Math 241

Vector Analysis

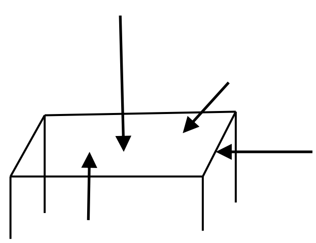
Books:
Vector analysis- by Spiegel-Second edition
Vector analysis- by Raisinghania-Third edition

Vector Algebra

The quantities are classified into two classes **Scalars** and **Vectors**

Scalars: The quantities having magnitude only are called Scalars. Examples- length, mass, temperature, time etc.

Vectors: The quantities having both magnitude and direction are called Vectors.
Examples- displacement, force, velocity, acceleration etc.
Apply a force on a table. You will ask me which direction I shall apply it.



Representation of vectors: A vector quantity can be represented by a directed line segment as shown in the fig.

$$A \xrightarrow{\mathbf{a} \text{ or } \vec{a} \text{ or } \underline{a}} B$$

The vector from the initial point A to the terminal point B is denoted by **AB**. Its length is AB or $|AB|$ called the magnitude or the modulus and its direction is from A to B as shown by arrow head.

$|\vec{a}| = a$

Unit vector-Unit vector is that vector whose magnitude is unity. Unit vector is obtained when a given vector is divided by its length. Along the vector $\vec{a}$ it is denoted by $\hat{a}$

$$\hat{a} = \frac{\vec{a}}{a} \qquad |\hat{a}| = \frac{|\vec{a}|}{a} = \frac{a}{a} = 1$$

$\vec{a} = a\hat{a}$=magnitude multiplied by the unit vector along that direction

AB=$\vec{a}$

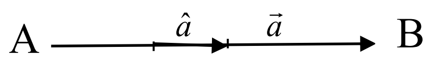


Fig1.

Equal vector- Equal vectors are those vectors which have equal magnitude and same direction. **a=b**

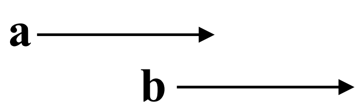


Fig2.

Like vectors-Like vectors are those vectors which have same directions but magnitude may be different.

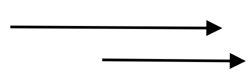


Fig3.

Unlike vectors-Unlike vectors are those vectors which have opposite directions but magnitude may be different.

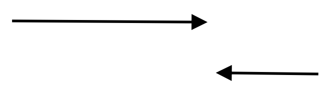


Fig4.

Negative vector- Negative vector is a vector whose magnitude is equal to that of the given vector but having opposite direction.

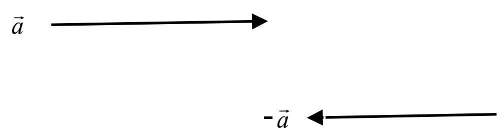


Fig5.

Zero vector or Null vector

The vector whose length or modulus is zero is called a Zero vector or Null vector and it is denoted by **0**

Addition of vectors

Triangle law of addition

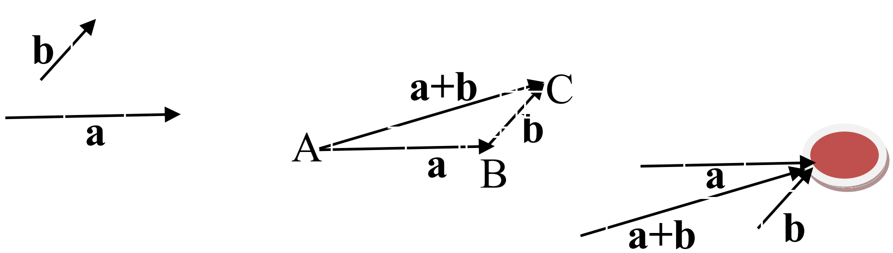


Fig6

If a particle is displaced from A to B and then B to C ultimately it is from A to C. This suggests that **AB+BC=AC**

Parallelogram law of addition

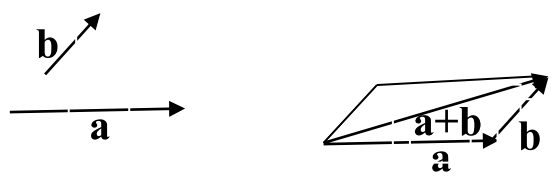


Fig7

Polygon law of addition

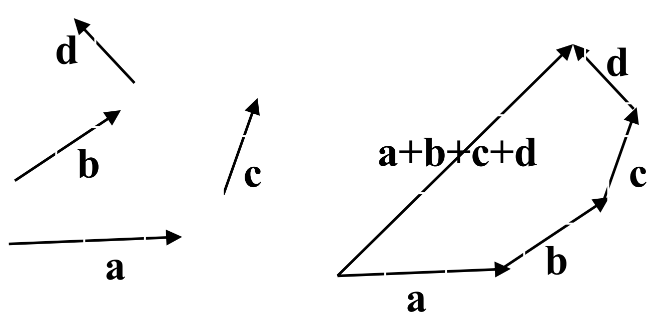


Fig8.

Position vector-Let O be the origin of the coordinate system. Then the vector **r** joining the origin to any point P is called the position vector of the point P. OP=**r** is the position vector of the point P. Sometimes denoted as P(**r**).

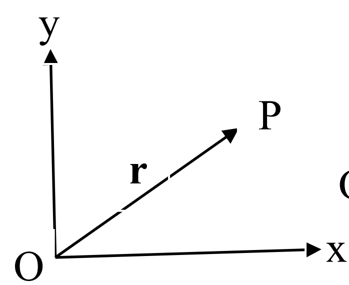


Fig9.

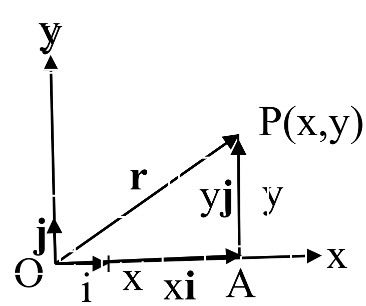
i, j, k or $\hat{i},\ \hat{j},\ \hat{k}$

Vectors in rectangular cartesian coordinates-

Let **i**, **j** be the unit vectors along x and y respectively

Then **r**=x**i**+y**j**. In three dimensions **r**=x**i**+y**j**+z**k**

$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$



OA=x**i**

AP=y**j**

OP=OA+AP

r=x**i**+y**j**

$|\vec{r}| = r = \sqrt{x^2 + y^2}$

Fig10.

AB in cartesian form-Let O be the origin of the coordinate system and the coordinates of A and B are
(a_1, a_2, a_3) and (b_1, b_2, b_3) respectively.
OA=a= a_1**i**+ a_2**j**+ a_3**k**
OB=b= b_1**i**+ b_2**j**+ b_3**k**
OA+AB=OB
AB=OB-OA= b-a=(b_1**i**+ b_2**j**+ b_3**k**)-(a_1**i**+ a_2**j**+ a_3**k**)
= **(**b_1-a_1)**i**+ (b_2- a_2)**j**+ (b_3-a_3)**k**

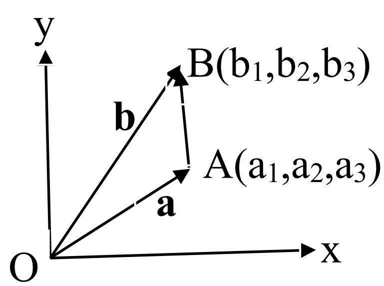
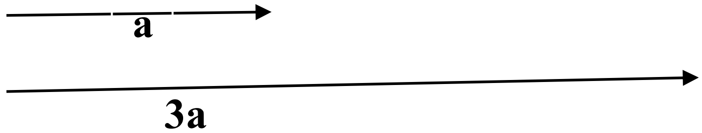


Fig11.

Scalar multiplication of vector-Let **a** be a vector and λ is a scalar then λ**a** is a vector parallel to **a** whose magnitude is λ times magnitude of **a**



if $\lambda = 3$

Fig12.

Product of vectors

1. Scalar or dot product

Def. $\mathbf{a}.\mathbf{b}$=abcosθ

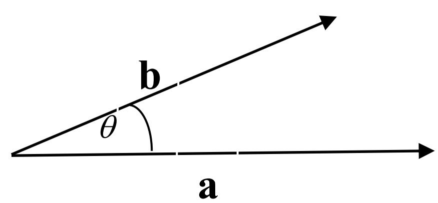


Fig12a.

$\mathbf{b}.\mathbf{a}$=bacosθ= abcosθ=$\mathbf{a}.\mathbf{b}$

If parallel $\theta = 0$
$\mathbf{a}.\mathbf{b}$=ab
If perpendicular $\theta = 90^o$
$\mathbf{a}.\mathbf{b}$=0
$\mathbf{i}.\mathbf{i}=\mathbf{j}.\mathbf{j}=\mathbf{k}\mathbf{k}=1$
$\mathbf{i}.\mathbf{j}=0$, $\mathbf{i}.\mathbf{k}=0$

Analytic form of the dot product

$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ $\quad\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$
$\mathbf{a}.\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}).(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})=a_1b_1+a_2b_2+a_3b_3$
$\mathbf{a}.\mathbf{b}$=abcosθ

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab}$$

$$\cos\theta = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

Geometrical interpretation <u>of dot product</u>

a.b=abcosθ=a(bcosθ)=$|\vec{a}|$xProjection$_{\mathbf{a}}$**b**

a.b=bacosθ=b(acosθ)=$|\vec{b}|$xProjection$_{\mathbf{b}}$**a**

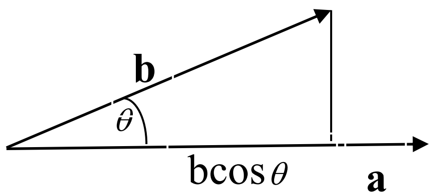


Fig13.

Physical interpretation of the dot product:
W=Force x displacement=Fd

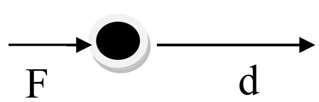


Fig14.

W=Fcosθx d=**F.d**   Fcosθ is the effective force

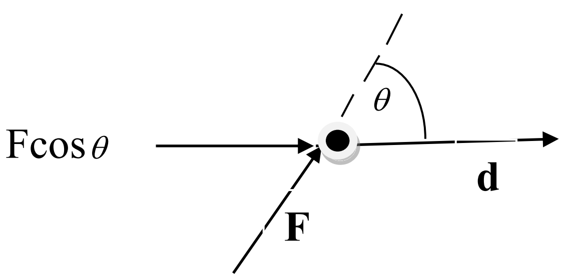
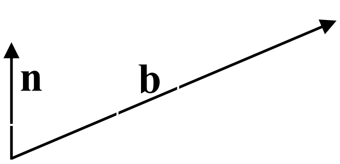


Fig15.

2. Vector or Cross product

Def. **axb**=absinθ**n**



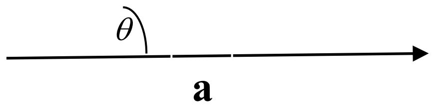
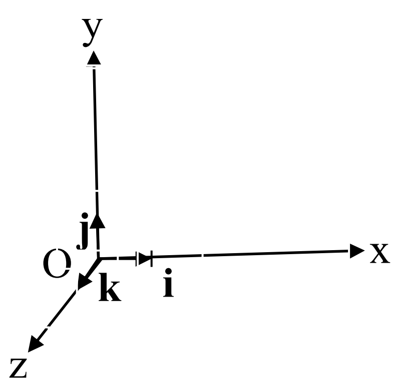


Fig 16

bxa=-absinθ**n**
axb=- bxa
if θ=0 **a**x**b**=**0** **a**x**a**=**0**
ix**i**=**j**x**j**=**k**x**k**=**0** **i**x**j**=**k** **j**x**i**=**-k**



Analytic form of the cross product
a=a_1i+a_2j +a_3k **b**=b_1i+b_2j +b_3k
axb= (a_1i+a_2j+a_3k)x(b_1i+b_2j +b_3k)
= a_1 b_2 k- a_1 b_3j-a_2b_1k+a_2b_3i+a_3b_1j-a_3b_2i
=i(a_2b_3-a_3b_2)+j(a_3b_1- a_1 b_3)+k(a_1 b_2--a_2b_1)

$$=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \left|\vec{a}x\vec{b}\right|=\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}$$

$\left|\vec{a}x\vec{b}\right|$=absin$\theta$

$$\sin\theta=\frac{\left|\vec{a}x\vec{b}\right|}{ab}$$

$$\sin\theta=\frac{\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^{\ 2}+a_2^{\ 2}+a_3^{\ 2}}\sqrt{b_1^{\ 2}+b_2^{\ 2}+b_3^{\ 2}}}$$

Geometrical interpretation of the cross product

Area of a parallelogram

If **a, b** are the adjacent sides of a parallelogram then $|\vec{a}x\vec{b}|$ represents the area of the parallelogram

$|\vec{a}x\vec{b}|$=absinθ=Area of the parallelogram ABCD having **a** and **b** as the adjacent sides.

axb=absinθ**n** = Vector area of the parallelogram ABCD having **a** and **b** as the adjacent sides.

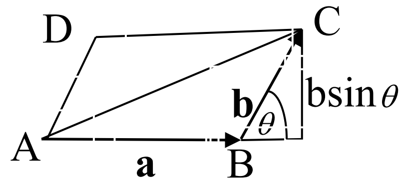


Fig17.

Area of a triangle

If **a, b** are the adjacent sides of a triangle then $\frac{1}{2}|\vec{a}x\vec{b}|$ represents the area of the triangle.

$|\vec{a}x\vec{b}|$=absinθ

Area of the triangle=$\frac{1}{2}$absinθ=$\frac{1}{2}|\vec{a}x\vec{b}|$

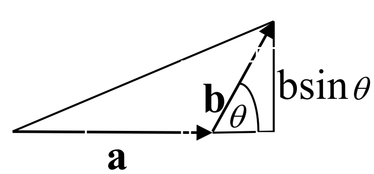


Fig18.

ABC16-11-21

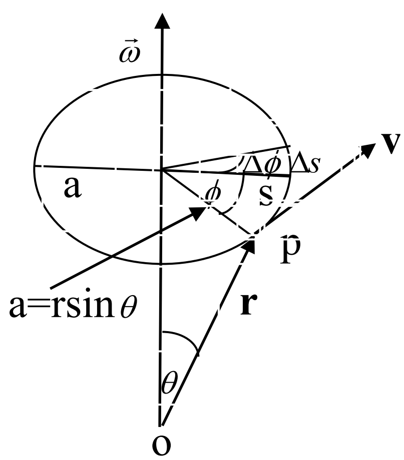
Physical interpretation of the cross product of two vectors

Let a particle be moving in a circular path about an axis passing through the centre of the circle and perpendicular to the plane of the circle.
Let the particle at P describes an arc Δs in time Δt. If $\Delta\phi$ is the central angle subtended by Δs then

$\omega = \frac{d\phi}{dt}$ =angular speed

$v = \frac{ds}{dt}$

s=aϕ



$\frac{ds}{dt} = a\frac{d\phi}{dt}$

$v = a\omega$

v is along the tangent to the circle

$\vec{\omega}$ is perpendicular to the plane of the circle

Let **r** be the position vector of P inclined at an angle θ with the direction of ω

a=rsinθ

$v = a\omega$

$v = \omega$ rsinθ

v=$\vec{\omega}$x**r**

As from the figure it is seen that **v** has the direction of ωx**r**

Moment=M=rxF

===

Triple product of vectors

1.Scalar triple product

Analytic form of the Scalar triple product

a=a_1i+a_2j +a_3k **b**=b_1i+b_2j +b_3k **c**=c_1i+c_2j +c_3k

bxc=i(b_2c_3-b_3c_2)+j(b_3c_1- b_1 c_3)+k(b_1 c_2-b_2c_1)

a.(bxc)= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1- b_1 c_3)+a_3 (b_1 c_2-b_2c_1)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Problem

Prove that in scalar triple product dot and cross can be interchanged.

a.(bxc)= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1- b_1 c_3)+a_3 (b_1 c_2-b_2c_1)

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

b.(cxa)= $\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

c.(axb)= $\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

a.(bxc)= b.(cxa)= c.(axb)= (axb).c
a.(bxc)= (axb).c
a.bxc= axb.c proved

Geometrical interpretation of the Scalar triple product
If **a, b, c** are the edges of a parallelepiped then **a.bxc** represents the volume of the parallelepiped

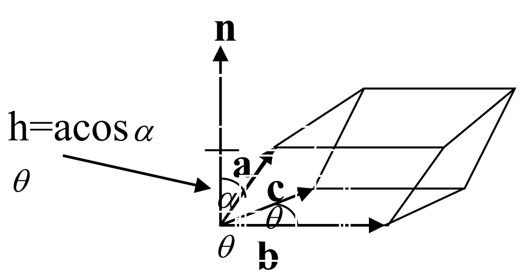


Fig 20

If A is area of the base and h is the height of the parallelepiped then volume of the parallelepiped V=Ah
Area of the base=A= bcsinθ h=acosα
bxc=bcsinθ**n**
a.(bxc)= a. bcsinθ**n**
= bcsinθ **a.n**
= bcsinθ acosα=Ah=V=Volume of the parallelepiped

For this reason it is also known as box product.

a.(bxc)=[abc] **[aab]**=0 [ijk]=ixj.k=1

[abc]=[bca]=[cab]
[abc]=-[acb]
Coplanarity condition [**abc**]=0
Problems

1. If three vectors **a**=i-j+k, **b**=2i+j-k and **c**=λi-j+λk are coplanar find the value of λ.

If a, b, c are coplanar then [**abc**]=0

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ \lambda & -1 & \lambda \end{vmatrix} = 0 \qquad 3\lambda-3=0, \quad \lambda=1$$

2. If the volume of the parallelepiped with edges **a**=2i-j-k, **b**=3i+2j+2k and **c**= 5i-λj+3λk is 4 find λ.

here [**abc**]=4 $\begin{vmatrix} 2 & -1 & -1 \\ 3 & 2 & 2 \\ 5 & -\lambda & 3\lambda \end{vmatrix} = 4 \quad 28\lambda=4 \quad \lambda = \frac{1}{7}$

Volume of a tetrahedron

If A is area of the base triangle and h is the height of a tetrahedron then volume of the tetrahedron V=$\frac{1}{3}$Ah

Area of the base triangle=A= $\frac{1}{2}$bcsinθ h=acosα

bxc=bcsinθ<u>n</u>
a.(bxc)= a. bcsinθ**n**
= bcsinθ **a.n**
= bcsinθ acosα
=2Ah
= 6$\frac{1}{3}$Ah
=6V

$V = \frac{1}{6}\,\mathbf{a.(bxc)} = \frac{1}{6}\,\mathbf{[abc]}$

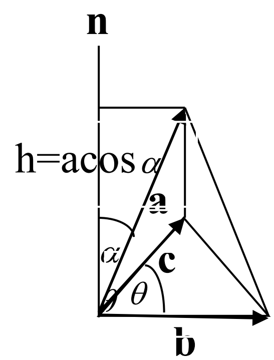


Fig 21.

Problem
Prove that [(a+b)(b+c)(c+a)] = 2[abc]
Or, Prove that the volume of the parallelepiped whose edges are the vectors **a+b, b+c, c+a** is twice the volume whose edges are **a, b, c**

[a+b b+c c+a]
=(a+b).(b+c)x(c+a)
= (a+b).(bxc+bxa+cxc+cxa)
=(a+b).(bxc+bxa+cxa)=a.bxc+a.bxa+a.cxa+b.bxc+b.bxa+b.cxa
= a.bxc+b.cxa
=[abc]+[bca]
=[abc]+[abc]
= 2[abc]

2.Vector triple product

Prove that ax(bxc)=(a.c)b- (a.b)c
hence show that ax(bxc)+ bx(cxa)+ cx(axb)=0

$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}$ $\mathbf{c}=c_1\mathbf{i}$

$\mathbf{bxc}=(b_1i+b_2j)\text{x } c_1i=-b_2c_1k$

$\mathbf{ax(bxc)}=(a_1i+a_2j+a_3k)\text{x}(-b_2c_1k)=a_1b_2c_1j-a_2b_2c_1i$

$\mathbf{(a.c)b-(a.b)c}=(a_1c_1)(b_1i+b_2j)-(a_1b_1+a_2b_2)c_1i=a_1b_2c_1j-a_2b_2c_1i$

ax(bxc)=(a.c)b- (a.b)c

----

ax(bxc)+ bx(cxa)+ cx(axb)

= (a.c)b- (a.b)c +(b.a)c-(b.c)a+(c.b)a-(c.a)b

=0

(axb)xc

=-cx(axb)

=-{(c.b)a-(c.a)b}

=(c.a)b-(c.b)a

(axb)xc=(c.a)b-(c.b)a

Product of four vectors

1. Scalar product of four vectors

(axb).(cxd)

= axb.(cxd)

= a.bx(cxd)

=a.{(b.d)c-(b.c)d}   **ax(bxc)=(a.c)b- (a.b)c**

= (b.d)(a.c)-(b.c)(a.d)

$$=\begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}$$

2. Vector product of four vectors

(axb)x(cxd) suppose axb=e
=ex(cxd) ax(bxc)=(a.c)b- (a.b)c
=(e.d)c-(e.c)d
=(axb.d)c-(axb.c)d
=[abd]c-[abc]d

(axb)x(cxd)
=-(cxd)x(axb) suppose cxd=e
=-ex(axb)
=-{(e.b)a-(e.a)b}
=-{(cxd.b)a-(cxd.a)b}
=[cda]b-[cdb]a

Problem
Prove that **[bxc cxa axb] = $[abc]^2$**
Or, Prove that the volume of the parallelepiped whose edges are the vectors **axb, bxc, cxa** is square of the volume whose edges are **a, b, c**

[(bxc)(cxa)(axb)]
=(bxc).{(cxa)x(axb)}
=(bxc).{[cab]a-[caa]b
=[cab]{(bxc).a}
=[cab][bca] [abc]=[bca]=[cab]
=[abc] [abc]
= $[abc]^2$

Problem:
Prove that **d.[ax{bx(cxd)}]=(b.d)[acd]**
d.[ax{bx(cxd)}]
= d.[ax {(b.d)c- (b.c)d}] ax(bxc)=(a.c)b- (a.b)c
=d.[(b.d)(axc)- (b.c)(axd)]

= (b.d)(d.axc)- (b.c)(d.axd)}
=(b.d)[acd]

Prove that **(bxc)x(axd)+ (cxa)x(bxd) +(axb)x(cxd)=-2[abc]d**
(bxc)x(axd)- (bxd)x(cxa)+ (axb)x(cxd)
=[bcd]a-[bca]d-[bda]c+[bdc]a+[abd]c-[abc]d
=[bcd]a-[abc]d-[bda]c-[bcd]a+[bda]c-[abc]d
=-2[abc]d

Reciprocal vectors

If $\vec{a}_1 = \frac{\vec{b}_2 x \vec{b}_3}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$ $\vec{a}_2 = \frac{\vec{b}_3 x \vec{b}_1}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$ $\vec{a}_3 = \frac{\vec{b}_1 x \vec{b}_2}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$ then the set $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are called reciprocal set of $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ ab=1 b=1/a, b is the reciprocal of a

Now

$\vec{a}_1 . \vec{b}_1 = \frac{\vec{b}_2 x \vec{b}_3 . \vec{b}_1}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]} = 1$ similarly $\vec{a}_2 . \vec{b}_2 = 1$ $\vec{a}_3 . \vec{b}_3 = 1$

$$[\vec{a}_1 \vec{a}_2 \vec{a}_3] = \frac{[\vec{b}_2 x \vec{b}_3 \ \ \vec{b}_3 x \vec{b}_1 \ \ \vec{b}_1 x \vec{b}_2]}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]^3} = \frac{(\vec{b}_2 x \vec{b}_3).\{(\vec{b}_3 x \vec{b}_1) x (\vec{b}_1 x \vec{b}_2)\}}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]^3}$$

$$= \frac{(\vec{b}_2 x \vec{b}_3).\{[\vec{b}_3 \vec{b}_1 \vec{b}_2]\vec{b}_1 - [\vec{b}_3 \vec{b}_1 \vec{b}_1]\vec{b}_2}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]^3} = \frac{[\vec{b}_1 \vec{b}_2 \vec{b}_3]^2}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]^3} = \frac{1}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$$

$[\vec{a}_1 \vec{a}_2 \vec{a}_3]\,[\vec{b}_1 \vec{b}_2 \vec{b}_3] = 1$

Problem:
Obtain the set of vectors reciprocal to the set of vectors –**i**+**j**+**k**, **i**-**j**+**k** and **i**+**j**-**k**
Let $\vec{b}_1$=–i+j+k, $\vec{b}_2$=i-j+k and $\vec{b}_3$=i+j-k

$\vec{a}_1 = \frac{\vec{b}_2 x \vec{b}_3}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$ $\vec{a}_2 = \frac{\vec{b}_3 x \vec{b}_1}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$ $\vec{a}_3 = \frac{\vec{b}_1 x \vec{b}_2}{[\vec{b}_1 \vec{b}_2 \vec{b}_3]}$

$$[\vec{b}_1\vec{b}_2\vec{b}_3]=\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}=4$$

$$\vec{b}_2 x \vec{b}_3=\begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}=2j+2k \qquad \vec{a}_1=\frac{\vec{b}_2 x \vec{b}_3}{[\vec{b}_1\vec{b}_2\vec{b}_3]}=\frac{1}{4}(2j+2k)=\frac{1}{2}(j+k)$$

Similarly $\vec{a}_2=\frac{1}{2}(i+k)$ $\vec{a}_3=\frac{1}{2}(i+j)$