Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

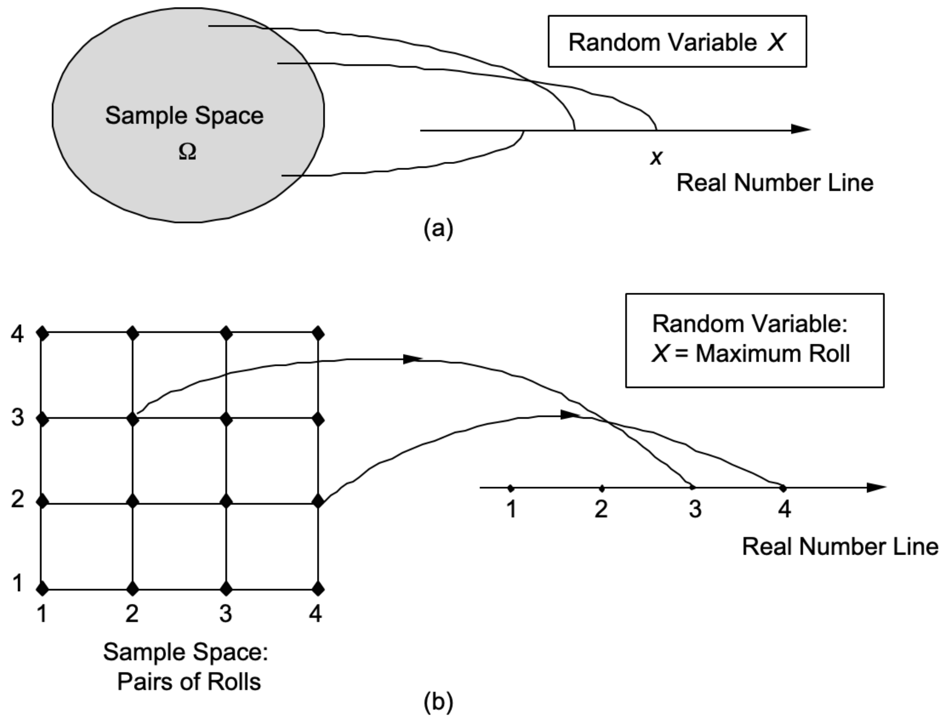
April 17, 2021

Outline

- Random Variable: Discrete
- PMF (Probability Mass Function)
- Representative Discrete Random Variables
- Expectation and Variance
- Functions of Random Variables
- Conditioning and Independence for Random Variables

Random Variable: Idea

- In reality, many outcomes are , e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



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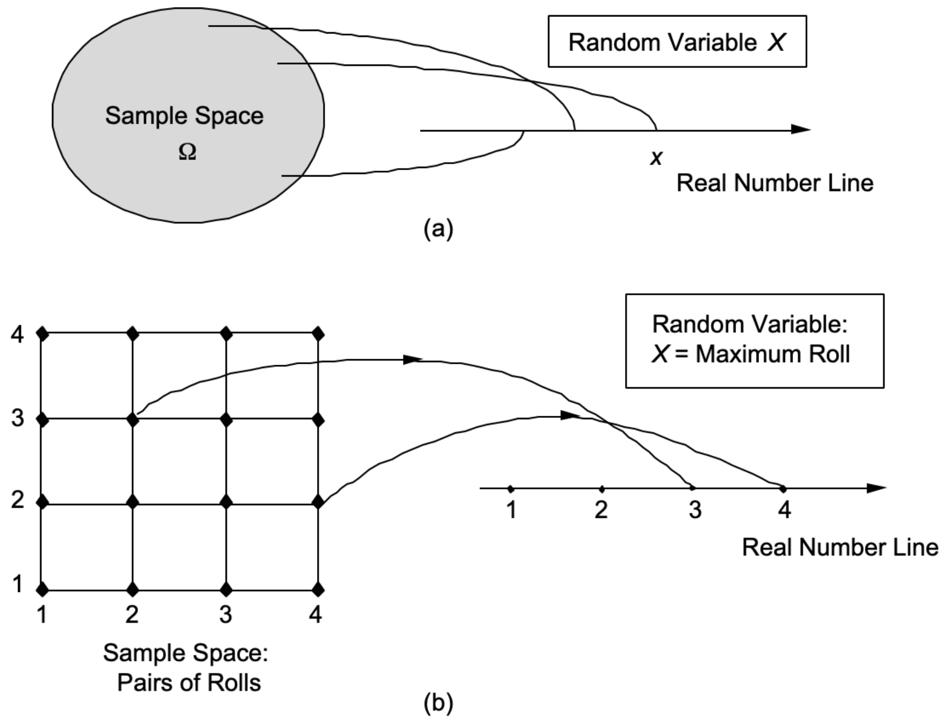

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- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}\Big(\{\omega \in \Omega \mid X(w) = x\}\Big)$$

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Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
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$$X = \begin{cases} 0, & \text{w.p.}^2 \quad 1 - p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

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- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an indicator rv of an event A. Define a rv 1_A as:

$$1_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[2] with probability

Uniform X with parameter a, b

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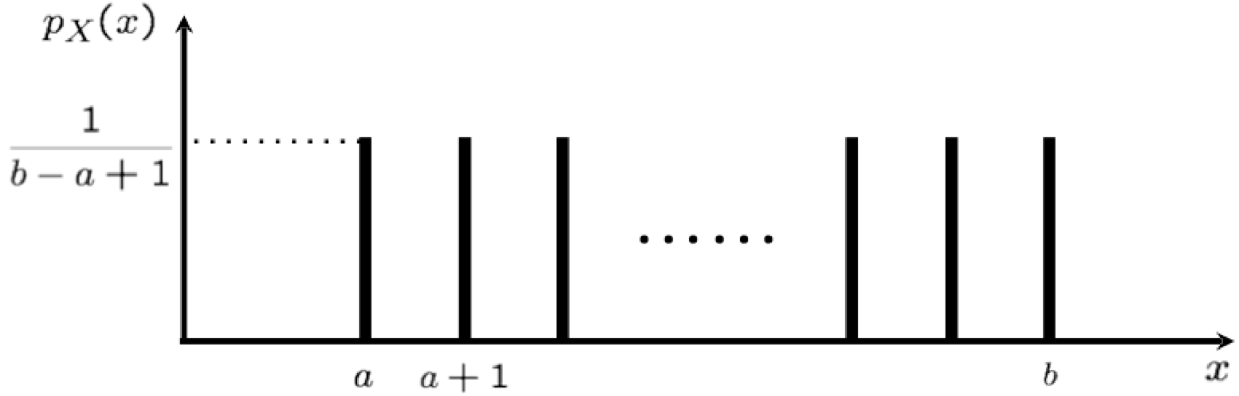

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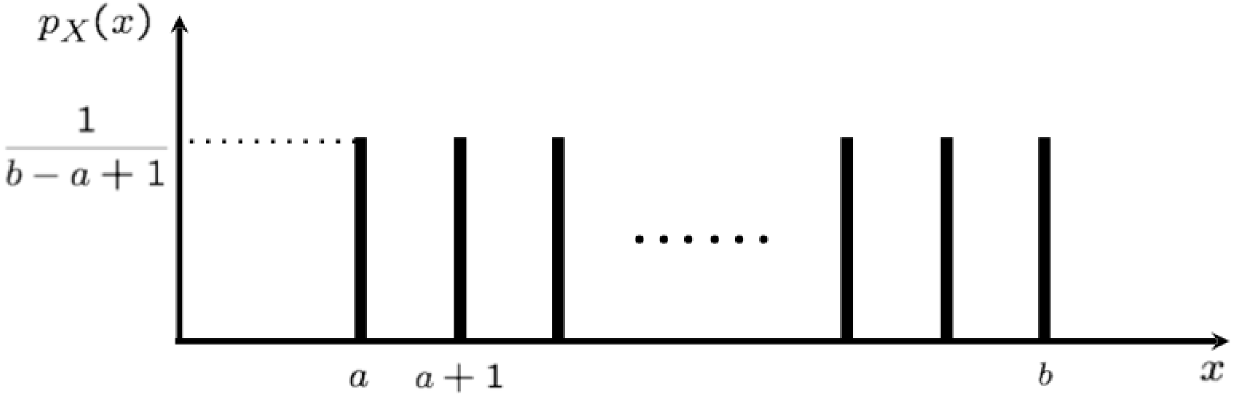
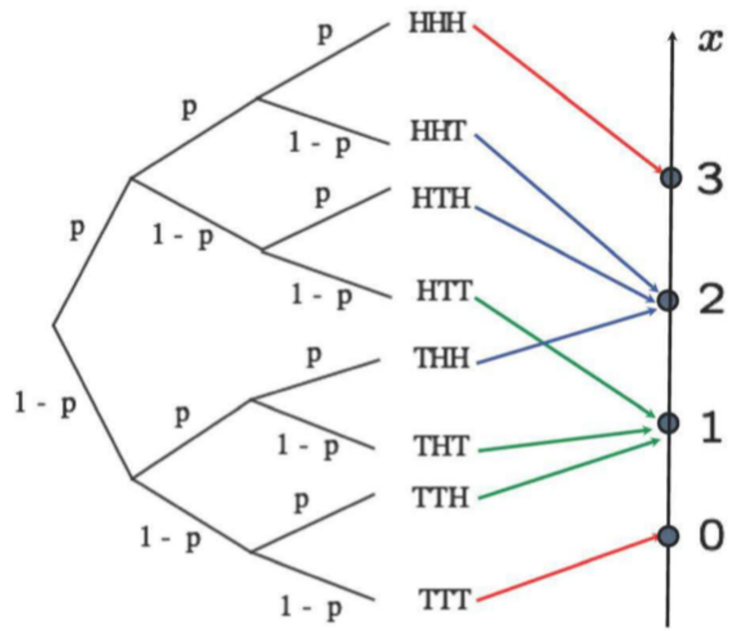


- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p



Binomial X with parameter n, p

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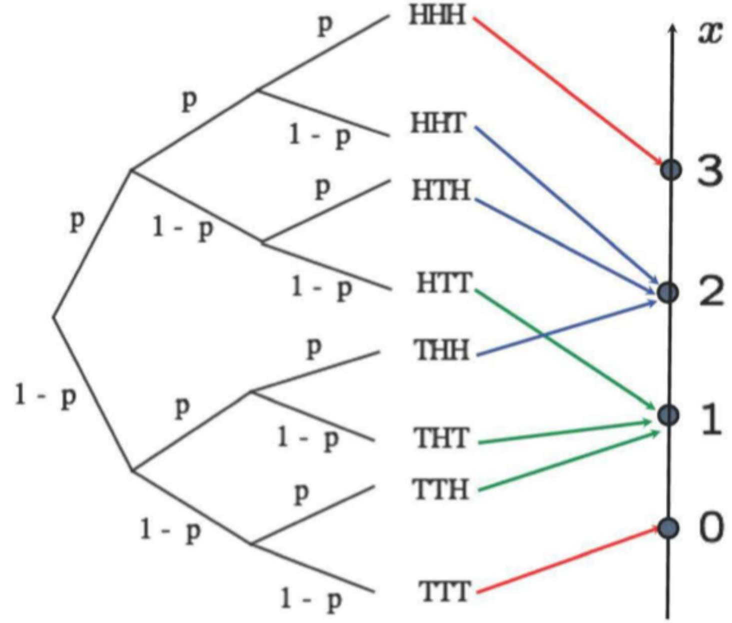

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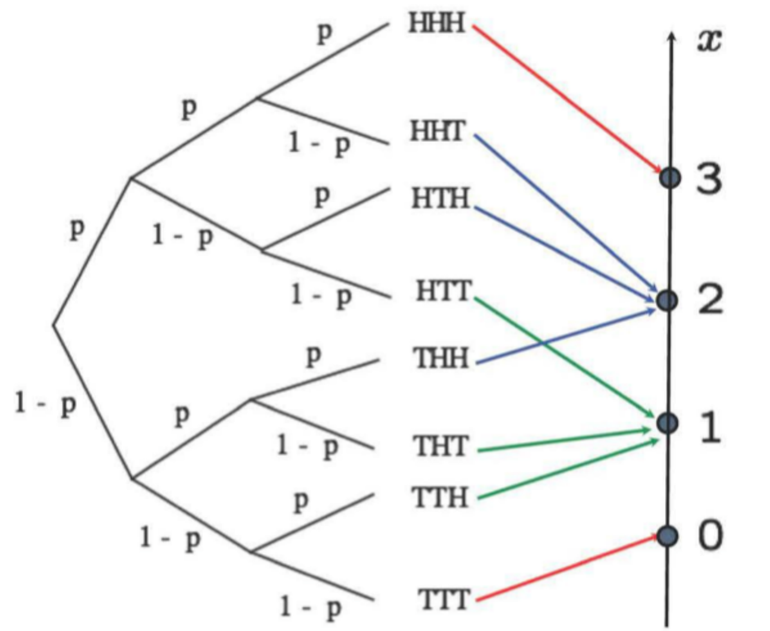
- Models the number of successes in a given number of independent trials
- n independent trials, where one trial has the success probability p.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



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- Is this a legitimate PMF?

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!}\ldots\right) = e^{-\lambda}e^{\lambda} = 1$$

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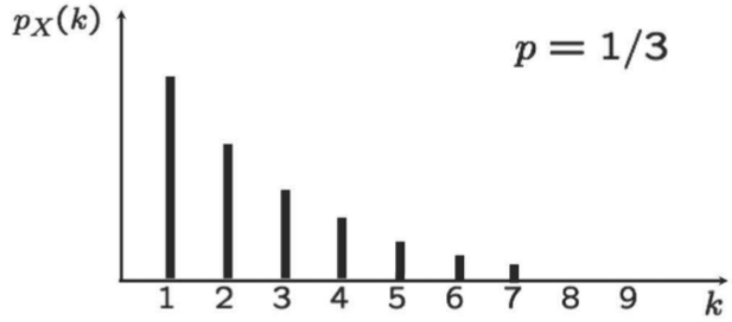
- Prove this:

$$\lim_{n\to\infty} p_X(k) = \binom{n}{k}(1/n)^k(1-1/n)^{n-k} = e^{-\lambda}\frac{\lambda^k}{k!}$$

Geometric X with parameter p

- Experiment: infinitely many independent Bernoulli trials, where each trial has success probability p



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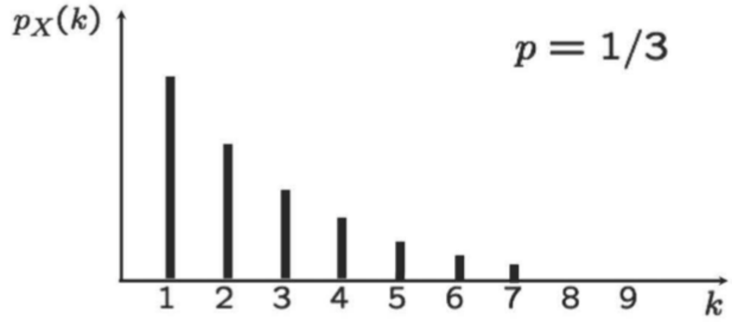
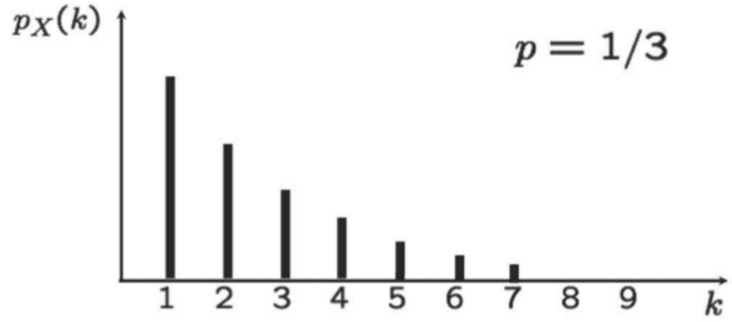


Geometric X with parameter p

- Experiment: infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the first success.
- Models waiting times until something happens.

$$p_X(k) = (1-p)^{k-1}p$$



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- Average.

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- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example 1: Bernoulli r.v. with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p_X(1)$$

Properties of Expectation

Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.

- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.

- For a constant c, $\mathbb{E}[c] = c$.

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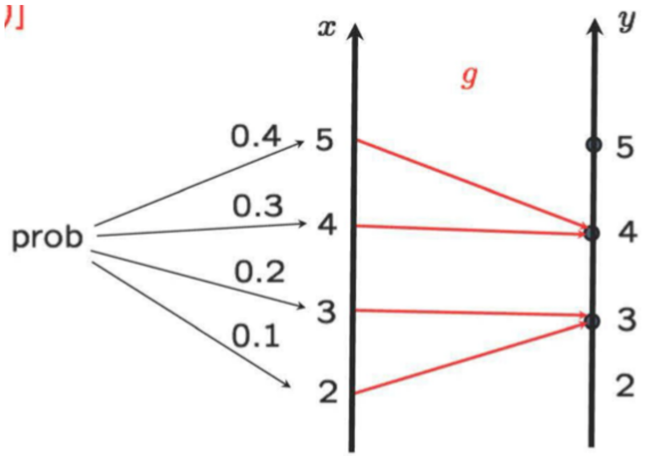

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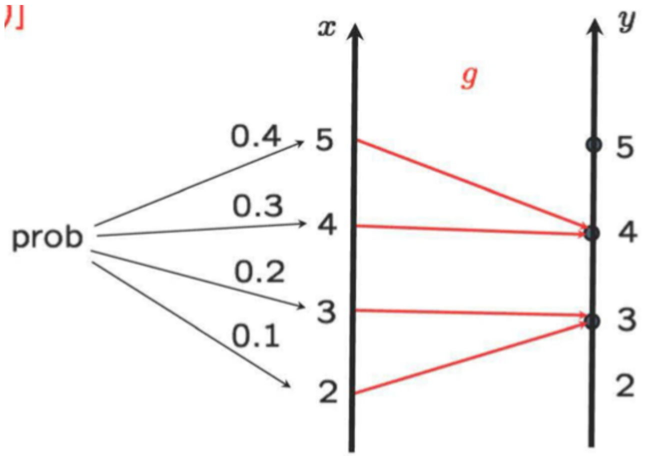

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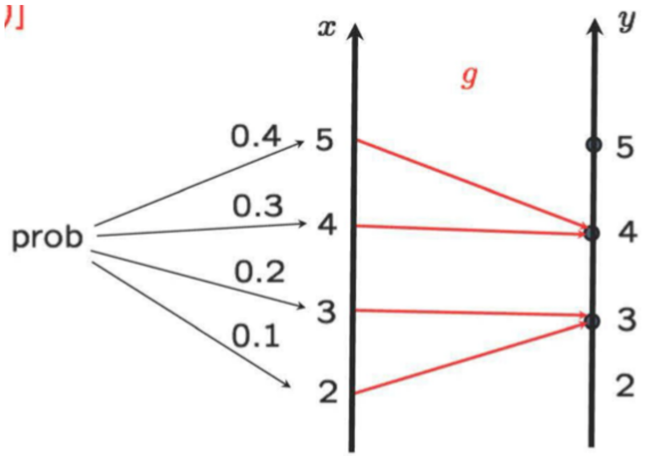

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Linearity of Expectation

$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

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Variance, Standard Deviation

$$\begin{aligned} \mathrm{var}[X] &= \mathbb{E}[(X-\mu)^2] \\ \sigma_X &= \sqrt{\mathrm{var}[X]} \end{aligned}$$

Variance: Useful Property

- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$

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Example: Variance of a Bernoulli rv (p)

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$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p$$
$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1-p) = p$$
$$\text{var}[X] = \mathbb{E}[X^2] - \mu^2 = p - p^2$$
$$= p(1-p)$$

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Joint PMF

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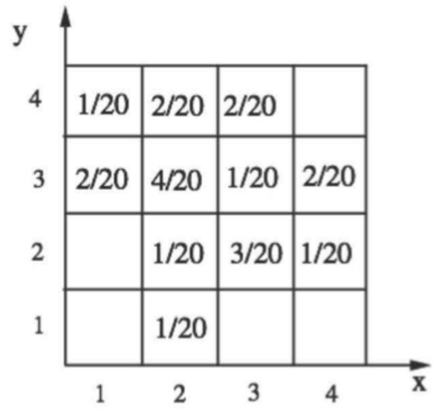

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$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

Joint PMF

- For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

$$\mathbb{P}\Big(\{X = x\} \cap \{Y = y\}\Big)$$

Example.

y \ x	1	2	3	4
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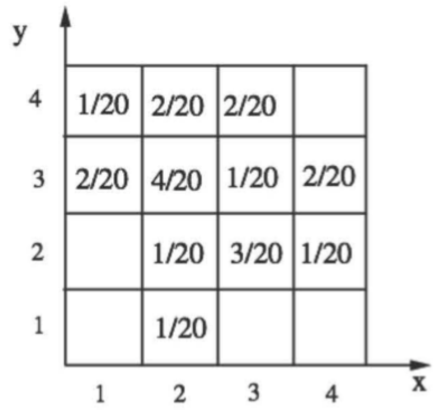

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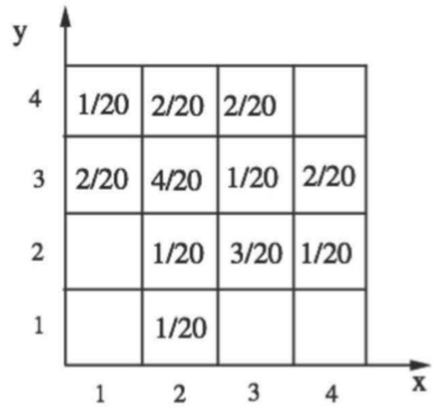

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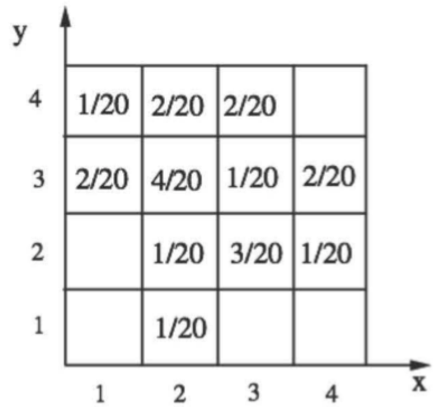

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- $\boxed{\text{Marginal PMF.}}$

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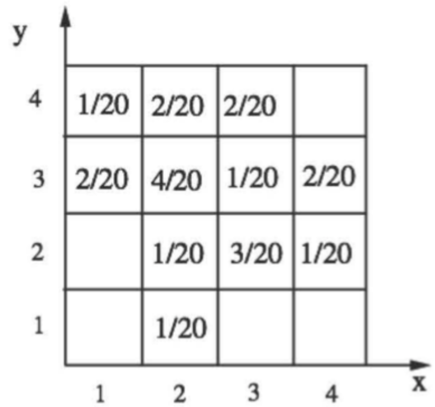
Example.



$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

- Consider a rv $Z = g(X, Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] =$$

Functions of Multiple R.V.s

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$$p_Z(z) = \mathbb{P}(g(X, Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x, y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

Linearity of Expectation for multiple rvs

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- Example. Mean of a binomial rv Y with (n, p)
- Y: number of successes in n Bernoulli trials with p
- $Y = X_1 + \ldots X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

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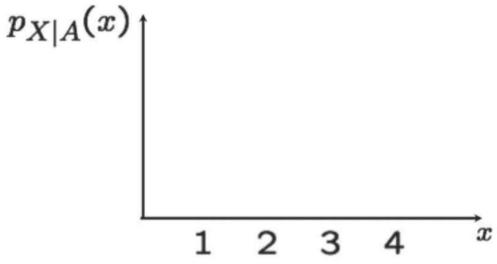
- Note. $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

Example: Conditional PMF

$A = \{X \geq 2\}$



$\mathbb{E}[X] =$

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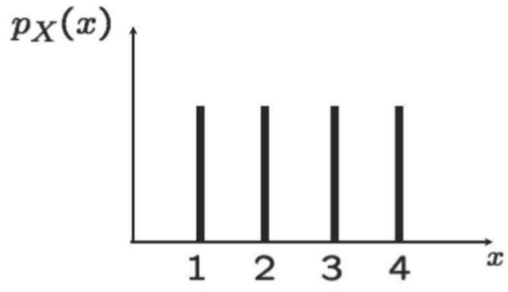

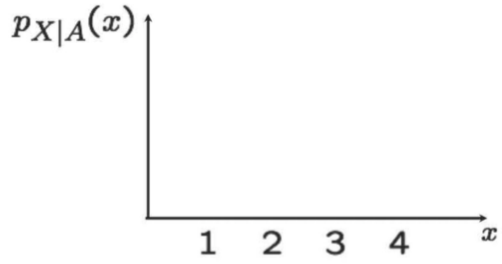

$$\mathbb{E}[X] = \frac{1}{4}(1 + 2 + 3 + 4) = 2.5$$

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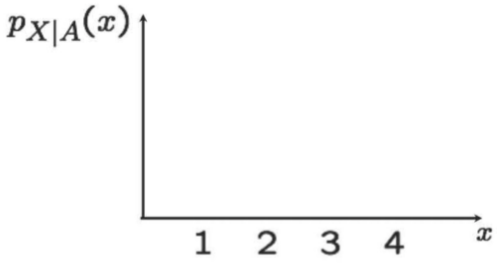

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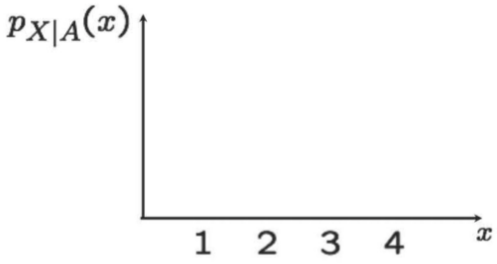

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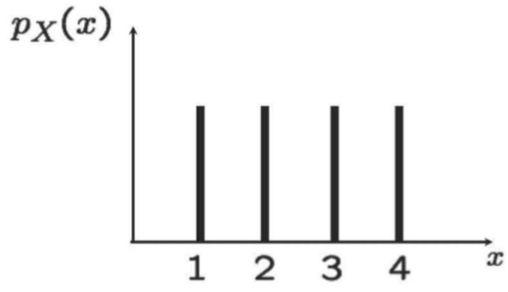

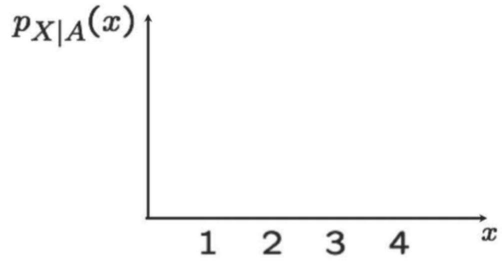

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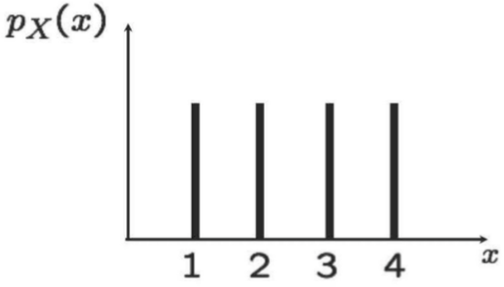

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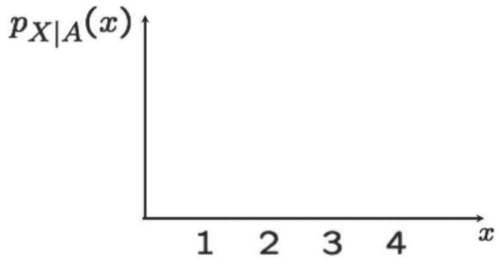

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Conditional PMF

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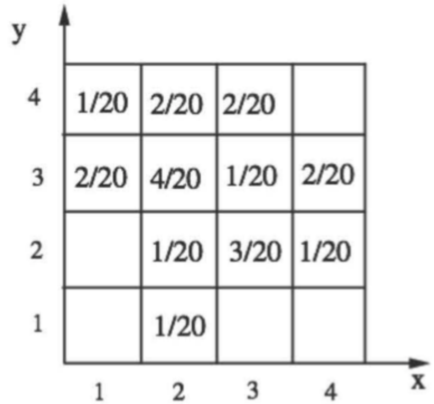
- Multiplication rule.

  $p_{X,Y}(x, y) =$

- $p_{X,Y,Z}(x, y, z) =$



$p_{X|Y}(2|2) = \frac{1}{1+3+1}$

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$$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

for y such that $p_Y(y) > 0$.

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y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
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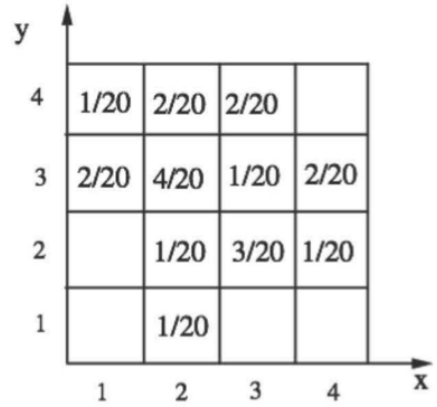

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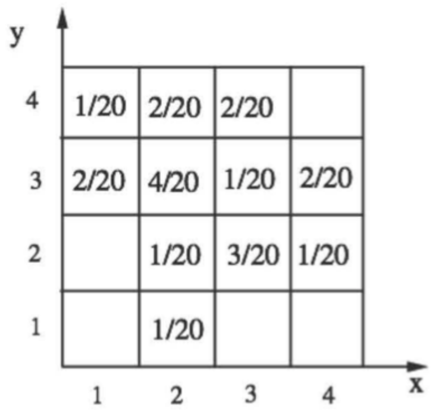

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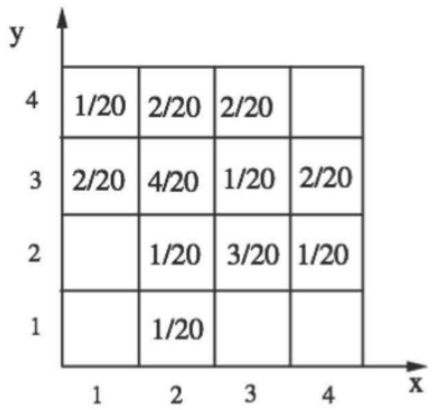

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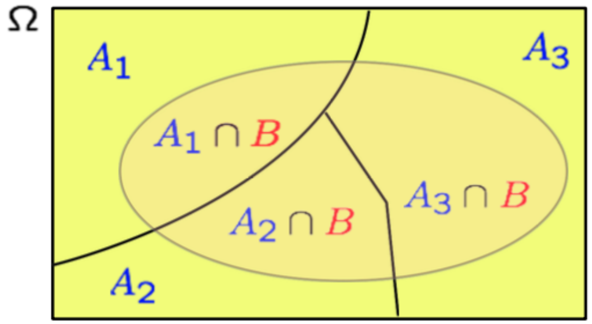
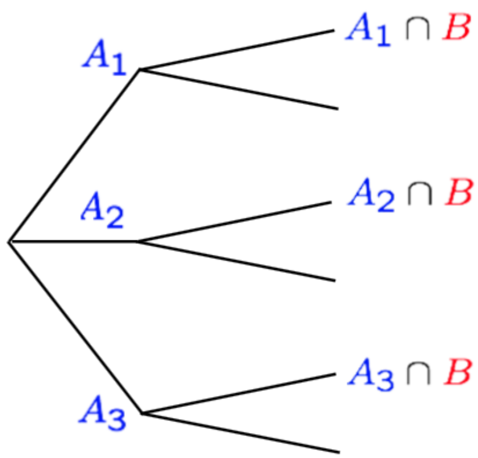
Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$? (probability of result)

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$



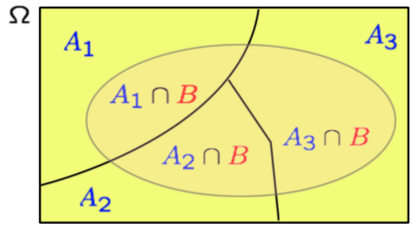


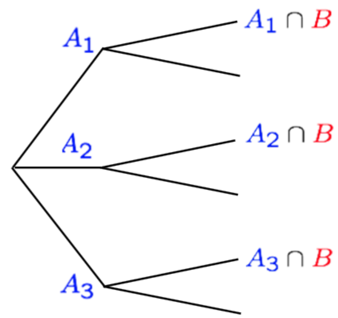
Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i) p_{X|A_i}(x)$$





Total Expectation Theorem for $\{A_i\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$

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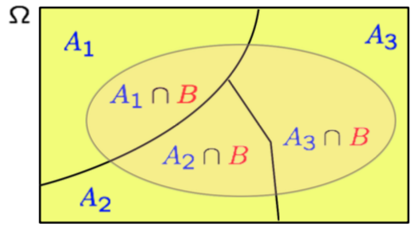

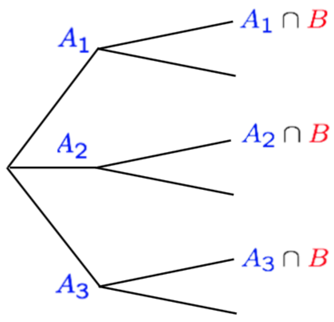

Total Expectation Theorem for $\{Y = y\}$

- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

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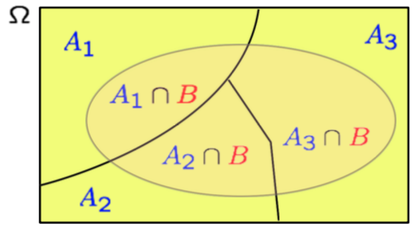

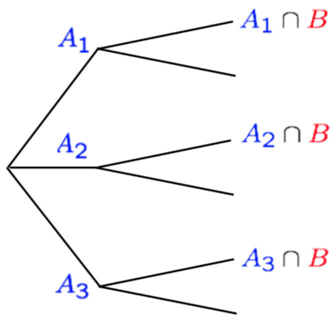

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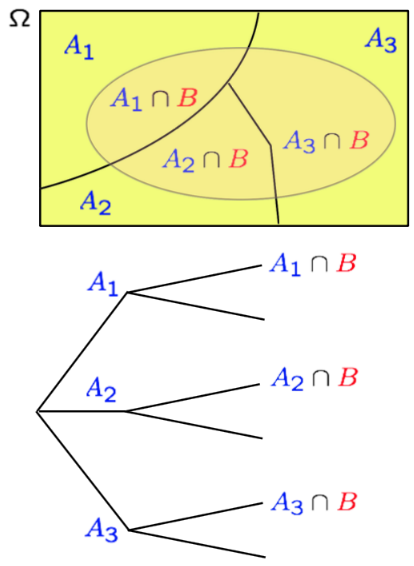

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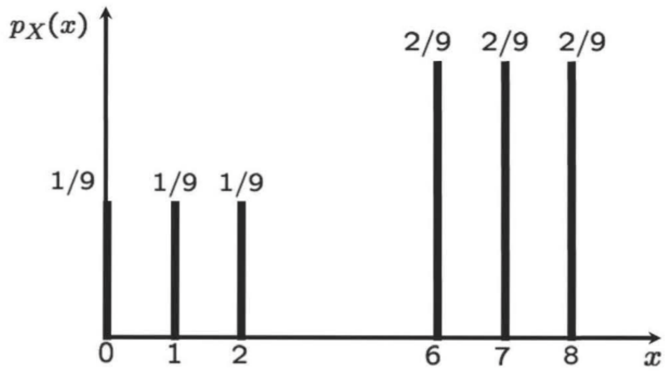
Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y] = \sum_y p_Y(y)\mathbb{E}[X|Y = y]$$

Example 1: Total Expectation Theorem

- $A_1 = \{X \in \{0, 1, 2\}\}$, $A_2 = \{X \in \{6, 7, 8\}\}$



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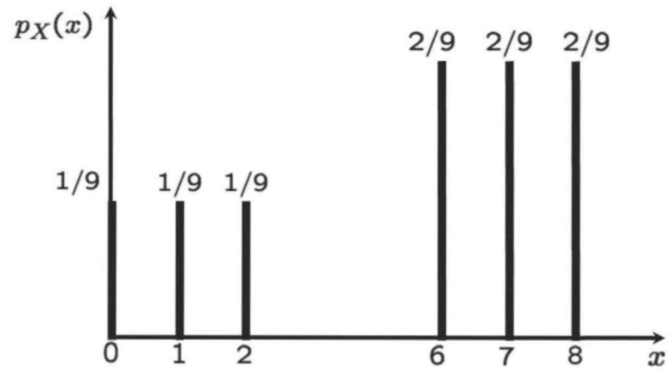
- Using TET,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i] \\ &= 1/3 \cdot 1 + 2/3 \cdot 7 = 7\end{aligned}$$



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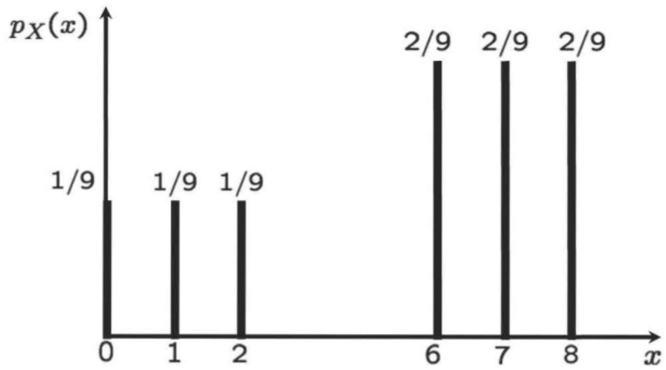
- Without using TET,

$$\mathbb{E}[X] = \frac{1}{9}(0 + 1 + 2) + \frac{2}{9}(6 + 7 + 8)$$



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Background: Memoryless Property of Geometric rv (1)

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- Remind. Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1-p)^{k-1} p$$

$$\mathbb{P}(X > k) = 1 - \sum_{k'=1}^{k} (1-p)^{k'-1} p = (1-p)^k$$

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- **Meaning.** Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

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$$\begin{aligned}
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$$\mathbb{E}[X] = 1 + (1-p)\frac{1}{p} = 1/p.$$

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$
$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

Independence, Conditional Independence

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$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$

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$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

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$$p_{X,Y|Z}(x, y) = p_{X|Z}(x) \cdot p_{Y|Z}(y)$$

Example

- $X \perp\!\!\!\perp Y$?

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

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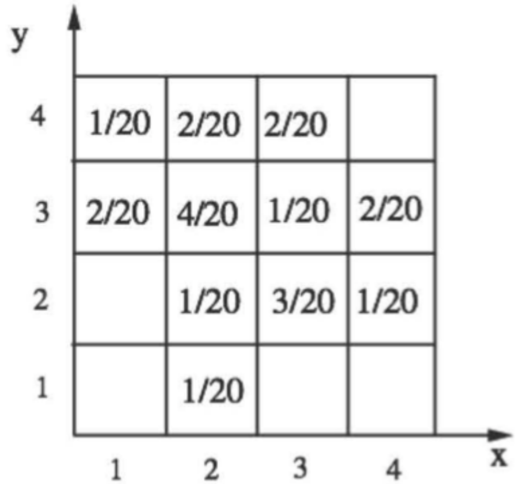

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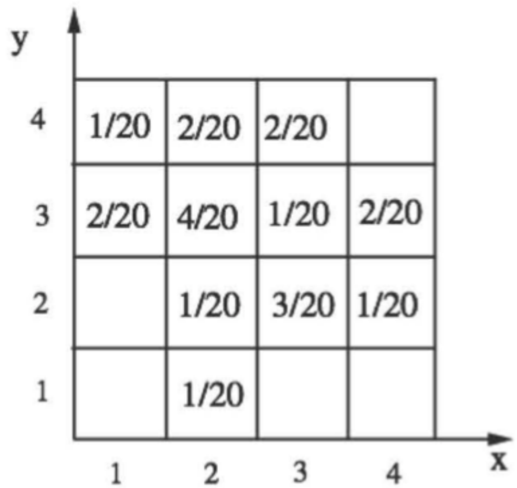
  \- Yes.



$Y = 4\ (1/3)$	$1/9$	$2/9$
$Y = 3\ (2/3)$	$2/9$	$4/9$
	$X = 1\ (1/3)$	$X = 2(2/3)$

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- Proof.

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  - $X = -Y \implies \text{var}[X + Y] = 0$

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  - $X=Y \implies \text{var}[X+Y]=4\text{var}[X]$
  - $X=-Y \implies \text{var}[X+Y]=0$
  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X-3Y]=\text{var}[X]+9\text{var}[Y]$

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\mathrm{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\
&= \mathbb{E}[X^2 + Y^2 + 2XY] - \left((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right) \\
&= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)
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- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

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- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (symmetry)

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Questions?

Review Questions

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFS?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?