博弈论第九次作业

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[Title 1]:

Consider an arbitrary game with two players in which best response are guaranteed to terminate from any starting state. Suppose further that in this game best response are always unique. Prove that best response always reach an equilibrium in at most$O(m)$ steps, where m is the number of strategies of each player. (Note that $O(m^2)$ is trivial.)

(Bonus) If we remove the assumption that best response are unique, can we still obtain an algorithm which finds a pure Nash equilibrium in $O(m)$ steps?

[证明]:

First, we know that an arbitrary game with two players who has m strategies has a $m \times m$ Decision Matrix, which could be like this(D_{ij} represents that player 1 choose strategy i and player 2 choose strategy 2):

D_{11}	...	D_{1m}
...	...	...
D_{m1}	...	D_{mm}

表 1: Decision Matrix

Then, we prove a player will never choose a strategy that was chosen before.

We assume a player can choose a strategy that was chosen before.

We can assume that player 1 chooses strategy i, player 2 chooses strategy j at first.

If strategy j is player 2's best strategy, he will not choose the other strategies.

If strategy j is not player 2's best strategy, we can assume he will choose strategy j' by the best response algorithm when player choose strategy i.

So, assuming that after several times of best response algorithm player 1 choose strategy i again, player 2 will also choose strategy j' again which will create a loop and best response algorithm cannot terminate.

So, our assuming is not true that means a player will never choose a strategy that was chosen before.

Based on the conclusion that a player will never choose a strategy that was chosen before, we know our Decision Matrix will reduce a row after player 1 experiences a best response algorithm and reduce a column after player 2 experiences a best response algorithm. So, after player 1 and player 2 both experience a best response algorithm, our Decision Matrix change from $(m \times m)$ to $(m - 1 \times m - 1)$.

For example, if starting state is D_{11} and two players both choose another strategy, Decision Matrix will be like this:

$\cancel{D_{11}}$	~~...~~	$\cancel{D_{1m}}$
~~...~~	...	...
$\cancel{D_{m1}}$	...	D_{mm}

表 2: Decision Matrix

So, we can search the whole $m \times m$ Decision Matrix at most 2m steps. That means we can always reach an equilibrium at most $O(m)$

(Bonus)If best response is not unique, we know we can reach best response at a faster speed. So we can obtain an algorithm which finds a pure Nash equilibrium in $O(m)$ steps.

[Title 2]:

Consider a n-player game and a product distribution $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$. Show that σ is a correlated equilibrium of the game if and only if $\sigma_1, \cdot\cdot\cdot, \sigma_n$ form a mixed Nash Equilibrium of the game.

[证明]:

Let $\sigma = \sigma_1 \times \cdots \times \sigma_n$, $\sigma_i \in \Delta(\mathcal{S}_i)$.

Mixed Nash Equilibrium:

$$\forall i \in \mathcal{N}, \forall s_i' \in \mathcal{S}_i,\ \sum_{s \in \mathcal{S}} \sigma(s) u_i(s) \geq \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i', s_{-i}). \tag{1}$$

Correlated Equilibrium:

$$\forall i \in \mathcal{N},\ \forall s_i \in \mathcal{S}_i,\ \forall s_i' \in \mathcal{S}_i,\ \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma(s_i, s_{-i}) u_i(s_i', s_{-i}). \tag{2}$$

Observe that $\sigma(s_i, s_{-i}) = \sigma_i(s_i)\sigma_{-i}(s_{-i})$.

(In fact, for semantic correctness, $\sigma(s_{-i} \mid s_i)$ must used on both sides of the inequality above, but technically these are equivalent inequalities.)

1. MNE $\to$ CE

First, we make an important observation that for a given player i, all actions $s_i \in \mathcal{S}_i$ that are in support of σ_i (i.e., $\sigma_i(s_i) > 0$) must have the same expected utility $U = \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i})$. Otherwise, i would be better off by removing actions with lower utility, meaning σ is not an equilibrium. This also means that the expected utility of the mixed strategy σ_i itself $\sum_{s \in \mathcal{S}} \sigma(s) u_i(s)$ also equals U.

Now, since expected utility of a mixed strategy equals expected utilities of action in its support, from (1) we can deduce

$$\forall i \in \mathcal{N}, \forall s_i \in \mathcal{S}_i : \sigma_i(s_i) > 0,\ \forall s_i' \in \mathcal{S}_i,\ \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in \mathcal{S}_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i', s_{-i}). \tag{3}$$

Multiplying both sides of (3) by $\sigma_i(s_i)$ gives us the inequality from (2). For actions that are not in the support of σ_i, it holds trivially.

2. CE $\to$ MNE

Simply summing both sides of (2) over all $s_i \in \mathcal{S}_i$ gives

$$\forall i \in \mathcal{N},\ \forall s_i' \in \mathcal{S}_i,\ \sum_{s\in\mathcal{S}} \sigma(s)u_i(s) \geq \sum_{s_{-i}\in\mathcal{S}_{-i}} \sigma_{-i}(s_{-i})u_i(s_i', s_{-i}) \underbrace{\sum_{s_i\in\mathcal{S}_i} \sigma_i(s_i)}_{=1}, \tag{4}$$

which is exactly (1).

[Title 3]:

Prove that every correlated equilibrium of a n-player game is also a coarse correlated equilibrium.

[证明]:

A strategy profile $\sigma \in \Delta(\mathcal{S})$ is a correlated equilibrium when:

$$\forall i \in \mathcal{N},\ \forall s_i \in \mathcal{S}_i,\ \forall s_i' \in \mathcal{S}_i,\quad \sum_{s_{-i}\in\mathcal{S}_{-i}} \sigma(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in\mathcal{S}_{-i}} \sigma(s_i, s_{-i})u_i(s_i', s_{-i}).$$

(In fact, for semantic correctness, $\sigma(s_{-i} \mid s_i)$ must used on both sides of the inequality above, but technically these are equivalent inequalities.)

Summing both sides of each group of inequalities over s_i gives

$$\forall i \in \mathcal{N},\ \forall s_i' \in \mathcal{S}_i,\quad \sum_{s\in\mathcal{S}} \sigma(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s\in\mathcal{S}} \sigma(s_i, s_{-i})u_i(s_i', s_{-i}).$$

which is precisely the condition that σ is course correlated equilibrium. So, every CE is automatically CCE.