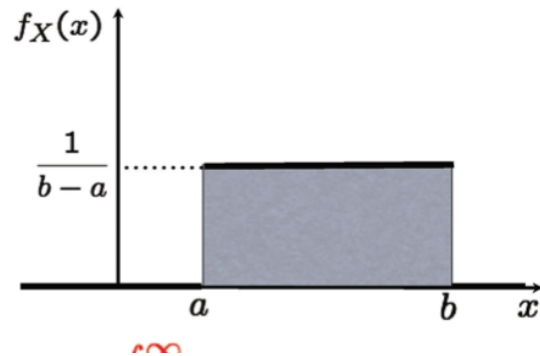
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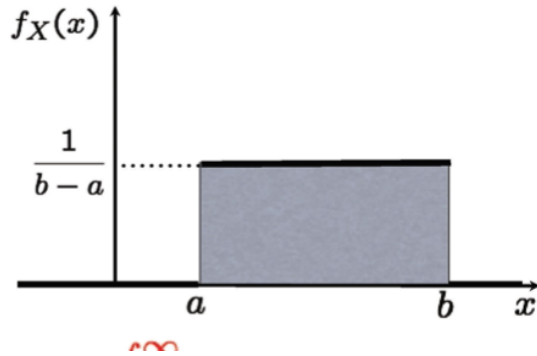
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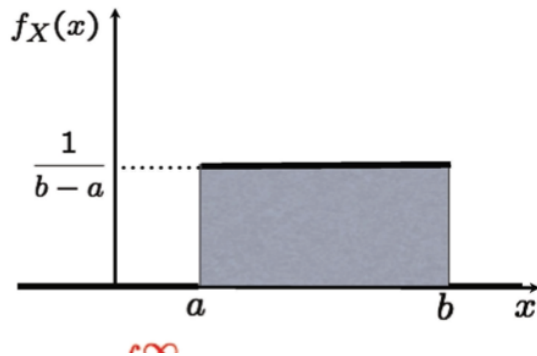




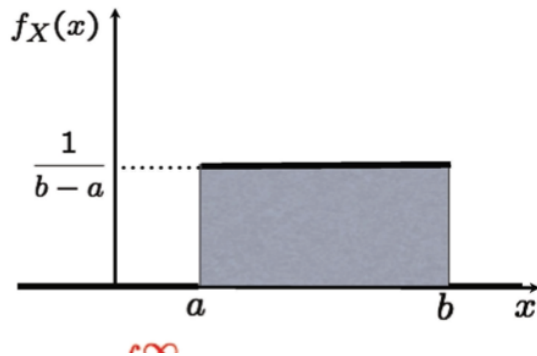
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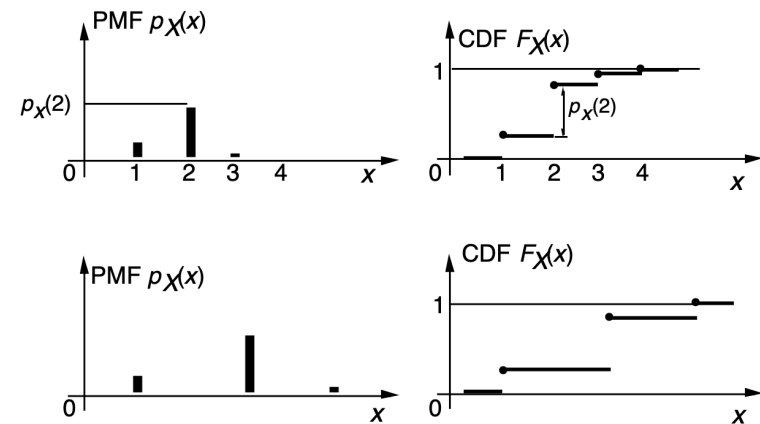
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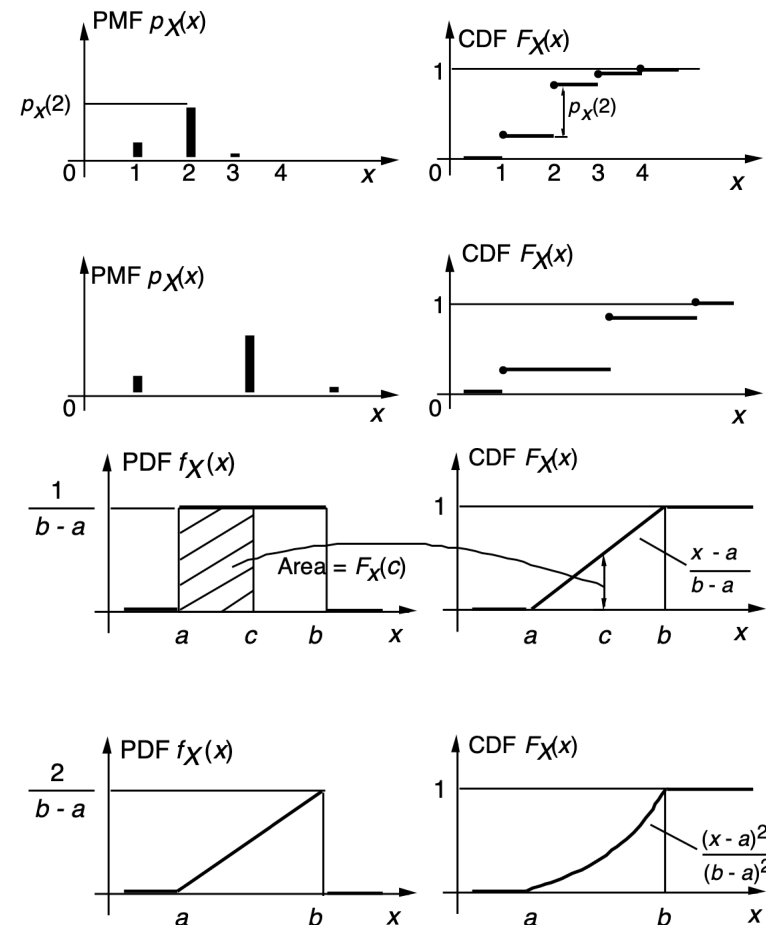
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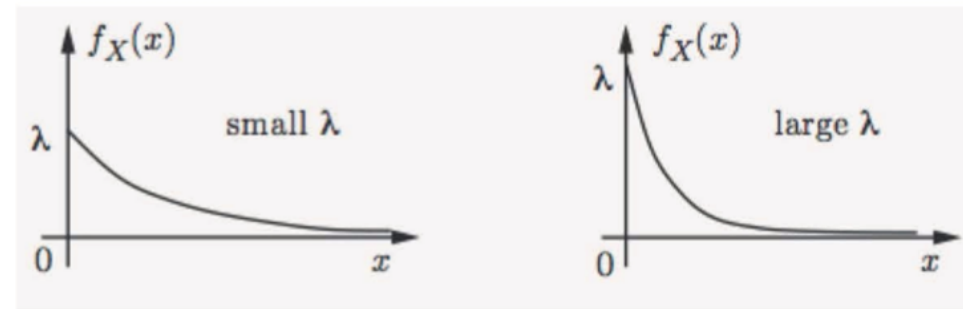
Now, let's look at famous continuous random variables popularly used in our life.

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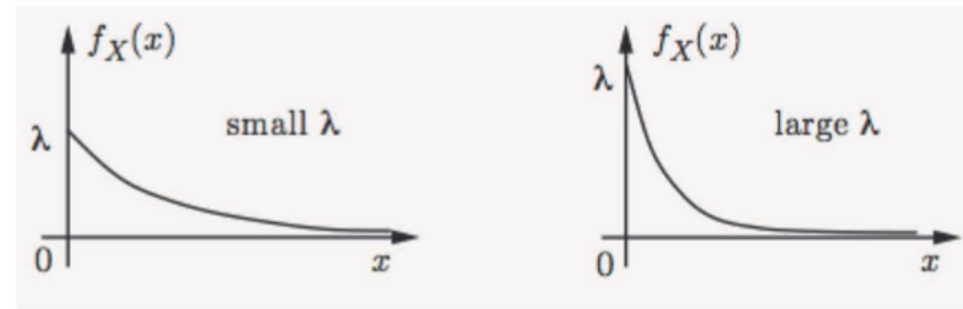


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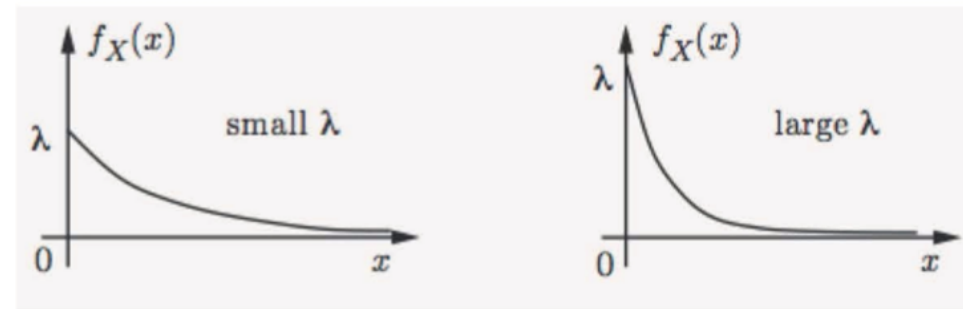


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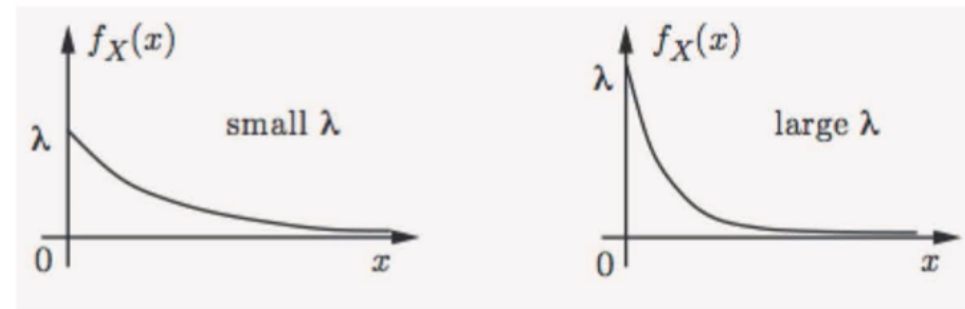


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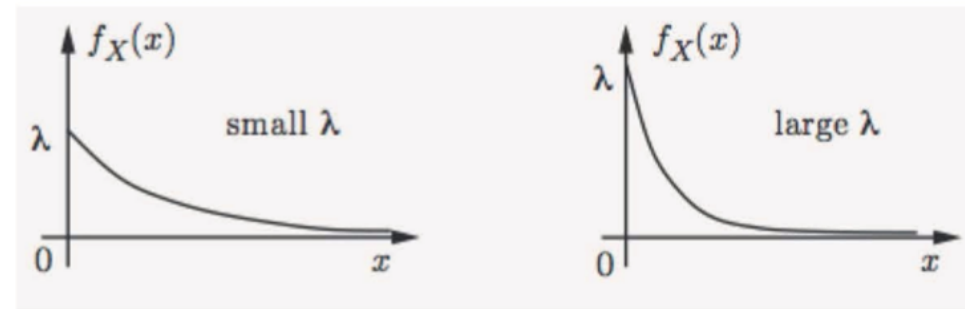


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- (Q) What is the discrete rv which models a waiting time?



Modeling Waiting Time? A Discrete Twin (1)

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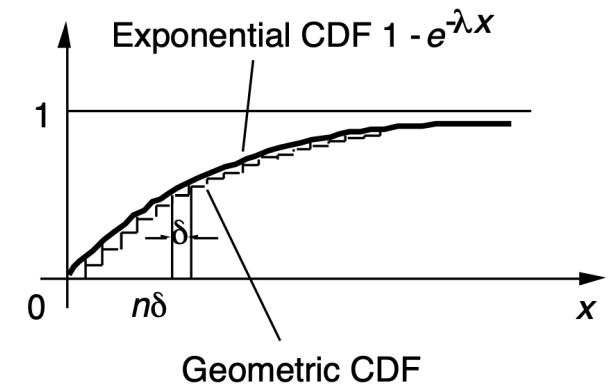
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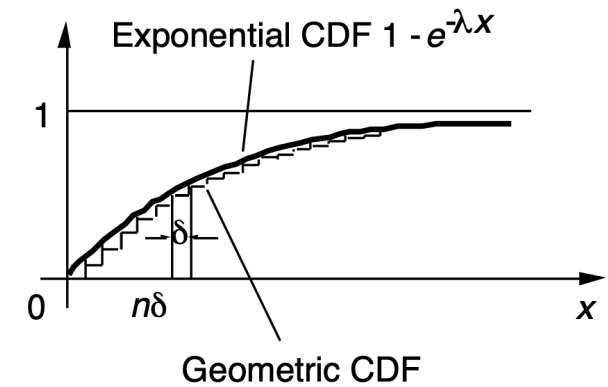
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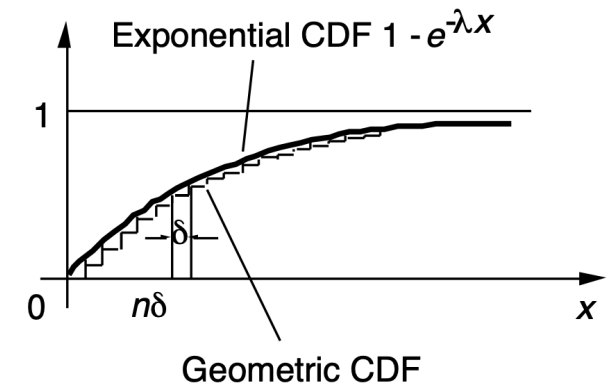
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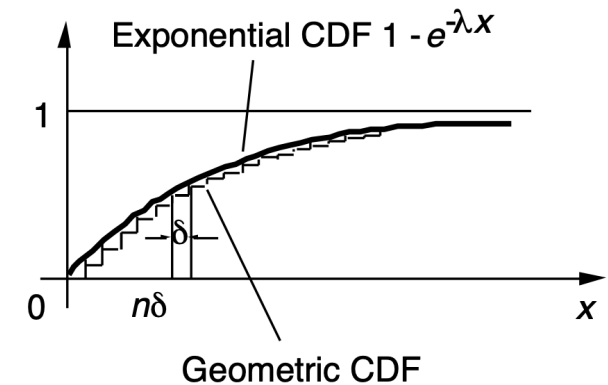
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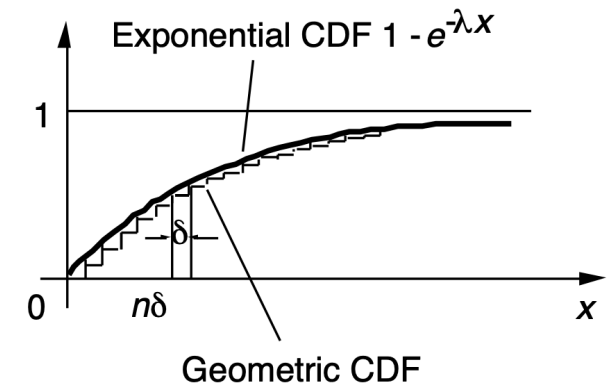
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- The CDF values of exponential and n -th geometric rvs become equal whenever $x = \delta, 2\delta, 3\delta, \dots$, i.e.,

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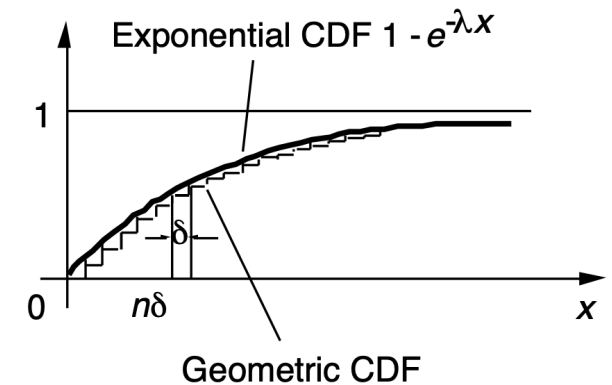
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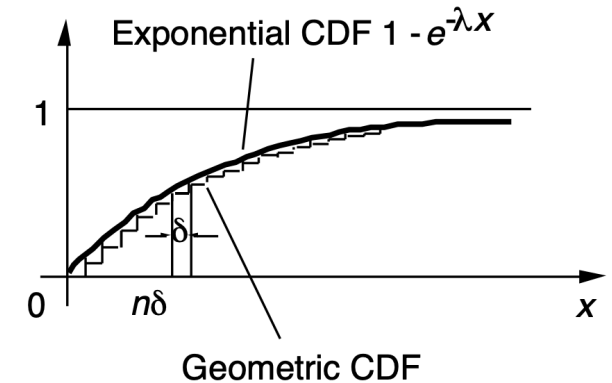
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- As n grows, the number of slots grows, but the success probability over one slot decreases, so that everything is balanced up.
- As n grows, $F_{\text{geo}}^n(n)$ approaches $F_{\text{exp}}(n\delta)$.

Why important?

- Central limit theorem (중심극한정리)
 - One of the most remarkable findings in the probability theory
- Convenient analytical properties
- Modeling aggregate noise with many small, independent noise terms

- Standard Normal $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

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Need to check:

- a legitimate PDF or not
- expectation/variance

- Linear transformation preserves normality

Linear transformation of Normal

If $X \sim \text{Norm}(\mu, \sigma^2)$, then for $a \neq 0$ and b $Y = aX + b \sim \text{Norm}(a\mu + b, a^2\sigma^2)$.

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If $X \sim \text{Norm}(\mu, \sigma^2)$, then $Y = \frac{X - \mu}{\sigma} \sim \text{Norm}(0, 1)$

- Thus, we can make the **table** which records the following CDF values:

$$\Phi(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

Example

- Annual snowfall X is modeled as $Norm(60, 20^2)$. What is the probability that this year's snowfall is at least 80 inches?

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
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- $Y = \frac{X-60}{20}$.

$$\begin{aligned}\mathbb{P}(X \geq 80) &= \mathbb{P}\left(Y \geq \frac{80 - 60}{20}\right) \\ &= \mathbb{P}(Y \geq 1) = 1 - \Phi(1) \\ &= 1 - 0.8413 = 0.1587\end{aligned}$$

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0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
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- Famous discrete random variables used in the community
 - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
 - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

** Continuous counterparts are intuitively understandable. So, we will be quick at reviewing them.

Jointly Continuous

Two continuous rvs are if a non-negative function $f_{X,Y}(x,y)$ (called joint PDF) satisfies: for subset B of the two dimensional plane,

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y) dx dy$$

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1. The joint PDF is used to calculate probabilities

$$\mathbb{P}((X, Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x, y) dx dy$$

Our particular interest: $B = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$

2. The marginal PDFs of X and Y are from the joint PDF as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

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3. The joint CDF is defined by $F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y)$, and determines the joint PDF as:

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{x,y}}{\partial x \partial y}(x, y)$$

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4. A function $g(X, Y)$ of X and Y defines a new random variable, and

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy$$

* Conditional PDF, given an event

* Conditional PDF, given $X \in B$

* Conditional PDF, given an event

- $f_X(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta)$
 $f_{X|A}(x) \cdot \delta \approx \mathbb{P}(x \leq X \leq x + \delta | A)$

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- $\mathbb{P}(X \in B) = \int_B f_X(x) dx$
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Note: A is an event, but B is a subset that includes the possible values which can be taken by the rv X .

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$$\mathbb{P}(x \leq X \leq x + \delta | X \in B) \approx f_{X|X \in B}(x) \cdot \delta$$

$$f_{X|X \in B}(x) = \begin{cases} 0, & \text{if } x \notin B \\ \frac{f_X(x)}{\mathbb{P}(B)}, & \text{if } x \in B \end{cases}$$

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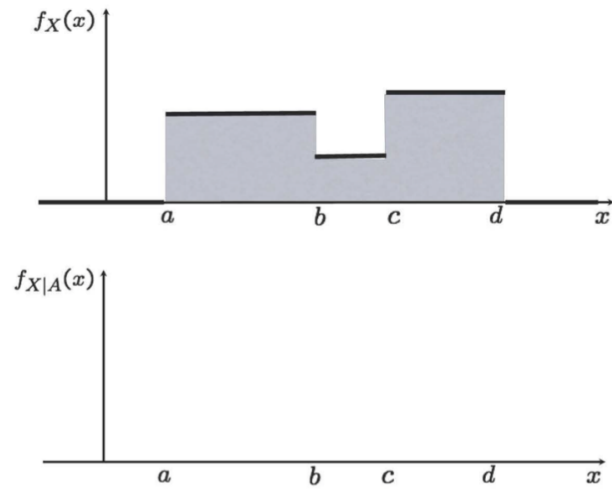
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(Q) In the discrete, we consider the event $\{X = x\}$, not $\{X \in B\}$. Why?

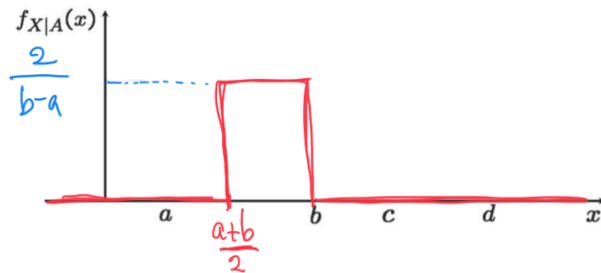
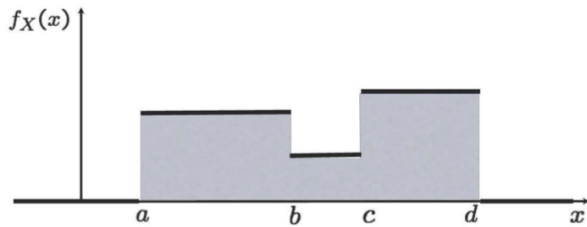
Continuous: Conditional Expectation

$$A = \left\{ \frac{a+b}{2} \leq X \leq b \right\}$$



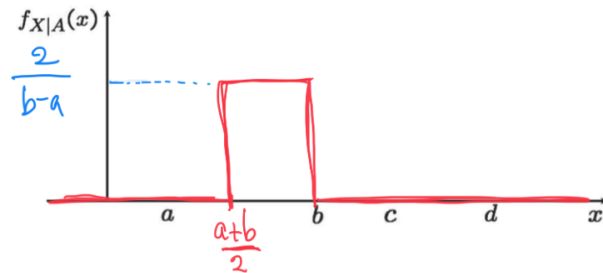
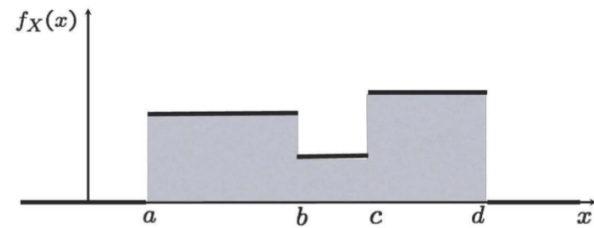
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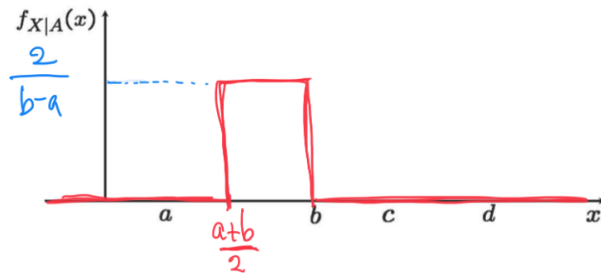
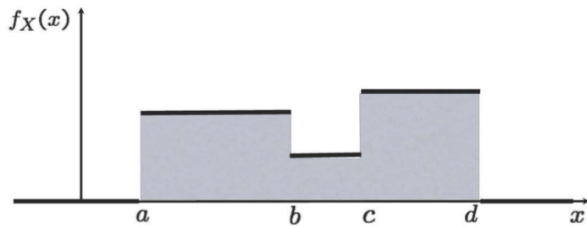
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- $\mathbb{E}[X] = \int x f_X(x) dx$
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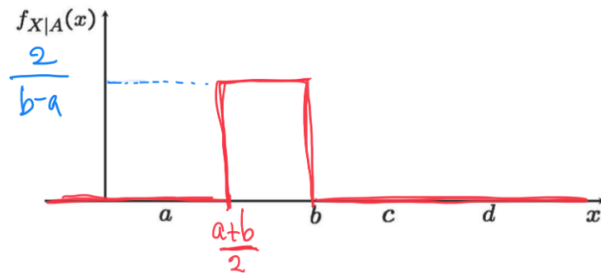
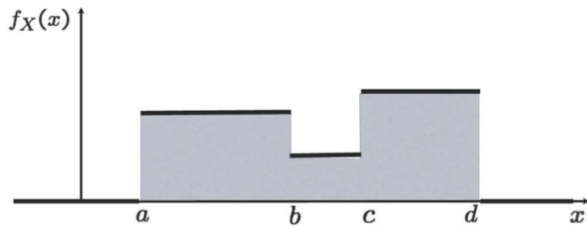
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- $\mathbb{E}[g(X)] = \int g(x) f_X(x) dx$
 $\mathbb{E}[g(X)|A] = \int g(x) f_{X|A}(x) dx$

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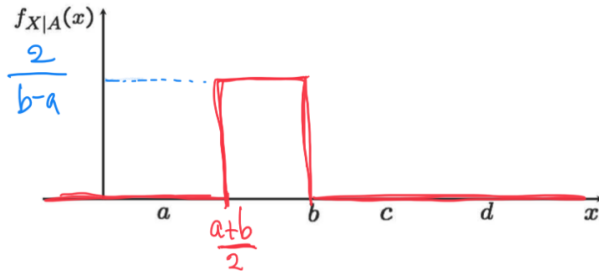
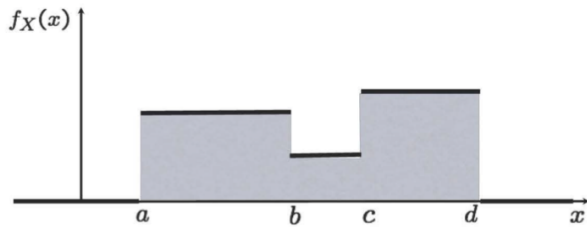
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$$\mathbb{E}[X|A] =$$

$$\mathbb{E}[X^2|A] =$$

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 $\mathbb{E}[g(X)|A] = \int g(x) f_{X|A}(x) dx$

$$\mathbb{E}[X|A] = \int_{(a+b)/2}^b x \frac{2}{b-a} dx = \frac{a}{4} + \frac{3b}{4}$$

$$\mathbb{E}[X^2|A] = \int_{(a+b)/2}^b x^2 \frac{2}{b-a} dx =$$

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Total Probability/Expectation Theorem

Partition of Ω into A_1, A_2, A_3, \dots

* Discrete case

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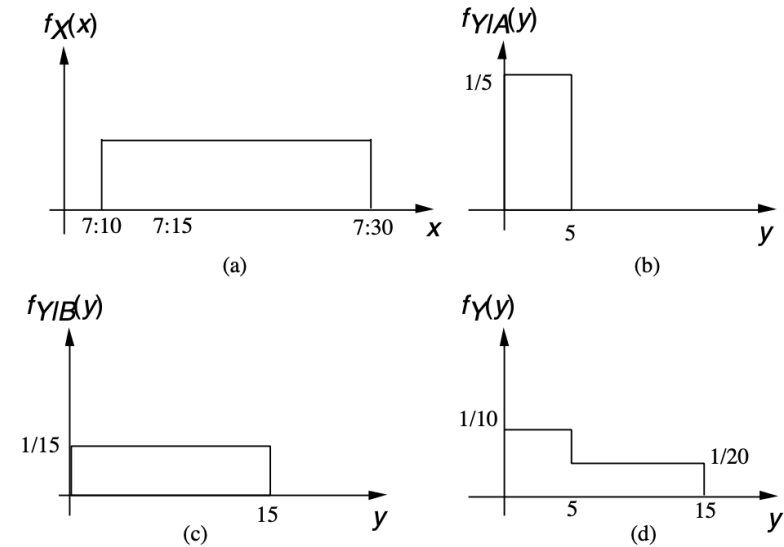
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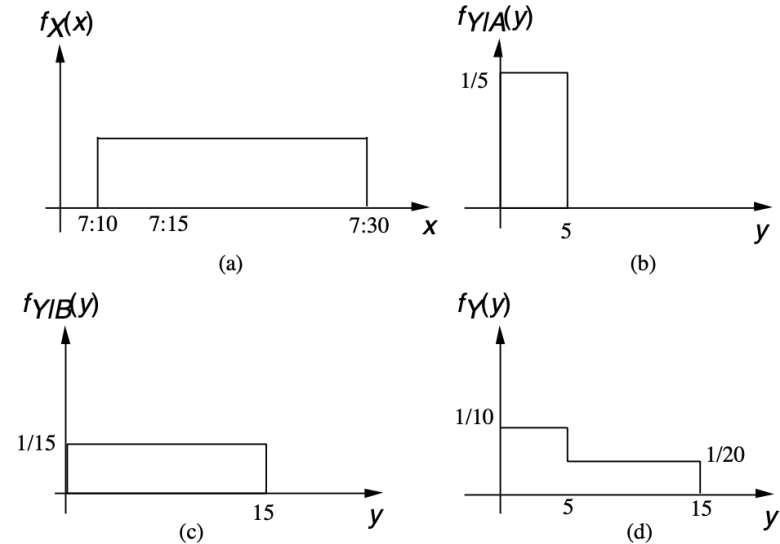
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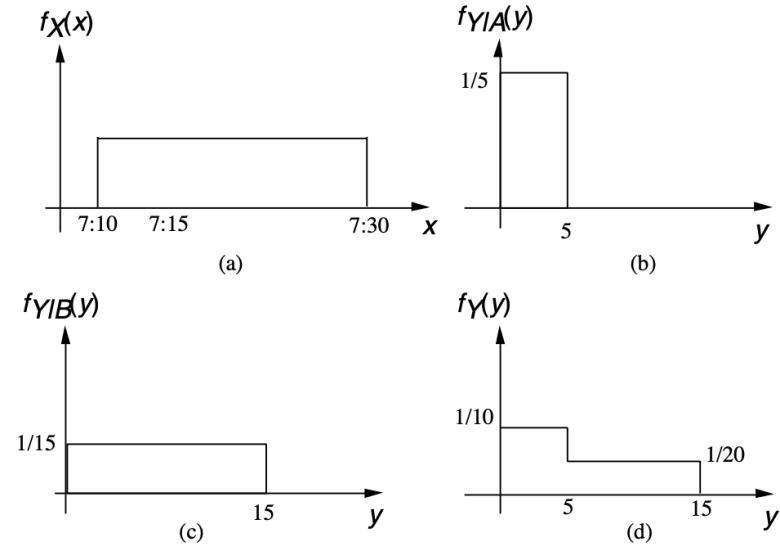
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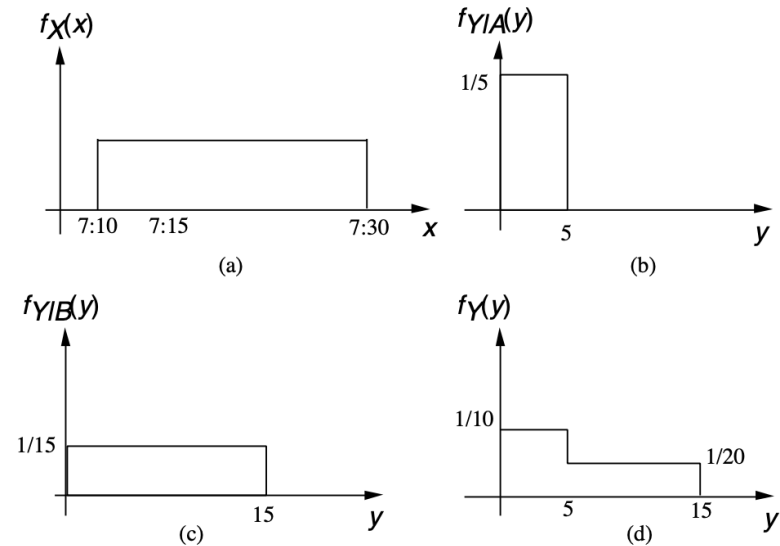


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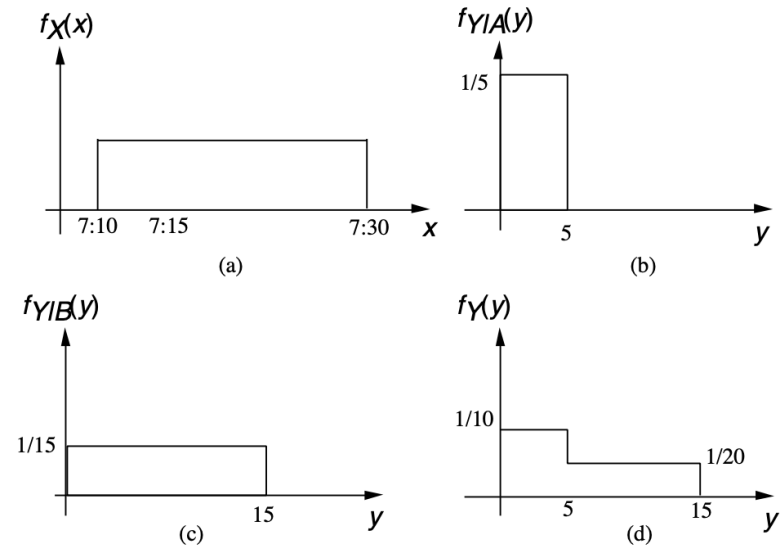


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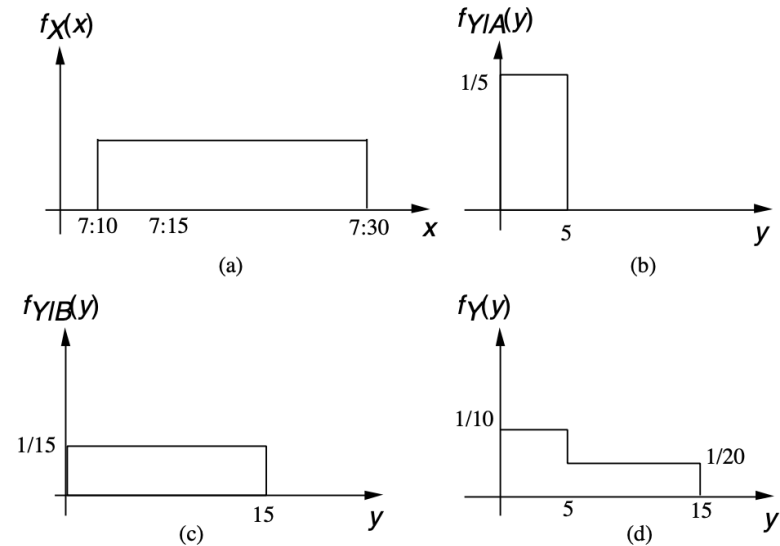
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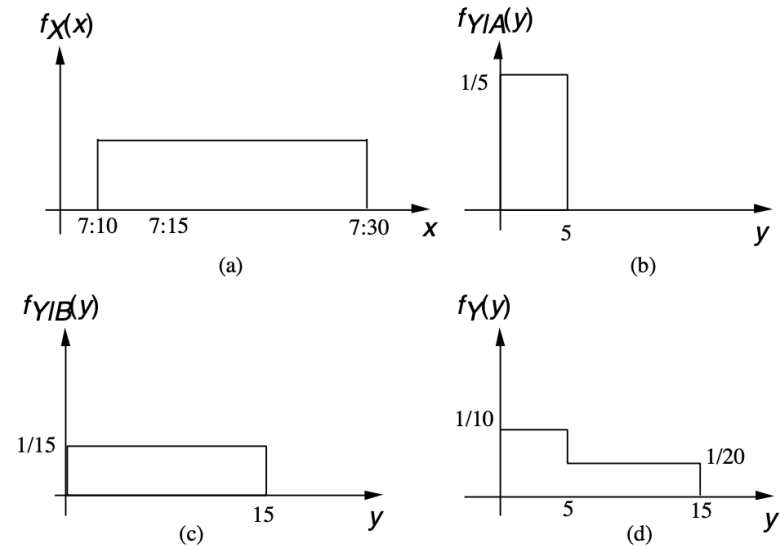
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- $f_X(x)$ and $f_{Y|X}(y|x)$ seems easy to compute. Thus,

$$f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{l} \cdot \frac{1}{x}$$

You can do many other things with the joint PDF.

- Famous discrete random variables used in the community
 - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
 - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- X : state/cause/original value \rightarrow Y : result/resulting action/noisy measurement
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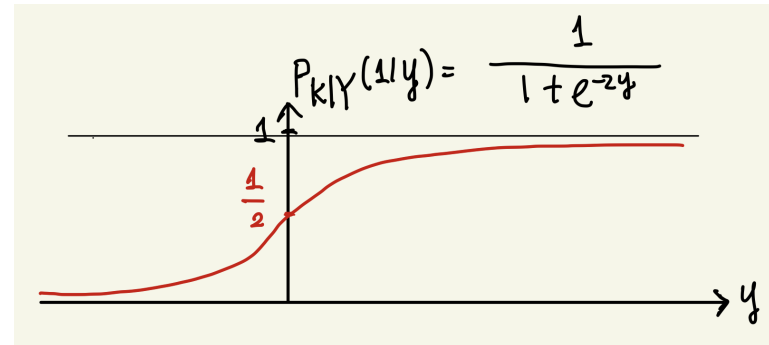
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Questions?

- 1) What is PDF and CDF?
- 2) Why do we need CDF?
- 3) What are joint/marginal/conditional PDFs?
- 4) Explain memorylessness of exponential random variables.
- 5) Explain the version of Bayes' rule for continuous and mixed random variables.