Lecture 9: Introduction to Statistical Inference

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EE210: Probability and Introductory Random Processes
KAIST EE

June 12, 2021

Roadmap

What is Statistical Inference?

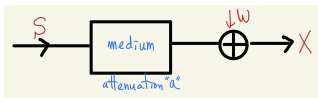- Examples

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  - When an original signal S is transmitted over the KAIST Wi-Fi connection, the received signal X becomes $X = aS + W$, where $0 < a < 1$ and $W \sim \mathcal{N}(0,1)$. If we have 10 samples of (S, X) values, what is the inferred value of a?
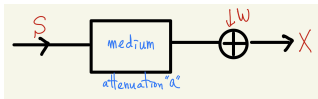
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- Process of extracting information about an unknown variable or an unknown model from noisy available data

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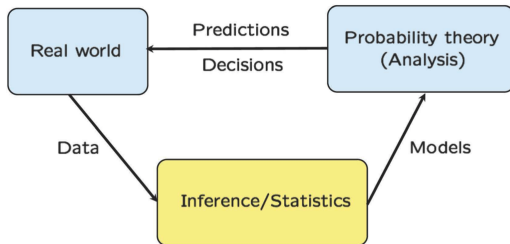
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2. There exists uncertainty (i.e., noise) as to how well the sample represents the unknown

3. How to obtain samples has impact on inference (e.g., when we need to pay for online surveys)

Source: Introduction to Probability course by MIT
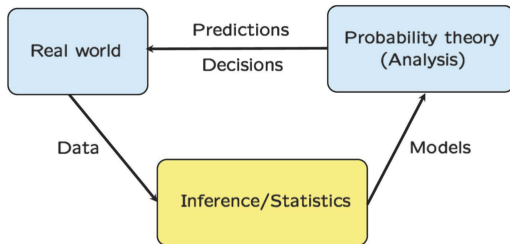
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- Inference
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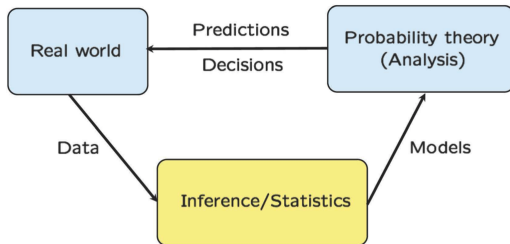
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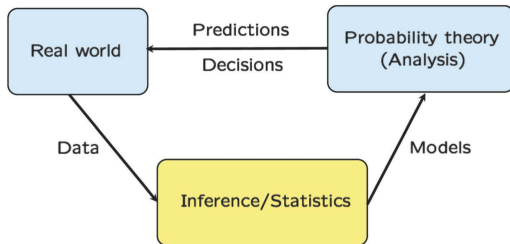
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- Inference
  - Using data, probabilistic models or parameters for models are determined.
- Why building up models?
  - Analysis is possible, so that predictions and decisions are made.
- Recently, deep learning
  - Connecting big data and big model building

- $X = aS + W$

What to Infer?: Unknown Model vs. Unknown Variable

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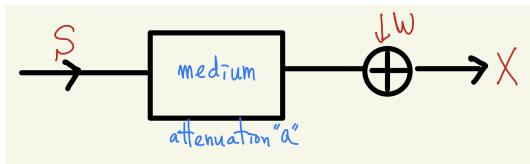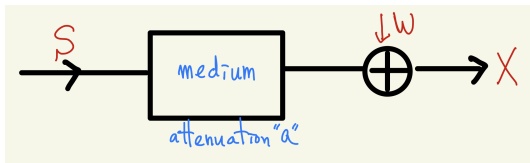
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  - know the original signal S, observe X
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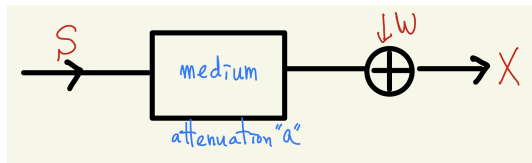
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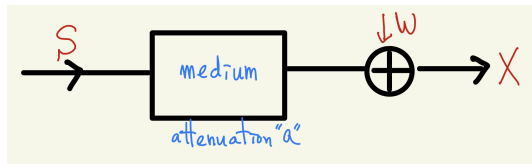
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- Same mathematical structure, because the parameters in models are variables in many cases

- Hypothesis testing

- Estimation

What Kind?: Hypothesis Testing vs. Estimation

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- Hypothesis testing
  ◦ Unknown: a few possible ones

- Estimation
  ◦ Unknown: a value included in an infinite, typically continuous set

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- Hypothesis testing
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  - (Note) If you have the candidate values of $\theta = \{1/4, 1/2, 3/4\}$, then it's a hypothesis testing problem

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- Use Bayes' rule and find the posterior:

$$\mathbb{P}\left[\theta = \frac{3}{4} \Big| (HHH)\right] = \frac{27}{28}, \ \mathbb{P}\left[\theta = \frac{1}{4} \Big| (HHH)\right] = \frac{1}{28}$$

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Different Views: Bayesian vs. Classical (1)

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(Note) There are other inference methods, and here we just show examples.

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- Fundamental difference about the nature of unknown models or variables

- Random variable or deterministic quantity

- Who is the winner? A century-long debate

- Example of debate: mass of the electron by noisy measurement
  - Classical. while unknown, it is a constant and there is no justification for modeling it as a random variable.

  - Bayesian. Prior distribution reflects our state of knowledge, e.g., some range of candidate values from our previous noisy measurements.

- Particular prior? too arbitrary vs. every statistical procedure's hidden choices

- Pratical issues: Bayesian approach is often computationally intractable (multi-dimensional integrals)

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator
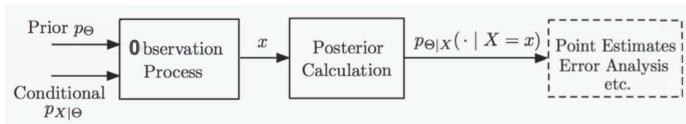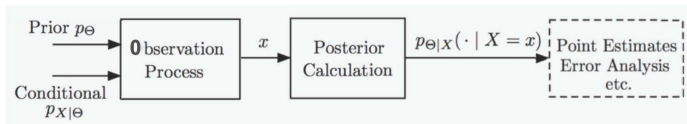
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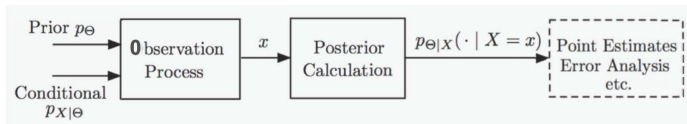
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- The posterior distribution is the complete answer of the Bayesian inference.

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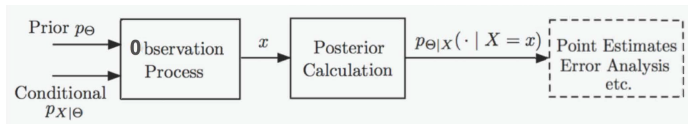

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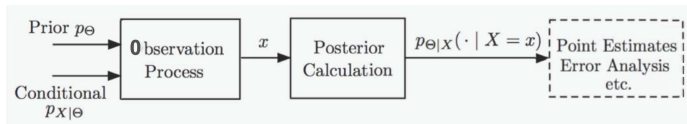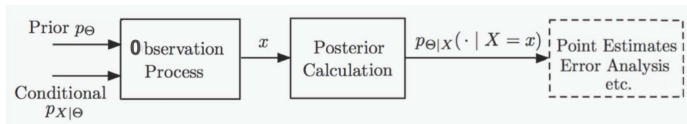

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- The posterior distribution is the complete answer of the Bayesian inference.
- However, one may use it for further processing, depending on what he/she wants, e.g., point estimation.
- Multiple observations and multiple parameters are possible
  - $X = (X_1, \ldots, X_n)$
  - $\Theta = (\Theta_1, \ldots, \Theta_n)$

- Θ: discrete, X: discrete

$$p_{\Theta|X}(\theta|x) = \frac{p_\Theta(\theta)p_{X|\Theta}(x|\theta)}{p_X(x)}$$

$$p_X(x) = \sum_{\theta'} p_\Theta(\theta')p_{X|\Theta}(x|\theta')$$

- Θ: continuous, X: continuous

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  $$f_\Theta(\theta) = \begin{cases} 1, & 0 \le \theta \le 1 \\ 0, & \text{otherwise} \end{cases},$$

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- Prior and observation model (likelihood)

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$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} =$$

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- Posterior

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x|\theta)}{\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x|\theta') d\theta'} = \begin{cases} \frac{1/\theta}{\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta |\log x|}, & x \leq \theta \leq 1, \\ 0, & \theta < x \text{ or } \theta > 1 \end{cases}$$

○ What happens if we have more observation samples?

- Romeo was late *n* times by $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, $X_i \sim \mathcal{U}[0, \theta]$.

- $X_1, \ldots, X_n$ are conditionally independent, given $\Theta = \theta$.

- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.

- Observation: Romeo was late *n* times by $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$

- See Example 8.2 at pp. 414 for more detailed treatment.

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- Observation model $p_{X_i|\Theta}(x_i|1)$ and $p_{X_i|\Theta}(x_i|2)$ are known.
- Assumption: Conditioned on Θ, X_i are independent.

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.
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- Posterior PMF

$$\mathbb{P}\Big[\Theta = m|(x_1, \ldots, x_n)\Big] = \frac{p_\Theta(m) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

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- We will look at the prior whose distribution is something called the Beta distribution.

Beta distribution

A continuous rv Θ follows a beta distribution with integer parameters $\alpha, \beta > 0$, if

$$f_\Theta(\theta) = \begin{cases} \theta^{\alpha-1}(1-\theta)^{\beta-1}, & 0 < \theta < 1, \\ 0, & \text{otherwise,} \end{cases}$$

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- A special case of Beta(1, 1) is $\mathcal{U}[0, 1]$

- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta|\{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

- In other words, Beta prior $\implies$ Beta posterior (why useful?)

Proof.

(a) First, the posterior pdf is given by:

$$f_{\Theta|X}(\theta|k) = c f_{\Theta}(\theta) p_{X|\Theta}(k|\theta) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k}, \ c \text{ the normalizing constant}$$

(b) Next, for $\text{Beta}(\alpha, \beta)$ prior, $f_{\Theta}(\theta) = \dfrac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$.

(c) Then, $f_{\Theta|X}(\theta|k) = c \binom{n}{k} f_{\Theta}(\theta) \theta^k (1-\theta)^{n-k} = \dfrac{d}{B(\alpha, \beta)} \cdot \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}$,

where $d = c \binom{n}{k}$.

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  ○ Thus, $\sigma^2 = \dfrac{1}{a}$ and $\dfrac{\mu}{\sigma^2} = b \implies \mu = b\sigma^2 = \dfrac{b}{a}$

Background: Product of Two Gaussian Densities

Theorem. The product of two Gaussian pdfs $\mathcal{N}(\mu_0, \nu_0)$ and $\mathcal{N}(\mu_1, \nu_1)$ is $\mathcal{N}\left(\dfrac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \dfrac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$.

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$$= \exp\left[-\frac{1}{2}\left(\left(\frac{1}{\nu_0} + \frac{1}{\nu_1}\right)x^2 - 2\left(\frac{\mu_0}{\nu_0} + \frac{\mu_1}{\nu_1}\right)x + c\right)\right]$$

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$$\implies \mathcal{N}\left(\nu\left(\frac{\mu_0}{\nu_0} + \frac{\mu_1}{\nu_1}\right), \overbrace{\frac{1}{\nu_0^{-1} + \nu_1^{-1}}}^{=\nu}\right) = \mathcal{N}\left(\frac{\nu_1\mu_0 + \nu_0\mu_1}{\nu_0 + \nu_1}, \frac{\nu_0\nu_1}{\nu_0 + \nu_1}\right)$$

Theorem. The product of $n + 1$ Gaussian pdfs $\mathcal{N}(\mu_0, \nu_0)$, $\mathcal{N}(\mu_1, \nu_1), \ldots,$ $\mathcal{N}(\mu_n, \nu_n)$, is $\mathcal{N}(\mu, \nu)$, where

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- n observations of θ: $W_i \sim \mathcal{N}(0, \sigma_i^2)$, and θ with the normal prior $\Theta \sim \mathcal{N}(x_0, \sigma_0^2)$

$$X_i = \theta + W_i, \quad W_i \sim \mathcal{N}(0, \sigma_i^2), \quad i = 1, \ldots, n$$

- $\Theta, W_1, \ldots, W_n$ are indendent and let $X = (X_1, \ldots, X_n)$, $x = (x_1, \ldots, x_n)$.
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- Numerator: $f_\Theta(\theta) f_{X|\Theta}(x|\theta) = c_1 c_2 \cdot \exp\left\{ -\sum_{i=0}^n \frac{(x_i - \theta)^2}{2\sigma_i^2} \right\}$, which can be reexpressed as the following, using the product of $n+1$ Gaussians:

$$c_1 c_2 \cdot \exp\left\{ -\sum_{i=0}^n \frac{(x_i - \theta)^2}{2\sigma_i^2} \right\} = d \cdot \exp\left\{ -\frac{(\theta - m)^2}{2v} \right\},$$

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- Denominator: just a constant, not a function of θ

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- In the example of parameter inference with the Normal prior, with the new observation $x_{n+1} \sim \mathcal{N}(x_{n+1}, \sigma_{n+1}^2)$, the posterior pdf is nothing but the Normal pdf of:
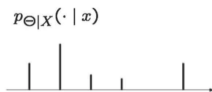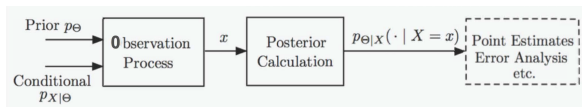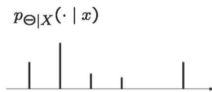
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- Recursive inference is possible.

- Suppose that after $X_1, \ldots, X_n$ are observed, an additional observation X_{n+1} is observed.

- Instead of solving the inference problem from scratch, we can view $f_{\Theta|X_1,\ldots,X_n}$ as our prior, use the new observation to obtain the new posterior $f_{\Theta|X_1,\ldots,X_n,X_{n+1}}$

- In the example of parameter inference with the Normal prior, with the new observation $x_{n+1} \sim \mathcal{N}(x_{n+1}, \sigma_{n+1}^2)$, the posterior pdf is nothing but the Normal pdf of:

$$\text{mean} = \frac{(m/v) + (x_{n+1}/\sigma_{n+1}^2)}{(1/v) + (1/\sigma_{n+1}^2)}, \qquad \text{variance} = \frac{1}{(1/v) + (1/\sigma_{n+1}^2)}$$

```
Prior p_Θ
 ──────────▶  ┌──────────────┐      x      ┌──────────────┐  p_{Θ|X}(· | X = x)  ┌─ ─ ─ ─ ─ ─ ─ ─┐
              │ Observation  │ ──────────▶ │  Posterior   │ ─────────────────▶  ┊ Point Estimates┊
              │  Process     │             │ Calculation  │                     ┊ Error Analysis ┊
 ──────────▶  └──────────────┘             └──────────────┘                     ┊     etc.       ┊
Conditional                                                                     └─ ─ ─ ─ ─ ─ ─ ─┘
  p_{X|Θ}
```

$p_{\Theta|X}(\cdot \mid x)$ $f_{\Theta|X}(\cdot \mid x)$
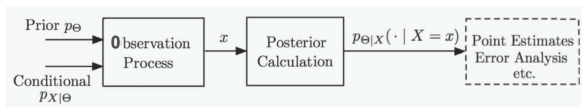
- Point Estimate
  - Given observation x, which single value θ are you going to choose as your inference result? People often want just the summary and a simple answer.
  - Very often, θ, our inference target, is by nature a single value, i.e., mass of the electron.

$p_{\Theta|X}(\cdot \mid x)$

$f_{\Theta|X}(\cdot \mid x)$

$$p_{\Theta|X}(\cdot \mid x) \qquad\qquad f_{\Theta|X}(\cdot \mid x)$$

M1. Choose the largest: Maximum a posteriori probability (MAP) rule

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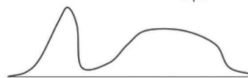
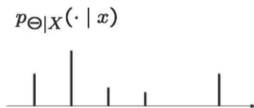$$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\Theta|X = x]$$

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- Why MAP and LMS are good? Not mathematically clear yet (We will discuss later)
- Notation: The community uses $\hat{\theta}$ to mean the estiamted value, i.e., hat for estimated value.

- Random observation: X

- Observation instance: x

- Estimate as a mapping from x to a number

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From now on we focus on the MAP estimate, mainly based on the examples that we've discussed in the previous section.

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
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- Given x, $f_{\Theta|X}(\theta|x)$ is decreasing in θ over $[x, 1]$. $\implies \hat{\theta}_{\mathrm{MAP}} = x$.

- E-mail: spam (1) or legitimate (2), $\Theta \in \{1, 2\}$, with prior $p_\Theta(1)$ and $p_\Theta(2)$.

- $\{w_1, w_2, \ldots, w_n\}$: a collection of words which suggest "spam".

- For each i, a Bernoulli $X_i = 1$ if w_i appears and 0 otherwise.

- Assumption: Conditioned on Θ, X_i are independent.

- Posterior PMF

$$\mathbb{P}\Big[\Theta = m | (x_1, \ldots, x_n)\Big] = \frac{p_\Theta(m) \prod_{i=1}^n p_{X_i|\Theta}(x_i|m)}{\sum_{j=1,2} p_\Theta(j) \prod_{i=1}^n p_{X_i|\Theta}(x_i|j)}, \quad m = 1, 2$$

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- MAP rule for this hypothesis testing problem. Decided that the message is spam if

$$p_\Theta(1) \prod_{i=1}^n p_{X_i|\Theta}(x_i|1) > p_\Theta(2) \prod_{i=1}^n p_{X_i|\Theta}(x_i|2)$$

Slide 21 for more details

- Biased coin with probability of head θ
- Unknown θ: modeled by Θ with some prior $f_\Theta(\theta)$
- Observation X: number of heads out of n tosses

  - If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta|\{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$
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- When $\alpha = \beta = 1$ (i.e., $\mathcal{U}[0, 1]$ prior), $\hat{\theta}_{\text{MAP}} = \frac{k}{n}$

- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\dfrac{(\theta - m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

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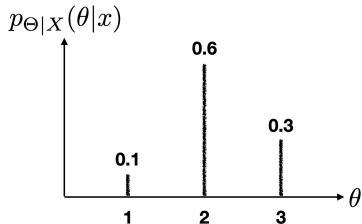
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$p_{\Theta|X}(\theta|x)$

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0.3

0.1

1   2   3 θ

◦ MAP: $\hat{\theta}_{\mathsf{MAP}} = 2$

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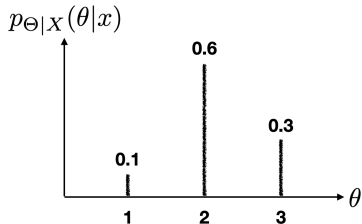
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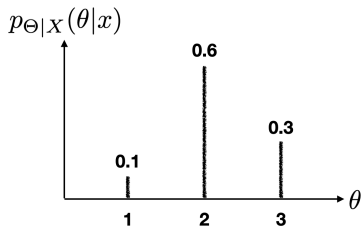
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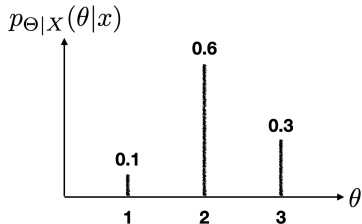

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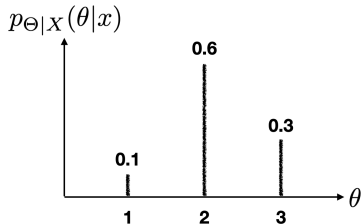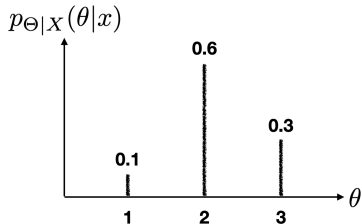$p_{\Theta|X}(\theta|x)$

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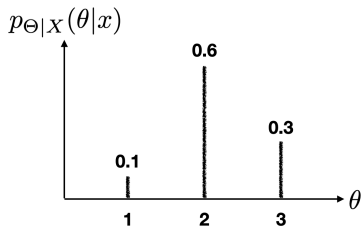
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$$= \sum_x \mathbb{P}(\hat{\theta} \neq \Theta | X = x) p_X(x)$$

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Thus, Claim 1 holds. We now take the expectation of the above equations, the law of iterated expectations leads to Claim 2.

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◦ Least Mean Square (LMS) Estimate

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- First, it makes sense, but, second, it also minimizes the mean squared error (MSE)

$$\min_{\hat{\theta}} \mathbb{E}\left[(\Theta - \hat{\theta})^2\right] = \min_{\hat{\theta}} \left(\text{var}(\Theta - \hat{\theta}) + \left(\mathbb{E}[\Theta - \hat{\theta}]\right)^2 \right) = \min_{\hat{\theta}} \left(\text{var}(\Theta) + \left(\mathbb{E}[\Theta - \hat{\theta}]\right)^2 \right)$$

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- First, it makes sense, but, second, it also minimizes the mean squared error (MSE)

$$\min_{\hat{\theta}} \mathbb{E}\left[(\Theta - \hat{\theta})^2\right] = \min_{\hat{\theta}} \left(\text{var}(\Theta - \hat{\theta}) + \left(\mathbb{E}[\Theta - \hat{\theta}]\right)^2\right) = \min_{\hat{\theta}} \left(\text{var}(\Theta) + \left(\mathbb{E}[\Theta - \hat{\theta}]\right)^2\right)$$

  - minimized when $\hat{\theta} = \mathbb{E}[\Theta]$.

- Unknown: θ modeled by Θ with prior $f_\Theta(\cdot)$.

- Observation $X = x$ with model $f_{X|\Theta}(x|\theta)$

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KAIST EE

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  ◦ Averaged over X: $\mathbb{E}\Big[(\Theta - \mathbb{E}[\Theta|X])^2\Big] = \mathbb{E}\Big[\mathrm{var}(\Theta|X)\Big]$

KAIST EE

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.
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  ∘ $\hat{\theta}_{\mathsf{MAP}} = x$.

Slides 17 and 35 for more details

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$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\theta|X=x] = \int_x^1 \theta \frac{1}{\theta|\log x|} d\theta =$

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- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

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◦ MAP estimate

$$\hat{\theta}_{\text{MAP}} = \frac{\alpha + k - 1}{\alpha + \beta - 2 + n}$$

◦ For $\alpha = \beta = 1$
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$$\hat{\theta}_{\text{MAP}} = \frac{k}{n}$$
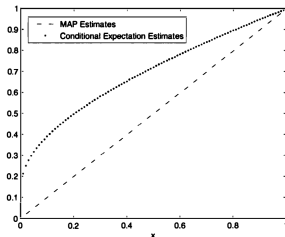
Slides 21 and 37 for more details

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○ LMS estimate:

$$\mathbb{E}[\Theta | X = k] = \frac{k + \alpha}{k + \alpha + n - k + \beta} = \frac{k + \alpha}{\alpha + \beta + n}$$

Example: Biased Coin with Beta Prior

- Biased coin with prob. of head θ. Unknown θ modeled by Θ with prior $f_\Theta(\theta)$.
- Observation X: number of heads out of n tosses
- If $\Theta \sim \text{Beta}(\alpha, \beta)$, then $\Theta | \{X = k\} \sim \text{Beta}(k + \alpha, n - k + \beta)$

◦ MAP estimate

$$\hat{\theta}_{\text{MAP}} = \frac{\alpha + k - 1}{\alpha + \beta - 2 + n}$$

◦ For $\alpha = \beta = 1$
($\mathcal{U}[0, 1]$ prior),

$$\hat{\theta}_{\text{MAP}} = \frac{k}{n}$$

◦ Fact. If $\Theta \sim \text{Beta}(\alpha, \beta)$,

$$\mathbb{E}[\Theta] = \frac{1}{B(\alpha, \beta)} \int_0^1 \theta \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\alpha}{\alpha + \beta}$$

◦ LMS estimate:
$$\mathbb{E}[\Theta | X = k] = \frac{k + \alpha}{k + \alpha + n - k + \beta} = \frac{k + \alpha}{\alpha + \beta + n}$$

◦ For $\alpha = \beta = 1$ ($\mathcal{U}[0, 1]$ prior): $\mathbb{E}[\Theta | X = k] = \frac{k + 1}{n + 2}$

- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\dfrac{(\theta-m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

- The pdf is normal, so it is maximized when $\theta = $ mean.
- Thus, $\hat{\theta}_{\text{MAP}} = m$.

Example: Parameter Inference with Normal Prior

- The posterior pdf $f_{\Theta|X}(\theta|x) = a \cdot \exp\left\{-\dfrac{(\theta - m)^2}{2v}\right\}$, where

$$m = \frac{\sum_{i=0}^{n} \frac{x_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}, \qquad v = \frac{1}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

- The pdf is normal, so it is maximized when $\theta = $ mean.
- Thus, $\hat{\theta}_{\mathsf{MAP}} = m$.
- What is the LMS esitmate?

$$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\Theta|X = x] = m$$

- Send signal θ with the uniform noise $W \sim \mathcal{U}[-1, 1]$. Observe X
- $X = \Theta + W$, where model θ with $\Theta \sim \mathcal{U}[4, 10]$

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$$f_{\Theta,X}(\theta, x) = f_{\Theta}(\theta)f_{X|\Theta}(x|\theta) = \begin{cases} \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}, & \text{if } 4 \leq \theta \leq 10, \ \theta - 1 \leq x \leq \theta + 1, \\ 0, & \text{otherwise} \end{cases}$$
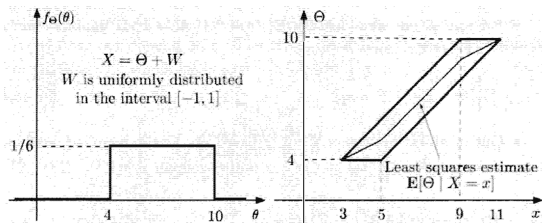
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$\hat{\theta}_{\mathsf{LMS}} = \mathbb{E}[\Theta | X = x]$: midpoint of the corresponding vertical section
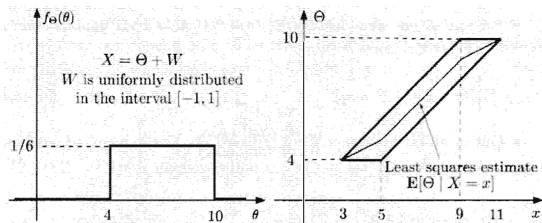
- What is conditional MSE? $\mathbb{E}\Big[(\Theta - \mathbb{E}[\Theta|X=x])^2|X=x\Big]$

- Given $X = 3$, it's the variance of $\mathcal{U}[4,4] = 0$

- Given $X = 5$, it's the variance of $\mathcal{U}[4,6] = (6-4)^2/12 = 1/3$

- The rising pattern between $X = 3$ and $X = 5$ is quadratic. This is because the expectation increases linearly, where the variance increases in a quadratic manner.

KAIST EE

$$f_{\Theta|X}(\theta|x) = \frac{f_\Theta(\theta)f_{X|\Theta}(x|\theta)}{f_X(x)}$$

$$f_X(x) = \int f_\Theta(\theta')f_{X|\Theta}(x|\theta')d\theta'$$

KAIST EE

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- Any alternative to LMS estimator?

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

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- We consider a restricted class of $g(X)$
  ◦ Estimator: $\hat{\Theta} = \boxed{}$.

  ◦ Estimate: Given $X = x$, $\hat{\theta} = \boxed{}$.

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- Our goal is to try our best within this restricted class:

$$\min_{a,b} \mathbb{E}\Big[(\Theta - aX - b)^2 | X = x\Big], \qquad \min_{a,b} \mathbb{E}\Big[(\Theta - aX - b)^2\Big]$$

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$$\min_{a,b} \mathbb{E}\Big[(\Theta - aX - b)^2|X = x\Big], \qquad \min_{a,b} \mathbb{E}\Big[(\Theta - aX - b)^2\Big]$$

- Linear models are always the first choice for a simple design in engineering.

LLMS

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \frac{\text{cov}(\Theta, X)}{\text{var}(X)}\Big(X - \mathbb{E}(X)\Big) = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big),$$

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∘ If $\rho > 0$:
  - Baseline ($\mathbb{E}[\Theta]$) + correction term
  - If $X > \mathbb{E}[X] \implies \hat{\Theta}_L > \mathbb{E}[\Theta]$
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LLMS Estimator: Result and Interpretation

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◦ If $\rho = 0$ (uncorrelated):
 - Just baseline ($\mathbb{E}[\Theta]$)
 - $\hat{\Theta}_L = \mathbb{E}[\Theta]$
 - No use of data X

- MSE $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$?

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)$$

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  - Assume $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$ (for simplicity). Then, MSE $= \mathbb{E}\left[(\Theta - \rho\frac{\sigma_\Theta}{\sigma_X}X)^2\right]$

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$$= \mathsf{var}(\Theta) + \left(\rho\frac{\sigma_\Theta}{\sigma_X}\right)^2\mathsf{var}(X) - 2\left(\rho\frac{\sigma_\Theta}{\sigma_X}\right)\mathsf{cov}(\Theta, X)$$

$$\boxed{\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)}$$

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KAIST EE

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- Uncertainty about Θ after observation decreases by the factor of $1 - \rho^2$

$$\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho\frac{\sigma_\Theta}{\sigma_X}\left(X - \mathbb{E}(X)\right)$$

- MSE $\mathbb{E}[(\hat{\Theta}_L - \Theta)^2]$?
  - Assume $\mathbb{E}[\Theta] = \mathbb{E}[X] = 0$ (for simplicity). Then, MSE $= \mathbb{E}\left[(\Theta - \rho \frac{\sigma_\Theta}{\sigma_X} X)^2\right]$
  - Note that $\text{var}[\Theta] = \sigma_\Theta^2 = \mathbb{E}(\Theta^2)$ and $\text{var}[X] = \sigma_X^2 = \mathbb{E}(X^2)$

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- What happens if $|\rho| = 1$ or $\rho = 0$?

$$\boxed{\hat{\Theta}_L = \mathbb{E}(\Theta) + \rho \frac{\sigma_\Theta}{\sigma_X}\left(X - \mathbb{E}(X)\right)}$$

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Linear LMS (LLMS) Estimator: Proof

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$$\mathbb{E}\Big[(Y - b)^2\Big], \quad Y = \Theta - aX$$

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- Using $\rho = \frac{\text{cov}(\Theta, X)}{\sigma_\Theta \sigma_X}$, we get:

$$a = \frac{\rho\sigma_\Theta\sigma_X}{\sigma_X^2} = \frac{\rho\sigma_\Theta}{\sigma_X}$$

- Then, we have (2).

- Romeo and Juliet start dating, where Romeo is late by $X \sim \mathcal{U}[0, \theta]$.

- Unknown: θ modeled by a rv $\Theta \sim \mathcal{U}[0, 1]$.

- Random observation: X

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- Question. What is the LLMS estimator $\hat{\Theta}_{\mathsf{L}}$?

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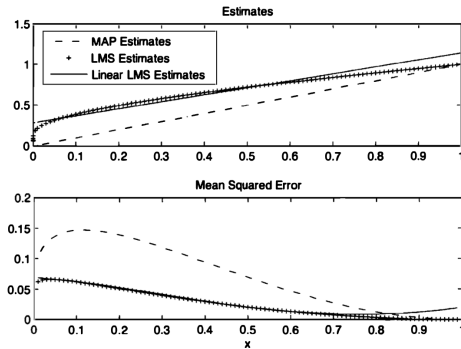- Using $\mathbb{E}[\Theta] = 1/2$ and $\mathbb{E}[\Theta^2] = 1/3$,

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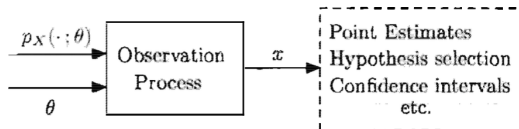$$\mathbb{E}[\Theta X] = \mathbb{E}[\mathbb{E}[\Theta X|\Theta]] = \mathbb{E}[\Theta \mathbb{E}[X|\Theta]]$$
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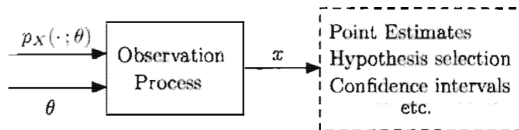$$\text{cov}(\Theta, X) = \frac{n}{3} - \frac{n}{4} = \frac{12}{n}$$

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- $\hat{\Theta}_L = \hat{\Theta}_{\text{LMS}}$! Intuitive?
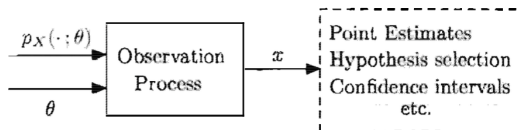- Yes, because the LMS esitmator was linear.

(1) Overview on Statistical Inference

(2) Bayesian Inference: Framework

(3) Examples

(4) MAP (Maximum A Posteriori) Estimator

(5) LMS (Least Mean Squares) Estimator

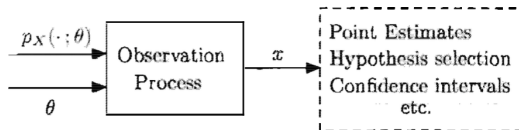(6) LLMS (Linear LMS) Estimator

(7) Classical Inference: ML Estimator

- Unknown θ

- Observations or measurements X

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  - deterministic (not random) quantity (thus, no prior distribution)
  - No prior, No posterior probabilities
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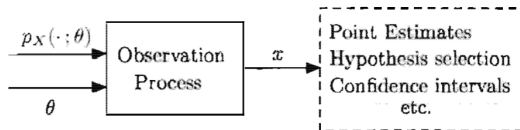 - Random observation X's distribution just depends on θ

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- Choosing one among multiple probabilistic models
  - Each θ corresponds to a probabilistic model

- Problem types
  - Estimation: θ: prob. of head?

  - Hypothesis testing: $\theta = 1/2$ or $\theta = 1/4$?

  - Significance testing: $\theta = 1/2$ or not?

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- Key inference methods
  ◦ ML (Maximum Likelihood) estimation

  ◦ Linear regression

  ◦ Likelihood ratio test

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- Just a taste in this course.

Maximum Likelihood Estimation

- Random observation $x = (x_1, x_2, \ldots, x_n)$ of $X = (X_1, X_2, \ldots, X_n)$
  - Assume a scalar θ and a vector of multiple observations in this lecture.

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- Very often, X_is are independent. Then, ML equals to maximizing the log-likelihood:

$$\log p_X(x_1, x_2, \ldots, x_n; \theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i; \theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i; \theta)$$

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$$\hat{\theta}_{\mathsf{MAP}} = \arg\max_{\theta} p_{\Theta|X}(\theta|x) = \arg\max_{\theta} \frac{p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{p_X(x)} = \frac{1}{p_X(x)} \arg\max_{\theta} p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)$$

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- Thus, when Θ is uniform (complete ignorance of Θ) in MAP, MAP == ML

- Romeo and Juliet start dating. Romeo: late by $X \sim U[0, \theta]$.

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- MAP: $\hat{\theta}_{\mathsf{MAP}} = x$

- LMS: $\hat{\theta}_{\mathsf{LMS}} = (1 - x)/|\log x|$

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Slides 17, 35, 45, and 56 for more details

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$$\arg \max_\theta f_X(x; \theta) = \arg \max_\theta \prod_{i=1}^n \theta e^{-\theta x_i} = \arg \max_\theta \left(n \log \theta - \theta \sum_{i=1}^n x_i \right)$$

Questions?

1) What is statistical inference?

2) Draw the building blocks of Bayesian inference and explain how it works.

3) What are MAP and LMS estimators and their underlying philosophies?

4) What is LLMS estimator and why is it useful?

5) Compare the classical and Bayesian inference.

6) What is the ML estimator and how is it related to the MAP estimator?