Vector Calculus

Vector Differentiation

If **F**(t)=ti+t^{2}j+t^{3}k find $\frac{d\overline{F}}{dt}$, $\frac{d^2\overline{F}}{dt^2}$, $\frac{d\overline{F}}{dt} \text{x} \frac{d^2\overline{F}}{dt^2}$

$\frac{d\vec{F}}{dt}$=i+2tj+3t^{2}k $\frac{d^2\vec{F}}{dt^2}$=2j+6tk

$$\frac{d\vec{F}}{dt} \text{x} \frac{d^2\vec{F}}{dt^2} = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 i + 6tj + 2k$$

Product formula

$$\frac{d(\vec{F}.\vec{G})}{dt} = \vec{F}.\frac{d\vec{G}}{dt} + \vec{G}.\frac{d\vec{F}}{dt}$$

$$\frac{d(\vec{F}x\vec{G})}{dt} = \vec{F}x\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}x\vec{G}$$

Problem

Show that $\frac{d(\vec{P}x\vec{P}')}{dt} = \vec{P}x\vec{P}''$

$$\frac{d(\vec{P}x\vec{P}')}{dt} = \vec{P}x\vec{P}'' + \vec{P}'x\vec{P}' = \vec{P}x\vec{P}''$$

Problem of partial differentiation

If a=$e^{st}i + (2s-t)j + t\sin sk$ find $\frac{\partial^2\overline{a}}{\partial s^2}$, $\frac{\partial^2\overline{a}}{\partial t^2}$, $\frac{\partial\overline{a}}{\partial s} x \frac{\partial\overline{a}}{\partial t}$

$$\frac{\partial\overline{a}}{\partial s} = te^{st}i + 2j + t\cos sk \qquad \frac{\partial^2\overline{a}}{\partial s^2} = t^2e^{st}i - t\sin sk$$

$$\frac{\partial\overline{a}}{\partial t} = se^{st}i - j + \sin sk \qquad \frac{\partial^2\overline{a}}{\partial t^2} = s^2e^{st}i$$

$$\frac{\partial\overline{a}}{\partial s} x \frac{\partial\overline{a}}{\partial t} = \begin{vmatrix} i & j & k \\ te^{st} & 2 & t\cos s \\ se^{st} & -1 & \sin s \end{vmatrix} = (2\sin s + t\cos s)i + j(se^{st}t\cos s - e^{st}\sin s) + k(-te^{st} - 2se^{st})$$

Problem

Find a unit vector in the direction of $i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$ at (1, 0, 2)

where $f(x,y,z)=4(x^2+y^2)-z^2$

$\frac{\partial f}{\partial x}=8x \quad \frac{\partial f}{\partial y}=8y \quad \frac{\partial f}{\partial z}=-2z$

$i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$=8xi+8yj-2zk=8i-4k

Unit vector along $i\frac{\partial f}{\partial x}+j\frac{\partial f}{\partial y}+k\frac{\partial f}{\partial z}$ is $\frac{8i-4k}{\sqrt{80}}=\frac{2i-k}{\sqrt{5}}$

Gradient, Divergence and Curl

The vector operator ∇ (del or nabla) is defined by $\nabla=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$

Definitions:

1. **Gradient-** If $\phi=\phi(x,y,z)$ is scalar point function then gradient of ϕ written as gradϕ or $\nabla\phi$ is defined as

$$\nabla\phi=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\phi$$
$$=i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$$

(vector quantity)

2. **Divergence-** If **v**= **v** (x,y,z)=v_1(x,y,z)i+v_2(x,y,z)j+v_3(x,y,z)k is a vector point function then divergence of **v** written as div**v** or ∇. **v** is defined as

$$\nabla.\,\mathbf{v}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(v_1 i+v_2 j+v_3 k)$$
$$=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

(scalar quantity) $\nabla.\mathrm{v} \neq \mathrm{v}.\nabla$

3. **Curl-** If **v**= **v** (x,y,z)=v_1(x,y,z)i+v_2(x,y,z)j+v_3(x,y,z)k is a vector point function then curl of **v** written as curl**v** or ∇x **v** is defined as

$$\nabla\,\mathrm{x}\mathbf{v}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\mathrm{x}(v_1 i+v_2 j+v_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + j\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \text{ (vector quantity)}$$

Problem

1. If $f(x,y,z)=3x^2y-y^3z^2$ find ∇f at the point P(1,-2,-1)

$$\nabla f = i\,\frac{\partial(3x^2y - y^3z^2)}{\partial x} + j\,\frac{\partial(3x^2y - y^3z^2)}{\partial y} + k\,\frac{\partial(3x^2y - y^3z^2)}{\partial z}$$

$=6xyi+(3x^2-3y^2z^2)+2y^3zk$

$\nabla f\,(1,-2,-1) =-12i-9j-16k$

2. Ex. If $\phi = \frac{1}{r}$ and $r = \sqrt{x^2 + y^2 + z^2}$ find $\nabla\phi$

$$\nabla\phi = \nabla\frac{1}{r} = i\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} + j\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y} + k\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial z}\ ,\quad r^2 = x^2 + y^2 + z^2 \quad 2r\,\frac{\partial r}{\partial x} = 2x$$

$$= -\frac{1}{r^3}xi - \frac{1}{r^3}yj - \frac{1}{r^3}zk \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$= -\frac{1}{r^3}(xi + yj + zk) \qquad = -\frac{1}{r^3}\mathbf{r}$$

3. Ex. If $\mathbf{A}=x^2z^2i-2y^2z^2j+xy^2zk$ find $\nabla.\mathbf{A}$ at P(1,-1,1)

$$\nabla.\mathbf{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2z^2) + \frac{\partial}{\partial y}(-2y^2z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$=2xz^2-4yz^2+xy^2$

$\nabla.\mathbf{A}(1,-1,1) =2+4+1=7$

4. Ex. If $\mathbf{A}=x^2z^2i-2y^2z^2j+xy^2zk$ find $\nabla \times \mathbf{A}$ at P(1,-1,1)

$$\nabla \times \mathbf{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2y^2z^2 & xy^2z \end{vmatrix}$$

$$=i\left(\frac{\partial(xy^2z)}{\partial y}-\frac{\partial(-2y^2z^2)}{\partial z}\right)+j\left(\frac{\partial(x^2z^2)}{\partial z}-\frac{\partial(xy^2z)}{\partial x}\right)+k\left(\frac{\partial(-2y^2z^2)}{\partial x}-\frac{\partial(x^2z^2)}{\partial y}\right)$$

$=(2xyz+4yz^2)i+(y^2z-2x^2z)j$

hence at P, $\nabla x\mathbf{A}=2i+j$

1. Prove that $\nabla.(\phi\mathbf{A}) = (\nabla\phi).\mathbf{A}+\phi(\nabla.\mathbf{A})$

   hence find $\nabla.(r^3\mathbf{r})$

   Let $\mathbf{A}=A_1i+A_2j+A_3k$

   $\nabla.(\phi\mathbf{A})= \nabla.(\phi A_1i+\phi A_2j+\phi A_3k)$

   $$=\frac{\partial}{\partial x}(\phi A_1)+\frac{\partial}{\partial y}(\phi A_2)+\frac{\partial}{\partial z}(\phi A_3)$$

   $$=\phi\frac{\partial A_1}{\partial x}+A_1\frac{\partial\phi}{\partial x}+\phi\frac{\partial A_2}{\partial y}+A_2\frac{\partial\phi}{\partial y}+\phi\frac{\partial A_3}{\partial z}+A_3\frac{\partial\phi}{\partial z}$$

   $$=\frac{\partial\phi}{\partial x}A_1+\frac{\partial\phi}{\partial y}A_2+\frac{\partial\phi}{\partial z}A_3+\phi\frac{\partial A_1}{\partial x}+\phi\frac{\partial A_2}{\partial y}+\phi\frac{\partial A_3}{\partial z}$$

   $$=\left(\frac{\partial\phi}{\partial x}i+\frac{\partial\phi}{\partial y}j+\frac{\partial\phi}{\partial z}k\right).(A_1i+A_2j+A_3k)+\phi\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

   $=(\nabla\phi).\mathbf{A}+\phi(\nabla.\mathbf{A})$

   $\nabla.(r^3\mathbf{r}) = (\nabla r^3).\mathbf{r}+r^3(\nabla.\mathbf{r})$

$$\nabla.\mathbf{r}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).(xi+yj+zk)$$

$$=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3$$

$$\nabla r^3=i\frac{\partial}{\partial x}(r^3)+j\frac{\partial}{\partial y}(r^3)+k\frac{\partial}{\partial z}(r^3)$$

$$=i\frac{\partial}{\partial r}(r^3)\frac{\partial r}{\partial x}+j\frac{\partial}{\partial r}(r^3)\frac{\partial r}{\partial y}+k\frac{\partial}{\partial r}(r^3)\frac{\partial r}{\partial z}\quad,\quad r^2=x^2+y^2+z^2\quad 2r\frac{\partial r}{\partial x}=2x$$

$$=3r^2\frac{x}{r}i+3r^2\frac{y}{r}j+3r^2\frac{z}{r}k\qquad \frac{\partial r}{\partial x}=\frac{x}{r}\quad\frac{\partial r}{\partial y}=\frac{y}{r}\quad\frac{\partial r}{\partial z}=\frac{z}{r}$$

$$=3r(xi+yj+zk)$$

$$=3r\,\mathbf{r}$$

$\nabla.(r^3\mathbf{r}) =(3r\mathbf{r}).\mathbf{r}+r^3(3)= 6r^3$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2\phi = \nabla\,.\,\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{Laplacian}$$

2. Prove that $\nabla \mathrm{x}(\phi \mathbf{A}) = \phi(\nabla \mathrm{x}\mathbf{A}) + \nabla\phi \mathrm{x}\mathbf{A}$

Let $\mathbf{A} = A_1 i + A_2 j + A_3 k$

$$\nabla \mathrm{x}(\phi \mathbf{A}) = \nabla \mathrm{x}(\phi A_1 i + \phi A_2 j + \phi A_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= i\left(\frac{\partial(\phi A_3)}{\partial y} - \frac{\partial(\phi A_2)}{\partial z}\right) + j\left(\frac{\partial(\phi A_1)}{\partial z} - \frac{\partial(\phi A_3)}{\partial x}\right) + k\left(\frac{\partial(\phi A_2)}{\partial x} - \frac{\partial(\phi A_1)}{\partial y}\right)$$

$$= i\left(\phi\frac{\partial A_3}{\partial y} + \frac{\partial\phi}{\partial y}A_3 - \phi\frac{\partial A_2}{\partial z} - \frac{\partial\phi}{\partial z}A_2\right)$$

$$+ j\left(\phi\frac{\partial A_1}{\partial z} + \frac{\partial\phi}{\partial z}A_1 - \phi\frac{\partial A_3}{\partial x} - \frac{\partial\phi}{\partial x}A_3\right) + k\left(\phi\frac{\partial A_2}{\partial x} + \frac{\partial\phi}{\partial x}A_2 - \phi\frac{\partial A_1}{\partial y} - \frac{\partial\phi}{\partial y}A_1\right)$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)i + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)j + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)k\right]$$

$$+ \left[\left(\frac{\partial\phi}{\partial y}A_3 - \frac{\partial\phi}{\partial z}A_2\right)i + \left(\frac{\partial\phi}{\partial z}A_1 - \frac{\partial\phi}{\partial x}A_3\right)j + \left(\frac{\partial\phi}{\partial x}A_2 - \frac{\partial\phi}{\partial y}A\right)k\right]$$

$$= \phi(\nabla \mathrm{x}\mathbf{A}) + \begin{vmatrix} i & j & k \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\nabla \mathrm{x}\mathbf{A}) + \nabla\phi \mathrm{x}\mathbf{A}$$

Alternative method

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = \sum i\frac{\partial\phi}{\partial x}$$

$$\nabla = \sum i\frac{\partial}{\partial x}$$

$$\overline{A} = A_1 i + A_2 j + A_3 k$$

$$\nabla.\overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$i.\frac{\partial\overline{A}}{\partial x} = i.\left[i\frac{\partial A_1}{\partial x} + j\frac{\partial A_2}{\partial x} + k\frac{\partial A_3}{\partial x}\right] = \frac{\partial A_1}{\partial x}$$

$$j.\frac{\partial\overline{A}}{\partial y} = j.\left[i\frac{\partial A_1}{\partial y} + j\frac{\partial A_2}{\partial y} + k\frac{\partial A_3}{\partial y}\right] = \frac{\partial A_2}{\partial y}$$

$$k.\frac{\partial\overline{A}}{\partial z} = \frac{\partial A_3}{\partial z}$$

$$\nabla.\overline{A} = i.\frac{\partial\overline{A}}{\partial x} + j.\frac{\partial\overline{A}}{\partial y} + k.\frac{\partial\overline{A}}{\partial z} = \sum i.\frac{\partial\overline{A}}{\partial x}$$

$$\nabla. = \sum i.\frac{\partial}{\partial x}$$

$$\mathbf{v} = v_1 i + v_2 j + v_3 k$$

$$\nabla\ \mathrm{x}\mathbf{v} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \mathrm{x}(\ v_1 i + v_2 j + v_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = i\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + j\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + k\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$i\,X\frac{\partial\overline{v}}{\partial x} = i\,X\frac{\partial}{\partial x}(\mathrm{v}_1\mathrm{i} + \mathrm{v}_2\mathrm{j} + \mathrm{v}_3\mathrm{k}) = \mathrm{k}\frac{\partial v_2}{\partial x} - j\frac{\partial v_3}{\partial x}$$

$$j\,X\frac{\partial\overline{v}}{\partial y} = j\,X\frac{\partial}{\partial y}(\mathrm{v}_1\mathrm{i} + \mathrm{v}_2\mathrm{j} + \mathrm{v}_3\mathrm{k}) = -\mathrm{k}\frac{\partial v_1}{\partial y} + i\frac{\partial v_3}{\partial y}$$

$$k\,X\frac{\partial\overline{v}}{\partial z} = k\,X\frac{\partial}{\partial z}(\mathrm{v}_1\mathrm{i} + \mathrm{v}_2\mathrm{j} + \mathrm{v}_3\mathrm{k}) = j\frac{\partial v_1}{\partial z} - i\frac{\partial v_2}{\partial z}$$

$$\nabla X\,\overline{v} = i\,X\frac{\partial\overline{v}}{\partial x} + j\,X\frac{\partial\overline{v}}{\partial y} + k\,X\frac{\partial\overline{v}}{\partial z} = \sum i\,X\frac{\partial\overline{v}}{\partial x}$$

$$\nabla X = \sum i\,X\frac{\partial}{\partial x}$$

Prove that

3.∇. **(FxG) = G**.curl**F-F**.curl**G**

$$\nabla.\ (\mathbf{FxG}) = \sum i.\frac{\partial}{\partial x}(\overline{F}X\overline{G}) = \sum i.\left[\frac{\partial \overline{F}}{\partial x}X\overline{G} + \overline{F}X\frac{\partial \overline{G}}{\partial x}\right]$$

$$= \sum i.\left(\frac{\partial \overline{F}}{\partial x}X\overline{G}\right) + \sum i.\left(FX\frac{\partial \overline{G}}{\partial x}\right)$$

$$= \sum G.\left(i\,X\frac{\partial \overline{F}}{\partial x}\right) + \sum F.\left(\frac{\partial \overline{G}}{\partial x}X\,i\right)$$

$$= \overline{G}.\sum i\,X\frac{\partial \overline{F}}{\partial x} - \overline{F}.\sum i\,X\frac{\partial \overline{G}}{\partial x}$$

=**G**.curl**F-F**.curl**G**

4.∇X **(FxG)=F**div**G-G**div**F+(G.** ∇)**F**-**(F.** ∇)**G**

5.∇ **(F.G)=F**xcurl**G+G**xcurl**F+(F.** ∇)**G**+**(G.** ∇)**F**

Vector analysis by Raisinghania

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Ex. 1. Prove that $\nabla \mathrm{x} \nabla\phi$=0 curlgrad$\phi$=0

$$\nabla \mathrm{x} \nabla\phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)i + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)j + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)k = 0$$

Remarks: If ∇x**v=0** then **v** must be $\nabla\phi$

Ex. 2. Prove that ∇. (∇x**A**)=0 divcurl**A=0**

∇ xA $= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)$ x(A_1i+A_2j+A_3k)

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + j\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \text{ (vector quantity)}$$

$$\nabla . (\nabla \mathrm{x}\mathbf{A}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left\{i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + j\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right\}$$

$$= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}\right)$$

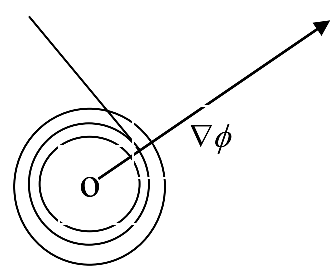
$$=0$$

Remarks: If $\nabla.\mathbf{v} = \mathbf{0}$ then **v** must be $\nabla \mathrm{x}\mathbf{A}$

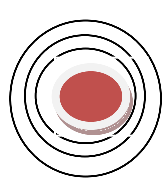
Definition:

Level surface: The family of surfaces f(x,y,z)=k is called iso-surface or level surface. For different values of c the surface such as $x^2 + y^2 + z^2 = c^2$ represents a family of concentric spheres with center at the origin and varying radius c and they constitute a level surface. The surfaces (i) of constant temperature known as isothermal surface (ii) of constant gravitational or electric potential known as equipotential surface are examples of level surfaces.

$x^2 + y^2 + z^2 = c^2$



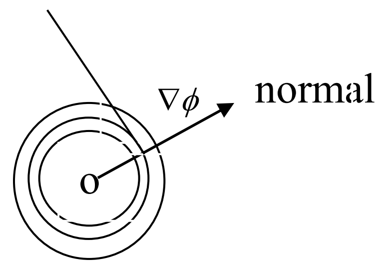
$x^2 + y^2 + z^2 = c^2$



Level curve: In two dimensions the family of curves f(x,y)=k is called level curve. For different values of c the curve such as

$x^2 + y^2 = c^2$ represents a family of concentric circles with centre at the origin and varying radius c and they constitute a level curve. The curve of constant temperature known as isothermal curve is an example of level curve.

$x^2 + y^2 = c^2$



Geometrical meaning of gradient of scalar ϕ

Theorem

Prove that ∇f is a vector perpendicular to the level surface $f(x, y, z) = c$ where c is a constant.

Let **r**=xi+yj+zk be the position vector to any point P(x, y, z) on the surface. Then d**r**=dxi+dyj+dzk lies in the tangent plane to the surface at P.

$f(x, y, z) = c$

$df = 0$

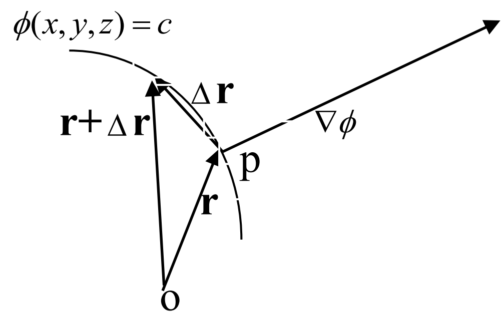
$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

$$\left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} \right).\,(dxi+dyj+dzk)=0$$

∇f .d**r** =**0**

So that ∇f is perpendicular to d**r** i.e. perpendicular to the tangent plane to the surface at (x,y,z).

Hence ∇f is a vector perpendicular to the surface $f(x, y, z) = c$ at any point (x, y, z).



Formula

The unit normal to the surface $f(x,y,z)=c$ at the point (x,y,z) is $\mathbf{n}=\frac{\nabla f}{|\nabla f|}$

Problem

Find the unit normal to the surface $x^2y+2xz=4$ at the point (2,-2, 3).

$$\nabla(x^2y+2xz)= i\frac{\partial(x^2y+2xz)}{\partial x}+j\frac{\partial(x^2y+2xz)}{\partial y}+k\frac{\partial(x^2y+2xz)}{\partial z}$$

$$=(2xy+2z)i+x^2j+2xk$$

$$=-2i+4j+4k$$

The unit normal to the surface is

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{-2i+4j+4k}{\sqrt{36}} = -\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k$$

Problem

Find the angle between the surfaces $xy^2z=3x+z^2$ and $3x^2-y^2+2z=1$ at (1, -2, 1)

Let f(x, y, z)= $xy^2z-3x-z^2$

g(x, y, z)= $3x^2-y^2+2z-1$

$$\nabla f=\nabla(xy^2z-3x-z^2)= i\frac{\partial(xy^2z-3x-z^2)}{\partial x}+j\frac{\partial(xy^2z-3x-z^2)}{\partial y}+k\frac{\partial(xy^2z-3x-z^2)}{\partial z}$$

$$=i(y^2z-3)+j2xyz+k(xy^2-2z)$$

$$=i-4j+2k$$

$\nabla g=6i+4j+2k$

$\nabla f.\ \nabla g=|\nabla f|.|\nabla g|\cos\theta$

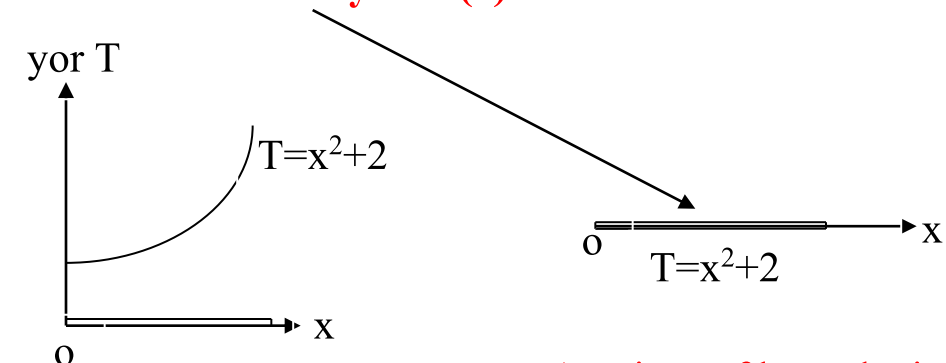
$$\cos\theta = \frac{\nabla f.\nabla g}{|\nabla f||\nabla g|} = \frac{(i-4j+2k).(6i+4j+2k)}{\sqrt{21}\quad 2\sqrt{14}} = \frac{\sqrt{6}}{14}$$

$$\theta = \cos^{-1}\frac{\sqrt{6}}{14}$$

==

Slope/gradient/rate of change in different dimensions in differential caculus

One dimensional body T=f(x)



A piece of heated wire

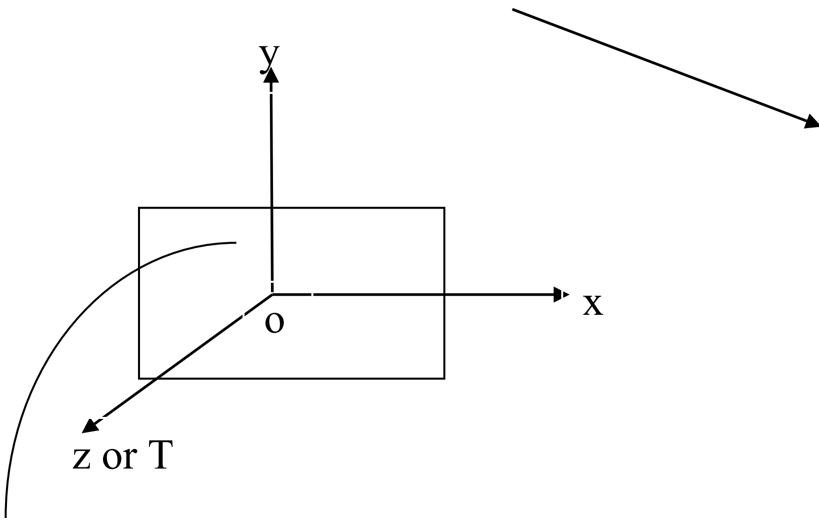
Geometrically y or T = f(x) is the equation of a curve

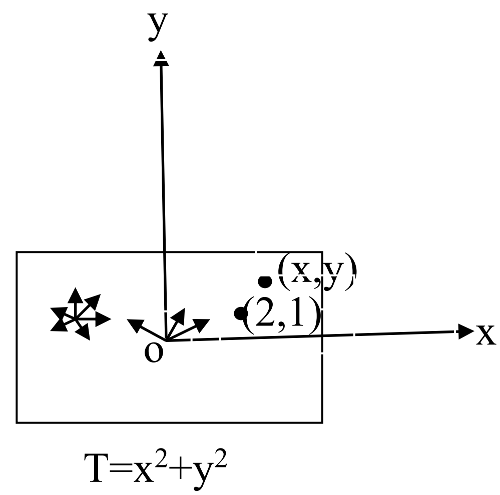
$\frac{dy}{dx}$ = rate of change of y along x-axis/directional derivative of y along x-axis/ gradient of y with respect to x,

$\frac{dy}{dx}$ =Slope of the tangent at P to the curve y = f(x)

Ex. T=f(x), $\frac{dT}{dx}$

Two dimensional body T=f(x,y)





A sheet of heated tin

Geometrically z = f(x,y) is the equation of a surface

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are the rate of changes of z along x and y-axes/directional derivative of z along x and y-axes/ gradient of z with respect to x and y,

Ex. T=f(x,y) $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$

Three dimensional body T=f(x,y,z)

u=f(x,y,z), (solid) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ are the rate of changes along x, y and z-axes/directional derivative along x, y and z-axes

Ex. T=f(x,y,z) $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$

What is the relation between Gradient of a scalar in vector calculus and Slope/gradient/rate of change in differential calculus

One dimension

$\nabla\phi = i\dfrac{\partial\phi}{\partial x}$ $\qquad \phi = \phi(x) \quad y = f(x) \quad T = T(x)$

$\nabla\phi.i = \dfrac{\partial\phi}{\partial x} =$ **Slope/gradient/rate of change of** ϕ with respect to x

in differential caculus $= \dfrac{dy}{dx} or \dfrac{dT}{dx}$

Two dimension

$\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y}$ $\qquad \phi = \phi(x, y), \; z = f(x, y), \; T = T(x, y)$

$\nabla\phi.i = \dfrac{\partial\phi}{\partial x} =$ **Slope/gradient/rate of change of** ϕ with respect to x

in differential caculus$= \dfrac{\partial z}{\partial x} or \dfrac{\partial T}{\partial x}$

$\nabla\phi.j = \dfrac{\partial\phi}{\partial y} =$ **Slope/gradient/rate of change of** ϕ with respect to y

in differential caculus $= \dfrac{\partial z}{\partial y} or \dfrac{\partial T}{\partial y}$

Three dimension

$\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$ $\qquad \phi = \phi(x, y, z) \quad u = f(x, y, z) \quad T = T(x, y, z)$

$\nabla\phi.i = \dfrac{\partial\phi}{\partial x} =$ **Slope/gradient/rate of change of** ϕ with respect to x

in differential caculus $= \dfrac{\partial u}{\partial x} or \dfrac{\partial T}{\partial x}$

$\nabla\phi.j = \dfrac{\partial\phi}{\partial y} =$ **Slope/gradient/rate of change of** ϕ with respect to y

in differential caculus $= \dfrac{\partial u}{\partial y} or \dfrac{\partial T}{\partial y}$

$\nabla\phi.k = \dfrac{\partial\phi}{\partial z} =$ **Slope/gradient/rate of change of** ϕ with respect to z

in differential caculus $= \dfrac{\partial u}{\partial z} or \dfrac{\partial T}{\partial z}$

Directional derivative

$\dfrac{\partial\phi}{\partial x}, \dfrac{\partial\phi}{\partial y}, \dfrac{\partial\phi}{\partial z}$ are called directional derivative in vector calculus

$$\nabla\phi . i = \frac{\partial\phi}{\partial x}$$

The directional derivative of ϕ along x-direction is obtained by taking dot product of $\nabla\phi$ and unit vector along x-direction

The directional derivative of ϕ along any direction is obtained by taking dot product of $\nabla\phi$ and unit vector along that direction.

The directional derivative/rate of change of ϕ along any vector **a** is $\nabla\phi . \hat{a}$ where $\hat{a}$ is unit vector along **a**.

Maximum rate of change

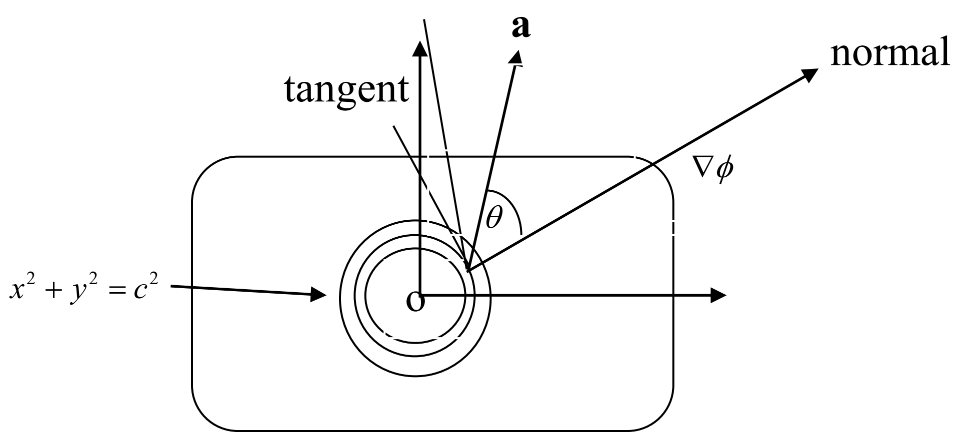
Geometrical meaning of $|\nabla\phi|$

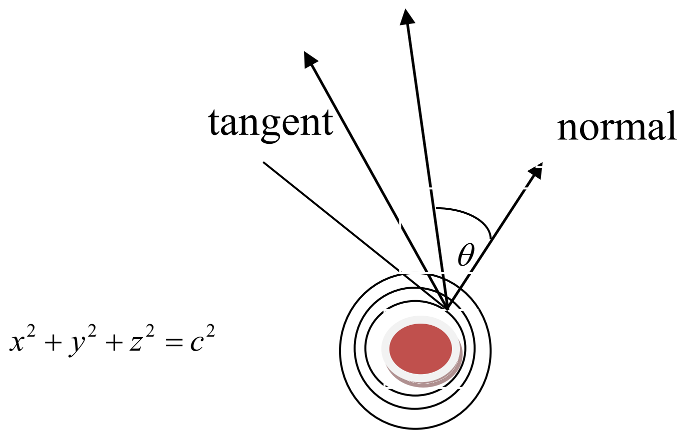
$\nabla\phi . \hat{a} = |\nabla\phi| 1 \cos\theta = |\nabla\phi| \cos\theta$ where $\hat{a}$ is unit vector along **a**

This rate of change is maximum when $\theta = 0$.

The rate of change is maximum along the normal to the surface $\phi(x, y, z) = c$ and it is equal to $|\nabla\phi|$

$$|\nabla\phi| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}$$





Problem

Find the directional derivative of ϕ=$x^2yz+4xz^2$ at P(1,-2,1) in the direction of the vector **a**=2i-j-k. Also find the maximum rate of change of ϕ.

ϕ=$x^2yz+4xz^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$$

$$=i\frac{\partial(x^2yz+4xz^2)}{\partial x}+j\frac{\partial(x^2yz+4xz^2)}{\partial y}+k\frac{\partial(x^2yz+4xz^2)}{\partial z}$$

$= i(2xyz+4z^2)+j(x^2z)+k(x^2y+8xz)$

$=j+6k$

The unit vector along **a** is $\hat{a}=\frac{\vec{a}}{|a|}=\frac{2i-j-k}{\sqrt{6}}$

the directional derivative is $\nabla\phi.\hat{a}=(j+6k).\frac{2i-j-k}{\sqrt{6}}=\frac{-7}{\sqrt{6}}$

The maximum rate of change of ϕ is

$$|\nabla\phi|=\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2}=\sqrt{0^2+1^2+6^2}=\sqrt{37}$$

==

Physical Interpretation of Curl of vector function

Ex If **v**=$\vec{\omega}$x**r**, Prove that $\vec{\omega}$=$\frac{1}{2}$curl**v** where $\vec{\omega}$ is a constant vector.

Let $\vec{\omega}=\omega_1$i+ω_2j+ω_3k be the angular velocity and **r**=xi+yj+zk

$$\omega \text{x}\mathbf{r}=\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}=(\omega_2 z-\omega_3 y)\text{i}+(\omega_3 x-\omega_1 z)\text{j}+(\omega_1 y-\omega_2 x)\text{k}$$

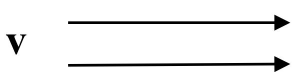
$$\text{curl}\mathbf{v}=\nabla\text{x}(\omega \text{x}\mathbf{r})=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x \end{vmatrix}$$

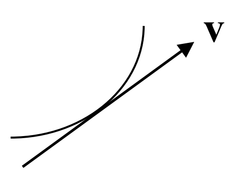
$$=2(\omega_1\text{i}+\omega_2\text{j}+\omega_3\text{k})$$

$$=2\vec{\omega}$$

$$\vec{\omega}=\frac{1}{2}\text{curl}\mathbf{v}$$

Thus physically interpreted, curl of linear velocity of any particle is twice the angular velocity of the particle.
Curl has the effect of rotation curl**A**=rot**A**





Condition of irrotatioal motion

curl**v**= $2\vec{\omega}$, If curl**v**=**0** the motion is irrotational
Also we know that $\nabla\text{x}\nabla\phi=0$,
If the motion is irrotational i. e. curl**v**=**0** then **v** must be $\nabla\phi$,
ϕ is called scalar potential function.

Ex. Determine the constant a, b, c so that vector
v=(-4x-3y+az)i+(bx+3y+5z)j+(4x+cy+3z)k is irrotational.
Find a scalar function ϕ so that **v**=$\nabla\phi$.

v is irrotational if ∇x**v**=**0**

$$\nabla \times \mathbf{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4x-3y+az & bx+3y+5z & 4x+cy+3z \end{vmatrix}$$

$=(c-5)i-(4-a)j+(b+3)k = \mathbf{0} = 0i+0j+0k$

$a=4, \quad b=-3, \quad c=5$

As $\nabla \times \mathbf{v} = \mathbf{0}$, $\mathbf{v}$ must be $\nabla\phi$

$\mathbf{v} = \nabla\phi$

$(-4x-3y+4z)i+(-3x+3y+5z)j+(4x+5y+3z)k = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = -4x-3y+4z$ ……. (1)

$\frac{\partial\phi}{\partial y} = -3x+3y+5z$ ……….(2)

$\frac{\partial\phi}{\partial z} = 4x+5y+3z$ ………….(3)

Integrating (1) with respect to x partially

$\phi = -2x^2-3xy+4xz+f(y,z)$ …..(4)

$\frac{\partial\phi}{\partial y} = -3x+\frac{\partial f(y,z)}{\partial y}$ ……….(5)

$\frac{\partial\phi}{\partial z} = 4x+\frac{\partial f(y,z)}{\partial z}$ …………(6)

Comparing (2) and (5) $\frac{\partial f(y,z)}{\partial y} = 3y+5z$ ….. (7)

Comparing (3) and (6) $\frac{\partial f(y,z)}{\partial z} = 5y+3z$ ….. (8)

Integrating (7) with respect to y, $f(y, z) = \frac{3}{2}y^2+5zy+g(z)$

$\frac{\partial f(y,z)}{\partial z} = 5y+g'(z)$ ….. (9)

Comparing (8) and (9) $g'(z)=3z \quad g(z)=\frac{3}{2}z^2+c$

therefore $f(y, z) = \frac{3}{2}y^2+5yz+\frac{3}{2}z^2+c$

hence $\phi = -2x^2-3xy+4xz+\frac{3}{2}y^2+5yz+\frac{3}{2}z^2+c$

Alternative method

$\mathbf{v}=\nabla\phi$

$(-4x-3y+4z)i+(-3x+3y+5z)j+(4x+5y+3z)k = i\dfrac{\partial\phi}{\partial x}+j\dfrac{\partial\phi}{\partial y}+k\dfrac{\partial\phi}{\partial z}$

$\dfrac{\partial\phi}{\partial x}= -4x-3y+4z \qquad \ldots\ldots. (1)$

$\dfrac{\partial\phi}{\partial y}= -3x+3y+5z \qquad \ldots\ldots\ldots.(2)$

$\dfrac{\partial\phi}{\partial z}=4x+5y+3z \qquad \ldots\ldots\ldots\ldots.(3)$

$\phi=-2x^2-3xy+4xz+f(y,z)$

$\phi= -3xy+\frac{3}{2}y^2+5yz+g(z,x)$

$\phi= 4xz + 5yz +\frac{3}{2}z^2 +h(x,y)$

$$
\begin{array}{llllllll}
\phi= & -2x^2 & -3xy+ & 4xz+ & + & + & & +f(y,z) \\
\phi= & & -3xy+ & + & \frac{3}{2}y^2 + & 5yz + & & +g(z,x) \\
\phi= & + & & +4xz+ & + & 5yz + & \frac{3}{2}z^2 & +h(x,y)
\end{array}
$$

$f(y,z)= \frac{3}{2}y^2+5yz+\frac{3}{2}z^2$

$g(z,x)= -2x^2+4xz+ \frac{3}{2}z^2$

$h(x,y)= -2x^2-3xy+ \frac{3}{2}y^2$

hence $\phi=-2x^2-3xy+4xz+\frac{3}{2}y^2+5yz+\frac{3}{2}z^2+c$

Physical interpretation of divergence of the vector function

Consider fluid motion in space. Let P(x, y, z) be any point of fluid at time t. Let $\mathbf{v}=ui+vj+wk$ be the fluid velocity at P. Construct a small rectangular box with edges of length

δx, δy, δz parallel to the respective coordinate axes, having P at one of the angular points as shown in the figure.

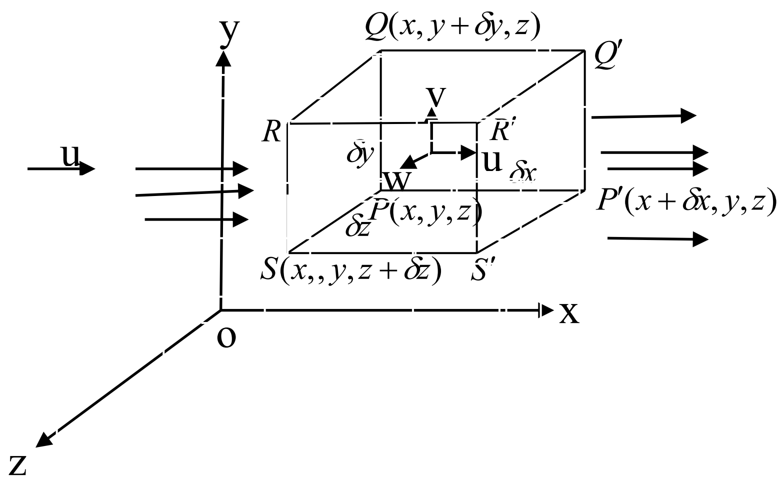


Fig18.

Then we have volume of the fluid that passes through the face $PQRS = (\delta y\,\delta z)u$ per unit time

$= f(x,y,z)$ per unit time (say)

Hence the fluid that passes out through the opposite face $P'Q'R'S' = f(x+\delta x, y, z)$ per unit time

$$= f(x,y,z) + \delta x \frac{\partial f(x,y,z)}{\partial x} + \text{terms containing higher powers of } \delta x \text{ [Taylore's theorem]}$$

[Face to face only x changes]

f(x+h)=f(x)+h $f'(x) + \frac{h^2}{2} f''(x) + \ldots\ldots\ldots$

The net outward flow from the rectangular box along x-axis per unit time$= f(x,y,z) + \delta x \frac{\partial f(x,y,z)}{\partial x} - f(x,y,z)$

$$= \delta x \frac{\partial f(x,y,z)}{\partial x} = \delta x \frac{\partial}{\partial x} u\,\delta y\,\delta z = \frac{\partial u}{\partial x}\delta x\,\delta y\,\delta z$$

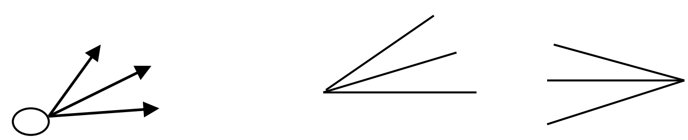
Similarly the net outward flow from the rectangular box along y-axis and z-axis per unit time will be $\frac{\partial v}{\partial y}\delta x\delta y\delta z$ and $\frac{\partial w}{\partial z}\delta x\delta y\delta z$ respectively.
Hence total net outward flow per unit time (rate of flow) through the rectangular box of volume $\delta x\delta y\delta z$ is $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta x\delta y\,\delta z$

The total rate of outward flow of the fluid through a unit volume

$$=\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta x\delta y\,\delta z}{\delta x\delta y\,\delta z}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla.\mathbf{v}$$

Thus physically interpreted div**v** represents the net outward flow of the fluid per unit volume per unit time.

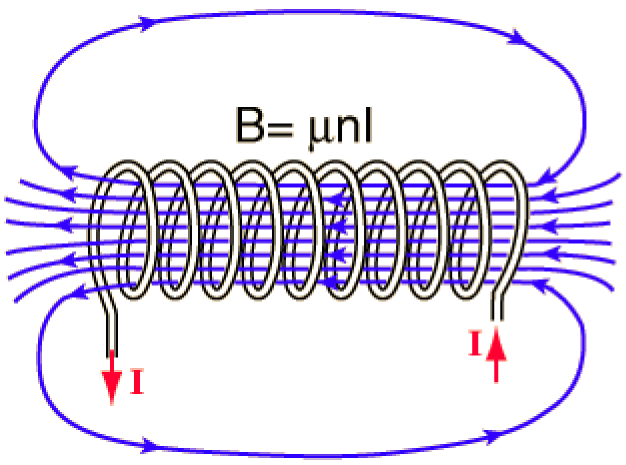
Divergence=Net outward flow



Remark: The name divergence originated in the above mentioned interpretation of div**v**

Equation of continuity of an incompressible fluid:
If the fluid is incompressible then fluid inflow =fluid outflow and there will be no net outward flow of fluid. Hence div**v=**0
This is known as condition of incompressibility of fluid or equation of continuity of incompressible fluid.

Solenoidal force field
Solen (Greek word) -pipe
Solenoid-shape of a pipe, strong magnet



The magnetic field is concentrated into a nearly uniform field in the center of a long solenoid. The field outside is weak and divergent.

Figure of solenoid

In a solenoid the magnetic force **F** is uniform, So div**F**=0
If div**F**=0 then **F** is a solenoidal
Or, if **F** is solenoidal then $\nabla.\ \mathbf{F}=0$

Problem

Ex. Determine the constant a so that vector
F =(-4x-6y+3z)i+(-2x+y-5z)j+(5x+6y+az)k is solenoidal.

F is solenoidal if $\nabla.\mathbf{F}=0$

$$\nabla.\ \mathbf{F}=\frac{\partial}{\partial x}(-4x-6y+3z)+\frac{\partial}{\partial y}(-2x+y-5z)+\frac{\partial}{\partial z}(5x+6y+az)=0$$

Or, -4+1-a=0 a=-3