

Lecture 7: Law of Large Numbers and Central Limit Theorem

Yi, Yung (이웅)

EE210: Probability and Introductory Random Processes
KAIST EE

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- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
 - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
 - Moment Generating Function (MGF)
- Strong Law of Large Numbers

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- $\mathbb{E}[X_i] = \mu, \text{var}[X_i] = \sigma^2$
- Our interest is to understand how the following sum behaves:

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- Take a certain **scaling** with respect to n that corresponds to a **new glass**, and investigate the system for large n
- First, consider the sample mean, and try to understand how it behaves:

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- Why important? If we take the scaling of S_n by $1/n$, it behaves like a deterministic number. This significantly simplifies how we understand the world.
- We call this **law of large numbers**.

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- Need to mathematically build up the concept of convergence for the sequence of random variables.

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Convergence in probability

For any $\epsilon > 0$, $\mathbb{P}(|Y_n - a| \geq \epsilon) \xrightarrow{n \rightarrow \infty} 0$.

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- Why "Weak"? There exists a stronger stronger version, which we call "strong" law of large numbers.
- Proof requires some knowledge about useful inequalities, which we cover later.

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- The answer is $\frac{1}{2}$

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- Interestingly, it converges to some **random variable** Z that we know very well.

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Central Limit Theorem

For every z ,

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- **Meaning from scaling perspective.**
 - LLN: Scaling S_n by $1/n$, you go to a deterministic world.
 - CLT: Scaling S_n by $1/\sqrt{n}$, you still stay at the random world, but not an arbitrary random world. That's the normal random world, not depending on the distribution of each X_i . Very interesting!

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

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¹Only unique mode. A single maximum or minimum.

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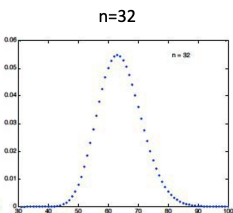
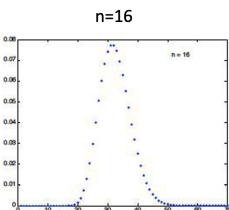
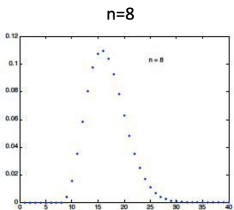
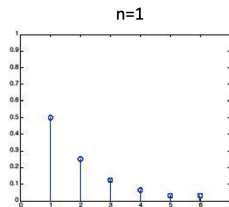
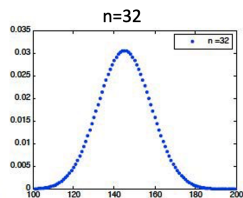
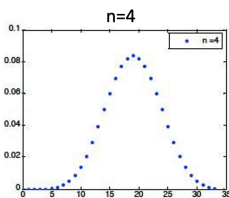
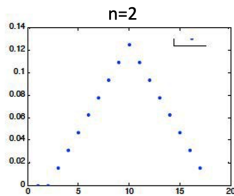
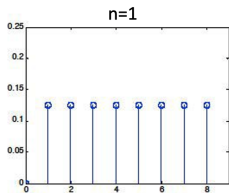
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- How large should n be?
 - A moderate n (20 or 30) usually works, which the power of CLT.
 - If X_i resembles a normal rv more, smaller n works: symmetry and unimodality¹

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Note that we have:

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- (Q) Knowing $\mathbb{E}(X)$, can we say something about the distribution of X ?
- Intuition: small $\mathbb{E}(X) \implies$ small $\mathbb{P}(X \geq a)$

Markov Inequality

If $X \geq 0$ and $a > 0$, then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.

Proof. For any $a > 0$, define Y_a as:

$$Y_a \triangleq \begin{cases} 0, & \text{if } X < a, \\ a, & \text{if } X \geq a \end{cases}$$

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- Both bounds are the ones that bound the probability of rare events.

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M_n converges to μ in probability.

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- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
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Ex1) $X \sim \exp(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$$\begin{aligned} M(s) &= \lambda \int_0^{\infty} e^{sx} e^{-\lambda x} dx \\ &= \lambda \left. \frac{e^{(s-\lambda)x}}{s-\lambda} \right|_0^{\infty} \quad (\text{if } s < \lambda) \\ &= \frac{\lambda}{\lambda - s} \end{aligned}$$

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Ex2) $X \sim N(0, 1)$ (homework problem)

$$M(s) = e^{s^2/2}$$

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4. MGF provides a convenient way of generating moments. That's why it is called moment generating function.

Inversion Property

The transform $M_X(s)$ associated with a random variable X uniquely determines the CDF of X , assuming that $M_X(s)$ is finite for all s in some interval $[-a, a]$, where a is a positive number.

- In easy words, we can recover the distribution if we know the MGF.
- Thus, each rv has its own MGF.

- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$

Proof.

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- Without loss of generality, assume $\mathbb{E}(X_i) = 0$ and $\text{var}(X_i) = 1$
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- If we apply l'hospital's rule twice (please check), we get

$$\lim_{y \rightarrow 0} \frac{\log M(ys)}{y^2} = \frac{s^2}{2}$$



- Most remarkable two results in probability theory history
- Weak Law of Large Numbers: Result and Meaning
- Central Limit Theorem: Result and Meaning
- Weak Law of Large Numbers: Proof
 - Inequalities: Markov and Chebyshev
- Central Limit Theorem: Proof
 - Moment Generating Function (MGF)
- Strong Law of Large Numbers (Optional)

Questions?

- 1) What's the practical value of LLN and CLT?
- 2) Explain LLN and CLT from the scaling perspective.
- 3) Why are LLN and CLT great?
- 4) Why do we need different concepts of convergence for random variables?
- 5) Explain what is convergence in probability.
- 6) Explain what is convergence in distribution.
- 7) Why is MGF useful?