Lecture 3: Random Variable, Part I

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EE210: Probability and Introductory Random Processes
KAIST EE

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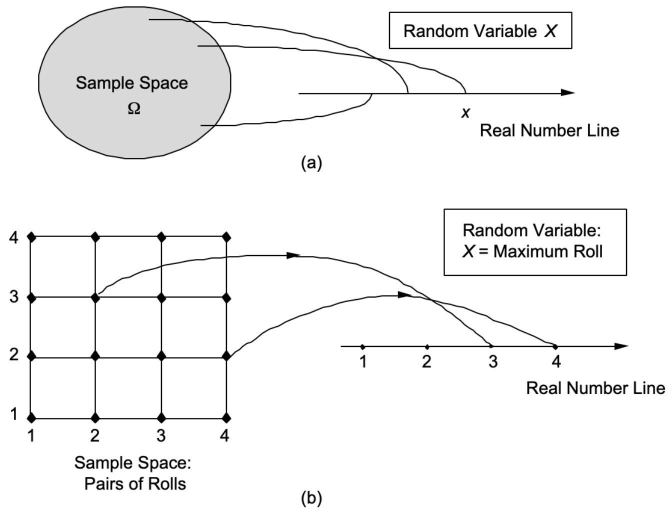
Outline

- Random Variable: Discrete
- PMF (Probability Mass Function)
- Representative Discrete Random Variables
- Expectation and Variance
- Functions of Random Variables
- Conditioning and Independence for Random Variables

Random Variable: Idea

- In reality, many outcomes are $\boxed{\text{numerical}}$, e.g., stock price.
- Even if not, very convenient if we map numerical values to random outcomes, e.g., '0' for male and '1' for female.



Random Variable: More Formally

- Mathematically, a random variable X is a $\boxed{\text{function}}$ which maps from Ω to $\mathbb{R}$.
- Notation. Random variable X, numerical value x.
- Different random variables X, Y,, etc can be defined on the same sample space.
- For a fixed value x, we can associate an $\boxed{\text{event}}$ that a random variable X has the value x, i.e., $\boxed{\{\omega \in \Omega \mid X(w) = x\}}$
- Assume that values x are discrete[1] such as $1, 2, 3, \ldots$.
  For notational convenience,

$$p_X(x) \triangleq \mathbb{P}(X = x) \triangleq \mathbb{P}\Big(\{\omega \in \Omega \mid X(w) = x\}\Big)$$

- For a discrete random variable X, we call $p_X(x)$ $\boxed{\text{probability mass function}}$ (PMF).

[1] Finite or countably infinite.

Roadmap

- Famous discrete random variables used in the community
  - Bernoulli, Uniform, Binomial, Geometric, Poisson, etc.
- Summarizing a random variable: Expectation and Variance
- Functions of a single random variable, Functions of multiple random variables
- Conditioning for random variables, Independence for random variables
- Continuous random variables
  - Normal, Uniform, Exponential, etc.
- Bayes' rule for random variables

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Bernoulli X with parameter $p \in [0, 1]$

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- Only binary values

$$X = \begin{cases} 0, & \text{w.p.}^2 \quad 1-p, \\ 1, & \text{w.p.} \quad p \end{cases}$$

In other words, $p_X(0) = 1 - p$ and $p_X(1) = p$ from our PMF notation.

- Models a trial that results in binary results, e.g., success/failure, head/tail
- Very useful for an $\boxed{\text{indicator rv}}$ of an event A. Define a rv 1_A as:

$$1_A = \begin{cases} 1, & \text{if } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}$$

[2] with probability

Uniform X with parameter a, b

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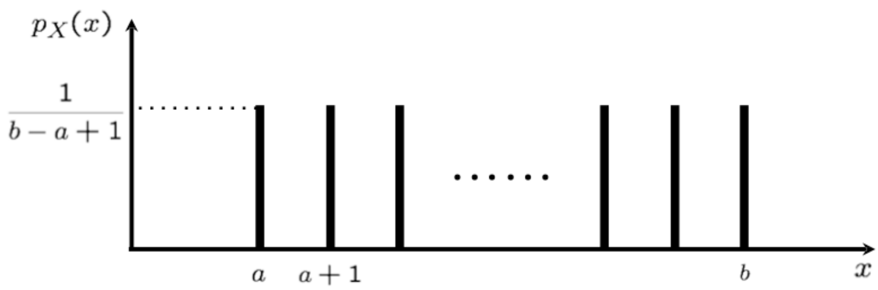
- integers a, b, where $a \le b$
- Choose a number of $\Omega = \{a, a+1, \dots, b\}$ uniformly at random.
- $p_X(i) = \frac{1}{b-a+1}$, $i \in \Omega$.

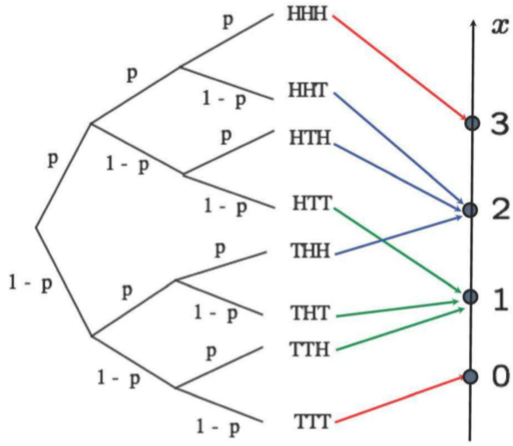


- Models complete ignorance (I don't know anything about X)

Binomial X with parameter n, p

- Models the number of successes in a given number of independent trials
- n independent trials, where one trial has the success probability p.

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$$



Poisson X with parameter λ

- *Binomial*(n, p): Models the number of successes in a given number of independent trials with success probability p.
- Very large n and very small p, such that $np = \lambda$

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

- Is this a legitimate PMF?

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \ldots \right) = e^{-\lambda} e^{\lambda} = 1$$

- Prove this:

$$\lim_{n \to \infty} p_X(k) = \binom{n}{k} (1/n)^k (1 - 1/n)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

Geometric X with parameter p

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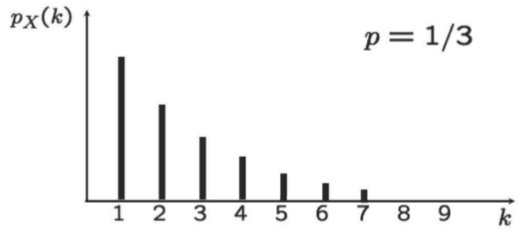
- Experiment: infinitely many independent Bernoulli trials, where each trial has success probability p
- Random variable: number of trials until the first success.
- Models waiting times until something happens.

$$p_X(k) = (1-p)^{k-1}p$$



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Expectation/Mean

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- Average.

Definition

$$\mathbb{E}[X] = \sum_{x} x p_X(x)$$

- $p_X(x)$: relative frequency of value x (trials with x/total trials)
- Example 1: Bernoulli r.v. with p

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p_X(1)$$

Properties of Expectation

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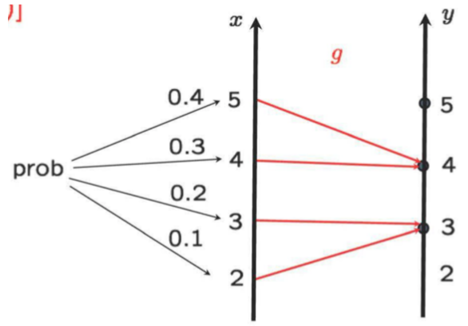
Not very surprising. Easy to prove using the definition.

- If $X \geq 0$, $\mathbb{E}[X] \geq 0$.
- If $a \leq X \leq b$, $a \leq \mathbb{E}[X] \leq b$.
- For a constant c, $\mathbb{E}[c] = c$.

Expectation of a function of a RV

- For a rv X, $Y = g(X)$ is also a r.v.
- $\mathbb{E}[Y] = \mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- Compute $\mathbb{E}[Y]$ for the following:



$$4 \times (0.4 + 0.3) + 3 \times (0.1 + 0.2)$$
$$= 2.8 + 0.9 = 3.7$$

Linearity of Expectation

$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Variance

- Measures how much the $\boxed{\text{spread}}$ of a PMF is.
- What about $\mathbb{E}[X - \mu]$, where $\mu = \mathbb{E}[X]$? Then, what about $\mathbb{E}[(X - \mu)^2]$?

Variance, Standard Deviation

$$\text{var}[X] = \mathbb{E}[(X - \mu)^2]$$
$$\sigma_X = \sqrt{\text{var}[X]}$$

Variance: Useful Property

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- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

  $\text{var}[X] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$
  $= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mu^2$

- $Y = X + b$, $\text{var}[Y] = \text{var}[X]$

  $\text{var}[Y] = \mathbb{E}[(X+b)^2] - (\mathbb{E}[X+b])^2$

- $Y = aX$, $\text{var}[Y] = a^2\text{var}[X]$

  $\text{var}[Y] = \mathbb{E}[a^2X^2] - (a\mathbb{E}[X])^2$

Example: Variance of a Bernoulli rv (p)

$$\mathbb{E}[X] = 1 \times p + 0 \times (1-p) = p$$

$$\mathbb{E}[X^2] = 1 \times p + 0 \times (1-p) = p$$

$$\text{var}[X] = \mathbb{E}[X^2] - \mu^2 = p - p^2 = p(1-p)$$

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Joint PMF

- $\boxed{\text{Joint PMF.}}$ For two random variables X, Y, consider two events $\{X = x\}$ and $\{Y = y\}$, and

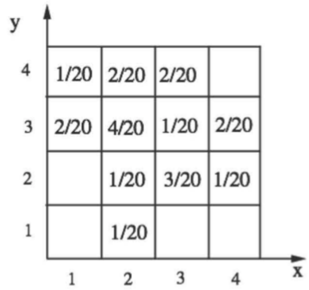
$$p_{X,Y}(x,y) \triangleq \mathbb{P}\Big(\{X = x\} \cap \{Y = y\}\Big)$$

- $\sum_x \sum_y p_{X,Y}(x,y) = 1$
- $\boxed{\text{Marginal PMF.}}$

$$p_X(x) = \sum_y p_{X,Y}(x,y),$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

Example.



$$p_{X,Y}(1,3) = 2/20$$

$$p_X(4) = 2/20 + 1/20 = 3/20$$

$$\mathbb{P}(X = Y) = 1/20 + 4/20 + 3/20 = 8/20$$

Functions of Multiple R.V.s

- Consider a rv $Z = g(X,Y)$. (Ex) $X + Y$, $X^2 + Y^2$. Then, PMF of Z is:

$$p_Z(z) = \mathbb{P}(g(X,Y) = z) = \sum_{(x,y):g(x,y)=z} p_{X,Y}(x,y)$$

- Similarly,

$$\mathbb{E}[Z] = \mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

Linearity of Expectation for multiple rvs

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- Remember: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Similarly,
$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$
(easy to prove, using the definition.)
- $\mathbb{E}[X_1 \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
- $\mathbb{E}[2X + 3Y - Z] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] - \mathbb{E}[Z]$

- Example. Mean of a binomial rv Y with (n, p)
- Y: number of successes in n Bernoulli trials with p
- $Y = X_1 + \ldots X_n$, where X_i is a Bernoulli rv.
- $\mathbb{E}[Y] = n\mathbb{E}[X_i] = n\mathbb{P}(X_i = 1) = np$

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Conditional PMF: Conditioning on an event

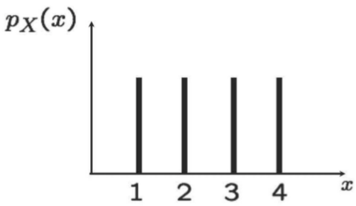
Remember two probability laws: $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot|A)$, for an event A.

- $p_X(x) = \mathbb{P}(X = x)$
- $\mathbb{E}[X] = \sum_x x p_X(x)$
- $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$
- $\text{var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$
- Note. $p_{X|A}(x)$, $\mathbb{E}[X|A]$, $\mathbb{E}[g(X)|A]$, and $\text{var}[X|A]$ are all just notations!

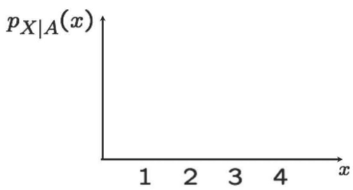
Example: Conditional PMF

$A = \{X \geq 2\}$



$$\mathbb{E}[X] = \frac{1}{4}(1 + 2 + 3 + 4) = 2.5$$

$$\begin{aligned}\text{var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{1}{4}(1 + 2^2 + 3^2 + 4^2) - 2.5^2\end{aligned}$$



$$\mathbb{E}[X|A] = \frac{1}{3}(2 + 3 + 4) = 3$$

$$\begin{aligned}\text{var}[X|A] &= \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2 \\ &= \frac{1}{3}(2^2 + 3^2 + 4^2) - 3^2 = 2/3\end{aligned}$$

Conditional PMF: Conditioning on a RV

What do we mean by "conditioning on a rv"? Consider $A = \{Y = y\}$ for a rv Y.

- $p_{X|A}(x) \triangleq \mathbb{P}(X = x|A)$
- $\mathbb{E}[X|A] \triangleq \sum_x x p_{X|A}(x)$
- $\mathbb{E}[g(X)|A] \triangleq \sum_x g(x) p_{X|A}(x)$
- $\text{var}[X|A] \triangleq \mathbb{E}[X^2|A] - (\mathbb{E}[X|A])^2$

- $p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y)$
- $\mathbb{E}[X|Y = y] \triangleq \sum_x x p_{X|Y}(x|y)$
- $\mathbb{E}[g(X)|Y = y] \triangleq \sum_x g(x) p_{X|Y}(x|y)$
- $\text{var}[X|Y = y] \triangleq$
  $\mathbb{E}[X^2|Y = y] - (\mathbb{E}[X|Y = y])^2$

Conditional PMF

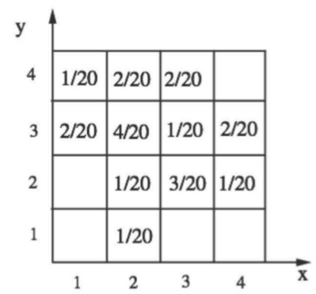
- Conditional PMF

  $$p_{X|Y}(x|y) \triangleq \mathbb{P}(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

  for y such that $p_Y(y) > 0$.
- $\sum_x p_{X|Y}(x|y) = 1$
- Multiplication rule.

  $$\begin{aligned} p_{X,Y}(x,y) &= p_Y(y) p_{X|Y}(x|y) \\ &= p_X(x) p_{Y|X}(y|x) \end{aligned}$$

- $p_{X,Y,Z}(x,y,z) =$
  $p_X(x) p_{Y|X}(y|x) p_{Z|X,Y}(z|x,y)$



$p_{X|Y}(2|2) = \frac{1}{1+3+1}$

$p_{X|Y}(3|2) = \frac{3}{1+3+1}$

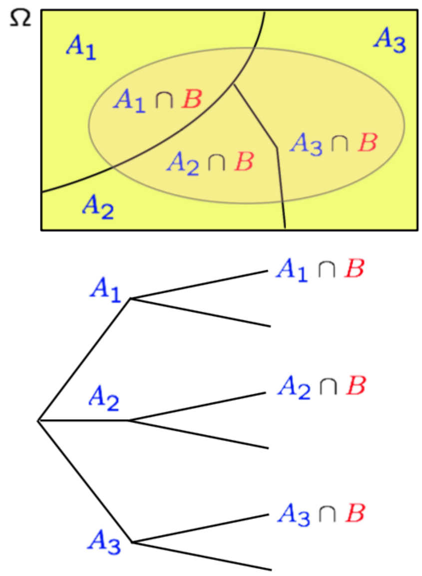
$\mathbb{E}[X|Y = 3] = 1(2/9) + 2(4/9) + 3(1/9) + 4(2/9)$

Remind: Total Probability Theorem (from Lecture 2)

- Partition of Ω into A_1, A_2, A_3
- Known: $\mathbb{P}(A_i)$ and $\mathbb{P}(B|A_i)$
- What is $\mathbb{P}(B)$? (probability of result)

Total Probability Theorem

$$\mathbb{P}(B) = \sum_i \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

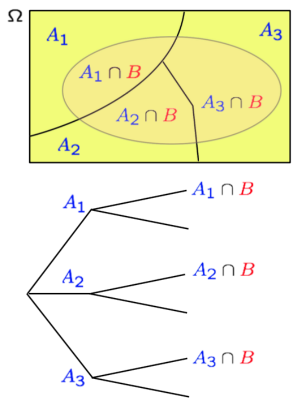


Total Probability Theorem: $B = \{X = x\}$

- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$



Total Expectation Theorem for $\{A_i\}$

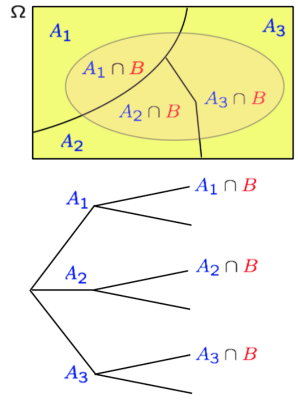
- Partition of Ω into A_1, A_2, A_3

Total Probability Theorem

$$p_X(x) = \sum_i \mathbb{P}(A_i)\mathbb{P}(X = x|A_i) = \sum_i \mathbb{P}(A_i)p_{X|A_i}(x)$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$



Total Expectation Theorem for $\{Y = y\}$

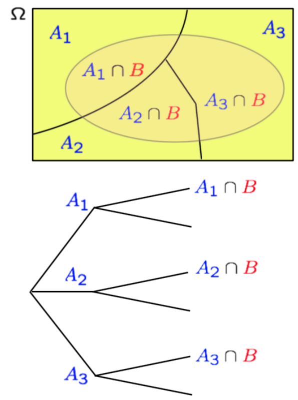
- Partition of Ω into A_1, A_2, A_3

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$

Total Expectation Theorem

$$\mathbb{E}[X] = \sum_y \mathbb{P}(Y = y)\mathbb{E}[X|Y = y] = \sum_y p_Y(y)\mathbb{E}[X|Y = y]$$



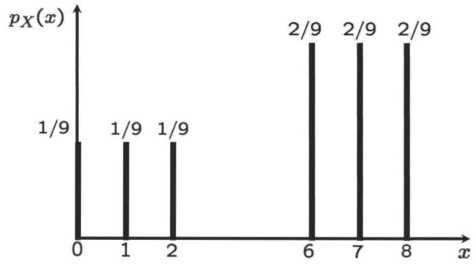
Example 1: Total Expectation Theorem

- $A_1 = \{X \in \{0,1,2\}\}$, $A_2 = \{X \in \{6,7,8\}\}$
- Using TET,

$$\mathbb{E}[X] = \sum_{i=1,2} \mathbb{P}(A_i)\mathbb{E}[X|A_i]$$
$$= 1/3 \cdot 1 + 2/3 \cdot 7 = 7$$

- Without using TET,

$$\mathbb{E}[X] = \frac{1}{9}(0+1+2) + \frac{2}{9}(6+7+8)$$



Background: Memoryless Property of Geometric rv (1)

- Some random variable often does not have memory.
- Definition. A random variable X is called $\boxed{\text{memoryless}}$ if, for any $n, m \geq 0$,

$$\mathbb{P}(X > n+m | X > m) = \mathbb{P}(X > n)$$

- Meaning. Conditioned on $X > m$, $X - m$'s distribution is the same as the original X.
- Remind. Geometric rv X with parameter p

$$\mathbb{P}(X = k) = (1-p)^{k-1}p$$

$$\mathbb{P}(X > k) = 1 - \sum_{k'=1}^{k} (1-p)^{k'-1}p = (1-p)^k$$

Background: Memoryless Property of Geometric rv (2)

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- Theorem. Any geometric random variable is memoryless.

$$
\begin{aligned}
\mathbb{P}(X > n+m | X > m) &= \frac{\mathbb{P}(X > n+m \text{ and } X > m)}{\mathbb{P}(X > m)} \\
&= \frac{\mathbb{P}(X > n+m)}{\mathbb{P}(X > m)} \\
&= \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = \mathbb{P}(X > n)
\end{aligned}
$$

- Meaning. Conditioned on $X > m$, $X - m$ is geometric with the same parameter.

Example 2: Mean and Variance of Geometric rv

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- Write softwares over and over, and each time w.p. p of working correctly (independent from prev. programs).
- X: number of tries until the program works correctly.
- Q) mean and variance of X
- X is geometric
- Direct computation is boring.

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

- Total expectation theorem and memorylessness nhelps a lot.

- $A_1 = \{X = 1\}$ (first try is success), $A_2 = \{X > 1\}$ (first try is failure).

$$
\begin{aligned}
\mathbb{E}[X] &= 1 + \mathbb{E}[X-1] \\
&= 1 + \mathbb{P}(A_1)\mathbb{E}[X-1|X=1] \\
&\quad + \mathbb{P}(A_2)\mathbb{E}[X-1|X>1] \\
&= 1 + (1-p)\mathbb{E}[X]
\end{aligned}
$$

$$\mathbb{E}[X] = 1 + (1-p)\tfrac{1}{p} = 1/p.$$

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Independence, Conditional Independence

- Two events

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$
$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A | C) \cdot \mathbb{P}(B | C)$$

- A rv and an event

$$\mathbb{P}(\{X = x\} \cap B) = \mathbb{P}(X = x) \cdot \mathbb{P}(B), \quad \text{for all } x$$
$$\mathbb{P}(\{X = x\} \cap B | C) = \mathbb{P}(X = x | C) \cdot \mathbb{P}(B | C), \quad \text{for all } x$$

- Two rvs

$$\mathbb{P}(\{X = x\} \cap \{Y = y\}) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y), \quad \text{for all } x, y$$
$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$$

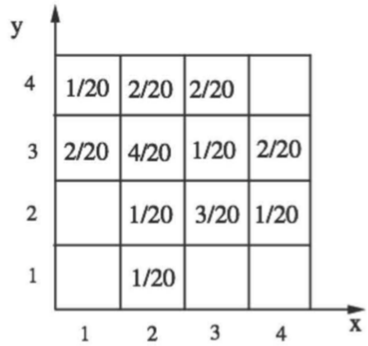
$$\mathbb{P}(\{X = x\} \cap \{Y = y\} | Z = z) = \mathbb{P}(X = x | Z = z) \cdot \mathbb{P}(Y = y | Z = z), \quad \text{for all } x, y$$
$$p_{X,Y|Z}(x, y) = p_{X|Z}(x) \cdot p_{Y|Z}(y)$$

Example

- $X \perp\!\!\!\perp Y$?

  $p_{X,Y}(1,1) = 0, \quad p_X(1) = 3/20$
  $p_Y 1 = 1/20.$

- $X \perp\!\!\!\perp Y | \{X \leq 2 \text{ and } Y \geq 3\}$?
  - Yes.



$Y = 4$ $(1/3)$	$1/9$	$2/9$
$Y = 3$ $(2/3)$	$2/9$	$4/9$
	$X = 1$ $(1/3)$	$X = 2(2/3)$

Expectation and Variance

- Always true.
  $\mathbb{E}[aX + b]$, $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Generally, $\mathbb{E}[g(X, Y)] \neq g(\mathbb{E}[X], \mathbb{E}[Y])$
- However, if $X \perp\!\!\!\perp Y$,

  $$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$
  $$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[g(Y)]$$

- Proof.

  $$\mathbb{E}[g(X)h(Y)] = \sum_x \sum_y g(x)h(y)p_{X,Y}(x,y)$$
  $$= \sum_x x p_X(x) \sum_y y p_Y(y)$$

- Always true.
  $\text{var}[aX] = a^2\text{var}[X]$, $\text{var}[X + a] = \text{var}[X]$
- Generally, $\text{var}[X + Y] \neq \text{var}[X] + \text{var}[Y]$
- However, if $X \perp\!\!\!\perp Y$,

  $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y]$

- Practice.
  - $X = Y \implies \text{var}[X + Y] = 4\text{var}[X]$
  - $X = -Y \implies \text{var}[X + Y] = 0$
  - $X \perp\!\!\!\perp Y \implies$
    $\text{var}[X - 3Y] = \text{var}[X] + 9\text{var}[Y]$

$\mathrm{var}[X+Y] \neq \mathrm{var}[X] + \mathrm{var}[Y]$

- Why not generally true?

$$\begin{aligned}\mathrm{var}[X+Y] &= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 \\ &= \mathbb{E}[X^2+Y^2+2XY] - \left((\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 + 2\mathbb{E}[X]\mathbb{E}[Y]\right) \\ &= \mathrm{var}[X] + \mathrm{var}[Y] + 2\left(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\right)\end{aligned}$$

- $\boxed{X \perp\!\!\!\perp Y}$ is a sufficient condition for $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Also, a necessary condition? we will see later, when we study covariance.

Example: The hat problem (1)

- n people throw their hats in a box and then pick one at random
- X: number of people with their own hat
- $\mathbb{E}[X]$? $\mathrm{var}[X]$?
- All permutations are equally likely as $1/n!$. Thus, this equals to picking one hat at a time.
- Key step 1. Define a rv $X_i = 1$ if i selects own hat and 0 otherwise.

$$X = \sum_{i=1}^{n} X_i.$$

- $\{X_i\}, i = 1, 2, \ldots, n$: identically distributed (symmetry)

Example: The hat problem (2)

- $\mathbb{E}[X] = n\mathbb{E}[X_1] = n\mathbb{P}(X_1 = 1) = n \times \frac{1}{n} = 1.$
- Key step 2. Are X_is are independent? If yes, easy to get $\text{var}(X)$.
- Assume $n = 2$. Then, $X_1 = 1 \to X_2 = 1$, and $X_1 = 0 \to X_2 = 0$. Thus, dependent.

$$\begin{aligned}
\text{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \mathbb{E}\Big[\sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j\Big] - (\mathbb{E}[X])^2
\end{aligned}$$

$$\mathbb{E}[X_i^2] = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

$$\mathbb{E}[X_i X_j] = \mathbb{E}[X_1 X_2] = 1 \times \mathbb{P}(X_1 X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 | X_1 = 1), \quad (i \neq j)$$

- $\mathbb{E}[X^2] = n\mathbb{E}[X_1^2] + n(n-1)\mathbb{E}[X_1 X_2] = n\frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2$
- $\text{var}(X) = 2 - 1 = 1$

Questions?

Review Questions

1) What is Random Variable? Why is it useful?
2) What is PMF (Probability Mass Function)?
3) Explain Bernoulli, Binomial, Poisson, Geometric rvs, when they are used and what their PMFs are.
4) What are joint and marginal PMFS?
5) Describe and explain the total probability/expectation theorem for random variables?
6) When is it useful to use total probability/expectation theorem?
7) What is conditional independence?