PARTIAL DIFFERENTIAL EQUATIONS

WHERE IS WHAT

CONTENTS

1

Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$$\partial z / \partial x + \partial z / \partial y = z + xy \qquad \ldots (1) \qquad (\partial z / \partial x)^2 + \partial^3 z / \partial y^3 = 2x(\partial z / \partial x) \qquad \ldots (2)$$

$$z(\partial z / \partial x) + \partial z / \partial y = x \qquad \ldots (3) \qquad \partial u / \partial x + \partial u / \partial y + \partial u / \partial z = xyz \qquad \ldots (4)$$

$$\partial^2 z / \partial x^2 = (1 + \partial z / \partial y)^{1/2} \qquad \ldots (5) \qquad y\left\{(\partial z / \partial x)^2 + (\partial z / \partial y)^2\right\} = z(\partial z / \partial y) \qquad \ldots (6)$$

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p = \partial z / \partial x, \qquad q = \partial z / \partial y, \qquad r = \partial^2 z / \partial x^2, \qquad s = \partial^2 z / \partial x \partial y \qquad \text{and} \qquad t = \partial^2 z / \partial y^2$$

In case there are n independent variables, we take them to be x_1, x_2....., x_n and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2$, $p_3 = \partial z/\partial x_3$, and $p_n = \partial z/\partial x_n$.

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x \partial y$ and so on.

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 2001; 2004]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $yx^2p + xy^2q = xyz + x^2y^3$ and $p + q = z + xy$

are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = (x^2z^2/y^2)$

are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $x^2zp + y^2zp = xy$ and $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\, q = z$ and $x^2\, p^2 + y^2\, q^2 = z^2$

are all non-linear partial differential equations.

1.8 Origin of partial differential equations. We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $$F(x, y, z, a, b) = 0, \qquad ...(1)$$

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\partial F/\partial x + p(\partial F/\partial z) = 0 \quad \text{and} \quad \partial F/\partial y + q(\partial F/\partial z) = 0 \qquad ..(2)$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \qquad ...(3)$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0, \qquad \partial F / \partial x = 0 \qquad \text{and} \qquad \partial F / \partial y = 0.$$

The equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $$z = ax + y, \qquad \text{... (1)}$$
where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $\partial z / \partial x = a$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\partial z / \partial y = 1$ ... (3)

Eliminating a between (1) and (2) yields $z = x(\partial z / \partial x) + y$ ... (4)

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $$az + b = a^2 x + y \qquad \text{... (1)}$$
Differentiating (1) partially w.r.t 'x' and 'y', we have

$$a(\partial z / \partial x) = a^2 \quad \text{... (2)} \qquad a(\partial z / \partial y) = 1 \quad \text{... (3)}$$

Eliminating a from (2) and (3), we have $(\partial z / \partial x)\,(\partial z / \partial y) = 1,$

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $$z = ax + by + cxy \qquad \text{... (1)}$$
Differentiating (1) partially w.r.t., 'x' and 'y', we have

$$\partial z / \partial x = a + c\,y \quad \text{... (2)} \qquad \partial z / \partial y = b + c\,x \quad \text{... (3)}$$

From (2) and (3), $$\partial^2 z / \partial x^2 = 0, \qquad \partial^2 z / \partial y^2 = 0 \qquad \text{... (4)}$$

and $$\partial^2 z / \partial x \partial y = c \qquad \text{... (5)}$$

Now, (2) and (3) $\Rightarrow$ $x(\partial z / \partial x) = ax + cxy$ and $y(\partial z / \partial y) = by + cxy$

$$\therefore \qquad x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$$

or $$x(\partial z / \partial x) + y(\partial z / \partial y) = z + xy(\partial^2 z / \partial x \partial y), \text{ using (1) and (5)} \qquad \text{... (6)}$$

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

1.10 SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $$z = ax + by + a^2 + b^2. \qquad \text{...(1)}$$
Differentiating (1) partially with respect to x and y, we get

$$\partial z / \partial x = a \qquad \text{and} \qquad \partial z / \partial y = b.$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from* $z = (x - a)^2 + (y - b)^2$ *to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $$z = (x - a)^2 + (y - b)^2. \qquad ...(1)$$

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x - a)$ and $\partial z/\partial y = 2(y - b)$.

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\ [(x - a)^2 + (y - b)^2]$$

or $$(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z, \text{ using (1).}$$

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(a) $z = a(x + y) + b$. (b) $z = ax + by + ab$. **[Bhopal 2010, Rewa 1996]**

(c) $z = ax + a^2y^2 + b$. **[Agra 2010]** (d) $z = (x + a)\ (y + b)$. **[Madurai Kamraj 2008]**

Sol. (a) Given $$z = a(x + y) + b \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = a$.

Eliminating a between these, we get $\partial z/\partial x = \partial z/\partial y$,

which is the required partial differential equation.

(b) Given $$z = ax + by + ab. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$ ....(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\mathrm{d}x)(\partial z/\partial y),$$

which is the required partial differential equation.

(c) Try yourself. **Ans.** $\partial z/\partial y = 2y(\partial z/\partial x)^2$.

(d) Try yourself. **Ans.** $z = (\partial z/\partial y)\ (\partial z/\partial x)$.

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b$. **[Meerut 2006]**

Sol. Given $$z = axe^y + (1/2) \times a^2e^y + b. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \qquad ...(2)$$

and $$\partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2. \qquad ...(3)$$

Substituting the value of ae^y from (2) in (3), we get $\partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2$.

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$. **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $$(x - h)^2 + (y - k)^2 + z^2 = \lambda^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$2(x - h) + 2z(\partial z/\partial x) = 0 \quad \text{or} \quad (x - h) = -z(\partial z/\partial x) \qquad ...(2)$$

and $$2(y - k) + 2z(\partial z/\partial y) = 0 \quad \text{or} \quad (y - k) = -z(\partial z/\partial y). \qquad ...(3)$$

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2 \quad \text{or} \quad z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2,$$

which is the required partial differential equation.

Ex. 5(*b*). *Find the differential equation of all spheres of radius* λ, *having centre in the xy-plane*. **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$$(x - h)^2 + (y - k)^2 + (z - 0)^2 = \lambda^2 \quad \text{or} \quad (x - h)^2 + (y - k)^2 + z^2 = \lambda^2, \qquad ...(1)$$

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z = (x^2 + a)(y^2 + b)$.

[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $$z = (x^2 + a)(y^2 + b). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2x(y^2 + b) \quad \text{or} \quad (y^2 + b) = (1/2x) \times (\partial z/\partial x) \qquad ...(2)$$

and $$\partial z/\partial y = 2y(x^2 + a) \quad \text{or} \quad (x^2 + a) = (1/2y) \times (\partial z/\partial y). \qquad ...(3)$$

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x) \quad \text{or} \quad 4xyz = (\partial z/\partial x)(\partial z/\partial y),$$

which the required partial differential equation.

Ex. 7. *Form differential equation by eliminating constants A and p from* $z = A\, e^{pt} \sin px$.

Sol. Given $$z = A\, e^{pt} \sin px. \qquad ...(1)$$

Differentiating (1) partially with respect to x and t, we get

$$\partial z/\partial x = Ap\, e^{pt} \cos px \qquad ...(2) \qquad\qquad \partial z/\partial t = Ap\, e^{pt} \sin px. \qquad ...(3)$$

Differentiating (2) and (3) partially with respect to x and t respectively gives

$$\partial^2 z/\partial x^2 = -Ap^2 e^{pt} \sin px. \qquad ...(4) \qquad\qquad \partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px. \qquad ...(5)$$

Adding (4) and (5), $$\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0,$$

which is the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha, \qquad ... (1)$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$$2x = 2(z - c)(\partial z/\partial x) \tan^2 \alpha \quad \text{and} \quad 2y = 2(z - c)(\partial z/\partial y) \tan^2 \alpha$$

$$\Rightarrow \quad y(z - c)(\partial z/\partial x) \tan^2 \alpha = xy \quad \text{and} \quad x(z - c)(\partial z/\partial y) \tan^2 \alpha = xy$$

$$\Rightarrow \quad y(z - c)(\partial z/\partial x) \tan^2 \alpha = x(z - c)(\partial z/\partial y) \tan^2 \alpha$$

Thus, $y(\partial z/\partial x) = x(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is* $px + qy = z$. *Verify that* $yz + zx + xy = 0$ *is a surface satisfying the above equation.*

[I.A.S. 1979, 2009]

Sol. The equation of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad ...(1)$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p = \partial z/\partial x$ and $q = \partial z/\partial y$)

$$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0 \quad \text{or} \quad ax + gz + hy + p(cz + gx + fy) = 0 \quad ...(2)$$

and $2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0$ or $by + fz + hx + q(cz + fy + gx) = 0$. ...(3)

Multiplying (2) by x and (3) by y and adding, we have

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0.$$

$$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0, \text{ using (1)}$$

or $(cz + fy + gx)(px + qy - z) = 0$ or $px + qy - z = 0$, ...(4)

which is required partial differential equation.

Second Part : Given surface is $yz + zx + xy = 0$ ...(5)

Differentiating (5) partially w.r.t. 'x' and 'y' by turn, we get

$yp + px + z + y = 0$ and $z + qy + xq + x = 0$. ...(6)

Solving (6) for p and q, $p = -(z + y)/(x + y)$ and $q = -(z + x)/(x + y)$.

$$\therefore \quad px + qy - z = -\frac{x(z+y)}{x+y} - \frac{y(z+x)}{x+y} - z = -\frac{2(xy+yz+zx)}{x+y} = 0, \text{ using (5)}$$

Hence (5) is a surface satisfying (4).

Ex. 10. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations:*

(a) $2z = x^2/a^2 + y^2/b^2$ **[Nagpur 1995; M.D.U. Rohtak 2006]**

(b) $2z = (ax + y)^2 + b$ **[Nagpur 1996; Delhi Maths (G) 2006; Pune 2010]**

Sol. (*a*) Given $2z = x^2/a^2 + y^2/b^2$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2(\partial z/\partial x) = 2x/a^2$ ... (2) $\qquad 2(\partial z/\partial y) = 2y/b^2$ ... (3)

From (2) and (3), $p = x/a^2$, $q = y/b^2$ $\Rightarrow$ $a^2 = x/p$, $b^2 = y/q$

Substituting these values of a^2 and b^2 in (1), we get

$2z = px + qy$, which is the required partial differential equation

(b) Given $2z = (ax + y)^2 + b$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2p = 2a(ax + y)$ ... (2) $\qquad 2q = 2(ax + y)$ ... (3)

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Dividing (2) by (3) yields $p/q = a$.

Substituting this value of a in (3), we get $q = (p/q)\, x + y$ or $px + qy = q^2$.

Ex. 11. *Eliminate a, b and c from $z = a(x + y) + b(x - y) + abt + c$* **[I.A.S. 1998]**

Sol. Given $z = a(x + y) + b(x - y) + abt + c$ ... (1)

Differentiating (1) partially w.r.t. 'x', 'y' and 't', we get

$\partial z/\partial x = a + b$ ... (2) $\qquad \partial z/\partial y = a - b$ ... (3) $\qquad \partial z/\partial t = ab$ ... (4)

We have the identity: $(a + b)^2 - (a - b)^2 = 4ab$

$$\therefore \quad (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t), \text{ using (2), (3) and (4)}$$

Ex. 12. *Form the partial differential equation by eliminating the arbitrary constants a and b from log $(az - 1) = x + ay + b$.* **[I.A.S. 2002]**

Sol. (*a*) Given $\log(az - 1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial x} = 1$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial y} = a$ ... (3)

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $a = \dfrac{1 + (\partial z / \partial y)}{z}$ ... (4)

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1 + (\partial z / \partial y)}{z(\partial z / \partial y)} \frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

Ex. 13. *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2000, 03, Jiwaji 2000, Vikram 2002, 04; Ravishanker 2010]

Sol. Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{dz}{dx} = 0 \qquad \text{... (2)}$$

and
$$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0. \qquad \text{... (3)}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{... (4)} \qquad c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0. \quad \text{... (5)}$$

From (2), $c^2 = -(a^2 z / x) \times (\partial z / \partial x)$ ... (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0. \qquad \text{... (7)}$$

Similarly, from (3) and (5), $zy\dfrac{\partial^2 z}{\partial y^2} + y\left(\dfrac{\partial z}{\partial y}\right)^2 - z\dfrac{\partial z}{\partial y} = 0.$ ... (8)

Differentiating (2) partially w.r.t. y, $0 + a^2\left\{(\partial z / \partial y)\,(\partial z / \partial x) + z(\partial^2 z / \partial x \partial y)\right\} = 0$

or
$$(\partial z / \partial x)\,(\partial z / \partial y) + z(\partial^2 z / \partial x \partial y) = 0 \qquad \text{... (9)}$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 14. *Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.*

Sol. Let $lx + my + nz = a$ ... (1)

be the equation of the given plane where l, m, n are direction colines of the normal to the plane so that $l^2 + m^2 + n^2 = 1$, l, m, n being parameters ... (2)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$l + np = 0$ ...(3) $m + nq = 0$, ... (4)

where $p = \partial z / \partial x$ and $q = \partial z / \partial y$. From (3) and (4), $l = -np$ and $m = -nq$. Substituting these values in (2), we have

$n^2(p^2 + q^2 + 1) = 1$ so that $n = (p^2 + q^2 + 1)^{-1/2}$ ... (5)

$\therefore$ $l = -np = -p(p^2 + q^2 + 1)^{-1/2}$ and $m = -nq = -q(p^2 + q^2 + 1)^{-1/2}$ ... (6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

$$-px\,(p^2+q^2+1)^{-1/2} - qy\,(p^2+q^2+1)^{-1/2} + z(p^2+q^2+1)^{-1/2} = a$$

or $\quad z = px + qy + a\,(p^2+q^2+1)^{1/2}$, which is the required partial differential equation.

Ex. 15. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and c from $z = ax + g(a)\,y + c$, where $g(a)$ is an arbitrary function of a, is free of the variables x, y, z.*

Sol. Differentiating $z = ax + g(a)\,y + c$ partially w.r.t. 'x' and 'y' yields $p = a$ and $q = g(a)$. Eliminating a between them leads to $q = g(p)$ or $f(p, q) = 0$, where f is an arbitrary function of p and q. Clearly, the resulting partial differential equation contains p and q but none of the variables x, y, z.

Ex. 16. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from $z = ax + by + f(a, b)$ is given by $z = px + qy + f(p, q)$.*

Sol. Differentiating $\quad z = ax + by + f(a, b) \quad$ ... (1)

partially with respect to 'x' and 'y', we get $\quad p = a \quad$ and $\quad q = b \quad$ ... (2)

Eliminating a and b from (1) and (2) yields $\quad z = px + qy + f(p, q)$

Ex. 17. *Form a partial differential equation by eliminating a, b and c from the relation $ax^2 + by^2 + cz^2 = 1$.* **[Mysore 2004]**

Sol. Given $\quad ax^2 + by^2 + cz^2 = 1. \quad$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$$2ax + 2cz(\partial z/\partial x) = 0 \quad \ldots (2) \qquad 2by + 2cz\,(\partial z/\partial y) = 0 \quad \ldots (3)$$

Differentiating (2) partially w.r.t. 'y', we get

$$0 + 2c\left\{(\partial z/\partial y)\,(\partial z/\partial x) + z(\partial^2 z/\partial y\partial x\right\} = 0 \quad \text{or} \quad (\partial z/\partial x)\,(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0\,, \quad \ldots (4)$$

since c is an arbitrary constant. (4) is the desired partial differential equation.

Again, differentiating partially (2) w.r.t. x and (3) w.r.t. y, we get

$$2a + 2c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2)\right\} = 0 \quad \ldots (5) \qquad 2b + 2c\left\{(\partial z/\partial y)^2 + z(\partial^2 z/\partial y^2)\right\} = 0 \quad \ldots (6)$$

From (2), $a = -(cz/x)\times(\partial z/\partial x)$. Putting this in (5), we get

$$-(cz/x)\times(\partial z/\partial x) + c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2\right\} = 0 \quad \text{or} \quad zx(\partial^2 z/\partial x^2) + x(\partial z/\partial x)^2 - z(\partial z/\partial x) = 0 \quad \ldots (7)$$

Similarly, from (3) and (6), we get $\quad zy(\partial^2 z/\partial y^2) + y(\partial z/\partial y)^2 - z(\partial z/\partial y) = 0\,. \quad$ ... (8)

(4), (7) and (8) are three possible forms of the required partial differential equations.

EXERCISE 1 (A)

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential equations.

1. $z = A\,e^{pt} \sin px$, (p and A). **Ans.** $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$.

2. $z = A\,e^{-p^2 t} \cos px$, ($p$ and A) **(Sagar 1999; Ranchi 2010)** **Ans.** $\partial^2 z/\partial x^2 = dz/\partial t$

3. $z = ax^3 + by^3$; (a, b) **Ans.** $x(\partial z/\partial x) + y(\partial z/\partial y) = 3z$

4. $4z = [ax + (y/a) + b]^2$; (a, b). **(Delhi B.A. (Prog) II 2011)** **Ans.** $z = (dz/\partial x)\,(\partial z/\partial y)$

5. $z = ax^2 + bxy + cy^2$, (a, b, c) **Ans.** $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 2z$

6. $z^2 = ax^3 + by^3 + ab$, (a, b)

Ans. $9x^2y^2z = 6x^3y^2(\partial z/\partial x) + 6x^2y^3(\partial z/\partial y) + 4z(\partial z/\partial x)\,(\partial z/\partial y)$

7. $e^{1/\{z-(x^2/y)\}} = \dfrac{ax^2}{y^2} + \dfrac{b}{y}, (a, b)$ **Ans.** $x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = 2(z - x^2/y)^2$

8. Find the differential equation of the family of spheres of radius 4 with centres on the xy-plane. **Ans.** $(x-y)^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = 16(\partial z/\partial x - \partial z/\partial y)^2$

9. Find the P.D.E of planes having equal x and y intercepts. **Ans.** $p - q = 0$

10. Find the partial differential equation of the family of spheres of radius 7 with centres on the plane $x - y = 0$. **Ans.** $(p^2 + q^2 + 1)(x - y)^2 = 49(p - q)$

11. Find the partial differential equation of all spheres whose centres lie on z-axis.

Ans. $xq - yp = 0$

1.11 Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

[Meerut 1995]

Proof. Given $\phi(u, v) = 0$. ...(1)

We treat z as dependent variable and x and y as independent variables so that

$\partial z/\partial x = p$, $\partial z/\partial y = q$, $\partial y/\partial x = 0$ and $\partial x/\partial y = 0$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or
$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or
$$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right). \qquad ...(3)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \qquad ...(4)$$

Eliminating ϕ with the help of (3) and (4), we get

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or
$$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or
$$Pp + Qq = R, \qquad ...(5)$$

where $P = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}$, $Q = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}$, $R = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$.

Thus we obtain a linear partial differntial equation of first order and of first degree in p and q.

Note. If the given equation between x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

1.12 SOLVED EXAMPLES BASED ON RULE II OF ART. 1.11.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. *What is the order of this partial differential equation ?*

[Bilaspur 2003; Indore 2003; Jiwaji 2003; Vikram 2001]

Sol. Given $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. ...(1)

Let $u = x + y + z$ and $v = x^2 + y^2 - z^2$. ...(2)

Then (1) becomes $$\phi(u, v) = 0. \qquad ...(3)$$

Differentiating (3) w.r.t., 'x' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0. \qquad ...(4)$$

From (2), $\dfrac{\partial u}{\partial x} = 1,\quad \dfrac{\partial u}{\partial z} = 1,\quad \dfrac{\partial v}{\partial x} = 2x,\quad \dfrac{\partial v}{\partial z} = -2z,\quad \dfrac{\partial u}{\partial y} = 1,\quad \dfrac{\partial v}{\partial y} = 2y.$ ..(5)

From (4) and (5), $$(\partial\phi/\partial u)(1 + p) + 2(\partial\phi/\partial v)(x - pz) = 0$$

or $$(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(x - pz)/(1 + p). \qquad ...(6)$$

Again, differentiating (3) w.r.t., 'y' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

or $$(\partial\phi/\partial u)(1 + q) + 2(\partial\phi/\partial v)(y - zq) = 0, \text{ by (5)}$$

or $$(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(y - qz)/(1 + q). \qquad ...(7)$$

Eliminating ϕ from (6) and (7), we obtain

$(x - pz)/(1 + p) = (y - qz)/(1 + q)$ or $(1 + q)(x - pz) = (1 + p)(y - qz)$

or $(y + z)p - (x + z)q = x - y$, which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$. **(Kanpur 2011)**

Sol. Given $$x + y + z = f(x^2 + y^2 + z^2). \qquad ...(1)$$

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp). \qquad ...(2)$$

and $$1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq). \qquad ...(3)$$

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq)$ or $(1 + p)(y + zq) = (1 + q)(x + zp)$

or $(y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 3. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

(Sagar 1997; Vikram 1995; Jabalpur 2002)

Sol. Given $$y = f(x - at) + F(x + at). \qquad ...(1)$$

From (1), $$\partial y/\partial x = f'(x - at) + F'(x + at)$$

and hence $$\partial^2 y/\partial x^2 = f''(x - at) + F''(x + at). \qquad ...(2)$$

Also, $$\partial y/\partial t = f'(x - at) . (-a) + F'(x + at) . (a)$$

and hence $$\partial^2 y/\partial t^2 = f''(x - at) . (-a)^2 + F''(x + at) . (a)^2$$

or $$\partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)]. \qquad ...(3)$$

Then, (2) and (3) $\Rightarrow$ $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$.

Ex. 4. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2)$. **[Bilaspur 1996; Sagar 1996; Bangalore 1995]**

(*ii*) $z = f(x^2 + y^2)$. **[Meerut 1995; Pune 2010]**

Sol. (*i*) Given $$z = f(x^2 - y^2). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x$ so that $f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(x^2 - y^2) \times (-2y)$ so that $f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y)$. ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \qquad \text{or} \qquad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 5. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. **[Sagar 2000]** (*ii*) $z = x^n f(y/x)$.

Sol. Given $z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x) \times (-y/x^2)$ or $f'(y/x) = -(x^2/y) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(y/x) \times (1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ...(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \qquad \text{or} \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = n\, x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2)$ ...(2)

and $\partial z/\partial y = x^n f'(y/x) \times (1/x)$. ...(3)

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = n\, x^n f(y/x) - yx^{n-1} f'(y/x)$. ...(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = y\, x^{n-1} f'(y/x)$. ...(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = n\, x^n f(y/x)$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = nz$, by (1)

Ex. 6. *Form a partial differential equation by eliminating the function* ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$. **[Ravishankar 2003; Vikram 2003]**

Sol. Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2) \times \{2x + 2z(\partial z/\partial x)\}$ ...(2)

and $m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2) \times \{2y + 2z(\partial z/\partial y)\}$ ...(3)

Dividing (2) by (3), we get $$\frac{l + n(\partial z/\partial x)}{m + n(\partial z/\partial y)} = \frac{2\{x + z(\partial z/\partial x)\}}{2\{y + z(\partial z/\partial y)\}}$$

or $(ny - mz)(\partial z/\partial x) + (lz - nx)(\partial z/\partial y) = mx - ly$, which is the required partial differential equation.

Ex. 7. *Form partial differential eqn. by eliminating the function f from* $z = e^{ax+by} f(ax - by)$.

Sol. Given $z = e^{ax+by} f(ax - by)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = e^{ax+by}\, a\, f'(ax - by) + a\, e^{ax+by} f(ax - by)$ ...(2)

and $\partial z/\partial y = e^{ax+by} \{-b\, f'(ax - by)\} + b\, e^{ax+by} f(ax - by)$. ...(3)

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by)$ or $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$, by (1)

Ex. 8. *Form a partial differential equation by eliminating the arbitrary functions f and F from* $z = f(x + iy) + F(x - iy)$, *where* $i^2 = -1$. **[Bilaspur 2004; Jiwaji 1998; Meerut 2010]**

Sol. Given $z = f(x + iy) + F(x - iy)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x + iy) + F'(x - iy)$ ...(2)

and $\partial z/\partial y = i f'(x + iy) - iF'(x - iy)$. ...(3)

Differentiating (2) and (3) partial w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = f''(x+iy) + F''(x-iy) \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = i^2 f''(x+iy) + i^2 F''(x-iy) = -\{f''(x+iy) + F''(x+iy)\}. \qquad ...(5)$$

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$, which is the required equation.

Ex. 9. *Form partial differential equation by eliminating arbitrary functions f and g from* $z = f(x^2 - y) + g(x^2 + y)$. **[Nagpur 1996 ; I.A.S. 1996; Kanpur 2011]**

Sol. Given $$z = f(x^2 - y) + g(x^2 + y). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}. \qquad ...(2)$$

and $$\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y). \qquad ...(3)$$

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y). \qquad ...(5)$$

Again, (2) $\Rightarrow$ $f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x). \qquad ...(6)$

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2} \qquad \text{or} \qquad x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2},$$

which is the required partial differential equation.

Ex. 10. *Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation.* **[I.A.S. 1997]**

Sol. From coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$$z = \phi[(x^2 + y^2)^{1/2}], \text{ where } \phi \text{ is an arbitrary function.} \qquad ...(1)$$

Differentating (1) partially with respect to x and y, we get

$$\partial z/\partial x = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2x \qquad ...(2)$$

and $$\partial z/\partial y = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2y. \qquad ...(3)$$

Dividing (2) by (3), $$\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{x}{y} \qquad \text{or} \qquad y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}.$$

Ex. 11. *Form a partial differential equation by eliminating the arbitrary functions f and g from* $z = y\ f(x) + x\ g(y)$. (*Guwahati 2007*)

Sol. Given $$z = y\,f(x) + x\,g(y). \qquad ...(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\partial z/\partial x = y\,f'(x) + g(y)$ ...(2) $\qquad \partial z/\partial y = f(x) + x\,g'(y)$. ...(3)

Differentiating (3) with respect to x, $\quad \partial^2 z/\partial x\partial y = f'(x) + g'(y)$. ...(4)

From (2) and (3), $$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] \qquad \text{and} \qquad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right].$$

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{x\,g(y) + y\,f(x)\} \qquad \text{or} \qquad xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z, \text{ by (2)}$$

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\qquad \phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0. \qquad$ ...(1)

Let $\quad u = x^2 + y^2 + z^2 \quad$ and $\quad v = z^2 - 2xy. \qquad$ ...(2)

Then, (1) becomes $\qquad \phi(u, v) = 0. \qquad$ ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \qquad \text{...(4)}$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x, \qquad \partial u/\partial y = 2y, \qquad \partial u/\partial z = 2z, \qquad \partial v/\partial x = -2y, \qquad \partial v/\partial y = -2x, \qquad \partial v/\partial z = 2z. \qquad$ ...(5)

Using (5), (4) reduces to $\qquad (\partial \phi/\partial u)(2x + 2pz) + (\partial \phi/\partial v)(-2y + 2pz) = 0$

or $\qquad (x + pz)(\partial \phi/\partial u) = (y - pz)(\partial \phi/\partial v). \qquad$ ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $\qquad (\partial \phi/\partial u)(2y + 2qz) + (\partial \phi/\partial v)(-2x + 2qz) = 0$, by (5)

or $\qquad (y + qz)(\partial \phi/\partial u) = (x - qz)(\partial \phi/\partial v). \qquad$ ...(7)

Dividing (6) by (7), $\qquad (x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $\qquad pz(y + x) - qz(y + x) = y^2 - x^2 \qquad$ or $\qquad (p - q)z = y - x.$

Ex. 13. *Eliminate the arbitrary function f and obtain the partial differential equation from* $z = e^y f(x + y)$ **[Madras 2005]**

Sol. Given $\qquad z = e^y f(x + y) \qquad$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$\partial z/\partial x = e^y f'(x + y) \qquad$ and $\qquad \partial z/\partial y = e^y f(x + y) + e^y f'(x + y) \qquad$ ... (2)

From (1) and (2), we have $\qquad \partial z/\partial y = z + \partial z/\partial x$

Ex. 14. *If* $z = f(x + ay) + \phi(x - ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2(\partial^2 z/\partial x^2)$

Hint. Refer solved Ex. 3. **[Madurai Kamraj 2008; Jabalpur 2002]**

Ex. 15. *Equation of any cone with vertex at P* (a, b, c) *is of the form* $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right) = 0.$ *Find the differential equation of the cone.*

Sol. Let $\quad (x - a)/(z - c) = u \quad$ and $\quad (y - b)/(z - c) = v \qquad$ ... (1)

Then, the equation of the given cone becomes $\qquad f(u, v) = 0 \qquad$ ... (2)

Differentiating (2) partially with respect to 'x', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial f}{\partial u}\left(\frac{1-0}{z-c} - \frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right) = 0, \text{ using (1)}$$

or $$\frac{\partial f}{\partial u}\left(\frac{1}{z-c} - p\frac{x-a}{(z-c)^2}\right) - \frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right) = 0, \qquad \text{where} \qquad p = \frac{\partial z}{\partial x} \qquad \text{... (3)}$$

Differentiating (2) partially with respect to 'y', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1-0}{z-c} - \frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right) = 0, \text{ using (1)}$$

or $$-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0, \quad \text{where} \quad q=\frac{\partial z}{\partial y} \qquad \ldots (4)$$

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \dfrac{1}{z-c}-p\dfrac{x-a}{(z-c)^2} & -p\dfrac{y-b}{(z-c)^2} \\ -q\dfrac{x-a}{(z-c)^2} & \dfrac{1}{z-c}-q\dfrac{y-b}{(z-c)^2} \end{vmatrix}=0$$

or $$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix}=0$$

or $$\{z-c-p(x-a)\}\{z-c-q(y-b)\}-pq(x-a)(y-b)=0$$

or $$(z-c)^2-p(x-a)(z-c)-q(y-b)(z-c)=0 \qquad \text{or} \qquad (x-a)p+(y-b)q=z-c.$$

which in the required partial differential equation of the given cone.

EXERCISE 1 (B)

Eliminate the arbitrary functions and hence obtain the partial differential equations:

1. $z=e^{mx}\phi(x+y)$. **Ans.** $p-q=mz$
2. $z=f(x+ay)$ **[Bilaspur 1997; Jabalpur 1999]** **Ans.** $q=ap$
3. $z=xy+f(x^2+y^2)$ **[Delhi B.A./B.Sc. (Maths) (Prog.) 2007]** **Ans.** $py-qx=y^2-x^2$
4. $z=x+y+f(xy)$ **[Delhi B.A. (Prog) II 2010]** **Ans.** $px-qy=x-y$
5. $z=f(xy/z)$ **[Nagpur 1995 KU Kurukshetra 2004]** **Ans.** $px-qy=0$
6. $z=f(x-y)$ **[Delhi B.A. (Prog.) II 2011]** **Ans.** $p+q=0$
7. $z=(x-y)\phi(x^2+y^2)$ **Ans.** $(x-y)yp-(x-y)xq=(x+y)z$
8. $z=f(x^2+2y^2)$ **Ans.** $xq-yp=x^2-y^2$
9. $x=f(z)+g(y)$ **Ans.** $ps-qr=0$
10. $z=f(y+ax)+g(y+bx)$, $a\neq b$. **Ans.** $r-(a+b)s+abt=0$
11. $f(x+y+z)=xyz$ **Ans.** $x(y-z)p+y(z-x)q=z(x-y)$
12. $z=(x+y)f(x^2-y^2)$ **Ans.** $yp+xq=z$
13. $z=f(x)+e^y g(x)$ **Ans.** $t-q=0$
14. $f(x+y+z,\, x^2+y^2-z^2)=0$ **(CDLU 2004)** **Ans.** $p(y+z)-(x+z)q=x-y$
15. $z=f(xy)+g(x/y)$ **Ans.** $x^2(\partial^2 z/\partial x^2)-y^2(\partial^2 z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=0$
16. $z=f(x-z)+g(x+y)$ **Ans.** $\dfrac{\partial z}{\partial y}\dfrac{\partial^2 z}{\partial x^2}+\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)\dfrac{\partial^2 z}{\partial x\partial y}-\left(1-\dfrac{\partial z}{\partial x}\right)\dfrac{\partial^2 z}{\partial y^2}=0$
17. $z=f(x\cos\alpha+y\sin\alpha-at)+\phi(x\cos\alpha+y\sin\alpha+at)$.

    Ans. $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2=(1/a^2)\times(\partial^2 z/\partial t^2)$
18. $y=f(x+at)+xg(x+at)$ **Ans.** $a^2(\partial^2 z/\partial x^2)-2a(\partial^2 z/\partial x\partial t)+(\partial^2 z/\partial t^2)=0$
19. $y=f(x-at)+xg(x-at)+x^2h(x-at)$. **(Jabalpur 1994)**

    Ans. $\partial^3 y/\partial t^3+3a(\partial^3 y/\partial x\partial t^2)+3a^2(\partial^3 y/\partial x^2\partial t)+a^3(\partial^3 y/\partial x^3)=0$
20. $z=f(xy)+g(x+y)$ **Ans.** $x(y-x)r-(y^2-x^2)s+y(y-x)t+(p-q)(x+y)=0$

1.13 CAUCHY'S PROBLEM FOR FIRST ORDER EQUATIONS

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given partial differential equation has a solution at all; the next step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are easily contained in the classic problem of Cauchy, which for the two independent variables can be stated as follows:

Cauchy's problem for first order partial differential equation

If (a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ *are functions which, together with their first derivatives, are continuous in the interval I defined by* $\mu_1 < \mu < \mu_2$.

(b) And if f (x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\Phi(x, y)$ *with the following properties :*

(i) $\Phi(x, y)$ *and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.*

(ii) For all values of x and y lying in R, the point {x, y, $\Phi(x, y)$, $\Phi_x(x, y)$, $\Phi_y(x, y)$*} lies in* $\frac{U}{x}$ *and* $f[x, y, \Phi(x, y), \Phi_x(x, y), \Phi_y(x, y)] = 0$.

(iii) For all μ *belonging to the interval I, the point* $\{x_0(\mu), y_0(\mu)\}$ *belongs to the region R, and* $\Phi\{x_0(\mu), y_0(\mu)\} = z_0$

Stated geometrically, what we wish to prove is that there exists a surface $z = \Phi(x, y)$ *which passes through the curve C whose parametric equations are given by* $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$ *and at every point of which the *direction (p, q, – 1) of the normal is such that* $f(x, y, z, p, q) = 0$

Problem 1. *State the properties of* $\Phi(x, y)$ *if there exists a surface* $z = \Phi(x, y)$ *which passes through the curve C with parametric equations* $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$ *and at every point of which the direction* $(p, q, -1)$ *of the normal is such that* $f(x, y, z, p, z) = 0$. (**Delhi B.Sc. (H) 2002**)

Sol. Hint. Refer conditions (*i*), (*ii*) and (*iii*) of the above Art. 1.13

Problem 2. *Solve the Cauchy's problem for* $zp + q = 1$, *when the initial data curve is* $x_0 = \mu,\ y_0 = \mu,\ z_0 = \mu/2,\ 0 \le \mu \le 1$. **[Bangalore 2003; I.A.S. 2004]**

Sol. Given $$f(x, y, z, p, q) = zp + q - 1 = 0 \qquad \text{... (1)}$$

Given inital data curve $$x_0 = \mu, \qquad y_0 = \mu, \qquad z_0 = \mu/2, \qquad 0 \le \mu \le 1 \qquad \text{... (2)}$$

From (1), $$\partial f/\partial p = z, \qquad \partial f/\partial q = 1,$$

and $$\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu \ne 0, \text{ for } 0 \le \mu \le 1.$$

Now, we have the following ordinary differential equations :

---

*Let $$z = \Phi(x, y) \qquad \text{... (1)}$$
be the equation of the given surface

Let $$F(x, y, z) = \Phi(x, y) - z. \qquad \text{... (2)}$$

From (1) and (2), $$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p, \qquad \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q, \qquad \frac{\partial F}{\partial z} = -1$$

Since ∇F is normal to the surface $F(x, y, z) = 0$, $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ i.e., $p, q - 1$ are direction ratios of the normal to $F(x, y, z) = 0$ or $z = \Phi(x, y)$.

$$\frac{dx}{dt} = \frac{\partial f}{\partial p}, \qquad \frac{dy}{dt} = \frac{\partial f}{\partial q} \qquad \text{and} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $\qquad dx/dt = z, \qquad dy/dt = 1 \qquad$ ... (3)

and $\qquad dz/dt = p(\partial f/\partial p) + q\,(\partial f/\partial q) = pz + q = 1$, by (1) $\qquad$ ... (4)

Integrating (3) and (4), $\qquad y = t + C_1 \quad$ and $\quad z = t + C_2 \qquad$ ... (5)

From (2), at $t = 0$, $\qquad x(\mu, 0) = \mu, \qquad y(\mu, 0) = \mu \quad$ and $\quad z(\mu, 0) = \mu/2 \qquad$ ... (6)

Using (6), (5) reduces to $\qquad y = t + \mu \quad$ and $\quad z = t + \mu/2 \qquad$ ... (7)

Then, from (3) and (7), $\quad dx/dt = t + \mu/2 \quad$ so that $\quad x = (1/2)\times t^2 + (1/2)\times \mu t + C_3 \qquad$ ... (8)

Using (6), (8) reduces to $\qquad x = (1/2)\times t^2 + (1/2)\times \mu t + \mu \qquad$ ... (9)

Solving $y = t + \mu$ with (9) for μ and t in terms of x and y, we get

$$t = \frac{y - x}{1 - (y/2)} \qquad \text{and} \qquad \mu = \frac{x - (y^2/2)}{1 - (y/2)}$$

Putting these values in $z = t + \mu/2$, the required solution passing through the initial data curve is $\qquad z = \{2(y - x) + x - y^2/2\}/(2 - y)$.

OBJECTIVE PROBLEMS ON CHAPTER 1

Indicate the correct answer by writing (a), (b), (c) or (d)

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order
 (*a*) 1 (*b*) 2 (*c*) 0 (*d*) none of these **[Agra 2005, 2008]**
2. Equation $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x \partial y) + (\partial z/\partial y)^2 = 0$ is of order
 (*a*) 1 (*b*) 2 (*c*) 3 (*d*) none of these **[Agra 2005, 2006]**
3. The equation $(2x + 3y)p + 4xq - 8pq = x + y$ is
 (*a*) linear (*b*) non-linear (*c*) quasi-linear (*d*) semi-linear **[Agra 2005, 06]**
4. $(x + y - z)(\partial z/\partial x) + (3x + 2y)(\partial z/\partial y) + 2z = x + y$ is
 (*a*) linear (*b*) quasi-linear (*c*) semi-linear (*d*) non-linear

Answers **1.** (*a*) **2.** (*b*) **3.** (*b*) **4.** (*b*)

MISCELLANEOUS EXAMPLES ON CHAPTER 1

Ex.1. *Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation* $(x + a)^2 + (y + b)^2 + z^2 = 1$. *Examine whether the partial differential equation is linear or non-linear. Also,* find its order and degreee. **[Delhi Maths (H) 2008]**

Hint. Proceed as in Ex. 5(a), page 1.6 with $\lambda = 1$. *Thus we get the partial differential equation* $z^2\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1\right\} = 1$, *which is non-linear partial differential equation of order one and degree two.*

Ex. 2. *Eliminate arbitrary constants a and b from the following equations :*

(*i*) $ax^2 + by^2 + z^2 = 1$ **(Delhi B.A. (Prog.) II 2010)**

(*ii*) $z = ax + (1 - a)y + b$ **(Lucknow 2010)**

Ans. (*i*) $z(z - px - qy) = 1$ (*ii*) $p + q = 1$, where $p = \partial z/\partial x, q = \partial z/\partial y$

Ex. 3. (*i*) *Eliminate the arbitrary function* ϕ *from* $p + x - y = \phi(q - x + y)$ **(Ranchi 2010)**

(*ii*) *State true or false with justification. Eliminating arbitrary function f from $z = f(x^2 + y^2)$, we get first order non-linear partial differential equation.* **(Pune 2010)**

Ans. (*i*) $(1+\partial^2 z/\partial x^2)(1+\partial^2 z/\partial y^2) = (\partial^2 z/\partial x\partial y - 1)^2$ (*ii*) False. see Ex. 4 (ii), page 1.21.

Ex. 4. (*i*) *Obtain the partial differential equation by eliminating arbitrary function of f and g from the equation* $v = \{f(r-at) + g(r+at)\}/r$ **(Nagpur 2010)**

Ans. Given $v = (1/r) \times \{f(r-at) + g(r+at)\}$ ...(1)

(1) $\Rightarrow \partial v/\partial t = (1/r) \times \{-af'(r-at) + ag'(r+at)\} = -(a/r) \times \{f'(r-at) - g'(r+at)\}$ ...(2)

(2) $\Rightarrow \partial^2 v/\partial t^2 = -(a/r)\times\{-af''(r-at) - ag''(r+at)\} = (a^2/r)\times\{f''(r-at) + g''(r+at)\}$ ...(3)

(1) $\Rightarrow \partial v/\partial r = (1/r)\times\{(f'(r-at) + g'(r+at)\} - (1/r^2)\times\{f(r-at) + g(r+at)\}$ ...(4)

(4) $\Rightarrow \partial^2 v/\partial r^2 = (1/r)\times\{f''(r-at) + g''(r+at)\} - (1/r^2) \times \{f'(r-at) + g'(r+at)\}$

$= -(1/r^2)\times\{f'(r-at) + g'(r+at)\} + (2/r^3) \times \{f(r-at) + g(r+at)\}$

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r^2) \times \{f'(r-at)\} + g'(r+at)\} + (2/r^2) \times v$, using (1) and (3)

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times [\partial v/\partial r + (1/r^2) \times \{f(r-at)\} + g(r+at)\} + (2/r^2)\times v$

[Since from (4), $(1/r)\times\{f'(r-at) + g'(r+at)\} = \partial v/\partial r + (1/r^2)\times\{f(r-at) + g(r+at)\}$]

Thus, $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial r^2) - (2/r) \times \{\partial v/\partial r + (1/r)\times v\} + (2/r^2)\times v$, using (1)

or $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times (\partial^2/\partial r)$, which is the required equation

2

Linear Partial differential equations of order one

2.1. LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For Example $xyp + yzq = zx$ is a Lagrange equation.

2.2. Lagrange's method of solving *Pp* + *Qq* = *R*, when *P*, *Q* and *R* are functions of *x*, *y*, *z* (Delhi Maths (H) 2009; Meerut 2003; Poona 2003, 10; Lucknow 2010)

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \qquad \text{... (1)}$$

is

$$\phi(u, v) = 0 \qquad \text{... (2)}$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1 \qquad \text{and} \qquad v(x, y, z) = c_2 \qquad \text{... (3)}$$

are two independent solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{... (4)}$$

Here, c_1 *and* c_2 *are arbitrary constants and at least one of* u, v *must contain* z*. Also recall that* u *and* v *are said to be independent if* u/v *is not merely a constant.*

Proof. Differentiating (2) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \qquad \text{... (5)}$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \qquad \text{... (6)}$$

Eliminating $\partial\phi/\partial u$ and $\partial\phi/\partial v$ between (5) and (6), we have

$$\begin{vmatrix} \partial u/\partial x + p(\partial u/\partial z) & \partial v/\partial x + p(\partial v/\partial z) \\ \partial u/\partial y + q(\partial u/\partial z) & \partial v/\partial y + q(\partial v/\partial z) \end{vmatrix} = 0$$

or

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=0$$

$$\therefore \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{... (7)}$$

Hence (2) is a solution of the equation (7)

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we get

$$(\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = 0 \qquad \text{... (8)}$$

and $$(\partial v/\partial x)dx + (\partial v/\partial y)dy + (\partial v/\partial z)dz = 0 \qquad \text{... (9)}$$

Since u and v are independent functions, solving (8) and (9) for the ratios $dx : dy : dz$, gives

$$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}} = \frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}} \qquad \text{... (10)}$$

Comparing (4) and (10), we obtain

$$\frac{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}}{P} = \frac{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}}{Q} = \frac{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}{R} = k, \text{ say}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = kQ \qquad \text{and} \qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = kR$$

Substituting these values in (7), we get $\quad k(Pp + Qq) = kR \quad$ or $\quad Pp + Qq = R$, which is the given equation (1).

Therefore, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system of differential equations $(dx)/P = (dy)/Q = (dz)/R$, then $\phi(u, v) = 0$ is a solution of $Pp + Qq = R$, ϕ being an arbitrary function. This is what we wished to prove.

Note. Equations (4) are called *Lagrange's auxillary* (*or subsidiary*) equations for (1).

2.3. Working Rule for solving *Pp* + *Qq* = *R* by Lagrange's method.

[Delhi Maths Hons. 1998]

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \qquad \text{...(1)}$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{...(2)}$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0, \qquad u = \phi(v) \qquad$ or $\qquad v = \phi(u)$, ϕ being an arbitrary function.

2.4. Examples based on working rule 2.3. In what follows we shall discuss four rules for getting two independent solutions of $(dx)/P = (dy)/Q = (dz)/R$. Accordingly, we have four types of problems based on $Pp + Qq = R$.

2.5. Type 1 based on Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

2.6. SOLVED EXAMPLES BASED ON ART. 2.5

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$. **[Indore 2004; Sagar 1994]**

Sol. Given $$(y^2z/x)p + xzq = y^2. \qquad \text{...(1)}$$

The Lagrange's auxiliary equations for (1) are $\quad \dfrac{dx}{(y^2z/x)} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}. \qquad$ ..(2)

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \qquad \text{or} \qquad 3x^2dx - 3y^2dy = 0, \qquad \text{...(3)}$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get

$xy^2dx = y^2zdz$ or $2xdx - 2zdz = 0.$ ...(5)

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is

$$\phi(x^3 - y^3, x^2 - z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* (*i*) $a(p + q) = z$. **[Bangalore 1997]** (*ii*) $2p + 3q = 1$. **[Bangalore 1995]**

Sol. (*i*) Given $ap + aq = z.$ ...(1)

The Lagrange's auxiliary equation for (1) are $(dx)/a = (dy)/a = (dz)/1.$ ...(2)

Taking the first two members of (1), $dx - dy = 0.$ ...(3)

Integrating (3), $x - y = c_1$, c_1 being an arbitrary constant ...(4)

Taking the last two members of (1), $dy - adz = 0.$ ...(5)

Integrating (5), $y - az = c_2$, c_2 being an arbitrary constant. ...(6)

From (4) and (6), the required solution is given by

$$\phi(x - y, y - az) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$. **[Madras 2005 ; Kanpur 2007]**

Sol. Given $(\tan x)p + (\tan y)q = \tan z.$ ...(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}.$ ...(2)

Taking the first two fractions of (2), $\cot x \, dx - \cot y \, dy = 0.$

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1.$ ...(3)

Taking the last two fractions of (2), $\cot y \, dy - \cot z \, dz = 0.$

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2.$ ...(4)

From (3) and (4), the required general solution is

$$\sin x/\sin y = \phi(\sin y/\sin z), \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $zp = -x$.

Sol. Given $zp + 0.q = -x.$ ...(1)

The Lagrange's subsidiary equations for (1) are $(dx)/z = (dy)/0 = (dz)/(-x)$ ...(2)

Taking the first and the last members of (2), we get

$-xdx = zdz$ or $2xdx + 2zdz = 0.$ ...(3)

Integrating (3), $x^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Next, the second fraction of (2) implies that $dy = 0$ giving $y = c_2$ ...(3)

From (4) and (5), the required solution is $x^2 + z^2 = \phi(y)$, ϕ being an arbitrary function.

Ex. 5. *Solve* $y^2p - xyq = x(z - 2y)$ **[Delhi Maths Hons. 1995, Delhi Meths(G) 2006]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}.$ ...(1)

Taking the first two fractions of (1) and re–writing, we get

$2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1.$ ...(2)

Now, taking the last two fractions of (1) and re–writing, we get

$\dfrac{dz}{dy} = -\dfrac{z-2y}{y}$ or $\dfrac{dz}{dy} + \dfrac{1}{y} z = 2$ ...(3)

which is linear in z and y. Its I.F. $= e^{\int (1/y)dy} = e^{\log y} = y$. Hence solution of (3) is

$$z\,.\,y = \int 2y\,dy + c_2 \qquad \text{or} \qquad zy - y^2 = c_2. \qquad ...(4)$$

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

Ex. 6. *Solve* $(x^2 + 2y^2)p - xyq = xz$ **[K.U. Kurukshetra 2005]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \qquad ...\ (1)$$

Taking the last two fractions of (2) and re–writing, we get

$(1/y)\,dy + (1/z)dz = 0$ so that $\log y + \log z = \log c_1$ or $yz = c_1$ ... (2)

Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy} \qquad \text{or} \qquad 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y \qquad ...\ (3)$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dx$, (3) yields

$dv/dx + (2/y)\,v = -\,4y$, which is a linear equation.

Its integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and hence its solution is

$$yv^2 = \int \left\{(-\,4y)xy^2\right\} dy + c_2 \qquad \text{or} \qquad y^2x^2 + y^4 = c_2 \qquad ...\ (4)$$

From (2) and (4), the required solution is $\phi(yz,\, y^2x^2 + y^4) = 0$, ϕ being an arbitrary function.

EXERCISE 2 (A)

Solve the following partial differential equations

1. $(-a + x)p + (-b + y)q = (-c + z)$. **Ans.** $\phi\{(x-a)/(y-b), (y-b)/(z-c)\} = 0$
2. $xp + yq = z$ **(Kanpur 2011)** **Ans.** $\phi(x/z,\, y/z) = 0$
3. $p + q = 1$ **Ans.** $\phi(x - y,\, x - z) = 0$
4. $x^2p + y^2p = z^2$ **[Bilaspur 2001, Jabalpur 2000, Sagar 2000, Vikram 1999]** **Ans.** $\phi(1/x - 1/y,\, 1/y - 1/z) = 0$
5. $x^2p + y^2q + z^2 = 0$ **Ans.** $\phi(1/x - 1/y,\, 1/y + 1/z) = 0$
6. $\partial z/\partial x + \partial z/\partial y = \sin x$ **[Meerut 1995]** **Ans.** $\phi(x - y,\, z + \cos x) = 0$
7. $yzp + 2xq = xy$ **[Nagpur 1996]** **Ans.** $\phi(x^2 - z^2,\, y^2 - 4z) = 0$
8. $xp + yq = z$ **[Bangalore 1995]** **Ans.** $\phi(x/y,\, x/z) = 0$
9. $yzp + zxq = xy$ **[M.S. Univ. T.N. 2007, Lucknow 2010, Revishankar 2004]** **Ans.** $\phi(x^2 - y^2,\, x^2 - z^2) = 0$
10. $zp = x$ **Ans.** $\phi(y,\, x^2 - z^2) = 0$
11. $y^2p^2 + x^2q^2 = x^2y^2z^2$ **Ans.** $\phi(x^3 - y^3,\, y^3 + 3z^{-1}) = 0$

2.7. Type 2 based on Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one integral of (1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

2.8. SOLVED EXAMPLES BASED ON ART. 2.7

Ex. 1. *Solve* $p + 3q = 5z + \tan(y - 3x)$.

[Agra 2006; Meerut 2003; Indore 2002; Ravishankar 2003]

Sol. Given $\quad p + 3q = 5z + \tan(y - 3x)$. ...(1)

The Lagrange's subsidiary equations for (1) are $\quad \dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z+\tan(y-3x)}$. ...(2)

Taking the first two fractions, $\quad dy - 3dx = 0$. ...(3)

Integrating (3), $y - 3x = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), from (2) we get $\quad \dfrac{dx}{1} = \dfrac{dz}{5z + \tan c_1}$. ...(5)

Integrating (5), $x - (1/5) \times \log(5z + \tan c_1) = (1/5) \times c_2$, c_2 being an arbitrary constant.

or $\quad 5x - \log[5z + \tan(y - 3x)] = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 2. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $\quad xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$. ...(1)

The Lagrange's subsidiary equations for (1) are $\quad \dfrac{dx}{xz(z^2+xy)} = \dfrac{dy}{-yz(z^2+xy)} = \dfrac{dz}{x^4}$. ...(2)

Cancelling $z(z^2 + xy)$, the first two fractions give

$(1/x)\,dx = -(1/y)\,dy \quad$ or $\quad (1/x)\,dx + (1/y)\,dy = 0$. ...(3)

Integrating (3), $\quad \log x + \log y = \log c_1 \quad$ or $\quad xy = c_1$. ...(4)

Using (4), from (2) we get $\quad \dfrac{dx}{xz(z^2+c_1)} = \dfrac{dz}{x^4}$

or $\quad x^3 dx = z(z^2 + c_1)dz \quad$ or $\quad x^3 dx - (z^3 + c_1 z)dz = 0$. ...(5)

Integrating (5), $\quad x^4/4 - z^4/4 - (c_1 z^2)/2) = c_2/4 \quad$ or $\quad x^4 - z^4 - 2c_1 z^2 = c_2$

or $\quad x^4 - z^4 - 2xy\, z^2 = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$$\phi(xy,\ x^4 - z^4 - 2xy\, z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $xyp + y^2 q = zxy - 2x^2$. **[Garhwal 2005]**

Sol. Given $\quad xyp + y^2 q = zxy - 2x^2$. ...(1)

The Lagrange's subsidiary equations for (1) are $\quad \dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$. ...(2)

Taking the first two fractions of (2), we have

$(dx)/xy = (dy)/y^2 \quad$ or $\quad (1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\quad \log x - \log y = \log c_1 \quad$ or $\quad x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence from second and third fractions of (2), we get

$\dfrac{dy}{y^2} = \dfrac{dz}{c_1 z y^2 - 2c_1^2 y^2} \quad$ or $\quad c_1 dy - \dfrac{dz}{z - 2c_1^2} = 0$. ...(5)

Integrating (5), $\quad c_1 y - \log(z - 2c_1^2) = c_2 \quad$ or $\quad x - \log[z - 2(x^2/y^2)] = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$$x - \log[z - 2(x^2/y^2)] = \phi(x/y), \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $xzp + yzq = xy$. **[Bhopal 1996; Jabalpur 1999; Jiwaji 2000; Punjab 2005; Agra 2007; Ravishanker 1996; Vikram 2000]**

Sol. Given $$xzp + yzq = xy. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}. \qquad ...(2)$$

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1 y^2)dz$ or $2c_1 y\, dy - 2z\, dz = 0$. ...(5)

Integrating (5), $c_1 y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $py + qx = xyz^2 (x^2 - y^2)$.

Sol. Given $$py + qx = xyz^2 (x^2 - y^2). \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}. \qquad ...(2)$$

Taking the first two fractions of (2), $2xdx - 2ydy = 0$. ...(3)

Integrating. $x^2 - y^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), the last two fractions of (2) give

$(dy)/x = (dz)/(xyz^2 c_1)$ or $2c_1 y\, dy - 2z^{-2} dz = 0$. ...(5)

Integrating (5), $c_1 y^2 + (2/z) = c_2$, c_2 being an arbitrary constant.

or $$y^2 (x^2 - y^2) + (2/z) = c_2, \text{ using (4)}. \qquad ...(6)$$

From (4) and (6), the required general solution is

$$y^2 (x^2 - y^2) + (2/z) = \phi(x^2 - y^2), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 6. *Solve* $xp - yq = xy$ **[Madras 2005]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$(dx)/x = (dy)/(-y) = (dz)/(xy) \qquad ...(1)$$

Taking the first two fractions of (1), $(1/x)dx + (1/y)dy = 0$

Intergrating, $\log x + \log y = c_1$ so that $xy = c_1$ ... (2)

Using (2), (1) yields $(1/x)dx = (1/c_1)\, dz$ so that $\log x - \log c_2 = z/c_1$

or $\log (x/c_2) = z/c_1$ or $\log (x/c_2) = z/(xy)$, by (2)

Thus, $x/c_2 = e^{z/(xy)}$ or $xe^{-z/(xy)} = c_2$, c_2 being an arbitrary constant. ... (3)

From (2) and (3), the required solution is $x e^{-z/(xy)} = \phi(xy)$, ϕ being an arbitrary function

Ex. 7. *Solve* $p + 3q = z + \cot (y - 3x)$. **[M.D.U Rohtak 2006]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot (y - 3x)} \qquad ...(1)$$

Taking the first two fractions of (1), $dy - 3\, dx = 0$ so that $y - 3x = c_1$ ... (2)

Taking the first and last fraction of (1), we have

$$dx = \frac{dz}{z + \cot (y - 3x)} \quad \text{or} \quad dx = \frac{dz}{z + \cot c_1}, \text{ using (2)}$$

Intergrating, $x = \log | z + \cot c_1| + c_2$, c_1 and c_2 being an arbitrary constants.

or $$x - \log|z + \cot(y - 3x)| = c_2, \text{ using (2)} \qquad \text{... (3)}$$

From (2) and (3), the required general solution is

$$x - \log|z + \cot(y - 3x)| = \phi(y - 3x), \quad \phi \text{ being an arbitrary function.}$$

Ex. 8. *Solve* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$**[Delhi B.Sc. II 2008; Delhi B.A. II 2010]**

Sol. Re–writing the given equation, we have

$$x(z - 2y^2)\, p + y\,(z - y^2 - 2x^3)\, q = z(z - y^2 - 2x^3) \qquad \text{... (1)}$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \text{... (2)}$$

Taking the last two fraction, we get $(1/z)dz = (1/y)dy$

Integrating, $\log z = \log y + \log a$ or $z/y = a$ ... (3)

where a is an arbitrary constant. Using (3), (2) yields

$$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)} \quad \text{so that} \quad (ay - y^2 - 2x^3)\,dx + x\,(2y - a)\,dy = 0 \qquad \text{... (4)}$$

Comparing (4) with $Mdx + Ndy = 0$, here $M = ay - y^2 - 2x^3$ and $N = x\,(2y - a)$. Then $\partial M/\partial y = a - 2y$ and $\partial N/\partial x = 2y - a$. Now, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(2y-a)} \times 2(a-2y) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

Hence, by usual rule, integrating factor of (1) $= e^{\int(-2/x)dx} = e^{-2\log x} = e^{x^{-2}} = x^{-2}$

Multiplying (4) by x^{-2}, we get exact equation $(ayx^{-2} - y^2x^{-2} - 2x)dx + x^{-1}(2y - a)\,dy = 0$

By the usual rule of solving an exact equation, its solution is

$$\int\left\{(ay-y^2)x^{-2} - 2x\right\}dx + \int x^{-1}(2y-a)dy = b$$

(Treating y as constant) (Integrating terms free from x)

or $(ay - y^2) \times (-1/x) - x^2 = b$ or $(y^2 - ax)/x - x^2 = b$

or $(y^2 - ax - x^3)/x = b$, where b is an arbitrary constant. ... (5)

From (3) and (5), required solution is $(y^2 - ax - x^3)/x = \phi(z/y)$, ϕ being an arbitrary function

EXERCISE 2 (B)

Solved the following differential equations:

1. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^2 \sin(y+2x) - z = \phi(y+2x)$
2. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$
3. $xy^2p - y^3q + axz = 0$. **Ans.** $\log z + (ax/3y^2) = \phi(xy)$
4. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. **Ans.** $(x^2+y^2+z^2)/z = \phi(y/z)$
5. *(a)* $z(p - q) = z^2 + (x + y)^2$. **(Meerut 2011)** **Ans.** $e^{2y}[z^2 + (x+y)^2] = \phi(x+y)$

   (b) $z(p+q) = z^2 + (x-y)^2$ **Ans.** $e^{2y}[z^2 + (x-y)^2] = \phi(x-y)$
6. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^3 \sin(y+2x) - z = \phi(y+2x)$
7. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

8. $zp - zq = x + y$. **Ans.** $2x(x+y) - z^2 = \phi(x+y)$

9. $xyp + y^2q + 2x^2 - xyz = 0$. **Ans.** $x - \log|z - (2x/y)| = \phi(x/y)$

2.9. Type 3 based on Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to $(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R)$. ...(2)

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.5 or rule II of Art. 2.7 as the case may be.

2.10. SOLVED EXAMPLES BASED ON ART. 2.9

Ex.1. *Solve* $\{(b - c)/a\}yzp + \{(c - a)/b\}zxq = \{(a - b)/c\}xy$.

Sol. Given $\{(b - c)/a\}yzp + \{(c - a)/b\}zxq = \{(a - b)/c\}xy$. ...(1)

The Lagrange's subsidiary equations of (1) are $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$. ...(2)

Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,xdx + by\,dy + cz\,dz}{xyz\,[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore$ $ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2xdx + b^2ydy + c^2\,zdz}{xyz\,[a(b-c) + b(c-a) + c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2\,zdz}{0}.$$

$\therefore$ $a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0, \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 2. *Solve* $z(x + y)p + z(x - y)q = x^2 + y^2$.

Sol. Given $z(x + y)p + z(x - y)q = x^2 + y^2$. ...(1)

The Langrange's subsidiary equations for (1) are $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. ...(2)

Choosing x, $-y$, $-z$, as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore$ $x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing y, x, $-z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$\therefore$ $y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2$, c2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(x^2 - y^2 - z^2,\ 2xy - z^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$. **[Patna 2003; Madras 2005; Delhi Maths Hons. 1 9 9 1 ; Bhopal 2004; Meerut 2008, 10; Sagar 2002; I.A.S. 1977; Kanpur 2005, 06]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad ...(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + n\,dz}{l(mx - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + n\,dz}{0}.$$

$\therefore\ ldx + mdy + n\,dz = 0$ so that $l\,x + m\,y + n\,z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$$\phi\,(x^2 + y^2 + z^2,\ l\,x + m\,y + n\,z) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Sol. The lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \qquad ...(1)$$

Choosing x, y, z, as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1$. ...(2)

Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}$$

$\Rightarrow\ (1/x)dx - (1/y)dy - (1/z)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\Rightarrow\ \log\{x/(yz)\} = \log c_2 \ \Rightarrow\ x/yz = c_2$. ...(3)

$\therefore$ The required solution is $\phi\,(x^2 + y^2 + z^2,\ x/yz) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $(y - zx)p + (x + yz)q = x^2 + y^2$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y - zx} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}. \qquad ...(1)$$

Choosing $x, -y, z$ as multipliers, each fraction of (1)

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\Rightarrow$ $2xdx - 2ydy + 2zdz = 0$ so that $x^2 - y^2 + z^2 = c_1$. ...(2)

Choosing $y, x, -1$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{d(xy) - dz}{0}$$

$\Rightarrow$ $d(xy) - dz = 0$ so that $xy - z = c_2$. ...(3)

$\therefore$ From (2) and (3) solution is $\phi\,(x^2 - y^2 + z^2, xy - z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$. **[I.A.S. 2004; Agra 2005 ; Delhi Maths (H) 2006; M.S. Univ. T.N. 2007; Indore 2003; Meerut 2009; Purvanchal 2007]**

Sol. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}. \qquad ...(1)$$

Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\Rightarrow$ $\log (xyz) = \log c_1$ $\Rightarrow$ $xyz = c_1$. ...(2)

Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\Rightarrow$ $x\,dx + y\,dy - z\,dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi\,(x^2 + y^2 - 2z, xyz) = 0$, ϕ is being an arbitrary function.

Ex. 7. *Solve* $(x + 2z)q + (4zx - y)q = 2x^2 + y$. **[Meerut 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{x + 2z} = \dfrac{dy}{4zx - y} = \dfrac{dz}{2x^2 + y}$. ...(1)

Choosing $y, x, -2z$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - 2zdz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{d(xy) - 2zdz}{0}$$

$\Rightarrow$ $d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(2)

Choosing $2x, -1, -1$ as multipliers, each fraction of (1)

$$= \frac{2xdx - dy - dz}{2x(x + 2z) - (4zx - y) - (2x^2 + y)} = \frac{2xdx - dy - dz}{0}$$

$\Rightarrow$ $2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi\,(xy - z^2, x^2 - y - z) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. **[Ranchi 2010; Meerut 1994]**

If the solution of the above equation represents a sphere, what will be the coordinates of its centre.

Sol. Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(y-z)dy = (y+z)dz$ or $2ydy - 2zdz - 2(zdy + ydz) = 0.$

Integrating, $y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing x, y, z as multipliers, each fractrion of (1)

$$= \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + xy(y+z) + xz(y-z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2xdx + 2ydy + 2zdz = 0$ so that $x^2 + y^2 + z^2 = c_2$. ...(3)

From (2) and (3), solution is $\phi\,(y^2 - z^2 - 2yz,\, x^2 + y^2 + z^2) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at (0,0,0), *i.e.*, origin.

Ex. 9. *Solve* $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^2 - y^3)$. **[Jabalpur 2004; M.S. Univ. T.N. 2007]**

Sol. Here Lagrange's auxiliary equations for the given equation are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}. \qquad ...(1)$$

Taking first two fractions of (1), we have $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy$

Dividing both sides by x^3y^3 gives $\left(\dfrac{2y}{x^3} - \dfrac{1}{y^2}\right)dx = \left(\dfrac{1}{x^2} - \dfrac{2x}{y^3}\right)dy$

or $\left(\dfrac{1}{x^2}dy - \dfrac{2y}{x^3}dx\right) + \left(\dfrac{1}{y^2}dx - \dfrac{2x}{y^3}dy\right) = 0$ or $d\left(\dfrac{y}{x^2}\right) + d\left(\dfrac{x}{y^2}\right) = 0.$

Integrating, $(y/x^2) + (x/y^2) = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/3z)dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_2$

$\Rightarrow$ $\log (xy\, z^{1/3}) = \log c_2$ $\Rightarrow$ $xyz^{1/3} = c_2$. ...(3)

From (2) and (3) solution is $\phi\,(xyz^{1/3},\, y/x^2 + x/y^2) = 0$, ϕ being an arbitrary function.

Ex. 10. *Solve* $x^2p + y^2q = nxy$. **[Ravishankar 1998; Bhopal 1998; Jabalpur 2002]**

Sol. Here Lagrange's auxiliary equations are $(dx)/x^2 = (dy)/y^2 = (dz)/nxy$ ... (1)

Taking the first two fractions of (1), we get $x^{-2}dx - y^{-2}dy = 0.$

Integrating, $-1/x + 1/y = -c_1$ so that $(y-x)/xy = c_1$. ...(2)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of (2)

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + y - x}, \text{ by (2)}$$

$$= \frac{(1/x)dx + (1/y)dy + (c_1/n)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0.$$

Integrating, $\log x - \log y + (c_1/n)\, z = (c_1/n)c_2$, c_2 being an arbitrary constant.

or $z - (n/c_1)\,(\log y - \log x) = c_2$ or $z - (n/c_1)\log (y/x) = c_2$

or $$z - \frac{nxy}{y-x}\,\log\frac{y}{x} = c_2, \text{ using (2)}. \qquad ...(3)$$

From (2) and (3), the required general solution is

$$\phi\left(\frac{y-x}{xy}, z-\frac{nxy}{y-x}\log\frac{y}{x}\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 11. *Solve* $(x-y)p + (x+y)q = 2xz$.

Sol. Here the Lagrange's subsidiary equations are $\qquad \dfrac{dx}{x-y} = \dfrac{dy}{x+y} = \dfrac{dz}{2xz}$. ...(1)

Taking the first two fractions of (1), $\qquad \dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1+(y/x)}{1-(y/x)}$. ...(2)

Let $\quad y/x = v \quad$ *i.e.*, $\quad y = xv$. ...(3)

From (3), $\quad (dy/dx) = v + x(dv/dx)$. ...(4)

Using (3) and (4), (2) gives $\qquad v + x\dfrac{dv}{dx} = \dfrac{1+v}{1-v}$

or $$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

or $\quad \dfrac{1-v}{1+v^2}\, dv = \dfrac{dx}{x} \quad$ or $\quad \left(\dfrac{2}{1+v^2} - \dfrac{2v}{1+v^2}\right)dv = \dfrac{2dx}{x}$

Integrating, $\quad 2\tan^{-1} v - \log(1+v^2) = 2\log x - \log c_1$

or $\quad \log x^2 - \log(1+v^2) - \log c_1 = 2\tan^{-1} v$

or $\quad \log\{x^2(1+v^2)/c_1\} = 2\tan^{-1} v \quad$ or $\quad x^2(1+v^2) = c_1 e^{2\tan^{-1} v}$

or $\quad x^2[1+(y^2/x^2)] = c_1 e^{2\tan^{-1}(y/x)}$, as $v = y/x$ by (3)

or $\quad (x^2+y^2)\, e^{-2\tan^{-1}(y/x)} = c_1$, c_1 being an arbitrary constant. ...(5)

Choosing $1, 1, -1/z$ as multipliers, each fraction of (1)

$$= \frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)} = \frac{dx+dy-(1/z)dz}{0}$$

$\Rightarrow \; dx + dy - (1/z)dz = 0 \quad$ so that $\quad x + y - \log z = c_2$. ...(6)

From (5) and (6), the required general solution is

$$\phi\left(x+y-\log z,\ (x^2+y^2)\, e^{-2\tan^{-1}(y/x)}\right) = 0, \quad \text{where } \phi \text{ is an arbitrary function.}$$

Ex. 12. *Solve* $y^2p + x^2q = x^2y^2z^2$.

Sol. Here Lagrange's auxiliary equations are $\qquad (dx)/y^2 = (dy)/x^2 = (dz)/x^2y^2z^2$. ...(1)

Taking the first two fractions of (1), we have

$3x^2dx - 3y^2dy = 0 \quad$ so that $\quad x^3 - y^3 = c_1$. ...(2)

Choosing $x^2, y^2, -2/z^2$ as multipliers, each fraction of (1) $= \{x^2dx + y^2dy - (2/z^2)dz\}/0$

so that $\quad 3x^2dx + 3y^2dy - (6/z^2)dz = 0$.

Integrating, $\quad x^3 + y^3 + (6/z) = c_2$, c_2 being an arbitrary constant. ...(3)

From (2) and (3), the required general solution is

$$\phi\,[x^3 - y^3,\ x^3 + y^3 + (6/z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $(3x+y-z)p + (x+y-z)q = 2(z-y)$. **[Bangalore 1992]**

Sol. Here Lagrange's auxiliary equations are $\qquad \dfrac{dx}{3x+y-z} = \dfrac{dy}{x+y-z} = \dfrac{dz}{2(z-y)}$ ...(1)

Choosing 1, –3, 1 as multipliers, each ratio of (1) $= \{dx - 3dy - dz\}/0$

so that $$dx - 3dy - dz = 0.$$

Integrating, $x - 3y - z = c_1$, c_1 being an arbitrary constant. ...(2)

From (2), $$z = c_1 - x + 3y. \quad ...(3)$$

Substituting the above value of z, the first two fractions of (2) reduce to

$$\frac{dx}{3x+y-(c_1-x+3y)} = \frac{dy}{x+y-(c_1-x+3y)} \quad \text{or} \quad \frac{dx}{2x+4y+c_1} = \frac{dy}{4y+c_1}. \quad ...(3)$$

Let $u = 4y + c_1$ so that $dy = (1/4) \times du$. ...(4)

Then, (3) $\Rightarrow$ $\dfrac{dx}{2x+u} = \dfrac{(1/4)du}{u}$ or $\dfrac{dx}{du} = \dfrac{1}{4}\dfrac{2x+u}{u}$ or $\dfrac{dx}{du} - \dfrac{1}{2u}x = \dfrac{1}{4}$, which is linear. ...(5)

Integrating factor of (5) $= e^{-\int (1/2u)du} = e^{-(1/2)\log u} = e^{\log(u)^{-1/2}} = u^{-1/2} = 1/\sqrt{u}$.

Hence solution of (5) is $$x \times \frac{1}{\sqrt{u}} = \int \frac{1}{4}\frac{1}{\sqrt{u}}\,du + c = \frac{1}{2}\sqrt{u} + c_2$$

or $$\frac{2x-u}{\sqrt{u}} = c_2 \quad \text{or} \quad \frac{2x-(4y+c_1)}{\sqrt{4y+c_1}} = c_2, \text{ by (4)}$$

or $$\frac{2x-4y-(x-3y-z)}{\sqrt{4y+x-3y-z}} = c_2, \text{ using (2)} \quad \text{or} \quad \frac{x-y+z}{\sqrt{x+y-z}} = c_2 \quad ...(6)$$

From (2) and (6), the required general solution is

$$\phi\left(x-3y-z, (x-y+z)/\sqrt{x+y-z}\right) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$. **[Delhi Maths Hons 95, 2000]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)} = \frac{dz}{2z(y^2-x^2)}. \quad ...(1)$$

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0.$$

Integrating, $\log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(2)

Taking the first two ratios of (1), $$\frac{dy}{dx} = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\left(\frac{y}{x}\right)\frac{3+(y/x)^2}{1+3(y/x)^2}. \quad ...(3)$$

Put $y/x = v$ or $y = xv$ so that $(dy/dx) = v + x(dv/dx)$. ...(4)

Using (4), (3) reduces to $v + x\dfrac{dv}{dx} = -v\dfrac{3+v^2}{1+3v^2}$ or $x\dfrac{dv}{dx} = -v\left[\dfrac{3+v^2}{1+3v^2}+1\right]$

or $$x\frac{dv}{dx} = -\frac{4(1+v^2)v}{1+3v^2} \quad \text{or} \quad 4\frac{dx}{x} + \frac{1+3v^2}{v(1+v^2)}dv = 0$$

or $$4\frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2}\right)dv, \text{ on resolving into partial fractions}$$

Integrating, $4\log x + \log v + \log(1 + v^2)$ or $x^4 v(1 + v^2) = c_2'$

$$x^4 (y/x)[1+(y/x)^2]=c_2' \quad \text{or} \quad xy(x^2+y^2)=c_2' \quad \text{or} \quad c_1 z(x^2+y^2)=c_2', \text{ by (2)}$$

or $\quad z(x^2+y^2) = c_2'/c_1 \quad$ or $\quad z(x^2+y^2)=c_2$, where $c_2 = c_2'/c_1$. ... (5)

$\therefore$ From (2) and (5) solution is $\phi(z(x^2+y^2), xy/z) = 0$, ϕ being an arbitrary function.

Ex. 15. *Solve* $(y-z)p + (z-x)q = x-y$. **[Agra 2010; Delhi Maths Hons. 1992]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$. ...(1)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)} = \frac{dx+dy+dz}{0}.$$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = c_2$ ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x+y+z, x^2+y^2+z^2) = 0$, ϕ being an arbitrary of function.

Ex. 16. *Solve the general solution of the equation* $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$.

[Delhi B.Sc. (Prog) II 2011; GATE 2001; Delhi Math Hons. 1997, 98]

Sol. Given $(y + zx)p - (x + yz)q = x^2 - y^2$. ...(1)

Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+zx} = \dfrac{dy}{-(x+yz)} = \dfrac{dz}{x^2-y^2}$. ...(2)

Choosing $x, y, -z$ as multipliers, each fraction of (2)

$$= \frac{xdx+ydy-zdz}{x(y+zx)-y(x+yz)-z(x^2-y^2)} = \frac{xdx+ydy-zdz}{0}$$

$\therefore$ $xdx + ydy - zdz = 0$ so that $2xdx + 2ydy - 2zdz = 0$.

Integrating, $x^2 + y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Choosing y, x, 1 as multipliers, each fraction of (2)

$$= \frac{ydx+xdy+dz}{y(y+zx)-x(x+yz)+x^2-y^2} = \frac{ydx+xdy+dz}{0}$$

$\therefore$ $ydx + xdy + dz = 0$ or $d(xy) + dz = 0$.

Integrating, $xy + z = c_2$, c_2 being an arbitrary constant. ...(4)

$\therefore$ The required solution is $\phi(x^2 + y^2 - z^2, xy + z) = 0$, ϕ being an arbitrary function.

Ex. 17. *Solve* $x(y-z)\,p + y\,(z-x)q = z(x-y)$, *i.e.*, $\{(y-z)/(yz)\}p + \{(z-x)/(zx)\}q = (x-y)/(xy)$. **[Delhi B.A (Prog) II 2010; I.A.S. 2005, M.S. Univ. T.N. 2007; Vikram 2003]**

Sol. Given $x(y-z)p + y(z-x)q = z(x-y)$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ ... (2)

Choosing $1/x$, $1/y$, $1/z$ as multipliers each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{(y-z)+(z-x)+(x-y)} = \frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore$ $\log (xyz) = c_1$ or $xyz = c_1$ ... (3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(xy - xz) + (yz - yx) + (zx - zy)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow$ $dx + dy + dz = 0$ so that $x + y + z = c_2$ ... (4)

From (3) and (4), solution is $\phi(x + y + z, xyz) = 0$, ϕ being an arbitrary function.

Ex. 18. *Solve* $2y(z - 3)p + (2x - z)q = y(2x - 3)$ **[Delhi Math (H) 1999]**

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x - z} = \frac{dz}{y(2x-3)} \quad \text{... (1)}$$

Taking the first and third fractions, $(2x - 3)dx = 2(z - 3)dz$.

Integrating, $x^2 - 3x = z^2 - 6z + C_1$ or $x^2 - 3x - z^2 + 6z = C_1$ ... (2)

Choosing 1, 2y, –2 as multipliers, each fraction of (1)

$$= \frac{dx + 2ydy - 2dz}{2y(z-3) + 2y(2x - z) - 2y(2x-3)} = \frac{dx + 2ydy - 2dz}{0}$$

$\therefore$ $dx + 2ydy - 2dz = 0$ so that $x + y^2 - 2z = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 - 3x - z^2 + 6z, x + y^2 - 2z) = 0$, ϕ being an arbitrary function.

Ex. 19. *Solve* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi Maths (H) 2001]**

Sol. Re-writing the given equation $x^2p + y^2q = (x + y)z$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$ ... (2)

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$.

Integrating, $-(1/x) + (1/y) = C_1$ or $1/y - 1/x = C_1$ ... (3)

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (2)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{x + y - (x + y)} = \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0}$$

$\therefore$ $(1/x)dx + (1/y)dy - (1/z)dz = 0$ so that $xy/z = C_2$ ... (4)

From (3) and (4), solution is $\Phi(1/y - 1/x, xy/z) = 0$, Φ being an arbitrary function.

Ex. 20. *Solve* $z(x + 2y)\, p - z(y + 2x)\, q = y^2 - x^2$ **[Vikram 1999]**

Sol. The Lagrange's subsidiary equations are $\dfrac{dx}{z(x+2y)} = \dfrac{dy}{-z(y+2x)} = \dfrac{dz}{y^2 - x^2}$ ... (1)

Taking the first two fraction of (1), we have

$(y + 2x)\, dx + (x + 2y)\, dy = 0$ or $2xdx + 2y\, dy + d\,(xy) = 0$

Integrating, $x^2 + y^2 + xy = C_1$, C_1 being an arbitrary constant ... (2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{(x^2z + 2xyz) - (y^2z + 2xyz) + (zy^2 - zx^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 + y^2 + z^2, x^2 + y^2 + xy) = 0$, ϕ being an arbitrary function

EXERCISE 2(C)

Solve the following partial differential equations:

1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ **Ans.** $\phi(x^2 + y^2 + z^2, xyz) = 0$

[Mysore 2004, Delhi B.Sc. (Prog). II 2007, M.S. Unit. T.N. 2007]

2. $z(xp - yq) = y^2 - x^2$ **Ans.** $\phi(x^2 + y^2 + z^2, xy) = 0$

3. $(y^2 + z^2)p - xyq + xz = 0$ **[I.A.S. 1990]** **Ans.** $\phi(x^2 + y^2 + z^2, y/z) = 0$

4. $yp - xq = 2x - 3y$ **[M.S. Univ. T.N. 2007]** **Ans.** $\phi(x^2 + y^2, 3x + 2y + z) = 0$

5. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ **Ans.** $\phi(xyz, 1/x + 1/y + 1/z) = 0$

[Meerut 2007, Bilaspur 2004, Rewa 2003]

2.11. Type 4 based on Rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to $(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R)$. ...(2)

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2 dx + Q_2 dy + R_2 dz)/(P_2 P + Q_2 Q + R_2 R) \qquad ...(3)$$

is such that its numerrator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of Art. 2.5 or rule 2 of Art. 2.7 or rule 3 of Art. 2.9.

2.12. SOLVED EXAMPLES BASED IN ART. 2.11

Ex. 1. *Solve* $(y + z)p + (z + x)q = x + y$. **[Indore 2000; Jabalpur 2000, Jiwaji 2002, Kanpur 2008; Purvanchal 2007, Ravishankar 2002, 2005; Delhi BA (Prog.) II 2011]**

Sol. Here the Lagrange's auxiliary equations are
$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}. \qquad ...(1)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{(y+z)-(z+x)} = \dfrac{d(x-y)}{-(x-y)}$. ...(2)

Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \dfrac{dy - dz}{(z+x)-(x+y)} = \dfrac{d(y-z)}{-(y-z)}$. ...(3)

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(4)$$

(2), (3) and (4) $\Rightarrow$
$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(5)$$

Taking the first two fractions of (5),
$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}.$$

Integrating, $\log(x - y) = \log(y - z) + \log c_1$, c_1 being an arbitrary constant.

or $\log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ...(6)

Taking the first and the third fractions of (5), $2\,\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2\log(x-y) + \log(x+y+z) = \log c_2$ or $(x-y)^2(x+y+z) = c_2$. ...(7)

From (6) and (7), the required general solution is

$$\phi\,[(x-y)^2(x+y+z),\ (x-y)/(y-z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ **[Delhi Maths Hons 1997; Nagpur 2010]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}. \qquad ...(1)$$

Taking the first two fractions of (1), $x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$.

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary as constant. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ...(4)

From (3) and (4), solution is $\phi\,(x^3+y^3,\ (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $(x^2-y^2-z^2)p + 2xyq = 2xz$ or $(y^2+z^2-x^2)p - 2xyq = -2xz$.

[Bangalore 1993, I.A.S. 1973; P.C.S. (U.P.) 1991; Bhopal 2010]

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dz}{-2xz} \quad \text{or} \quad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log(x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi\,(y/z,\ (x^2+y^2+z^2)/z) = 0$, ϕ being an arbitrary function.

Ex. 4. *Solve* $(1+y)p + (1+x)q = z$. **[M.S. Univ. T.N. 2007; Kanpur 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. ...(1)

Taking the first two fractions of (1), we have

$(1+x)dx = (1+y)dy$ or $2(1+x)dx - 2(1+y)dy = 0$.

Integrating, $(1+x)^2-(1+y)^2=c_1$, c_1 being an arbitrary constant. ...(2)

Taking 1, 1, 0 as multipliers, each fraction of (1) $=\dfrac{dx+dy}{1+y+1+x}=\dfrac{d(2+x+y)}{2+x+y}$. ...(3)

Combining the last fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y)}{2+x+y}=\frac{dz}{z} \quad \text{or} \quad \frac{d(2+x+y)}{2+x+y}-\frac{dz}{z}=0.$$

Integrating, $\log(2+x+y)-\log z=\log c_2$ or $(2+x+y)/z=c_2$. ...(4)

From (2) and (4), the required general solution is given by

$$\phi\,[(1+x)^2-(1+y^2),\ (2+x+y)/z]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 5. *Find the general integral of* $xzp+yzq=xy$.

Sol. Here the Lagrange's auxiliary equations are $(dx)/xz=(dy)/yz=(dz)/xy$ ... (1)

From the first two fractions of (1), $(1/x)dx=(1/y)dy$.

Integrating, $\log x=\log y+\log c_1$ or $x/y=c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1) $=\dfrac{(1/x)dx+(1/y)dy}{(1/x)xz+(1/y)yz}=\dfrac{ydx+xdy}{2xyz}$ ...(3)

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx+xdy}{2xyz}=\frac{dz}{xy} \quad \text{or} \quad ydx+xdy=2zdz \quad \text{or} \quad d(xy)=2zdz \quad \text{or} \quad d(xy)-2zdz=0$$

Integrating, $xy-z^2=c_2$, c_2 being an arbitrary constant. ...(4)

From (2) and (4) solution is $\phi\,(x/y,\ xy-z^2)=0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $(x^2-yz)p+(y^2-zx)q=z^2-xy$. **Delhi Math (H) 2005, 11, M.D.U. Rohtak 2005; Agra 2008, 09; Guwahati 2007; Meerut 2006; Sagar 2000; Ravishankar 2000; Lucknow 2010]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}$. ...(1)

Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

so that $\dfrac{dx-dy}{(x-y)(x+y+z)}=\dfrac{dy-dz}{(y-z)(y+z+x)}$ or $\dfrac{d(x-y)}{x-y}-\dfrac{d(y-z)}{y-z}=0.$

Integrating, $\log(x-y)-\log(y-z)=\log c_2$ or $(x-y)/(y-z)=c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}. \quad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}. \quad ...(4)$$

From (3) and (4), $\dfrac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz$

or $2(x+y+z)\,d(x+y+z)-(2xdx+2ydy+2zdz)=0.$

Integrating, $(x+y+z)^2-(x^2+y^2+z^2)=2c_2$

or $(x^2+y^2+z^2+2xy+2yz+2zx)-(x^2+y^2+z^2)=2c_2$

or $xy+yz+zx=c_2$, c_2 being an arbitrary constant. ...(5)

From (2) and (5), the required general solution is given by

$\phi\,[xy + yz + zx, (x - y)/(y - z)] = 0$, ϕ being an arbitrary function.

Ex. 7. *Solve* $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.

Sol. Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)} \qquad ...(1)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 - y^2 - yz) - (x^2 - y^2 - zx)} = \frac{dx - dy}{z(x - y)}. \qquad ...(2)$$

Choosing x, $-y$, 0 as multipliers each fraction of (1)

$$= \frac{xdx - ydy}{x(x^2 - y^2 - yz) - y(x^2 - y^2 - zx)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)}. \qquad ...(3)$$

From (1), (2), (3) we have

$$\frac{dz}{z(x - y)} = \frac{dx - dy}{z(x - y)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{2xdx - 2ydy}{2(x^2 - y^2)}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ so that $z - x + y = c_1$ ...(5)

Again, taking the first and third fractions of (4), $d(x^2 - y^2)/(x^2 - y^2) - (2/z)\,dz = 0$

Integrating, $\log (x^2 - y^2) - 2 \log z = c_2$ or $(x^2 - y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

Sol. Here the Lagrange's, auxiliary equations are $\dfrac{dx}{x^2 + y^2 + yz} = \dfrac{dy}{x^2 + y^2 - xz} = \dfrac{dz}{z(x + y)}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x + y)}. \qquad ...(2)$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)}. \qquad ...(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z(x + y)} = \frac{dx - dy}{z(x + y)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{xdx + ydy}{x^2 + y^2}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ or $dz - dx + dy = 0$.

Integrating, $z - x + y = c_1$, c_1 being an arbitrary constant. ...(5)

Taking the first and third fractions of (4), we have

$$\frac{2xdx + 2ydy}{x^2 + y^2} = 2\frac{dz}{z} \quad \text{or} \quad \frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0.$$

Integrating, $\log (x^2 + y^2) - 2\log z = \log c_2$ or $(x^2 + y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 9. *Solve* $\cos(x+y)p + \sin(x+y)q = z$. **[Garhwal 2010, Vikram 1998; Meerut 2007; Delhi Maths (H) 2007; Rajasthan 1994; Delhi B.A./B.Sc. (Prog.) Maths 2007]**

Sol. Here the Lagrange's auxiliary equations are $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad ...(1)$$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(2)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $$= \frac{dx-dy}{\cos(x+y)-\sin(x+y)}. \qquad ...(3)$$

From (1), (2) and (3), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}. \qquad ...(4)$$

Taking the first two fractions of (4), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(5)$$

Putting $x+y=t$ so that $d(x+y)=dt$, (5) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{(1/\sqrt{2})\cos t+(1/\sqrt{2})\sin t\right\}} = \frac{dt}{\sqrt{2}\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $$(\sqrt{2}/z)dz = \operatorname{cosec}\,(t+\pi/4)\ dt.$$

Integrating, $\sqrt{2}\log z = \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$, or $z^{\sqrt{2}} = c_1 \tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$

or $$z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1. \text{ as } t = x+y \qquad ...(6)$$

Taking the last two fraction of (4), $$dx-dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y). \qquad ...(7)$$

On R.H.S. of (7), putting $x+y=t$, so that $d(x+y)=dt$, (7) reduces to

$dx-dy = \dfrac{\cos t-\sin t}{\cos t+\sin t}dt.$ so that $x-y = \log(\sin t+\cos t) - \log c_2$

or $(\sin t+\cos t)/c_2 = e^{x-y}$ or $e^{-(x-y)}(\sin t+\cos t) = c_2$

or $$e^{y-x}[\sin(x+y)+\cos(x+y)] = c_2, \quad \text{as } t = x+y. \qquad ...(8)$$

From (6) and (8), the required general solution is

$$\phi\left[z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right] = 0,$$ where ϕ is an arbitrary function.

Ex. 10. *Solve* $\cos(x+y)p + \sin(x+y)q = z + (1/z)$. **[Delhi B.A. (Prog.) 2011]**

Sol. Do like Ex. 9. **Ans.** $$\phi\left[(z^2+1)^{1/\sqrt{2}}\tan\left(\frac{3\pi}{8}-\frac{x+y}{2}\right), e^{y-x}\{\cos(x+y)+\sin(x+y)\}\right] = 0$$

Ex. 11. *Solve* $xp + yq = z - a\sqrt{(x^2+y^2+z^2)}$. **[Meerut 1997; Jiwaji 1997; Rawa 1999]**

Sol. Here the lagrange's auxiliary equations are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}. \qquad ...(1)$$

Taking the first two fractions of (1), we have

$(1/x)dx = (1/y)dy$ or $(1/x)dx - (1/y)dy = 0.$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1) $= \dfrac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$ ...(3)

Combining first and third fractions of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{(x^2 + y^2 + z^2)}} = \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}. \quad ...(4)$$

Putting $x^2 + y^2 + z^2 = t^2$ so that $xdx + ydy + zdz = tdt$, (4) gives

$$\frac{dx}{x} = \frac{dz}{z - at} = \frac{tdt}{t^2 - azt} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{z - at} = \frac{dt}{t - az}. \quad ...(5)$$

Choosing 0, 1, 1 as multipliers, each fraction of (5) $= \dfrac{dz + dt}{(z + t) - a(t + z)} = \dfrac{d(z + t)}{(1 - a)(z + t)}$. ...(6)

Combining the first fraction of (5) with fraction (6), we get

$$\frac{dx}{x} = \frac{d(z + t)}{(1 - a)(z + t)} \quad \text{or} \quad (1 - a)\frac{dx}{x} - \frac{d(z + t)}{z + t} = 0.$$

Integrating, $(1 - a) \log x - \log (z + t) = \log c_2$, c_2 being an arbitrary constant.

or $$\frac{x^{a-1}}{z + t} = c_2 \quad \text{or} \quad \frac{x^{a-1}}{z + \sqrt{(x^2 + y^2 + z^2)}} = c_2, \quad \text{as} \quad t = (x^2 + y^2 + z^2)^{1/2} \quad ...(7)$$

From (2) and (7), the required general solution is

$$\phi\,[x^{a-1}/\{z + \sqrt{(x^2 + y^2 + z^2)}\}, x/y] = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 12. *Solve* $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x^3 + 3xy^2} = \dfrac{dy}{y^3 + 3x^2y} = \dfrac{dz}{2z(x^2 + y^2)}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx + dy}{x^3 + 3xy^2 + 3x^2y + y^3} = \dfrac{d(x + y)}{(x + y)^3}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \dfrac{d(x - y)}{(x - y)^3}$. ...(3)

From (2) and (3), $(x + y)^{-3}\, d(x + y) = (x - y)^{-3}\, d(x - y)$

or $u^{-3}du - v^{-3}dv = 0$, on putting $u = x + y$ and $v = x - y$.

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx + (1/y)\,dy}{(1/x) \times (x^3 + 3xy^2) + (1/y) \times (y^3 + 3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2 + y^2)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

From (4) and (6), the required general solution is given by

$$\phi\,[(x-y)^{-2}-(x+y)^{-2},\ (xy)/z^2]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $p+q=x+y+z$. **[Bhopal 2010, Bilaspur 2000, 02; I.A.S. 1975; Gulberge 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$. ...(1)

Taking the first two fractions of (1), $dx-dy=0$ so that $x-y=c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\dfrac{dx+dy+dz}{1+1+(x+y+z)}=\dfrac{d(2+x+y+z)}{2+x+y+z}$ ... (3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z)=dx$.

Integrating, $\log(2+x+y+z)-\log c_2=x$ or $(2+x+y+z)/c_2=e^x$

or $e^{-x}(2+x+y+z)=c_2$, c_2 being arbitrary function ...(4)

From (2) and (4), the required general solution is

$$\phi[x-y,\ e^{-x}(2+x+y+z)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2+2z^2-yz-zx-2xy$. **[Meerut 1996 ; I.A.S. 1992]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}. \quad ...(1)$$

Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}=\frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore\quad \frac{dx-dy}{(x-y)\,(x+y+z)}=\frac{dy-dz}{(y-z)\,(x+y+z)}=\frac{dz-dx}{(z-x)\,(x+y+z)}. \quad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx-dy)/(x-y)-(dy-dz)/(y-z)=0.$$

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ or $(x-y)/(y-z)=c_1$. ...(3)

Taking the last two fractions of (2), $(dy-dz)/(y-z)-(dz-dx)/(z-x)=0$.

Integrating, $\log(y-z)-\log(z-x)=\log c_2$ or $(y-z)/(z-x)=c_2$. ...(4)

From (3) and (4), the required general solution is

$$\phi[(x-y)/(y-z),\ (y-z)/(z-x)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 15. *Find the general solution of the partial differential equation* $px(x+y)-qy(x+y)+(x-y)(2x+2y+z)=0$. **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2006, 09, 11]**

Sol. Given $x(x+y)p-y(x+y)q=-(x-y)(2x+2y+z)$. ...(1)

Lagrange's auxiliary equations are $\dfrac{dx}{x\,(x+y)}=\dfrac{dy}{-y\,(x+y)}=\dfrac{dz}{-(x-y)\,(2x+2y+z)}$. ...(2)

Taking the first two fractions, $(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$.

Integrating, $\log x+\log y=\log c_1$ or $xy=c_1$. ...(3)

Again, each fraction of (2)

$$=\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$=\frac{dx+dy}{(x-y)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

Thus, $$\frac{dx+dy}{(x+y)} = \frac{dx+dy+dz}{x+y-(2x+2y+z)} = -\frac{dx+dy+dz}{x+y+z}$$

Thus, $\dfrac{dx+dy}{x+y} + \dfrac{dx+dy+dz}{x+y+z} = 0$, so that $\log(x+y) + \log(x+y+z) = \log c_2$

or $(x+y)(x+y+z) = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), solution is $\phi[xy, (x+y)(x+y+z)] = 0$, ϕ being an arbitrary function.

Ex. 16. *Solve* $\{my(x+y)-nz^2\}(\partial z/\partial x)-\{lx(x+y)-nz^2\}(\partial z/\partial y) = (lx-my)z$ **[I.A.S. 2001]**

Sol. Re-writing the given equation, $\{my(x+y)-nz^2\}\,p-\{lx(x+y)-nz^2\}q = (lx-my)z$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{my(x+y)-nz^2} = \dfrac{dy}{-lx(x+y)+nz^2} = \dfrac{dz}{(lx-my)z}$ ...(2)

Each fraction of (2) = $\dfrac{dx+dy}{(my-lx)(x+y)} = \dfrac{dz}{-(my-lx)z}$ so that $\dfrac{d(x+y)}{x+y} = -\dfrac{dz}{z}$

Integrating, $\log(x+y) = -\log z + \log C_1$ or $(x+y)z = C_1$ ... (3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lxdx+mydy+nzdz}{lx\,my(x+y)-lx\,nz^2-mylx(x+y)+mynz^2+nz^2(lx-my)} = \frac{lx\,dx+my\,dy+nz\,dz}{0}$$

$\therefore\ 2lx\,dx+2my\,dy+2nzdz = 0$ so that $lx^2+my^2+nz^2 = C_2$ ...(4)

From (3) and (4), solution is $\Phi(xz+yz,\ lx^2+my^2+nz^2) = 0$, Φ being an arbitrary function.

Ex. 17. *Solve* $px(z-2y^2) = (z-qy)(z-y^2-2x^2)$. **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation $x(z-2y^2)p+y(z-y^2-2x^2)q = z(z-y^2-2x^2)$ ... (1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(z-2y^2)} = \dfrac{dy}{y(z-y^2-2x^2)} = \dfrac{dz}{z(z-y^2-2x^2)}$ ...(2)

Taking the last two fractions, $(1/y)\,dy-(1/z)\,dz = 0$ so that $y/z = C_1$ ...(3)

Taking $0, -2y, 1$ as multipliers, each fraction of (2)

$$= \frac{-2ydy+dz}{-2y^2(z-y^2-2x^2)+z(z-y^2-2x^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad ... (4)$$

Combining fraction (4) with first fraction of (2), we get

$$\frac{dx}{x(z-2y^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{or} \quad \frac{d(z-y^2)}{dx} = \frac{z-y^2-2x^2}{x}$$

or $du/dx = (u-2x^2)/x$, taking $z-y^2 = u$ ... (5)

or $(du/dx)-(1/x)u = -2x$ which is an ordinary linear differential equation

whose I.F. $= e^{-\int(1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$$u.\frac{1}{x} = \int(-2x)\left(\frac{1}{x}\right)dx + C_2 \quad \text{or} \quad \frac{z-y^2}{x} = -2x+C_2, \text{ using (5)}$$

or $\quad (z-y^2)/x+2x=C_2 \qquad$ or $\qquad (z-y^2+2x^2)/x=C_2 \qquad \ldots (6)$

From (3) and (6), the required general solution of (1)

$$\Phi(y/z,\ (z-y^2-2x^2)/x)=0, \ \Phi \text{ being an arbitrary function.}$$

Ex. 18. *Solve* $px(z-2y^2)=(z-qy)\,(z-y^2-2x^3)$. **[I.A.S. 2006]**

Sol. Do like Ex. 17, **Ans.** $\Phi(y/z,\ (z-y^2+x^3)/x)=0$

For another method of solution, refer solved Ex. 8 of Art. 2.8.

Ex. 19. *Solve* $x(z+2a)p+(xz+2yz+2ay)q=z(z+a)$.

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \qquad \ldots (1)$$

Each fraction of (1) = $\dfrac{dx+dy}{2(x+y)(z+a)}=\dfrac{dz}{z(z+a)}$ $\qquad$ or $\qquad \dfrac{d(x+y)}{x+y}=\dfrac{2}{z}dz$

Integrating, $\quad \log(x+y)=2\log z+\log C_1 \qquad$ or $\qquad (x+y)/z^2=C_1 \qquad \ldots(2)$

Taking the first and third ratios of (4), $\dfrac{dx}{x}=\dfrac{z+2a}{z(z+a)}dz \qquad$ or $\qquad \dfrac{dx}{x}=\left(\dfrac{2}{z}-\dfrac{1}{z+a}\right)dz$

Integrating, $\log x=2\log z-\log(z+a)+\log C_2 \qquad$ or $\qquad x(z+a)/z^2=C_2 \qquad \ldots (3)$

From (2) and (3), solution is $\Phi\ \{(x+y)/z^2,\ x(z+a)/z^2)\}=0$. ϕ being an arbitrary function.

Ex. 20. *Solve* $2x\ (y+z^2)p+y(2y+z^2)q=z^3$ **[Delhi Maths (Hans.) 2007]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{2x(y+z^2)}=\frac{dy}{y(2y+z^2)}=\frac{dz}{z^3} \qquad \ldots(1)$$

Each fraction of (1) = $\dfrac{dx}{2x(y+z^2)}=\dfrac{z\,dy+y\,dz}{2yz(y+z^2)}=\dfrac{d(yz)}{2yz(y+z^2)}$

$\therefore \quad (1/x)\ dx+(1/yz)\ d(yz)=0 \qquad$ so that $\qquad x/(yz)=C_1 \qquad \ldots(2)$

From the last two fractions of (1), $\dfrac{dy}{dz}=\dfrac{y(2y+z^2)}{z^3}=\dfrac{2y^2}{z^3}+\dfrac{y}{z}$ $\quad$ or $\quad y^{-2}\dfrac{dy}{dz}-\dfrac{1}{z}y^{-1}=\dfrac{2}{z^3} \qquad \ldots(3)$

Putting $-y^{-1}=u$ and $(1/y^2)\times(dy/dz)=du/dz$ in (3), we get

$$(du/dz)+(1/z)\ u=2/z^3, \text{ which is an ordinary linear equation.}$$

Its I.F. = $e^{\int (1/z)dz}=e^{\log z}=z$ and solution is $u\,z=\int (2/z^3)\ z\ dz-C_2=-2\ z^{-1}-C_2$

or $\qquad -y^{-1}z-2z^{-1}=-C_2 \qquad$ or $\qquad z/y-2/z=C_2 \qquad \ldots (4)$

From (3) and (4), solution is $\Phi\ (x/yz,\ z/y-2/z)=0$, ϕ being arbitrary function.

Ex. 21. $xp+zq+y=0$. **[M.D.U. Rohtak 2004]**

Sol. Given equation is $\qquad xp+zq=-y$

Its Lagrange's auxiliary equation are $$\frac{dx}{x}=\frac{dy}{z}=\frac{dz}{-y} \qquad ...(1)$$

Taking the last two fractions of (2), $2ydy + 2zdz = 0$ so that $y^2 + z^2 = C_1$ ... (2)
Choosing 0, z, $-y$ as multipliers, each fraction of (1)

$$=\frac{zdy-ydz}{z^2+y^2}=\frac{(1/z)dy-(y/z^2)dz}{1+(y/z)^2}=\frac{d(y/z)}{1+(y/z)^2} \qquad ...(3)$$

Combining the first faction of (1) with fraction (3), we get

$$\frac{dx}{x}=\frac{d(y/z)}{1+(y/z)^2} \quad \text{or} \quad \frac{dx}{x}-d\left(\tan^{-1}\frac{y}{z}\right)=0$$

Integrating, $\log|x| - \tan^{-1}(y/z) = C_2$, c_2 being an arbitrary constant. ... (4)
From (2) and (4), the required general solution is

$$\log|x|-\tan^{-1}(y/z)=\phi(y^2+z^2), \ \phi \text{ being an arbitrary function.}$$

Ex. 22. *Find the general solution of the differential equation* $x^2(\partial z/\partial x)+y^2(\partial z/\partial y)=(x+y)z.$ **[Delhi B.A./B.Sc. (Prog.) Maths 2007)**

Sol. Let $p=\partial z/\partial x$ and $q=\partial z/\partial y$. Then, the given equation takes the form

$$x^2p = y^2q = z(x+y) \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are

$$(dx)/x^2 = (dy)/y^2 = (dz)/z(x+y) \qquad ...(2)$$

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$
Integrating, $-(1/x) + (1/y) = c_1$ or $(x - y)/xy = c_1$ ... (3)

Chossing 1, –1, 0 as multipliers, each fraction of (2) $=\dfrac{dx-dy}{x^2-y^2}$ ... (4)

Combining the last fraction of (2) with fraction (4), we have

$$\frac{dx-dy}{(x-y)(x+y)}=\frac{dz}{z(x+y)} \quad \text{or} \quad \frac{dx-dy}{x-y}-\frac{dz}{z}=0$$

Integrating, $\log(x - y) - \log z = \sin c^2$ or $(x - y)/z = c^2$ ... (5)
From (5), $x - y = c_2 z$ ... (6)
using (6), (3) becomes $(c_2z)/xy = a$ or $(xy)/z = c_2/c_1 = c_3$ say ... (7)
From (5) and (7), the required solution is $\phi\big((x,y)/z,(x-y)/z\big)=0.$

EXERCISE 2(D)

Solve the following partial differential equations:

1. $(x^2 + y^2)p + 2xy\, q = z(x + y)$ **Ans.** $(x+y)/z=\phi\big(y/(x^2-y^2)\big)$
2. $\{y(x + y) + az\}\, p + \{x(x + y) - az\}q = z(x + y)$ **Ans.** $(x+y)/z=\phi(x^2-y^2-2az)$
3. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$ **Ans.** $\phi\left(\dfrac{y-z}{x-y},\dfrac{x-z}{x-y}\right)=0$

2.13. Miscellaneous Examples on $Pp + Qq = R$

Ex. 1. *Solve* $(x + y - z)(p - q) + a(px - qy + x - y) = 0$.

Sol. Let $u = x + y$ and $v = x - y$. ...(1)

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(2)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(3)$$

From (2) and (3), we get $$p - q = 2(\partial z/\partial v). \qquad ...(4)$$

and $$px - qy = x\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} - y\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}$$

or $$px - qy = (x - y)\frac{\partial z}{\partial u} + (x + y)\frac{\partial z}{\partial v} = v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v}, \text{using (1)} \qquad ...(5)$$

Using (1), (4) and (5), the given equation reduces to

$$2(u - z)\frac{\partial z}{\partial v} + a\left(v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} + v\right) = 0$$

or $$av(\partial z/\partial u) + (2u - 2z + au)(\partial z/\partial v) = -av, \qquad ...(6)$$

which is Lagrange's linear equation. Its Lagrange's auxiliary equations are

$$\frac{du}{av} = \frac{dv}{2u - 2z + au} = \frac{dz}{-av}. \qquad ...(7)$$

Taking the first and third fractions of (7), we have

$du + dz = 0$ so that $u + z = c_1$. ...(8)

Considering the first two fractions of (7) and eliminating z with help of (8), we have

$$\frac{du}{av} = \frac{dv}{2u - 2(c_1 - u) + au} \quad \text{or} \quad avdv = (4u - 2c_1 + au)du.$$

Integrating, $(1/2) \times av^2 = 2u^2 - 2c_1 u + (1/2) \times au^2 + c_2/2$

$av^2 = 4u^2 - 4u(u + z) + au^2 + c_2$, or $av^2 + 4uz - au^2 = c_2$ (using (8) ....(9)

From (8) and (9), the required general solution is given by

$\phi(u + z, av^2 + 4uz - au^2) = 0$, where ϕ is an arbitrary function and u and v are given by (1).

Ex. 2 (*a*). *Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.*

(*b*) *Formulate partial differential equation for surfaces whose tangent planes form a tetrahedrom of constant volume with the coordinate planes.* **[I.A.S. 2005]**

Sol. (*a*) We know that the equation of the tangent plane at point (x, y, z) to a surface is given by $$p(X - x) + q(Y - y) = Z - z, \qquad ...(1)$$
where X, Y, Z denote current coodrinates of any point on the plane (1). Since (1) cuts an intercept k on the z–axis, it follows that (1) must pass through the point $(0, 0, k)$. Hence putting $X = 0$, $Y = 0$ and $Z = k$ in (1), we obtain

$$px + qy = z - k, \qquad ...(2)$$

which is well known Lagrange's linear equation. For (2), the Lagrange's auxiliary equations are

$$(dx)/x = (dy)/y = (dz)/(z - x). \qquad ...(3)$$

Taking the first two fractions of (3), $(1/x)dx - (1/y)dy = 0$. so that $x/y = c_1$. ...(4)

Again, taking the first and third fraction of (3), $[1/(z - k)]dz - (1/x)dx = 0$

Integrating, $\log(z - k) - \log x = \log c_2$ or $(z - k)/x = c_2$. ...(5)

From (4) and (5), the required surface (solution) is given by

$$\phi[y/x, (z - k)/x] = 0, \quad \phi \text{ being an arbitrary function.}$$

(*b*) Left as an exercise.

EXERCISE 2 (E)

Solve the following partial differential equations :

1. $p - qy \log y = z \log y$. **Ans.** $\phi(yz, e^x \log y) = 0$

2. $(p + q)(x + y) = 1$. **Ans.** $\phi(y - x, e^{-2z}y + x) = 0$

3. $x^2 p + y^2 q = x + y$. **Ans.** $\phi[(1/y) - (1/x), e^{-z}(x - y)] = 0$

4. $(x^2 + 2y^2)p - xyq = xz$. **Ans.** $\phi(x^2y^2 + y^4, yz) = 0$

5. $px - qy = (z - xy)^2$. **Ans.** $\phi[xy, xe^{1/(z-xy)}] = 0$

6. $zp + zq = z^2 + (x - y)^2$. **Ans.** $\log [z^2 + (x - y)^2] - 2x = \phi(x - y)$.

7. $x(y^n - z^n)p + y(z^n - x^n)q = z(x^n - y^n)$. **Ans.** $x^n + y^n + z^n = \phi(xyz)$.

8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$. **Ans.** $\phi(yz + x^2, 2xz - y^2) = 0$.

9. $xyp + y(2x - y)q = 2xz$. **Ans.** $\phi(xy - x^2, z/xy) = 0$.

2.14. Integral surfaces passing through a given curve. In the last article we obtained general integral of $Pp + Qq = R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

Method I. Let $$Pp + Qq = R \qquad \text{...(1)}$$
be the given equation. Let its auxiliary equations give the following two independent solutions
$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2. \qquad \text{...(2)}$$
Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by
$$x = x(t), \qquad y = (t), \qquad z = z(t), \qquad \text{...(3)}$$
where t is a parameter. Then (2) may be expressed as
$$u[x(t), y(t), z(t)] = c_1 \quad \text{and} \quad v[x(t), y(t), z(t)] = c_2. \qquad \text{...(4)}$$
We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

2.15. SOLVED EXAMPLES BASED ON ART. 2.14.

Ex. 1. *Find the integral surface of the linear partial differential equation* $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ *which contains the straight line* $x + y = 0$, $z = 1$. **[Delhi 2008; Pune 2010]**

Sol. Given $$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z. \qquad \text{...(1)}$$

Lagrange's auxiliary equations of (1) are $$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \qquad \text{...(2)}$$

Proceed as in solved Ex. 6, Art. 2.10 and show that
$$xyz = c_1 \quad \text{and} \quad x^2 + y^2 - 2z = c_2. \qquad \text{...(3)}$$
Taking t as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form $x = t$, $y = -t$, $z = 1$. ...(4)

Using (4), (3) may be re–written as $-t^2 = c_1$ and $2t^2 - 2 = c_2$. ... (5)

Eliminating t from the equations of (5), we have
$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0. \qquad \text{... (6)}$$
Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is
$$2xyz + x^2 + y^2 - 2z + 2 = 0.$$

Ex. 2. *Find the equation of the integral surface of the differential equation* $2y(z - 3)p + (2x - z)q = y(2x - 3)$, *which pass through the circle* $z = 0$, $x^2 + y^2 = 2x$. **[Meerut 2007]**

Sol. Given equation is $$2y(z - 3)p + (2x - z)q = y(2x - 3). \qquad \text{...(1)}$$
Given circle is $$x^2 + y^2 = 2x, \qquad z = 0. \qquad \text{...(2)}$$

Lagrange's auxiliary equations for (1) are $$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}. \qquad \text{...(3)}$$

Taking the first and third fractions of (3), $(2x-3)dx - 2(z-3)dz = 0.$

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 being an arbitrary constant. ...(4)

Choosing 1/2, y, –1 as multipliers, each fraction of (3)

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0.$

Integrating, $x + y^2 - 2z = c_2$, c2 being an arbitrary constant. ...(5)

Now, the parametric equations of given circle (2) are $x = t$, $y = (2t - t^2)^{1/2}$, $z = 0$. ...(6)

Substituting these values in (4) and (5), we have

$$t^2 - 3t = c_1 \quad \text{and} \quad 3t - t^2 = c_2. \qquad ...(7)$$

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0.$ ...(8)

Substituting the values of c_1 and c_2 from (4) and (5) in (8), the desired integral surface is

$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$ or $x^2 + y^2 - z^2 - 2x + 4z = 0.$

Method II. Let $Pp + Qq = R$ ...(1)

be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations

$$\phi(x, y, z) = 0 \quad \text{and} \quad \psi(x, y, z) = 0. \qquad ...(3)$$

We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

Ex. 3. *Find the integral surface of the partial differential equation* $(x - y)p + (y - x - z)q = z$ *through the circle* $z = 1$, $x^2 + y^2 = 1$. **(Nagpur 2002)**

Sol. Given $(x - y)p + (y - x - z)q = z.$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x-y} = \dfrac{dy}{y-x-z} = \dfrac{dz}{z}.$ ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) = $(dx + dy + dz)/0$

$\therefore\ dx + dy + dz = 0$ so that $x + y + z = c_1.$ ...(3)

Taking the last two fractions of (2) and using (3) we get

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z} \quad \text{or} \quad \frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0.$$

Integrating it, $\log\ (2y - c_1) - 2\log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$. ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1.$ ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1.$ ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2.$ ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0.$ ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

or $$z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0.$$

Ex. 4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1$, $y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$ ...(1)

Proceed as in solved Ex. 6, Art. 2.12 and show that

$$(x - y)/(y - z) = c_1 \qquad ...(2)$$

and $xy + yz + zx = c_2.$ ...(3)

The given curve is represented by $x = 1$ and $y = 0.$ ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1c_2$ or $c_1c_2 + 1 = 0.$ ...(5)

Putting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\,(xy + yz + zx) + 1 = 0$ or $(x - y)\,(xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1$, $x + z = 2.$ **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y.$ ...(1)

Given curve is given by $y^2 + z^2 = 1,$ and $x + z = 2.$ ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}.$ ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1.$ ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2.$ ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2,$ using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0.$

Ex. 6. *Find the general integral of the partial differential equation* $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ *and also the particular integral which passes through the line* $x = 1$, $y = 0$. **[I.A.S. 2008]**

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz).$ ...(1)

Given line is given by $x = 1$ and $y = 0.$ ...(2)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2x-2yz}.$ ...(3)

Taking z, 1, x as multipliers, each fraction of (3) $= (zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1.$ ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) $= \{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2.$ ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2.$ ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1.$

Ex. 7. *Find the integral surface of* $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ *which passes through the hyperbola* $xy = x + y$, $z = 1$. **[I.A.S. 1994, 2009]**

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2.$ ...(1)

Given curve is given by $xy = x + y$ and $z = 1.$ ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2).$ ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0$.

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1$. ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0$.

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2$. ...(5)

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z} = c_1 + c_2$ or $\frac{x+y}{xy}+\frac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2$, using (2) or $c_1 + c_2 = 3$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz$.

Ex. 6. *Find the integral surface of the linear first order partial differential equation* $yp + xq = z - 1$ *which passes through the curve* $z = x^2 + y^2 + z,\ y = 2x$

Sol. Given equation is $yp + xq = z - 1$ ... (1)

and the given curve is given by $z = x^2 + y^2 + 1$ and $y = 2x$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$ ... (3)

Taking the first two fractions, $2ydy - 2xdx = 0$

Integrating, it, $y^2 - x^2 = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first and the last fractions of (3) and using (4), we get

$\frac{dx}{(x^2 + C_1)^{1/2}} = \frac{dz}{z-1}$ so that $\log(z-1) - \log\{x + (x^2 + C_1)^{1/2}\} = \log C_2$

or $\log(z - 1) - \log(x + y) = \log C_2$, by (4) or $(z - 1)/(x + y) = C_2$ ... (5)

The parametric form of the given curve (2) is $x = t$, $y = 2t$, $z = 5t^2 + 1$ ... (6)

Substituting these values in (4) and (5), we get $3t^2 = C_1$ and $5t/3 = C_2$ ... (7)

Eliminating t from the above equations (7), we get $5\sqrt{C_1}/3\sqrt{3} = C_2$ ... (8)

Substituting the values of C_1 and C_2 from (4) and (5) in (8), the required surface is given by

$$5(y^2 - x^2)^{1/2}/3\sqrt{3} = (z-1)/(x+y).$$

Ex. 7. *Find the integral surface of the partial differential equation* $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ *passing through the curve* $xz = a^3,\ y = 0$.

Sol. Given equation is $(x-y)y^2p + (y-x)x^2q = (x^2 + y^2)z$ ... (1)

and the given curve is given by $xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ... (3)

Each fraction of (3) = $\frac{dx - dy}{(x-y)\,(y^2 + x^2)} = \frac{dz}{(x^2+y^2)z}$ so that $\frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary contant ... (4)

Taking the first two fractions, $3x^2dx + 3y^2dy = 0$

Integrating it, $x^3 + y^3 = C_2$, C^2 being an arbitrary constant. ... (5)

The parameteric form of the given curve (2) is $z = t$, $x = a^3/t$, $y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$$a^3/t^2 = C_1 \quad \text{so that} \quad t^2 = a^3/C_1 \qquad \text{... (7)}$$

and $$(a^3/t)^3 = C_2 \quad \text{so that} \quad t^3 = a^9/C_2 \qquad \text{... (8)}$$

Squaring both sides of (8), $t^6 = a^{18}/C_2^2$ or $(t^2)^3 = a^{18}/C_2^2$

or $$(a^3/C_1)^3 = a^{18}/C_2^2, \quad \text{since} \quad t^2 = a^3/C_1, \text{ by (7)}$$

or $$a^9/C_1^3 = a^{18}/C_2^2, \quad \text{or} \quad C_2^2 = a^9\ C_1^3 \qquad \text{... (9)}$$

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$$(x^3+y^3)^2 = a^9\ (x-y)^3/z^3 \quad \text{or} \quad z^3(x^3+y^3)^2 = a^9\ (x-y)^3.$$

EXERCISE 2(F)

1. *Find particular integrals of the following partial differential equations to represent surfaces passing through the given curves :*

(*i*) $p + q = 1$; $x = 0$, $y^2 = z$. **Ans.** $(y - x)^2 = z - x$.

(*ii*) $xp + yq = z$; $x + y = 1$, $yz = 1$. **Ans.** $yz = (x + y)^2$.

(*iii*) $(y - z)p + (z - x)q = x - y$; $z = 0$, $y = 2x$ **Ans.** $5(x + y + z)^2 = 9(x^2 + y^2 + z^2)$.

(iv) $x(y - z)p + y(z - x)q = z(x - y)$; $x = y$; $x = y = z$. **Ans.** $(x + y + z)^3 = 27xyz$.

(*v*) $yp - 2xyq = 2xz$; $x = t$, $y = t^2$, $z = t^3$. **Ans.** $(x^2 + y^2)^5 = 32y^2z^2$.

(*vi*) $(y - z)\ [2xyp + (x^2 - y^2)q] + z(x^2 - y^2) = 0$; $x = t^2$, $y = 0$, $z = t^3$.

Ans. $x^3 - 3xy^2 = z^2 - 2yz$.

2. Find the general solution of the equation $2x(y + z^2)p + y(2y + z^2)q = z^2$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.

3. Find the general solution of $x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$.

Find also the integral surfaces which pass through the curves :

(*i*) $y = 0$, $z^2 = 4ax$. (*ii*) $y = 0$, $z^3 + x(z + a)^2 = 0$.

4. Solve $xp + yq = z$. Find a solution representing a surface meeting the parabola $y^2 = 4x$, $z = 1$. **Ans.** General solution $\phi\ (x/2, y/2) = 0$; surface $y^2 = 4xz$.

2.16. SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

Let $$f(x, y, z) = C \qquad \text{...(1)}$$

represents a system of surfaces where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

Let the surface $$z = \phi(x, y) \qquad \text{...(2)}$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Since normals at (x, y, z) to (1) and (2) are at right angles, we have

$$p(\partial f/\partial x) + q(\partial f/\partial y) - (\partial f/\partial z) = 0 \quad \text{or} \quad p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z \qquad \text{...(3)}$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

2.17. SOLVED EXAMPLES BASED ON ART. 2.16.

Ex. 1. *Find the surface which intersects the surfaces of the system* $z(x + y) = c(3z + 1)$ *orthogonally and which passes through the circle* $x^2 + y^2 = 1$, $z = 1$. **[I.A.S. 1999]**

Sol. The given system of surfaces is $f(x, y, z) \equiv \{z(x+y\}/(3z+1) = C.$ ...(1)

$\therefore\ \dfrac{\partial f}{\partial x} = \dfrac{z}{3z+1}, \qquad \dfrac{\partial f}{\partial y} = \dfrac{z}{3z+1}, \qquad \dfrac{\partial f}{\partial z} = (x+y)\dfrac{(3z+1)-z\times 3}{(3z+1)^2} = \dfrac{x+y}{(3z+1)^2}.$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \qquad \text{or} \qquad \frac{z}{3z+1}p+\frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $$z(3z + 1)p + z(3z + 1)q = x + y. \qquad \text{...(2)}$$

Lagrange's auxiliary equations for (2) are $\dfrac{dx}{z(3z+1)} = \dfrac{dy}{z(3z+1)} = \dfrac{dz}{x+y}$. ...(3)

Taking the first two fractions of (3), we get $dx - dy = 0$ so that $x - y = C_1$. ...(4)

Choosing $x, y, -z(3z + 1)$ as multipliers, each fraction of (3) = $[xdx + ydy - z(3z + 1)dz]/0$

$\therefore \quad xdx + ydy - 3z^2dz - zdz = 0 \quad$ or $\quad 2xdx + 2ydy - 6z^2dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z^2 = C_2$, C_2 being an arbitrary constant. ...(5)

Hence any surface which is orthogonal to (I) has equation of the form

$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y)$, ϕ being an arbitrary function ...(6)

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, $z = 1$ we must choose $\phi(x - y) = -2$. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$.

Ex. 2. *Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by* $x(x^2+y^2+z^2)=C_1y^2$. **[I.A.S. 2001]**

Sol. Given family of surfaces is $\quad x(x^2+y^2+z^2)/y^2 = C_1$

Let $$f(x,y,z) = x(x^2+y^2+z^2)/y^2 = C_1 \qquad \text{... (1)}$$

Then the surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x}=\frac{dy}{\partial f/\partial y}=\frac{dz}{\partial f/\partial z} \qquad \text{or} \qquad \frac{dx}{(3x^2+y^2+z^2)/y^2}=\frac{dy}{-2x(x^2+z^2)/y^3}=\frac{dz}{2x/y^2z}$$

or $$\frac{dx}{y(3x^2+y^2+z^2)}=\frac{dy}{-2x(x^2+z^2)}=\frac{dz}{2xyz} \qquad \text{... (2)}$$

Taking x, y, z as multipliers, each fraction of (2)

$$= \frac{xdx+ydy+zdz}{xy(3x^2+y^2+z^2)-2xy(x^2+z^2)+2xyz} = \frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} \qquad \text{... (3)}$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)}=\frac{dz}{2xyz} \qquad \text{or} \qquad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{\mathrm{dz}}{\mathrm{z}}$$

Integrating, $\quad \log(x^2+y^2+z^2)=\log z+\log C_2 \quad$ or $\quad (x^2+y^2+z^2)/z=C_2$ ...(4)

Taking $4x$, $2y$, 0 as multipliers, each fraction of (2)

$$= \frac{4xdx+2ydy}{4xy(3x^2+y^2+z^2)-4xy(x^2+y^2)} = \frac{4xdx+2ydy}{4xy(2x^2+y^2)} \qquad \text{... (5)}$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx+2ydy}{4xy(2x^2+y^2)}=\frac{dz}{2xyz} \qquad \text{or} \qquad \frac{4xdx+2ydy}{2x^2+y^2}=\frac{2dz}{z}$$

Integrating, $\quad \log(2x^2+y^2)=2\log z+\log C_3 \quad$ or $\quad (2x^2+y^2)/y^2=C_3$ ... (6)

From (4) and (5), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form $(x^2+y^2+z^2)/z = \phi\,\{(2x^2+y^2)/z^2\}$, *i.e.*,

or $\quad x^2+y^2+z^2 = z\,\phi\,\{(2x^2+y^2)/z^2\}$, where ϕ is an arbitrary function.

Ex. 3. *Find the surface which is orthogonal to the one parameter system* $z = cxy(x^2+y^2)$ *which passes through the hyperbola* $x^2-y^2=a^2$, $z=0$

Sol. The given system of surfaces is $\qquad f(x,y,z) = z/(x^3y+xy^3) = C \qquad$ ... (1)

$$\frac{\partial f}{\partial x} = -\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of $\qquad p(\partial f/\partial x)+q(\partial f/\partial y) = \partial f/\partial z$

or
$$-\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}p - \frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q = \frac{1}{x^3y+xy^3}$$

or
$$\{(3x^2+y^2)/x\}p + \{(3y^2+x^2)/y\}q = -(x^2+y^2)/z \qquad \text{... (2)}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{(3x^2+y^2)/x} = \frac{dy}{(3y^2+x^2)/y} = \frac{dz}{-(x^2+y^2)/z} \qquad \text{... (3)}$$

Taking the first two fractions of (3), $\quad 2xdx - 2ydy = 0 \quad$ so that $\quad x^2-y^2 = C_1$

Choosing x, y, $4z$ as multipliers, each fraction of (3) = $(xdx+ydy+4zdz)/0$

$\therefore \qquad 2xdx+2ydy+8zdz = 0 \qquad$ so that $\qquad x^2+y^2+4z^2 = C_2$

Hence any surface which is orthogonal to (1) is of the form

$$x^2+y^2+4z^2 = \Phi\,(x^2-y^2),\ \Phi \text{ being an arbitrary function.} \qquad \text{... (4)}$$

For the particular surface passing through the hyperbola $x^2-y^2=a^2$, $z=0$ we must take $\Phi(x^2-y^2) = a^4(x^2+y^2)/(x^2-y^2)^2$. Hence, the required surface is given by

$$(x^2+y^2+4z^2)^2\,(x^2-y^2)^2 = a^4(x^2+y^2)$$

Ex. 4. *Find the equation of the system of surfaces which cut orthogonally the cones of the system* $x^2+y^2+z^2 = cxy$.

2.18 (*a*). Geometrical description of the solutions of *Pp + Qq = R* and of the system of equations *dx/P = dy/Q = dz/R* and to establish relationship between the two.

[G.N.D.U. Amritsar 1998; Meerut 1997; Kanpur 1996]

Proof. Consider $\qquad Pp+Qq = R. \qquad$ ....(1)

and $\qquad (dx)/P = (dy)/Q = (dz)/R, \qquad$ ...(2)

where P, Q and R are functions of x.

Let $\qquad z = \phi(x, y) \qquad$ ...(3)

represent the solution of (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Also we know that the simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp+Qq+R(-1) = 0, \qquad \text{...(4)}$$

showing that the normal to surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence the member must touch the surface at that point. Since this holds for each point on (3), we conclude that the curves (2) lie completely on the surface (3) whose differential equation is (1).

2.18 (b). Another geomertical interpretation of Lagrange's equation Pp + Qq = R.

To show that the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(1)$$

are orthogonal to the system of the surfaces whose equation satisfies

$$Pdx + Qdy + Rdz = 0. \qquad ...(2)$$

Again from Art 2.18 (*a*) the curves of (1) lie completely on the surface represented by

$$Pp + Qq = R. \qquad ...(3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

2.19. SOLVED EXAMPLES BASED ON ART 2.18(*a*) AND ART. 2.18 (*b*)

Ex. 1. *Find the family orthogonal to $\phi\,[z(x + y)^2, x^2 - y^2] = 0$.*

Sol. Given $\qquad \phi[z(x + y)^2, x^2 - y^2] = 0. \qquad ...(1)$

Let $\qquad u = z(x + y)^2 \qquad$ and $\qquad v = x^2 - y^2 \qquad ...(2)$

Then (1) becomes $\qquad \phi(u, v) = 0. \qquad ...(3)$

Differentiating (3) w.r.t. x and y partially by turn, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0 \qquad ...(4)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0. \qquad ...(5)$$

From (2), $\qquad (\partial u/\partial x) = 2z(x + y), \qquad (\partial u/\partial y) = 2z(x + y), \qquad (\partial u/\partial z) = (x + y)^2,$

$\qquad (\partial v/\partial x) = 2x, \qquad (\partial v/\partial y) = -2y, \qquad (\partial v/\partial z) = 0.$

Putting these values in (4) and (5), we get

$$(\partial\phi / \partial u)\left[2z(x+y)+p(x+y)^2\right]+(\partial\phi / \partial v)\,(2x+0) = 0 \qquad ...(6)$$

and

$$(\partial\phi / \partial u)\left[2z(x+y)+q(x+y)^2\right]+(\partial\phi / \partial v)\,(-2y+0) = 0 \qquad ...(7)$$

Evaluating the values of $-\dfrac{\partial\phi / \partial u}{\partial\phi / \partial v}$ from (6) and (7) and then equating these, we get

$$-\frac{\partial\phi / \partial u}{\partial\phi / \partial v} = \frac{2x}{2z(x+y)+p(x+y)^2} = \frac{-2y}{2z(x+y)+q(x+y)^2}$$

or $x(x + y)[2z + q(x + y)] = -y(x + y)[2z + p(x + y)] \qquad$ or $\qquad 2xz + qx(x + y) + 2yz + py(x + y) = 0$

or $\qquad py(x + y) + qx(x + y) = -2z(x + y) \qquad$ or $\qquad py + qx = -2z \qquad ...(8)$

which is differential equation of the family of surfaces given by (1). So the differential equation of the family of surfaces orthogonal to (8) is given by [use Art. 2.18 (b)]

$ydx + xdy - 2zdz = 0 \qquad$ or $\qquad d(xy) - 2zdz = 0. \qquad ...(9)$

Integrating (9), $\qquad xy - z^2 = C,$

which is the desired family of orthogonal surfaces, C being parameter

Ex. 2. *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y + z)p + (z + x)q = x + y$.*

Sol. Let $\qquad P = y + z, \qquad Q = z + x \qquad$ and $\qquad R = x + y. \qquad ...(1)$

Then, the given differential equation can be written as $\qquad Pp + Qq = R. \qquad ...(2)$

Now, the differential equation of the family of surfaces orthogonal to the given family is

$$Pdx + Qdy + Rdz = 0 \quad \text{or} \quad (y + z)dx + (z + x)dy + (x + y)dz = 0$$

or $$(ydx + xdy) + (ydz + zdy) + (zdx + xdz) = 0.$$

Integrating, $$xy + yz + zx = C,$$

which is the required family of surfaces, C being a parameter.

2.20. The linear partial differential equation with n independent variables and its solution. Let $x_1, x_2 ..., x_n$ be the n independent variables and let $p_1 = \partial z/\partial x_1, p_2 = \partial z/\partial x_2, ..., p_n = \partial z/\partial x_n$, where z is the dependent variable. Consider the general linear partial differential equation with n independent variables

$$P_1 p_1 + P_2 p_2 + .. + P_n p_n = R, \qquad ...(1)$$

where $P_1, P_2,.., P_n$ are functions of $x_1, x_2..., x_n$. Let $u_1 = c_1$, $u_2 = c_2$, ..., $u_n = c_n$ be any n independent integrals of the auxiliary equations

$$(dx_1)/P_1 = (dx_2)/P_2 = ... = (dx_n)/P_n. \qquad ...(2)$$

Then the general solution of (1) is given by $$\phi(u_1, u_2, ...u_n) = 0. \qquad ...(3)$$

Note that the above procedure is generalization of Lagrange's method.

2.21. SOLVED EXAMPLES BASED ON ART. 2.20

Ex. 1. *Solve* $x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 + x_1x_2x_3 = 0$.

Sol. Re–writing the given equation in standard form, we have

$$x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 = -x_1x_2x_3. \qquad ...(2)$$

The auxiliary equations for (2) are $$\frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{-x_1x_2x_3}. \qquad ...(3)$$

Taking the first and the fourth fractions of (3), $x_1dx_1 + dz = 0$ so that $x_1^2 + 2z = c_1$. ...(4)

Taking 1st and 2nd fractions of (3), $x_1dx_1 = x_2dx_2$ so that $x_1^2 - x_2^2 = c_2$. ...(5)

Finally, 2nd and 3rd fractions of (3) give $x_2dx_2 = x_3dx_3$ so that $x_2^2 - x_3^2 = c_3$. ...(6)

Hence the required general integral is

$$\phi(x_1^2 + 2z, x_1^2 - x_2^2, x_2^2 - x_3^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = az + \frac{xz}{t}$

Sol. Here auxiliary equations for the given equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + xy/t}. \qquad ...(1)$$

From the first two fractions of (1), $(1/x)dx - (1/y)dy = 0$ so that $x/y = C_1$. ...(2)

From the first and third fractions of (1), $(1/x)dx - (1/t)dt = 0$ so that $x/t = C_2$. ...(3)

Dividing (3) by (2), we have $$y/t = C_2/C_1. \qquad ...(4)$$

Taking the first and third fractions of (1) and using (4), we get

$$\frac{dx}{x} = \frac{dz}{az + (C_2/C_1)x} \quad \text{or} \quad \frac{dz}{dx} = \frac{az + (C_2/C_1)x}{x}$$

or $$\frac{dz}{dx} - \left(\frac{a}{x}\right)z = \left(\frac{C_2}{C_1}\right), \text{ which is linear.} \qquad ...(5)$$

I.F. of (5) = $e^{-\int (a/x)\,dx} = e^{-a\log x} = e^{\log x^{-a}} = x^{-a}$ and so solution of (5) is given by

$$zx^{-a} = C_3 + \frac{C_2}{C_1}\int x^{-a}dx = C_3 + \frac{C_2}{C_1}\frac{x^{1-a}}{1-a} = C_3 + \frac{y}{t}\frac{x^{1-a}}{1-a}, \text{ using (4)}$$

$$\therefore \quad zx^{-a} - \frac{y}{t}\frac{x^{1-a}}{1-a} = C_3, \ C_3 \text{ being an arbitrary constant.} \qquad ...(6)$$

From (2), (3) and (6), the required general solution is

$$\phi\left(\frac{x}{y}, \frac{x}{t}, zx^{-a} - \frac{y}{t}\frac{x^{1-a}}{1-a}\right) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z) = xyz$. **[Bhopal 1995, 98; I.A.S. 1999]**

Sol. Here the auxiliary equations for the given equation are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}. \quad ...(1)$$

Taking the first two fractions of (1), $(1/x)dx - (1/y)dy = 0$.

Integrating it, $\log x - \log y = \log C_1$ or $x/y = C_1$. ...(2)

Taking the first and third fractions of (1), $(1/x)dx - (1/z)dz = 0$

Integrating it, $\log x - \log z = \log C_2$ or $x/z = C_2$. ...(3)

Choosing yz, zx, xy as multipliers, each fraction of (1) $= \dfrac{yzdx + zxdy + xydz}{xyz + xyz + xyz} = \dfrac{d(xyz)}{3xyz}$. ...(4)

Combining the fourth fraction of (1) with fraction (4), we get

$\dfrac{du}{xyz} = \dfrac{d(xyz)}{3xyz}$ or $d(xyz) - 3du = 0$ so that $xyz - 3u = C_3$. ...(5)

From (2), (3) and (5), the required general solution is

$$\phi(x/y, x/z, xyz - 3u) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $(y + z + w)(\partial w/\partial x) + (z + x + w)(\partial w/\partial y) + (x + y + w)(\partial w/\partial z) = (x + y + z)$.

[Ravishanker 2004 ; I.A.S. 1995; Indore 1998, Kanpur 2004]

Sol. Here the auxiliary eqautions of the given equation are

$$\frac{dx}{y+z+w} = \frac{dy}{z+x+w} = \frac{dz}{x+y+w} = \frac{dw}{x+y+z}. \quad ...(1)$$

Each fraction of (1) $= \dfrac{dw - dx}{-(w-x)} = \dfrac{dw-dy}{-(w-y)} = \dfrac{dw-dz}{-(w-z)} = \dfrac{dw+dx+dy+dz}{3(w+x+y+z)}$. ...(2)

Taking the first and the fourth fractions of (2), $\dfrac{dw+dx+dy+dz}{3(w+x+y+z)} + \dfrac{dw-dz}{w-x} = 0$.

Integrating, $(1/3) \times \log(w + x + y + z) + \log(w - x) = \log C_1$

or $(w + x + y + z)^{1/3}(w - x) = C_1$. ...(3)

Similarly, $(w + x + y + z)^{1/3}(w - y) = C_2$. ...(4)

and $(w + x + y + z)^{1/3}(x - z) = C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$\phi[(w + x + y + z)^{1/3}(w - x), (w + x + y + z)^{1/3}(w - y), (w + x + y + z)^{1/3}(w - z)] = 0$,

where ϕ is an arbitrary function.

Ex. 5. *Solve* $p_1 + p_2 + p_3 = 4z$. **Ans.** $\phi(ze^{-4x_1}, ze^{-4x_2}, ze^{-4x_3}) = 0$

Ex. 6. *Solve* $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 + x_1x_2x_3 = 0$.

Sol. Putting the given equation in standard form, we have

$$x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = -x_1x_2x_3. \quad ...(1)$$

Here the auxiliary equations for (1) are $\dfrac{dx_1}{x_2x_3} = \dfrac{dx_2}{x_3x_1} = \dfrac{dx_3}{x_1x_2} = \dfrac{dz}{-x_1x_2x_3}$. ...(2)

Taking the first and second fractions of (2), we have

$2x_1dx_1 - 2x_2dx_2 = 0$ so that $x_1^2 - x_2^2 = C_1$. ...(3)

Taking the first and third fractions of (2), we have

$2x_1dx_1 - 2x_3dx_3 = 0$ so that $x_1^2 - x_3^2 = C_2$. ...(4)

Taking the first and fourth fractions of (2), we have

$2x_1dx_1 + 2dz = 0$ so that $x_1^2 + 2z = C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$$\phi(x_1^2 - x_2^2,\ x_1^2 - x_3^2,\ x_1^2 + 2z) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 7. *Solve* $p_1 + x_1p_2 + x_1x_2p_3 = x_1x_2x_3\sqrt{z}$. **Ans.** $\phi\,(x_1^2 - 2x_2,\ x_2^2 - 2x_3,\ x_3^2 - 4\sqrt{z}) = 0$

Ex. 8. *Solve* $(x_3 - x_2)p_1 + x_2p_2 - x_3p_3 + x_2^2 - (x_2x_1 + x_2x_3) = 0$.

Sol. Re–writing the given equation in the standard form, we get

$$(x_3 - x_2)p_1 + x_2p_2 - x_3p_3 = x_2x_1 + x_2x_3 - x_2^2. \qquad ...(1)$$

Here the auxiliary equations for (1) are $\dfrac{dx_1}{x_3 - x_2} = \dfrac{dx_2}{x_2} = \dfrac{dx_3}{-x_3} = \dfrac{dz}{x_2x_1 + x_2x_3 - x_2^2}$. ...(2)

Taking the second and the third fractions of (2), we have

$(1/x_2)dx + (1/x_3)dx_2 = 0$ so that $\log x_2 + \log x_3 = \log C_1$ or $x_2x_3 = C_1$....(3)

Each fraction of (2) $= \dfrac{dx_1 + dx_2 + dx_3}{(x_3 - x_2) + x_2 - x_3} = \dfrac{dx_1 + dx_2 + dx_3}{0}$.

$\therefore\ dx_1 + dx_2 + dx_3 = 0$ so that $x_1 + x_2 + x_3 = C_2$. ...(4)

Each fraction of (2) $= \dfrac{x_2dx_1 + x_1dx_2}{x_2(x_3 - x_2) + x_1x_2} = \dfrac{d(x_1x_2)}{x_1x_2 + x_2x_3 - x_2^2}$. ...(5)

Combining the last fraction of (2) with fraction (5), we have

$\dfrac{dz}{x_1x_2 + x_2x_3 - x_2^2} = \dfrac{d(x_1x_2)}{x_1x_2 + x_2x_3 - x_2^2}$ or $dz - d(x_1x_2) = 0$.

Integrating, $z - x_1x_2 = C_3$, C_3 being an arbitrary constant. ...(6)

From (3), (4) and (5), the required general solution is

$$\phi(x_2x_3,\ x_1 + x_2 + x_3,\ z - x_1x_2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 9. *If u is a function of x, y and z which satisfies* $(y - z)(\partial u/\partial x) + (z - x)(\partial u/\partial y) + (x - y)(\partial u/\partial z) = 0$, *show that u contains x, y, z only in combinations of* $x + y + z$ and $x^2 + y^2 + z^2$. **(Nagpur 2002, 05)**

Sol. Here auxiliary equations for given equation are $\dfrac{dx}{y - z} = \dfrac{dy}{z - x} = \dfrac{dz}{x - y} = \dfrac{du}{0}$. ...(1)

$$\text{Each fraction of (1)} = \frac{dx + dy + dz}{(y - z) + (z - x) + (x - y)} = \frac{xdx + ydy + zdz}{x(y - z) + y(z - x) + z(x - y)} = \frac{du}{0}$$

$$= \frac{dx + dy + dz}{0} = \frac{xdx + ydy + zdz}{0} = \frac{du}{0}.$$

$\therefore\ dx + dy + dz = 0,$ $2xdx + 2ydy + 2zdz = 0$ and $du = 0$

Integrating, $x + y + z = C_1,$ $x^2 + y^2 + z^2 = C_2$ and $u = C_3$.

Hence the required general solution is

$$u = f(x + y + z,\ x^2 + y^2 + z^2), \quad f \text{ being an arbitrary function.}$$

Ex. 10. *Prove that if* $x_1^3 + x_2^3 + x_3^3 = 1$ *when* $z = 0$, *the solution of the equation* $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ can be given in the form $s^3\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\}^4 = (x_1 + x_2 + x_3 - 3z)^3$, where $s = x_1 + x_2 + x_3 + z$ and $p_i = \partial z/\partial x_i,\ i = 1, 2, 3.$ **[I.A.S. 2000]**

Sol. Given $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ ... (1)

where $s = x_1 + x_2 + x_3 + z$ ... (2)

The auxiliary equations for (2) are $\dfrac{dx_1}{s-x_1}=\dfrac{dx_2}{s-x_2}=\dfrac{dx_3}{s-x_3}=\dfrac{dz}{s-z}$

or $$\frac{dx_1}{x_2+x_3+z}=\frac{dx_2}{x_3+x_1+z}=\frac{dx_3}{x_1+x_2+z}=\frac{dz}{x_1+x_2+x_3}, \text{ using (2)} \quad \text{... (3)}$$

Each fraction of (3) = $\dfrac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}=\dfrac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}$ ... (4)

Again, each fraction of (3) = $\dfrac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)}=\dfrac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)}$ ... (5)

Then, (4) and (5) give $\dfrac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}=\dfrac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)}$ ... (5)

or $$\frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z}+3\frac{d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z}=0$$

Integrating, $\log(x_1+x_2+x_3+z)+3\log(x_1+x_2+x_3-3z)=\log a$

or $(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3=a,$ where a is an arbitrary constant. ... (6)

Given that $x_1^3+x_2^3+x_3^3=1$ when $z=0$ ... (7)

Hence (6) gives $a=(x_1+x_2+x_3)^4$. Then (6) reduces to

$$(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3=(x_1+x_2+x_3)^4 \quad \text{... (8)}$$

Now, each fraction of (3) = $\dfrac{dx_1-dz}{-(x_1-z)}=\dfrac{3(x_1-z)^2\,d(x_1-z)}{-3(x_1-z)^3}=\dfrac{d(x_1-z)^3}{3(x_1-z)^3}$ ... (9)

By symmetry, each fraction of (3) is also = $\dfrac{d(x_2-z)^3}{-3(x_2-z)^3}=\dfrac{d(x_3-z)^3}{-3(x_3-z)^3}$ ...(10)

Using (9) and (10), we find that each fraction of (3)

$$=\frac{d(x_1-z)^3}{-3(x_1-z)^3}=\frac{d(x_2-z)^3}{-3(x_2-z)^3}=\frac{d(x_3-z)^3}{-3(x_3-z)^3}=\frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{-3[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]} \quad \text{... (11)}$$

Then, from (4) and (11), we have

$$\frac{3d(x_1+x_2+x_3-3z)}{(x_1+x_2+x_3-3z)}=\frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}$$

Integrating it, $3\log\,(x_1+x_2+x_3-3z)+\log b=\log\,\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}$

or $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3=b\,(x_1+x_2+x_3-3z)^3$ where b is an arbitrary constant. ...(12)

Putting $z=0$, (12) gives $x_1^3+x_2^3+x_3^3=b\,(x_1+x_2+x_3)^3$

or $1=b\,(x_1+x_2+x_3)^3$, using (7) so that $b=1/(x_1+x_2+x_3)^3$

$\therefore$ (12) $\Rightarrow$ $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3=(x_1+x_2+x_3-3z)^3/(x_1+x_2+x_3)^3$ ... (13)

Raising both sides of (8) to power 3, we have

$$(x_1+x_2+x_3+z)^3\,(x_1+x_2+x_3-3z)^9=(x_1+x_2+x_3)^{12} \quad \text{... (14)}$$

Raising both sides of (13) to power 4, we have

$$\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^{12}/(x_1+x_2+x_3)^{12} \quad \text{... (15)}$$

Multiplying the corresponding sides of (14) and (15), we have

$$(x_1+x_2+x_3+z)^3\,\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^3$$

or $$s^3\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^3, \text{ using (2)}$$

EXERCISE 2 (H)

Ex. 1. *Solve* $p_2 + p_3 = 1 + p_1$. **Ans.** $\phi(x_1 + x_2, x_1 + x_3, x_1 + z) = 0$

Ex. 2. *Solve* $zx_2x_3p_1 + zx_3x_1p_2 + zx_1x_2p_3 = x_1x_2x_3$. **Ans.** $\phi(x_1^2 - x_2^2, x_1^2 - x_2^2, x_1^2 - z^2) = 0]$

Ex. 3. *Solve* $x_1p_1 + 2x_2p_2 + 3x_3p_3 + 4x_4p_4 = 0$. **Ans.** $\phi(x_1^2/x_2, x_1^3/x_3, x_1^4/x_4, z) = 0$

Ex. 4. *Solve* $p_1 + p_2 + p_3 \{1 - (z - x_1 - x_2 - x_3)^{1/2}\} = 3$.

Ans. $\phi[z - 3x_1, z - 3x_2, z + 6(z - x_1 - x_2 - x_3)^{1/2}] = 0$

Ex. 5. $p_1 + p_2 + p_3 \{1 + (z + x_1 + x_2 + x_3)^{1/2}\} + 3 = 0$.

Ans. $\phi[z + 3x_1, z + 3x_2, z + 6(z + x_1 + x_2 + x_3)^{1/2}] = 0$

Ex. 6. $x_1 p_1 + x_2 p_2 + x_3 p_3 = az + (x_1x_2)/x_3$. **[Delhi Maths (H) 1998]**

[**Hint.** This is same as Ex. 2 of Art. 2.21. Here $x = x_1$, $y = x_2$, $t = x_3$, $\partial z/\partial x = \partial z/\partial x = p_1$, $\partial z/\partial y = \partial z/\partial x_2 = p_2$, $\partial z/\partial t = \partial z/\partial x_3 = p_3$]

OBJECTIVE PROBLEMS ON CHAPTER 2

Select correct answer by writing (a), (b), (c) or (d).

1. The equation $Pp + Qq = R$ is known as (*a*) Charpit's equation (*b*) Lagrange' equation (*c*) Bernoulli's equation (*d*) Clairaut's equation.

[Agra 2005, 06, 08]

2. The Lagrange's auxiliary equations for the partial differential equation $Pp + Qq = R$ are (*a*) $(dx)/P = (dy)/Q = (dz)/R$ (*b*) $(dx)/P = (dy)/Q$ (*c*) $(dx)/P = (dz)/R$. (*d*) none of these. **[Garhwal 2005]**

3. The general solution of $(y - z)\, p + (z - x)\, q = x - y$ is

(*a*) $\phi(x + y + z, x^2 + y^2 + z^2) = 0$, (*b*) $\phi(xyz, x + y + z) = 0$

(*c*) $\phi(xyz, x^2 + y^2 + z^2) = 0$ (*d*) $\phi(x^2 - y^2 - z^2, x - y - z) = 0$ **[M.S. Univ. T.N. 2007]**

[**Hint :** Refer Ex. 15, Art 2.10]

4. Subsidiary equations for equation $(y^2z/x) + zxy = y^2$ are

(*a*) $(dx)/y^2z = (dy)/(zx) = (dz)/y^2$ (*b*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$

(*c*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$ (*d*) $(dx)/(1/x^2) = (dy)/(1/y^2) = (dz)/(1/zx)$

[Kanpur 2004]

5. The general solution of the linear partial differential equation $Pp + Qq = R$ is

(*a*) $\phi(u,v) = 1$ (*b*) $\phi(u,v) = -1$ (*c*) $\phi(u,v) = 0$ (d) None of these **[Agra 2007]**

Answers. **1.** (*b*) **2.** (*a*) **3.** (*a*) **4.** (*d*) **5.** (c)

MISCELLANEOUS EXAMPLES ON CHAPTER 2

Ex. 1. *Tranform the equation $yz_x - xz_y = 0$ into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution.* **(I.A.S. 2007)**

Sol. Let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow$ $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ ... (1)

$$\Rightarrow 2r\ (\partial r/\partial x) = 2x,\ 2r\ (\partial r/\partial y) = 2y \quad \Rightarrow \quad \partial r/\partial x = \cos\theta,\ \partial r/\partial y = \sin\theta \qquad ... (2)$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial\theta}{\partial y} = \frac{1}{1 + y^2/x^3}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad ... (3)$$

Now, $$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}, \text{ using (2) and (3)} \quad ...(4)$$

and $$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta} \quad \text{using (2) and (3)} \quad ...(5)$$

Using (1), (4) and (5), the given equation $yz_x - xz_y = 0$ reduce to

$$r\sin\theta\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right) - r\cos\theta\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right) = 0 \quad \text{or} \quad \frac{\partial z}{\partial \theta} = 0 \quad ...(6)$$

Integrating (6) w.r.t. 'θ', $z = f(r)$, where f is an arbitrary function .... (7)

Clearly (7) represents surfaces of revolution, as required.

Ex.2. Solve $(y + z)\, p - (x + z)\, q = x - y$ **(Agra 2010)**

Hint. Do like Ex. 15, page 2.14. **Ans.** $\Phi(x + y + z,\ x^2 + y^2 - z^2) = 0$

Ex. 3. The integral surface satisfying equation $y(\partial z/\partial x) - x(\partial z/\partial y) = x^2 + y^2$ and passing through the curve $x = 1 - t$, $y = 1 + t$, $z = 1 + t^2$ is

(*a*) $z = xy + (x^2 - y^2)/2$ (*b*) $z = xy + (x^2 - y^2)^2/8$

(*c*) $z = xy + (x^2 - y^2)^2/4$ (*d*) $z = xy + (x^2 - y^2)^2/16$ **[GATE 2009]**

Ex. 4. Find the partial differential equation whose surfaces are orthogonal to the surface $z(x + y) = 3z + 1$ **[Pune 2010]** **Ans.** $z(p + q) = x + y - 3$

Ex. 5. if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are integral curves of $(dx)/P = (dy)/Q = (dz)/R$, then show that $F(u, v) = 0$ is general solution of $Pp + Qq = R$, where F is an arbitrary function. **[Pune 2010]**

3

Non–linear partial differential equations of order one

3.1. Explanation of terms : complete integral (or complete solution), particular integral, singular integral (or singular solution), and general integral (or general solution) as applied to solutions of first order partial differential equations

[I.A.S. 1995; Meerut 1997; Delhi Maths Hons. 1995]

A *solution* or *integral* of a differential equation is a relation between the variables, by means of which and the derivatives obtained there from the equation is satisfied. Let us now discuss various classes of integrals of a partial differential equation of order one.

Complete Integral (C. I.). or complete solution (C.S.) **[Sagar 1995]**

Let us consider a relation $\phi(x, y, z, a, b) = 0$ ...(1)

in which x, y, z are variables such that z is dependent on x and y. Differentiating (1) partially w.r.t x and y respectively, we obtain

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z} p = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z} q = 0, \qquad \text{...(2)}$$

Since there are two arbitrary constants (namely a and b) connected by the above three equations, these can be eliminated and there will appear a relation of the form

$$f(x, y, z, p, q) = 0, \qquad \text{...(3)}$$

which is a partial differential equation of order one.

Suppose, now, that (1) has been derived from (3), by using some method; then the integral (1), which has as many arbitrary constants as there are independent variables, is called the *complete integral* of (3).

Particular Integral : A particular integral of (3) is obtained by giving particular values to a and b in (1) which is the complete integral of (3).

Singular Integral (S.I.) or singular solution (S.S.) **[Delhi 2009; Sagar 1995]**

We know that the locus of all the points whose co–ordinates along with the values of p and q satisfy (3), represent the doubly infinite system of surfaces given by (1). The system is doubly infinite, since there are two constants a and b and each of these can take an infinite number of values. Since the envelope of all the surfaces given by (1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope along with the values of p and q belonging to the envelope at that point must also satisfy (3). Hence we conclude that the equation of the envelope is a solution of (3). The envelope of the surfaces given by (2) is obtained by eliminating a and b between the equations

$$\phi(x, y, z, a, b) = 0, \qquad \partial \phi / \partial a = 0 \qquad \text{and} \qquad \partial \phi / \partial b = 0. \quad \text{...(4)}$$

The relation between x, y, and z so obtained is called the *singular integral*. In general, it is distinct from the complete integral. However, in exceptional cases it may be contained in the complete integral, that is, singular integral may be obtained by giving particular values to the constants in the complete integral. Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of singular integral satisfies the given differential equation.

General Integral (G.I.) or General Solution (G.S.).

Assume that in (1), one of the constants is a function of the other, say $b = F(a)$, then (1) becomes
$$\phi(x, y, z, a, F(a)) = 0. \qquad ...(5)$$

Now (5) represents one of the families of surfaces given by the system (1). As before, the equation of the envelope of the family of surfaces given by (5) must also satisfy (3). Again the equation so obtained will be distinct from that of the envelope of the surfaces, and it is not a particular integral. It is known as the *general integral* and is obtained by eliminating a between

$$\phi(x, y, z, a\ F(a)) = 0 \qquad \text{and} \qquad \partial\,\phi\,/\partial a = 0. \qquad ...(6)$$

Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of general integral satisfies the given differential equation.

Important Note. While solving a non–linear equation, we must not only obtain the complete integral but should also find the singular and general integrals. In absence of details of singular and general integrals, merely the complete solution is considered to be incomplete solution of the given partial differential equation. However, for reason of space, we have found complete integral only in some problems. The students are advised to find singular and general integrals also for such problems. Note that there is always a simple routine method for the same.

Also, if you are asked to find complete integral of a given equation, then you need not give singular and general integrals. Again, if examiner wants singular integral/general integral, then you must find them.

3.2. Geometrical interpretation of three types of integrals of f(x, y, z, p, q) = 0.

(*i*) **Complete integral.**

A complete integral, being a relation between x, y and z represents equation of a surface. Since it involves two arbitrary parameters, it belongs to a double infinite system of surfaces or to a single infinite system of family of surfaces.

(*ii*) **General integral.**

Let a complete solution of $f(x, y, z, p, q) = 0$ be
$$\phi(x, y, z, a, b) = 0. \qquad ...(1)$$

A general integral is obtained by eliminating 'a' between (1) and the equations
$$b = \psi(a) \qquad ...(2)$$
$$(\partial\phi/\partial a) + (\partial\phi/\partial b)\ \psi'(a) = 0. \qquad ...(3)$$
where ψ is an arbitrary function.

The operation of elimination is equivalent to selecting from the system of families of surfaces a representative family and finding the envelope. Equations (1), (2) and (3) together represent a curve drawn on the surface of the family whose parameter is 'a' whereas the equation obtained by eliminating 'a' between them is the envelope of the family. It follows that the envelope touches the surface represented by (1) and (2) along the curve represented by (1), (2) and (3). This curve is known as *characteristic of the envelope* and the general integral thus represents the envelope of a family of surfaces considered as composed of its characteristics.

(*iii*) **Singular Integral.**

The singular integral is obtained by eliminating 'a' and 'b' between equation (1)
$$\partial\,\phi\,/\partial a = 0 \qquad ...(4)$$
and
$$\partial\,\phi\,/\partial b = 0. \qquad ...(5)$$

The operation of elimination is equivalent to finding the envelope of all the surfaces included in the complete integral. (1), (4) and (5) give the point of contact of the particular surfaces represented by (1) with the general envelope. It follows that the singular integral represents the general envelope of all surfaces included in the complete integral.

3.3. Method of getting singular integral directly from the partial differential equation of first order.

Let the given partial differential equation be $f(x, y, z, p, q) = 0,$ ...(1)

whose complete integral is of the form $\phi(x, y, z, a, b) = 0,$ ...(2)

where 'a' and 'b' are arbitrary constants.

The singular integral of (1) is obtained by eliminating 'a' and 'b' between equation (2)

$\partial\phi/\partial a = 0$ ...(3) and $\partial\phi/\partial b = 0.$ ...(4)

The values of z, p, q derived from (2) when substituted in (1) will reduce it into an identity and the substitution of the values of p and q (but not of z) will in general render (1) equivalent to the integral equation. By using this substitution p and q are replaced by functions of x, y, z, a and b in (1). It follows that the singular integral is given by (1) and the equations obtained on differentiating (1) partially w.r.t. 'a' and 'b', namely the equations

$$\frac{\partial f}{\partial p}\frac{\partial p}{\partial a}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial a} = 0 \quad ...(5) \qquad \frac{\partial f}{\partial p}\frac{\partial p}{\partial b}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial b} = 0. \quad ...(6)$$

If $\dfrac{\partial f}{\partial p} \neq 0$ and $\dfrac{\partial f}{\partial q} \neq 0$, (5) and (6) hold if $\dfrac{\partial p}{\partial a}\dfrac{\partial q}{\partial b}-\dfrac{\partial p}{\partial b}\dfrac{\partial q}{\partial a} = 0,$

showing that there exists a functional relation between p and q which does not contain a and b explicitly. Let this functional relation be

$$\psi(p, q) = 0. \qquad ...(7)$$

If both the constants a and b occur in p and q (which does nolt always happen), then (7) shows that one of them is a function of the other and the equations using them give general integral which is not now required.

Equations (5) and (6) are also true if

$\partial f/\partial p = 0$ ...(8) and $\partial f/\partial q = 0.$ ... (9)

Elimination of p and q from (1), (7) and (8) will yield a relation between x, y, z free from 'a' and 'b'. If this relation satisfies the given differential equation (1), it must be the singular integral.

3.4. COMPATIBLE SYSTEM OF FIRST–ORDER EQUATIONS

[Delhi Maths (H) 2007; Pune 2010]

Consider first order partial differential equations

$$f(x, y, z, p, q) = 0 \qquad ...(1)$$

and

$$g(x, y, z, p, q) = 0. \qquad ...(2)$$

Equations (1) and (2) are known as compatible when every solution of one is also a solution of the other.

To find condition for (1) and (2) to be compatible. [Delhi 2008; Pune 2011]

Let J = Jacobian of f and $g \equiv \partial(f,g)/\partial(p,q) \neq 0.$ ...(3)

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$p = \phi(x, y, z)$ and $q = \psi(x, y, z).$ ...(4)

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable, *i.e.*, that the equation

$dz = pdx + qdy$ or $\phi\, dx + \psi dy - dz = 0$, using (4) ...(5)

should be integrable. (5) is integrable if*

$$\phi\left(\frac{\partial \psi}{\partial z}-0\right)+\psi\left(0-\frac{\partial \phi}{\partial z}\right)+(-1)\left(\frac{\partial \phi}{\partial y}-\frac{\partial \psi}{\partial x}\right) = 0$$

which is equivalent to $$\frac{\partial \psi}{\partial x}+\phi\frac{\partial \psi}{\partial z} = \frac{\partial \phi}{\partial y}+\psi\frac{\partial \phi}{\partial z}. \qquad ...(6)$$

Substituting from equations (4) in (1) and differentiating *w.r.t.* 'x' and 'z' respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial \psi}{\partial x} = 0 \qquad ...(7)$$

and $$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial \psi}{\partial z} = 0. \qquad ...(8)$$

From (7) and (8), $$\frac{\partial f}{\partial x}+\phi\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\left(\frac{\partial \phi}{\partial x}+\phi\frac{\partial \phi}{\partial z}\right)+\frac{\partial f}{\partial q}\left(\frac{\partial \psi}{\partial x}+\phi\frac{\partial \psi}{\partial z}\right) = 0. \qquad ...(9)$$

Similarly (2) yields $$\frac{\partial g}{\partial x}+\phi\frac{\partial g}{\partial z}+\frac{\partial g}{\partial p}\left(\frac{\partial \phi}{\partial x}+\phi\frac{\partial \phi}{\partial z}\right)+\frac{\partial g}{\partial q}\left(\frac{\partial \psi}{\partial x}+\phi\frac{\partial \psi}{\partial z}\right) = 0. \qquad ...(10)$$

Solving (9) and (10), $$\frac{\partial \psi}{\partial x}+\phi\frac{\partial \psi}{\partial z} = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\}. \qquad ...(11)$$

Again, substituting from equations (4) in (1) and differentiating w.r.t. 'y' and 'z' and proceeding as before, we obtain $$\frac{\partial \phi}{\partial y}+\psi\frac{\partial \phi}{\partial z} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \qquad ...(12)$$

Substituting from equations (11) and (12) in (1) and replacing ϕ, ψ by p, q respectively, we obtain

$$\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}\right\} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}\right\} \quad \text{or} \quad [f, g] = 0, \qquad ...(13)$$

where $$[f, g] \equiv \frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} \qquad ...(14)$$

3.5. A PARTICULAR CASE OF ART. 3.4.

To show that first order partial differential equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible if and only if $\partial P/\partial y = \partial Q/\partial x$. **[Delhi Maths (H) 2009; Pune 2010]**

Proof. Given $\partial z/\partial x = p = P(x,y)$ and $\partial z/\partial z = q = Q(x,y)$ ... (1)

Since $$dz = (\partial z/\partial x)dx + (\partial z/\partial y)dy = pdx + qdy, \qquad ... (2)$$

it follows that the given partial differential equations (1) are compatible if and only if the single differential equation

$$dz = Pdx + Qdy \qquad ... (3)$$

is integrable.

Since P and Q are functions of two variables x and y, hence $Pdx + Qdy$ is an exact differential if and only if $\partial P/\partial y = \partial Q/\partial x$. Therefore (3) is integrable if and only if $\partial P/\partial y = \partial Q/\partial x$

Remark 1. If $\partial P/\partial y = \partial Q/\partial x$, then the system of two given partial differential equations (1) is compatible and hence these will possess a common solution.

---

*$Pdx + Qdy + Rdz = 0$ is integrable if $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right) = 0.$

Remark 2. If $\partial P/\partial y \neq \partial Q/\partial x,$ then the system of two given partial differential equations (1) is not compatible and hence these equations possess no solution.

3.6. SOLVED EXAMPLES BASED ON ART. 3.4 AND ART. 3.5.

Ex. 1. (*a*) *Show that the differential equations* $\partial z/\partial x = 5x - 7y$ *and* $\partial z/\partial y = 6x + 8y$ *are not compatible.*

(*b*) $\partial z/\partial x = 5x - 7y,\ \partial z/\partial y = 6x + 8y$ *possess* (*i*) *common solution* (*ii*) *No common solution* (*iii*) *No solution* (*iv*) *None of these. Point out correct choice.* **[Agra 2005, 06]**

(*c*) *Show that the differential equations* $p = x^2 - ay,\ q = y^2 - ax$ *are compactible and find their common solution.*

(*d*) *Show that the differential equations* $\partial z/\partial x = (x+y)^2,\quad \partial z/\partial y = x^2 + 2xy - y^2$ *are compatible and solve them*

(*e*) *Show that* $p = x - y/(x^2 + y^2),\ q = y + x/(x^2 + y^2)$ *are compatible and find their solution.*

(*f*) Show that $p = 1 + e^{x/y},\ q = e^{x/y}(1 - x/y)$ *are compatible and find their solution.*

Sol. (a) Given $dz/dx = p = 5x - 7y$ and $dz/dy = q = 6x + 8y$ ...(1)

Comparing (1) with $p = P(x, y)$ and $q = Q(x, y)$ ... (2)

here $p = 5x - 7y$ and $Q = 6x + 8y$ ... (3)

We know that $p = P(x, y)$ and $q = Q(x, y)$ are compatible if $\partial P/\partial y = \partial Q/\partial x.$ Hence the system (1) is compatible if $\partial P/\partial y = \partial Q/\partial x.$

From (3), $\partial P/\partial y = -7$ and $\partial Q/\partial x = 6$ and so $\partial P/\partial y \neq \partial Q/\partial x$

Therefore, the given system (1) is not compatible.

(*b*) **Ans.** (*iii*) As in part (a), the given system is not compatible. Hence the given equations have no solution (refer Art. 3.5).

(*c*) We known that the system of equations $p = P(x, y),$ $q = Q(x, y)$ ... (1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x.$

Comparing $p = x^2 - ay,$ and $q = y^2 - ax$ ... (2)

with (1), here $P = x^2 - ay,$ and $Q = y^2 - ax$ ... (3)

From (3), $\partial P/\partial y = -a = \partial Q/dx$ and so equations (2) are compatible

To find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 - ay)\,dx + (y^2 - ax)\,dy = x^2dx + y^2dy - a\,d(xy)$$

Integrating, $z = (x^3 + y^3)/3 - axy + c$, c being an arbitrary contant ... (4)

(4) is the required common solution of the given equation (2).

(*d*) We know that the system of equations $p = P(x, y),$ $q = Q(x, y)$ ...(1)

is compatible if and only if $\partial P/\partial y = \partial Q/\partial x.$

Comparing $\partial z/\partial x = p = (x+y)^2,$ and $\partial z/\partial y = q = x^2 + 2xy - y^2$ ... (2)

with (1), here $P = (x + y)^2 = x^2 + 2xy + y^2$ and $Q = x^2 + 2xy - y^2$ ... (3)

From (3), $\partial P/\partial y = 2x + 2y = \partial Q/\partial y$ and hence equations (2) are compatible.

The find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 + 2xy + y^2)dx + (x^2 + 2xy - y^2)\,dy \qquad ... (4)$$

Integrating (4) and noting that R.H.S. of (4) must an exact differential, we have, by method of solving an exact equation

$$z = \underset{\text{(Treating y as a constant)}}{\int (x^2 + 2xy + y^2)dx} + \underset{\text{(Integrating terms free from } x)}{\int (x^2 + 2xy - y^2)} + c$$

or $\quad z = x^3/3 + x^2 y + y^2 x - y^3/3 + c,$ c being an arbitrary constant

(*e*) We know that the system of equation $\quad p = P(x, y), \quad q = Q(x, y)$ ... (1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad p = x - y/(x^2 + y^2), \quad$ and $\quad q = y + x/(x^2 + y^2)$ ... (2)

with (1), here $\quad P = x - y/(x^2 + y^2) \quad$ and $\quad Q = y + x/(x^2 + y^2)$ ... (3)

From (3),
$$\frac{\partial P}{\partial y} = 0 - \frac{1 \cdot (x^2 + y^2) - 2y \cdot y}{(x^2 - y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \text{... (4)}$$

and
$$\frac{\partial Q}{\partial x} = 0 + \frac{1 \cdot (x^2 + y^2) - 2x \cdot x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 - y^2)^2} \qquad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow \quad$ The system (2) is compatible.

To find the solution of the system (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = \{x - y/(x^2 + y^2)\}dx + \{y + x/(x^2 + y^2)\}dy \qquad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \underset{\text{(Treating } y \text{ as a constant)}}{\int \left\{x - y/(x^2 + y^2)\right\} dx} + \underset{\text{(Integrating terms free from } x)}{\int \left\{y + x/(x^2 + y^2)\right\} dy} + c$$

or $\quad z = x^2/2 - y \times (1/y) \times \tan^{-1}(x/y) + y^2/2 + c = (x^2 + y^2)/2 - \tan^{-1}(x/y) + c,$

which is the required solution, c being an arbitrary constant.

(*f*) We know that the system of equations $\quad p = P(x, y), \quad$ and $\quad q = Q(x, y)$ ... (1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad p = 1 + e^{x/y} \quad$ and $\quad q = e^{x/y}(1 - x/y)$ ... (2)

with (1), here $\quad P = 1 + e^{x/y} \quad$ and $\quad Q = e^{x/y}(1 - x/y)$ ... (3)

$$(3) \Rightarrow \partial P/\partial y = 0 + e^{x/y}(-x/y^2) = -(x/y^2)\, e^{x/y} \qquad \text{... (4)}$$

and
$$\partial Q/\partial x = e^{x/y} \times (1/y) \times (1 - x/y) + e^{x/y} \times (-1/y) = -(x/y^2)e^{x/y} \qquad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow \quad$ The system (2) is compatible.

To find the solution of (2). Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy \qquad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \underset{\text{(Treating } y \text{ as constant)}}{\int (1 + e^{x/y})dx} + \underset{\text{(Integrating terms free from } x)}{\int e^{x/y}(1 - x/y)\, dy} + c$$

or $\quad z = x + y\, e^{x/y} + c,$ c being an arbitrary constant.

Ex. 2. *Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.*

[Delhi Maths (Hons) 2005, 07, 11]

Sol. Let $$f(x, y, z, p, q) = xp - yq = 0 \qquad ...(1)$$
and $$g(x, y, z, p, q) = z(xp + yq) - 2xy = 0 \qquad ...(2)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy$$

and $$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2q.$$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 2xy - x^2p^2 - xyqp - 2xy + xypq + y^2q^2$$
$$= -xp(xp + yq) + yq(xp + yq) = -(xp - yq)(xp + yq) = 0, \text{ using (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $\quad p = y/z \quad$ and $\quad q = x/z. \quad ...(3)$

Using (3) in $dz = pdx + qdy$, we have $\quad dz = (y/z)dx + (x/z)dy \quad$ or $\quad z\,dz = d(xy)$.

Integrating, $\quad z^2/2 = xy + c/2 \quad$ or $\quad z^2 = 2xy + c$, where c is an arbitrary constant.

Ex. 3. *Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution.* **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2007]**

Sol. Let $$f(x, y, z, p, q) = xp - yq - x = 0. \qquad ...(1)$$
and $$g(x, y, z, p, q) = x^2p + q - xz = 0. \qquad ...(2)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = (p-1)x^2 - x(2xp - z).$$

Similarly, $\quad \dfrac{\partial(f,g)}{\partial(z,p)} = x^2, \qquad \dfrac{\partial(f,g)}{\partial(y,q)} = -q, \qquad \dfrac{\partial(f,g)}{\partial(z,q)} = -xy.$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = (p-1)x^2 - x(2xp - z) - px^2 - q - xyq$$
$$= -x^2 + zx - q - xyq = -x^2 + x^2p - qxy, \text{ by (2)}$$
$$= x(-x + xp - yq) = 0, \text{ by (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $\quad p = (1 + yz)/(1 + xy) \quad$ and $\quad q = x(z - x)/(1 + xy). \quad ...(3)$

Using (3) in $dz = pdx + qdy$, $\quad dz = [(1 + yz)/(1 + xy)]\,dx + [x(z - x)/(1 + xy)]dy$

or $\quad (1 + xy)dz = (1 + yz)dx + x(z - x)dy \quad$ or $\quad (1 + xy)dz - z(ydx + xdy) = dx - x^2dy$

or $$\frac{(1+xy)\,dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2} = \frac{(dx/x^2) - dy}{(y+1/x)^2} \quad \text{or} \quad d\left(\frac{z}{1+xy}\right) = \frac{-d\,(y+1/x)}{(y+1/x)^2}.$$

Integrating it, $$\frac{z}{1+xy} = \frac{1}{(y+1/x)} + c \quad \text{or} \quad \frac{z}{1+xy} = \frac{x}{1+xy} + c$$

or $\quad z - x = c(1 + xy)$, $\;c$ being an arbitrary constant.

Ex. 4. *Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in x, y, z.* **[Delhi Maths, Hons. 2001, 06, 10]**

Sol. Given that differential equation $\qquad f(x, y, z, p, q) = 0 \qquad$ ...(1)

is homogeneous in x, y, z. Then, clearly $f(x, y, z, p, q)$ will be a homogeneous function in variables x, y, z; say of degree n. Then, by Euler's theorem on homogeneous function, we have

$$x(\partial f/\partial x) + y(\partial f/\partial y) + z(\partial f/\partial z) = nf \quad \text{so that} \quad x(\partial f/\partial x) + z(\partial f/\partial z) = 0, \text{ by (1)} \qquad \text{... (2)}$$

We take $\qquad g(x, y, z, p, q) = px + qy - z = 0 \qquad$ ... (3)

Then, using (3), we have

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ p & x \end{vmatrix} = x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ -1 & x \end{vmatrix} = x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ q & y \end{vmatrix} = y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q}$$

and
$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ -1 & y \end{vmatrix} = y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}$$

$$\therefore \qquad [f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,q)}{\partial(z,q)}$$

$$= x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p} + p\left(x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\right) + y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q} + q\left(y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}\right)$$

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + (px+qy)\frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}, \text{ using (3)}$$

$$= 0, \text{ using (2)}$$

Hence, the differential equation $z = px + qy$ is compatible with any differential equation $f(x, y, z, p, q)$ that is homogeneous in x, y, z.

Ex. 5. *If $u_1 = \partial u/\partial x$, $u_2 = \partial u/\partial y$, $u_3 = \partial u/\partial z$, show that the equations $f(x, y, z, u_1, u_2, u_3) = 0$ and $g(x, y, z, u_1, u_2, u_3) = 0$ are compatible if* $\dfrac{\partial(f,g)}{\partial(x,u_1)} + \dfrac{\partial(f,g)}{\partial(y,u_2)} + \dfrac{\partial(f,g)}{\partial(z,u_3)} = 0.$ **(Delhi Maths (H) 2004)**

Sol. Treating z as constant, given equations are compatible if

$$\frac{\partial(f,g)}{\partial(x,u_1)} + u_1\frac{\partial(f,g)}{\partial(u,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + u_2\frac{\partial(f,g)}{\partial(u,u_2)} = 0. \qquad \text{...(1)}$$

Since f and g do not contain u, we have $\qquad \partial f/\partial u = 0 \qquad$ and $\qquad \partial g/\partial u = 0$. ...(2)

$$\therefore \qquad \frac{\partial(f,g)}{\partial(u,u_1)} = 0 \qquad \text{and} \qquad \frac{\partial(f,g)}{\partial(u,u_2)} = 0. \qquad \text{...(3)}$$

$$\therefore \qquad \text{(1) reduces to} \qquad \frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} = 0. \qquad \text{...(4)}$$

Similarly treating x and y constant respectively, given equtions are compatible if

$$\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0 \qquad ...(5) \qquad\qquad \frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0. \qquad ...(6)$$

We know that the given equations are compatible when they remain compatible even when any variable is taken as constant, *i.e.*, (4), (5) and (6) hold simultaneously. Hence adding (4), (5) and (6), the required condition for given equations to be compatible is

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0.$$

Ex. 6. *Show that the equations* $f(x,y,p,q)=0$, $g(x,y,p,q)=0$ *are compatible if* $\partial(f,\,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$

Verify that the equations $p=P(x,y)$, $q=Q\,(x,y)$ *are compatible if* $\partial P/\partial y=\partial Q/\partial x$.

Sol. We know that $\quad f(x,y,z,p,q)=0 \qquad$ and $\qquad g(x,y,z,p,q)=0 \qquad ...\ (1)$

are compatible if
$$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0 \qquad ...\ (2)$$

First part: Comparing the given equations $f(x,y,p,q)=0$ and $g(x,y,p,q)=0$ with (1), we find that z is absent in given equations and so

$$\partial f/\partial z=0 \qquad \text{and} \qquad \partial g/\partial z=0 \qquad ...\ (3)$$

Now,
$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial p\\ 0 & \partial g/\partial p\end{vmatrix}=0$$

and
$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial q\\ 0 & \partial g/\partial q\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$$\partial(f,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$$

Second Part. Let $\qquad f=P(x,y)-p \qquad$ and $\qquad g=Q(x,y)-q \qquad ...\ (4)$

Comparing (4) with (1), we find that z and q are absent in f and z and p are absent in g and so $\quad \partial f/\partial z=0, \quad \partial f/\partial q=0, \quad \partial g/\partial z=0 \quad$ and $\quad \partial g/\partial p=0 \qquad ...\ (5)$

$\therefore$
$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\partial f/\partial x & \partial f/\partial p\\ \partial g/\partial x & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}\partial P/\partial x & -1\\ \partial Q/\partial x & 0\end{vmatrix}=\frac{\partial Q}{\partial x}$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & -1\\ 0 & 0\end{vmatrix}=0$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\partial f/\partial y & \partial f/\partial q\\ \partial g/\partial y & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}\partial P/\partial y & 0\\ \partial Q/\partial y & -1\end{vmatrix}=-\frac{\partial P}{\partial y}$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$$\partial Q/\partial x-\partial P/\partial y=0 \qquad \text{or} \qquad \partial P/\partial y=\partial Q/\partial x\,.$$

Ex. 7. *Show that* $p^2+q^2=1$ *and* $(p^2+q^2)x=pz$ *are compatible and solve them.*

Hint. Proceed as in solved Ex. 1 Ans. $z^2=x^2+(y+c)^2$.

Ex. 8. *Solve completey the simultaneous equations:* $z = px + qy$ *and* $2xy(p^2 + q^2) = z(yp + xq)$. **[Delhi Math (H) 2006, 10]**

Sol. Given $$z = px + qy \qquad \text{... (1)}$$

and $$2xy(p^2 + q^2) - z(yp + xq) = 0 \qquad \text{... (2)}$$

Let $f(x, y, z, p, q) = 2xy\,(p^2 + q^2) - z\,(yp + xq) = z^2\,\{2(x/z)\,(y/z)\,(p^2 + q^2) - (y/z)p - (x/z)q\}$,

showing that $f(x, y, z, p, q)$ is homogeneous in x, y, z.

We know that (refer solved example 4) the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogenous in x, y, z. Hence (1) and (2) are compatible.

From (1), we have $$q = (z - px)/y \qquad \text{... (3)}$$

Using (3), (2) gives $$2x(x^2 + y^2)p^2 - z(3x^2 + y^2)\;p + xz^2 = 0$$

or $$(2xp - z)\;\{(x^2 + y^2)p - xz\} = 0$$

so that $$p = z/2x,\; xz/(x^2 + y^2) \qquad \text{... (4)}$$

Using (4), (3) gives $$q = z/2y,\; yz/(x^2 + y^2) \qquad \text{... (5)}$$

Using the corresponding values $p = z/2x,\ q = z/2y$ in $dz = px + qdy$, we get

$dz = (z/2x)dx + (z/2y)dy$ or $2(1/z)dz = (1/x)dx + (1/y)dy$

Integrating, $2\log z = \log x + \log y + \log C_1$ or $z^2 = C_1xy$ ...(6)

Similarly, using the corresponding values $p = xz/(x^2 + y^2)$ and $q = yz/(x^2 + y^2)$ in $dz = pdx + qdy$, we get

$$dz = \frac{xzdx}{x^2 + y^2} + \frac{yzdy}{x^2 + y^2} \quad \text{or} \quad \frac{2dz}{z} = \frac{2(xdx + ydy)}{x^2 + y^2}$$

Integrating, $2\log z = \log(x^2 + y^2) + \log C_2$ or $z^2 = C_2(x^2 + y^2)$ ... (7)

(6) and (7) give two two common solutions of (1) and (2)

EXERCISE 3(A)

1. Show that $\partial z/\partial x = 7x + /8y - 1$ and $\partial z/\partial y = 9x + 11y - 2$ are not compatible.

2. Show that the partial differential equations $p = 6x - 4y + 1$ and $q = 4x + 6y + 1$ do not possess any common solution.

3. Show that the following system of partial differential equations are compatible and hence solve them

(*i*) $p = 6x + 3y,\ q = 3x - 4y$ **Ans.** $z = 3x^2 + 3xy - 2y^2 + c$

(*ii*) $p = ax + hy + g,\ q = hx + by + f$ **Ans.** $z = (ax^2 + by^2)/2 + hxy + gx + fy + c$

(*iii*) $\partial z/\partial x = y(2ax + by),\ \partial z/\partial y = x(ax + 2by)$ **Ans.** $z = ax^2y + bxy + c$

(*iv*) $p = x^4 - 2xy^2 + y^4,\ q = 4xy^3 - 2x^2y - \sin y$ **Ans.** $z = x^5/5 - x^2y^2 + xy^4 + \cos y + c$

(*v*) $p = (e^y + 1)\cos x,\ q = e^y \sin x$ **Ans.** $z = (e^y + 1)\sin x + c$

(*vi*) $p = y\,(1 + 1/x) + \cos y,\ q = x + \log x - x\sin y$ **Ans.** $z = y\,(x + \log x) + x\cos y + c$

(*vii*) $p = y^2 e^{xy^2} + 4x^3,\ q = 2xy\, e^{xy^2} - 3y^2$ **Ans.** $z = e^{xy^2} + x^4 - y^3 + c$

(*viii*) $p = \sin x \cos y + e^{3x},\ q = \cos x \sin y + \tan y$

Ans. $x = (1/3) \times e^{3x} - \cos x \cos y + \log \sec x + c$

3.7. Charpit's method.* (General method of solving partial differential equations of order one but of any degree. **[Agra 2003; Delhi Maths (H) 2000, 05, 06, 08-11; Kanpur 1998; Meeerut 2003, 05; Nagpur 2002, 04, 06, 08; Rohilkhand 2001, 04]**

Let the given partial equation differential of first order and non–linear in p and q be

$$f(x, y, z, p, q) = 0. \qquad ...(1)$$

We know that
$$dz = p\, dx + q\, dy. \qquad ...(2)$$

The next step consists in finding another relation $F(x, y, z, p, q) = 0$ ...(3)
such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad ...\ (4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad ...\ (5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad ...\ (6)$$

and
$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0. \qquad ...\ (7)$$

Eliminating $\partial p/\partial x$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or
$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0. \qquad ...(8)$$

Similarly, eliminating $\partial q/\partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0. \qquad ...(9)$$

Since $\partial q/\partial x = \partial^2 z/\partial x\partial y = \partial p/\partial y$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).

Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0. \qquad ...(10)$$

This is a linear equation of the first order to obtain the desired function F. As in Art 2.20 of chapter 2, integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{(\partial f/\partial x) + p(\partial f/\partial z)} = \frac{dq}{(\partial f/\partial y) + q(\partial f/\partial z)} + \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dF}{0}. \quad ...(11)$$

---

*This is general method for solving equations with two independent variables. Since the solution by this method is generally more complicated, this method is applied to solve equations which cannot be reduced to any of the standard forms which will be discussed later on. Thus, Charpit's method is used in two situations (*i*) When you are asked to solve a problem by Charpit's method (*ii*) when the given equation is not of any four standard forms given in Articles 3.10, 3.12, 3.14 and 3.17.

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\partial f / \partial x = f_x, \qquad \partial f / \partial y = f_y, \qquad \partial f / \partial z = f_z, \qquad \partial f / \partial p = f_p, \qquad \partial f / \partial q = f_q.$$

Therefore, Charpit's auxiliary equations (11) may be re–written as

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \qquad \text{... (11)}'$$

3.8A. WORKING RULE WHILE USING CHARPIT'S METHOD

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (11) or (11)′.

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x$, $\partial f/\partial y$ ..., *i.e.*, f_x, f_y, ... etc. occuring in step 2 and put these in Charpit's equations (11) or (11)′.

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.

Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

3.8.B. SOLVED–EXAMPLES BASED ON ART. 3.8A.

Ex. 1. *Find a complete integral of* $z = px + qy + p^2 + q^2$.

[Bilaspur 2000l; Bhopal 1996, I.A.S. 1996; Indore 2000; Jabalpur 2000; K.U. Kurukshetra 2005; Ravishankar 2000; 04; Meerut 2010; Garhwal 2010]

Sol. Let $\qquad f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \qquad$ ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q} \qquad$ ... (2)

From (1), $\quad f_x = -p, \quad f_y = -q, \quad f_z = 0, \quad f_p = -x - 2p \quad$ and $\quad f_q = -y - 2q \qquad$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \qquad \text{... (4)}$$

Taking the first fraction of (4), $\quad dp = 0 \quad$ so that $\quad p = a \qquad$ ... (5)

Taking the second fraction of (4), $\quad dq = 0 \quad$ so that $\quad q = b \qquad$ ... (6)

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$$z = ax + by + a^2 + b^2, \ a, b \text{ being arbitrary constants.}$$

Ex. 2. *Find a complete integral of* $q = 3p^2$. **[Agra 2006]**

Sol. Here given equation is $\qquad f(x, y, z, p, q) \equiv 3p^2 - q = 0. \qquad$ ...(1)

$\therefore$ Charpit's auxiliary equations are $\quad \dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $$\frac{dp}{0+p.0}=\frac{dq}{0+q.0}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1}, \text{ using (1)} \qquad ...(2)$$

Taking the first fraction of (1), $dp = 0$ so that $p = a$. ...(3)

Substituting this value of p in (1), we get $q = 3a^2$. ...(4)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$ so that $z = ax + 3a^2y + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 3. *Find the complete integral of* $zpq = p + q$ **[Nagpur 2010; Meerut 2006]**

Sol. Let $f(x, y, z, p, q) = zpq - p - q = 0$ ... (1)

Here Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \qquad ... (2)$$

From (1), $f_x = 0$, $f_y = 0$, $f_z = pq$, $f_p = zq - 1$ and $f_q = zp - 1$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{p^2q}=\frac{dq}{pq^2}=\ldots\ldots \quad \text{or} \quad \frac{dp}{p}=\frac{dq}{q} \quad \text{so that} \quad p = aq \qquad ... (4)$$

Solving (1) and (2), $p = (1 + a)/z$ and $q = (1 + a)/az$.

$\therefore$ $dz = pdx + qdy = [(1 + a)/z]dx + [(1 + a)/az]dy$ or $2zdz = 2(1 + a)\,[dx + (1/a)dy]$

Integrating, $z^2 = 2(1 + a)\,[x + (1/a)y] + b$, a, b being arbitrary constants

Ex. 4. *Find a complete integral of* $p^2 - y^2q = y^2 - x^2$. **[M.D.U. Rohtak 2006]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2 - y^2q - y^2 + x^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2x}=\frac{dq}{-2qy-2y}=\frac{dz}{-p(2p)-q(-q^2)}=\frac{dx}{-2p}=\frac{dy}{y^2}, \text{ using (1)} \qquad ...(2)$$

Taking the first and fourth fractions, $pdp + xdx = 0$ so that $p^2 + x^2 = a^2$ ... (3)

Solving (1) and (3) for p and q, $p = (a^2 - x^2)^{1/2}$, $q = a^2y^{-2} - 1$.

$\therefore$ $dz = pdx + qdy = (a^2 - x^2)^{1/2}dx + (a^2y^{-2} - 1)dy$.

Integrating, $z = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a) - (a^2/y) - y + b$.

Ex. 5. *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997; Meerut 2007]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dq}{-f_q}$$

or $$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}, \text{ by (1)} \qquad ... (2)$$

Taking the first two fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$.

Solving (1) and (2) for p and q, $$p = \frac{a}{z(a^2z^2+1)^{1/2}}, \qquad q = \frac{1}{z(a^2z^2+1)^{1/2}}.$$

$\therefore$ $dz = pdx + qdy = (a\,dx + dy)/z\,(a^2z^2 + 1)^{1/2}$ or $adx + dy = z(a^2z^2 + 1)^{1/2}dz$.

Integrating, $ax + y = \int (a^2z^2 + 1)^{1/2} . zdz$. ...(3)

Putting $a^2z^2 + 1 = t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$ax + y = \int (1/a^2)\, t . t\, dt$ or $ax + y + b = (1/3a^2)t^3$, where $t = (a^2z^2 + 1)^{1/2}$

or $ax + y + b = (1/3a^2) \times (a^2z^2 + 1)^{3/2}$ or $9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3$,

which is a complete integral, a and b being arbitrary constants.

Ex. 6. *Find a complete integral of* $px + qy = pq$. **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+p.0} = \frac{dq}{q+q.0}, \text{ by (1)} \quad ...(2)$$

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq$.

Integrating, $\log p = \log q + \log a$ or $p = aq$. ...(3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y$, as $q \neq 0$ ...(4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y$. ...(5)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\, dy$ or $adz = (ax + y)\, (adx + dy)$

or $adz = (ax + y)\, d(ax + y) = udu$, where $u = ax + y$.

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 7. *Find the complete integrals of following equations:*

(*i*) $q = (z + px)^2$ **[Indore 2004; Ravishanker 2005]**

(*ii*) $p = (z + qy)^2$ **[Meerut 2008, 09; Agra 2001; Delhi B.Sc. (Prog) 2008; Kurukshetra 2005]**

Sol. (*i*). Here given equations is $f(x, y, z, p, q) = (z + px)^2 - q = 0$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{2p(z+px)+2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{0}, \text{ by (1)}$$

Taking the second and fourth fractions, $(1/q)dq = -(1/x)dx$.

Integrating, $\log q = \log a - \log x$ so that $q = a/x$. ...(2)

Substituting the above value of q in (1), we have

$(z + px)^2 = a/x$ or $px = \sqrt{a}/\sqrt{x} - z$ or $p = \sqrt{a}/x\sqrt{x} - z/x$. ...(3)

$$\therefore \quad dz = pdx + qdy = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right) dx + \frac{a}{x} dy, \text{ by (2) and (3)}$$

or $xdz = \sqrt{a}\, x^{-1/2}dx - zdx + ady$ or $xdz + zdx = \sqrt{a}\, x^{-1/2}dx + ady$

or $d(xz) = \sqrt{a}\, x^{-1/2}dx + ady$.

Integrating, $xz = 2\sqrt{a}\sqrt{x} + ay + b$, a, b being arbitrary constants

(*ii*) **Sol.** Do as in part (1). **Ans.** $yz = ax + \sqrt{ay} + b$.

Ex. 8. *Find a complete integral of* $yzp^2 - q = 0$.

Sol. Here $$f(x, y, z, p, q) = yzp^2 - q = 0.$$

Charpit's uxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-2yzp^2 + q} = \frac{dx}{-2yzp} = \frac{dy}{1}, \text{ by (1)} \qquad ...(2)$$

Taking the first and fifth fractions, $(1/yp^3)\ dp = dy$

or $p^{-3}dp = ydy$ or $-2p^{-3}dp = -2ydy$.

Integrating, $p^{-2} = a^2 - y^2$ so that $p = 1/(a^2 - y^2)^{1/2}$. ...(3)

Using (3), (1) $\Rightarrow$ $q = yzp^2$ $\Rightarrow$ $q = yz/(a^2 - y^2)$. ...(4)

$$\therefore \qquad dz = pdx + qdy = \frac{dx}{(a^2 - y^2)^{1/2}} + \frac{yzdy}{(a^2 - y^2)}$$

or $$(a^2 - y^2)^{1/2}\, dz - \frac{yzdy}{(a^2 - y^2)^{1/2}} = dx \qquad \text{or} \qquad d\,[z\,(a^2 - y^2)^{1/2}] = dx.$$

Integrating, $z\,(a^2 - y^2)^{1/2} = x + b$ or $z^2(a^2 - y^2) = (x + b)^2$, a, b being arbitrary constants.

Ex. 9. *Find a complete integral of* $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$. **[I.A.S. 1994]**

Sol. Given equation is $$f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)} = \frac{dz}{-p(32pz^2) - q(18qz^2)} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}.$$

Taking the first and second fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$ ... (2)

Solving (1) and (2) for p and q, we have

$$q = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}} \qquad \text{and} \qquad p = \frac{2a(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}. \qquad ...(3)$$

Hence, $$dz = pdx + qdy = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}\ (adx + dy), \text{ using (3)}$$

or $$(1/2) \times (16a^2 + 9)^{1/2}\,(1 - z^2)^{-1/2}\,(-2zdz) = -\,2(adx + dy). \qquad ...(4)$$

Putting $1 - z^2 = t$ so that $-2zdz = dt$, (4) becomes

or $$(1/2) \times (16a^2 + 9)^{1/2}\, t^{-1/2}\, dt = -\,2\,(adx + dy).$$

Integrating, $(16a^2 + 9)^{1/2}\, t^{1/2} = -\,2(ax + y) + b$, a, b being arbitrary constants.

or $$(16a^2 + 9)^{1/2}\sqrt{(1 - z^2)} + 2(ax + y) = b, \text{ as } t = 1 - z^2.$$

Ex. 10(a). *Find a complete integral of* $(p^2 + q^2)x = pz$.

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation* $(p^2 + q^2)x = pz$ *and deduce the solution which passes through the curve* $x = 0$, $z^2 = 4y$. **[Meerut 2007]**

Sol. Let $$f(x, y, q, p, q) = (p^2 + q^2)x - pz = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or $2pdp + 2qdq = 0$.

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant. ...(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$. ...(3)

$\therefore$ $dz = pdx + qdy = \dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2-a^2x^2)}\,dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2-a^2x^2)}} = ady.$

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2xdx) = dt$, we get

$(1/2\sqrt{t})dt = ady$ or $(1/2)\times t^{-1/2} = ady.$

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2-a^2x^2)} = ay + b,$ as $t = \sqrt{z^2 - a^2x^2}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$. ... (4)

(*b*) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve $x = 0$, $z^2 = 4y$ are given by

$x = 0,$ $y = t^2,$ $z = 2t$ ... (5)

Therefore the intersections of (1) and (2) are determined by

$4t^2 = (at^2 + b)^2$ or $a^2t^4 + 2(ab - 2)t^2 + b^2 = 0$ ... (6)

Equation (6) has equal roots if its discriminant = 0, *i.e.*, if

$4(ab - 2)^2 - 4a^2b^2 = 0$ or $a^2b^2 = 1$ so that $b = 1/a$

Hence from (4), the appropriate one paremeter sub–system is given by

$z^2 = a^2x^2 + (ay + 1/a)^2$ or $a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0,$

which is a quatratic equation in parameter 'a'. Therefore, this has for its envelope surface

$(2y - z^2)^2 - 4(x^2 + y^2) = 0$ or $(2y - z^2)^2 = 4(x^2 + y^2)$ ... (7)

The desired solution is given by the function z defined by equation (7).

Ex. 10(c). *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.
[Guwahati 2007; Agra 2001; Bilaspur 1998; Delhi Maths (H) 2003, 05; Garhwal 2005; Meerut 2010, 11; K.V. Kurukshetra 2004; Kanpur 2005; Rohilkhand 2001; Pune 2010]

Sol. Here the given equation is $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-2p^2y + qz - 2q^2y} = \dfrac{dx}{-2py} = \dfrac{dy}{-2qy + z}$, by (1) ...(2)

Taking the first two fractions, we get $2pdp + 2qdq = 0$ so that $p^2 + q^2 = a$ ... (3)

Using (3), (1) gives $a^2y = qz$ or $q = a^2y/z.$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2 - q^2)} = \sqrt{a^2 - (a^4y^2/z^2)} = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,.$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$dz = \dfrac{a}{z}\sqrt{(z^2 - a^2y^2)}\,dx + \dfrac{a^2y\,dy}{z}dy$ or $\dfrac{zdz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = a\,dx.$

Integrating, $(z^2 - a^2y^2)^{1/2} = ax + b$ or $z^2 - a^2y^2 = (ax + b)^2$, ...(4)
which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2\,(ax + b)x \qquad ...(5)$$

and
$$0 = 2(ax + b). \qquad ...(6)$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \qquad ...(7)$$

Differentiating (7) partially w.r.t. a, $\quad -2ay^2 = 2[ax + \phi(a)]\,.\,[x + \phi'(a)]$. ...(8)

General integral is obtained by eliminating a from (7) and (8).

Ex. 11. *Find a complete integral of* $p\,(1 + q^2) + (b - z)q = 0$. **[Agra 1996]**

Sol. Here given equation is $\quad f(x, y, z, p, q) \equiv p(1 + q^2) + (b - z)q = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or
$$\frac{dp}{pq}=\frac{dq}{p^2} = \frac{dz}{-p(1+q^2)-(b-z)q} = \frac{dx}{-(q^2+1)}=\frac{dy}{-2pq-(b-z)}, \text{ by (1)}$$

First two fractions give $\quad (1/p)dp = (1/q)dq \quad$ so that $\quad q = pc$.

Putting $q = pc$ in (1), we have $\quad p = \sqrt{[c(z-b)-1]}\,/c$.

$\therefore \quad q = pc \quad$ gives $\quad q = \sqrt{[c(z-b)-1]}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{[c(z-b)-1]}\,\left(\frac{dx}{c}+dy\right) \quad \text{or} \quad \frac{cdz}{\sqrt{[c(z-b)-1]}} = dx + c\,dy.$$

Integrating, $\quad 2\sqrt{[c(z-b)-1]} = x + cy + a \quad$ or $\quad 4\{c(z - b) - 1\} = (x + cy + a)^2$
which is a complete integral, a and c being arbitrary constants.

Ex. 12. *Find a complete and singular integrals of* $2xz - px^2 - 2q\,xy + pq = 0$. **[I.A.S. 1991, 93, 2007, 2008; Delhi Hons. 2001, 01, 05; Kanpur 2001, 03; Meerut 2005; Bhopal 2004, 10; Indore 1999; M.D.U. Rohtak 2004, Ravishanker 2004; Rajasthan 2000, 03, 05, 10]**

Sol. Here given equation is $\quad f(x, y, z, p, q) = 2xz - px - 2qxy + pq = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or
$$\frac{dp}{2z-2qy}=\frac{dq}{0} = \frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}, \text{ by (1)}$$

The second fraction gives $\quad dq = 0 \quad$ so that $\quad q = a$

Putting $\quad q = a$ in (1), we get $\quad p = 2x(z - ay)/(x^2 - a)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy \quad \text{or} \quad \frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}.$$

Integrating, $\quad \log\,(z - ay) = \log\,(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a) \quad$ or $\quad z = ay + b(x^2 - a), \quad$ ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$$0 = y - b \quad \text{and} \quad 0 = x^2 - a. \qquad ...(3)$$

Solving (3) for a and b, $\quad a = x^2 \quad$ and $\quad b = y. \quad$ ...(4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2 y$, which is the required singular integral.

Ex. 13. *Find a complete integrals of the following partial differential equations:*

(*i*) $q = px + p^2$. **[Sagar 2003; Meerut 1994]**

(*ii*) $q = -px + p^2$.

Sol. (*i*) Here given equation is $\quad f(x, y, z, p, q) \equiv q - px - p^2 = 0. \quad$ ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{-p} = \dfrac{dq}{0} = \dfrac{dz}{-p(-x-2p)-q} = \dfrac{dx}{-(-x-2p)} = \dfrac{dy}{-1}$, by (1)

The 2nd fraction gives $\quad dq = 0 \quad$ so that $\quad q = a.$

Putting $q = a$ in (1) gives $p^2 + px - a = 0$ so that $\quad p = (1/2)\times\left[-x \pm (x^2+4a)^{1/2}\right]$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = -(x/2) \times dx \pm (1/2)\times(x^2+4a)^{1/2}\, dx + a\, dy.$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2+4a)^{1/2} + 2a\log\{x+(x^2+4a)^{1/2}\}\right] + ay + b,$$

Part (*ii*). Proceed like part (*i*) yourself. Complete integral is

$$z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2+4a)^{1/2} + 2a\log\{x+(x^2+4a)^{1/2}\}\right] + ay + b.$$

Ex. 14. *Find a complete integral of* $pxy + pq + qy = yz$. **[Delhi B.A. (Prog) H 2010]** **[Garhwal 2001; Rohilkhand 1999; Meerut 2001, 02; Kanpur 2005]**

Sol. Given $\quad f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0. \quad$ ...(1)

Charpit's auxiliary equation are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{0} = \dfrac{dq}{(px+q)+qy} = \dfrac{dz}{-p(xy+q)-q(p+y)} = \dfrac{dx}{-(xy+q)} = \dfrac{dy}{-(p+y)}$, by (1)

The first fraction gives $\quad dp = 0 \quad$ so that $\quad p = a.$

Putting $p = a$ in (1) gives $\quad axy + aq + qy = yz \quad$ so that $\quad q = y(z - ax)/(a + y).$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = adx + \frac{y(z-ax)}{a+y}dy \quad \text{or} \quad \frac{dz - adx}{z-ax} = \frac{y\,dy}{a+y} = \left(1 - \frac{a}{a+y}\right)dy.$$

Integrating, $\quad \log(z - ax) = y - a\log(a + y) + \log b$, a, b, being arbitrary constants.

or $\quad \log(z - ax) + \log(a + y)^a - \log b = y \quad$ or $\quad (z - ax)(y + a)^a = be^y$

Ex. 15. *Find a complete integral* $p^2 + q^2 - 2px - 2qy + 1 = 0$.

[Patna 2003; Meerut 99, 2003; Delhi Maths Hons 91; Ravishanker 2010]

Sol. Given $$f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 1 = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x)-q(2q-2y)} = \frac{dx}{-(2p-2y)} = \frac{dy}{-(2q-2y)}, \text{ by (1)}$$

The first two fractions give $(1/p)dp = (1/q)dq$ so that $p = aq$.
Putting $p = aq$ in (1), $a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$ or $(a^2 + 1)q^2 - 2(ax - y)q + 1 = 0$.

$$\Rightarrow \; q = \frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}, \; p = aq = a\,\frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}.$$

Putting these values of p and q in $dz = p\,dx + y\,dy$, we get

$$dz = \frac{(ax+y) \pm \sqrt{\{(ax+y)^2 - (a^2+1)\}}}{(a^2+1)}(adx + dy). \qquad ...(2)$$

Put $ax + y = v$ so that $a\,dx + dy = dv$. Then (2) gives

$$(a^2 + 1)dz = [v \pm \sqrt{\{v^2 - (a^2+1)\}}]dv.$$

Integrating, $$(a^2 + 1)\, z = v^2/2 \pm [\,(v/2) \times \sqrt{\{v^2 - (a^2+1)\}}$$
$$- (1/2) \times (a^2 + 1) \log (v + \sqrt{\{v^2 - (a^2+1)\}})] + b$$

is the complete integral, where $v = ax + b$ and a, b are arbitrary constants.

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $$f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives $$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $dp + dq = dx + dy$ *i.e.*, $dp - dx + dq - dy = 0$.

Integrating, $$(p - x) + (q - y) = a \qquad ...(2)$$

Re–writing (1), $$(p - x)^2 + (q - y)^2 = (x - y)^2. \qquad ...(3)$$

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2$ or $2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0$.

$$\therefore \; p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \quad \Rightarrow \quad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}],$$

$$\therefore \text{ (2) gives} \quad q = a + y - p + x \quad \text{or} \quad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}].$$

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or
$$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2}.(dx - dy).$$

Integrating, the desired complete integral is

$$z = \frac{x^2+y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} \;-\frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $\quad f(x, y, z, p, q) = p^2x + q^2y - z = 0.$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} - \dfrac{dy}{-f_q}$

or
$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2) $= \dfrac{2px\,dp + p^2dx}{2px(-p+p^2) + p^2(-2px)} = \dfrac{2qy\,dq + q^2dy}{2qy(-q+q^2) + q^2(-2qy)}$

or
$$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\quad \log(p^2x) = \log(q^2y) + \log a \quad$ or $\quad p^2x = q^2ya.$ ...(3)

Form (1) and (3), $\quad aq^2y + q^2y = z \quad$ or $\quad q = [z/(1+a)]^{1/2}.$ ...(4)

Form (3) and (4), $\quad p = q\left(\dfrac{ya}{x}\right)^{1/2} = \left\{\dfrac{za}{(1+a)x}\right\}^{1/2}.$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy \qquad \text{or} \qquad (1+a)^{1/2}z^{-1/2}\,dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $\quad (1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 18. *Find a complete integral of* $2z + p^2 + qy + 2y^2 = 0$. **[I.F.S. 2005; Meerut 2000; Rohilkhand 1993; Bilaspur 2004, M.D.U Rohtak 2005; Rawa 1999; Ranchi 2010]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 2z + p^2 + qy^2 + 2y^2 = 0.$ ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dp}{-f_p} = \dfrac{dq}{-f_q}$

or
$$\frac{dp}{0+2p} = \frac{dq}{(q+4y)+2q} = \frac{dz}{-p\times(2p) - qy} = \frac{dx}{-2p} = \frac{dy}{-y}, \text{ by (1)}$$

Taking the first and fourth fractions, $\quad dp = -\,dx.$

Integrating, $\quad p = a - x \quad$ or $\quad p = -(x - a).$ ...(2)

Using (2), (1) becomes $\quad 2z + (a-x)^2 + qy + 2y^2 = 0$

$\therefore \quad q = -[2z + (x-a)^2 + 2y^2]/y.$ ...(3)

$\therefore$ $dz = p\,dx + q\,dy = -(x-a)\,dx - [\{2z + (x-a)^2 + 2y^2\}/y]dy$, by (2) and (3)

Multiplying both sides by $2y^2$ and re–writing, we have

$$2y^2dz = -2(x-a)y^2\,dx - 4zydy - 2y(x-a)^2dy - 4y^3dy$$

or $$2(y^2dz + 2zy\,dy) + [2(x-a)^2y^2dx + 2y(x-a)^2dy] + 4y^3dy = 0$$

or $$2d(y^2z) + d[y^2(x-a)^2] + 4y^3dy = 0.$$

Integrating, $2y^2z + y^2(x-a)^2 + y^4 = b$, a, b being arbitrary constants

Ex. 19(a). *Find a complete integral of* $2(z + px + qy) = yp^2$.

[Delhi B.A. (Prog.) II 2007, 10; CDLU 2004; Delhi Maths Hons. 1998, 2008]

Sol. Given equation is $f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dp}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-q\times 2y} = \frac{dx}{-(2x-2yp)} = \frac{dy}{-2y}, \text{ by (1)}$$

Taking the first and the last fractions, $\dfrac{dp}{4p} = \dfrac{dy}{-2y}$ or $\dfrac{dp}{p} + 2\dfrac{dy}{y} = 0.$

Integrating, $\log p + 2\log y = \log a$ or $py^2 = a$. ...(2)

Solving (1) and (2) for p and q, $p = \dfrac{a}{y^2}$ and $q = -\dfrac{z}{y} - \dfrac{ax}{y^3} + \dfrac{a^2}{2y^4}$.

$$\therefore \qquad dz = p\,dx + q\,dy = \frac{a}{y^2}dx + \left[-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right]dy$$

Multiplying both sides by y and re–arranging, we get

$$(ydz + zdy) - a\left(\frac{y\,dx - x\,dy}{y^2}\right) - \frac{a^2}{2y^3}\,dy = 0 \qquad \text{or} \qquad d(yz) - ad\left(\frac{x}{y}\right) - \frac{a^2}{2}y^{-3}\,dy = 0.$$

Integrating, $yz - a(x/y) + (a^2/4y^2) = b$, a, b being arbitrary constants. ... (3)

Ex. 19(b). *Find the complete integral, general integral and the singular integral of* $2(z + xp + yq) = yp^2$ **[Delhi B.Sc. (H) 1998, 2008]**

Sol. Proceed as in solved Ex. 19(a) to get the complete integral (3).

General integral. Replacing b by $\phi(a)$ in (3), we get

$$yz - a(x/y) + (a^2/4y^2) = \phi(a) \qquad \text{... (4)}$$

Differentiating (4) partially w.r.t. 'a', $-(x/y) + (a/2y^2) = \phi'(a)$ ... (5)

Then the general integral is obtained by eliminating a from (4) and (5).

Singular integral. Differentiating (3) partially w.r.t. 'a' and 'b' by turn, we get

$-(x/y) + (a/2y^2) = 0$ ... (6) $\qquad 0 = 1$ ... (7)

Relation (7) is absurd and hence there is no singular solution of the given equation.

Ex. 20. *Find a complete integral of* $z^2 = pqxy$. **[Delhi B.A. (Prog) II 2010]**

[Delhi Maths (H) 2004 Jabalpur 2004; Meerut 2006; Lucknow 2010]

Sol. The given equation is $f(x, y, z, p, q) = z^2 - pqxy = 0.$ ...(1)

Charpits's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-pqy+2pz}=\frac{dq}{-pqx+2qz}=\frac{dz}{-p(-qxy)-q(-pxy)}=\frac{dx}{qxy}=\frac{dy}{pxy}, \text{ by (1)} \qquad ...(2)$$

Each fraction of (2) $= \dfrac{x\,dp+p\,dx}{x(-pqy+2pz)+pqxy} = \dfrac{y\,dq+q\,dy}{y(-pqx+2qz)+pqxy}$

or $$\frac{xdp+pdx}{2pxz}=\frac{ydq+qdy}{2qyz} \quad \text{or} \quad \frac{d(xp)}{xp}=\frac{d(yq)}{yq}.$$

Integrating, $\log(xp) = \log(yq) + \log a^2$ or $xp = a^2yq$. ...(3)

Solving (1) and (2) for p and q, $p = (az)/x$ and $q = z/(ay)$.

$\therefore$ $dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)\,dy$ or $(1/z)\,dz = (a/x)\,dx + (1/ay)\,dy$.

Integrating, $\log z = a \log x + (1/a)\log y + \log b$ or $z = x^a y^{1/a}\, b$.

Ex. 21. *Using Charpit's method, find three complete integrals of $pq = px + qy$.*

(Kanpur 2004; Meerut 2002; Rajasthan 2001)

Sol. Here given equation is $f(x, y, z, p, q) = pq - px - qy = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-p(q-x)-p(p-y)}=\frac{dx}{-(q-x)}=\frac{dy}{-(p-y)}, \text{ by (1)} \qquad ...(2)$$

To find first complete integral. Taking the first two fractions of (2), we get

$(1/p)dp = (1/q)dq$ so that $\log p = \log q + \log a$ or $p = aq$. ...(3)

Using (3), (1)$\Rightarrow$ $aq^2 = q(ax + y)$ $\Rightarrow$ $q = (ax + y)/a$. ...(4)

Hence, from (3), we have $p = ax + y$. ...(5)

$\therefore$ $dz = p\,dx + q\,dy = (ax + y)dx + [(ax + y)/a]dy = (1/a)(ax + y)(a\,dx + y)$.

Putting $ax + y = t$ so that $adx + dy = dt$, we get

$dz = (1/a)\times t\,dt$ so that $z = (1/2a) \times t^2 + b$ or $z = (1/2a) \times (ax + y)^2 + b$, as $t = ax + y$.

To find second complete integral. Taking the second and the fourth ratios in (2), we get

$dx/(q - x) = dq/q$ or $q\,dx + x\,dq = q\,dq$.

Integrating, $qx = q^2/2 + a/2$ or $q^2 - 2xq + a = 0$.

$\therefore$ $q = [2x \pm 2(x^2-a)^{1/2}]/2$ so that $q = x + (x^2-a)^{1/2}$. ...(6)

Using (6), (1) $\Rightarrow$ $p[x + (x^2-a)^{1/2}] - px - y[x + (x^2-a)^{1/2}] = 0$

so that $$p = \left\{1+x/(x^2-a)^{1/2}\right\}y. \qquad ...(7)$$

$$\therefore \quad dz = p\,dx + q\,dy = \left\{1+x/(x^2-a)^{1/2}\right\}ydx + [x + (x^2-a)^{1/2}]dy$$

or $$dz = (y\,dx + x\,dy) + \left[\frac{xy\,dy}{(x^2-a)^{1/2}}+(x^2-a)^{1/2}dy\right] \quad \text{or} \quad dz = d(xy) + d[y(x^2-a)^{1/2}].$$

Integreing, $z = xy + y(x^2-a)^{1/2} + b$, a, b being arbitrary constants.

To find third complete integral. Taking the first and the fifth ratios of (2) and proceeding as above third complete integral is $z = xy + x(y^2-a)^{1/2} + b$.

Ex. 22. *Find complete integral of $xp + 3yq = 2(z - x^2q^2)$.* **[Delhi B.Sc. (Prog) II 2009; Delhi B.Sc. (Hons) II 2010;]**

Sol. Given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p + 4xq^2} = \frac{dq}{q} = \frac{dz}{-p\,x - q(3y + 4x^2 q)} = \frac{dx}{-x} = \frac{dy}{-3y - 4x^2 q}, \text{ by (1)} \qquad ...(2)$$

(2) $\Rightarrow$ $\dfrac{dq}{q} = \dfrac{dx}{-x}$ $\Rightarrow$ $\log q = \log a - \log x$ $\Rightarrow$ $qx = a$ $\Rightarrow$ $q = \dfrac{a}{x}$. ...(3)

Using (3), (1) $\Rightarrow$ $xp + 3y(a/x) - 2z + 2x^2(a^2/x^2) = 0$ $\Rightarrow$ $p = \dfrac{2(z - a^2)}{x} - \dfrac{3ay}{x^2}$. ...(4)

$\therefore$ $$dz = p\,dx + q\,dy = \left\{\frac{2(z - a^2)}{x} - \frac{3ay}{x^2}\right\} dx + \frac{a}{x} dy$$

or $x^2 dz = 2x(z - a^2)dx - 3ay\,dx + ax\,dy$ or $x^2 dz - 2x(z - a^2)\,dx = -3ay\,dx + ax\,dy$

or $\dfrac{x^2 dz - 2x(z - a^2)dx}{x^4} = -\dfrac{3ay\,dx}{x^4} + \dfrac{a\,dy}{x^3}$ or $d\left(\dfrac{z - a^2}{x^2}\right) = d\left(\dfrac{ay}{x^3}\right)$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$ or $z = a(a + y/x) + bx^2$.

Ex. 23. *Find complete integrals of the following equations* :

(*i*) $(p^2 + q^2)^n\,(qx - py) = 1$.

(*ii*) $qx + py = (p^2 - q^2)^n$.

Sol. (*i*) Given equation is $f(x, y, z, p, q) = (p^2 + q^2)^n\,(qx - py) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $\dfrac{dp}{q(p^2 + q^2)^n} = \dfrac{dq}{-p(p^2 + q^2)^n} = \dots$ or $\dfrac{dp}{q} = \dfrac{dq}{-p}$ or $pdp + qdq = 0$.

Integrating, $p^2 + q^2$ = constant = $(1/a^2)$, say ...(2)

Using (2), (1) $\Rightarrow qx - py = a^{2n}$ or $qx = py + a^{2n}$. ...(3)

Using (3), (2) $\Rightarrow$ $p^2 + (p^2y^2 + a^{4n} + 2a^{2n}yp)/x^2 = 1/a^2$

or $p^2(x^2 + y^2) + 2a^{2n}yp + \{a^{4n} - (x^2/a^2)\} = 0$ so that

$$p = \frac{-ya^{2n} + \sqrt{\left\{a^{4n}y^2 - (x^2 + y^2)\left(a^{4n} - x^2/a^2\right)\right\}}}{x^2 + y^2} = \frac{-ya^{2n} + x\sqrt{\left[\left\{(x^2 + y^2)/a^2\right\} - a^{4n}\right]}}{x^2 + y^2} \qquad ...(4)$$

$\therefore$ (3) $\Rightarrow$ $$q = \frac{xa^{2n} + y\sqrt{\left[\left\{(x^2 + y^2)/a^2\right\} - a^{4n}\right]}}{x^2 + y^2}. \qquad ...(5)$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = a^{2n}\left(\frac{x\,dy - y\,dx}{x^2 + y^2}\right) + \frac{x\,dx + y\,dy}{x^2 + y^2}\sqrt{\left\{\left(\frac{x^2 + y^2}{a^2}\right) - a^{4n}\right\}}.$$

Integrating, $z + b = a^{2n}\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\displaystyle\int \frac{1}{u}(ua^{-2} - a^{4n})^{1/2}\,du$, where $u = x^2 + y^2$.

Part (ii). Proceed as in part (i). If $u = x^2 + y^2$, then complete integral is

$$z + b = -\frac{1}{2}a^{2n}\log\frac{x-y}{x+y} - \frac{1}{2}\int\frac{1}{u}\sqrt{(a^{4n}+a^2u)}\,du.$$

Ex. 24. *Find complete integral of* $p^2 + q^2 - 2pq \tanh 2y = \text{sech}^2\, 2y$.

Sol. Given $\quad f(x, y, z, p, q) = p^2 + q^2 - 2pq \tanh 2y - \text{sech}^2\, 2y = 0.$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{0} = \frac{dq}{-4pq\,\text{sech}^2 2y + 4\,\text{sech}^2 2y \tanh 2y} = \cdots, \text{ by (1)}$$

Then, first fraction $\Rightarrow$ $dp = 0$ $\Rightarrow$ p = constant = a, say. ...(2)

Using (2), (1) $\Rightarrow$ $q^2 - (2a \tanh 2y)q + a^2 - \text{sech}^2\, 2y = 0$

$\Rightarrow$ $$q = [2a \tanh 2y \pm 2\sqrt{(a^2\tanh^2 2y - a^2 + \text{sech}^2\, 2y)}\,]/2$$

$\Rightarrow$ $$q = a \tanh 2y + \sqrt{(1-a^2)}\,.\,\text{sech}\, 2y. \qquad ...(3)$$

[Note that $\text{sech}^2 2y = 1 - \tanh^2 2y$]

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = a\,dx + \{a \tanh 2y + \sqrt{(1-a^2)}\,\text{sech}\, 2y\}dy$$

Integrating, $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\int\frac{2dy}{e^{2y}+e^{-2y}}$$

or $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\int\frac{2e^{2y}dy}{1+(e^{2y})^2}$$

or $$z + b = ax + \frac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\tan^{-1}(e^{2y}),$$

$$\left[\because \text{ on putting } e^{2y} = t \text{ and } 2e^{2y}\,dy = dt,\quad \int\frac{2e^{2y}dy}{1+(e^{2y})^2} = \int\frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} e^{2y}\right]$$

Ex. 25. *Find complete integral of the equation* $q = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px)^2$.

Sol. Let $\quad f(x, y, z, p, q) = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px^2) - q = 0.$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{\{(1+p^2)/(1+y^2)\} - 2yp^2(z-px) + 2yp^2(z-px)} = \frac{dy}{1} = \text{.........}, \text{ by (1)}$$

or $\quad \dfrac{dp}{1+p^2} = \dfrac{dy}{1+y^2}$ so that $\tan^{-1} p - \tan^{-1} y$ = constant = $\tan^{-1} a$

$\Rightarrow$ $(p - y)/(1 + py) = a$ $\Rightarrow$ $p = (y + a)/(1 - ay).$ ...(2)

Using (2), (1) $\Rightarrow$ $$q = \frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2. \qquad ...(3)$$

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = \frac{y+a}{1-ay}dx + \left[\frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2\right]dy$$

or $dz = d\left(\dfrac{y+a}{1-ay}x\right)+\dfrac{y(y+a)}{(1-ay)^3}\{z(1-ay)-x(y+a)\}^2 dy$ or $dz = du + \dfrac{y(y+a)}{1-ay}(z-u)^2 dy$, ...(4)

where $u = x(y + a)/(1 - ay)$. ...(5)

(4) $\Rightarrow$ $\dfrac{dz-du}{(z-u)^2} = \dfrac{y(y+a)}{1-ay}\text{dy}.$ or $\dfrac{d(z-u)}{(z-u)^2}=\left\{-1-\dfrac{1}{a^2}(1+ay)+\dfrac{a^2+1}{a^2}\dfrac{1}{1-ay}\right\}dy.$

Integrating, $b-\dfrac{1}{z-u} = -y-\dfrac{1}{a^2}\left(y+\dfrac{ay^2}{2}\right)-\dfrac{a^2+1}{a^3}\log(1-ay)$, where u is given by (5).

Ex. 26. Find complete integral of $xp - yq = xq\,f(z - px - qy)$.

Sol. Let $F(x, y, z, p, q) = xp - yq - xq\,f(z - px - qy) = 0.$ ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x+p(\partial F/\partial z)}=\frac{dq}{\partial F/\partial y+q(\partial F/\partial z)}=\frac{dz}{-p(\partial F/\partial p)-q(\partial F/\partial q)}=\frac{dx}{-(\partial F/\partial p)}=\frac{dy}{-(\partial F/\partial q)}$$

or $$\frac{dp}{p-qf+xqpf'-pqxf'}=\frac{dq}{-q+xq^2f'-xq^2f'}=\cdots\cdots,\ \text{by (2)} \qquad ...(3)$$

Each ratio of (3) = $\dfrac{x\,dp+y\,dq}{xp-yq-qxf}=\dfrac{x\,dp+y\,dq}{0}$, by (2)

$\Rightarrow$ $x\,dp + y\,dq = 0$ $\Rightarrow$ $x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow$ $dz - d(xp) - d(yq) = 0$, as $dz = pdx + qdy$

Integrating, $z - xp - yq =$ constant $= a$, say ...(4)

$\therefore$ $xp + yq = z - a.$ ...(5)

Using (4), (1) becomes $x\,p - y\,q = x\,q\,f(a).$ ...(6)

Subtracting (6) from (5), $2yq = z - a - xqf(a)$ $\Rightarrow$ $q = (z - a)/\{2y + xf(a)\}$ ...(7)

Using (7), (5) $\Rightarrow$ $p=\dfrac{(z-a)\{y+xf(a)\}}{x\{2y+xf(a)\}}.$ ...(8)

Using (7) and (8), $dz = p\,dx + q\,dy$ reduces to

$$dz = (z-a)\left[\frac{\{y+xf(a)\}\,dx}{x\{2y+xf(a)\}}+\frac{dy}{2y+xf(a)}\right]$$

or $$\frac{2dz}{z-a}=\frac{2y\,dx+2xf(a)dx+2x\,dy}{x\{2y+xf(a)\}}=\frac{2d(xy)+2xf(a)\,dx}{2xy+x^2f(a)}.$$

Integrating, $2\log(z-a) = \log\{2xy + x^2f(a)\} + \log b$ or $(z-a)^2 = b\,\{2xy + x^2f(a)\}$.

Ex. 27. *Find a complete integral of* $px + qy = z(1 + pq)^{1/2}$

[Meerut 2001, 02; Kanpur 1995, I.A.S. 1992]

Sol. Given $f(x, y, z, p, q) = px + qy - z(1 + pq)^{1/2} = 0.$ ...(1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-pf_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

or $\dfrac{dp}{p-p(1+pq)^{1/2}}=\dfrac{dq}{q-q(1+pq)^{1/2}}=\ldots\ldots\ldots$ so that $\dfrac{dp}{p}=\dfrac{dq}{q}$, by (1)

$\Rightarrow$ $\log p = \log a + \log q$ $\Rightarrow$ $p = aq.$ ...(2)

Using (2), (1) $\Rightarrow$ $q(ax + y) = z\,(1 + aq^2)^{1/2}$ or $q^2\,[(ax + y)^2 - az^2] = z^2.$

$\therefore\ q = \dfrac{z}{[(ax+y)^2 - az^2]^{1/2}}$ and $p = aq = \dfrac{az}{[(ax+y)^2 - az^2]^{1/2}}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{z(a\,dx+dy)}{\sqrt{\{(ax+y)^2-az^2\}}} \quad \text{or} \quad \frac{dz}{z} = \frac{a\,dx+dy}{\sqrt{\{(ax+y)^2-az^2\}}}. \qquad \text{... (3)}$$

Let $ax + y = \sqrt{a}\,u$ so that $a\,dx + dy = \sqrt{a}\,du$.

$$\therefore\ (3) \Rightarrow \frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \quad \text{or} \quad \frac{du}{dz} = \frac{\sqrt{(u^2-z^2)}}{z} = \sqrt{\left\{\left(\frac{u}{z}\right)^2 - 1\right\}}, \qquad \text{... (4)}$$

which is linear homogeneus equation. To solve it, we put

$\dfrac{u}{z} = v$ or $u = vz$ so that $\dfrac{du}{dz} = v + z\dfrac{dv}{dz}$.

$\therefore$ (4) yields $v + z\dfrac{dv}{dz} = (v^2-1)^{1/2}$. or $\dfrac{dz}{z} = \dfrac{dv}{(v^2-1)^{1/2} - v}$

or $(1/z)dz = -\left[(v^2-1)^{1/2} + v\right]dv$, on rationalization.

Integrating, $\log z = -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\}\right] - \dfrac{v^2}{2} + b$, where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex. 28. *Find complete integral of* $(x^2 - y^2)\,pq - xy(p^2 - q^2) = 1$.

Sol. Let $f(x, y, z, p, q) = (x^2 - y^2)\,pq - xy(p^2 - q^2) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{2pqx - z(p^2-q^2)} = \dfrac{dq}{-2pqy - x(p^2-q^2)} = \dfrac{dx}{-(x^2-y^2)y + 2pxy} = \dfrac{dy}{-(x^2-y^2)p - 2pxy}$, by (1)

Using x, y, p, q as multipliers, each fraction $= \dfrac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \dfrac{d(xp) + d(yq)}{0}$

$\Rightarrow\ d(xp + qy) = 0 \ \Rightarrow\ xp + yq = a \ \Rightarrow\ p = (a - qy)/x$. ...(2)

Using (2), (1) $\Rightarrow$ $(x^2 - y^2)\left(\dfrac{a-qx}{x}\right)q - xy\left[\left(\dfrac{a-qy}{x}\right)^2 - q^2\right] - 1 = 0$

or $\dfrac{a-qy}{x}\{(x^2-y^2)q - (a-qy)y\} + xyq^2 - 1 = 0$ or $\{(a - qy)/x\}\,(x^2q - ay) + xyq^2 - 1 = 0$

or $(a - qy)\,(x^2q - ay) + x^2yq^2 - x = 0$ or $aq(x^2 + y^2) = a^2y + x$

$\therefore\ q = \dfrac{a^2y + x}{a(x^2+y^2)}$ and $p = \dfrac{1}{x}\left[a - \dfrac{(a^2y+x)y}{a(x^2+y^2)}\right] = \dfrac{a^2x - y}{a(x^2+y^2)}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)dx + (a^2y + x)dy}{a(x^2+y^2)} = a\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{x\,dy - y\,dx}{a(x^2+y^2)}.$$

Integrating, $z = (a/2) \times \log(x^2 + y^2) + (1/a) \times \tan^{-1}(y/x) + b$.

Ex. 29. *Find a complete integral of* $2(pq + yp + qx) + x^2 + y^2 = 0$. **[Kanpur 1993]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0. \quad$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $\quad \dfrac{dp}{2q+2x}=\dfrac{dq}{2p+2y}=\dfrac{dz}{-p(2q+2y)-q(2p+2x)}=\dfrac{dx}{-(2q+2y)}=\dfrac{dy}{-(2p+2x)}$, by (1)

Each of these above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$
$= (dp + dq + dx + dy)/0$

This $\Rightarrow \quad dp + dq + dx + dy = 0 \quad$ so that $\quad (p + x) + (q + y) = a. \quad$ ...(2)

Re–writing (1), $2(p + x)(q + y) + (x - y)^2 = 0 \quad$ or $\quad (p + x)(q + y) = -(x - y)^2/2. \quad$ ...(3)

Now, $\quad (p + x) - (q + y) = \sqrt{\{(p+x)^2+(q+y)\}^2-4(p+x)(q+y)}$

$\therefore \quad (p + x) - (q + y) = \sqrt{a^2+2(x-y)^2}$, using (2) and (3) $\quad$ ...(4)

Adding (2) and (4), $\quad 2(p + x) = a + \sqrt{a^2+2(x-y)^2}$.

Substracting (4) from (2), $\quad 2(q + y) = a - \sqrt{a^2+2(x-y)^2}$.

These give $\quad p = -x + \dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}, \quad q = -y + \dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$

Substituting the above values of p and q, $dz = p\,dx + q\,dy$ becomes

$$dz = -(x\,dx + y\,dy) + (a/2) \times (dx + dy) + (1/2)\times\sqrt{a^2+2(x-y)^2}\ (dx - dy)$$

or $$dz = -\frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\ d(x - y) \quad ...(5)$$

Put $\quad x - y = t \quad$ so that $\quad d(x - y) = dt$. Then (5) becomes

$$dz = -(1/2) \times d(x^2 + y^2) + (a/2) \times d(x + y) + (1/\sqrt{2}) \times \sqrt{(a/\sqrt{2})^2+t^2}\ dt.$$

$$\therefore \quad z = -\frac{x^2+y^2}{2}+a\frac{x+y}{2}+\frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{(a/\sqrt{2})^2+t^2}+\frac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

Putting the value of t, the required complete integral is

$$z = -\frac{x^2+y^2}{2}+\frac{a(x+y)}{2}+\frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}+\frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right]+b.$$

Ex. 30. *Solve* $z = (1/2)\times(p^2+q^2)+(p-x)(q-y)$ **[I.A.S. 2002]**

Sol. Given $\quad z = (1/2)\times(p^2+q^2)+(p-x)(q-y)$

Re–writing (1), $\quad f(x,y,z,p,q) = (1/2)\times(p^2+q^2)+pq-xq-yp+xy-z=0 \quad$ ... (2)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

or $$\frac{dp}{-q+z-p}=\frac{dq}{-p+x-q}=\frac{dz}{-p(p+q-y)-q(p+q-x)}=\frac{dx}{-(p+q-y)}=\frac{dy}{-(p+q-x)}$$

Taking the first and the fourth fractions, we have

$dp = dx$ so that $p = x + a$, a being an arbitratry constant. ... (3)

Taking the second and the fifth fractions, we have

$dq = dy$ so that $q = y + b$, b being an arbitrary constant ... (4)

Putting $p = x + a$ and $q = y + b$ in (1), the required solution is

$z = (1/2)\times\{(x+a)^2+(y+b)^2\}+ab$, a and b being arbitrary constants.

Ex. 31 *Find a complete integral of $z = pq$.* **[Sagar 2004, Ravishankar 2003, Rewa 2003]**

Sol. Here given equation is $f(x, y, z, p, q) = z - pq = 0$ ... (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-2pq}=\frac{dx}{-q}=\frac{dy}{-p}\text{, by (1)} \qquad \text{... (2)}$$

Taking the first and last fractions of (2), $dp = dy$

Integrating $p = y + a$, a being an arbitrary constant ... (3)

Similarly, taking the second and fourth fractions of (2), we get

$dq = dx$ so that $q = x + b$, b being an arbitrary constant. ... (4)

Putting values of p and q given by (3) and (4) in (1), we get

$z = (x+b)(x+a)$, which is the required complete integral.

Ex. 32. *Use Charpit's method to find the complete integral of* $2x\,\{z^2(\partial z/\partial y)^2+1\} = z(\partial z/\partial x)$.

[I.A.S. 1998]

Sol. Given $2x(z\partial z/\partial y)^2+2x-(z\partial z/\partial x)=0$ ... (1)

Let $z\,dz = dZ$ so that $z^2 = 2Z$ ... (2)

Then (1) becomes $2x(\partial Z/\partial y)^2+2x-(\partial Z/\partial x)=0$ or $2xQ^2+2x-P=0$

where $P=\partial Z/\partial x$ and $Q=\partial Z/\partial y$ ... (3)

Let $f(x, y, Z, P, Q) = 2x\,Q^2 + 2x - P = 0$ ... (4)

Charpit's auxiliary equations are $$\frac{dP}{f_x+Pf_Z}=\frac{dQ}{f_y+Qf_Z}=\frac{dZ}{-Pf_P-Qf_Q}=\frac{dx}{-f_P}=\frac{dy}{-f_Q}$$

giving $$\frac{dP}{2Q^2+2}=\frac{dQ}{O}=\text{..., by (4)} \qquad \text{so that} \qquad dQ=0.$$

Integrating, $Q = a$, a being an arbitrars constant ... (5)

Using $Q = a$, (4) gives $P = 2x\,(a^2 + 1)$, $Q = a$ ... (6)

$\therefore$ $dZ = P\,dx + Q\,dy = 2x(a^2+1)dx + a\,dy$, by (5) and (6)

Integrating, $Z = x^2(a^2+1)+ay+b/2$, or $z^2/2 = x^2(a^2+1)+ay+b/2$, using (2) or $\qquad z^2 = 2x^2(a^2+1)+2ay+b$, which is complete integral of (1)

Ex. 33. *Solve by Charpit's method the partial differential equation.*

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0. \qquad \textbf{[I.A.S. 2000]}$$

Sol. Let $\quad f(x, y, z, p, q) = p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0 \qquad \ldots (1)$

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q} \quad \ldots (2)$

From (1), $\qquad f_x = p^2(2x-1)+2pqy-2pz, \qquad f_y = 2pqx+q^2(2y-1)-2qz,$

$f_z = -2px-2qy+2z, \qquad f_p = 2px(x-1)+2qxy-2xz; \qquad f_q = 2pxy+2qy(y-1)-2yz$

and so $\quad f_x+p\,f_z=-p^2, \qquad f_y+q\,f_z=-q^2$. Then (2) becomes

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)} \qquad \ldots (3)$$

$$\text{Each fraction of (3)} = \frac{(1/p)\,dp}{-p}=\frac{(1/q)\;dq}{-q}=\frac{(1/p)\,dp-(1/q)dq}{-p+q} \qquad \ldots (4)$$

$$\text{Also, each fraction of (3)} = \frac{(1/x)\;dx-(1/y)\;dy}{-2px+2p-2qy+2z+2px+2qy-2q-2z} \qquad \ldots (5)$$

$\therefore \qquad$ (4) and (5) $\Rightarrow \dfrac{(1/p)\,dp-(1/q)\;dq}{-(p-q)}=\dfrac{(1/x)\;dx-(1/y)\,dy}{2(p-q)}$

or $\qquad (1/2)\times\{(1/x)\;dx-(1/y)dy\}=(1/q)\;dq-(1/p)\;dp$

Integrating, $\qquad (1/2)\times\{\log x-\log y\}=\log q-\log p+\log a \qquad$ or $\qquad (x/y)^{1/2}=aq/p$

or $\qquad p=(ay^{1/2}q)/x^{1/2}$, a being an arbitrary constant. $\qquad \ldots (5)$

Re–writing (1), $(px+qy-z)^2=p^2x+q^2y \qquad$ or $\qquad px+qy-z=\pm(p^2x+qy^2)^{1/2} \qquad \ldots (6)$

Taking + ive sign in (7), $\qquad px+qy-z=(p^2x+q^2y)^{1/2} \qquad \ldots (7)$

[The case of – ve sign in (7) can be discussed similarly]

Substituting the value of p given by (6) in (8), $\quad aqy^{1/2}x^{1/2}+qy-z=(a^2q^2y+q^2y)^{1/2}$

or $\quad q\{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}\}=z \quad$ so that $\quad q=z/y^{1/2}\{y^{1/2}+a\;x^{1/2}-(1+a^2)^{1/2}\} \qquad \ldots (9)$

Then (6) gives $\qquad p=az/x^{1/2}\;\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\} \qquad \ldots (10)$

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{az\;dx}{x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}+\frac{z\;dy}{y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}}$$

Integrating, $$\log z = 2\log\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\} + \log b$$

or $z = b\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}^2$, a and b being an arbitrary constants.

Ex. 34. *Find the complete integral of* $(p+q)\ (px+qy) = 1$.

[Meerut 2007; Delhi Maths (H) 2007, Purvanchal 2007]

Sol. Let $$f(x, y, z, p, q) = (p+q)\ (px+qy) - 1 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations $$\frac{dp}{f_x + p\ f_z} = \frac{dq}{f_y + q\ f_z} = \frac{dz}{-p\,f_p - q\ f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

give $\dfrac{dp}{p(p+q)} = \dfrac{dq}{q(p+q)} = ...$ so that $\dfrac{dp}{p} = \dfrac{dq}{q}$, using (1)

Integrating, $p = aq$, a being an arbitrary constant ... (2)

Putting $p = aq$ in (2) gives $(aq+q)\ (aqx+qy) - 1 = 0$ or $q^2\ (1 + a)\ (ax + y) = 1$ ... (3)

$\therefore$ From (2) and (3), $q = 1/(1+a)^{1/2}\ (ax+y)^{1/2}$, $p = a/(1+a)^{1/2}(ax+y)^{1/2}$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a\ dx}{(1+a)^{1/2}(ax+y)^{1/2}} + \frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{d\,(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b$, a, b being arbitrary constants.

Ex. 35. *Find the complete integral of the following partial differential equations*

(a) $px^5 - 4q^2x^2 + 6x^2z - 2 = 0$. **[Delhi B.Sc. (H) 2002; Delhi B.A. (Proj) II 2011]**

(b) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

Sol. (a) Let $$f(x, y, z, p, q) = px^5 - 4q^2x^2 + 6x^2z - 2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_y} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{5px^4 - 8q^2x + 12xz + 6px^2} = \frac{dq}{6qx^2} = \frac{dz}{-px^5 + 8q^2x^2} = \frac{dx}{-x^5} = \frac{dy}{8qx^2}\text{, by (1)}$$

Taking the second and the last fractions, $4\,dq = 3dy$

Integrating, $4q = 3y + 3a$ or $q = 3(y+a)/4$ ... (2)

Using (2), (1) gives $$p = \{(9/4) \times (y+a)^2 - 6x^2z + 2\}/x^5 \qquad \text{... (3)}$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (9/4x^3)\ (y+a)^2\ dx - (6z/x^3)\ dx + (2/x^5)dx + (3/4)\ (y+a)dy$$

or $$(6z/x^3)dx + dz = \{(9/4x^3)\ (y+a)^2 dx + (3/4)\ (y+a)\ dy\} + (2/x^5)dx \qquad \text{... (4)}$$

The total differential equation (4) is always integrable. To solve (4), we first proceed to find the integrating factor of the L.H.S. of (4). Comparing L.H.S. of (4) with $M\,dx + N\,dz$ (here on L.H.S. we have variable x, z in place of usual variables x, y), we have $M = 6z/x^3$ and $N = 1$.

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial z}-\dfrac{\partial N}{\partial x}\right)=\dfrac{6}{x^3}$, which is function x alone and so I.F. $= e^{\int (6/x^3)dx} = e^{-3/x^2}$.

Multiplying both sides of (4) by I.F. e^{-3/x^2}, we get

$$(6z/x^3)\, e^{-3/x^2} dx + e^{-3/x^2} dz = (3/8)\times\{(6/x^3)\,(y+a)^2\, e^{-3/x^2} dx + 2(y+a)\, e^{-3/x^2} dy\} + (2/x^5)\, e^{-3/x^2} dx$$

or $$d(z\, e^{-3/x^2}) = (3/8)\times d\{(y+a)^2\, e^{-3/x^2}\} + (2/x^5)\times e^{-3/x^2} dx$$

Integrating, $$z\, e^{-3/x^2} = (3/8)\times(y+a)^2 e^{-3/x^2} + 2\int (1/x^2)\, e^{-3/x^2}\,(1/x^3)dx$$

or $z\, e^{-3/x^2} = (3/8)\times(y+a)^2\, e^{-3/x^2} - (1/9)\times\int u\, e^u du$, putting $(-3/x^2) = u$ so that $(6/x^3)dx = du$

or $$z\, e^{-3/x^2} = (3/8)\times(y+a)^2\, e^{-3/x^2} - (1/9)\times(ue^u - e^u) + b$$

or $$z\, e^{-3/x^2} = (3/8)\times(y+a)^2\, e^{-3/x^2} - (1/9)\times(-3/x^2)e^{-3/x^2} + (1/9)\times e^{-3/x^2} + b$$

or $$z = (3/8)\times(y+a)^2 + (1/3x^2) + (1/9) + be^{3/x^2}$$, a, b being arbitrary constants.

(*b*) Proceed exacts as in part (*a*) **Ans.** $z = (2/3)\times(y+a)^{3/2} + (1/3x^2) + (1/9) + b\, e^{3/x^2}$

Ex. 36. *Find the complete integral of* $(p+y)^2 + (q+x)^2 = 1$

Sol. Let $$f(x,y,z,p,q) = (p+y)^2 + (q+x)^2 - 1 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{2(q+x)} = \frac{dq}{2(p+y)} = \frac{dz}{-2(p^2+q^2+py+qx)} = \frac{dx}{-2(p+y)} = \frac{dy}{-2(q+x)}, \text{ by (1)}$$

Taking the first and the last fractions, $dp + dy = 0$ so that $p + y = a$ ... (2)

Using (2), (1) gives $a^2 + (q+x)^2 - 1 = 0$ or $q + x = (1-a^2)^{1/2}$ ... (3)

Using (2) and (3) in $dz = p\,dx + q\,dy$, we get

$$dz = (a-y)dx + \{(1-a^2)^{1/2} - x\}dy = a\,dx - (1-a^2)^{1/2} dy - (y\,dx + x\,dy)$$

Integrating, $z = ax - (1-a^2)^{1/2} y - xy + b$, a, b being arbitrary constants.

Ex. 37. *Find the complete integral of* $2\,(y + zq) = q(xp + yq)$ **[Nagpur 2003, 06; Delhi B.Sc. (Prog) II 2011; Delhi B.Sc. (Hons) 2011]**

Sol. Let $$f(x, y, z, p, q) = 2y + 2zq - xpq - yq^2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_x - q\,f_y} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad \text{... (2)}$$

$$\frac{dp}{-pq+2pq}=\frac{dq}{2-q^2+2q^2}=\frac{dz}{2pqx+2qy-2qz}=\frac{dx}{qx}=\frac{dy}{xp+2yq-2z}, \text{ by (1)}$$

Taking the first and fourth fractions, $(1/pq)dp = (1/qx)dx$ or $(1/p)dp = (1/x)dx$

Integrating, $\log p = \log a + \log x$ or $p = ax$, ... (3)

where a is an arbitrary constant. Substituting the value of p given by (3) in (1), we have

$2y + 2zq - ax^2q - yq^2 = 0$ or $yq^2 + q(ax^2 - 2z) - 2y = 0.$

$$\Rightarrow \qquad q=[-(ax^2-2z)\pm\{(ax^2-2z)^2+8y^2\}^{1/2}]/(2y) \qquad \text{... (4)}$$

Substituting the values of p and q given by (3) and (4) in $dz = p\,dx + q\,dy$, we obtain

$$dz = ax\,dx+(1/2y)\times[2z-ax^2\pm\{(2z-ax^2)^2+8y^2\}^{1/2}]dy$$

or
$$\frac{2dz-2ax\,dx}{(2z-ax^2)\pm\{(2z-ax^2)^2+8y^2\}^{1/2}}=\frac{dy}{y} \qquad \text{... (5)}$$

Putting $2z - ax^2 = u$ and $2dz - 2\,ax\,dx = du$, (5) yields

$$\frac{du}{u\pm(u^2+8y^2)^{1/2}}=\frac{dy}{y} \quad \text{or} \quad \frac{du}{dy}=\frac{u}{y}\pm\left\{\left(\frac{u}{y}\right)^2+8\right\}^{1/2}, \qquad \text{... (6)}$$

which is linear homogeneous differential equation. To solve it, we put $u/y = v$, *i.e.*, $u = yv$ so that $du/dy = v + y(dv/dy)$ and so (6) reduces to

$$v+y\frac{dv}{dy}=v\pm(v^2+8)^{1/2} \quad \text{or} \quad \frac{dv}{(v^2+8)^{1/2}}=\frac{dy}{y},$$

taking positive sign. Integrating it, we have

$$\log\{v+(v^2+8)^{1/2}\}=\log y+\log b \quad \text{or} \quad v+(v^2+8)^{1/2}=by$$

or $$u/y+\left\{(u/y)^2+8\right\}^{1/2}=by \quad \text{or} \quad u+(u^2+8y^2)^{1/2}=by^2$$

or $2z-ax^2+\{(2z-ax^2)^2+8y^2\}^{1/2}=by^2$, as $u = 2z - ax^2$; a, b being arbitrary constants

EXERCISE 3(B)

Using Charpit's method, find a complete integral of the following equations :

1. $z = px + qy + pq$. **[Mysore 2004]** **Ans.** $z = ax + by + ab$
2. $pq = xz$. **Ans.** $z = (a + x^2/2)(b + y)$
3. $p^2 + px + q = z$. **Ans.** $z = ax + a^2 + be^y$
4. $(p + q)(z - px - qy) = 1$. **Ans.** $(a + b)(z - ax - by) = 1$
5. $px + qy + pq = 0$ **Ans.** $az = -(1/2) \times (y + ax)^2 + b$
6. $q = px + q^2$ **Ans.** $z = (a - a^2)\log x + ay + b$
7. $p - 3x^2 = q^2 - y$ **Ans.** $z = x^3 - (1/3) \times (a - x)^3 + ay - xy + b$
8. $x^2p^2 + y^2q^2 = 4$ **Ans.** $z = a\log x + (4 - a^2)^{1/2}\log y + b$
9. $xpq + yq^2 = 1$ **Ans.** $(z + b)^2 = 4(ax + b)$
10. $p + q = 3pq$ **Ans.** $az = b - (1/2) \times (y + ax)^2$
11. $pq + x(2y + 1)p + (y^2 + y)q - (2y + 1)z = 0$ **Ans.** $z = ax + b(a + y + y^2)$
12. $z^2(p^2 + q^2) = x^2 + e^{2y}$.

Ans. $$\frac{z^2}{2}=\frac{x\sqrt{(x^2+a)}}{2}+\frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}}+\sqrt{(e^{2y}-a)}-\sqrt{a}\tan^{-1}\left(\frac{e^{2y}-a}{a}\right)+b$$

13. $p^2 - y^2 q = x^2 - y^2$ **[Madurai Kamraj 2008]**

$$\textbf{Ans. } z = \frac{x\sqrt{(x^2+b)}}{2} + \frac{b}{2}\sinh^{-1}\frac{x}{\sqrt{b}} - \frac{b}{2y^2} + \log y + c$$

14. $p^2 q\,(x^2 + y^2) = p^2 + q$ **Ans.** $z = \log[x + \sqrt{(x^2+a)}] + \dfrac{1}{2\sqrt{a}}\log\dfrac{y-\sqrt{a}}{y+\sqrt{a}} + b$

15. $yp = 2xy + \log q$. **[Lucknow 2010]** **[Ans.** $z = (a+2x)^2/4 + (1/a)\times e^{ay} + b$]

3.9. Special methods of solutions applicable to certain standard forms:

We now consider equations in which p and q occur other than in the first degree, that is non–linear equations. We have already discussed the general method (*i.e.*, Charpit's method — see Art. 3.7). We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

3.10. Standard Form I. Only p and q present. **[Nagpur 2002; Bhopal 2010]**

Under this standard form, we consider equations of the form $f(p, q) = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

giving $\dfrac{dp}{0} = \dfrac{dq}{0}$, by (1)

Taking the first ratio, $dp = 0$ so that p = constant = a, say ...(2)

Substituting in (1), we get $f(a, q) = 0$, giving q = constant = b, say, ... (3)

where b is such that $f(a, b) = 0.$ ...(4)

Then, $dz = p\,dx + q\,dy = adx + bdy$, using (2) and (3).

Integrating, $z = ax + by + c,$ ...(5)

where c is an arbitrary constant. (5) together with (4) give the required solution.

Now solving (4) for b, suppose we obtain $b = F(a)$, say.

Putting this value of b in (5), the *complete integral* of (1) is

$$z = ax + yF(a) + c, \qquad ...(6)$$

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

The *singular integral* of (1) is obtained by eliminating a and c between the complete integral (6) and the equations obtained by differentiating (6) partially w.r.t. a and c ; *i.e.*, between

$z = ax + yF(a) + c,$ $0 = x + yF'(a)$ and $0 = 1.$ ...(7)

Since the last equation in (7) is meaningless, we conclude that the equations of standard form I have no singular solution.

In order to find the general integral of (1), we first take $c = \phi(a)$ in (6), ϕ being an arbitrary function and obtain $z = ax + yF(a) + \phi(a).$ ...(8)

Now, we differentiate (8) partially with respect to a and get

$$0 = x + yF'(a) + \phi'(a). \qquad ....(9)$$

Eliminating a between (8) and (9), we get the general solution of (1).

Remark. Sometimes change of variables can be employed to transform a given equation to standard form I.

3.11. SOLVED EXAMPLES BASED ON ART. 3.10

Ex. 1. (a) *Solve $pq = k$, where k is a constant.* **[M.S. Univ. T.N. 2007; Meerut 1995]**

(b) Solve pq = 1 by standard from I **[Bhopal 2010]**

Sol. Given that $pq = k$. ...(1)

Since (1) is of the form $f(p, q) = 0$, its solution is $z = ax + by + c$, ...(2)

where $ab = k$ or $b = k/a$, on putting a for p and b for q in (1).

$\therefore$ From (2), the complete integral is $z = ax + (k/a)y + c$, ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - (k/a^2)y$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + (k/a)y + \phi(a). \qquad ...(4)$$

Differentiating (4) partially with respect to 'a', we get $0 = x - (k/a^2)y + \phi'(a)$. ...(5)

Eliminating a from (4) and (5), we get the required general solution.

(b) Do like part (a) taking $k = 1$

Ex. 2. *Solve* (a) $p^2 + q^2 = m^2$, *where m is a constant.* **[Kanpur 1993]**

(b) $p^2 + q^2 = 1$ **[Meerut 2011]**

Sol. (a) Given that $p^2 + q^2 = m^2$. ...(1)

Since (1) is of the form $f(b, q) = 0$, its solution is $z = ax + by + c$, ...(2)

where $a^2 + b^2 = m^2$ or $b = (m^2 - a^2)^{1/2}$, on putting a for p and b for b in (1).

$\therefore$ From (2), the complete integral is $z = ax + y(m^2 - a^2)^{½} + c$, ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - ay/(m^2 - a^2)^{1/2}$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + y(m^2 - a^2)^{1/2} + \phi(a). \qquad ...(4)$$

Differentiating (4) partially wit respect to 'a', we get

$$0 = a - ay/(m^2 - a^2)^{1/2} + \phi'(a). \qquad ...(5)$$

Eliminating a from (4) and (5), we get the required general solution.

(b) **Hint.** Do like part (a) by taking $m = 1$

EQUATIONS REDUCIBLE TO STANDARD FORM I

Ex. 3. *Find the complete integral of $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0$.*

Sol. The given equation can be rewritten as

$$z^2y\,(\partial z/\partial x)^2 + 6zxy(\partial z/\partial x) + 2zx^2(\partial z/\partial y) + 4x^2y = 0$$

or $$\left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2 + 6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right) + 2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right) + 4 = 0, \text{ dividing by } x^2y \qquad ...(1)$$

Put $x\,dx = dX$, $y\,dy = dY$ and $z\,dz = dZ$. ...(2)

so that $x^2/2 = X$, $y^2/2 = Y$ and $z^2/2 = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + 6\,(\partial Z/\partial X) + 2(\partial Z/dY) + 4 = 0$

or $P^2 + 6P + 2Q + 4 = 0$, where $P = \partial Z/\partial X$, $Q = \partial Z/\partial Y$. ...(4)

Equation (4) is of the form $f(P, Q) = 0$. Note that now we have P, Q, X, Y, Z in place of p, q, x, y, z in usual equations. Accordingly, solution of (4) is

$$Z = aX + bY + c, \qquad \text{...(5)}$$

where $a^2 + 6a + 2b + 4 = 0$ or $b = -(a^2 + 6a + 4)/2$, on putting a for P and b for Q in (4). So, from (5), the required complete integral is

$$Z = aX - \{(a^2 + 6a + 4)/2\}Y + c, \text{ where } a \text{ and } c \text{ are arbitrary constants.}$$

or $$z^2/2 = a(x^2/2) - (a^2 + 6a + 4) \times (y^2/4) + c, \text{ using (3)}$$

or $$z^2 = ax^2 - (2 + 3a + a^2/2)y^2 + c', \text{ where } c' = 2c.$$

Ex. 4. *Find the complete integral of*

(*i*) $x^2p^2 + y^2q^2 = z$ **[Delhi Maths (H) 2004]**

(*ii*) $p^2x + q^2y = z.$ **[Meerut 1994]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x\,\partial z}{\sqrt{z}\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{\sqrt{z}\,\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $\quad (1/x)dx = dX, \quad (1/y)dy = dY \quad$ and $\quad (1/\sqrt{z})dz = dZ \qquad$ ...(2)

so that $\quad \log x = X, \quad \log y = Y \quad$ and $\quad 2\sqrt{z} = Z. \qquad$ ...(3)

Using (2), (1) becomes $\quad (\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1 \quad$ or $\quad P^2 + Q^2 = 1, \qquad$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $\quad Z = aX + bY + c, \qquad$ ...(5)

where $\quad a^2 + b^2 = 1 \quad$ or $\quad b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = a \log x + \log y \,.\, \sqrt{1-a^2} + c, \text{ by (3)}$$

or $$\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}, \text{ taking } c = -\log c'$$

or $$\log \{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z} \quad \text{or} \quad x^a\, y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$$

where a and c' are two arbitrary constants.

(*ii*) The given equation can be re–written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $\quad \left(1/\sqrt{x}\right)dx = dX, \quad \left(1/\sqrt{y}\right)dy = dY \quad$ and $\quad \left(1/\sqrt{z}\right)dz = dZ \qquad$ ...(2)

so that $\quad 2\sqrt{x} = X, \quad 2\sqrt{y} = Y \quad$ and $\quad 2\sqrt{z} = Z. \qquad$ ...(3)

Using (2), (1) becomes $\quad (\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1 \quad$ or $\quad P^2 + Q^2 = 1, \qquad$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $\quad z = aX + bY + c, \qquad$ ...(5)

where $\quad a^2 + b^2 = 1 \quad$ or $\quad b = \sqrt{1-a^2}$, putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c, \text{ by (3)}$$

where a and c are two arbitrary constants.

Ex. 5. *Solve* $x^2p^2 + y^2q^2 = z^2$. **[Jabalpur 2000, 03; Gulbarga 2005; Bilaspur 1997; Meerut 2008, Sagar 2004, Vikram 1999; Ravi Shanker 1994, 96; Rohitkhand 2004]**

Sol. The given equation can be rewritten as

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(\frac{x\,\partial z}{z\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{z\,\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $\quad (1/x)dx = dX, \quad (1/y)dy = dY \quad$ and $\quad (1/z) = dZ \qquad ...(2)$

so that $\quad \log x = X, \quad \log y = Y \quad$ and $\quad \log z = Z. \qquad ...(3)$

Using (2), (1) becomes $\quad (dZ/dX)^2 + (dZ/dY)^2 = 1 \quad$ or $\quad P^2 + Q^2 = 1, \qquad ... (4)$

where $P = dZ/dX$ and $Q = dZ/dy$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $\qquad Z = aX + bY + c, \qquad ... (5)$

where $\quad a^2 + b^2 = 1 \quad$ or $\quad b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $Z = X \cos\alpha + Y \sin\alpha + \log c'$, taking $a = \cos\alpha$ and $c = \log c'$

or $\quad \log z = \cos\alpha \log x + \sin\alpha \log y + \log c' \quad$ or $\quad z = c'\, x^{\cos\alpha}\, y^{\sin\alpha}. \qquad ...(6)$

To determine singular integral. Differentiating (6) partially w.r.t. α and c' successively, we obtain $\quad 0 = c' \cos\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log y - c' \sin\alpha \,.\, x^{\cos\alpha}\, y^{\sin\alpha} \log x \qquad ...(7)$

and $\qquad 0 = x^{\cos\alpha}\, y^{\sin\alpha}. \qquad ...(8)$

Eliminating α and c' from (6), (7) and (8), the singlar solution is $z = 0$.

To determine general integral. Putting $c' = \phi(\alpha)$, where ϕ is an arbitrary function, (4) gives

$$z = \phi(\alpha) \sin\alpha\, x^{\cos\alpha}\, y^{\sin\alpha}. \qquad ...(9)$$

Differentiating (9), partially, w.r.t. 'α', we get

$$0 = \phi'(\alpha)\, x^{\cos\alpha}\, y^{\sin\alpha} + \phi(\alpha)\, \{x^{\cos\alpha}\, y^{\sin\alpha} \cos\alpha - y^{\sin\alpha}\, x^{\cos\alpha} \sin\alpha\}. \qquad ...(10)$$

The required general integral is obtained by eliminating α from (9) and (10).

Ex. 6. *Find a complete integral of* (*i*) $pq = x^m y^n z^{2l}$ **[Delhi B.Sc. (Prog) II 2007]**

(*ii*) $pq = x^m y^n z^l$ **[I.A.S. 1989, 94]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{z^{-l}z^{-l}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad ...(1)$$

Put $\quad x^m dx = dX, \quad y^n dy = dY \quad$ and $\quad z^{-l}dz = dZ \qquad ...(2)$

so that $\quad \dfrac{x^{m+1}}{m+1} = X, \quad \dfrac{y^{n+1}}{n+1} = Y \quad$ and $\quad \dfrac{z^{1-l}}{1-l} = Z. \qquad ...(3)$

Using (2), (1) becomes $\quad (\partial Z/\partial X)\,(\partial Z/\partial Y) = 1 \quad$ or $\quad PQ = 1, \qquad ...(4)$

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c, \qquad ... (5)$

where $\quad ab = 1 \quad$ or $\quad b = 1/a$, on putting a from P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-l}}{1-l} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

(ii) The given equation can be rewritten as

$$\frac{z^{-l/2}z^{-l/2}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}=1 \qquad \text{or} \qquad \left(\frac{z^{-l/2}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l/2}}{y^n}\frac{\partial z}{\partial y}\right)=1. \qquad ...(1)$$

Put $\quad x^m dx = dX, \qquad y^n dy = dY \qquad$ and $\qquad z^{-l/2}dz = dZ \qquad ...(2)$

so that $\quad \dfrac{x^{m+1}}{m+1}=X, \qquad \dfrac{y^{n+1}}{n+1}=Y \qquad$ and $\qquad \dfrac{z^{1-(l/2)}}{1-(l/2)}=Z. \qquad ...(3)$

Using (2), (1) becomes $\qquad (\partial Z/\partial X)\,(\partial Z/\partial Y) = 1 \qquad$ or $\qquad PQ = 1, \qquad ...(4)$

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c, \qquad ...\ (5)$

where $\quad ab = 1 \qquad$ or $\qquad b = 1/a$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-(l/2)}}{1-(l/2)} = a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

Ex. 7. *Find complete integral of* $p^m\sec^{2m}x + z^l q^n \operatorname{cosec}^{2n}y = z^{lm/(m-n)}$

Sol. The given equation can be re–written as

$$\frac{1}{z^{lm/(m-n)}}\left(\frac{1}{\cos^2 x}\frac{\partial z}{dx}\right)^m+\frac{z^l}{z^{lm/(m-n)}}\left(\frac{1}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1 \text{ or } \left(\frac{z^{-l/(m-n)}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^m+\left(\frac{z^{-l/(m-n)}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n=1. \quad ...(1)$$

Put $\qquad \cos^2 x\, dx = dX, \qquad \sin^2 y\, dy = dY \qquad$ and $\qquad z^{-l/(m-n)}dz = dZ \qquad ...(2)$

i.e., $\{(1 + \cos 2x)/2\}dx = dX, \qquad \{(1 - \cos 2y)/2\}dy = dY \qquad$ and $\qquad z^{-l/(m-n)}\, dz = dZ$

so that $\quad \dfrac{1}{2}(x + \dfrac{1}{2}\sin 2x) = X, \quad \dfrac{1}{2}(y - \dfrac{1}{2}\sin 2y) = Y \quad$ and $\quad \dfrac{(m-n)z^{(m-n-l)/(m-n)}}{m-n-l} = Z. \qquad ...(3)$

Using (2), (1) becomes $\quad (\partial Z/\partial X)^m + (\partial Z/\partial Y)^n = 1 \qquad$ or $\qquad P^m + Q^n = 1, \qquad ...(4)$

where $P = \partial Z/dX$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad Z = aX + bY + c, \qquad ...\ (5)$

where $\quad a^m + b^n = 1 \qquad$ or $\qquad b = (1 - a^m)^{1/n}$, on putting a for P and b for Q is (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + (1 - a^m)^{1/n}Y + c$, $\;a$ and c being two arbitrary constants.

or $$\frac{m-n}{m-n-l}z^{(m-n-l)/(m-n)} = \frac{a}{4}(2x + \sin 2x) + \frac{(1-a^m)^{1/n}}{4}(2y - \sin 2y) + c, \text{ by (3).}$$

Ex. 8. *Find the complete integral of* $(1 - x^2)\, yp^2 + x^2 q = 0$.

Sol. The given equation can be rewritten as

$$\frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{1}{y}\frac{\partial z}{\partial y}=0 \qquad \text{or} \qquad \left(\frac{(1-x^2)^{1/2}}{x}\frac{\partial z}{\partial x}\right)^2+\left(\frac{1}{y}\frac{\partial z}{\partial y}\right)=0. \qquad ...(1)$$

Put $\qquad \{x/(1 - x^2)^{1/2}\}dx = dX \qquad$ and $\qquad y\, dy = dY \qquad ...(2)$

so that $\quad X = \displaystyle\int\frac{x\,dx}{(1-x^2)^{1/2}}=-\frac{1}{2}\int(1-x^2)^{-1/2}(-2x)dx = -(1 - x^2)^{1/2} \quad$ and $\quad Y = \dfrac{y^2}{2} \qquad ...(3)$

Using (2), (1) becomes $\quad (\partial z/\partial X)^2 + (\partial z/\partial Y) = 0 \qquad$ or $\qquad P^2 + Q = 0, \qquad ...(4)$

where $\quad P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Note carefully that here the old variable z remains unchanged even after transformation (2). Here (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad \text{... (5)}$$

where $a^2 + b = 0$ or $b = -a^2$, on putting a for P and b for Q in (4),

$\therefore$ from (5), the required complete integral is

$z = aX - a^2Y + c$ or $z = -a(1 - x^2)^{1/2} - (a^2 y^2)/2 + c$, by (3).

Ex. 9. *Find the complete integral of* $(y - x)(qy - px) = (p - q)^2$. **[Delhi Maths (H) 2005; Ravishankar 2010; Meerut 1995, 97; Agra 1999; Kanpur 2001, 04, 07, 08]**

Sol. Let X and Y be two new variables such that

$$X = x + y \quad \text{and} \quad Y = xy. \qquad \text{...(1)}$$

Given equation is $$(y - x)(qy - px) = (p - q)^2. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of p and q in (2), we have

$$(y - x)\left[y\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right) - x\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)\right] = \left[\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right) - \left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $$(y - x)^2\frac{\partial z}{\partial X} = (y - x)^2\left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad \frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad P = Q^2, \qquad \text{...(5)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad \text{...(6)}$$

where $a = b^2$, on putting a for P and b for Q in (5).

$\therefore$ from (6), the required complete integral is

$z = b^2X + bY + c$ or $z = b^2(x + y) + bxy + c$, by (1).

Ex. 10. *Find the complete integral of* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. **[I.A.S. 1991; Kanpur 2006; Meerut 1997]**

Sol. Let X and Y be two new variables such that

$$X^2 = x + y \quad \text{and} \quad Y^2 = x - y. \qquad \text{...(1)}$$

Given equation is $$(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1/2X$ and $\partial Y/\partial x = 1/2Y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1/2X$ and $\partial Y/\partial y = -1/2Y$]

(3) and (4) $\Rightarrow$ $$p + q = \frac{1}{X}\frac{\partial z}{\partial X} \quad \text{and} \quad p - q = \frac{1}{Y}\frac{\partial z}{\partial Y}. \qquad \text{...(5)}$$

Using (1) and (5), (2) reduces to

$$X^2 \times \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \times \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \qquad \text{...(6)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad ...(7)$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (6).

$\therefore$ from (7), the required complete integral is

$$z = aX + Y\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\sqrt{x+y} + \sqrt{x-y}\sqrt{1-a^2} + c, \text{ by (1).}$$

Ex. 11. *Find a complete integral of* $(x^2 + y^2)(p^2 + q^2) = 1$.

[Agra 2008; Indore 2004; Vikram 2000; Meerut 1995; Rohitkhand 1994]

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$. ...(1)

Then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. ...(2)

Differentiating (2) partially with respect to x and y, we get

$$2r(\partial r/dx) = 2x \qquad \text{and} \qquad 2r(\partial r/\partial y) = 2y$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{r\sin\theta}{r} = \sin\theta. \qquad ...(3)$$

and $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\times\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \qquad ...(4)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\times\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \qquad ...(5)$$

Given equation is $$(x^2 + y^2)(p^2 + q^2) = 1. \qquad ....(6)$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}, \text{ by (3) and (4)}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos 0}{r}\frac{\partial z}{\partial\theta}, \text{ by (3) and (5).}$$

Hence $$p^2 + q^2 = (\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2. \qquad ...(7)$$

$\therefore$ (6) becomes $$r^2[(\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2] = 1, \text{ using (2) and (7)}$$

or $$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1. \qquad ...(8)$$

Let R be a new variable such that $(1/r)dr = dR$ so that $\log r = R$. ...(9)

Then (8) becomes $(\partial z/\partial R)^2 + (\partial z/\partial\theta)^2 = 1$ or $P^2 + Q^2 = 1$, ...(10)

where $P = \partial z/\partial R$ and $Q = \partial z/\partial\theta$. (10) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $$z = aR + b\theta + c, \qquad ...(11)$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (10)

$\therefore$ from (11), the required complete integral is

$$z = aR + \theta\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\log r + \theta\sqrt{1-a^2} + c,$$

or $$z = a\log(x^2 + y^2)^{1/2} + \tan^{-1}(y/x)\,.\,\sqrt{1-a^2} + c, \text{ by (2)}$$

or $z = (a/2)\times\log(x^2 + y^2) + \sqrt{1-a^2}\tan^{-1}(y/x) + c$, a and c being arbitrary constants.

Ex. 12. *Find the complete integral of* $z^2 = pqxy$ **[Meerut 2007; Punjab 2005]**

Sol. The given equation can be re–written as

$$\frac{xy}{z^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)\left(\frac{y}{z}\frac{\partial z}{\partial y}\right) = 1 \qquad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)\,dy = dY$ and $(1/z)dz = dZ$

so that $\log x = X$, $\log y = Y$ and $\log z = Z$ ... (2)

Then (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1$ ... (3)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Then, solution of (3) is

$$Z = aX + bY + C', \quad \text{where} \quad ab = 1 \quad \text{so that} \quad b = 1/a.$$

$\therefore$ $\log z = a\log x + (1/a)\log y + \log C$, taking $C' = \log C$ and using (2)

or $z = x^a\ y^{1/a}C$, a and C being arbitrary constants.

Ex. 13. *Find the complete integral of* $(x/p)^n + (y/q)^n = z^n$.

Sol. The given can be re–written as $(x/zp)^n + (y/zq)^n = 1$ ... (1)

Let $X = x^2/2$, $Y = y^2/2$, $Z = z^2/2$, $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$ ... (2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial Z}\dfrac{\partial Z}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{x}{z}P$. Similarly, $q = \dfrac{y}{z}Q$, using (2)

Hence $x/zp = 1/P$ and $y/zq = 1/Q$ and so (1) reduces to $P^{-n} + Q^{-n} = 1$, whose solution is

$Z = aX + bY + C'$, where $a^{-n} + b^{-n} = 1$ so that $b = (1 - a^{-n})^{-1/n}$ ... (3)

$\therefore$ (2) and (3) $\Rightarrow$ $z^2/2 = a(x^2/2) + (1 - a^{-n})^{-1/n}\,(y^2/2) + C/2$, taking $C' = C/2$

or $z^2 = ax^2 + (1 - a^{-n})^{-1/n} y^2 + C$, a and C being arbitrary constants.

Ex. 14. *Find the complete integral of* $p^3 \sec^6 x + z^2 q^2 \operatorname{cosec}^4 y = z^6$

Sol. The given equation can be re–written as

$$\frac{z^{-6}}{\cos^6 x}\left(\frac{\partial z}{\partial x}\right)^3 + \frac{z^{-4}}{\sin^4 y}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{z^{-2}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-2}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{... (1)}$$

Let $\cos^2 x\,dx = dX$, $\sin^2 y\,dy = dY$, $z^{-2}dz = dZ$ ... (2)

$\Rightarrow$ $X = (1/2)\displaystyle\int (1 + \cos 2x)dx$, $Y = (1/2)\displaystyle\int (1 - \cos 2y)dy$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,\{x + (1/2)\sin 2x\}$, $y = (1/2)\,\{y - (1/2)\sin 2y\}$, $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,(x + \sin x\cos x)$, $y = (1/2)\,(y - \sin y\cos y)$, $Z = -(1/z)$ ... (3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$ ... (4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Now, solution of (4) is

$Z = aX + bY + C/2$, where $a^2 + b^2 = 1$ so that $b = (1 - a^2)^{1/2}$ ... (5)

$\therefore$ $-(1/z) = a(1/2)\,(x + \sin x\cos x) + (1 - a^2)^{1/2}(1/2)(y - \sin y\cos y) + C/2$, by (3) and (5)

or $(2/z) + a(x + \sin x\cos x) + (1 - a^2)^{1/2}(y - \sin y\cos y) + C = 0$.

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \qquad \text{or} \qquad \frac{\partial z}{x\partial x} + \frac{\partial z}{y\partial y} = \left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \qquad \text{... (1)}$$

Put $x\,dx = dX$, $y\,dy = dY$ so that $x^2/2 = X$, $y^2/2 = Y$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)\ (\partial z/\partial Y)$ or $P + Q = PQ$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, where $a + b = ab$ so that $b = a/(a-1)$ ... (4)

or $z = a(x^2/2) + a(a-1)^{-1}\ (y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 16. *Find the complete integral of* $p^2x^2 + px = q$

Sol. The given equation can be re–written as

$$x^2\left(\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad \text{or} \qquad \left(x\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad \text{... (1)}$$

Putting $(1/x)dx = dX$ so that $\log x = X$, (1) gives

$(\partial z/\partial X)^2 + \partial z/\partial X = q$ or $P^2 + P = q$, where $P = \partial z/\partial X$.

Its solution is $z = aX + by + c$ where $a^2 + a = b$

or $z = a\log x + (a^2 + a)y + c$, a and c being arbitrary constants. $[\because\ X = \log x]$

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \qquad \text{or} \qquad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \qquad \text{... (1)}$$

Putting $2x\ dx = dX$, $2y\ dy = dY$ so that $x^2 = X$, $y^2 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = 1$ or $PQ = 1$ whose solution is

$z = aX + bY + d$, where $ab = 1$ so that $b = 1/a$.

$$\therefore \qquad z = ax^2 + (1/a)y^2 + d \qquad \text{... (2)}$$

is complete integral of (1) containing two arbitrary constants a and d.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\ y^2 + \phi(a) \qquad \text{... (3)}$$

Differenting (3) partially w.r.t. 'a', $0 = x^2 - (1/a^2)\ y^2 + \phi'(a)$ ... (4)

Then general integral is obtained by eliminating a from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$$0 = x^2 + (-1/a^2)\, y^2 \quad \text{... (5)} \qquad\qquad 0 = 1 \quad \text{... (6)}$$

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$$\boldsymbol{z = 2xy + C,\ C \text{ being an arbitrary constant}} \quad \text{... (7)}$$

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.,* if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$$z = (y/x)\, x^2 + (x/y)\, y^2 + C \qquad \text{or} \qquad z = 2xy + C,$$

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write

$$z = ax^2 + (1/a)\, y^2 + C \quad \text{... (8)}$$

Differenting (8) partially w.r.t. 'a', we get

$$0 = x^2 - (1/a^2) y^2 \qquad \text{or} \qquad a = y/x \quad \text{... (9)}$$

Eliminating a from (8) and (9), we get $z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$. **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \quad \text{... (1)}$$

Let X, Y and Z be new variables such that

$$dX = dx, \quad dY = dy \quad \text{and} \quad dZ = z^{-1/2} dz \quad \text{so that} \quad X = x, \quad Y = y, \quad Z = 2z^{1/2} \quad \text{... (2)}$$

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \quad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $Z = aX + by + c$, ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$$Z = aX \pm (a^2 - 1)^{1/2}\, Y + c \qquad \text{or} \qquad 2z^{1/2} = ax \pm (a^2 - 1)^{1/2}\, y + c,$$

which is the complete integral, a and c being arbitrary constants and $|a| \ge 1$.

EXERCISE 3(C)

Solve the following partial differential equations (1 – 10)

1. $p^2 - q^2 = 1$ **Ans. C.I.** $z = ax + (a^2 - 1)^{1/2} + c$, a and c are arbitrary constants and $|a| \ge 1$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - 1)^{1/2} y - \psi(a) = 0$, $-x - a(a^2 - 1)^{-1/2}\, y - \psi'(a)$, where ψ is an arbitrary function.

2. $p^2 - q^2 = \lambda$ **Ans. C.I.** $z = ax + (a^2 - \lambda)^{1/2} y + c,$ where a and c are arbitrary constants and $\lambda \le a^2$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - \lambda)^{1/2} y - \psi(a) = 0,$

$$-x - a(a^2 - \lambda)^{1/2} y - \psi'(a) = 0$$

3. $p + q = pq$ **[Mysore 2004; Gulberga 2005; Kanpur 2011; Pune 2010]**

Ans. C.I. $z = ax + \{a/(a - 1)\}y + c$, where a and c are arbitrary constants and $a \neq 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{a/(a-1)\}y - \psi(a) = 0$ and $-x - y(a-1)^{-2} - \psi'(a) = 0,$ where ψ is an arbitrary function

4. $p + q + pq = 0$. **Ans. C.I.** $z = ax - \{a/(a + 1)\}y + c$, where a and c are arbitrary constants and $a \neq -1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax + \{a/(a+1)\}y - \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2 + a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

5. $p^2 + q^2 = npq$. **(M.S. Univ. T.N. 2007), Ans. C.I.** $z = ax + (a/2) \times \{n + (n^2 - 4)^{1/2}y + c$, where a and c are arbitrary constants and $n^2 \ge 4$; **S.S.** Does not exists; **G.S.** It is given by $z - ax - (a/2) \times \{n + (n^2 - 4)^{1/2}\}y - \psi(a) = 0,$ $-x - (1/2) \times \{n + (n^2 - 4)^{1/2\}} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

6. $p = 2q^2 + 1$. **Ans. C.I.** $z = ax + \{(a-1)/2\}^{1/2} y + c,$ where a and c are arbitrary constants and $a \ge 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{(a-1)/2\}^{1/2} y - \psi(a) = 0,$

$-x - (2\sqrt{2}\sqrt{a-1})^{-1} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

7. $p = e^q$. **Ans. C.I.** $z = ax + y \log a + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - y \log a - \psi(a) = 0, - x - (y/a) - \psi'(a) = 0,$ where ψ is an arbitrary function.

8. $p^2q^3 = 1$ **Ans. C.I.** $z = ax + a^{-2/3}y + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - a^{-2/3} y - \psi(a) = 0,$ $-x + (2/3) \times a^{-5/3} y - \psi(a) = 0,$ where ψ is an arbitrary function.

9. $p^2 + p = q^2$. **Ans. C.I.** $z = ax + (a^2 + a)^{1/2}y + c$, where a and c are arbitrary constants and $a \in \boldsymbol{R} - (-1, 0)$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - (a^2 + a)^{1/2} y - \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2 + a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

10. $p^2 + 6p + 2q + 4 = 0$. **C.I.** $z = ax - (2 + 3a + a^2/2)y + c$, where a and c are arbitrary constants; **S.S.** Does not exist; **G.S.** It is given by $z - ax + (2 + 3a + a^2/2)y - \psi(a) = 0,$

$-x + (a+3)y - \psi'(a) = 0,$ where ψ is an arbitrary function.

Find the complete integral (solution) of the following equations (Ex. 11—18).

11. $zy^2p = x(y^2 + z^2q^2)$. **Ans.** $z^2 = ax^2 \pm y^2(a - 1)^{1/2} + c$, where $a \ge 1$

12. $z^2(p^2/x^2 + q^2/y^2) = 1$. **Ans.** $z^2 = ax^2 \pm y^2(1 - a^2)^{1/2} + c$, where $-1 \le a \le 1$

13. $yp + x^2q^2 = 2x^2y$. **Ans.** $(3z - ax^3 - b)^2 = 4(2 - a)y^2$

14. $(1 - y^2)\, xq^2 - y^2p = 0$. **Ans.** $(2z - ax^2 - b)^2 = a(1 - y^2)$

15. $p^2y(1 + x^2) = qx^2$. **Ans.** $z = a(1 + x^2)^{1/2} + (1/2) \times a^2y^2 + c$

16. $x^4p^2 + y^2zq - z^2 = 0$. **Ans.** $xy \log z = ay + (a^2 - 1)x + bxy$

17. $p^2 + q^2 = z$. **[Bangalore 1995]** **Ans.** $2z^{1/2} = ax \pm (1 - a^2)^{1/2}y + c$, where $-1 \le a \le 1$

18. $x^2p^2 + y^2q^2 = 4z^2$. **Ans.** $\log z = a \log x + (4 - a^2)^{1/2} + c$, $-2 \le a \le 2$

3.12. Standard form II. Clairaut equation. [Meerut 2009; Nagpur 2002]

A first order partial differential equation is said to be of Clariaut form if it can be written in the form

$$z = px + qy + f(p, q). \quad \text{...(1)}$$

Let

$$F(x, y, z, p, q) \equiv px + qy + f(p, q) - z. \quad \text{...(2)}$$

Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \frac{dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}} = \frac{dz}{-p\dfrac{\partial F}{\partial p} - q\dfrac{\partial F}{\partial q}} = \frac{dx}{-\dfrac{\partial F}{\partial p}} = \frac{dy}{-\dfrac{\partial F}{\partial q}}$$

or

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px - qy - p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-x - (\partial f/\partial p)} = \frac{dy}{-y - (\partial f/\partial q)}, \text{ by (1)}$$

Then, first and second fractions $\Rightarrow$ $dp = 0$ and $dq = 0 \Rightarrow$ $p = a$ and $q = b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.

Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

3.13. SOLVED EXAMPLES BASED ON ART. 3.12

Ex. 1. *Solve* $z = px + qy + pq$. **[Ravishanker 1997; Bangalore 2005; Sagar 1995, 96]**

Sol. The complete integral is $z = ax + by + ab$, a, b being arbitrary constants ...(1)

Singular integral. Differentiating (1) partially w.r.t. a and b, we have

$$a = x + b \quad \text{and} \quad 0 = y + a. \quad \text{...(2)}$$

Eliminating a and b between (1) and (2), we get $z = -xy - xy + xy$ *i.e.*, $z = -xy$, which is the required singular solution, for it satisfies the given equation.

General Integral. Take $b = \phi(a)$, where ϕ denotes an arbitrary function.

Then (1) becomes $z = a x + \phi(a) y + a \phi(a)$. ...(3)

Differentiating (3) partially w.r.t. a, $0 = x + \phi'(a)y + \phi(a) - a \phi'(a)$. ...(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 2. *Prove that complete integral of the equations* $(px + qy - z)^2 = 1 + p^2 + q^2$ is $ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$. **[I.A.S. 1989]**

Sol. Re–writting the given equation, we have

$$px + qy - z = \pm\sqrt{(1 + p^2 + q^2)} \quad \text{or} \quad z = px + qy \pm\sqrt{(1 + p^2 + q^2)}$$

which is of standard form II and so its complete integral is

$$z = Ax + By \pm (1 + A^2 + B^2)^{1/2}. \quad \text{...(1)}$$

To get the desired form of solution we take +ve sign in (1) and set $A = -a/c$ and $B = -b/c$. Then (1) becomes $z = -(ax + by)/c + (c^2 + a^2 + b^2)^{1/2}/c$

or

$$ax + by + cz = (a^2 + b^2 + c^2)^{1/2}.$$

Ex. 3. *Solve* $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$. **[I.A.S. 1989; Meerut 1998]**

Sol. The complete integral of the given equation is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)}, \ a, b \text{ being arbitrary constants.} \quad \text{...(1)}$$

Singular Integral. Differentiating (1) partially w.r.t. a and b, we get

$0 = x + ac/\sqrt{(1+a^2+b^2)}$ ...(2) $\qquad 0 = y + bc/\sqrt{(1+a^2+b^2)}$. ...(3)

$\therefore$ From (2) and (3), $\qquad x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2)$.

$$\therefore \qquad c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2+b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

so that $\qquad 1 + a^2 + b^2 = c^2/(c^2 - x^2 - y^2)$. ...(4)

From (2), $\qquad a = -\dfrac{x\sqrt{(1+a^2+b^2)}}{c} = -\dfrac{x}{\sqrt{(c^2-x^2-y^2)}}$, by (4)

Similarly from (3) and (4), we obtain $\qquad b = -y/\sqrt{c^2-x^2-y^2}$.

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}$$

or $\quad z = (c^2 - x^2 - y^2)^{1/2}$ $\quad$ or $\quad z^2 = c^2 - x^2 - y^2$ $\quad$ or $\quad x^2 + y^2 + z^2 = c^2$. ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take $b = \phi(a)$, where ϕ is an arbitrary function.

Then, (1) yeilds $\qquad z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$. ...(6)

Differentiating both sides of (6) partially w.r.t. 'a', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\,\phi'(a)]. \qquad ...(7)$$

Eliminating a from (6) and (7), we get the general integral.

Ex. 4. *Find the complete and singular integrals of the following equations:*

(*i*) $z = px + qy + \log(pq)$ **[Indore 2004; K.U. Kurukshetra 2006]**

(*ii*) $z = px + qy - 2\sqrt{pq}$. **[Bangalore 1993; Lucknow 2010]**

Sol. (*i*) The complete integral is $\qquad z = ax + by + \log(ab)$

or $\qquad z = ax + by + \log a + \log b$, a, b being arbitrary constants ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x + (1/a)$ $\quad$ and $\quad$ $0 = y + (1/b)$ $\quad$ so that $\quad a = -1/x$ and $b = -1/y$. ...(2)

Eliminating a and b from (1) and (2), the required singular integral is

$$z = -1 - 1 + \log(1/xy) \qquad \text{or} \qquad z = -2 - \log(xy).$$

(*ii*) The complete integral is $\qquad z = ax + by - 2\sqrt{ab}$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x - \dfrac{2b}{2\sqrt{ab}}$ $\quad$ and $\quad$ $0 = y - \dfrac{2a}{2\sqrt{ab}}$ $\quad$ so that $\quad x = \sqrt{\dfrac{b}{a}}$ and $y = \sqrt{\dfrac{a}{b}}$. ...(2)

Now, using (1) $\qquad x - z = x - (ax + by - 2\sqrt{ab}) = \sqrt{\dfrac{b}{a}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$, using (2)

$$\therefore \qquad x - z = \sqrt{(b/a)}. \qquad ...(3)$$

Similarly, using (1) $\quad y - z = y - (ax + by - 2\sqrt{ab})$, $= \sqrt{\dfrac{a}{b}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$

$$\therefore \qquad y - z = \sqrt{(a/b)}. \qquad ...(4)$$

From (3) and (4), $\qquad (x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 5. *Prove that the complete integral of* $z = px + qy - 2p - 3q$ *represents all possible planes through the point* (2, 3, 0). *Also find the envelope of all planes represented by the complete integral* (*i.e.*, *find the singular integral*). **(M.D.U. Rohtak 2006)**

Sol. Given that $$z = px + qy - 2p - 3q, \qquad ...(1)$$
which is of the form $z = px + qy + f(p, q)$ and so its complete integral is
$$z = ax + by - 2a - 3b,\ a, b \text{ being arbitrary constants} \qquad ...(2)$$
Since (2) is a linear equation in x, y, z, it follows that (2) represents planes for various values of a and b. Again putting $x = 2$, $y = 3$, $z = 0$ in (2), we have
$$0 = 2a + 3b - 2a - 3b \qquad i.e., \qquad 0 = 0,$$
showing that coordinates of the point (2, 3, 0) satisfy (2). Hence the complete integral (2) of (1) represents all possible planes passing through the point (2, 3,0).

Differentiating (2) partially with respect to a and b, we get
$$0 = x - 2 \quad \text{and} \quad 0 = y - 3 \quad \text{so that} \quad x = 2 \quad \text{and} \quad y = 3.$$
Substituting these values in (2), we get $z = 0$ as the required envelope (*i.e.*, singular integral).

Ex. 6. *Prove that the complete integral of* $z = px + qy + [pq/(pq - p - q)]$ *represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.*

Sol. Since the given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is
$$z = ax + by + [ab/(ab - a - b)],\ a \text{ and } b \text{ being arbitrary constants.} \qquad ...(1)$$
Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b. We now rewrite (1) in the intercept form of a plane as follows :
$$ax + by - z = ab/(a + b - ab)$$
or
$$\frac{x}{[b/(a+b-ab)]} + \frac{y}{[a/(a+b-ab)]} + \frac{z}{[-ab/(a+b-ab)]} = 1.$$
$\therefore$ The algebric sum of the intercepts on three coordinate axes
$$= \frac{b}{a+b-ab} + \frac{a}{a+b-ab} + \frac{(-ab)}{a+b-ab} = \frac{b+a-ab}{a+b-ab} = 1, \text{ as required.}$$

Ex. 7. *Show that the complete integral of the equation* $z = px + qy + (p^2 + q^2 + 1)^{1/2}$ *represents all planes at unit distance from the origin.*

Sol. Given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is
$$z = ax + by + (a^2 + b^2 + 1)^{1/2},\ a, b \text{ being an arbitrary constants.}$$
or
$$ax + by - z + (a^2 + b^2 + 1)^{1/2} = 0. \qquad ...(1)$$
Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b.

The perpendicular distance of (1) from origin (0, 0, 0)
$$= \frac{a.0 + b.0 - 0 + \sqrt{a^2 + b^2 + 1}}{\sqrt{\{a^2 + b^2 + (-1)^2\}}} = \frac{\sqrt{a^2 + b^2 + 1}}{\sqrt{a^2 + b^2 + 1}} = 1, \text{ as required}$$

Ex. 8. *Find the complete integral of the following equations:*

(*i*) $(p+q)(z - px - qy) = 1$ **[Pune 2010]**

(*ii*) $pqz = p^2(xq + p^2) + q^2(yp + q^2)$ **[Delhi B.A. (Prog) II 2008, 10]**

Sol. (*i*) Re–writing the given equation in the standard form $z = px + qy + f(p, q)$, we get
$$z - px - qy = 1/(p+q) \qquad \text{or} \qquad z = px + qy + 1/(p+q)$$

$\therefore$ Its complete integral is $z = ax + by + 1/(a+b)$, where a and b are arbitrary constants.

(*ii*) Dividing both sides of the given equation by pq, $\quad z = px + qy + (p^4 + q^4)/pq$,

Its complete integral is $\quad z = ax + by + (a^4 + b^4)/ab$, a, b being arbitrary constants.

Ex. 9. (*a*) *Find the complete integral the equation* $\quad 2(y + zq) = q(xp + yq)$.

[Delhi Maths (H) 1999]

Sol. Re–writing the given equation, we have

$$2zq = xpq + yq^2 - 2y \qquad \text{or} \qquad z = (1/2)px + (1/2)qy - (y/q)$$

or
$$z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1} \qquad \text{... (1)}$$

Putting $\quad 2x\,dx = dX$ and $\quad 2y\,dy = dY \quad$ so that $\quad x^2 = X$ and $y^2 = Y$, $\quad$ (1) gives

$$z = X\,(\partial z/\partial X) + Y\,(\partial z/\partial Y) - 1/\{2\,(\partial z/\partial Y)\} \qquad \text{or} \qquad z = PX + QY - (1/2Q),$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. The above equation is of the form $z = PX + Qy + f(P,Q)$ and hence its complete integral is

$z = aX + bY - (1/2b) \quad$ or $\quad z = ax^2 + by^2 - (1/2b)$, a and b being arbitrary constants.

Ex. 9. (*b*) *Find the complete integral of* $2q(z - px - qy) = 1 + q^2$.

Sol. Re–writing the given equation in the form $z = px + qy + f(p,q)$, we have

$$z - px - qy = (1+q^2)/2q \qquad \text{or} \qquad z = px + qy + (1+q^2)/2q,$$

Its complete integral is $z = ax + by + (1+b^2)/2b$, a and b being arbitrary constants.

Ex. 10. *Find the compelte integral of* $p^2x + q^2y = (z - 2px - 2qy)^2$.

Sol. Taking positive root, the given equation reduces to

$$z - 2px - 2qy = (p^2x + q^2y)^{1/2} \qquad \text{or} \qquad z = 2px + 2qy + (p^2x + q^2y)^{1/2}$$

or
$$z = \sqrt{x}\,\frac{\partial z}{(1/2\sqrt{x})\partial x} + \sqrt{y}\,\frac{\partial z}{(1/2\sqrt{y})\partial y} + \frac{1}{2}\left[\left(\frac{\partial z}{(1/2\sqrt{x})\partial x}\right)^2 + \left(\frac{\partial z}{(1/2\sqrt{y})\partial y}\right)^2\right]^{1/2} \qquad \text{... (1)}$$

Put $\quad (1/2\sqrt{x})\,dx = dX \quad$ and $\quad (1/2\sqrt{y})dy = dY \quad$ so that $\quad \sqrt{x} = X \quad$ and $\quad \sqrt{y} = Y$ ... (2)

Using (2), (1) gives $\quad z = (\partial z/\partial X)X + (\partial z/\partial Y)Y + (1/2) \times \{(\partial z/\partial X)^2 + (\partial z/\partial Y)^2\}^{1/2}$

or $\quad z = PX + QY + (1/2) \times (P^2 + Q^2)^{1/2}, \qquad$ where $\quad P = \partial z/\partial X \quad$ and $\quad Q = \partial z/\partial Y$.

It is of the Clairaut's form $z = Px + Qy + f(P,Q)$ and so its complete integral is given by

$$z = aX + bY + (1/2) \times (a^2 + b^2)^{1/2} \qquad \text{or} \qquad z = a\sqrt{x} + b\sqrt{y} + (1/2) \times (a^2 + b^2)^{1/2}$$

Ex. 11. *Find a complete and the singular integral of* $4xyz = pq + 2px^2y + 2qxy^2$

Sol. The given equation can be rewritten as

$$z = \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right). \qquad ... (1)$$

Put $2x\,dx = dX$ and $2y\,dy = dY$ ... (2)

so that $x^2 = X$ and $y^2 = Y.$ ... (3)

Using (2), (1) becomes $z = (\partial z/\partial X)(\partial z/\partial Y) + X(\partial z/\partial X) + Y(\partial z/\partial Y)$

or $$z = XP + YQ + PQ, \qquad ... (4)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $z = XP + YQ + f(P, Q)$.

$\therefore$ Solution of (4) is $z = aX + bY + ab$, a, b being arbitrary constants.

or $z = ax^2 + by^2 + ab$, which is complete integral. ... (5)

Differentiating (5) partially w.r.t a and b, we have

$0 = x^2 + b$ and $0 = y^2 + b$ so that $b = -x^2$ and $a = -y^2$ .... (6)

Eliminating a and b between (5) and (6), the required singular integral is

$z = -x^2y^2 - x^2y^2 + x^2y^2$ or $z = -x^2y^2$.

Ex. 12. *Find the complete and singular solutions of* $z = px + qy + p^2q^2$. **[Jabalpur 2000; Sagar 1995; Rewa 2003; Ravishankar 2004]**

Sol. Given $z = px + qy + p^2q^2$ ... (1)

Since (1) is in Clairaut's form, its complete solution is

$z = ax + by + a^2b^2$, a, b being arbitrary constants ... (2)

To find singular solution of (1). Differentiating (2) partially w.r.t. 'a' and 'b' successively,

$0 = x + 2ab^2$ and $0 = y + 2a^2b$ ... (3)

From (3), $a = -(y^2/2x)^{1/3}$ and $b = -(x^2/2y)^{1/3}$ ... (4)

Substituting the values of a and b given by (4) in (2), we get

$z = -x(y^2/2x)^{1/3} - y(x^2/2y)^{1/3} + (x^2y^2/16)^{1/3}$ or $z = -(3/4) \times 4^{1/3}\, x^{2/3}\, y^{2/3}$,

which is the required singular solution of (1)

EXERCISE 3 (D)

Solve the following partial differential equations : (1 – 9)

1. $z = px + qy - 2p - 3q$. **[M.D.U. Rohtak 2006]**

Ans. C.I. $z = ax + by - 2a - 3b$; **S.S.** $z = 0$; **G.S.** It is given by $z - ax - \psi(a)y + 2a + 3\psi(a) = 0$, $x + (y-3)\psi'(a) - 2 = 0$

2. $z = px + qy + 5pq$. **Ans. S.I.** $z = ax + by + 5ab$; **S.S.** $5z + xy = 0$

G.S. $z - ax - \psi(a)y - 5a\,\psi(a) = 0$, $x + 5\psi(a) + (y + 5a)\,\psi'(a) = 0$

3. $z = px + qy + p^2 - q^2$ **[Purvanchal 2007]** **Ans. S.I.** $z = ax + by + a^2 - b^2$;

S.S. $x^2 - y^2 + 4z = 0$; **G.S.** $z - ax - \psi(a)y - a^2 + \{\psi(a)\}^2 = 0$; $x + 2a + \{y - 2\psi(a)\}\psi'(a) = 0$;

4. $z = px + qy + (q/p) - p$. **[Madras 2005]**

Ans. C.I. $z = ax + by + (b/a) - a$; **S.S.** $yz = 1 - x$; **G.S.** It is given by

$z - ax + \psi(a)y + (1/a)\times\psi(a) - a$, $-x + \psi'(a)y - (1/a^2)\,\psi(a) + (1/a)\times\psi'(a) = 0$

5. $z = px + qy + p/q$ **Ans. C.I.** $= ax + by + a/b$; **S.S.** $xz + 4 = 0$;

G.S. $z - ax - \psi(a)y - a/\psi(a) = 0$; $x + \psi'(a)y + 1/\psi(a) - \{a\psi'(a)\}/\{\psi(a)\}^2 = 0$

6. $z = px + qy + 2\sqrt{pq}$ **[Bangalore 1994]**

Ans. C.I. $z = ax + by + 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$; **G.S.** $z - ax - \psi(a)y - 2\sqrt{a\psi(a)} = 0$,

$$x + \psi'(a) + \{\psi(a) + a\psi'(a)\}/2\sqrt{a\psi(a)} = 0$$

7. $z = px + qy - 2\sqrt{pq}$. **Ans. C.I.** $z = ax + by - 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$;

G.S. $z - ax - \psi(a)y + 2\sqrt{a\psi(a)} = 0,\ x + \psi'(a)y - \{\psi(a) + a\psi'(a)\}/\sqrt{a\psi(a)} = 0$

8. $z = px + qy + p^2 + pq + q^2$. **[Ranchi 2010]**

Ans. C.I. $z = ax + by + a^2 + ab + b^2$; **S.S.** $x^2 + y^2 - xy + 3z = 0$, **G.S.** $z - ax - \psi(a)y - a^2 - a\psi(a) - \{\psi(a)\}^2 = 0,\quad x + \{y + a + 2\psi(a)\}\psi'(a) + 2a + \psi(a) = 0$

9. $z = px + qy + (\alpha p^2 + \beta q^2 + 1)^{1/2}$. **Ans. C.I.** $z = ax + by + (\alpha a^2 + \beta b^2 + 1)^{1/2}$;

S.S. $x^2/\alpha + y^2/\beta + z^2 = 1$; **G.S.** $z - ax - \psi(a)y - [a\alpha^2 + \beta\{\psi(a)\}^2 + 1]^{1/2} = 0; x + \psi'(a)y$

$$+\{a\alpha + \beta\psi(a)\psi'(a)\}/[a\alpha^2 + \beta(\psi(a)\}^2 + 1]^{1/2} = 0$$

10. Find the complete integral of $z = px + qy - \sin(pq)$ **[GATE 2003]**

Ans. $z = ax + by - \sin(ab)$ a, b being arbitrary constants.

11. Find the complete integral and singular integral of the differential equation $z = px + qy + p^2 - q^2$. Find also a developable surface belonging to the general integral of this differential equation. **[I.A.S 1983]**

Ans. Complete integral is $z = ax + by + a^2 - b^2$; singular integral is $4z = 3(x^2 - y^2)$

3.14. Standard form III. Only p, q and z present. [Nagpur 2003; Delhi Maths (H) 2006]

Under this standard form we consider differential equation of the form

$$f(p, q, z) = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $$\frac{dp}{p(\partial f/\partial z)} = \frac{dq}{q(\partial f/\partial z)} = \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}, \text{ using (1)}$$

Taking the first two ratios, $(1/p)dp = (1/q)dq$

Integrating, $q = ap$, a being an arbitrary constant. ...(2)

Now, $dz = p\,dx + q\,dy = p\,dx + ap\,dy$, using (2)

or $dz = p(dx + a dy) = pd(x + ay) = p\,du$, ...(3)

where $u = x + ay$. ...(4)

Now, (3) $\Rightarrow$ $p = dz/du$ and so by (2) $q = ap = a(dz/du)$.

Substituting these values of p and q in (1), we get $f\left(\dfrac{dz}{du}, a\dfrac{dz}{du}, z\right) = 0$, ...(5)

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x + ay)$.

3.15. Working rule for solving equations of the form $f(p,q, z) = 0$. ...(1)

Step I. Let $u = x + ay$, where a is an arbitrary constant. ...(2)

Step II. Replace p and q by dz/du and $a(dz/du)$ respectively in (1) and solve the resulting ordinary differential equation of first order by usual methods.

Step III. Replace u by $x + ay$ in the solution obtained in step II.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form III.

Remark 2. Singular and general integrals are obtined by well known methods.

3.16. SOLVED EXAMPLES BASED ON ART 3.15.

Ex. 1. *Find a complete integral of* $9(p^2z + q^2) = 4$.

[Delhi Maths (H) 2006; Banglore 1995; I.A.S. 1988; Meerut 1996; Rohilkhand 1995]

Sol. Given equation is $$9(p^2z + q^2) = 4, \qquad ...(1)$$

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \qquad \text{or} \qquad \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}.$$

or $\qquad du = \pm (3/2) \times (z + a^2)^{1/2} dz$, separating variables u and z.

Integrating, $\quad u + b = \pm (3/2) \times [(z + a^2)^{3/2}/(3/2)] \quad$ or $\quad u + b = \pm (z + a^2)^{3/2}$

or $\quad (u + b)^2 = (z + a^2)^3 \quad$ or $\quad (x + ay + b)^2 = (z + a^2)^3$, as $u = x + ay$

which is a complete integral contaning two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p^2 = qz$. **[Bilaspur 1996; Sagar 2004]**

Sol. Given equation is $$p^2 = qz, \qquad ...(1)$$

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \qquad \text{or} \qquad \frac{dz}{du} = az \qquad \text{or} \qquad \frac{dz}{z} = a\,du.$$

Integrating, $\quad \log z - \log b = au \quad$ or $\quad z = be^{au} \quad$ or $\quad z = be^{a(x + ay)}$,

which is a complete integral containing two arbitrary constants a and b.

Ex. 3.(a) *Find a complete integral of* $z = pq$. **[Meerut 1994]**

Sol. Given equation is $$z = pq, \qquad ... (1)$$

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z = a\left(\frac{dz}{du}\right)^2 \qquad \text{or} \qquad \frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{a}} \qquad \text{or} \qquad \pm\sqrt{a}\, z^{-1/2} dz = du.$$

Integrating, $\quad \pm 2\sqrt{az} = u + b \quad$ or $\quad 4(az) = (x + ay + b)^2$, as $u = x + ay$

Ex. 3.(b) *Find a complete integral of* $pq = 4z$.

Sol. Proceed as in Ex. 3.(a). **Ans.** $(x + ay + b)^2 = az$

Ex. 4.(a) *Find a complete integral of* $p(1 + q^2) = q(z - \alpha)$. **[Meerut 1999; Bilaspur 2002; Jiwaji 2003; Ravishanker 2005 Rewa 1998, Vikram 2004]**

(b) *Find a complete integral of* $p(1 + q^2) = q(z - 1)$. **[M.S. Univ. T.N. 2007]**

Sol. (*a*) Given equation is $$p(1 + q^2) = q(z - \alpha), \qquad ...(1)$$

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\frac{dz}{du}\left\{1 + \left(a\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}(z - \alpha) \qquad \text{or} \qquad 1 + a^2\left(\frac{dz}{du}\right)^2 = a(z - \alpha)$$

$$\text{or} \qquad \frac{dz}{du} = \pm\frac{\sqrt{\{a(z-\alpha)-1\}}}{a} \qquad \text{or} \qquad du = \pm\frac{a\,dz}{\sqrt{\{a(z-\alpha)-1\}}}.$$

Integrating, $u + b = \pm 2\sqrt{\{a(z-\alpha)-1\}}$ or $(u + b)^2 = 4\{a(z - \alpha) - 1\}^2$

or $(x + ay + b)^2 = 4\{a(z - \alpha) - 1\}^2$, a and b being arbitrary constants.

(*b*) Proceed as in part (a) by taking $\alpha = 1$.

Ex. 5.(a) *Find a complete integral of* $pz = 1 + q^2$. **[Meerut 1996]**

Sol. Given equation is $$pz = 1 + q^2, \qquad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z\frac{dz}{du} = 1 + a^2\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad a^2\left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0.$$

$$\therefore \frac{dz}{du} = \frac{z \pm (z^2 - 4a^2)^{1/2}}{2a^2} \quad \text{or} \quad \frac{dz}{z \pm (z^2 - 4a^2)^{1/2}} = \frac{du}{2a^2}$$

or $$\frac{[z \mp (z^2 - 4a^2)^{1/2}]dz}{[z \pm (z^2 - 4a^2)^{1/2}][z \mp (z^2 - 4a^2)^{1/2}]} = \frac{du}{2a^2} \quad \text{or} \quad \frac{z \mp (z^2 - 4a^2)^{1/2}}{4a^2} = \frac{du}{2a^2}$$

or $$[z \mp (z^2 - 4a^2)^{1/2}]\, dz = 2du.$$

Integrating, $$\frac{z^2}{2} \mp \left[\frac{z}{2}(z^2 - 4a^2)^{1/2} - \frac{4a^2}{2}\log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 2u + \frac{b}{2}$$

or $$z^2 \mp \left[z(z^2 - 4a^2)^{1/2} - 4a^2 \log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 4(x + ay) + b.$$

Ex. 5. (b) *Find a complete integral of* $1 + p^2 = qz$.

Sol. Proceed as in Ex. 5.(*a*). The requied complete integral is

$$a^2z^2 \mp \left[az\sqrt{(a^2z^2 - 4)} - 4\log\left\{az + \sqrt{(a^2z^2 - 4)}\right\}\right] = 4(x + ay) + b.$$

Ex. 6. *Find complete integrals of the following partial differential equations*

(*i*) $p(z + p) + q = 0$

(*ii*) $p(1 + q) = qz$. **[Gulbarga 2005]**

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}\left(z + \frac{dz}{du}\right) + a\frac{dz}{du} = 0 \quad \text{or} \quad \frac{dz}{du} = -(z + a) \quad \text{or} \quad \frac{dz}{z + a} = -du.$$

Integrating, $\log(z + a) - \log b = -u$ or $z + a = be^{-u}$ or $z + a = be^{-(x + ay)}$.

(*ii*) Proceed as in part (*i*). **Ans.** $az - 1 = be^{x + ay}$.

Ex. 7. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$. **[I.A.S. 1991]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation,

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0 \quad \text{or} \quad (1 + a^3)\frac{dz}{du} = 3az \quad \text{or} \quad \frac{1 + a^3}{z}dz = 3au.$$

Integrating $(1 + a^3)\log z = 3au + b$ or $(1 + a^3)\log z = 3a(x + ay) + b$.

Ex. 8. *Find a complete integrals of* (*i*) $p + q = z/c$.

(*ii*) $p + q = z$.

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du} + a\frac{dz}{du} = \frac{z}{c} \quad \text{or} \quad (1+a)\frac{dz}{du} = \frac{z}{c} \quad \text{or} \quad \frac{c(1+a)}{z}dz = du.$$

Integrating, $c(1 + a) \log z = u + b$ or $c(1 + a) \log z = x + ay + b$.

(*ii*) Proceed as in part (*i*). **Ans.** $(1 + a) \log z = x + ay + b$.

Ex. 9. *Find a complete integral of* $p^2 = z^2(1 - pq)$. **[Jiwaji 1998; Meerut 2001]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^2 = z^2\left\{1 - a\left(\frac{dz}{du}\right)^2\right\} \quad \text{or} \quad \left(\frac{dz}{du}\right)^2 (1 + az^2) = z^2$$

or $$\frac{dz}{du} = \pm \frac{z}{\sqrt{(1+az^2)}} \quad \text{or} \quad \pm\, du = \frac{\sqrt{(1+az^2)}}{z}dz = \pm \frac{(1+az^2)dz}{z\sqrt{(1+az^2)}}.$$

or $$\pm\int du \pm b = \int \frac{dz}{z\sqrt{(1+az^2)}} + \frac{1}{2}\int \frac{2az\, dz}{\sqrt{(1+az^2)}}. \qquad ...(1)$$

Now, $$\int \frac{dz}{z\sqrt{(1+az^2)}} = \int \frac{(-1/t^2)\,dt}{(1/t)\times\sqrt{\{(1+(a/t^2)\}}}, \text{ putting } z = 1/t \text{ so that } dz = -(1/t^2)dt$$

$$= -\int \frac{dt}{(t^2+a)^{1/2}} = -\sinh^{-1}\frac{t}{\sqrt{a}} = -\sinh^{-1}\frac{1}{z\sqrt{a}}, \text{ as } t = \frac{1}{z} \qquad ...(2)$$

and $$\frac{1}{2}\int \frac{2az\, dz}{(1+az^2)^{1/2}} = \frac{1}{2}\int \frac{2v\, dv}{v}, \text{ putting } 1 + az^2 = v^2 \text{ and } 2az\, dz = 2v\,dv$$

$$= v = (1 + az^2)^{1/2}.$$

Using (2) and (3), (1) reduces to $\pm (u + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}$

or $$\pm (x + ay + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}.$$

Ex. 10. *Find complete and singular integrals of* $4(1 + z^3) = 9z^4pq$.

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$4(1 + z^3) = 9z^4 a\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad \pm\frac{3\sqrt{a}z^2}{(1+z^3)^{1/2}}\,dz = 2\,du$$

or $$\pm\left(\sqrt{a}/t\right) \times 2t\,dt = 2\,du, \text{ putting } 1 + z^3 = t^2 \quad \text{so that} \quad 3z^2dz = 2t\,dt$$

Integrating, $\pm\sqrt{a}\, t = u + b$ or $\pm\sqrt{a}\,(1 + z^3)^{1/2} = x + ay + b$

or $$a(1 + z^3) = (x + ay + b)^2, \qquad ...(1)$$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. a and b by turn, we get

$$1 + z^3 = 2y(x + ay + b) \qquad ...(2)$$

and $$0 = 2(x + ay + b). \qquad ...(3)$$

Eliminating a and b from (1), (2) and (3), the singular integral is $1 + z^3 = 0$. ...(4)

From (4), $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Thus these values of p and q together with $1 + z^3 = 0$ satisfy the given equation. Hence $1 + z^3 = 0$ is the required singular integral.

Ex. 11. *Find complete and singular integrals of* $q^2 = z^2p^2(1 - p^2)$.

[Madurai Kamraj 2008; CDLU 2004]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right] \qquad \text{or} \qquad a^2 = z^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

or $$\left(\frac{dz}{du}\right)^2 = \frac{z^2-a^2}{z^2} \qquad \text{or} \qquad du = \pm\,\frac{zdz}{(z^2-a^2)^{1/2}}.$$

Integrating, $\quad u + b = \pm\,(z^2 - a^2)^{1/2} \quad$ or $\quad (x + ay + b)^2 = z^2 - a^2, \quad \ldots(1)$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$-2a = 2y(x + ay + b) \qquad \ldots(2)$$

and $$0 = 2(x + ay + b). \qquad \ldots(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

Ex. 12. *Find complete, singular and general integral of* $p^3 + q^3 = 27z$. **[Ravishankar 2005]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 = 27z \qquad \text{or} \qquad \frac{dz}{du}(1 + a^3)^{1/3} = 3z^{1/3}$$

or $$du = (1/3) \times (1 + a^3)^{1/3}\, z^{-1/3}\, dz.$$

Integrating, $\quad u + b = (1/3) \times (1 + a^3)^{1/3} \times [z^{2/3}/(2/3)] \quad$ or $\quad 2(u + b) = (1 + a^3)^{1/3}z^{2/3}$

or $\quad 8(u + b)^3 = (1 + a^3)z^2 \quad$ or $\quad 8(x + ay + b)^3 = (1 + a^3)z^2, \quad \ldots(1)$

which is a complete integral containing two arbitrary contants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$24y(x + ay + b)^2 = 3a^2z^2 \qquad \ldots(2)$$

and $$24(x + ay + b) = 0. \qquad \ldots(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

General integral. Let $b = \phi(a)$, where ϕ is an arbitrary function. Then (1) becomes

$$8[x + ay + \phi(a)]^3 = z^2(1 + a^3). \qquad \ldots(4)$$

Differentiating (4) partially w.r.t. 'a', $\quad 24[x + ay + \phi(a)]^2\,[y + \phi'(a)] = 3a^2z^2. \quad \ldots(5)$

General integral is obtained by eliminating a from (4) and (5).

Ex. 13. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

[Delhi Maths Hons 2005; Meerut 2003]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \qquad \text{or} \qquad z^2(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

or $$du = \pm\, z(z^2 + a^2)^{1/2}\, dz = \pm\,(1/2) \times (z^2 + a^2)^{1/2}\,(2zdz)$$

Integrating, $$u + b = \pm\,(1/2) \times [(z^2 + a^2)^{3/2}/(3/2)]$$

or $\quad 9(u + b)^2 = (z^2 + a^2)^3 \quad$ or $\quad 9(x + ay + b)^2 = (z^2 + a^2)^3, \quad \ldots(1)$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially, w.r.t. 'a' and 'b', we get

$$18(x + ay + b)y = 3(z^2 + a^2) \times 2a \qquad ...(2)$$

and
$$18(x + ay + b) = 0. \qquad ...(3)$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ do not satisfy the given equation. Hence $z = 0$ is not a singular solution of the given equation.

Ex. 14. (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

[Jabalpur 2004; Banglore 1993; I.A.S. 1996; Meerut 1997]

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$. **[I.A.S. 1979]**

Sol. (*i*) The given equation is of the form $f(p, q\ z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = k^2 \qquad \text{or} \qquad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

or
$$\pm (1 + a^2)^{1/2}\frac{z}{(k^2 - z^2)^{1/2}}\,dz = du \qquad \text{or} \qquad \pm\frac{1}{2}(1 + a^2)(k^2 - z^2)^{-1/2}(-2zdz) = du.$$

Integrating, $\pm (1 + a^2)^{1/2}\ (k^2 - z^2)^{1/2} = u + b$ or $(1 + a^2)(k^2 - z^2) = (u + b)^2$

or
$$(1 + a^2)(k^2 - z^2) = (x + ay + b)^2.$$

(*ii*) Here $k = 1$. Proceed as in part (*i*) and get complete integral

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2. \qquad ...(1)$$

Differentiating (1) partially w.r.t. a and b, we get

$$2a(1 - z^2) = 2(x + ay + b) \times y \qquad ...(2)$$

and
$$0 = 2(x + ay + b). \qquad ...(3)$$

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Now, $p = 0$, $q = 0$ and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z^2 = 1$.

Ex. 15. *Find a complete integral of* (*i*) $q^2y^2 = z(z - px)$ **[Meerut 1997]**

(*ii*) $p^2x^2 = z(z - qy)$.

Sol. (*i*) Given equation can be rewritten as
$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right). \qquad ...(1)$$

We choose new variables X and Y such that $(1/x)dx = dX$ and $(1/y)dy = dY$. ...(2)

so that $\log x = X$ and $\log y = Y$. ...(3)

Using (2), (1) becomes
$$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right) \qquad \text{or} \qquad Q^2 = z\,(z - P), \qquad ...(4)$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, where a is an arbitrary constant. Replacing P by dz/du and Q by $a(dz/du)$ in (4), we get

$$a^2\left(\frac{dz}{du}\right)^2 = z\left(z - \frac{dz}{du}\right) \qquad \text{or} \qquad a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0.$$

$$\therefore \qquad \frac{dz}{du} = \frac{-z \pm (z^2 + 4a^2z^2)^{1/2}}{2a^2} = \frac{-1 \pm (1 + 4a^2)^{1/2}}{2a^2}z = kz, \qquad ...(5)$$

where $$k = [-1 \pm (1 + 4a^2)^{1/2}]/2a^2. \qquad ...(6)$$

From (5), $(1/kz)dz = du$ so that $(1/k) \log z = u + \log b$

or $\log z^{1/k} = X + aY + \log b = \log x + a \log y + \log b = \log (xby^a)$.

$\therefore$ $z^{1/k} = xby^a$ is complete integral containing two arbitrary constants a and b and an absolute constant k given by (6).

(*ii*) Proceed as in part (*i*). **Ans.** $xby^a = z^{1/k}$, where $k = [-1 \pm (a^2 + 4)^{1/2}]/2$

Ex. 16. *Solve* $p^2 + q^2 = z$.

Sol. Given equation is $$p^2 + q^2 = z, \qquad ... (1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we have
$$(du/dz)^2 + a^2(du/dz)^2 = z \quad \text{or} \quad (dz/du)^2 = z/(1 + a^2)$$

or $$\frac{dz}{du} = \pm \frac{z^{1/2}}{(1+a^2)^{1/2}} \quad \text{or} \quad \pm z^{-1/2}(1+a^2)^{1/2} dz = du$$

Integrating, $$\pm 2z^{1/2}(1+a^2)^{1/2} = u+b \quad \text{or} \quad \pm 2z^{1/2}(1+a^2)^{1/2} = x+ay+b$$

Thus, $4z(1 + a^2) = (x + ay + b)^2$, a, b being arbitrary constants ... (2)

(2) is the complete integral of the given equation (1).

Differentiating (2) partially w.r.t. 'a' and 'b', we get
$$8az = 2y(x + ay + b) \quad \text{or} \quad 4az = y(x + ay + b) \qquad ... (3)$$
$$0 = 2(x + ay + b) \quad \text{or} \quad x + ay + b = 0 \qquad ... (4)$$

Substituting the value of $x + ay + b$ from (4) in (3), we have
$$4az = 0 \quad \text{or} \quad z = 0,$$ which is the singular solution.

In order to get the general solution, put $b = \psi(a)$ in (2) and get
$$4z(1+a^2) - \{x+ay+\psi(a)\}^2 = 0 \qquad ... (5)$$

Differentiating (5) partially w.r.t. 'a', $$8az - 2\{x+ay+\psi(a)\} \times \{y+\psi'(a)\} = 0 \qquad ... (6)$$

The required general solution is given by (5) and (6)

Ex. 17. *Find the complete integral of* $16p^2z^2 + 9q^2z^2 + 4(z^2-1) = 0$

Sol. Given equation is of the form $f(p,q,z) = 0$

Let $$u = x + ay, \ a \text{ being an arbitrary constant.} \qquad ... (1)$$

Now replacing p and q by dz/du and $a(dz/du)$ respectively in the given equation, we have
$$16z^2 (dz/du)^2 + 9a^2z^2 (dz/du)^2 + 4(z^2-1) = 0$$

or $$(16+9a^2)z^2\left(\frac{dz}{du}\right)^2 = 4(1-z^2) \quad \text{or} \quad \frac{dz}{du} = \frac{2(1-z^2)^{1/2}}{z(16+9a^2)^{1/2}}$$

or $$(-1/2) \times (16+9a^2)^{1/2} (1-z^2)^{-1/2}(-2z)dz = du$$

Integrating, $$-(16+9a^2)^{1/2} (1-z^2)^{1/2} = u+b = x+ay+b, \text{ by (1)}$$

or $(16+9a^2)(1-z^2) = (x+ay+b)^2$ is the complete integral, a, b being arbitrary constants

Ex. 18. *Find the complete integral of* $pq = x^4y^3z^2$

Sol. Re–writing the given equation, we get $$(\partial z/x^4\partial x)(\partial z/y^3\partial y) = z^2 \qquad ... (1)$$

Putting $x^4 dx = dX$, $y^3 dy = dY$ so that $x^5/5 = X$, $y^4/4 = Y$, (1) gives

$$(\partial z/\partial X)\ (\partial z/\partial Y) = z^2 \quad \text{or} \quad PQ = z^2 \qquad \text{... (2)}$$

which is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, a being an arbitrary constant. Replacing P and Q be dz/du and $a(dz/du)$ respectively in (2), we get

$$a\ (dz/du)^2 = z^2 \quad \text{giving} \quad (\sqrt{a}/z)dz = du$$

Integrating $\sqrt{a}\log z = u + b = X + aY + b$, as $u = X + aY$

or $\sqrt{a}\ \log z = x^5/5 + (ay^4)/4 + b$, a, b being arbitrars constants

EXERCISE 3(E)

Find the complete integral of the following equations (1–7)

1. $z = p^2 - q^2$. **Ans.** $x + ay + b = 4(1 - a^2)z$

2. $zpq = p + q$. **(Delhi 2007)** **Ans.** $x + ay + b = (4az)/(1 - a)$

3. $z^2p^2 + q^2 = p^2q$. **Ans.** $z = a \tan(x + ay + b)$

4. $p^2z^2 + q^2 = 1$. **[Delhi B.A. (Prog) II 2010, 11; M.S. Univ. T.N. 2007, Nagpur 2001; Meerut 2008]** **Ans.** $x + ay + b = \pm[(z/2) \times (z^2 + a^2)^{1/2} + (a^2/2) \times \sinh^{-1}(z/a)]$

5. $p^3 = qz$. **(Delhi B.Sc. (Prog) II 2007)** **Ans.** $4z = (x + ay + b)^2$

6. $16z^2p^2 + 25z^2a^2 + 9z^2 - 81 = 0$. **Ans.** $(16 + 25a^2)\ (9 - z^2) = 9(x + ay + b)^2$

7. $z^2 = 1 + p^2 + q^2$. **Ans.** $z = \cosh\{x + ay + b)/(1 + a^2)^{1/2}\}$

8. Using Charpit's method, discuss how to solve equations of the form $f(z, p, q) = 0$. Hence find complete integral of the equation $9(p^2z + q^2) = 4$. **[Delhi Maths (H) 2006]**

Hint : Refer Art. 3.14 and Ex. 1 of Art. 3.15

Solve the following partial differential equations (9 – 14)

9. $z^2(p^2 + q^2 + 2) = 1$ **Ans. C.I.** $(1 + a^2)\ (1 - 2z^2) = 4(x + ay + b)^2$; **S.S.** $2z^2 - 1 = 0$;

G.S. It is given by $(1 + a^2)\ (1 - 2z^2) - \{x + ay + \psi(a)\}^2 = 0,\quad a(1 - 2z^2)$

$-4\{x + ay + \psi(a)| \times \{y + \psi'(a)\} = 0$

10. $z = pq$. **Ans. C.I.** $4az = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** It is given by

$4az - \{x + ay + \psi(a)\}^2,\quad 2z - \{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0$

11. $p(1 - q^2) = q(1 - z)$. **Ans. C.I.** $4(1 - a + az) = (x + ay + b)^2$; **S.S.** Does not exist; **G.S.** It is given by $4(1 - a + az) - \{x + ay + \psi(a)\}^2 = 0,\quad 2z - 2 - \{x + ay + \psi(a)\}\{y + \psi'(a)\} = 0$

12. $p^2 + pq = 4z$. **Ans. C.I.** $(1 + a)z = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** $(1 + a)z - \{x + ay + \psi(a)\}^2 = 0$,

$z - 2\{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0$

13. $p^3 + q^3 = 3pqz$, $z > 0$. **Ans. C.S.** $(1 + a^3)\log z = 3a(x + ay) + b$; **S.S.** Does not exist. **G.S.** It is given by $(1 + a^3)\log z - 3a(x + ay) - \psi(a) = 0,\quad 3a^2 \log z - 3x - 6ay - \psi'(a) = 0$

14. $p^2 + q^2 = 4z$. **Ans. C.I.** $4(1+a^2)z - (x + ay + b)^2 = 0$; **S.S.** $z = 0$; **G.S.** It is given by

$(1 + a^2)z - \{x + ay + \psi(a)\}^2 = 0,\quad az - \{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0.$

3.17. Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$. *i.e.*, a form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side. **[Bhopal 2010; Ravishankar 1999]**

Let $$F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0. \qquad ...(1)$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or $$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p(\partial f_1/\partial p) + q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}, \text{ by (1)}$$

Taking the first and the fourth ratios, we have

$(\partial f_1 / \partial p)\, dp + (\partial f_1 / \partial x) dx = 0$ or $df_1 = 0.$

Integrating, $f_1 = a$, a being an arbitrary constant.

$\therefore$ (1) $\Rightarrow$ $f_1(x, p) = f_2(y, q) = a.$ ...(2)

Now, (2) $\Rightarrow$ $f_1(x, p) = a$ and $f_2(y, q) = a.$ ...(3)

From (3), on solving for p and q respectively, we get

$p = F_1(x, a)$, say and $q = F_2(y, a)$, say ...(4)

Substituting these values in $dz = p\, dx + q\, dy$, we get $dz = F_1(x, a)\, dx + F_2(y, a)\, dy.$

Integrating, $$z = \int F_1(x,a)\;dx + \int F_2(y,a)dy + b,$$

which is a complete integral containing two arbitrary constants a and b.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form IV.

Remark 2. Singular and general integral are obtained by well known methods.

3.18. SOLVED EXAMPLES BASED ON ART 3.17

Ex. 1. *Find a complete integral of* $x(1 + y)p = y(1 + x)q$. **[Agra 1991]**

Sol. Separating p and x from q and y, the given equation reduces to

$$(xp)/(1 + x) = (yq)/(1 + y)$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a \quad \text{so that} \quad p = a\left(\frac{1+x}{x}\right) \quad \text{and} \quad q = a\left(\frac{1+y}{y}\right).$$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy \quad \text{or} \quad dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy.$$

Integrating, $z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b,$
which is a complete integral containing two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p - 3x^2 = q^2 - y$. **[Meerut 1996]**

Sol. Equating each side to an arbitrary constant a, we get

$p - 3x^2 = a$ and $q^2 - y = a$ so that $p = a + 3x^2$ and $q = (a + y)^{1/2}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (a + 3x^2)dx + (a + y)^{1/2}\, dy$ so that $z = ax + x^3 + (2/3) \times (a + y)^{3/2} + b.$

Ex. 3. *Find a complete integral of* $yp = 2yx + \log q$. **[Ravishankar 2005]**

Sol. Rewritting the given equation, $p = 2x + (1/y) \log q$ or $p - 2x = (1/y) \log q$.

Equating each side to an arbitrary constant a, we get

$p - 2x = a$ and $(1/y) \log q = a$ so that $p = a + 2x$ and $q = e^{ay}$.

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$dz = (a + 2x)dx + e^{ay} dy$ so that $z = (ax + x^2) + (1/a) \times e^{ay} + b$.

Ex. 4. *Find a complete integral of* $q = px + p^2$.

[Agra 1995; Meerut 1994; Bilaspur 2004; Jabalpur 1998]

Sol. Equating each side of the given equation to an arbitrary constant a, we have

$q = a$ and $px + p^2 = a$ or $q = a$ and $p^2 + px - a = 0$.

$\therefore$ $q = a$ and $p = [-x \pm (x^2 + 4a)^{1/2}]/2$.

Putting thse values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = (1/2) \times [-x \pm (x^2 + 4a)^{1/2}]dx + a\, dy.$$

Integrating, $$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{(x^2+4a)} + 2a \log\left\{x + \sqrt{(x^2+4a)}\right\}\right] + ay + b.$$

Ex. 5. *Solve* $py + qx + pq = 0$. **[Kurukshetra 2004; I.A.S 1990]**

Sol. Given $py + q(x + p) = 0$. or $p/(p + x) = -q/y$.

Equating each side to an arbitrary constant a, we get

$p/(p + x) = a$ and $-q/y = a$ $\Rightarrow$ $p = (xa)/(1 - a)$ and $q = -ay$.

Putting thse values of p and q in $dz = p\, dx + q\, dy$, we get

$dz = \{a/(1 - a)\}\, x\, dx - ay\, dy$ so that $z = \{a/(1 - a)\} \times (x^2/2) - a \times (y^2/2) + b/2$

or $2z = \{a/(1 - a)\}x^2 - ay^2 + b$, a, b being arbitrary constants.

Ex. 6. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

[Agra 2006; Jabalpur 2004; Rewa 2002 Sagar 1999; Vikram 1996 Delhi Maths Hons 1990; I.A.S. 1989; Kanpur 1994; Meerut 2003]

Sol. Given $$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{or} \quad \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \quad ...(1)$$

Let $z\, dz = dz$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side of the above equation to an arbitrary constant a^2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, we have

$$dZ = (a^2 + x^2)^{1/2}\, dx + (y^2 - a^2)^{1/2}\, dy.$$

Integrating, $$Z = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$
$$+ (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$$

or $$z^2 = x^2(a^2 + x^2)^{1/2} + a^2 \log [x + (a^2 + x^2)^{1/2}\} + y(y^2 - a^2)^{1/2} - a^2\log\{y + (y^2 - a^2)^{1/2}\} + b$$

$[\because$ From (2), $Z = z^2/2]$

Ex. 7. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

[Bilaspur 2003; Indore 2002, 02; Jiwaji 2000; Bangalore 1995; I.A.S 1989]

Sol. Re–writting the given equation, $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y.$ ...(1)

Let $\sqrt{z}\, dz = dZ$ so that $(2/3) \times z^{3/2} = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial y)^2 = x - y$ or $P^2 - Q^2 = x - y,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y. \qquad ...(3)$$

Equating each side to an arbitrary constant a, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (x + a)^{1/2}dx + (y + a)^{1/2}\,dy.$

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$, as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$, a, b being arbitrary constants.

Ex. 8. *Find a complete integral of* $z(xp - yq) = y^2 - x^2$.

Sol. Re–writting the given equation, we have

$$xz\frac{\partial z}{\partial x} - yz\frac{\partial z}{\partial y} = y^2 - x^2 \quad \text{or} \quad x\left(z\frac{\partial z}{\partial x}\right) - y\left(z\frac{\partial z}{\partial y}\right) = y^2 - x^2. \qquad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $x(\partial Z/\partial x) - y(\partial Z/\partial y) = y^2 - x^2$ or $xP - yQ = y^2 - x^2,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$xP + x^2 = yQ + y^2.$$

Equating each side to an arbitrary constant a, we have

$xP + x^2 = a$ and $yQ + y^2 = a$ so that $P = a/x - x$ and $Q = a/y - y$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a/x - x)dx + (a/y - y)dy.$

Integrating, $Z = a \log x - (x^2/2) + a \log y - (y^2/2) + b/2$

or $z^2/2 = a(\log x + \log y) - (x^2 + y^2 - b)/2$ or $z^2 = 2a \log (xy) - x^2 - y^2 + b.$

Ex. 9. *Find a complete integral of* $p^2 + q^2 = z^2(x + y)$. **[Agra 2010; M.S. Univ. T.N. 2007]**

Sol. Given $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2(x + y)$ or $\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x + y$ ....(1)

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x + y$ or $P^2 + Q^2 = x + y,$
where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = y - Q^2.$$

Equating each side to an arbitrary constant a, we have

$P^2 - x = a$ and $y - Q^2 = a$ so that $P = (a + x)^{1/2}$ and $Q = (y - a)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a + x)^{1/2}dx + (y - a)^{1/2}dy.$

Integrating, $Z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2}] + (2/3) \times b$

$\therefore$ $\log z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2} + b]$ is a complete integral, using $Z = \log z$

Ex. 10. *Find a complete integral of* $p^2 + q^2 = (x^2 + y^2)z$. **[Delhi Maths Hons. 1995]**

Sol. The given equation can be rewritten as

$$\frac{1}{z}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2 + y^2 \quad \text{or} \quad \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \qquad ...(1)$$

Let $\left(1/\sqrt{z}\right)dz = dZ$ *i.e.,* $z^{-1/2}dz = dZ$ so that $2\sqrt{z} = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$, where $P = \partial Z/\partial x$ *and* $Q = \partial Z/\partial y$. Separting P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side to an arbitrary constant a^2, we have

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a^2 + x^2)^{1/2}dx + (y^2 - a^2)^{1/2}dy.$

Integrating, $$Z = \frac{x}{2}(x^2 + a^2)^{1/2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}(y^2 - a^2)^{1/2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + \frac{b}{2}$$

or $4z^{1/2} = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$, as $Z = 2\sqrt{z}$

Ex. 11. *Find a complete integral of* $(p^2/x) - (q^2/y) = (1/z) \times [(1/x) + (1/y)]$.

[Delhi B.Sc. Hons. 1996]

Sol. The given equation can be re–written as

$$\frac{z}{x}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z}{y}\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{1}{x}\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \frac{1}{y}\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{....(1)}$$

Let $\sqrt{z}\,dz = dZ$ so that $(2/3) \times z^{3/2} = Z.$ ...(2)

Using (2), (1) becomes

$$\frac{1}{x}\left(\frac{\partial Z}{\partial x}\right)^2 - \frac{1}{y}\left(\frac{\partial Z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{P^2}{x} - \frac{Q^2}{y} = \frac{1}{x} + \frac{1}{y},$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$(P^2 - 1)/x = (Q^2 + 1)/y.$$

Equating each side to an arbitrary constant a, we have

$(P^2 - 1)/x = a$ and $(Q^2 + 1)/y = a$ so that $P = (1 + ax)^{1/2}$ and $Q = (ay - 1)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (1 + ax)^{1/2}dx + (ay - 1)^{1/2}dy.$

Integrating, $Z = (2/3a) \times (1 + ax)^{3/2} + (2/3a) \times (ay - 1)^{3/2} + (2/3a) \times b$

or $az^{3/2} = (1 + ax)^{3/2} + (ay - 1)^{3/2} + b$, as $Z = (2/3) \times z^{3/2}$.

Ex. 12. *Find a complete integral of* $yzp^2 = q$. **[M.S. Univ. T.N. 2007]**

Sol. Given $$yz^2\left(\frac{\partial z}{\partial x}\right)^2 = z\frac{\partial z}{\partial y} \quad \text{or} \quad y\left(z\frac{\partial z}{\partial x}\right)^2 = \left(z\frac{\partial z}{\partial y}\right). \quad \text{...(1)}$$

Let $z\,dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $y(\partial Z/\partial x)^2 = \partial Z/\partial y$ or $yP^2 = Q$, ...(3)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P from y and Q, we get

$P^2 = Q/y = a^2$, (say) ; a being an arbitrary constant. Hence $P = a$ and $Q = ya^2$.

Then, $dZ = P\,dx + Q\,dy$ reduces to $dZ = a\,dx + ya^2dy$ so that $Z = ax + (a^2/y^2)/2 + b/2$

or $z^2/2 = ax + (a^2y^2)/2 + b/2$ or $z^2 = 2ax + a^2y + b$.

Ex. 13. *Find a complete integral of* $zpy^2 = x(y^2 + z^2q^2)$.

Sol. Given $$y^2\left(z\frac{\partial z}{\partial x}\right) = x\,y^2 + x\left(z\frac{\partial z}{\partial y}\right)^2 \quad \text{...(1)}$$

Let $z\,dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $y^2(\partial Z/\partial x) = xy^2 + x\,(\partial Z/\partial y)^2$ or $y^2P = x(y^2 + Q^2)$, where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from, Q and y, we get

$$P/x = (y^2 + Q^2)/y^2.$$

Equating each side to an arbitrary constant a, we get

$P/x = a$ and $1 + (Q^2/y^2) = a$ so that $P = ax$ and $Q = \pm (a-1)^{1/2}y$.

$\therefore$ $$dZ = P\,dx + Q\,dy = ax\,dx \pm (a-1)^{1/2}y\,dy$$

Integrating, $Z = (ax^2/2) \pm (a-1)^{1/2}(y^2/2) + b/2$ or $z^2 = ax^2 \pm (a-1)^{1/2}y^2 + b$, as $Z = z^2/2$.

Ex. 14. *Find the complete integral of the partial differential equation*

$$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2+y^2)$$ **[I.A.S. 2001]**

Sol. Re–writing the given equation, we have

$2q^2(p^2 - 4x^4) = x^2y^2(8q^2 - 3)$ or $(p^2 - 4x^4)/x^2 = y^2\ (8q^2-3)/2q^2 = 4a^2$, say

where a is an arbitrary constant. Then, $p^2 = 4x^2(a^2 + x^2)$ and $8q^2(y^2 - a^2) = 3y^2$

so that $p = 2x(a^2 + x^2)^{1/2}$ and $q = (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}$

Substituting these values in $dz = p\,dx + q\,dy$, we get

$$dz = 2x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (y/2) \times (y^2-a^2)^{-1/2}dy$$

Integrating, $z = 2\int x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (1/2) \times \int y(y^2-a^2)^{-1/2}dy + b$ ... (1)

Put $x^2 + a^2 = u$ and $y^2 - a^2 = v$ so that $2xdx = du$ and $2ydy = dv$ ... (2)

i.e., $xdx = (1/2) \times du$ and $ydy = (1/2) \times dv$. Then (1) reduces to

$$z = \int u^{1/2}du + (3/2)^{1/2} \times (1/4) \times \int v^{-1/2}dv + b$$

or $$z = (2/3) \times u^{3/2} + (3/2)^{1/2} \times (1/4) \times 2v^{1/2} + b$$

or $$z = (2/3) \times (x^2+a^2)^{3/2} + (3/2)^{1/2} \times (1/2) \times (y^2-a^2)^{1/2} + b,$$

which is the required complete integral containing a and b as arbitrary constants.

Ex. 15. *Find the complete integral of the partial differential equation* $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$ **[Delhi Maths (H) 2002; Agra 2005]**

Sol. Re–writing, $p^2/x^2 + y^2/q^2 = x^2 + y^2$ or $(p^2/x^2) - x^2 = y^2 - (y^2/q^2) = a^2$, say

$\Rightarrow$ $p = x(x^2+a^2)^{1/2}$, and $q = y/(y^2 - a^2)^{1/2}$

$\therefore$ $dz = p\,dx + q\,dy$ becomes $dz = x(x^2+a^2)^{1/2}dx + y(y^2-a^2)^{-1/2}dy$

Integrating, $$z = (1/3) \times (x^2+a^2)^{1/2} + (y^2-a^2)^{1/2} + b,$$

which is complete integral with a and b as arbitrary constants.

Ex. 16. *Find the complete integral of* $(1-x^2)yp^2 + x^2q = 0$

Sol. Re–writing, we have $(x^2-1)p^2/x^2 = q/y = a^2$, say

$\therefore$ $p = ax/(x^2-1)^{1/2}$ and $q = a^2y$. Hence $dz = p\,dx + q\,dy$ becomes

$dz = ax(x^2-1)^{-1/2}dx + a^2ydy$ so that $z = a(x^2-1)^{1/2} + (a^2y^2)/2 + b$.

Ex. 17. *Find the the complete integral of* $p + q - 2px - 2qy + 1 = 0$.

Sol. Re–writing, $p - 2px = 2qy - q - 1 = a$, say

$\therefore$ $p = \dfrac{a}{1-2x}$, $q = \dfrac{a+1}{2y-1}$ and so $dz = p\,dx + q\,dy = \dfrac{a\,dx}{1-2x} + \dfrac{(a+1)dy}{2y-1}$

Integrating, $z = -(a/2) \times \log|1-2x| + (1/2) \times (a+1) \log|2y+1| + b$.

Ex. 18. *Find the complete integral of* $2x(z^2q^2+1) = pz$

Sol. Re–writing the given equation, we have $2x\,\{(z\partial z/\partial y)^2+1\} = (z\partial z/\partial x)$ ... (1)

Putting $z\,dz = dZ$ so that $z^2/2 = Z$, (1) reduces to

$2x\{(\partial Z/\partial y)^2+1\} = \partial Z/\partial x$ or $2x(Q^2+1) = P$, ... (2)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Re–writing (2), we have

$(P/2x) - 1 = Q^2 = a^2$, say so that $P = 2x(1+a^2)$, $Q = a$

$\therefore\; dZ = P\,dx + Q\,dy$ becomes $dZ = 2x(1+a^2)dx + a\,dy$

Integrating, $Z = (1+a^2)x^2 + ay + b$ or $z^2/2 = (1+a^2)x + ay + b$.

EXERCISE 3 (F)

Find a complete integral of the following equations (1 – 9)

1(a). $p^2 = q + x$. **Ans.** $z = (2/3) \times (a+x)^{3/2} + ay + b$.

(b). $p^2y(1+x^2) = qx^2$. **[Delhi B.A (Prog) II 2011]** **Ans.** $z = a(1+x^2)^{1/2} + (a^2y/2) + b$.

2. $p^2 + q^2 = x + y$. **[Agra 2009; Meerut 2007]** **Ans.** $3z = 2(x+a)^{3/2} + 2(y-a)^{3/2} + b$.

3. $p^2 + q^2 = x^2 + y^2$. **[Jiwaji 1999; Ravishankar 2003]**

Ans. $2z = x(x^2+a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2-a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$.

4. $pe^y = qe^x$. **[Jiwaji 1996]** **Ans.** $z = ae^x + ae^y + b$.

5. $p^{1/3} - q^{1/3} = 3x - 3y$. **Ans.** $z = 3x^3 - 3ax^2 + a^2x + 2y^4 - 4ay^3 + 3a^2y^2 - a^3y + b$.

6. $q = 2yp^2$. **Ans.** $z = ax + a^2y^2 + b$.

7. $p^2 - y^3q = x^2 - y^2$. **Ans.** $2z = x(x^2+a^2)^{1/2} + a^2\sinh^{-1}(x/a) - (a^2/2) + \log y^2 + b$.

8. $z^2(p^2+q^2) = x^2 + e^{2y}$. **[Delhi Maths (H) 2005]**

Ans. $z^2 = x(x^2+a)^{1/2} + a\sinh^{-1}(x/\sqrt{a}) + 2(e^{2y}-a)^{1/2} - \sqrt{a}\,\tan^{-1}\{(e^{2y}-a)/a\}^{1/2} + b$

9. $p + q = px + qy$. **[Bangalore 1996]** **Ans.** $z = -a\log(1-x) + a\log(y-1) + b$.

Solve the following partial differential equations: (10 – 17)

10. $pq = xy$ **Ans. C.I.** $2z = ax^2 + y^2/a + b$; **S.S.** Does not exist **G.S.** $2z - ax^2 - y^2/a - \psi(a) = 0$,

$x^2 - y^2/a^2 + \psi'(a) = 0$

11. $\sqrt{p} + \sqrt{q} = 2x$. **Ans. C.I.** $z = (2x-a)^3/6 + a^2y + b$; **S.S.** Does not exit; **G.S.**

$z - (2x-a)^3/6 - a^2y - \psi(a) = 0,\;\; (2x-a)^2/2 - 2ay - \psi'(a) = 0$

12. $q(p - \cos x) = \cos y$. **Ans.** $z = ax + \sin x + (1/a) \times \sin y + b$; **S.S.** Does not exist

G.S. $z - ax - (1/a) \times \sin y - \psi(a) = 0,\;\; -x - (1/a^2) \times \sin y + \psi'(a) = 0$

13. $q = xyp^2$ **Ans. C.I.** $2z = 4\sqrt{ax} + ay^2 + b$; **S.S.** Does not exist; **G.S.** $2z - 4\sqrt{ax} - ay^2 - \psi(a) = 0$,

$2\sqrt{(x/a)} + y^2 + \psi'(a) = 0$

14. $x^2p^2 = q^2y$. **Ans. C.I.** $z = \sqrt{a}\log x + 2\sqrt{ay} + b$; **S.S.** Does not exist.

G.S. $z - \sqrt{a}\log x - 2\sqrt{ay} - \psi(a) = 0,\;\; \log x + 2\sqrt{y} + 2\sqrt{a}\,\psi'(a) = 0$

15. $p - q = x^2 + y^2$. **Ans. C.I.** $z = (x^3 - y^3)/3 + a(x+y) + b$; **S.S.** Does not exit;

G.S. $z - (x^3 - y^3)/3 - a(x+y) - \psi(a) = 0,\;\; x + y + \psi'(a) = 0$

16. $p^2 - x = q^2 - y$ **Ans. C.I.** $3z = 2(x+a)^{3/2} + 2(y+a)^{3/2} + b$; **S.S.** Does not exist

G.S. $3z - 2(x+a)^{3/2} - 2(y+a)^{3/2} - \psi(a) = 0, 3(x+a)^{1/2} + 3(y+a)^{1/2} + \psi'(a) = 0$

17. $px + q = p^2$. **Ans. C.I.** $z = (1/4) \times \left\{x^2 + x(x^2+4a)^{1/2}\right\} + a \log \{x + (x^2 + 4a^2)^{1/2}\}\ ay$ $+ b$; **S.S.** Does not exist **G.S.** $z - (1/4) \times \{x^2 + x(x^2+4a)^{1/2}\} - a\log\{x + (x^2+4a)^{1/2}\} - ay$ $-\psi(a) = 0$, $(x/2) \times (x^2 + 4a)^{-1/2} + \log \{x + (x^2 + 4a)^{1/2}\} + (2a)/[\{x + (x^2 + 4a)^{1/2}\} \times (x^2 + 4a)]$ $+ y + \psi'(a) = 0$

3.19. JACOBI'S METHOD [Himachel 2005; Meerut 2005, 06, 08; Pune 2010]

This method is used for solving partial differential equations involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables. We begin with the case of three independent variables. The results arrived at are, however, general and will be used with suitable modification for the case of four independent variables and so on.

Let $p_1 = \partial z/\partial x_1$, $p_2 = \partial z/\partial x_2$ and $p_3 = \partial z/\partial x_3$.

Consider a partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$, ...(1)

where the dependent variable z does not occur except by its partial differential coefficients with respect to the three independent variables x_1, x_2, x_3.

The main idea in Jacobi's method is to get two additional partial differential equations of the first order $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$, ...(3)

where a_1 and a_2 are two arbitrary constants such that (1), (2) and (3) can be solved for p_1, p_2, p_3 in terms of x_1, x_2, x_3 which when substituted in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3, \quad \text{...(4)}$$

makes it integrable, for which the conditions are

$$\partial p_2/\partial x_1 = \partial p_1/\partial x_2, \qquad \partial p_3/\partial x_2 = \partial p_2/\partial x_3, \qquad \text{and} \qquad \partial p_1/\partial x_3 = \partial p_3/\partial x_1 \quad \text{...(5)}$$

Differentiating (1) and (2) partially, w.r.t. x_1, we have

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0 \quad \text{...(6)}$$

and

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0. \quad \text{...(7)}$$

Eliminating $\partial p_1/\partial x_1$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1} = 0. \quad \text{...(8)}$$

Similarly, differentiating (1) and (2) partially w.r.t. x_2 and then eliminating $\partial p_2/\partial x_2$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2} = 0. \quad \text{...(9)}$$

Again, differentiating (1) and (2) partially w.r.t. x_3 and then eliminating $\partial p_3/\partial x_3$ from the resulting equation, we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right) + \left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3} + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3} = 0. \quad \text{...(10)}$$

Adding (8), (9) and (10) and using the relations (5), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0. \quad ...(11)$$

The L.H.S. of (11) is generally denoted by (f, F_1). Then, (11) becomes

$$(f, F_1)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0. \quad ...(11)'$$

Starting with (1) and (3) in place of (1) and (2) and proceeding as above, we have a similar relation

$$(f, F_2)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(12)$$

Again, starting with (2) and (3) in place of (1) and (2) and proceeding as above, we again have a similar relation

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(13)$$

(11) [or (11)′] and (12) are linear equtions of first order with x_1, x_2, x_3, p_1, p_2, p_3 as independent variables and F_1, F_2 as dependent variables respectively. For both of these equations, Lagrange's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}, \quad ...(14)$$

which are known as *Jacobi's auxiliary equations*.

We try to find two independent integrals $F_1(x_1, x_2, x_3, p_1, p_2, p_3)=a_1$ and $F_2(x_1, x_2, x_3, p_1, p_2, p_3)=a_2$ with help of (14). If these relations satisfy (13), these are the required two additional relations (2) and (3).

We now solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Substituting these values in (4) and then integrating the resulting equation, we shall obtain a complete integral of the given equation containing three arbitrary constants of integration.

3.20. Working rules for solving partial differential equations with three or more independent variable. Jacobi's method

Step 1 : Suppose the given equation with three independent variables is

$$f(x_1, x_2, x_3, p_1, p_2, p_3)=0. \quad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}.$$

Solving these equation we obtain two additional equations

$F_1(x_1, x_2, x_3, p_1, p_2, p_3)=a_1$ ...(2) $\qquad$ $F_2(x_1, x_2, x_3, p_1, p_2, p_3)=a_2$. ...(3)

where a_1 and a_2 are arbitrary constants.

While obtaining (2) and (3), try to select simple equations so that later on solutions of (1), (2) and (3) may be as easy as possible.

Step III. Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(4)$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Their substitution in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3$$

and subsequent integration leads to a complete integral of the given equation.

Remark 1. Sometime, change of variables can be employed to reduce the given equation in a form solvable by Jacobian method.

Remark 2. While solving a partial differential equation with four independent variables, we modify the above working rule as follows :

Step I. Suppose the given equation with four independent variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0. \qquad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4} = \frac{dx_4}{-\partial f/\partial p_4}$$

Solving these equations we obtain three additional equations

$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1$, ...(2) $\qquad F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2$, ...(3)

and $\qquad F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3$, ...(4)

where a_1, a_2 and a_3 are arbitrary constants.

Step IV. Verify that relations (2), (3) and (4) satisfy following three conditions:

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0, \ ...(4) \qquad (F_2, F_3) = \sum_{r=1}^{4}\left(\frac{\partial F_2}{\partial x_r}\frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r}\frac{\partial F_3}{\partial x_r}\right) = 0 \ ...(5)$$

and
$$(F_3, F_1) = \sum_{r=1}^{4}\left(\frac{\partial F_3}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right) = 0. \qquad ...(6)$$

If (4), (5) and (6) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4. Their substitution in $\quad dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$ and subsequent integration leads to a complete integral of the given equation.

3.21 SOLVED EXAMPLES BASED ON ART 3.20.

Ex. 1. *Find a complete integral of* $p_1^3 + p_2^2 + p_3 = 1$. **[I.A.S. 1997; Meerut 2006]**

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}, \text{ using (1)}$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $\qquad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$. ...(2)

and $\qquad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

Now,
$$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or
$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

or $\quad (F_1, F_2) = (0)(0) - (1)(0) + (0)(1) - (0)(0) + (0)(0) - (0)(0) = 0$, by (3) and (4).

Thus, we have verified that for relations (2) and (3), $(F_1, F_2) = 0$. Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3.$$

Integrating, $\quad z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2)\, x_3 + a_3,$

which is a complete integral of given equation containing three arbitrary constants a_1, a_2, and a_3.

Ex. 2. *Find a complete integral of* $x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0$. **[Delhi Maths (H) 2006]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{0} = \frac{dx_1}{-(2p_1 x_3^2 p_2^2 p_3^2 + 2p_1 p_2^2)} = \frac{dp_2}{0} = \frac{dx_2}{-(2p_2 x_3^2 p_1^2 p_3^2 + 2p_2 p_1^2)} = \text{.....}, \text{ by (1)}$$

From first and third fractions, $\quad dp_1 = 0$ and $dp_2 = 0 \quad$ so that $\quad p_1 = a_1$ and $\quad p_2 = a_2$.

$\therefore$ Here $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1, \quad$ ...(2)

and $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2. \quad$ ...(3)

As in Ex. 1, verify that for relations (2) and (3), $\quad (F_1, F_2) = 0$.

Hence (2) and (3) may be taken as the additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, we have $p_1 = a_1,\; p_2 = a_2,\; p_3 = \pm a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = a_1 dx_1 + a_2 dx_2 \pm \left\{ a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)} \right\} dx_3, \text{ whose integration gives}$$

$$z = a_1 x_1 + a_2 x_2 \pm \sin^{-1}(a_1 a_2 x_3) + a_3, \; a_1, a_2, a_3 \text{ being arbitrary constants.}$$

Ex. 3. *Find a complete integral of* $p_1 x_1 + p_2 x_2 = p_3^2$. **[Meerut 2007]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 x_1 + p_2 x_2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{2p_3}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first two fractions of (2), $(1/x_1)dx + (1/p_1)dp_1 = 0. \Rightarrow \log x_1 + \log p_1 = \log a_1$.

$\therefore x_1 p_1 = a_1 \quad$ and $\quad$ let $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1 = a_1$. ...(3)

Taking the third and fourth fractions of (2), $\quad (1/x_2)dx_2 + (1/p_2)dp_2 = 0$.

$\therefore x_2 p_2 = a_2 \quad$ and $\quad$ let $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2 p_2 = a_2$. ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0$.

Solving (1), (3) and (4) for p_1, p_2, p_3, $\quad p_1 = a_1/x_1, \quad p_2 = a_2/x_2 \quad$ and $\quad p_3 = (a_1 + a_2)^{1/2}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = (a_1/x_1)dx_1 + (a_2/x_2)dx_2 + (a_1 + a_2)^{1/2}\, dx_3.$$

Integrating, $\quad z = a_1 \log x_1 + a_2 \log x_2 + x_3 (a_1 + a_2)^{1/2} + a_3.$

Ex. 4. *Find complete integral of* $2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0$.

[I.A.S. 1998, Meerut 1999]

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{2p_1x_3}=\frac{dx_1}{-2x_1x_3}=\frac{dp_2}{0}=\frac{dx_2}{-3x_3^2-2p_2p_3}=\frac{dp_3}{2p_1x_1+6p_2x_3}=\frac{dx_3}{-p_2^2}, \text{ by (1)} \quad ...(2)$$

Taking the first two fractions of (2), $(1/p_1)dp_1 + (1/x_1)dx_1 = 0.$ so $p_1x_1 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 = a_1.$ ...(3)

From the third fraction of (2), $dp_2 = 0$ so that $p_2 = a_2.$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2.$ ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0.$

Solving (1), (3) and (4) for p_1, p_2, p_3, $p_1 = a_1/x_1$, $p_2 = a_2$, $p_3 = -(2a_1x_3 + 3a_2x_3^2)/a_2^2.$

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_2$, we have

$dz = (a_1/x_1)dx_1 + a_2dx_2 - \{(2a_1x_3 + 3a_2x_3^2)/a_2^2\}dx_3$, whose integration gives

$z = a_1 \log x_1 + a_2x_2 - (a_1x_3^2 + a_2x_3^3)/a_2^2 + a_3.$ which is required complete integral

Ex. 5. *Find a complete integral of* $p_3x_3(p_1 + p_2) + x_1 + x_2 = 0.$

Sol. Given $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3(p_1 + p_2) + x_1 + x_2 = 0.$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{1}=\frac{dx_1}{p_3x_3}=\frac{dp_2}{1}=\frac{dx_2}{-p_3x_3}=\frac{dp_3}{p_3(p_2+p_3)}=\frac{dx_3}{-x_3(p_1+p_2)}, \text{ by (1)} \quad ...(2)$$

Taking the two fractions of (2), $dp_1 - dp_2 = 0$ so $p_1 - p_2 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - p_2 = a_1.$ ...(3)

Taking the fifth and sixth fractions of (2), $(1/p_3)dp_3 + (1/x_3)dx_3 = 0$ giving $p_3x_3 = a_3$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3 = a_2.$ ...(4)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

$$=\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$= (0)(0) - (1)(0) + (0)(0) - (-1)(0) + (0)(x_3) - (0)(p_3) = 0$ by (3) and (4)

Thus, we have verified that for the relations (3) and (4), $(F_1, F_2) = 0.$

From (1) and (4), $a_2(p_1 + p_2) + x_1 + x_2 = 0$ or $p_1 + p_2 = -(x_1 + x_2)/a_2.$ ...(5)

Solving (3) and (5), $p_1 = \dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}$ and $p_2 = -\dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}.$ ...(6)

Again, from (4), $p_3 = a_2/x_3.$ ...(7)

Putting the values of p_1, p_2, p_3 given by (6) and (7) in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = \frac{a_1}{2}(dx_1 - dx_2) - \frac{(x_1+x_2)}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3.$$

Integrating, $z = (a_1/2) \times (x_1 - x_2) - (1/4a_2) \times (x_1 + x_2)^2 + a_2 \log x_3 + a_3.$

Ex. 6. *Find a complete integral of* $(p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 = 3(x_1 + x_2 + x_3)$.

Sol. Let the given partial differential equation be re–written as

$$f(x_1, x_3, x_3, p_1, p_2, p_3) = (p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 - 3(x_1 + x_2 + x_3) = 0. \quad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}, \text{ giving}$$

$$\frac{dp_1}{2(p_1+x_1)-3} = \frac{dx_1}{-2(p_1+x_1)} = \frac{dp_2}{2(p_2+x_2)-3} = \frac{dx_2}{-2(p_2+x_2)} = \frac{dp_3}{2(p_3+x_3)-3} = \frac{dx_3}{-2(p_3+x_3)}. \quad ...(2)$$

$$\text{Each fraction of (2)} = \frac{dp_1 + dx_1}{-3} = \frac{dp_2 + dx_2}{-3} = \frac{dp_3 + dx_3}{-3} \quad ...(3)$$

Then (3) $\Rightarrow$ $dp_1 + dx_1 = dp_2 + dx_2$ and $dp_3 + dx_3 = dp_2 + dx_2$

Integrating, $p_1 + x_1 = p_2 + x_2 + a_1$ and $p_3 + x_3 = p_2 + x_2 + a_2$,

where a_1 and a_2 are arbitrary constants

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 + p_1 - x_2 - p_2 = a_1$. ...(4)

and $F_2(x_1, x_2, x_2, p_1, p_2, p_3) = x_3 + p_3 - x_2 - p_2 = a_2$. ...(5)

As in Ex. 1, verify that for relations (4) and (5), the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) may be taken as two additional equations.

With help of (4) and (5), (1) reduces to

$$(x_2 + p_2 + a_1)^2 + (x_2 + p_2)^2 + (x_2 + p_2 + a_2)^2 = 3(x_1 + x_2 + x_3)$$

or $$3(p_2 + x_2)^2 + 2(p_2 + x_2)(a_1 + a_2) + a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3) = 0.$$

$$\therefore \quad p_2 + x_2 = \left[-2(a_1 + a_2) \pm \sqrt{[4(a_1+a_2)^2 - 12\{a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3)\}}\right] \Big/ 6$$

$$\Rightarrow \quad p_2 = -x_2 + \left[-(a_1 + a_2) \pm \sqrt{\{9(x_1 + x_2 + x_3) - 2a_1^2 - 2a_2^2 + 2a_1 a_2\}}\right] \Big/ 3$$

For sake of simplification, we take $a_1 = 3c_1$ and $a_2 = 3c_2$. Then, we get

$$p_2 = -x_2 - (c_1 + c_2) \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}. \quad ...(6)$$

$\therefore$ From (4), $p_1 = x_2 + p_2 + 3c_1 - x_1$

$$\Rightarrow \quad p_1 = -x_1 + 2c_1 - c_2 \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{by (6)}$$

Again, from (5), $p_3 = x_2 + p_2 + 3c_2 - x_3$

$$\Rightarrow \quad p_3 = -x_3 + 2c_2 - c_1 \pm \sqrt{\{(x_1 + x_2 + x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{ by (6)}$$

Substituting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = -(x_1 dx + x_2 dx_2 + x_3 dx_3) + [(2c_1 - c_2)dx_1 - (c_1 + c_2)dx_2 + (2c_2 - c_1)dx_3]$$
$$\pm (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{1/2} \; (dx_1 + dx_2 + dx_3).$$

Integrating, $$z = -(1/2) \times (x_1^2 + x_2^2 + x_3^2) + (2c_1 - c_2)x_1 - (c_1 + c_2)x_2 + (2c_2 - c_1)x_3$$
$$\pm (2/3) \times (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{3/2} + c_3,$$

which is a complete integral containing c_1, c_2, c_3 as arbitrary constants.

Ex. 7. *Find a complete integral of* $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. **[Delhi B.Sc. (Hons) III 2011]**

Sol. Given $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1), we have

$$(x_2 + x_3)\left(\frac{1}{z}\frac{\partial z}{\partial x_2} + \frac{1}{z}\frac{\partial z}{\partial x_3}\right)^2 + \frac{1}{z}\frac{\partial z}{\partial x_1} = 0. \qquad \text{...(2)}$$

Let $(1/z)dz = dZ$ so that $Z = \log z$. ...(3)

Then, (2) $\Rightarrow$ $(x_2 + x_3)\,(\partial Z/\partial x_2 + \partial Z/\partial x_3)^2 + \partial Z/\partial x_1 = 0.$ ...(4)

Let $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$, $P_3 = \partial Z/\partial x_3$. Then (4) becomes

$$(x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$$

So here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv (x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$ ...(5)

Jacobi's auxiliary equations take the form

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dx_1}{-1} = \frac{dP_1}{0} = \frac{dx_2}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_2}{(P_2+P_3)^2} = \frac{dx_3}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_3}{(P_2+P_3)^2}. \qquad \text{...(6)}$$

Taking second ratio of (6), we have $dP_1 = 0$ $\Rightarrow$ $P_1 = -a_1$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 = -a_1$. ...(7)

Taking the fourth and sixth ratios in (6), we get $dP_2 = dP_3$ $\Rightarrow$ $P_2 - P_3 = a_2$.

Let $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - P_3 = a_2$. ...(8)

Using (7), (5) $\Rightarrow P_2 + P_3 = \pm\ \{a_1/(x_2 + x_3)\}^{1/2}$. ...(9)

Solving (8) and (9) for P_2 and P_3, we have

$$P_2 = \frac{1}{2}\left[a_2 \pm \left(\frac{a_1}{x_2+x_3}\right)^{1/2}\right] \quad \text{and} \quad P_3 = \frac{1}{2}\left[\pm\left(\frac{a_1}{x_2+x_3}\right)^{1/2} - a_2\right]. \qquad \text{...(10)}$$

Using (7) and (10), $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$ becomes

$$dZ = -a_1dx_1 + \frac{1}{2}\left[a_2 \pm \frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}}\right]dx_2 + \frac{1}{2}\left[\pm\frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}} - a_2\right]dx_3$$

or
$$dZ = -\,a_1dx_1 + (1/2) \times a_2dx_2 - (1/2) \times a_2dx_3 \pm (1/2)\times\sqrt{a_1}\,(x_2 + x_3)^{-1/2}(dx_2 + dx_3).$$

Integrating and noting that $dZ = (1/z)dz$, complete integral is given by

$$\log z = -\,a_1x_1 + (a_2/2) \times (x_2 - x_3) \pm \sqrt{a_1}\,(x_2 + x_3)^{1/2} + a_3.$$

Ex. 8. *Find a complete integral of* $p_1p_2p_3 = z^3x_1x_2x_3$. **[Meerut 1998]**

i.e., $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. **[Delhi Maths (H) 2000, 10; I.A.S. 1995]**

Sol. Given $p_1p_2p_3 = z^3x_1x_2x_3$ or $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3$. ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1) we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1x_2x_3. \qquad \text{...(2)}$$

Let $(1/z)dz = dZ$ so that $\log z = Z$. Then (2) becomes

$(\partial Z/\partial x_1)(\partial Z/\partial x_2)\,(\partial Z/\partial x_3) = x_1x_2x_3$ or $P_1P_2P_3 = x_1x_2x_3$.

$\therefore$ Here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1P_2P_3 - x_1x_2x_3 = 0.$ ...(3)

$\therefore$ Jacobi's auxilliary equations are

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-P_2P_3} = \frac{dP_2}{-x_1x_3}=\frac{dx_2}{-P_1P_3} = \frac{dP_3}{-x_1x_2}=\frac{dx_3}{-P_1P_2}, \text{ by (3)}$$

Since from (3), $P_2P_3 = (x_1\ x_2\ x_3)/P_1$, hence first and second fractions give

$$\frac{dP_1}{-x_2x_3} = \frac{dx_1}{-(x_1x_2x_3/P_1)} \qquad \text{or} \qquad \frac{dP_1}{P_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log P_1 = \log x_1 + \log a_1$ or $P_1 = a_1x_1$.

Thus, here we have $F_1(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1 - a_1x_1 = 0.$ ...(4)

Similarly, $F_2(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_2 - a_2x_2 = 0.$ ...(5)

As in Ex. 1, verify that for (4) and (5) the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) can be taken as two additional equations. Solving (3), (4) and (5) for P_1, P_2, P_3, we have

$P_1 = a_1x_1$, $P_2 = a_2x_2$ and $P_3 = x_3/(a_1a_2)$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = a_1x_1dx + a_2x_2dx_2 + \{x_3/(a_1a_2)\}dx_3.$$

Integrating, $Z = (1/2) \times a_1x_1^2 + (1/2) \times a_2x_2^2 + \{1/(2a_1a_2)\}x_3^2 + a_3/2$

or $2 \log z = a_1x_1^2 + a_2x_2^2 + \{1/(a_1a_2)\}x_3^2 + a_3$, as $Z = \log z$

Ex. 9. *Find a complete integral of* $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ **[Delhi Maths (H) 2009]**

Sol. Given equation is $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Dividing each term by z^2, (1) can be re–written as

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right)-\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right) - \left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = 0. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(3)

Using (3), (2) becomes $P_1^2 + P_2P_3 - P_2 - P_3 = 0,$ ...(4)

Let us write $f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1^2 + P_2P_3 - P_2 - P_3 = 0.$ ...(5)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dP_1}{0}=\frac{dx_1}{-2P_1} = \frac{dP_2}{0}=\frac{dx_2}{-P_3+1} = \frac{dP_3}{0}=\frac{dx_3}{-P_2+1}, \text{ by (5)}$$

Taking the third and fifth fractions, $dP_2 = 0$ and $dP_3 = 0$ so that $P_2 = a_1$ and $P_3 = a_2$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 = a_1.$ ...(6)

and $F_3(x_1, x_2, x_3, P_1, P_2, P_3) = P_3 = a_2.$ ...(7)

As in Ex. 1, verify that for (6) and (7), the condition $(F_1, F_2) = 0$ is satisfied. Hence (6) and (7) can be taken as two additional equations. Solving (4), (6) and (7) for P_1, P_2, P_3, we have

$P_2 = a_1$, $P_3 = a_2$, $P_1 = (a_1 + a_2 - a_1a_2)^{1/2}$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1 + a_2 - a_1a_2)^{1/2}dx_1 + a_1dx_2 + a_2dx_3.$$

Integrating, $Z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3$. Then, the complete integral is

$$\log z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3, \text{ using (3).}$$

Ex. 10. *Find a complete integral of* $2x_1x_3zp_1p_3 + x_2p_2 = 0.$

Sol. Given equation is $2x_1x_3zp_1p_3 + x_2p_2 = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Multiplying each term by z, (1) can be re–written as

$$2x_1x_3\left(z\frac{\partial z}{\partial x_1}\right)\left(z\frac{\partial z}{\partial x_3}\right)+x_2\left(z\frac{\partial z}{\partial x_2}\right) = 0. \qquad ...(2)$$

Let $\quad zdz = dZ \quad$ so that $\quad z^2/2 = Z. \qquad ...(3)$

Using (3), (2) becomes $\quad 2x_1x_3P_1P_3 + x_2P_2 = 0, \qquad ...(4)$

where $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$ and $P_3 = \partial Z/\partial x_3$. We re–write (4) as

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 2x_1x_3P_1P_3 + x_2P_2 = 0. \qquad ...(5)$$

$\therefore$ Jacobi's auxilairy equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dP_1}{2x_3P_1P_3}=\frac{dx_1}{-2x_1x_3P_3} = \frac{dP_2}{P_2}=\frac{dx_2}{-x_2} = \frac{dP_3}{2x_1P_1P_3}=\frac{dx_3}{-2x_1x_3P_2}, \text{ by (5)} \qquad ...(6)$$

Taking the first and second fractions of (6) and simplifying, we get

$(1/P_1)dP_1 + (1/x_1)dx_1 = 0 \quad$ so that $\quad \log P_1 + \log x_1 = \log a_1 \quad$ or $\quad P_1x_1 = a_1$

So here $\quad F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1x_1 = a_1. \qquad ...(7)$

Taking the fifth and sixth fractions of (6) and simplifying, we get

$(1/P_3)dP_3 + (1/x_3)dx_3 = 0 \quad$ so that $\quad \log P_3 + \log x_3 = \log a_3 \quad$ or $\quad P_3x_3 = a_2$

So here $\quad F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_3x_3 = a_2. \qquad ...(8)$

As in Ex. 1, verify that for (7) and (8), the condition $(F_1, F_2) = 0$ is satisfied. Hence (7) and (8) can be taken as additional equations. Solving (5), (7) and (8) for P_1, P_2, P_3, we have

$$P_1 = a_1/x, \qquad P_3 = a_2/x_3, \qquad P_2 = -(2a, a_2)/x_2.$$

Putting these values in $dZ = P_1dx + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1/x_1)dx_1 - \{(2a_1a_2)/x_2\}dx_2 + (a_2/x_3)dx_3.$$

Integrating, $\quad Z = a_1\log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$

or $\quad z^2/2 = a_1 \log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$, by (3).

Ex. 11. *Find a complete integral of* $p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0$.

Sol. [In the present problem we have four independent variables in places of three. According we shall use modified working as explained in remark 2 of Art 3.20]

The given equation can be written as

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4}=\frac{dx_4}{-\partial f/\partial p_4}, \text{ giving}$$

$$\frac{dp_1}{p_4^3x_2x_3x_4^3}=\frac{dx_1}{-p_2p_3} = \frac{dp_2}{p_4^3x_1x_3x_4^3}=\frac{dx_2}{-p_1p_3} = \frac{dp_3}{p_4^3x_1x_2x_4^3}=\frac{dx_3}{-p_1p_2} = \frac{dp_4}{3p_4^3x_1x_2x_3x_4^2}=\frac{dx_4}{-3p_4^3x_1x_2x_3x_4^3}$$

Since from (1), $p_4^3x_2x_3x_4^3 = -p_1p_2p_3/x_1$, the first two fractions give

$$\frac{dp_1}{-(p_1p_2p_3/x_1)} = \frac{dx_1}{-p_2p_3} \qquad \text{or} \qquad \frac{dp_1}{p_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log p_1 = \log x_1 + \log a_1$ or $p_1 = a_1x_1$.

Let $F_1(x_1, x_2, x_3, x_4\, p_1, p_2, p_3, p_4) = p_1 - a_1x_1 = 0.$ ...(2)

Similarly, $F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_2 - a_2x_2 = 0$ ...(3)

and $F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_3 - a_3x_3 = 0.$ ...(4)

With these values of F_1, F_2 and F_3, we can verify that

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0.$$

Similarly, we see that $(F_2, F_3) = 0$ and $(F_3, F_1) = 0$. Hence (2), (3) and (4) can be taken as the three desired additional equations. Now solving (1), (2) (3) and (4) for p_1, p_2, p_3 and p_4, we get

$$p_1 = a_1x_1, \qquad p_2 = a_2x_2, \qquad p_3 = a_3x_3 \qquad \text{and} \qquad p_4 = (a_1a_2a_3)^{1/3}/x_4.$$

Putting these in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$ and integrating the desired complete integral is $z = (1/2) \times (a_1x_1^{\,2} + a_2x_2^{\,2} + a_3x_3^{\,2}) - (a_1a_2a_3)^{1/2} \log x_4 + a_4/2$

or $2z = a_1x_1^{\,2} + a_2x_2^{\,2} + a_3x_3^{\,2} - 2(a_1a_2a_3)^{1/2} \log x_4 + a_4,$

Ex. 12. *Find a complete integral by Jacobi's method of the equation* $2x^2y(\partial u/\partial x)^2\,(\partial u/\partial z)$ $= x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Delhi Maths (H) 2001]**

Sol. Let $x = x_1$, $y = x_2$, $z = x_3$, $\partial u/\partial x = p_1$, $\partial u/\partial y = p_2$, and $\partial u/\partial z = p_3$

Then given equation becomes $2x_1^2x_2p_1^2p_3 = x_1^2p_2 + 2x_2p_1^2$

Dividing by $x_1^2\ x_2$, $2p_1^2p_3 = (p_2/x_2) + (2p_1^2/x_1^2)$, which can be written as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1^2(p_3 - 1/x_1^2) - p_2/x_2 = 0 \qquad \text{... (1)}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{4p_1^2/x_1^3} = \frac{dx_1}{-4p_1(p_3 - 1/x_1^2)} = \frac{dp_2}{p_2/x_2^2} = \frac{dx_2}{1/x_2} = \frac{dp_3}{0} = \frac{dx_3}{-2p_1^2}, \text{ by (1)}$$

Taking the fifth fraction, $dp_3 = 0$ so that $p_3 = a_1$

Taking the second and fourth fractions, $(1/p_2)\ dp_2 = (1/x_2)\ dx_2$

Integrating, $\log p_2 = \log x_2 + \log(2a_2^2)$ or $p_2/x_2 = 2a_2^2$

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_3 = a_1$ ... (2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2/x_2 = 2a_2^2$ ... (3)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or $$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= (0)\ (0) - (0)\ (0) + (0)\ \ (0) - (0)\ \ (0) + (0)\ \ (0) - (1)\ (0) = 0$$

Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_2x_1/(a_1x_1^2-1)^{1/2}$, $\quad p_2 = 2a_2^2x_2$, $\quad p_3 = a_1$

Putting these in $du = p_1dx_1 + p_2dx_2 + p_3dx_3 = a_2x_1(a_1x_1^2-1)^{-1/2}dx_1 + 2a_2^2x_2dx_2 + a_1dx_3$.

Integrating, $$u = (a_2/a_1)\times(a_1x_1^2-1)^{1/2} + a_2^2x_2^2 + a_1x_3 + a_3,$$
which is the complete integral with a_1, a_2, a_3 as arbitrary constants.

Ex. 13. *Show that a complete integral of the equation* $f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z) = 0$ *is* $u = ax + by + \theta(a, b)z + c$, *where a, b and c are arbitrary constants and* $f(a, b, \theta) = 0$

(*b*)*Find a complete integral of the equation* $\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = (\partial u/\partial x)(\partial u/\partial y)(\partial u/\partial z)$. **[Allahabad 2004, 06; Meerut 2004, 06; Purvanchal 2003]**

Sol. (*a*) Let $\quad \partial u/\partial x = p_1, \quad \partial u/\partial y = p_2 \quad$ and $\quad \partial u/\partial z = p_3$.

Then gievn equation becomes $$f(p_1, p_2, p_3) = 0 \qquad \text{... (1)}$$

We shall now proceed as in Ex. 1, Art. 3.21. Here Jacobi's auxiliary equations are given by
$$\frac{dp_1}{\partial f/\partial x} = \frac{dx}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial y} = \frac{dy}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial z} = \frac{dz}{-\partial f/\partial p_3}$$
$$\Rightarrow \quad \frac{dp_1}{0} = \frac{dp_2}{0}, \text{ using (1)} \quad \Rightarrow \quad dp_1 = 0 \quad \text{and} \quad dp_2 = 0$$

Integrating, $\quad p_1 = a, \quad p_2 = b, \quad$ *a* and *b* being arbitrary constants ... (2)

Putting $p_1 = a$ and $p_2 = b$ in (1), $f(a, b, p_3) = 0$ so that
$$p_3 = \text{a function of } a, b = \theta(a, b), \text{ say} \qquad \text{... (3)}$$

Now, we have $$du = (\partial u/\partial x)\,dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = p_1dx + p_2dy + p_3dz$$
or $$du = a\,dx + b\,dy + \theta(a, b)\,dz, \text{ by (2) and (3)}$$

Integrating, $$u = ax + by + \theta\,(a,b)\,z + c, \qquad \text{... (4)}$$
where c is an arbitrary constant and a, b, θ are connected by relation
$$f(a, b, \theta\,(a, b)) = 0, \text{ by (1), (2) and (3)} \qquad \text{... (5)}$$

(***b***) Given $$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z - (\partial u/\partial x)(\partial u/\partial y)(\partial u/\partial z) = 0 \qquad \text{... }(i)$$

Let $\quad p_1 = \partial u/\partial x, \quad p_2 = \partial u/\partial y \quad$ and $\quad p_3 = \partial u/\partial z$. Then, (*i*) gives
$$p_1 + p_2 + p_3 - p_1\,p_2\,p_3 = 0 \qquad \text{... }(ii)$$

Comparing (*ii*) with (1) of part (*a*), here
$$f(p_1, p_2, p_3) = p_1 + p_2 + p_3 - p_1\,p_2\,p_3 \qquad \text{... }(iii)$$

Hence required complete integral is given by (4) and (5) of part (*a*) *i.e.*,
$$u = a\,x + b\,y + \theta\,(a,b)\,z + c, \qquad \text{... }(iv)$$
where $$a + b + \theta\,(a,b) - ab\,\theta(a, b) = 0 \qquad \text{... }(v)$$

From (*v*), $$\theta(a, b) = (a+b)/(ab-1) \qquad \text{... }(vi)$$

From (*iv*) and (*vi*), $\quad u = ax + by + \{(a + b)/(ab - 1)\} + c,$
which is the required complete integral of (*i*), *a*, *b*, *c* being arbitrary constants.

EXERCISE 3(G)

Find the complete integral of the following equation: (1 – 5)

1. $f(p_1, p_2, p_3) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $f(a_1, a_2, a_3) = 0$

2. $p_1 + p_2 + p_3 - p_1p_2p_3 = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $a_1 + a_2 + a_3 - a_1a_2a_3 = 0$

3. $p_1x_1^2 - p_2^2 - ap_3^2 = 0$ **Ans.** $z = -(a_1^2 + a\,a_2^2)x_1^{-1} + a_1x_2 + a_2a_3x_3 + a_3$

4. $x_3(x_3 + p_3) = p_1^2 + p_2^2$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1^2 + a_2^2)\log x_3 - x_3^2/2 + a_3$

5. $x_3 + 2p_3 - (p_1 + p_3^2) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1 + a_2)^2 \times (x_3/2) - (x_3^2/4) + a_3$

6. $x_1 + p_1^2 + x_2 + p_2^2 - x_3p_3^2 = 0$ **Ans.** $z = 2\,(a_1x_1)^{1/2} + 2\,(a_2x_2)^{1/2} + 2\,\{(a_1 + a_2)\,x_3\}^{1/2} + a_3$

7. *Show how to solve, by Jacobi method, a partial differential equation of the type* $f(x, \partial u/\partial x, \partial u/\partial z) = g(y, \partial u/\partial y, \partial u/\partial z)$ *and illustrate the method by finding a complete integral of equation* $2x^2y(\partial u/\partial x)^2(\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Meerut 2005]**

Sol. Try yourself **Ans.** $u = (ax^2 - b)^{1/2} + ay^2 + (z/b) + c$

8. *Prove that an equation of the "Clairaut" form* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z) = f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z)$ *is always solvable by Jacobi's method. Hence solve*

$$(\partial u/\partial x + \partial u/\partial y + \partial u/\partial z)\,\{x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)\} = 1$$

3.22. Jacobi's method for solving a non-linear first order partial differential equation in two independent variables.

[Delhi Maths (H) 1997; Amaravati 2001; Himanchal 2003, 05]

Let $$F(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

be the non-linear first order equation in two independent variables x, y.

Then we know that a solution of (1) is of the form $$u(x, y, z) = 0 \qquad \text{... (2)}$$

showing that u can be treated as a dependent variable and x, y, z as three independent variables.

Differentiating (2) partially w.r.t. 'x' and 'y', respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$p_1 + p_3 p = 0 \quad \text{and} \quad p_2 + p_3\ q = 0 \qquad \text{... (3)}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $p_1 = \partial u/\partial x = \partial u/\partial x_1$, $p_2 = \partial u/\partial y = \partial u/\partial x_2$, $p_3 = \partial u/\partial z = \partial u/\partial x_3$

by taking $$x = x_1, \quad y = y_2 \quad \text{and} \quad z = x_3 \qquad \text{... (4)}$$

From (3), $$p = -(p_1/p_3) \quad \text{and} \quad q = -(p_2/p_3) \qquad \text{... (5)}$$

Using (4) and (5), (1) reduces to $$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad \text{... (6)}$$

We now solve (6) by Jacobi's method as usual (refer Art. 3.20) to get the complete integral of (6). Finally, putting $x_1 = x$, $x_2 = y$, $x_3 = z$, we obtain solution of (6) containing original variables x, y, z and new dependent variable u. The solution so obtained will contain three arbitrary constants a_1, a_2, a_3 (say). However, for the given equation in the form (1), we need only two arbitrary constants in the final solution. The required solution $u = 0$ of (1) is obtained by making different choices of our third arbitrary constant.

Ex. 1. *Solve* $p^2x+q^2y=z$ *by Jacobi's method.* **[Nagpur 2002; Himanchal 2003, 05]**

Sol. Given $p^2x+q^2y=z$.

Let a solution of (1) be of the form $u(x,y,z)=0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variables, differentiation of (2) partially w.r.t 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0 \quad i.e. \quad p_1+p_3p=0 \quad \text{and} \quad p_2+p_3\,q=0$$

so that $p=-p_1/p_3$ and $q=-p_2/p_3$ .... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1$, $p_2=\partial u/\partial y=\partial u/\partial x_2$, $p_3=\partial u/\partial z=\partial u/\partial x_3$, $p=\partial z/\partial x, q=\partial z/\partial y$

by taking $x=x_1$, $y=x_2$ and $z=x_3$ ... (4)

Using (3) and (4), (1) $\Rightarrow$ $x_1(p_1/p_3)^2+x_2(p_2/p_3)^2=x_3$. $\Rightarrow$ $x_1p_1^2+x_2p_2^2-x_3p_3^2=0$

Let $f(x_1, x_2, x_3, p_1, p_2, p_3)=x_1p_1^2+x_2p_2^2-x_3\ p_3^2=0$ ... (5)

Now, the Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{p_1^2}=\frac{dx_1}{-2p_1x_1}=\frac{dp_2}{p_2^2}=\frac{dx_2}{-2p_2x_2}=\frac{dp_3}{-p_3^2}=\frac{dx_3}{2p_3x_3}, \text{ by (5)}$$

Taking the first two fractions, $(2/p_1)\ dp_1+(1/x)\,dx=0$.

Integrating, $2\log p_1+\log x_1=\log a_1$ so that $x_1p_1^2 = a_1$ or $p_1=(a_1/x_1)^{1/2}$

Similarly, the third and fourth fractions give $p_2=(a_2/x_2)^{1/2}$

Substituting these values of p_1 and p_2 in (5), we get $p_3=\{(a_1+a_2)/x_3\}^{1/2}$.

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1^{1/2}\ x_1^{-1/2}dx_1+a_2x_2^{-1/2}dx_2+(a_1+a_2)^{1/2}x_3^{-(1/2)}dx_3.$$

Integrating, $u=2(a_1x_1)^{1/2}+(a_2x_2)^{1/2}+2(a_1+a_2)^{1/2}x_3^{1/2}+a_3$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$2(a_1x)^{1/2}+2y^{1/2}+2(a_1+1)^{1/2}z^{1/2}+a_3=0,$$

which is the complete integral containing two arbitrary constants a_1 and a_3.

Ex. 2. *Solve* $p^2+q^2=k^2$ by Jacobi's method **[Delhi B.A./B.Sc. (Prog) Maths 2007]**

Sol. Given $p^2+q^2=k^2$ ... (1)

Let a solution of (1) be of the form $u(x, y, z)=0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variable, differentiation of (2) partially w.r.t. 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0 \quad i.e., \quad p_1 + p_3 p = 0 \quad \text{and} \quad p_2 + p_3 q = 0$$

so that $\qquad p = -(p_1/p_3) \qquad$ and $\qquad q = -(p_2/p_3) \qquad$ ... (3)

where $p_1 = \partial u/\partial x = \partial u/\partial x_1$, $p_2 = \partial u/\partial y = \partial u/\partial x_2$, $p_3 = \partial u/\partial z = \partial u/\partial x_3$, $p = \partial z/\partial x$, $q = \partial z/\partial y$

by taking $\qquad x = x_1, \qquad y = x_2 \qquad$ and $\qquad z = x_3 \qquad$ ... (4)

Using (3) and (4), (1) reduces to $\quad p_1^2/p_3^2 + p_2^2/p_3^2 = k^2 \qquad$ or $\qquad p_1^2 + p_2^2 = k^2 p_3^2$

Let $\qquad f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^2 + p_2^2 - k^2 p_3^2 = 0 \qquad$ ... (5)

Now, the Jacobi auxilliary equations are given by

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or

$$\frac{dp_1}{0} = \frac{dx_1}{-2p_1} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{2k^2 p_3}, \text{ using (5)}$$

From the first and third fractions of (5), $\qquad dp_1 = 0 \qquad$ and $\qquad dp_2 = 0$

Integrating, $\quad p_1 = a_1$ and $p_2 = a_2$, a_1 and a_2 being arbitrary constants

With $\qquad p_1 = a_1 \qquad$ and $\qquad p_2 = a_2, \qquad$ (5) gives $\ p_3 = (a_1^2 + a_2^2)^{1/2}/k$

Putting the above values of p_1, p_2 and p_3 in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$du = a_1 dx_1 + a_2 dx_2 + \{(a_1^2 + a_2^2)^{1/2}/k\} dx_3$$

Integrating, $\qquad u = a_1 x + a_2 x_2 + \{(a_1^2 + a_2^2)^{1/2}/k\} x_3 + a_3 \qquad$ ... (6)

Taking $a_2 = 1$ and using (4), the required solution $u = 0$ is given by

$$a_1 x + x_2 + \{(a_1^2 + 1)^{1/2}/k\} x_3 + a_3 = 0,$$

which is the complete integral of (1) containing two arbitrary constants a_1 and a_3.

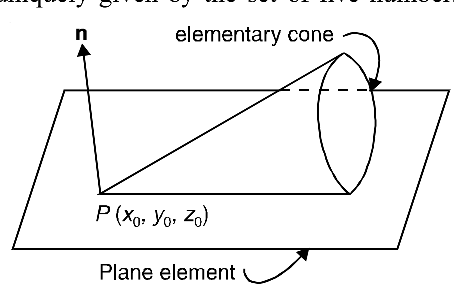
Ex. 3. *Solve the following partial differential equations by Jacobi's method:*

(*i*) $p = (z + qy)^2 \qquad$ (*ii*) $(p^2 + q^2)x = pz \qquad$ (*iii*) $xpq + yq^2 = 1 \qquad$ **[Nagpur 2005]**

Hint. Proceed as in the above solved Ex. 1

3.23 Cauchy's method of characteristics for solving non-linear partial differential equation $\quad f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0 \quad i.e., \quad f(x, y, z, p, q) = 0 \quad$ ... (1)

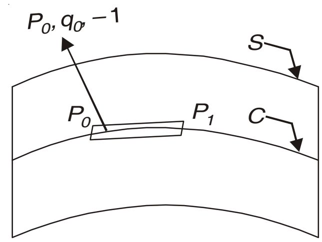
We know that the plane passing through the point $P(x_0, y_0 z_0)$ with its normal parallel to the direction **n** whose direction ratios are $p_0, q_0, -1$ is uniquely given by the set of five numbers $D(x_0, y_0, z_0, p_0, q_0)$ and conversely any such set of five numbers defines a plane in three dimensional space. In view of this fact a set of five numbers $D(x, y, z, p, q)$ is known as a *plane element* of a three dimensional space. As a special case a plane element $(x_0, y_0, z_0, p_0, q_0)$ whose components satisfy (1) is known as an *integral element* of (1) at *P*. Solving (1) for q, suppose we get



$$q = F(x, y, z, p).$$

which gives a value of q corresponding to known values of x, y, z and p. Then, keeping x_0, y_0 and z_0 fixed and varying p, we shall arrive at a set of plane elements $\{x_0, y_0, z_0, p, \; G(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies, we get a set of plane elements all of which pass through the point P. Hence the above mentioned set of plane elements envelop a cone with vertex P. The cone thus obtained is known as the *elementary cone of* (1) at the point P.

Consider a surface S with equation $\qquad z = g(x, y) \qquad$ ... (2)



If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous in a certain region R of the xy-plane, then the tangent plane at each point of S determines a plane element of the form $\{x_0, y_0, g(x_0, y_0), \; g_x(x_0, y_0), \; g_y(x_0, y_0)\}$ which will be referred as *the tangent element* of the surface S at the point $\{x_0, y_0, g(x_0, y_0)]$.

Consider a curve C with parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad \text{t being the parameter.} \qquad \text{... (3)}$$

Then curve C lies on (2) provided $\qquad z(t) = g\{x(t), y(t)\} \qquad$ ... (4)

holds good for all values of t in the appropriate interval I. Let P_0 be a point on curve C corresponding to $t = t_0$. Now, the direction ratios of the tangent line $P_0 \; P_1$ are $x'(t_0), \; y'(t_0), \; z'(t_0)$ where $x'(t_0), \; y'(t_0) \; z'(t_0)$ denote the values of $dx/dt, \; dy/dt, \; dz/dt$ respectively at $t = t_0$

This direction will be perpendicular to direction of normal $\boldsymbol{n}$ (with direction ratios $p_0, q_0, -1$)

if $\qquad p_0 x'(t_0) + q_0 \; y'(t_0) + (-1) \; z'(t_0) = 0 \qquad$ or $\qquad z'(t_0) = p_0 \; x'(t_0) + q_0 \; y'(t_0)$

It follows that any set $\qquad \{x(t), \; y(t), z(t), p(t), \; q(t)\} \qquad$ ...(5)

of five real functions satisfying the condition that $\qquad z'(t) = p(t) \; x'(t) + q(t) \; y'(t) \qquad$ ... (6)

defines a strip at the point (x, y, z) of the curve C. When such a strip is also an integral element of (1), then the strip under consideration is known as an *integral strip* of (1). In other words, the set of functions (5) is known as an integral strip of (1) provided these satisty (6) and the following additional condition

$$f\{x(t), \; y(t), \; z(t), \; p(t), \; q(t)\} \; = 0, \quad \text{for all } t \text{ in I.}$$

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a *characteristic strip*.

Derivation of the equations determining a *characteristic strip*

Clearly, the point $(x + dx, \; y + dy, \; z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = p\,dx + q \; dy \qquad \text{... (7)}$$

where p, q satisfy (1). Differentiation (7) w.r.t. 'p', we get

$$0 = dx + (dq/dp) \; dy \qquad \text{... (8)}$$

Again, differentiating (1) partially w.r.t. 'p', we have

$$\partial f/\partial p + (\partial f/\partial q) \; (dq/dp) = 0 \qquad \textit{i.e.,} \qquad f_p + f_q (dq/dp) = 0 \qquad \text{... (9)}$$

Here, $\qquad \partial f/\partial p = f_p \qquad$ and $\qquad \partial f/\partial q = f_q$

Solving (7), (8) and (9) for the ratios of dy, dz to dx, we get

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{p\,f_p+q\,f_q}. \qquad \text{... (10)}$$

Hence along a characteristic strip $x'(t),\ y'(t),\ z'(t)$ will be proportional to $f_p,\ f_q,\ p\,f_p+q\,f_q$ respectively. If the parameter t be selected satisfying the relations

$$x'(t)=f_p \qquad \text{and} \qquad y'(t)=f_q.$$

then, we have $$z'(t)=p\,f_p+q\,f_q$$

Since along a characteristic strip p is a function of t, hence

$p'(t)=(\partial p/\partial x)\ (dx/dt)+(\partial p/\partial y)\ (dy/dt)=(\partial p/\partial x)\ \ (\partial f/\partial p)+(\partial p/\partial y)\ (\partial f/\partial q)$, using (11)

Thus, $$p'(t)=(\partial p/\partial x)\ (\partial f/\partial p)+(\partial q/\partial x)\ (\partial f/\partial q) \qquad \text{... (12)}$$

$$\left[\because\ \frac{\partial p}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial q}{\partial x}\right]$$

Now, differentiating (1) partially w.r.t. 'x', gives $$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0$$

or $$f_x+p\ f_z+p'(t)=0, \text{ using (12)}$$

Hence on a characteristic strip, $$p'(t)=-f_x-p\,f_z \qquad \text{... (13)}$$

Similarly, we have $$q'(t)=-f_y-q\,f_z \qquad \text{... (14)}$$

Here $f_x=\partial f/\partial x,\qquad f_y=\partial f/\partial y,\qquad f_z=\partial f/\partial z$

From (11), (13) and (14), we get the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t)=f_p,\quad y'(t)=f_q,\quad z'(t)=p\ f_p+q\,f_q,\quad p'(t)=-f_x-p\ f_z \quad \text{and} \quad q'(t)=-f_y-q\,f_z \qquad \text{...(15)}$$

The above equations are called the *characteristic equations* of (1). In view of a well known result if the functions which are involved in (15) satisfy a Lipschitz condition, there exists a unique solution of (15) for given set of initial values of the variables. It follows that the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_o of t.

Working rule for solving Cauchy's problem. [Meerut 2005]

Suppose we wish to find the integral surface of (1) which passes through a given curve with parametric equation $x=f_1(\lambda),\ y=f_2(\lambda),\quad z=f_3(\lambda),\quad \lambda$ being the parameter ... (16)

then in the solution $$x=x\ (p_0, q_0, x_0, y_0, t_0, t) \text{ etc.} \qquad \text{... (17)}$$

of the characteristic equations (15), we shall assume that

$$x_0=f_1(\lambda), \qquad y_0=f_2(\lambda), \qquad z_0=f_3(\lambda)$$

are the initial values of x, y, z respectively. Then the corresponding initial values of p_0, q_0 can be obtained by the following relations

$$f_3'(\lambda)=p_0\ f_1'(\lambda)+q_0\ f_3'(\lambda\ ) \qquad \text{and} \qquad f\{f_1(\lambda),\ f_2(\lambda),\ f_3(\lambda),\ p_0,\ q_0\}=0$$

When the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted in (17), we shall be able to express x, y, z involving the two parameters t and λ of the form

$$x = \phi_1\ (t, \lambda), \qquad y = \phi_2\ (t, \lambda) \qquad \text{and} \qquad z = \phi_3(t, \lambda) \qquad \dots (18)$$

which are known as characteristics of (1)

Finally, by eliminating λ and t from (18), we arrive at a relation of the form $G\ (x,y,z) = 0$, which is the required equation of the integral surface of (1) passing through the given curve (16).

3.24 Some Theorems:

Theorems 1. *A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.*

Proof. Using geometrical considerations and Art 3.23, complete the proof yourself.

Theorem II. *Along every characteristic strip of the partial differential equation $f(x, y, z, p, q) = 0$ the function $f(x, y, z, p, q)$ is a constant.*

Proof. Along a characteristic strip, we have

$$\frac{d}{dt} f\{x(t),\ y(t),\ z(t),\ p(t), q(t)\} = f_x\ x'(t) + f_y\ y'(t) + f_z\ z'(t) + f_p\ p'(t) + f_q\ q'(t)$$

$$= f_x f_p + f_y f_q + f_z\ (p\ f_p + q\ f_q) - f_p(f_x + p\ f_z) - f_q(f_y + q\ f_z)$$

$= 0$, using the characteristic equation (15) of Art. 3.23

showing that $f(x, y, z, p, q) = K$, a constant along the strip.

Corollary to theorem II. *If a characteristic strip contains at least one integral element of $f(x,y,z,p,q) = 0$ it is an integral strip of the equation $f(x, y, z,\ \partial z/\partial x,\ \partial z/\partial y) = 0$*

Proof. Left as an exercise.

3.25 SOLVED EXAMPLES BASED ON ART. 3.23

Ex. 1. *Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0$, $y^2 = z$.* **[Meerut 2005; I.A.S. 1999]**

Sol. Given equation is $$pq = z \qquad \dots (1)$$

We are to find its integral surface which passes through the given parabola given by

$$x = 0, \qquad \text{and} \qquad y^2 = z \qquad \dots (2)$$

Re-writing (2) in parametric form, we have

$$x = 0, \qquad y = \lambda, \qquad z = \lambda^2, \qquad \lambda \text{ being a parameter} \qquad \dots (3)$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = 0, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = \lambda^2 \qquad \dots (4A)$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$$p_0 q_0 = z_0, \qquad \text{or} \qquad p_0\ q_0 = \lambda^2, \text{ by (4A)} \qquad \dots (5)$$

Also, we have $$z_0'(\lambda) = p_0\ x_0'(\lambda) + q_0\ y_0'(\lambda)$$

so that $$2\lambda = p_0 \times 0 + q_0 \times 1 \qquad \text{or} \qquad q_0 = 2\lambda, \text{ by (4A)} \qquad \dots (6)$$

Solving (5) and (6), $$p_0 = \lambda/2 \qquad \text{and} \qquad q_0 = 2\lambda \qquad \dots (4B)$$

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = 0, \quad y_0 = \lambda, \quad z_0 = \lambda^2, \quad p_0 = \lambda/2, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Re-writing (1), let $f(x, y, z, p, q) = pq - z = 0$ ... (8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = p \qquad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \qquad \text{... (12)}$$

and
$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q \qquad \text{... (13)}$$

From (9) and (13), $(dx/dt) - (dq/dt) = 0$, so that $x - q = C_1$, ... (14)

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$x_0 - q_0 = C_1$ or $0 - 2\lambda = C_1$ or $C_1 = -2\lambda$, Then (14) becomes

$x - q = -2\lambda$ or $x = q - 2\lambda$, ... (15)

From (10) and (12), $(dy/dt) - (dp/dt) = 0$ so that $y - p = C_2$, ... (16)

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2$ or $\lambda - (\lambda/2) = C_2$ or $C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2$ or $y = p + (\lambda/2)$ ... (17)

From (12), $(1/p)\, dp = dt$ so that $\log p - \log C_3 = t$ or $p = C_3\, e^t$ ... (18)

Using initial values (7), (18) gives $p_0 = C_3 e^0$ or $\lambda/2 = C_3$

Hence (18) reduces to $p = (\lambda/2) \times e^t$ ... (19)

From (13), $(1/q)\, dq = dt$ so that $\log q - \log C_4 = t$ or $q = C_4\, e^t$ ... (20)

Using initial values (7), (20) gives $q_0 = C_4\, e^0$ or $2\lambda = C_4$

Hence (20) reduces to $q = 2\lambda\, e^t$ ... (21)

Using (21), (15) becomes $x = 2\lambda\, e^t - 2\lambda$ or $x = 2\lambda\,(e^t - 1)$ ... (22)

Using (19), (17) becomes $y = (\lambda/2)\, e^t + \lambda/2$ or $y = (\lambda/2) \times (e^t + 1)$ ... (23)

Substituting values of p and q from (19) and (21) in (11), we get

$dz/dt = 2\{(\lambda/2) \times e^t\} \times \{2\lambda\, e^t\}$ or $dz = 2\lambda^2 e^{2t}\, dt$.

Integrating, $z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant .... (24)

Using initial values (7), (24) gives $z_0 = \lambda^2 e^0 + C_5$ or $\lambda^2 = \lambda^2 + C_5$ or $C_5 = 0$

Then, (24) gives $$z = \lambda^2 e^{2t} \quad \text{or} \quad z = \lambda^2 (e^t)^2 \qquad \text{... (25)}$$

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$$e^t = (x+4y)/(4y-x) \quad \text{and} \quad \lambda = (4y-x)/4$$

Substituting these values of e^t and λ in (25), we have

$$z = \{(4y-x)^2/16\} \times \{(x+4y)/(4y-x)\}^2 \quad \text{or} \quad 16z = (4y+x)^2,$$

which is the required integral surface of (1) passing through (2).

Ex. 2. *Find the solution of the equation* $z = (p^2+q^2)/2 + (p-x)(q-y)$ *which passes through the x-axis.* **[Himachal 1996; 2004; I.A.S. 2002]**

Sol. Given equation is $$z = (p^2+q^2)/2 + (p-x)(q-y) \qquad \text{... (1)}$$

We are to find its integral surface which passes through x-axis which is given by equations

$$y = 0 \quad \text{and} \quad z = 0 \qquad \text{... (2)}$$

Re-writing (2) in parametric form, $x = \lambda, \quad y = 0, \quad z = 0$, λ being the parameter ... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = \lambda, \quad y_0 = y_0(\lambda) = 0, \quad z = z_0(\lambda) = 0 \qquad \text{... (4A)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$$z_0 = (p_0^2+q_0^2)/2 + (p_0-x_0)(q_0-x_0) \quad \text{or} \quad 0 = (p_0^2+q_0^2)/2 + q_0(p_0-\lambda), \text{ by (4A)}$$

or $$p_0^2 + q_0^2 + 2q_0 p_0 - 2q_0\lambda = 0 \qquad \text{... (5)}$$

Also, we have $$z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$$

so that $$0 = p_0 \times 1 + q_0 \times 0 \quad \text{or} \quad p_0 = 0, \text{ by (4A)} \qquad \text{... (6)}$$

Solving (5) and (6), $$p_0 = 0 \quad \text{and} \quad q_0 = 2\lambda \qquad \text{... (4B)}$$

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Let $$f(x, y, z, p, q) = (p^2+q^2)/2 + pq - py - qx + xy - z = 0 \qquad \text{... (8)}$$

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = p + q - y \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = q + p - x \qquad \text{... (10)}$$

$$dz/dt = p\,(\partial f/\partial p) + q(\partial f/\partial q) = p(p+q-y) + q(q+p-x), \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y \qquad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q\,(\partial f/\partial z) = p + q - x \qquad \text{... (13)}$$

From (9) and (12), $$(dx/dt) - (dp/dt) = 0 \quad \text{so that} \quad x - p = C_1 \qquad \text{... (14)}$$

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $x - p = \lambda$ or $x = p + \lambda$ ... (15)

From (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = C_2$, ... (16)

where C_2 is an arbitrary constant.

Using initial conditions (7), (16) gives $0 - 2\lambda = C_2$ or $C_2 = -2\lambda$.

Hence (16) reduces to $y - q = -2\lambda$ or $y = q - 2\lambda$ ... (17)

$$\therefore \quad \frac{d(p+q-x)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - y + p + q - x - (p + q - y), \text{ using (9), (12) and (13)}$$

or $\dfrac{d(p+q-x)}{dt} = p + q - x$ or $\dfrac{d(p+q-x)}{p+q-x} = dt\cdot$

Integrating, $\log(p+q-x) - \log C_3 = t$ or $p + q - x = C_3\, e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = C_3$ or $C_3 = \lambda$.

Hence (18) reduces to $p + q - x = \lambda e^t$ ... (19)

Now, $\dfrac{d(p+q-y)}{dt} = \dfrac{dp}{dt} + \dfrac{dq}{dt} - \dfrac{dy}{dt} = p + q - y + p + q - x - (q + p - x)$, by (10), (12) and (13)

or $\dfrac{d(p+q-y)}{dt} = p + q - y$ or $\dfrac{d(p+q-y)}{b+q-y} = dt$.

Integrating, $\log(p+q-y) - \log C_4 = t$ or $p + q - y - C_4\, e^t$ ... (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0 + 2\lambda - 0 = C_4$ or $C_4 = 2\lambda$.

Hence (20) reduces to $p + q - y = 2\lambda\, e^t$ ... (21)

From (9) and (21), $dx/dt = 2\lambda\, e^t$ so that $x = 2\lambda\, e^t + C_5$ ... (22)

where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda = 2\lambda + C_5$ or $C_5 = -\lambda$.

Hence (22) reduces to $x = 2\lambda\, e^t - \lambda$ or $x = \lambda\,(2e^t - 1)$ ... (23)

From (10) and (19), $dy/dt = \lambda\, e^t$ so that $y = \lambda\, e^t + C_6$ ... (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0 = \lambda + C_6$ or $C_6 = -\lambda$.

Hence (24) reduces to $y = \lambda e^t - \lambda$ or $y = \lambda\,(e^t - 1)$ ... (25)

Substituting value of y from (17) in (12), we get

$dp/dt = p + q - (q - 2\lambda)$ or $(dp/dt) - p = 2\lambda$, ... (26)

which is a linear equation whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$p\, e^{-t} = \int (2\lambda)\, e^{-t} dt + C_7 = -2\lambda\, e^{-t} + C_3$ or $p = -2\lambda + C_3\, e^t$ ... (27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7$ or $C_7 = 2\lambda$.

Hence (27) reduces to $p = -2\lambda + 2\lambda\, e^t$ or $p = 2\lambda\,(e^t - 1)$ ... (28)

Substituting value of x from (15) in (13), we get

$$dq/dt = p+q-(p+\lambda) \quad \text{or} \quad dq/dt - q = -\lambda, \qquad \text{... (29)}$$

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$$qe^{-t} = \int(-\lambda)\, e^{-t}dt + C_8 = \lambda e^{-t} + C_8 \quad \text{or} \quad q = \lambda + C_8\, e^{t} \qquad \text{... (30)}$$

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.

Hence (30) reduces to $\quad q = \lambda + \lambda\, e^{t} \quad$ or $\quad q = \lambda\,(1+e^{t})$ ... (31)

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda\, e^{t}) + q(\lambda\, e^{t}) = 2\lambda\,(e^{t}-1)\,(2\lambda\, e^{t}) + \lambda(1+e^{t})\,(\lambda\, e^{t})$$

[on putting values of p and q with help of (28) and (31)]

or $\quad dz/dt = 5\lambda^2 e^{2t} - 3\lambda^2 e^{t} \quad$ or $\quad dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^{t})dt$.

Integrating, $\quad z = (5/2) \times \lambda^2 e^{2t} - 3\lambda^2 e^{t} + C_9$ ... (32)

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives $0 = (5/2) \times \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to

$$z = (5/2) \times \lambda^2(e^{2t}-1) - 3\lambda^2(e^{t}-1) \qquad \text{... (33)}$$

Solving (23) and (25) for λ and e^t, $\quad \lambda = x - 2y \quad$ and $\quad e^{t} = (x-y)/(x-2y)$ ... (34)

Eliminating λ and e^t from (33) and (34), we have

$(p^2+q^2)x = pz$

$$z = \frac{5}{2}(x-2y)^2\left\{\left(\frac{x-y}{x-2y}\right)^2 - 1\right\} - 3(x-2y)^2\left(\frac{x-y}{x-2y} - 1\right)$$

or $\quad z = (5/2) \times \{(x-y)^2 - (x-2y)^2\} - 3\ \{(x-2y)\,(x-y) - (x-2y)^2\}$

or $\quad z = (y/2) \times (4x - 3y)$, on simplification.

Ex. 3. *Determine the characteristics of the equation* $z = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4z + x^2 = 0,\ y = 0$. **[Himachal 2000, 05]**

Sol. Do yourself, the required characteristics are $x = 2\lambda(2-e^{-t})$, $y = 2\sqrt{2}\lambda\,(e^{-t}-1)$, $z = -\lambda^2 e^{-2t}$, λ being parameter. Solution is $4z + (x + y\sqrt{2})^2 = 0$.

Ex. 4. *Determine the characteristics of the equation* $p^2 + q^2 = 4z$ *and find the solution of this equation which reduces to* $z = x^2 + 1$ *when* $y = 0$.

Miscellaneous Problem on Chapter 3

1. Show that the envelope of the family of surfaces touch each member of the family at all points of its characteristics. **[Meerut 2008]**

2. Find a complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the surface solution which passes through the curve $x = 0,\ z^2 = 4y$. **[Meerut 2007]**

3. Solve $p^2 y + p^2 y x^2 = q x^2$ **[Pune 2010]**

Ans. Complete integral is $z = a\,(1+x^2)^{1/2} + (a^2y^2)/2 + b$.

4. Given thta $(x-a)^2 + (y-b)^2 + z^2 = 1$ is complete integral of $z^2\,(1+p^2+q^2) = 1$. Find its singular integral. **[Pune 2010]**

Hint. Use definition on page 3.1. **Ans.** $z^2 = 1$

4

Homogenous Linear Partial Differential Equations with Constant Coefficients

4.1. Homogeneous and Non–homogeneous linear equations with constant coefficients. A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a *linear partial differential equation*. The general form of such an equation is

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \ldots \quad + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right) + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right)$$
$$+ N_0 z = f(x, y), \quad \ldots(1)$$

where the coefficients $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are constants or functions of x and y. If $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are all constants, then (1) is called a *linear partial differential equation with constant coefficients*.

For convenience $\partial/\partial x$ and $\partial/\partial y$ will be denoted by D (or D_x) and D' (or D_y) respectively. Then (1) can be rewritten as

$$[(A_0 D^n + A_1 D^{n-1} D' + \ldots + A_n D'^n) + (B_0 D^{n-1} + B_1 D^{n-2} D' + \ldots + B_{n-1} D'^{n-1})$$
$$+ (M_0 D + M_1 D') + N_0] z = f(x, y), \quad \ldots(2)$$

or, briefly,
$$F(D, D') z = f(x, y). \quad \ldots(3)$$

When all the derivatives appearing in (1) are of the same order, then the resulting equation is called a *linear homogeneous partial differential equation with constant coefficients* and it is then of the form

$$(A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n) z = f(x, y). \quad \ldots(4)$$

On the other hand, when all the derivatives in (1) are not of the same order, then it is called a *non–homogeneous linear partial differential equation with constant coefficients*.

In this chapter we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely, (4)

4.2. Solution of a homogeneous linear partial differential equation with constant coefficients, namely,
$$(A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n) z = f(x, y), \quad \ldots(1)$$

where $A_0, A_1, \ldots, A_n$ are constants. (1) may rewritten as

$$F(D, D') z = f(x, y), \quad \ldots(2)$$

where
$$F(D, D') = A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n. \quad \ldots(3)$$

As in the case of linear ordinary differential equation with constant coefficients, we start with the following basic theorems.

Theorem I. *If u is the complementary function and z' a particular integral of a linear partial differential equation $F(D, D')z = f(x, y)$, then $u + z'$ is a general solution of the equation.*

Proof. Given
$$F(D, D') z = f(x, y). \quad \ldots(1)$$

The complementary function u of (1) is the most general solution of

$$F(D, D') z = 0. \quad \ldots(2)$$

$$\therefore \qquad F(D, D')u = 0. \qquad ...(3)$$

Note that the complementary function must contain as many arbitrary constants as is the order of equation (2).

Any solution z' of (1) is called a particular integral of (1). Note that particular integral does not contain any arbitrary constant. Thus, by definition, we have

$$F(D, D')z' = f(x, y). \qquad ...(4)$$

Adding (3) and (4), $\quad F(D, D')(u + z') = f(x, y),$

showing that $u + z'$ is a solution of (1). Since (1) and (2) are of the same order, the general solution $u + z'$ will contain as many arbitrary constants as the general solution of (1) requires.

Theorem II. *If $u_1, u_2, ..., u_n$ are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then $\sum_{r=1}^{n} c_r u_r$ is also a solution, where $C_1, C_2, ..., C_r, ..., C_n$ are arbitrary constants.*

Proof. Given equation is $\quad F(D, D')z = 0. \qquad ...(1)$

We have $\quad F(D, D')(c_r u_r) = c_r F(D, D')u_r \qquad ...(2)$

and
$$F(D, D')\sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D, D')v_r \qquad ...(3)$$

for any set of functions v_r. Using results (2) and (3), we get

$$F(D, D')\sum_{r=1}^{n}(c_r u_r) = \sum_{r=1}^{n} F(D, D')(c_r u_r) = c_r \sum_{r=1}^{n} F(D, D')u_r \qquad ...(4)$$

Since u_r is solution of (1) for $r = 1, 2, ..., n,$ so $F(D, D')u_r = 0$ for $r = 1, 2, 3, ..., n.$

$\therefore$ (4) gives $F(D, D')\sum_{r=1}^{n}(c_r u_r) = 0$, which proves the required result.

Note. For convenience we shall denote complementary function by C.F. and particular integral by P.I.

4.3. Method of finding the complementary function (C.F.) of the linear homogeneous partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$

i.e.,
$$(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = f(x, y), \qquad ...(1)$$

where $A_0, A_1, ..., A_n$ are all constants.

The complementary function of (1) is the general solution of

$$(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = 0. \qquad ...(2)$$

or
$$[(D - m_1 D')(D - m_2 D') \dots (D - m_n D')]z = 0. \qquad ...(3)$$

where $m_1, m_2, ..., m_n$ are some constants.

Clearly, the solution of any one of the equations

$$(D - m_1 D')z = 0, \qquad (D - m_2 D')z = 0, \; ...., \qquad (D - m_n D')z = 0 \quad ...(4)$$

is also a solution of (3).

We now show that the general solution of $\quad (D - mD')z = 0 \quad$ is $\quad z = \phi(y + mx),$ where ϕ is an arbitrary function.

We have, $(D - mD')z = 0$ or $(\partial z/\partial x) - m(\partial z/\partial y) = 0$ or $p - mq = 0, \quad ...(5)$

which is in Lagrange's form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (5) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}. \qquad ...(6)$$

Taking the first two fractions of (6), $dy + mdx = 0$ so that $y + mx = c_1$ ... (7)

From the third fraction of (6), $dz = 0$ so that $z = c_2$. ...(8)

Hence from (7) and (8), the general solution of (5) is $z = \phi(y + mx)$, where ϕ is an arbitrary function. So, we assume that a solution of (2) is of the form

$$z = \phi(y + mx). \qquad ...(9)$$

From (9), $Dz = \partial z/\partial x = m\phi'(y + mx)$,

$D^2 z = \partial^2 z/\partial x^2 = m^2\phi''(y + mx)$,

....

and $D^n z = \partial^n z/\partial x^n = m^n \phi^{(n)}(y + mx)$.

Again, $D'z = \partial z/\partial y = \phi'(y + mx)$,

$D'^2 z = \partial^2 z/\partial y^2 = \phi''(y + mx)$,

....

and $D'^n z = \partial^n z/\partial y^n = \phi^{(n)}(y + mx)$.

Also, in general, $D^r D'^s z = \partial^{r+s} z/\partial x^r \partial y^s = m^r \phi^{(r+s)}(y + mx)$.

Substituting these values in (2) and simplifying, we get

$$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n)\phi^{(n)}(y + mx) = 0,$$

which is true if m is a root of the equation

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0. \qquad ...(10)$$

The equation (10) is known as the *auxiliary equation* (*A.E.*) and is obtained by putting $D = m$ and $D' = 1$ in $F(D, D') = 0$.

Let $m_1, m_2, \dots, m_n$ be n roots of A.E. (10). Two cases arise.

Case I. When $\mathbf{m_1, m_2, m_3, \dots m_n}$ are distinct. Then the part of C.F. corresponding to $m = m_r$ is $z = \phi_r(y + m_r x)$ for $r = 1, 2, 3, \dots, n$. Since (2) is linear, the sum of the solutions is also a solution.

$\therefore$ C.F. of (2) $= \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$, ...(11)

where $\phi_1, \phi_2, \dots, \phi_n$ are arbitrary functions.

Case II. Repeated roots. Let m be repeated root of (10) and so consider

$$(D - mD')(D - mD')z = 0. \qquad ...(12)$$

Let $(D - mD')z = v$. ...(13)

Then, (12) $\Rightarrow$ $(D - mD')v = 0$ or $(\partial v/\partial x) - m(\partial v/\partial y) = 0$, ...(14)

which is in Lagrange's form. Hence Lagrange's auxiliary equations for (14) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dv}{0}. \qquad ...(15)$$

As before, two independent integrals of (15) are $y + mx = c_3$ and $v = c_4$.

$\therefore$ $v = \phi(y + mx)$ ...(16)

is a solution of (14), ϕ being as arbitrary function.

Using (16), (13) becomes $(\partial z/\partial x) - m(\partial z/\partial y) = \phi(y + mx)$ ...(17)

which is in Lagrange's form. Its Lagrange's auxiliary equations for (7) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)} \qquad ...(18)$$

Taking the first two fractions of (18), $dy + mdx = 0$ so that $y + mx = c_5$. ... (19)

Taking the first and third fractions of (18) and using (19), we get

$(dx)/1 = (dz)/\phi(c_5)$ so that $dz - \phi(c_5)dx = 0$.

Integrating, $z - x\,\phi(c_5) = c_6$ or $z - x\,\phi(y + mx) = c_6$, using (19). ...(20)

From (19) and (20), the general solution of (12) is

$z - x\phi(y + mx) = \psi(y + mx)$ or $z = \psi(y + mx) + x\phi(y + mx)$, ...(21)

where ϕ and ψ are arbitrary functions. (21) is a part of C.F. corresponding to the two times repeated root m. In general, if a root 'm' is repeated 'r' times, the corresponding part of C.F. is

$$\phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \ldots + x^{r-1}\phi_r(y + mx).$$

4.4.A. Working rule for finding C.F. of linear homogeneous partial differential equation with constant coefficients

Step 1. Put the given equation in standard form $(A_0D^n + A_1D^{n-1}D' + \ldots A_nD'^n)z = f(x, y)$....(1)

Step 2. Replacing D by m and D' by 1 in the coefficients of z, we obtain auxiliary equation (A.E.) for (1) as $A_0m^n + A_1m^{n-1} + \ldots + A_n = 0$. ...(2)

Step 3. Solve (2) for m. Two cases will arise :

Case (*i*) Let $m = m_1, m_2, \ldots, m_n$ (different roots). Then

C.F. $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \ldots + \phi_n(y + m_nx)$, where $\phi_1, \phi_2 \ldots, \phi_n$ are arbitrary functions.

If in the above case (*i*), $m = a_1/b_1, a_2/b_2, \ldots, a_n/b_n$, then

$$\text{C.F.} = \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x) + \ldots + \phi_n(b_ny + a_nx)$$

Further if $m = -(a_1/b_1), -(a_2/b_2), \ldots, -(a_n/b_n)$, then

$$\text{C.F.} = \phi_1(b_1y - a_1x) + \phi_2(b_2y - a_2x) + \ldots + \phi_n(b_ny - a_nx)$$

Case (*ii*) Let $m = m'$ (repeated n times). Then corresponding to these roots

$$\text{C.F.} = \phi_1(y + m'x) + x\,\phi_2(y + m'x) + x^2\phi_2(y + m'x) + \ldots + x^{n-1}\phi_n(y + m'x).$$

In the above case (*ii*), if $m = a/b$ (repeated n times), Then corresponding to these n roots,

$$\text{C.F.} = \phi_1(by + ax) + x\,\phi_2(by + ax) + x^2\phi_2(by + ax) + \ldots + x^{n-1}\phi_n(by + ax)$$

And, if $m = -(a/b)$, (repeated n times), then

$$\text{C.F.} = \phi_1(by - ax) + x\,\phi_2(by - ax) + x^2\phi_2(by - ax) + \ldots + x^n\phi_n(by - ax)$$

Case (*iii*) Corresponding to a non–repeated factor D on L.H.S. of (1), the part of C.F. is taken as $\phi(y)$.

Case (*iv*) Corresponding to a repeated factor D^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \ldots + x^{m-1}\phi_m(y).$$

Case (*v*) Corresponding to a non–repeated factor D' on L.H.S. of (1), the part of C.F. is taken as $\phi(x)$.

Case (*vi*) Corresponding to a repeated factor D'^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \ldots + y^{m-1}\phi_m(x).$$

4.4.B. Alternative working rule for finding C.F.

Let the given partial differential equation be $F(D, D')z = f(x, y)$. Factorize $F(D, D')$ into linear factors of the form $(bD - aD')$. Then we use the following results :

(*i*) Corresponding to each non–repeated factor $(bD - aD')$, the part of C.F. is taken as

$$\phi(by + ax).$$

(*ii*) Corresponding to a repeated factor $(bD - aD')^m$, the part of C.F. is taken as

$$\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \ldots + x^{m-1}\phi_m(by + ax).$$

(*iii*) Corresponding to a non–repeated factor D, part of C.F. is taken as $\phi(y)$.

(*iv*) Corresponding to a repeated factor D^m, the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \ldots + x^{m-1}\phi_m(y).$$

(*v*) Corresponding to a non–repeated factor D', part of C.F. is taken as $\phi(x)$.

(*vi*) Corresponding to a repeated factor D'^m, the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \ldots + y^{m-1}\phi_m(x).$$

4.5. Solved examples based on articles 4.4A and 4.4B

[Notations $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$ and $t = \partial^2 z/\partial y^2$ will be used]

Ex. 1. *Solve* (*a*) $r = a^2 t$. **[I.A.S. 1987; Meerut 1991]**

(*b*)$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = 0$.

(*c*) $(D^2 - 3aDD' + 2a^2 D'^2)z = 0$. **[Kanpur 2007; Meerut 2007]**

Sol. (*a*) Given equation is $\partial^2 z/\partial x^2 = a^2(\partial^2 z/\partial y^2)$ or $(D^2 - a^2 D'^2)z = 0$. ...(1)

The auxiliary equation of (1) is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ The general solution of (1) is $z = \text{C.F.} = \phi_1(y + ax) + \phi_2(y - ax)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y - x)$

(*c*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + ax) + \phi_2(y + 2ax)$

Ex. 2. *Solve* (*a*) $(D^3 - 6D^2 D' + 11DD'^2 - 6D'^3)z = 0$. **[Agra 2005]**

(*b*) $(\partial^3 z/\partial x^3) - 7(\partial^3 z/\partial x\partial y^2) + 6(\partial^3 z/\partial y^3) = 0$. **[Bhopal 2010]**

(*c*) $(D^3 - 3D^2 D' + 2DD'^2)z = 0$. **[Meerut 2008; Lucknow 2010]**

Sol. (*a*) The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$

or $(m - 1)(m - 2)(m - 3) = 0$ so that $m = 1, 2, 3$.

$\therefore$ The general solution of the given equation is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*b*) The given equation can be written as $(D^3 - 7DD'^2 + 6D'^3)z = 0$. ...(1)

Its auxiliary equation is $m^3 - 7m + 6 = 0$ or $(m - 1)(m - 2)(m + 3) = 0$.

Hence $m = 1, 2, -3$ and so the general solution of (1) is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*c*)Proceed as above. **Ans.** $z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$.

Ex. 3. (*a*) *Solve* $2r + 5s + 2t = 0$. **[Meerut 2011]**

(*b*) $2(\partial^2 z/\partial x^2) - 3(\partial^2 z/\partial x\partial y) - 2(\partial^2 z/\partial y^2) = 0$.

Sol. (*a*) Now, $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/Dx\partial y = DD'z$ and $t = \partial^2 z/\partial y^2 = D'^2 z$. Hence the given equation can be re–written as $(2D^2 + 5\,DD' + 2D'^2)y = 0$. ...(1)

Its auxiliary equation is $2m^2 + 5m + 2 = 0$ or $(2m + 1)(m + 2) = 0$.

So $m = -1/2, -2$ and hence the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Alternative method : (1) can be re–written as $(2D + D')(D + 2D') = 0$.

So by using the alternative working rule 4.4B, the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 ϕ_2 being arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(2y - x) + \phi_2(y + 2x)$

Ex. 4. *Solve* (*a*) $r + t + 2s = 0$ **[Kanpur 2009]**

(*b*) $25r - 40s + 16t = 0$ **[Bilaspur 1996; Jabalpur 2002; Sagar 2004];**

(*c*) $(4D^2 + 12DD' + 9D'^2)z = 0$. **[Kanpur 2008; Indore 2004]**

Sol. (*a*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $t = \partial^2 z/\partial y^2 = D'^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$.

So the given equation becomes $(D^2 + D'^2 + 2DD')z = 0$. ...(1)

Its auxiliary equation is $m^2 + 1 + 2m = 0$ or $(m + 1)^2 = 0$ or $m = -1, -1$.

So the general solution of (1) is $z = \phi_1(y - x) + x\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

(*b*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$, $t = \partial^2 z/\partial y^2 = D'^2 z$. So the given equation becomes $(25D^2 - 40DD' + 16D'^2)z = 0$. ...(1)

Its auxiliary equation is $25m^2 - 40m + 16 = 0$ or $(5m - 4)^2 = 0$ so that $m = 4/5, 4/5$.

Hence the general solution of (1) is $z = \phi_1(5y + 4x) + x\phi_2(5y + 4x)$.

Alternative method (1) may be written as $(5D - 4D')^2 z = 0$.

Using result (*ii*) of Art 4.4B, the solution of (1) is

$$z = \phi_1(5y + 4x) + x\phi_2(5y + 4x),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

(*c*) Proceed as in part (b). **Ans.** $z = \phi_1(2y - 3x) + x\phi_2(2y - 3x)$

Ex. 5. *Solve* (*a*) $(D^3 - 4D^2D' + 4DD'^2)z = 0$. **[Bhopal 2000, 03]**

(*b*) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$. **[Bilaspur 2004]**

(*c*) $(D^4 + D'^2 - 2D^2D'^2)z = 0$.

(*d*) $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.

Sol. (*a*) The auxiliary equation of the given equation is

$m^3 - 4m^2 + 4m = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$

Hence the general solution of the given equation is

$$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x),\ \phi_1,\ \phi_2,\ \phi_3 \text{ being arbitrary functions.}$$

(*b*) The auxiliary equation of the given equation is

$m^4 - 2m^3 + 2m - 1 = 0$ or $(m + 1)(m - 1)^3 = 0$ so that $m = -1, 1, 1, 1$.

Hence the general solution of the given equation is

$z = \phi_1(y - x) + \phi_2(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*c*) Try yourself. **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - x) + x\phi_4(y - x)$

(*d*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$

Ex. 6. *Solve* (*a*) $(D^3D'^2 + D^2D'^3)z = 0$. (*b*) $(D^3D' - 4D^2D'^2 + 4DD'^3)z = 0$.

Sol. (*a*) The given equation can be re–written as $D^2D'^2(D + D')z = 0$. ...(1)

Hence using the alternative method 4.4B for C.F., the general solution is

$z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(y - x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 are arbitrary functions.

(*b*) The given equation can be re–written as

$DD'(D^2 - 4DD' + 4D'^2) = 0$ *Or* $DD'(D - 2D')^2 = 0$.

Using the working rule 4.4B, the required general solution is

$z = \phi_1(y) + \phi_2(x) + \phi_3(y + 2x) + x\phi_4(y + 2x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

Ex. 7. *Solve* (*a*) $(\partial^4 z/\partial x^4) - (\partial^4 z/\partial y^4) = 0$ (*b*) $(D^4 + D'^4)z = 0$.

Sol. (*a*) Rewriting, the given equation is $(D^4 - D'^4)z = 0$. ...(1)

Its auxiliary equation is $m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0$ $\Rightarrow$ $m = 1, -1\ i, -i$.

Hence the general solution of (1) is $z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + ix) + \phi_4(y - ix)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*b*) The auxiliary equation of the given equation is $m^4 + 1 = 0$ or $(m^2 + 1)^2 - 2m^2 = 0$

or $(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$ or $(m^2 + 1 + \sqrt{2}m)(m^2 + 1 - \sqrt{2}m) = 0$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0$ $\Rightarrow$ $m = (-1 \pm i)/\sqrt{2}, (1 \pm i)/\sqrt{2}$.

Let $z_1 = (-1 + i)/\sqrt{2}$ and $z_2 = (1 + i)/\sqrt{2}$, then, $m = z_1, \bar{z}_1, z_2, \bar{z}_2$,

where $\bar{z}_1$ and $\bar{z}_2$ denote complex conjugates of z_1 and z_2 respectively.

Hence the general solution of the given equation is

$z = \phi_1(y + z_1x) + \phi_2(y + \bar{z}_1x) + \phi_3(y + z_2x) + \phi_4(y + \bar{z}_2x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are arbitrary functions.

4.6. Particular integral (P.I.) of homogeneous linear partial differential equation given by $F(D, D')y = f(x, y)$ ... (1)

The inverse operator $1/F(D, D')$ of the operator $F(D, D')$ is defined by the following identity

$$F(D, D')\left(\frac{1}{F(D, D')} f(x, y)\right) = f(x, y)$$

$\therefore$ Particular integral (P.I.) of (1) $= \dfrac{1}{F(D, D')} f(x, y)$

In what follows we shall treat the symbolic functions of D and D' as we do for the symbolic functions of D alone in ordinary differential equations. Thus it will be factorized and resolved into partial fractions or expanded in an infinite series as the case may be. The reader is advised to note carefully the following results :

(*i*) D, D^2, ... will stand for differentiating partially with respect to x once, twice and so on.

For example, $\quad Dx^4y^5 = \dfrac{\partial}{\partial x} x^4y^5 = 4x^3y^5$; $\quad D^2x^4y^5 = \dfrac{\partial^2}{\partial x^2} x^4y^5 = 12x^2y^5$.

(*ii*) D', D'^2... will stand for differentiating partially with respect to y once, twice and so on.

For example, $\quad D' x^4y^5 = \dfrac{\partial}{\partial y} x^4y^5 = 5x^4y^4$; $\quad D'^2x^4y^5 = \dfrac{\partial^2}{\partial y^2} x^4y^5 = 20x^4y^3$.

(*iii*) $1/D$, $1/D^2$, ... will stand for integrating partially with respect to x once, twice and so on.

For example, $\quad \dfrac{1}{D} x^4y^5 = \displaystyle\int x^4y^5\, dx = \dfrac{x^5y^5}{5}$; $\quad \dfrac{1}{D^2} x^4y^5 = \displaystyle\iint x^4y^5\, dx\, dx = \dfrac{x^6y^5}{30}$

(*iv*) $1/D'$, $1/D'^2$, ... will stand for integrating partially with respect to y once, twice and so on.

For example, $\quad \dfrac{1}{D'} x^4y^5 = \displaystyle\int x^4y^5\, dy = \dfrac{x^4y^6}{6}$; $\quad \dfrac{1}{D'^2} x^4y^5 = \displaystyle\iint x^4y^5\, dy\, dy = \dfrac{x^4y^7}{42}$.

4.7. Short methods of finding the P.I. in certain cases.

Before taking up the general method for finding P.I. of $\quad F(D, D')z = f(x, y)$
we begin with cases when $f(x, y)$ is in two special forms. The methods corresponding to these forms are much shorter than the general methods to be discussed in Art. 4.12.

4.8. A Short Method I. When f(x, y) is of the form f(ax + by).

The method under consideration is based on the following theorem.

Theorem I. *If $F(D, D')$ be homogeneous function of D and D' of degree n, then*

$$\frac{1}{F(D, D')} \phi^{(n)}(ax + by) = \frac{1}{F(a, b)} \phi(ax + by),$$

provided $F(a, b) \neq 0$, $\phi^{(n)}$ being the nth derivative of ϕ w.r.t. $ax + by$ as a whole.

Proof. By direct differentiation, we have $\quad D^r\phi(ax + by) = a^r\phi^{(r)}(ax + by)$,

$D'^s\phi(ax + by) = b^s\phi^{(s)}(ax + by) \quad$ and $\quad D^rD'^s\phi(ax + by) = a^rb^s\phi^{(r+s)}(ax + by)$.

Since $F(D, D')$ is homogeneous function of degree n, so we have

$$F(D, D')\phi(ax + by) = F(a, b)\phi^{(n)}(ax + by). \qquad ...(1)$$

Operating both sides of (1) by $1/F(D, D')$, we have

$$\phi(ax + by) = F(a, b)\frac{1}{F(D, D')}\phi^{(n)}(ax + by). \qquad ...(2)$$

Since $F(a, b) \neq 0$, dividing both sides of (2) by $F(a, b)$, we get

$$\frac{1}{F(D, D')} \phi^{(n)}(ax + by) = \frac{1}{F(a, b)} \phi(ax + by). \qquad ...(3)$$

An important deduction from result (3) : Putting $ax + by = v$, (3) gives

$$\frac{1}{F(D, D')}\phi^{(n)}(v) = \frac{1}{F(a, b)}\phi(v)\,. \qquad ...(4)$$

Integrating both sides of (4) n times w.r.t. 'v', we have

$$\frac{1}{F(D, D')}\phi(v) = \frac{1}{F(a, b)}\int\int\ldots\int\phi(v)\,dv\,dv\ldots dv, \qquad \text{where} \qquad v = ax + by.$$

Exceptional case when F(a, b) = 0. When $F(a, b) = 0$, then the above theorem does not hold good. In such a case the new method is based on the following theorem. Note that $F(a, b) = 0$ if and only if $(bD - aD')$ is a factor $F(D, D')$.

Theorem II. $$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

Proof. Consider the equation $\qquad (bD - aD')z = x^r\phi(ax + by) \qquad ...(1)$

or $$bp - aq = x^r\phi(ax + by). \qquad ...(2)$$

Lagrange's subsidiary equations for (2) are $$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r\phi(ax + by)}. \qquad ...(3)$$

Taking the first two fractions of (3), $\quad adx + bdy = 0 \quad$ so that $\quad ax + by = c_1 \quad ...(4)$

Taking the first and third members of (4) and using (4), we get

$$\frac{dx}{b} = \frac{dz}{x^r\phi(c_1)} \qquad \text{or} \qquad dz = \frac{x^r\phi(c_1)}{b}dx.$$

Integrating, $$z = \frac{x^{r+1}\phi(c_1)}{b(r+1)} = \frac{x^{r+1}\phi(ax + by)}{b(r+1)}, \text{ by (4).} \qquad ...(5)$$

(5) is a solution of (1).

Now, from (1), $$z = \frac{1}{(bD - aD')}x^r\phi(ax + by). \qquad ...(6)$$

From (5) and (6), $$\frac{1}{(bD - aD')}x^r\phi(ax + by) = \frac{x^{r+1}}{b(r+1)}\phi(ax + by). \qquad ...(7)$$

Hence, if $\quad z = \dfrac{1}{(bD - aD')^n}\phi(ax + by), \quad$ then we have

$$z = \frac{1}{(bD - aD')^{n-1}}\left[\frac{1}{(bD - aD')}x^0\phi(ax + by)\right], \text{ as } x^0 = 1$$

$$= \frac{1}{(bD - aD')^{n-1}}\frac{x}{b}\phi(ax + by), \text{ using (7) for } r = 0$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\left[\frac{1}{(bD - aD')}x\phi(ax + by)\right]$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\frac{x^2}{2b}\phi(ax + by), \text{ using (7) for } r = 1$$

$$= \frac{1}{2!b^2}\frac{1}{(bD - aD')^{n-2}}x^2\phi(ax + by)$$

$$= \frac{1}{n!b^n}\frac{1}{(bD - aD')^{n-n}}x^n\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by)$$

[after repeated use of (7) for $n - 2$ times more]

Working rule for finding particular integral where f(x, y) = ϕ(ax + by).

The following rules will used depending upon the situation in hand.

Formula (i). When $F(a, b) \neq 0$ and $F(D, D')$ is a homogeneous function of degree n, then

$$\text{P.I.} = \frac{1}{F(D, D')}\phi(ax + by) = \frac{1}{F(a, b)}\int\int\ldots\int f(v)\, dv\, dv \ldots dv, \qquad \text{where} \qquad v = ax + by$$

Note that R.H.S. contains a multiple integral of nth order.

Formula (ii). When $F(a, b) = 0$, we have

$$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

4.9. Solved Examples based on Short Method I of Art. 4.8

Ex. 1. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$. **[I.A.S. 1986, Meerut 2005, 07, 09, 10]**

Sol. The auxiliary equation of the given equation is $m^2 + 3m + 2 = 0$ giving $m = -1, -2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$

$$= \frac{1}{1^2 + 3\times 1\times 1 + 2\times 1^2}\int\int v\, dv\, dv, \text{ where } v = x + y, \text{ using formula (i) of working rule}$$

$$= \int (v^2/2)\, dv = (1/6) \times (v^3/6) = (1/36) \times (x + y)^3.$$

Hence the required general solution is z = C.F. + P.I., *i.e.*,

$$z = \phi_1(y - x) + \phi_2(y - 2x) + (1/36) \times (x + y)^3.$$

Ex. 2. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$. **[Vikram 2000]**

(*b*) $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = 12(x + y)$ **[Nagpur 2010]**

(*c*) $(D^2 + D'^2)z = 30(2x + y)$. **[Bhopal 1996, Sagar 2004]**

(*d*) $(\partial^2 z/\partial x^2) + 2(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) = 2x + 3y$. **[Kumaun 1992]**

(*e*) $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$. **[Kurukshetra 2005]**

(*f*) $r + s - 2t = (2x + y)^{1/2}$. **[Lucknow 2010]**

Sol. (*a*) The auxiliary of the given equation is $2m^2 - 5m + 2 = 0$ giving $m = 1/2$, 2.

$\therefore$ C.F. $= \phi_1(2y + x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}24(y - x) = 24\frac{1}{2D^2 - 5DD' + 2D'^2}(y - x)$$

$$= \frac{24}{2\times(-1)^2 - 5\times(-1)\times 2 + 2\times 2^2}\int\int v\, dv\, dv, \text{ where } v = y - x, \text{ using formula } (i) \text{ of}$$

working rule

$$= (24/20)\times\int (v^2/2)\, dv = (6/5)\times(v^3/6) = (1/5)\times(y - x)^3.$$

Hence the required general solution is $z = \phi_1(2y + x) + \phi_2(y + 2x) + (y - x)^3/5$.

(*b*) The given equation can be written as $(D^2 + D'^2)V = 12(x + y)$. ...(1)

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{D^2 + D'^2}12(x + y) = 12\frac{1}{D^2 + D'^2}(x + y) = \frac{12}{1^2 + 1^2}\int\int v\, dv\, dv$$

$$= 6\int (v^2/2)\,dv = v^3 = (x+y)^3 .$$

Hence the required general solution is $V = \phi_1(y+ix) + \phi_2(y-ix) + (x+y)^3$.

(c) Try yourself. **Ans.** $z = \phi_1(y+ix) + \phi_2(y-ix) + (2x+y)^3$

(d) Try yourself. **Ans.** $z = \phi_1(y-x) + x\phi_2(y-x) + (1/150) \times (2x+3y)^3$

(e) Proceed as in part (d). **Ans.** $y = \phi_1(y-x) + \phi_2(y-2x) + (1/240) \times (2x+3y)^3$

(f) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x\partial y,\ t = \partial^2 z/\partial y^2$, the given equation can be re–written as

$\partial^2 z/\partial x^2 + \partial^2 z/\partial x\,\partial y - 2\,(\partial^2 z/\partial y^2) = (2x+y)^{1/2}$ or $(D^2 + DD' - 2D'^2)z = (2x+y)^{1/2}$.

Its auxiliary equation is $m^2 + m - 2 = 0$ so that $m = 1, -2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2}(2x+y)^{1/2} = \frac{1}{2^2 + 2\times 1 - 2\times 1^2}\int\!\!\int v^{1/2}\,dv\,dv\,, \text{ where } v = 2x+y$$

$$= \frac{1}{4}\int \frac{2}{3} v^{3/2}\,dv = \frac{1}{4}\times\frac{2}{3}\times\frac{2}{5}v^{5/2} = \frac{1}{15}(2x+y)^{5/2}$$

Hence the required general solution is $z = \phi_1(y+x) + \phi_2(y-2x) + (1/15)(2x+y)^{5/2}$.

Ex. 3. *Solve* (a) $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$. **[Bhopal 2010; Indore 1998; Jabalpur 1998; Purvanchal 2007, Sagar 1999; K.V. Kurkshetra 2005]**

(b) $(D^2 - 2DD' + D'^2)z = e^{x+2y}$. **[Bhopal 1997, 98, Kanpur 2005]**

(c) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$.

Sol. (a) Here auxiliary equation is $m^2 + 2m + 1 = 0$ so that $m = -1, -1$.

$\therefore$ C.F. $= \phi_1(y-x) + x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2}e^{2x+3y} = \frac{1}{(D+D')^2}e^{2x+3y} = \frac{1}{(2+3)^2}\int\!\!\int e^v\,dv\,dv\,, \text{ where } v = 2x+3y$$

$$= (1/25)\times\int e^v\,dv = (1/25)\times e^v = (1/25)\times e^{2x+3y}$$

$\therefore$ Solution is $z = \text{C.F.} + \text{P.I.} = \phi_1(y-x) + x\phi_2(y-x) + (1/25) \times e^{2x+3y}$.

(b) Proceed as in part (a). **Ans.** $z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y}$

(c) Here auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$ giving $m = 1, 2, 3$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 6D^2D' + 11DD'^2 - 6D'^3}e^{5x+6y} = \frac{1}{(D-D')(D-2D')(D-3D')}e^{5x+6y}$$

$$= \frac{1}{(5-6)(5-12)(5-18)}\int\!\!\int\!\!\int e^v\,dv\,dv\,dv, \text{ where } v = 5x+6y$$

$$= \frac{1}{-91}\int\!\!\int e^v\,dv\,dv = -\frac{1}{91}\int e^v\,dv = -\frac{1}{91}e^v = -\frac{1}{91}e^{5x+6y}$$

Hence the required solution is $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x) - (1/91) \times e^{5x+6y}$.

Ex. 4. *Solve* (a) $r - 2s + t = \sin(2x+3y)$. **[Meerut 2007; Indore 2002; Vikram 1996;]**

(b) $(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x+2y)$. **[Kanpur 2008; I.A.S. 2006]**

(c) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+3y)$.

(d) $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y)$. **[Delhi Maths Hons. 1992]**

Sol. (a) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x\partial y,\ t = \partial^2 z/\partial t^2$, the given equation becomes

$\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = \sin(2x+3y)$ or $(D^2 - 2DD' + D'^2)z = \sin(2x + 2y)$.

Its auxiliary equation is $m^2 - 2m + 1 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-D')^2}\sin(2x+3y) = \frac{1}{(2-3)^2}\int\int \sin v\,dv\,dv, \text{ where } v = 2x+3y$$

$$= -\int \cos v\,dv = -\sin v = -\sin(2x+3y).$$

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) - \sin(2x + 3y)$.

(*b*) The auxiliary equation of the given equation is $m^3 - 4m^2 + 4m = 0$.

or $m(m^2 - 4m + 4) = 0$ or $m(m-2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2}\,2\sin(3x+2y)$$

$$= 2\times\frac{1}{3^3 - 4\times 3^2\times 2 + 4\times 3\times 2^2}\int\int\int \sin v\,dv\,dv, \text{ where } v = 3x+2y$$

$$= (2/3)\times\int\int(-\cos v)dv\,dv = -(2/3)\times\int \sin v\,dv = (2/3)\times\cos v = (2/3)\times\cos(3x+2y)$$

The required general solution is $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + (2/3)\times\cos(3x + 2y)$.

(*c*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - (1/32)\times\sin(2x + 3y)$

(*d*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + 2x) + (1/27)\times\sin(x + 2y)$

Ex. 5. *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \cos ny$. **[Kanpur 2007; Nagpur 2010]**

Sol. Given equation can be written as $(D^2 + D'^2)z = \cos mx \cos ny$.

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + D'^2}\cos mx\cos ny = \frac{1}{D^2+D'^2}\frac{\cos(mx+ny)+\cos(mx-ny)}{2}$$

$$= \frac{1}{2}\frac{1}{D^2+D'^2}\cos(mx+ny) + \frac{1}{2}\frac{1}{D^2+D'^2}\cos(mx-ny)$$

$$= \frac{1}{2}\frac{1}{m^2+n^2}\int\int\cos v\,dv\,dv + \frac{1}{2}\frac{1}{m^2+(-n)^2}\int\int\cos u\,du\,du,$$

where $v = mx + ny$ and $u = mx - ny$

$$= \frac{1}{2}\frac{1}{m^2+n^2}\int\sin v\,dv + \frac{1}{2}\frac{1}{m^2+n^2}\int\sin u\,du = \frac{1}{2}\frac{1}{m^2+n^2}[-\cos v - \cos u]$$

$$= -\frac{1}{2(m^2+n^2)}[\cos(mx+ny) + \cos(mx-ny)], \text{ as } v = mx+ny,\ u = mx-ny$$

$$= -\frac{1}{2(m^2+n^2)}\times 2\cos mx\cos ny = -(m^2+n^2)^{-1}\cos mx\cos ny.$$

Hence the required general solution is $z = \phi_1(y + ix) + \phi_2(y - ix) - (m^2 + n^2)^{-1}\cos mx \cos ny$.

Ex. 5. (*b*) *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \sin ny$. **[Ravishankar 1999, 2001]**

Sol. Do like Ex. 5(*a*) **Ans.** $z = \phi_1(y+ix) + \phi_2(y-ix) + (\sin mx + \sin ny)/(m^2+n^2)$

Ex. 6. *Solve the following partial differential equations :*

(*a*) $(D^2 - 2DD' + D'^2)z = \tan(y + x)$ or $(D - D')^2 z = \tan(y + x)$ **[Jiwaji 1996]**

(*b*) $(D^2 - 2aDD' + a^2D'^2)z = f(y + ax)$ or $(D - aD')^2 z = f(y + ax)$.

(*c*) $4r - 4s + t = 16 \log(x + 2y)$. **[Agra 2009; Meerut 2009; Ravishankar 2000]**

Sol. (*a*) Here auxiliary equation is $(m - 1)^2 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Now,
$$\text{P.I.} = \frac{1}{(D - D')^2}\tan(y + x) = \frac{x^2}{1^2 \times 2!}\tan(y + x) = \frac{x^2}{2}\tan(y + x)$$

[Using formula (*ii*) of working rule with $a = 1$, $b = 1$, $m = 2$]

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times \tan(y + x)$.

(*b*) Here auxiliary equation is $(m - a)^2 = 0$ so that $m = a, a$.

$\therefore$ C.F. $= \phi_1(y + ax) + x\phi_2(y + ax)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - aD')^2} f(y + ax) + \frac{x^2}{1^2 \times 2!} f(y + ax) = \frac{x^2}{2} f(y + ax).$$

[using formula (*ii*) of working rule with $a = a$, $b = 1$, $m = 2$]

$\therefore$ General solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times f(y + ax)$.

(*c*) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x \partial y,\ t = \partial^2 z/\partial y^2$, the given equation becomes

$4(\partial^2 z/\partial x^2) - 4(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) = 16 \log(x + 2y)$ or $(4D^2 - 4DD' + D'^2)z = 16 \log(x + 2y)$

Its auxiliary equation is $4m^2 - 4m + 1 = 0$ so that $m = 1/2.\ 1/2$.

$\therefore$ C.F. $= \phi_1(2y + x) + x\phi_2(2y + x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Now,
$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x + 2y) = 16 \times \frac{x^2}{2^2 \times 2!}\log(x + 2y) = 2x^2 \log(x + 2y)$$

[using formula (*ii*) of working rule with $a = 1$, $b = 2$, $m = 2$]

$\therefore$ The required solution is $z = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2 \log(x + 2y)$.

Ex. 7. *Solve the following partial differential equations :*

(*a*) $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x + y)$. **[M.D.U. Rohtak 2005]**

(*b*) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$. **[Meerut 2006, 08]**

(*c*) $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$. **[Bhopal 2000, 03, Meerut 2007; Jabalpur 2004]**

(*d*) $r - t = x - y$.

(*e*) $2r - s - 3t = 5e^x/e^y$. **[Indore 2003; Jiwaji 2003, Vikram 1998]**

(*f*) $r + 5s + 6t = (y - 2x)^{-1}$ or $(\partial^2 z/\partial x^2) + 5(\partial^2 z/\partial x\partial y) + 6(\partial^2 z/\partial y^2) = 1/(y - 2x)$

[Agra 2009; Indore 2000; I.A.S. 1991; Garhwal 2005]

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now,
$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x + y) = 5\frac{1}{(D - 2D')}\left[\frac{1}{(2D - D')}\sin(2x + y)\right]$$

$$= 5\frac{1}{D - 2D'}\frac{1}{(2 \times 2) - 1}\int \sin v\, dv$$, where $v = 2x + y$, using formula (*i*) of working rule

$$= \frac{5}{3}\frac{1}{D-2D'}(-\cos v) = -\frac{5}{3}\frac{1}{D-2D'}\cos(2x+y) = -\frac{5}{3}\times\frac{x}{1^1\times 1!}\cos(2x+y)$$

[Using formula (*ii*) with $a = 2$, $b = 1$, $m = 2$]

$\therefore$ The required general solution is $z = \phi_1(y + 2x) + \phi_2(2y + x) - (5x/3) \times \cos(2x + y)$.

(*b*) Do as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y + 4x) - (x/3) \times \cos(4x + y)$

(*c*) Here auxiliary equation is $m^3 - 2m^2 - m + 2 = 0$ or $m^2(m - 2) - (m - 2) = 0$

or $(m^2 - 1)(m - 2) = 0$ so that $m = 2, 1, -1$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y + x) + \phi_3(y - x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3}e^{x+y} = \frac{1}{(D-D')}\left\{\frac{1}{(D^2 - DD' - 2D'^2)}e^{x+y}\right\}$$

$= \dfrac{1}{D-D'}\dfrac{1}{1^2 - (1\times 1) - (2\times 1)^2}\displaystyle\int e^v\,dv$, where $v = x + y$, using formula (*i*) of working rule

$$= \frac{1}{D-D'}\left(-\frac{1}{2}\right)e^v = -\frac{1}{2}\frac{1}{D-D'}e^{x+y} = -\frac{1}{2}\times\frac{x}{1^1\times 1!}e^{x+y} = -\frac{x}{2}e^{x+y}$$

[Using formula (*ii*) with $a = 1$, $b = 1$, $m = 1$]

$\therefore$ Required solution is $z = \phi_1(y + 2x_1) + \phi_2(y + x) + \phi_3(y - x) - (x/2) \times e^{x+y}$.

(*d*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$ or $(D^2 - D'^2)z = x - y$. ...(1)

Its auxiliary equation is $m^2 - 1 = 0$ so that $m = 1, -1$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^2 - D'^2}(x-y) = \dfrac{1}{D+D'}\cdot\dfrac{1}{D-D'}(x-y) = \dfrac{1}{D+D'}\cdot\dfrac{1}{1-(-1)}\displaystyle\int v\,dv$, where $v = x - y$.

$$= \frac{1}{2}\frac{1}{D+D'}\frac{v^2}{2} = \frac{1}{4}\frac{1}{D+D'}(x-y)^2 = -\frac{1}{4}\frac{1}{[(-1)\times D - 1\times D']^1}(x-y)^2$$

$= -\dfrac{1}{4}\dfrac{x}{(-1)^1\times 1!}(x-y)^2 = \dfrac{x}{4}(x-y)^2$, by formula (*ii*) with $a = 1$, $b = -1$, $m = 1$.

$\therefore$ The required solution is $z = \phi_1(y + x) + \phi_2(y - x) + (x/4) \times (x - y)^2$.

(*e*) The given equation can be re–written as

$2(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) - 3(\partial^2 z/\partial y^2) = 5e^{x-y}$ or $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$

or $(D + D')(2D - 3D')z = 5e^{x-y}$.

C.F. $= \phi_1(y - x) + \phi_2(2y + 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{(D+D')}\times\dfrac{1}{(2D-3D')}5e^{x-y} = 5\dfrac{1}{D+D'}\times\dfrac{1}{(2\times 1) - 3\times(-1)}\displaystyle\int e^v\,dv$, where $v = x - y$,

$$= \frac{1}{D+D'}e^v = -\frac{1}{[(-1)\times D - 1\times D']}e^{x-y} = -\frac{x}{(-1)^1\times 1!}e^{x-y} = xe^{x-y},$$

[using formula (*ii*) with $a = 1$, $b = -1$, $m = 1$]

$\therefore$ The required solution is $z = \phi_1(y - x) + \phi_2(2y + 3x) + xe^{x-y}$.

(*f*) Given equation can be rewritten as $(D^2 + 5DD' + 6D'^2)z = (y - 2x)^{-1}$. ...(1)

Its auxiliary equation is $m^2 + 5m + 6 = 0$ so that $m = -2, -3$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^2 + 5DD' + 6D'^2}(y-2x)^{-1} = \dfrac{1}{D+2D'}\left[\dfrac{1}{D+3D'}(y-2x)^{-1}\right]$

$= \dfrac{1}{D+2D'} \times \dfrac{1}{-2+(3\times 1)} \int v^{-1}\, dv$, where $v = y - 2x$, by formula (i)

$= \dfrac{1}{D+2D'} \log v = \dfrac{1}{D+2D'} \log(y-2x) = \dfrac{1}{[1\times D - (-2)\times D']^1} \log(y-2x)$

$= \dfrac{x}{1^1 \times 1!} \log(y-2x)$, by formula (ii) with $a = -2$, $b = 1$, $m = 1$

$\therefore$ The general solution is $z = \phi_1(y-2x) + \phi_2(y-3x) + x\log(y-2x)$.

Ex. 8. *Solve the following partial differential equation :*

(a) $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$.

[Bilaspur 1995; Indore 2004;Rewa 2000, 01; MDU Rohtak 2004]

(b) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+y)$.

(c) $(D^3 - 4D^2D' + 4DD'^2)z = \sin(y+2x)$.

Sol. (a) Here the auxiliary equation is $m^3 - 4m^2 + 4m = 0$ or $m(m^2 - 4m + 4) = 0$

or $m(m-2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^3 - 4D^2D' + 4DD'^2} 4\sin(2x+y) = 4\dfrac{1}{(D-2D')^2}\left\{\dfrac{1}{D}\sin(2x+y)\right\}$

$= 4\dfrac{1}{(D-2D')^2}\left\{-\dfrac{1}{2}\cos(2x+y)\right\}$, since $1/D$ stands for integration w.r.t. x treating y as constant.

$= -2\dfrac{1}{(D-2D')^2}\cos(2x+y) = -2\dfrac{x^2}{1^2\times 2!}\cos(2x+y)$,

[Using formula (ii) with $a = 2$, $b = 1$, $m = 2$]

So the required solution is $y = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - x^2\cos(2x+y)$.

(b)Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + (x^2/4)\times\sin(2x+y)$

(c)Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - (x^2/4)\times\cos(2x+y)$

Ex. 9. *Solve the following partial differential equations :*

(a) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x+2y)$.

(b) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + \cos(x+2y)$ **[Delhi Maths (H) 2006]**

(c) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{1/2}$.

(d) $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$. **[I.A.S. 1995]**

Sol. (a) Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, P.I. corresponds to e^{2x-y}

$= \dfrac{1}{D^2 - 3DD' + 2D'^2}e^{2x-y} = \dfrac{1}{2^2 - 3\times 2\times(-1) + 2\times(-1)^2}\iint e^v\, dv\, dv$, where $v = 2x - y$

$= (1/12)\times\int e^v dv = (1/12)\times e^v = (1/12)\times e^{2x-y}$. ...(2)

P.I. corresponding to e^{x+y} $= \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{x+y} = \dfrac{1}{D-D'}\left\{\dfrac{1}{D-2D'} e^{x+y}\right\}$

$= \dfrac{1}{D-D'}\left\{\dfrac{1}{1-(2\times 1)}\int e^v\, dv\right\}$, where $v = x + y$, using formula (i)

$= -\dfrac{1}{D-D'} e^v = -\dfrac{1}{(D-D')^1} e^{x+y} = -\dfrac{x}{1^1 \times 1!} e^{x+y} = -xe^{x+y}$. ...(3)

[Using formula (ii) with $a = b = 1$, $m = 1$]

Finally, P.I. corresponding to $\cos(x + 2y)$

$= \dfrac{1}{D^2 - 3DD' + 2D'^2}\cos(x+2y) = \dfrac{1}{1^2 - (3\times 1\times 2) + (2\times 2^2)}\iint \cos v\, dv\, dv$, where $v = x + 2y$

$= (1/3)\times \int \sin v dv = -(1/3)\times \cos v = -(1/3)\times \cos(x+2y)$ ...(4)

From (1), (2), (3) and (4), the required solution is z = C.F. + P.I.

or $z = \phi_1(y + x) + \phi_2(y + 2x) + (1/12) \times e^{2x-y} - xe^{x+y} - (1/3) \times \cos(x + 2y)$.

(b) This problem is same as part (a) except that the term e^{x+y} is missing on R.H.S. So, now you need not compute P.I. corresponding to e^{x+y}. Therefore the solution will take the form

$$y = \phi_1(y+x) + \phi_2(y+2x) + (1/12)\times e^{2x-y} - (1/3)\times\cos(x+2y)$$

(c) Here auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$ giving $m = 1, 1, 2$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x)$., ϕ_1, ϕ_2, ϕ_3 being arbitrary function ...(1)

Now, P.I. corresponding to e^{y+2x}

$= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{y+2x} = \dfrac{1}{D-2D'}\left\{\dfrac{1}{(D-D')^2} e^{y+2x}\right\}$

$= \dfrac{1}{D-2D'}\dfrac{1}{(2-1)^2}\iint e^v\, dv\, dv$, where $v = y + x$, by formula (i)

$= \dfrac{1}{D-2D'}\int e^v\, dv = \dfrac{1}{D-2D'} e^v = \dfrac{1}{(1\times D - 2\times D')^1} e^{y+2x} = \dfrac{x}{1\times 1!} e^{y+x} = xe^{y+x}$, ...(2)

[using formula (ii) with $a = 2$, $b = 1$, $m = 1$]

Finally, P.I. corresponding to $(y + x)^{1/2}$

$= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3}(y+x)^{1/2} = \dfrac{1}{(D-D')^2}.\left\{\dfrac{1}{D-2D'}(y+x)^{1/2}\right\}$

$= \dfrac{1}{(D-D')^2}\times\dfrac{1}{1-(2\times 1)}\int v^{1/2}\, dv$, where $v = y + x$, using formula (i)

$= -\dfrac{1}{D-D'}\times\dfrac{2}{3}v^{3/2} = -\dfrac{2}{3}\dfrac{1}{(D-D')^2}(y+x)^{3/2} = -\dfrac{2}{3}\times\dfrac{x^2}{1^2\times 2!}(y+x)^{3/2}$

$= -(x^2/3) \times (y + x)^{3/2}$, using formula ($ii$) with $a = b = 1$, $m = 2$ ...(3)

From (1), (2) and (3), the required general solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x) + xe^{y+x} - (x^2/3) \times (y + x)^{3/2}.$$

(d) Here note that D_x and D_y stand for D and D' respectively.

$\therefore$ Auxiliary equation is $\quad m^3 - 7m - 6 = 0 \quad$ so that $\quad m = -1, -2, 3.$

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. corresponding to $\sin(x + 2y)$

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}\sin(x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (6\times 2^3)}\iiint \sin v\, dv\, dv\, dv, \text{where } v = x + 2y$$

$$= -(1/75)\times\iint(-\cos v)\, dv\, dv = -(1/75)\times\int(-\sin v)\, dv = -(1/75)\times\cos v = -(1/75)\times\cos(x+2y)$$

and P.I. corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{D_x - 3D_y}\left[\frac{1}{(D_x + D_y)(D_x + 2D_y)}e^{3x+y}\right]$$

$$= \frac{1}{D_x - 3D_y}\cdot\frac{1}{(3+1)(3+2)}\iint e^v\, dv\, dv, \text{ where } v = 3x + y, \text{ by formula } (i)$$

$$= \frac{1}{20}\frac{1}{D_x - 3D_y}\int e^v\, dv = \frac{1}{20}\frac{1}{D_x - 3D_y}e^v = \frac{1}{20}\frac{1}{(D_x - 3D_y)^1}e^{3x+y}$$

$$= \frac{1}{20}\times\frac{x}{1^1\times 1!}e^{3x+y} = \frac{x}{20}e^{3x+y}, \text{ using formula } (ii) \text{ with } a = 2,\ b = 1,\ m = 1.$$

Hence the required general solution is $\quad z = \text{C.F.} + \text{P.I.}$

or $\quad z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - (1/75)\times\cos(x + 2y) + (1/20)\times x\, e^{3x+y}.$

Ex. 10. *Solve (i)* $(D^2 - 6DD' + 9D'^2)z = \tan(y + 3x)$ **[Delhi 2007; Ravishankar 2004]**

(ii) $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

Sol (i) Here auxiliary equation is $\quad (m - 3)^2 = 0 \quad$ so that $\quad m = 3, 3.$

$\therefore$ C.F. $= \phi_1(y + 3x) + x\,\phi_2(y + 3x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 3D')^2}\tan(y + 3x) = \frac{x^2}{1^2\times 2!}\tan(y + 3x) = \frac{x^2}{2}\tan(y + 3x)$$

$\therefore$ The required solution is $\quad z = \phi_1(y + 3x) + x\,\phi_2(y + 3x) + (x^2/2)\times\tan(y + 3x)$.

(ii) Re-writing the given equation reduces to $\quad (D - 3D')^2 z = 2(3x + y)$

$\therefore$ C.F. $= \phi_1(y + 3x) + x\phi_1(y + 3x)$, ϕ_1, ϕ_2 being arbitrars constants.

Now, $$P.I. = \frac{1}{(D - 3D')^2}\, 2(3x + y) = 2\,\frac{x^2}{1^2\times 2!}(3x + y) = x^2(3x + y)$$

$\therefore$ The required solution is $\quad z = \phi_1(y + 3x) + x\phi_2(y + 3x) + 3x^3 + x^2 y.$

Ex. 11. *Solve (i)* $(D - 3D')^2(D + 3D')z = e^{3x+y}$ **[Delhi Maths (Hons) 2000, Agra 2005]**

(ii) $(D - 2D')(D + D')^2 z = \cos(2x + y)$

Sol. (i) C.F. $= \phi_1(y + 3x) + x\,\phi_2(y + 3x) + \phi_3(y - 3x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 3D')^2}\frac{1}{D + 3D'}e^{3x+y} = \frac{1}{(D - 3D')^2}\frac{1}{3 + (3\times 1)}\int e^v dv, \text{ where } v = 3x + y$$

$$= \frac{1}{6}\frac{1}{(D-3D')^2}e^v = \frac{1}{6}\frac{1}{(D-3D')^2}e^{3x+y} = \frac{1}{6}\frac{x}{1\times 2!}e^{3x+y}$$

The required solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + (x^2/12)\times e^{3x+y}$

(*ii*) Try your self **Ans.** $z = \phi_1\ (y+2x) + \phi_2(y-x) + x\ \phi_3(y-x) - (x/9)\times\cos(2x+y)$

Ex. 12. Solve (*i*) $r + s - 2t = e^{x+y}$

(*ii*) $(D^3 - 7DD'^2 - 6D'^3)y = \sin(x+2y)$

(*iii*) $(D^3 - 3DD'^2 + 2D'^3)y = (x-2y)^{1/2}$ **[Delhi Maths (H) 2009]**

Sol. (*i*) Re-writing given equation becomes $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial x\,\partial y) - 2\,(\partial^2 z/\partial y^2) = e^{x+y}$

or $(D^2 + DD' - 2D'^2)\ z = e^{x+y}$ or $(D-D')\ (D+2D')z = e^{x+y}$.

Its C.F. $= \phi(y+x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D-D'}\frac{1}{D+2D'}e^{x+y} = \frac{1}{D-D'}\frac{1}{1+(2\times 1)}\int e^v dv, \text{ where } v = x+y$$

$$= \frac{1}{3}\frac{1}{D-D'}e^v = \frac{1}{3}\frac{1}{D-D'}e^{x+y} = \frac{1}{3}\frac{x}{1!}e^{x+y}$$

$\therefore$ The required solution is $z = \phi_1\ (y+x) + \phi_2(y-2x) + (x/3)\times e^{x+y}$.

(*ii*) Here the auxiliary equation is $m^3 - 7m - 6 = 0$ giving $m = -1, -2, 3.$

$\therefore$ C.F. $= \phi_1\ (y-x) + \phi_2\ (y-2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin\ (x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (16\times 2^3)}\int\!\!\iint \sin v (dv)^3, \text{ where } v = x + 2y$$

$$= -\frac{1}{75}\iint(-\cos v)dv\ dv = -\frac{1}{75}\int\ (-\sin v)\ dv = -\frac{1}{75}\cos v = -\frac{1}{75}\cos\ (x+2y)$$

$\therefore$ Required solution $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - (1/75)\times\cos(x+2y)$

(*iii*) The auxiliary equation is $m^3 - 3m + 2 = 0$ giving $m = 1, 1, 2.$

$\therefore$ C.F. $= \phi_1(y+x) + x\ \phi_2(y+x) + \phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

$$\text{P.I.} = \frac{1}{D^3 - 3DD'^2 + 2D'^3}(x-2y)^{1/2} = \frac{1}{1^3 - 3\times 1\times(-2)^2 + 2\times(-2)^3}\iiint v^{1/2}(dv)^3, \text{ where } v = x - 2y$$

$$= -\frac{1}{27}\iint\frac{v^{3/2}}{(3/2)}dv\ dv = -\frac{1}{27}\int\frac{v^{5/2}}{(3/2)\times(5/2)}dv = -\frac{1}{27}\frac{v^{7/2}}{(3/2)\times(5/2)\times(7/2)}$$

$$= -(8/2835)\times v^{7/2} = -(8/2835)\times(x-2y)^{7/2}$$

General solution is $z = \phi_1\ (y+x) + x\ \phi_2\ (y+x) + \phi_3(y+2x) - (8/2835)\times(x-2y)^{7/2}$.

Ex. 13. *Solve* $(D^2 - 3DD' + 2D'^2)z = \cos\ (x+2y)$

Sol. The auxiliary equation $m^2 - 3m + 2 = 0$ gives $m = 1, 2.$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2}\cos(x+2y) = \frac{1}{1^2 - 3\cdot 1\cdot 2 + 2\cdot 2^2}\iint \cos v (dv)^2, \text{ where } v = x + 2y$$

$$= (1/3)\times\int \sin v\, dv = -(1/3)\times\cos v = -(1/3)\times\cos(x+2y)$$

$\therefore$ Solution is $\quad z = \phi_1(y+x) + \phi_2(y+2x) - (1/3)\times\cos(x+2y)$

Ex. 14. *Solve* $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$. **[I.A.S. 2000]**

Sol. The auxiliary equation $\quad m^2 - m - 2 = 0 \quad$ giving $\quad m = 2, -1$.

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. corresponding to $(2x + 3y)$

$$= \frac{1}{D^2 - DD' - 2D'^2}(2x+3y) = \frac{1}{2^2 - (2\times 3) - (2\times 3^2)}\iint v\,(dv)^2, \text{ where } v = 2x+3y$$

$$= -\frac{1}{20}\int \frac{v^2}{2}\,dv = -\frac{1}{20}\left(\frac{v^3}{2\times 3}\right) = -\frac{1}{60}(2x+3y)^3$$

P.I. corresponding to e^{3x+4y}

$$= \frac{1}{D^2 - DD' - 2D'^2}e^{3x+4y} = \frac{1}{3^2 - (3\times 4) - (2\times 4^2)}\iint e^v (dv)^2, \text{ where } v = 3x+4y$$

$$= -(1/35)\times e^v = -(1/35)\times e^{3x+4y}.$$

$\therefore$ General solution is $\quad z = \phi_1(y+2x) + \phi_2(y-x) - (1/60)\times(2x+3y)^3 - (1/35)\times e^{3x+4y}$.

EXERCISE 4(A)

Solve the following partial differential equations:

1. $(D^2 - DD' - 6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x) + \phi_2(y-2x) - (1/4)\times\cos(2x+y)$ ϕ_1, ϕ_2, being arbitrary functions.

2. $r - 4s + 4t = e^{2x+y}$ **[Agra 2010]**

Ans. $z = \phi_1(y+2x) + x\phi_2(y+2x) + (x^2/2)\times e^{2x+y}$ ϕ_1, ϕ_2 being arbitrary functions

3. $(D^3 - 4D^2D'^2 + 4DD'^2)z = 6\sin(3x+2y)$

Ans. $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + 2\cos(3x+2y)$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4. $(D - 3D')^2(D + 3D')z = e^{3x+y}$ **[Agra 2005]**

Ans. $z = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + (x/12)\times e^{3x+y}$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4.10. Short Method II. When f(x, y) is of the form $x^m y^n$ or a rational integral algebraic function of *x* and *y*.

Then the particular integral (P.I.) is evaluated by expanding the symbolic function $1/f(D, D')$ in an infinite series of ascending powers of D or D'. In solved examples 1 and 2 of Art. 4.11, we have shown that P.I. obtained on expanding $1/f(D, D')$ in ascending powers of D is different from that obtained on expanding $1/f(D, D')$ in ascending powers of D'. Since to get the required general solution of given differential equation any P.I. is required, any of the two methods can be used. The difference in the two answers of P.I. is not material as it can be incorporated in the arbitrary functions occuring in C.F. of that given differential equation.

Remark : If $n < m$, $1/f(D, D')$ should be expanded in powers of D'/D whereas if $m < n$, $1/f(D, D')$ should be expanded in powers of D/D'.

4.11 SOLVED EXAMPLES BASED ON SHORT METHOD II

Ex. 1. *Solve* $(D^2 - a^2D'^2)z = x$ or $(\partial^2 z/\partial x^2) - a^2(\partial^2 z/\partial y^2) = x$.

Sol. Here auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ C.F. $= \phi_1(y + ax) + \phi_2(y - ax)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

$$\text{Now, P.I.} = \frac{1}{D^2 - a^2D'^2}x = \frac{1}{D^2[1 - a^2(D'^2/D^2)]}x = \frac{1}{D^2}\left(1 - a^2\frac{D'^2}{D^2}\right)^{-1}x$$

$$= \frac{1}{D^2}\left(1 + a^2\frac{D'^2}{D^2} + \dots\right)x = \frac{1}{D^2}x = \frac{x^3}{6}. \qquad \text{...(2)}$$

Alternatively, we can compute P.I. on follows :

$$\text{P.I.} = \frac{1}{D^2 - a^2D'^2}x = \frac{1}{-a^2D'^2[1 - (D^2/a^2D'^2)]}x = -\frac{1}{a^2D'^2}\left(1 - \frac{D^2}{a^2D'^2}\right)^{-1}x$$

$$= -\frac{1}{a^2D'^2}\left(1 + \frac{D^2}{a^2D'^2} + \dots\right)x = -\frac{1}{a^2D'^2}x = -\frac{1}{a^2}\times\frac{xy^2}{2}. \qquad \text{...(3)}$$

Hence the required general solution is $z =$ C.F. + P.I. that is,

$z = \phi_1(y + ax) + \phi_2(y - ax) + x^3/6$, using (1) and (2).

or $z = \phi_1(y + ax) + \phi_2(y - ax) - (xy^2)/(2a^2)$, using (1) and (3).

Ex. 2. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$, *by expanding the particular integral in ascending powers of D as well as in ascending powers of D'.*

[Bhopal 2000, 03; Indore 1999; Jiwaji 1995; Rewa, 2002, 03; I.A.S. 1994]

Sol. Here auxiliary equation is $m^2 + 3m + 2 = 0$ so that $m = -2, -1$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, by expanding in ascending powers of D, we have

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \frac{1}{2D'^2\left[1 + \left(\dfrac{D^2}{2D'^2} + \dfrac{3}{2}\dfrac{D}{D'}\right)\right]}(x + y)$$

$$= \frac{1}{2D'^2}\left[1 + \left(\frac{D^2}{2D'^2} + \frac{3}{2}\frac{D}{D'}\right)\right]^{-1}(x + y) = \frac{1}{2D'^2}\left(1 - \frac{3}{2}\frac{D}{D'} + \dots\right)(x + y)$$

$$= \frac{1}{2D'^2}\left(x + y - \frac{3}{2}y\right) = \frac{1}{2D'^2}\left(x - \frac{y}{2}\right) = \frac{xy^2}{4} - \frac{y^3}{24}. \qquad \text{...(2)}$$

Again, by expanding in ascending powers of D, P.I. of given equation is given by

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \frac{1}{D^2\left[1 + \left(\dfrac{3D'}{D} + \dfrac{2D'^2}{D^2}\right)\right]}(x + y) = \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x + y)$$

$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D} + \dots\right)(x + y) = \frac{1}{D^2}(x + y - 3x) = \frac{yx^2}{2} - \frac{x^3}{3}.$$

Hence the required general solution is given by $z =$ C.F. + P.I., *i.e.*,

$z = \phi_1(y - 2x) + \phi_2(y - x) + (1/4)\times xy^2 - (1/24)\times y^3$, using (1) and (2)

or $z = \phi_1(y - 2x) + \phi_2(y - x) + (1/2)\times yx^2 - (1/3)\times x^3$, using (1) and (3).

Ex. 3. *Solve* $(\partial^3 z/\partial x^3) - (\partial^3 z/\partial y^3) = x^3y^3$ or $(D^3 - D'^3)z = x^3y^3$. **[I.A.S. 1997]**

Sol. Here auxiliary equation is $m^3 - 1 = 0$ so that $m = 1, \omega, \omega^2$, where ω and ω^2 are complex cube roots of unity.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^3 - D'^3}x^3y^3 = \frac{1}{D^3[1-(D'^3/D^3)]}x^3y^3 = \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1}x^3y^3$$

$$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3} + \ldots\right)x^3y^3 = \frac{1}{D^3}\left(x^3y^3 + \frac{1}{D^3}6x^3\right) = \frac{1}{D^3}\left(x^3y^3 + 6\times\frac{x^6}{4\times 5\times 6}\right)$$

$$= (1/120)\times x^6y^3 + (1/10080)\times x^9.$$

Hence the required general solution is $z =$ C.F. + P.I.

or $z = \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x) + (1/120)\times x^6y^3 + (1/10080)\times x^9$.

Ex. 4. *Solve* $r + (a + b)s + abt = xy$. **[Indore 1998; Vikram 1998, 2000; Rewa 1998]**

Sol. Given equation can be written as $[D^2 + (a + b)DD' + abD'^2] = xy$.

Its auxiliary equation is $m^2 + (a + b)m + ab = 0$ or $(m + a)(m + b) = 0$ so that $m = -a, -b$.

$\therefore$ C.F. $= \phi_1(y - ax) + \phi_2(y - bx)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^2 + (a+b)DD' + abD'^2}xy = \frac{1}{D^2\left[1 + (a+b)\dfrac{D'}{D} + ab\dfrac{D'^2}{D^2}\right]}xy$$

$$= \frac{1}{D^2}\left[1 + (a+b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right]^{-1}xy = \frac{1}{D^2}\left[1 - (a+b)\frac{D'}{D} + \ldots\right]xy$$

$$= \frac{1}{D^2}\left\{xy - \frac{a+b}{D}D'(xy)\right\} = \frac{1}{D^2}\left\{xy - \frac{a+b}{D}x\right\} = \frac{1}{D^2}\left\{xy - \frac{(a+b)x^2}{2}\right\}$$

$$= y\times\frac{x^3}{2\times 3} - \frac{a+b}{2}\times\frac{x^4}{3\times 4} = \frac{x^3y}{6} - \frac{(a+b)x^4}{24}.$$

Required general solution $z = \phi_1(y - ax) + \phi_2(y - bx) + (1/6)\times x^3y - (a + b)\times(x^4/24)$,

Ex. 5. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

(*b*) $(D^2 + 3DD' + 2D'^2)z = x + y$. **[Meerut 1996]**

(*c*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x\partial y) + 2(\partial^2 z/\partial y^2) = 2x + 3y$.

(*d*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x\partial y) + 2(\partial^2 z/\partial y^2) = 6(x + y)$.

(*e*) $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$.

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}24(y - x) = 24\frac{1}{2D^2\left(1 - \dfrac{5D'}{2D} + \dfrac{D'^2}{D^2}\right)}(y - x)$$

$$= \frac{12}{D^2}\left(1 - \frac{5D'}{2D} + \frac{D'^2}{D^2}\right)^{-1}(y-x) = \frac{12}{D^2}\left(1 + \frac{5D'}{2D} + \ldots\right)(y-x) = \frac{12}{D^2}\left\{(y-x) + \frac{5}{2D}D'(y-x)\right\}$$

$$= \frac{12}{D^2}\left(y - x + \frac{5}{2D}1\right) = \frac{12}{D^2}\left(y - x + \frac{5x}{2}\right) = \frac{12}{D^2}\left(y + \frac{3x}{2}\right) = \left(12y \times \frac{x^2}{2}\right) + 18 \times \left(\frac{x^3}{2 \times 3}\right)$$

Hence the required general solution is $z = \phi_1(y+2x) + \psi_2(2y+x) + 6x^2y + 3x^3$.

(*b*)Try as in part (*a*). **Ans.** $z = \phi_1(y-2x) + \phi_2(y-x) - (1/3) \times x^3 + (1/2) \times x^2 y$

(*c*) Try yourself **Ans.** $z = \phi_1(y-x) + \phi_2(y-2x) + (3/2) \times x^2 y - (7/6) \times x^3$

(*d*) Try as in part (*c*). [**Ans.** $z = \phi_1(y-x) + \phi_2(y-2x) + 3x^2 y - 2x^3$

(*e*) Try yourself. **Ans.** $z = \phi_1(y+x) + \phi_2(y-x) + (1/6) \times x^3 - (1/2) \times x^2 y$

Ex. 6. *Solve* $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$. **[Meerut 1994, Bhuj 1999, Jabalpur 2003]**

Sol. Re-writing the given equation, we get $(D - 3D')^2 z = 12(x^2 + 3xy)$.

Its auxiliary equation is $(m-3)^2 = 0$ so that $m = 3, 3$.

$\therefore$ C.F. = $\phi_1(y+3x) + x\phi_2(y+3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I.$= \dfrac{1}{(D-3D')^2} 12(x^2+3xy) = 12\dfrac{1}{D^2(1-3D'/D)^2}(x^2+3xy)$

[Take D common as power of y is less than that of x]

$$= \frac{12}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}(x^2+3xy) = \frac{12}{D^2}\left(1 + 6\frac{D'}{D} + \ldots\right)(x^2+3xy)$$

[Retain upto D' as maximum power of y in (x^2+3xy) is one]

$$= \frac{12}{D^2}\left\{x^2 + 3xy + \frac{6}{D}D'(x^2+3xy)\right\} = \frac{12}{D^2}\left\{x^2 + 3xy + \frac{6}{D}(3x)\right\} = \frac{12}{D^2}\left\{x^2 + 3xy + 18 \times \frac{x^2}{2}\right\}$$

$$= \frac{12}{D^2}(10x^2 + 3xy) = 120\left(\frac{x^4}{3 \times 4}\right) + 36y\left(\frac{x^3}{2 \times 3}\right) = 10x^4 + 6x^3 y.$$

Hence the required general solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + 10x^4 + 6x^3 y$.

Ex. 7. (*a*) Solve $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = -4\pi(x^2+y^2)$.

(*b*) Find a real function V of x and y, satisfying $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = -4\pi(x^2+y^2)$ and reducing to zero, when $y = 0$. **[Nagpur 2005; I.A.S. 1998]**

Sol. (*a*) Given equation can be rewritten as $(D^2 + D'^2)V = -4\pi(x^2+y^2)$. ...(1)

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = i, -i$.

$\therefore$ C.F. $=\phi_1(y+ix) + \phi_2(y-ix)$, where ϕ_1, ϕ_2 are arbitrary functions.

P.I. $= \dfrac{1}{D^2 + D'^2}[-4\pi(x^2+y^2)] = -4\pi\dfrac{1}{D^2+D'^2}(x^2+y^2) = -4\pi\dfrac{1}{D^2(1+D'^2/D^2)}(x^2+y^2)$

$$= -4\pi \frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(x^2+y^2) = -\frac{4\pi}{D^2}\left(1-\frac{D'^2}{D^2}+\ldots\right)(x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}D'^2(x^2+y^2)\right\} = -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}.2\right\}$$

$$= -\frac{4\pi}{D^2}\left(x^2+y^2-2\times\frac{x^2}{2}\right) = -\frac{4\pi}{D^2}y^2 = -4\pi y^2 \times \frac{x^2}{2} = -2\pi^2 x^2 y^2 .$$

Hence the required general solution is $V = \phi_1(y+ix)+\phi_2(y-ix)-2\pi^2x^2y^2$. ...(2)

(b)Proceed as in part (a) upto equation (2).Since we want real function $V(x, y)$ satisfying (1) and reducing to zero when $y = 0$, it follows $\phi_1(y + ix) = \phi_2(y - ix) = 0$ in (2) and hence the required solution is $V = -2\pi^2x^2y^2$.

Ex. 8. *Solve* $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$. **[Bhopal 1995, 97, 98; Lucknow 2010]**

Sol. Given equation is $(D - D')^2 z = e^{x+2y} + x^3$. ...(1)

Its auxiliary equation is $(m-1)^2 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}\int\int e^v \, dv \, dv, \text{ where } v = x+2y$$

$= \int e^v\, dv = e^v = e^{x+2y}$, as $v = x + 2y$, using formula (i) of working rule of Art. 4.8 and P.I. corresponding to x^3

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2(1-D'/D)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3 = \frac{1}{D^2}(1+\ldots)x^3 = \frac{x^5}{20}$$

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) + e^{x+2y} + x^5/20$.

Ex. 9. *Solve* $\partial^2 z/\partial x^2 - a^2(\partial^2 z/\partial y^2) = x^2$. **[Ravishanker 2000, 04; Vikram 1995]**

Sol. Let $D \equiv \partial/\partial x$ and $D' = \partial/\partial y$. Then, the given equation becomes $(D^2 - a^2D'^2)y = x^2$

The auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ $C.F. = \phi_1(y+ax)+\phi_2(y-ax)$, ϕ_1, ϕ_2 being arbitrary functions

Now,
$$P.I. = \frac{1}{(D^2-a^2D'^2)}x^2 = \frac{1}{D^2\{1-a^2(D'^2/D^2)\}}x^2 = \frac{1}{D^2}\left(1-a^2\frac{D'^2}{D^2}\right)^{-1}x^2$$

$$= \frac{1}{D^2}\left(1+a^2\frac{D'^2}{D^2}+\ldots\right)x^2 = \frac{1}{D^2}x^2 = \frac{1}{D}\frac{x^3}{3} = \frac{x^4}{12}$$

Hence, the required general solution is $z = C.F. + P.I.$ i.e., $z = \phi_1(y+ax)+\phi_2(y-ax)+x^4/12$.

Ex. 10. *Solve* $\partial^3 z/\partial x^2\partial y - 2(\partial^3 z/\partial x\partial y^2) + \partial^3 z/\partial y^3 = 1/x^2$ **(Vikram 1994)**

Sol. Let $D = \partial/\partial x$ and $D' = \partial/\partial y$. Then the given equation becomes

$(D^2D' - 2DD'^2 + D'^3)z = 1/x^2$ or $(D-D')^2D'z = 1/x^2$ ...(1)

Corresponding to repeated factor $(D-D')^2$, the part of C.F. is $\phi_1(y+x)+x\phi_2(y+x)$. Again corresponding to factor D', the part of $C.F.$ is $f_1(x)$.

$\therefore \qquad C.F.$ of (1) $= \phi_1(y+x)+x,\ \phi_2(y+x)+\phi_3(x)$

$$\text{P.I.}=\frac{1}{(D-D')^2 D'}\frac{1}{x^2}=\frac{1}{(D-D')^2}\int\frac{1}{x^2}dy=\frac{1}{(D-D')^2}\frac{y}{x^2}=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}\frac{y}{x^2}$$

$$=\frac{1}{D^2}\left(1+\frac{2D'}{D}+\frac{3D'^2}{D^2}+\dots\right)\frac{y}{x^2}=\frac{1}{D^2}\left\{\frac{y}{x^2}+\frac{2}{D}\left(\frac{1}{x^2}\right)\right\}=y\frac{1}{D^2}\frac{1}{x^2}+\frac{2}{D^3}\frac{1}{x^2}$$

$$=y\frac{1}{D}\left(-\frac{1}{x}\right)=-y\log x,$$

where we have omitted a function of x as it can be included in the term $\phi_3(x)$ of C.F.

$\therefore$ Required general solution is $\qquad z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(x)-y\log x,$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 11. *Solve* $(D^3-7DD'^2-6D'^3)z=x^2+xy^2+y^3+\cos(x-y)$ **[K.U. Kurukshetra, 2004]**

Sol. Given $\qquad (D^3-7DD'^2-6D'^3)\,z=x^2+xy^2+y^3+\cos(x-y) \qquad$ ...(1)

Here auxiliary equation is $\qquad m^3-7m-6=0 \qquad$ so that $\qquad m=-1,-2,3.$

$\therefore\ C.F.=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$, ϕ_1,ϕ_2,ϕ_3 being arbitrary functions

P.I. Corresponding to $(x^2+xy^2+y^3)$

$$=\frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3)=\frac{1}{D^3}\left\{1-\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)\right\}^{-1}(x^2+xy^2+y^3)$$

$$=\frac{1}{D^3}\left\{1+\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)+\dots\right\}(x^2+xy^2+y^3)=\frac{1}{D^3}(x^2+xy^2+y^3)+\frac{7}{D^5}(2x+6y)+\frac{36}{D^6}1$$

$$=(x^5/60+x^4y^2/24+x^3y^3/6)+7(x^6/360+x^5y/20)+36\times(x^6/720)$$

$$=5x^6/72+x^5/60+7x^5y/20+x^4y^2/24+x^3y^3/6$$

P.I. Corresponding to $\cos(x-y)$

$$=\frac{1}{D^3-7DD'^2-6D'^3}\cos(x-y)=\frac{1}{(D+D')}\frac{1}{(D^2-DD'-6D'^2)}\cos(x-y)$$

$$=\frac{1}{D+D'}\frac{1}{1^2-4\times1\times(-1)-6\times(-1)^2}\iint\cos v\,dv\,dv,\ \text{where } v=x-y$$

$$=\frac{1}{D+D'}\times\frac{1}{(-4)}\times(-\cos v)=\frac{1}{4}\frac{1}{D+D'}\cos(x-y)-\frac{1}{4}\frac{1}{(-1)\times D-1\times D'}\cos(x-y)=-\frac{1}{4}\frac{x}{(-1)^1\times1!}\cos(x-y)$$

$$=(x/4)\times\cos(x-y)$$

Hence the required general solution is $\qquad z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

$$+(5/72)\times x^6+x^5/60+(7/20)\times x^5y+(1/24)\times x^4y^2+(1/6)\times(x^3y^3)+(x/4)\times\cos(x-y),$$

Ex 12. *Solve* $(D^2 - 2DD' - 15D'^2)z = 12xy$. **(K.U. Kurukshetra 2004; Meerut 2006, 2011)**

Sol. Here auxiliary equation is $m^2 - 2m - 15 = 0$ so that $m = 5, -3$.

$\therefore$ $C.F. = \phi_1(y+5x) + \phi_2(y-3x)$, ϕ_1, ϕ_2 being arbitrary functions

Now,
$$\text{P.I.} = \frac{1}{D^2 - 2DD' - 15D'^2}(12xy) = \frac{1}{D^2(1 - 2D'/D - 15D'^2/D^2)}(12xy)$$

$$= \frac{12}{D^2}\left\{1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right\}^{-1}(xy) = \frac{12}{D^2}\left(1 + \frac{2D'}{D} + \frac{15D'^2}{D^2} + \dots\right)(xy)$$

$$= \frac{12}{D^2}\left\{xy + \frac{2}{D}D'(xy) + \frac{15}{D^2}D'^2(xy) + \dots\right\} = \frac{12}{D^2}\left(xy + \frac{2}{D}x\right)$$

$$= 12y\frac{1}{D^2}x + \frac{24}{D^3}x = 12y\left(\frac{x^3}{6}\right) + 24\left(\frac{x^4}{24}\right) = 2x^3y + x^4$$

Hence the required general solution is $z = \phi_1(y+5x) + \phi_2(y-3x) + 2x^3y + x^4$

Ex. 13. *Solve* $\partial^3 u/\partial x^3 + \partial^3 u/\partial y^3 + \partial^3 u/\partial z^3 - 3(\partial^3 u/\partial x\,\partial y\,\partial z) = x^3 + y^3 + z^3 - 3xyz$.

Sol. Let $D = \partial/\partial x$, $D' = \partial/\partial y$, $D'' = \partial/\partial z$. Then the given equation can be re-written as

$$(D^3 + D'^3 + D''^3 - 3DD'D'')u = x^3 + y^3 + z^3 - 3xyz.$$

or
$$(D + D' + D'')(D + \omega D' + \omega^2 D'')(D + \omega^2 D' + \omega D'')u = x^3 + y^3 + z^3 - 3xyz,$$

where ω is a complex cube root of unity. ...(1)

For C.F., let us consider $(D + \omega^2 D' + \omega D'')u = 0.$ ...(2)

Subsidiary equations of (2) are $\dfrac{dx}{1} = \dfrac{dy}{\omega^2} = \dfrac{dz}{\omega} = \dfrac{du}{0}$. ...(3)

Three independent integrals of (3) are

$y - \omega^2 x = \text{constant}$, $z - \omega x = \text{constant}$ and $u = \text{constant}$.

Hence, general solution of (2) is $u = \phi_1(y - \omega^2 x, z - \omega x)$.

Similarly, the contributions to complementary function corresponding to other factors in (1) are $\phi_2(y - \omega x, z - \omega^2 x)$ and $\phi_3(y - x, z - x)$ and hence

C.F. $= \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x) + \phi_3(y - x, z - x)$,

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

P.I. corresponding to x^3

$$= \frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}x^3 = \frac{1}{D^3}\left\{1 + \left(\frac{D'^3}{D^3} + \dots\right)\right\}^{-1}x^3 = \frac{1}{D^3}x^3 = \frac{x^6}{120}.$$

Similarly, P.I. corresponding to $y^3 = y^6/120$ and P.I. corresponding to $z^3 = z^6/120$.

Finally, P.I. corresponding to $(-3xyz)$

$$= -3\frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}xyz = \frac{1}{DD'D''}\left\{1 + \frac{D^3 + D'^3 + D''^3}{3DD'D''}\right\}^{-1}xyz$$

$$= \frac{1}{DD'D''}\left\{1 - \frac{D^3 + D'^3 + D''^3}{3DD'D''} + \dots\dots\right\}xyz = \frac{1}{DD'D''}xyz = \frac{x^2y^2z^2}{8}.$$

Hence the required general solution of (1) is $z = \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x)$
$+ \phi_3(y - x, z - x) + (1/120) \times (x^6 + y^6 + z^6) + (1/8) \times x^2 y^2 z^2$.

Ex.14. *Solve* $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial^3 u}{\partial y^3} + \dfrac{\partial^3 u}{\partial z^3} - 3\dfrac{\partial^3 u}{\partial x \partial y \partial z} = x^3 - 3xyz$.

Sol. Proceed as in Ex. 13. Its general solution is $z = \phi(y - w^2 x, z - wx) + \phi_2(y - wx, z - w^2 x) + \phi_3(y - x, z - x) + (1/120)x^6 + (1/8)x^2 y^2 z^2$.

EXERCISE 4(B)

Solve the following partial differential equations:

1. $(D^2 - 2DD' + D'^2)z = 12xy$ **(Jiwaji 1998; Ravishanker 1999; Vikram 1995, 97)**

Ans. $z = \phi_1(y + x) + x\,\phi_2(y + x) + 2x^3 y + x^4$, ϕ_1, ϕ_2 being arbitrary functions

2. $(D^2 - DD' - 6D'^2)z = xy$ **(Sagar 2003, Vikram 1995)**

Ans. $z = \phi_1(y - 2x) + \phi_2(y + 3x) + (1/6) \times x^3 y + (1/24) \times x^4$

4.12. A general method of finding the particular integral of linear homogeneous equation with constant coefficients.

Let the given equation be $\quad F(D, D')z = f(x, y)$, ...(1)

where $F(D, D')$ is a homogeneous function of D and D' of degree n, (say) so that

$$F(D, D') = (D - m_1 D')(D - m_2 D') \ldots. (D - m_n D').$$

$\therefore$ P.I. of (1) $= \dfrac{1}{F(D, D')} f(x, y) = \dfrac{1}{(D - m_1 D')(D - m_2 D')\ldots.(D - m_n D')} f(x, y)$. ...(2)

In order to evaluate P.I. given by (2), we consider a solution of the following equation :

$(D - mD')z = f(x, y)$ or $p - mq = f(x, y)$, ...(3)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equation for (3) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)}. \qquad \text{...(4)}$$

Taking the first two fractions of (4), $\quad dy + mdx = 0 \quad$ so that $\quad y + mx = c$. ...(5)

Next, taking the first and the last fractions of (4), we have

$$dz = f(x, y)dx = f(x, c - mx)dx, \text{ as from (5), } y = c - mx$$

Integrating, $\quad z = \int f(x, c - mx)\, dx$.

Thus, $\quad z = \dfrac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx$, ...(6)

where after integration the constant c must be replaced by $y + mx$ since the P.I. does not contain any arbitrary constant.

Hence the P.I. given by (2) can be obtained by applying the operation (6) by the factors, in succession, starting from the right.

Working rule for finding P.I. (General method) of F(D, D′)z = f(x, y).

$$\text{P.I.} = \frac{1}{(D - m_1 D')(D - m_2 D')\ldots(D - m_n D')} f(x, y) \qquad \text{...(7)}$$

We shall use one of the following formulas :

Formula I : $\dfrac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx$, where $c = y + mx$. ...(8)

Formula II : $\dfrac{1}{D + mD'} f(x, y) = \int f(x, c + mx)\, dx$, where $c = y - mx$. ...(9)

Hence in order to evaluate P.I. (7), we apply (8) or (9) depending on the factor $D - mD'$ and $D + mD'$. Note that result (9) can be obtained from (8) by replacing m by $-m$.

4.13 SOLVED EXAMPLES BASED ON GENERAL METHOD

Ex. 1. *Solve* $(\partial z/\partial x) + (\partial z/\partial y) = \sin x$.

Sol. Rewritting, the given equation is $(D + D')z = \sin x$. ...(1)

Its auxiliary equation is $m + 1 = 0$ so that $m = -1$.

$\therefore$ C.F. $= \phi(y - x)$, where ϕ is an arbitrary function.

and P.I. $= \dfrac{1}{D + D'} \sin x = \int \sin x \, dx = -\cos x$

Hence the required solution is $z = \text{C.F.} + \text{P.I.} = \phi(y - x) - \cos x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

[Delhi Maths (H) 2004, 10; Bhopal 2004; Jiwaji 2000; Rewa 2003; Vikram 2002, 04]

(*b*) $(D - D')(D + 2D')z = (y + 1)e^x$. **[Delhi Maths (H) 1993; I.A.S. 2004]**

Sol. (*a*) Here given $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$ or $(D + D')(D - 2D')z = (y - 1)e^x$.

Its auxiliary equation is $(m + 1)(m - 2) = 0$ so that $m = -1, 2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D + D')(D - 2D')}(y - 1)e^x = \frac{1}{D + D'}\left\{\frac{1}{D - 2D'}(y - 1)e^x\right\}$$

$= \dfrac{1}{D + D'}\int (c - 2x - 1)e^x \, dx$, by formula I of working rule of Art. 4.12. and taking $c = y + 2x$

$= \dfrac{1}{D + D'}\left[(c - 2x - 1)e^x - \int (-2)e^x \, dx\right]$, integrating by parts

$= \dfrac{1}{D + D'}[(c - 2x - 1)e^x + 2e^x] = \dfrac{1}{D + D'}(c - 2x + 1)e^x$

$= \dfrac{1}{D + D'}\{(y + 2x) - 2x + 1\}e^x$, replacing c by $y + 2x$ $= \dfrac{1}{D + D'}(y + 1)e^x$

$= \int (c' + x + 1)e^x \, dx$, by formula II of working rule of Art. 4.12 and taking $c' = y - x$

$= (c' + x + 1)\, e^x - \int (1.e^x)\, dx = (c' + x + 1)\, e^x - e^x = ye^x$, since $c' = y - x$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ or $z = \phi_1(y - x) + \phi_2(y + 2x) + ye^x$.

(*b*) Here auxiliary equation is $(m - 1)(m + 2) = 0$ so that $m = 1, -2$.

$\therefore$ C.F.$= \phi_1(y + x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $= \dfrac{1}{(D - D')(D + 2D')}(y + 1)\, e^x = \dfrac{1}{D - D'}\left\{\dfrac{1}{D + 2D'}(y + 1)\, e^x\right\}$

$= \dfrac{1}{D - D'}\int (c + 2x + 1)\, e^x \, dx = \dfrac{1}{(D - D')}\{(c + 2x + 1)\, e^x - 2e^x\}$, where $c = y - 2x$

$= \dfrac{1}{D - D'}(y - 1)\, e^x = \int (c' - x - 1)\, e^x \, dx$, where $c' = y + x$

$= (c' - x - 1)e^x + e^x = ye^x$ as $c' = y + x$.

$\therefore$ General solution is $y = \phi_1(y + x) + \phi_2(y - 2x) + ye^x$.

Ex. 3. $(D^2 - 4D'^2)z = (4x/y^2) - (y/x^2)$. **[Delhi Maths (H) 2004, 08; Meerut 1992, Bhopal 2010]**

Sol. Here auxiliary equation is $m^2 - 4 = 0$ so that $m = 2, -2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I .$= \dfrac{1}{(D + 2D')(D - 2D')}\left(\dfrac{4x}{y^2} - \dfrac{y}{x^2}\right) = \dfrac{1}{D + 2D'}\int\left\{\dfrac{4x}{(c - 2x)^2} - \dfrac{c - 2x}{x^2}\right\} dx$, where $c = y + 2x$.

$$= \frac{1}{D+2D'}\int\left\{-\frac{2}{c-2x}+\frac{2c}{(c-2x)^2}-\frac{c}{x^2}+\frac{2}{x}\right\}dx = \frac{1}{D+2D'}\left\{\log(c-2x)+\frac{c}{c-2x}+\frac{c}{x}+2\log x\right\}$$

$$= \frac{1}{D+2D'}\left[\log y+\frac{y+2x}{y}+\frac{y+2x}{x}+2\log x\right]$$

$$= \int\left\{\log(c'+2x)+1+2\frac{x}{c'+2x}+\frac{c'+2x}{x}+2+2\log x\right\}dx, \text{ taking } c'=y-2x$$

$= x\log(c'+2x)+5x+c\log x+2x\log x-2x \quad = x\log y+y\log x+3x$, as $c'=y-2x$

$\therefore$ The required solution $\qquad z=\phi_1(y+2x)+\phi_2(y-2x)+x\log y+y\log x+3x$

Ex. 4. *Solve* $(D^2-DD'-2D'^2)z=(2x^2+xy-y^2)\sin xy-\cos xy$.

[I.A.S 2009; Meerut 1994; Delhi Maths (Hons.) 2007]

Sol. Here auxiliary equation is $\quad m^2-m-2=0 \quad$ so that $\quad m=2,-1$.

$\therefore$ C.F. $=\phi_1(y+2x)+\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')}\frac{1}{D+D'}\{(2x^2+xy-y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\frac{1}{D+D'}\{(2x-y)(x+y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin x(c+x)-\cos x(c+x)\}\,dx, \text{ taking } c=y-x$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin(cx+x^2)-\cos(cx+x^2)\}\,dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos(cx+x^2)+\int\cos(cx+x^2)\,dx-\int\cos(cx+x^2)\,dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy, \text{ as } c=y-x$$

$= \int(c'-4x)\cos(c'x-2x^2)\,dx$, where $c'=y+2x$

$= \int\cos t\,dt = \sin t$, putting $c'x-2x^2=t$ so that $(c'-4x)dx=dt$

$= \sin(c'x-2x^2)=\sin xy$, as $c'=y+2x$.

$\therefore$ Required solution is $\qquad z=\phi_1(y+2x)+\phi_2(y-x)+\sin xy$.

Ex. 5. *Solve* (*a*) $r+s-6t=y\cos x$. $\quad$ or $\qquad (D^2+DD'-6D'^2)z=y\cos x$

[Bilaspur 2002, Indore 2002, Jabalpur 1999 Meerut 2000, 02, Ravishankar 1994, Jiwaji 1999, Garhwal 2005, 10; I.A.S.1992, 2008; Vikram 1999, Delhi Maths (Hons) 2007, Purvanchal 2007; Kanpur 2011]

(*b*) $(D^2+DD'-6D'^2)z=y\sin x$.

Sol. (*a*) Since $r=\partial^2 z/\partial x^2,\ s=\partial^2 z/\partial x\partial y,\ t=\partial^2 z/\partial y^2$, the given equation becomes

$\partial^2 z/\partial x^2+\partial^2 z/\partial x\,\partial y-6(\partial^2 z/\partial y^2)=y\cos x \qquad$ or $\qquad (D^2+DD'-6D'^2)z=y\cos x$....(1)

Its auxiliary equation is $\quad m^2+m-6=0 \quad$ so that $\quad m=2,-3$.

$\therefore$ C.F. $=\phi_1(y+2x)+\phi_2(y-3x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2+DD'-6D'^2}y\cos x = \frac{1}{(D-2D')(D+3D')}y\cos x$$

$$= \frac{1}{D-2D'}\int(3x+c)\cos x\,dx, \text{ where } c=y-3x$$

$= \dfrac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\, dx\right]$, integrating by parts

$= \dfrac{1}{D-2D'}[y\sin x + 3\cos x]$, as $c = y - 3x$

$= \int [(c'-2x)\sin x + 3\cos x]\, dx$, where $c' = y + 2x$

$= (c'-2x)(-\cos x) - \int (-2)(-\cos x)\, dx + 3\sin x$, integrating by parts

$= y(-\cos x) - 2\sin x + 3\sin x$, as $c' = y + 2x$

$= \sin x - y\cos x.$

$\therefore$ General solution is $\qquad z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y+2x) + \phi_2(y-3x) - y\sin x - \cos x$

Ex. 6. *Solve* $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$. **[Agra 2009; Meerut 1999; Bilaspur 2002; Indore 2004; Jabalpur 1999; Rewa 2002; Ranchi 2010]**

Sol. Given equation is $\qquad (D+D')^2 z = 2\cos y - x\sin y.$ ...(1)

Its auxiliary equation is $\quad (m+1)^2 = 0 \quad$ so that $\quad m = -1, -1.$

$\therefore$ C.F. $= \phi_1(y-x) = x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{D+D'}\dfrac{1}{D+D'}(2\cos y - x\sin y) = \dfrac{1}{D+D'}\int [2\cos(x+c) - x\sin(x+c)]\, dx$, where $c = y - x$

$= \dfrac{1}{D+D'}\left[2\int\cos(x+c)\, dx - \int x\sin(x+c)\, dx\right] = \dfrac{1}{D+D'}\left[2\sin(x+c) - \left\{-x\cos(x+c) + \int\cos(x+c)\, dx\right\}\right]$

$= \dfrac{1}{D+D'}[2\sin(x+c) + x\cos(x+c) - \sin(x+c)] = \dfrac{1}{D+D'}(\sin y + x\cos y)$, as $c = y - x$

$= \int [\sin(x+c') + x\cos(x+c')]\, dx = -\cos(x+c') + x\sin(x+c') - \int \{1.\sin(x+c')\}\, dx$, where $c' = y - x$

$= -\cos(x+c') + x\sin(x+c') + \cos(x+c') = x\sin y$, as $c' = y - x$.

So the required solution is $\qquad z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y.$

Ex. 7. *Solve* $r - t = \tan^3 x\tan y - \tan x\tan^3 y$ or $(D^2 - D'^2)z = \tan^3 x\tan y - \tan x\tan^3 y$ **[Agra 2010; Delhi Maths (G) 2006]**

Sol. Given equation is $\qquad (D^2 - D'^2)z = \tan^3 x\tan y - \tan x\tan^3 y$

or $\qquad (D+D')(D-D')z = \tan^3 x\tan y - \tan x\tan^3 y.$ ...(1)

Its auxiliary equation is $\quad (m+1)(m-1) = 0 \quad$ so that $\quad m = -1, 1.$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D+D')(D-D')}(\tan^3 x\tan y - \tan x\tan^3 y)$

$= \dfrac{1}{D+D'}\int [\tan^3 x\tan(c-x) - \tan x\tan^3(c-x)]\, dx$, where $c = y + x$

$= \dfrac{1}{D+D'}\int [\tan x\tan(c-x)(\sec^2 x - 1) - \tan x\tan^2(c-x)\{\sec^2(c-x) - 1\}]\, dx$

$= \dfrac{1}{D+D'}\int [\tan x\sec^2 x\tan(c-x) - \tan(c-x)\sec^2(c-x)\tan x]\, dx$

$$= \frac{1}{D+D'}\left[\frac{\tan^2 x}{2}\tan(c-x) - \int\frac{\tan^2 x}{2}\sec^2(c-x).(-1)\, dx - \left\{\frac{\tan^2(c-x)}{2\times(-1)}\tan x - \int\frac{\tan^2(c-x)}{2\times(-1)}\sec^2 x\, dx\right\}\right]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x)]$$
$$+ \int (\sec^2 x - 1)\sec^2(c-x)\,dx - \int \{\sec^2(c-x) - 1\}\sec^2 x\,dx]$$
$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) - \int \sec^2(c-x)\,dx + \int \sec^2 x\,dx]$$
$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) + \tan(c-x) + \tan x]$$
$$= \frac{1}{2(D+D')}(\tan^2 x \tan y + \tan x \tan^2 y + \tan y + \tan x), \text{ as } c = y + x$$
$$= \frac{1}{2(D+D')}[\tan y(\tan^2 x + 1) + \tan x(\tan^2 y + 1)] = \frac{1}{2(D+D')}(\tan y \sec^2 x + \tan x \sec^2 y)$$
$$= \frac{1}{2}\left[\int \tan(c'+x)\sec^2 x\,dx + \int \tan x \sec^2(c'+x)\,dx\right], \text{ where } c' = y - x$$
$$= \frac{1}{2}\left[\tan(c'+x)\tan x - \int \sec^2(c'+x)\tan x\,dx + \int \tan x \sec^2(c'+x)\,dx\right]$$

[On integrating the first integral by parts keeping the second integral unchanged]

$= (1/2) \times \tan(c'+x)\tan x = (1/2) \times \tan y \tan x$, as $c' = y - x$.

$\therefore$ The required solution is $\quad z = \phi_1(y-x) + \phi_2(y+x) + (1/2) \times \tan y \tan x$.

Ex. 8. *Solve* $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$. **[Meerut 1994]**

Sol. Re-writing the given equation is $\quad (D+3D')(D-2D')z = x^2 \sin(x+y)$. ...(1)

$\therefore$ C.F. $= \phi_1(y-3x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+3D')(D-2D')} x^2 \sin(x+y) = \frac{1}{D+3D'} \cdot \left\{\frac{1}{D-2D'}, x^2 \sin(x+y)\right\}$$
$$= \frac{1}{D+3D'}\int x^2 \sin(x+c-2x)\,dx = \frac{1}{D+3D'}\int x^2 \sin(c-x)\,dx, \text{ where } c = y+2x$$
$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \int 2x\cos(c-x)\,dx\right], \text{ integrating by parts}$$
$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x) + \int 2\sin(c-x)\,dx\right\}\right], \text{ integrating by parts}$$
$$= \frac{1}{D+3D'}[x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)]$$
$$= \frac{1}{D+3D'}[(x^2-2)\cos(x+y) + 2x\sin(x+y)], \text{ as } c = y+2x$$
$$= \int [(x^2-2)\cos(x+c'+3x) + 2x\sin(x+c'+3x)]\,dx, \text{ where } c' = y-3x$$
$$= \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$
$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\frac{\sin(4x+c')}{4}\,dx + 2\int x\sin(4x+c')\,dx$$

[Integrating by part 1st integral and keeping the second integral unchanged]

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx = \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x\cos(4x+c')}{4} + \int \frac{\cos(4x+c')}{4}\,dx\right]$$
$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$
$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x), \text{ as } c' = y-3x$$

$= \left(x^2/4 - 13/32\right)\sin(x+y) - (3x/8)\times\cos(x+y)$, on simplification

The solution is $z = \phi_1(y-3x) + \phi_2(y+2x) + [(x^2/4) - (13/32)]\sin(x+y) - (3x/8)\times\cos(x+y)$.

Ex. 9. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3)\, z = e^y \cos 2x$

Sol. Here $D^3 + D^2D' - DD'^2 - D'^3 = D^2(D+D') - D'^2(D+D') = (D+D')^2(D-D')$

So the given equation reduces to $(D-D')\,(D+D')^2 z = e^y \cos 2x$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. $= \dfrac{1}{(D-D')\,(D+D')}\dfrac{1}{D+D'}\, e^y \cos 2x = \dfrac{1}{(D-D')\,(D+D')}\displaystyle\int e^{a+x}\cos 2x\, dx$, where $y - x = a$

$$= \frac{1}{(D-D')(D+D')} e^a \int e^x \cos 2x\; dx = \frac{1}{(D-D')(D+D')} e^{y-x}\, \frac{1}{1^2+2^2} e^x(\cos 2x + 2\sin 2x)^*$$

$= \dfrac{1}{5}\dfrac{1}{(D-D')\,(D+D')} e^y(\cos 2x + 2\sin 2x) = \dfrac{1}{5}\dfrac{1}{D-D'}\displaystyle\int e^{x+a}(\cos 2x + 2\sin 2x)\, dx$, where $y - x = a$

$$= \frac{1}{5}\frac{1}{D-D'}\, e^a \left\{\int e^x \cos 2x\, dx + 2\int e^x \sin 2x\, dx\right\}$$

$$= \frac{1}{5}\frac{1}{D-D'}\, e^{y-x}\left\{\frac{e^x}{1^2+2^2}(\cos 2x + 2\sin 2x) + \frac{2e^x}{1^2+2^2}(\sin 2x - 2\cos 2x)\right\}$$

$= \dfrac{1}{25}\dfrac{1}{D-D'}\, e^y(4\sin 2x - 3\cos 2x) = \dfrac{1}{25}\displaystyle\int e^{b-x}(4\sin 2x - 3\cos 2x)\, dx$, where $b = y + x$

$$= \frac{1}{25}\, e^b\left\{4\int e^{-x}\sin 2x\; dx - 3\int e^{-x}\cos 2x\; dx\right\}$$

$$= \frac{1}{25}\, e^{y+x}\left\{\frac{4e^{-x}}{1^2+2^2}(-\sin 2x - 2\cos 2x) - \frac{3e^{-x}}{1^2+2^2}(-\cos 2x + 2\sin 2x)\right\}$$

$= -(1/25)\times e^y\,(\cos 2x + 2\sin 2x)$

$\therefore$ Required solution is $z = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x) - (e^y/25)\times(\cos 2x + 2\sin 2x)$

Ex. 10. *Find the solution of the equation* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$ *which* $\to 0$ *as* $x \to \infty$ *and has the value cos y when x = 0.* **[I.A.S. 1999]**

Sol. Given $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$ or $(D^2 + D'^2)z = e^{-x}\cos y$ ... (1)

$$\text{P.I.} = \frac{1}{D^2+D'^2}\, e^{-x}\cos y = \frac{1}{D^2+D'^2}\, e^{(-1)x+0.y}\cos y = e^{(-1)x+0.y}\,\frac{1}{(D-1)^2+(D'+0)^2}\cos y$$

$$= e^{-x}\,\frac{1}{D^2+D'^2-2D-1}\cos y = e^{-x}\,\frac{1}{0^2+(-1)^2-2D+1}\cos y = e^{-x}\,\frac{1}{-2D}\cos y = -\frac{1}{2}x e^{-x}\cos y.$$

Now, the general solution of (1) is $z =$ C.F. + P.I.

where C.F. is solution of $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ ... (2)

---

* We shall use the following results of Integral calculus directly

$$\int e^{ax}\sin bx\; dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx), \text{ and } \int e^{ax}\cos bx\; dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Since $$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0,$$
we observe that P.I. $\to$ 0 as $x \to \infty$. Also, we have P.I. = 0 when $x = 0$

Here we are to solve (1) satisfying the conditions $z \to 0$ as $x \to \infty$ and $z = \cos y$ when $x = 0$. It follows that C.F. of (1), that is solution of (2) must satisfy the conditions C.F. $\to 0$ as $x \to \infty$ and C.F. = cos y when $x = 0$. In other words, we now * solve (2) subject to conditions :

$$z(x, y) \to 0 \text{ as } x \to \infty \qquad \text{... (3)}$$

and $$z(x, y) = \cos y \text{ when } x = 0 \qquad \text{... (4)}$$

Let a solution of (2) be $$z(x, y) = X(x), Y(y) \qquad \text{... (5)}$$

From (5), $\partial^2 z / \partial x^2 = X''(x)\, Y(y)$ and $\partial^2 z / \partial y^2 = X(x)\, Y''(y)$,

where prime denotes the derivative w.r.t. to the relevent variable. Substituting these in (2), we get

$$X''(x)\;\; Y(y) + X(x)\;\; Y''(y) = 0 \qquad \text{or} \qquad X''/X = -Y''/Y \qquad \text{... (6)}$$

Since x and y are independent variables, (6) is true if each side is equal to a constant, say n^2. Since condition (4) involves trigonometric function cos y, we choose n as positive integer.

$\therefore$ (6) $\Rightarrow$ $X''/X = n^2$ and $-Y''/Y = n^2$

or $d^2X/dx^2 - n^2 X = 0$ and $d^2Y/dy^2 + n^2 Y = 0$

Solving these, $$X(x) = A\, e^{nx} + Be^{-nx} \qquad \text{... (7)}$$

and $$Y(y) = C \cos ny + D \sin ny \qquad \text{... (8)}$$

Take $A = 0$ in (7) for otherwise $X(x) \to \infty$ as $x \to \infty$ and so $z(x, y) \to \infty$ as $x \to \infty$ which contradicts (3) Using (7) and (8), (5) reduces to

$$z(x, y) = e^{-nx} (E \cos ny + F \sin ny) \qquad \text{... (9)}$$

where E (= BC) and F (= BD) are new arbitrary constants.

Now, putting $x = 0$ in (9) and using (4), we get cos y = E cos ny + F sin ny which holds if we choose $n = 1$, $E = 1$ and $F = 0$. Hence from (9), the C.F. of (1) is given by $e^{-x} \cos y$. Keeping in mind this C.F. of (1) and P.I. already obtained, the required solution of (1) is

$$z = e^{-x} \cos y - (x/2) \times e^{-x} \cos y = (1/2) \times (2 - x)\, e^{-x} \cos y.$$

EXERCISE 4(C)

Solve the following partial differential equations:

1. $(D^3 + 2D^2 D - DD'^2 - 2D'^3)\; z = (y+2)e^x$ **Ans.** $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x) + ye^y$

2. $(D^3 - 3DD'^2 - 2D'^3) z = \cos(x+2y) - e^y(3+2x)$

Ans. $z = \phi_1(y-x) + x\,\phi_2(y-x) + \phi_3(y+2x) + (1/27) \times \sin(y+2x) + xe^x$.

3. $(D^3 - D^2 D' - 2DD'^2) z = e^{x+2y}(x^2 + 4y^2)$.

Ans. $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x)\; -(1/81) \times (9x^2 + 36y^2 - 18x - 72y + 76)\; e^{x+2y}$

* We shall use the method of separation for solving partial differential equation. For details refer part III "Boundary value problems" in author's "Advanced Differential Equations."

4.14. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS:

We have seen that solutions obtained in above methods involve arbitrary functions of x and y. We shall now determine these under the given geometrical conditions. This will lead to required surface satisfying the given differential equation under the given geometrical conditions.

4.15. SOLVED EXAMPLES BASED ON ART. 4.14.

Ex. 1. *Find a surface passing through the two lines* $z = x = 0$, $z - 1 = x - y = 0$ *satisfying* $r - 4s + 4t = 0$. **[Meerut 97,2000; I.A.S. 1996; Bhopal 2010]**

Sol. The given equation may be written as $\partial^2 z/\partial x^2 - 4(\partial^2 z/\partial x\partial y) + 4(\partial^2 z/\partial y^2) = 0$

or $(D^2 - 4DD' + 4D'^2)z = 0$ or $(D - 2D')^2 z = 0$.

Its solution is $z = \phi_1(y + 2x) + x\phi_2(y + 2x)$. ϕ_1, ϕ_2 being arbitrary functions ...(1)

Since (1) passes through $z = x = 0$, we have $0 = \phi_1(y)$ which gives $\phi_1(y + 2x) = 0$.

$\therefore$ (1) becomes $z = x\,\phi_2(y + 2x)$. ...(2)

Since (2) passes through $z - 1 = x - y = 0$, *i.e.* $z = 1$ and $y = x$, we get

$1 = x\,\phi_2(3x)$ or $\phi_2(3x) = 3/(3x)$ so that $\phi_2(y + 2x) = 3/(y + 2x)$.

$\therefore$ from (2), we have $3x = z(y + 2x)$, which is the required surface.

Ex. 2. *Find the surface satisfying the equation* $r + t - 2s = 0$ *and the conditions that* $bz = y^2$ *when* $x = 0$ *and* $az = x^2$ *when* $y = 0$.

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 - 2(\partial^2 z/\partial x\partial y) = 0$

or $(D^2 - 2DD' + D'^2) = 0$ or $(D - D')^2 z = 0$.

Its solution is $z = \phi_1(y + x) + x\,\phi_2(y + x)$. ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Since $z = y^2/b$ when $x = 0$, (1) gives $y^2/b = \phi_1(y)$, $\Rightarrow$ $\phi_1(y + x) = (y + x)^2/b$. ...(2)

Again since $z = x^2/a$ when $y = 0$, (1) gives $x^2 a = x\phi_2(x) + \phi_1(x)$. ...(3)

Since from (2), $\phi_2(x) = x^2/b$, (3) becomes $\dfrac{x^2}{a} = x\phi_2(x) + \dfrac{x^2}{b}$ *i.e.* $\phi_2(x) = \dfrac{b - a}{ab}x$

which gives $\phi_2(y + x) = \dfrac{b - a}{ab}(y + x)$. ...(4)

Using (2) and (4) in (1), the required surface is

$$z = \frac{b-a}{ab}x(y + x) + \frac{(y + x)^2}{b} = \frac{y+x}{b}\left(\frac{b-a}{a}x + y + x\right) \quad \text{or} \quad z = (y + x)\left(\frac{x}{a} + \frac{y}{b}\right)$$

Ex. 3. *Find a surface satisfying* $r - 2s + t = 6$ *and touching the hyperbolic paraboloid* $z = xy$ *along its section by the plane* $y = x$. **[Meerut 1997]**

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = 6$

or $(D^2 - 2DD' + D'^2)z = 6$ or $(D - D')^2 z = 6$. ... (1)′

Its C.F. $= \phi_1(y + x) + x\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{Now, P.I.} = \frac{1}{(D - D')^2}.6 = \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} 6 = \frac{1}{D^2}\left(1 + \frac{2D'}{D} + ...\right) 6 = \frac{1}{D^2} 6 = 3x^2.$$

$\therefore$ General solution of (1)′ is $z = \text{C.F.} + \text{P.I.} = \phi_1(y + x) + x\phi_2(y + x) + 3x^2$. ...(1)

Since the required surface (1) touches the given surface $z = xy$ ...(2)

along the section $y = x$, the values of p and q for the two surfaces must be equal for any point on the plane $\qquad y = x.$ ...(3)

Now equating the values of p and q from (1) and (2), we have

$$p = \phi_2(y+x) + x\phi_2'(y+x) + \phi_1'(y+x) + 6x = y \qquad ...(4)$$

and $$q = x\phi_2'(y+x) + \phi_1'(y+x) = x. \qquad ...(5)$$

Subtracting (5) from (4) and using (3), we get

$$\phi_2(2x) = -6x = -3 \times (2x)$$

which gives $$\phi_2(y+x) = -3(y+x). \qquad ...(6)$$

From (6), $\qquad \phi_2'(y+x) = -3$. Then (5) becomes

$-3x + \phi_1'(y+x) = x \qquad$ so that $\qquad \phi_1'(2x) = 2 \times (2x), \qquad$ as $\qquad y = x$

Now, $$\phi_1'(2x) = 2(2x) \Rightarrow \phi_1'(x) = 2x. \qquad ...(7)$$

Integrating (7), $\quad \phi_1(x) = x^2 + c \qquad$ which gives $\qquad \phi_1(y+x) = (y+x)^2 + c. \qquad ...(8)$

Putting the values of $\phi_2(y+x)$ and $\phi_1(y+x)$ given by (6) and (8) in (1), we get

$z = x\{-3(y+x)\} + (y+x)^2 + c + 3x^2 \qquad$ or $\qquad z = x^2 - xy + y^2 + c.$
...(9)

Equating the values of z from (2) and (9), we get

$xy = x^2 - xy + y^2 + c \qquad$ or $\qquad x^2 = x^2 - x^2 + x^2 + c$, using (3)

giving $c = 0$. Hence the required surface is $\qquad z = x^2 - xy + y^2.$

Ex. 4. *A surface is drawn satisfying $r + t = 0$ and touching $x^2 + z^2 = 1$ along its section by $y = 0$. Obtain its equation in the form $x^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$.* **[Meerut 1998]**

Sol. Given equation $\qquad (D^2 + D'^2)z = 0 \qquad$ *i.e.* $\qquad (D + iD)(D - iD')z = 0.$

$\therefore$ Its solution is $\qquad z = \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

The given surface is $\qquad x^2 + z^2 = 1 \qquad$ or $\qquad z = (1 - x^2)^{1/2}.$ ...(2)

Since (1) and (2) touch along their common section by $\qquad y = 0,$ ...(3)

the values of p and q from (1) and (2) must be the same.

$$\therefore \quad p = i\,\phi_1'(y+ix) - i\,\phi_2'(y-ix) = -\frac{x}{(1-x^2)^{1/2}} \quad \text{and} \quad q = \phi_1'(y+ix) + \phi_2'(y-ix) = 0.$$

Using (3), these reduce to

$$\phi_1'(ix) - \phi_2'(-ix) = \frac{ix}{(1+x^2i^2)^{1/2}} \qquad \text{and} \qquad \phi_1'(ix) + \phi_2'(-ix) = 0, \quad \text{noting that } i^2 = -1$$

Solving these for $\phi_1'(ix)$ and $\phi_2'(-ix)$, $\qquad \phi'_1(ix) = \dfrac{ix}{2(1+x^2i^2)^{1/2}}, \qquad \phi'_2(-ix) = \dfrac{-ix}{2(1+x^2i^2)^{1/2}}.$

Writing $ix = X$ and $-ix = Y$, these give $\qquad \phi_1'(X) = \dfrac{X}{2(1+X^2)^{1/2}} \qquad \phi_2'(Y) = \dfrac{Y}{2(1+Y^2)^{1/2}}$

Integrating, $\qquad \phi_1(X) = (1/2) \times (1 + X^2)^{1/2} + c_1, \qquad \phi_2(Y) = (1/2) \times (1 + Y^2)^{1/2} + c_2$

These give $\phi_1(y+ix) = (1/2) \times \{1 + (y+ix)^2\}^{1/2} + c_1$, $\phi_2(y-ix) = (1/2) \times \{1 + (y-ix)^2\}^{1/2} + c_2$.

Putting these in (1) and writing $c_1 + c_2 = c$, $z = (1/2) \times [\sqrt{\{1 + (y+ix)^2\}} + \sqrt{\{1 + (y-ix)^2\}}] + c$ ...(4)

Now equating two values of z from (2) and (4) at $y = 0$, we get

$(1/2) \times [\sqrt{(1-x^2)} + \sqrt{(1-x^2)}] + c = \sqrt{(1-x^2)} \qquad$ so that $\qquad c = 0.$

Then, (4) gives $2z = \sqrt{\{1+(y+ix)^2\}} + \sqrt{\{1+(y-ix)^2\}}$. Squaring its both sides gives

or $$4z^2 = \{1+(y+ix)^2\} + (1+(y-ix)^2\} + 2\sqrt{[\{1+(y+ix)^2\}\ \{1+(y-ix)^2\}]},$$

or $$2z^2 = (1+y^2-x^2) + \sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}. \qquad ...(5)$$

Squaring both sides of (5), we get

$$4z^4 = (1+y^2-x^2)^2 + \{1+(y+ix)^2\}\{1+(y-ix)^2\} + 2(1+y^2-x^2)\sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}$$

or $$4z^4 = (1+y^2-x^2)^2 + \{(1+y^2-x^2)+2ixy\}\{(1+y^2-x^2)-2ixy\}$$
$$+2(1+y^2-x^2)\{2z^2-(1+y^2-x^2)\}, \text{ using (5)}$$

or $$4z^4 = (1+y^2-x^2)^2 + (1+y^2-x^2)^2 + 4x^2y^2 + 4z^2(1+y^2-x^2) - 2(1+y^2-x^2)^2$$

or $$4z^4 = 4x^2y^2 + 4z^2(1+y^2-x^2) \quad \text{or} \quad z^2(x^2+z^2-1) = y^2(x^2+z^2).$$

Ex.5. *Find a surface satisfying the equation* $D^2 z = 6x+2$ *and touching* $z = x^3+y^3$ *along its section by the plane* $x+y+1=0$. **Ans.** $z = x^3 + y^3 + (x+y+1)^2$

MISCELLANEOUS PROBLEMS ON CHAPTER 4

1. Solution of differential equation $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ is

(*a*) $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y-3x)$ (*b*) $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

(*c*) $z = \phi_1(y-x) + \phi_2(y+2x) + \phi_3(y+3x)$ (*d*) $z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x)$

Sol. Ans (*a*). Refer solved Ex. 2 (*a*) of Art. 4.5. **[Agra 2005]**

2. P.I. of the equation $r - 2s + t = \cos(2x+3y)$ is

(*a*) $-\cos(2x+3y)$ (*b*) $\cos(2x+3y)$ (*c*) $\sin(2x+3y)$ (*d*) None of these **[Kanpur 2004]**

Sol. Ans (*a*). Proceed like Ex. 4, Art. 4.9.

3. Auxialary equation of $r - 2s + t = \sin(2x+3y)$ is

(*a*) $m^2 - 2m + 1 = \sin(2x+3y)$ (*b*) $m^2 + 2m + 1 = \sin(2x+3y)$

(*c*) $(m-1)^2 = 0$ (*d*) $(m+1)^2 = 0$ **[Bhopal 2010]**

Ans. (*c*)

4. Solve $(D^2 - DD' - 6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x) + \phi_2(y-2x) - (1/4) \times \cos(2x+y)$

5

Non–homogeneous Linear Partial Differential Equations with Constant Coefficients

5.1. Non–homogeneous linear partial differential equations with constant coefficients.

Definition. A linear partial differential equation with constant coefficients is known as non–homogeneous linear partial differential equation with coefficients if the orders of all the partial derivatives involved in the equation are not equal.

For example, $(\partial^2 z/\partial x^2)-(\partial^3/\partial y^3)+\partial z/\partial x+z=x+y$ is a non–homogenous partial differential equation with constant coefficients.

5.2. Reducible and irreducible linear differential operators.

A linear differentiable operator $\mathrm{F}(D,\ D')$ is known as *reducible*, if it can be written as the product of linear factors of the form $aD+bD'+c$ with a, b and c as constants.

$F(D,\ D')$ is known as *irreducible*, if it is not reducible.

For example, the operator $D^2-D'^2$ which can be written in the form $(D+D')(D-D')$ is reducible, whereas the operator $D^2-D'^3$ which cannot be decomposed into linear factors is irreducible.

5.3. Reducible and irreducible linear partial differential equations with constant coefficients. [Delhi Maths (H) 2001, 2004, 09]

A linear partial differential equation with constant coefficients $F(D,\ D')z=f(x,y)$ is known as reducible, if $F(D,D')$ is reducible.

$F(D,D')z=f(x,y)$ is known as irreducible if $F(D,D')$ is irreducible.

For example, $(D^2-D'^2)z=x^2y^3$ is a reducible partial differential equation, with constant coefficients, since $D^2-D'^2=(D+D')\ (D-D')$ whereas $(D^2-D'^3)\ z=x^2y^3$ is an irreducible partial differential equation with constant coefficients, since $D^2-D'^3$ cannot be decomposed into linear factors.

5.4. Theorem. *If the operator* $F(D,\ D')$ *is reducible, the order in which the linear factors occur is unimportant.*

Proof. In order to prove the theorem we must show that

$$(a_rD+b_r\ D'+c_r)\ (a_sD+b_sD'_r+c_s)=(a_sD+b_sD'+c_s)\ (a_rD+b_rD'+c_r) \quad \dots (1)$$

for any reducible operator can be written in the form

$$F(D,\ D')=\prod_{r=1}^{n}(a_rD+b_rD'+c_r) \quad \dots (2)$$

The proof of (1) is immediate, since both sides are equal to

$$a_ra_sD^2+(a_sb_r+a_rb_s)\ DD'+b_rb_sD'^2+(c_sa_r+c_ra_s)D+(c_sb_r+c_rb_s)D'+c_sc_r$$

5.5. Determination of complementary function (C.F.) of a reducible non–homogeneous linear partial differential equation with constant coefficients given by

$$F(D, D')z = 0 \qquad \text{... (1)}$$

Let $F(D, D') = (b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)\ldots\ldots (b_nD - a_nD' - c_n)$, ...(2)

where a's, b's and c's are constants. Then (1) becomes

$$(b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)\ldots(b_nD - a_nD' - c_n)\, z = 0. \qquad \text{...(3)}$$

Equation (3) shows that any solution of the equation

$$(b_rD - a_rD' - c_r)z = 0,\ r = 1, 2, \ldots n \qquad \text{...(4)}$$

is a solution of (3) *i.e.* $b_rp - a_rq = c_rz$ which is Lagrange's equation.

Its Lagrange's auxiliary equations are $\dfrac{dx}{b_r} = \dfrac{dy}{-a_r} = \dfrac{dz}{c_rz}$. ...(5)

Proceeding as usual two independent integrals of (5) are $b_ry + a_rx = c_1$

and $z = c_2e^{(c_r/b_r)x}$, if $b_r \neq 0$ or $z = c_2'e^{-(c_r/a_r)y}$, if $a_r \neq 0$

$\therefore$ the general solution of (4) is $z = e^{(c_r/b_r)x}\phi_r(b_ry + a_rx)$, if $b_r \neq 0$ ...(6)

or $z = e^{-(c_r/a_r)y}\psi_r(b_ry + a_rx)$, if $a_r \neq 0$, ...(7)

where ϕ_r and ψ_r are arbitrary functions.

The general solution of (3) is the sum of the solutions of the equations of the form (4) corresponding to each factor in (2).

Case of repeated factors. Let two times repeated factor of (2), be $bD - aD' - c$.

Consider the equation $(bD - aD' - c)(bD - aD' - c)z = 0$. ...(8)

Let $(bD - aD' - c)z = v$. ...(9)

Then (8) reduces to $(bD - aD' - c)v = 0$. ...(10)

As before, the general solution of (10) is $v = e^{(c/b)x}\phi(by + ax)$, if $b \neq 0$. ...(11)

or $v = e^{-(c/a)y}\psi(by + ax)$, $a \neq 0$. ...(12)

where ϕ and ψ are arbitrary functions

Substituting from (11) is (9), we have

$(bD - aD' - c)z = e^{(c/b)x}\phi(by + ax)$ or $bp - aq = cz + e^{(c/b)x}\phi(by + ax)$. ...(13)

Lagrange's auxiliary equations of (13) are $\dfrac{dx}{b} = \dfrac{dy}{-a} = \dfrac{dz}{cz + e^{(c/b)x}\phi(by + ax)}$. ...(14)

Taking the first two fractions of (14), $adx + bdy = 0$ so that $by + ax = \lambda$, (say). ...(15)

where λ is an arbitrary constant.

Taking first and third fractions of (14), we get

$$\frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(by + ax) \quad \text{or} \quad \frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(\lambda), \text{ using (15)}$$

This is linear equation. Its I.F. $= e^{-\int (c/b)dx} = e^{-(c/b)x}$ and solution is given by

$$ze^{-(c/b)x} = \int \frac{1}{b}e^{(c/b)x}\phi(\lambda)e^{-(c/b)x}\, dx, \text{ that is,}$$

$ze^{-(c/b)x} - (x/b)\,\phi(\lambda) = \mu$ or $ze^{-(c/b)x} - (x/b)\,\phi(by + ax) = \mu$, by (15) ...(16)

where μ is an arbitrary constant.

From (15) and (16), the general solution of (13) or (8) is

$ze^{-(c/b)x} - (x/b)\phi(by + ax) = \phi_1(by + ax)$ or $z = e^{(c/b)x}[\phi_1(by + ax) + x\,\phi_2(by + ax)]$, if $b \neq 0$ ...(17)

where ϕ_1 and ϕ_2 are arbitrary functions.

Taking (12) and (9), we obtain as before.

$$z = e^{-(c/a)y}[\psi_1(by + ax) + y\,\psi_2(by + ax)], \text{ if } a \neq 0. \qquad ...(18)$$

where ψ_1 and ψ_2 are arbitrary functions.

In general, if $(bD - aD' - c)$ is repeated r times, then

$$z = c^{(c/b)x} \sum_{i=1}^{r} x^{i-1}\phi_i(by + ax), \text{ if } b = 0 \qquad \text{or} \qquad z = c^{-(c/a)y} \sum_{i=1}^{r} y^{i-1}\psi_i(by + ax), \text{ if } a \neq 0.$$

5.6. Working rule for finding C.F. of reducible non–homogeneous linear partial differential equation with constant coefficients. For proofs refer Art. 5.5.

Let the given differential equation be $\qquad F(D, D') = f(x, y)$.

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non–repeated factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{(cx/b)}\phi(by + ax)$, if $b \neq 0$

We now have three particular cases of Rule I

Rule IA. Take $c = 0$ in rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax),\ b \neq 0$.

Rule IB. Take $a = 0$ in rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is $e^{(cx/b)}\phi(by),\ b \neq 0$.

Rule IC. Take $a = c = 0$ and $b = 1$ in rule I. Hence corresponding to linear factor D, the part of C.F. is $\phi(y)$.

Rule II. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$$e^{(cx/b)}\ [\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax)], \text{ if } b \neq 0$$

We now have three particular cases of Rule II.

Rule II A. Take $c = 0$ in rule II. Hence corresponding to each repeated factor $(bD - aD')^r$, the part of C.F. is $\qquad \phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax),\ b \neq 0$.

Rule II B. Take $a = 0$ in rule II. Hence corresponding to a repeated factor $(bD - c)^r$, the part of C.F. is $\qquad e^{(cx/b)}\ [\phi_1(by) + x\phi_2(by) + x^2\phi_3(by) + ... + x^{r-1}\phi_r(by)],\ b \neq 0$

Rule II C. Take $a = c = 0$ and $b = 1$ in rule II. Hence corresponding to repeated factor D^r, the part of C.F. is $\qquad \phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + ... + x^{r-1}\phi_r(y)$.

Rule III. Corresponding to each non–repeated linear factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{-(cy/a)}\phi(by + ax)$, if $a \neq 0$.

We now have three particular cases of rule III.

Rule III A. Take $c = 0$ in rule III. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax),\ a \neq 0$.

Rule III B. Take $b = 0$ in rule III. Hence corresponding to each linear factor $(aD' + c)$, the part of C.F. is $e^{-(cy/a)}\phi(ax),\ a \neq 0$.

Rule III C. Take $b = c = 0$ and $a = 1$ in rule III. Hence corresponding to linear factor D', the part of C.F. is $\phi(x)$

Rule IV. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$$e^{-(cy/a)}\,[\phi_1\,(by+ax) + y\,\phi_2(by+ax) + y^2\phi_3(by+ax) + \dots + y^{r-1}\phi_r\,(by+ax)\,], \text{ if } a \neq 0$$

We now have three particular cases of rule IV.

Rule IV A. Take $c = 0$ in rule IV. Hence corresponding to repeated factor $(bD - aD')^r$, the part of C.F. is
$$\phi_1(by+ax) + y\;\phi_2\,(by+ax) + y^2\phi_3\,(by+ax) + \dots + y^{r-1}\phi_r\,(by+ax)\,,\;\; a \neq 0\,.$$

Rule IV B. Take $b = 0$ in rule IV. Hence corresponding to a repeated factor $(aD' + c)^r$, the part of C.F. is
$$e^{-(cy/a)}[\phi_1(ax) + y\;\phi_2(ax) + y^2\phi_3(ax) + \dots + y^{r-1}\phi_r(ax)]\,,\;\; a \neq 0$$

Rule IV C. Take $b = c = 0$ and $a = 1$ in rule IV. Hence corresponding to repeated factor D'^r, the part of C.F. is
$$\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \dots + y^{r-1}\phi_r(x)\,.$$

5.7. SOLVED EXAMPLES BASED ON ART. 5.6.

Ex. 1. *Solve* $(D^2 - D'^2 + D - D')z = 0$.

Sol. The given equation can be re-written as $\qquad (D - D')(D + D' + 1)z = 0.$

Here R.H.S. = 0 $\Rightarrow$ P.I. = 0. Hence the required solution is $\qquad z =$ C.F.

or $\qquad z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Ex. 2. *Solve* $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$.

Sol. The given equation can be re–written as

$[(D + aD')(D - aD') + 2ab(D + aD')]z = 0 \qquad$ or $\qquad (D + aD')(D - aD' + 2ab)z = 0.$

Hence general solution is $\qquad z =$ C.F. $= \phi_1(y - ax) + e^{-2abx}\phi_2(y + ax)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 3. *Solve* $r + 2s + t + 2p + 2q + z = 0$.

Sol. The given equation can be re–written as

$$(\partial^2 z/\partial x^2) + 2(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) + 2(\partial z/\partial x) + 2(\partial z/\partial y) + z = 0$$

or $$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$$

or $$[(D + D')^2 + 2(D + D') + 1]z = 0 \qquad \text{or} \qquad (D + D' + 1)^2 z = 0.$$

There are repeated linear factors. So the required general solution is

$$z = \text{C.F.} = e^{-x}[\phi_1(y - x) + x\phi_2(y - x)],\;\; \phi_1, \phi_2 \text{ being arbitrary functions.}$$

Ex. 4. *Solve* $DD'(D - 2D' - 3)z = 0$.

Sol. Using rules I, IC, III C of Art. 5.6, the required solution is $\qquad z =$ C.F., i.e.,

$z = \phi_1(y) + \phi_2(x) + e^{3x}\phi_3(y + 2x)$, where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 5. *Solve* $(2D - 3)\;(3D - 5D' - 7)^2 z = 0$

Sol. Using rules IB and II of Art. 5.6, the required solution is $\qquad z =$ C.F., i.e.,

$z = e^{3x/2}\phi_1(2y) + e^{7x/3}\{\phi_2(3y + 5x) + x\,\phi_3(3y + 5x)\}$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Ex. 6. *Solve* $(3D - 5)\,(7D' + 2)\,DD'(2D + 3D' + 5)z = 0$

Sol. Using rules IB, IIIB, IC, IIIC and I, the required solution is $\qquad z =$ C.F. i.e.,

$$z = e^{5x/3}\phi_1(3y) + e^{-(2y/7)}\phi_2(7x) + \phi_3(y) + \phi_4(x) + e^{-(5x/2)}\phi_5(2y - 3x)$$

Ex. 7. *Solve the partial differential equation $t + s + q = 0$.*

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y + \partial z/\partial y = 0$

or $(D^2 + DD' + D')z = 0$ or $D'(D + D' + 1) = 0$,

whose general solution is $z = \phi_1(x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

EXERCISE 5(A)

Solve the following partial differential equations :

1. $(D + D' - 1)(D + 2D' - 2)z = 0$ **Ans.** $z = e^x\phi_1(y - x) + e^{2x}\phi_2(y - 2x)$
2. $(D - D' + 1)(D + 2D' - 3)z = 0$ **Ans.** $z = e^{-x}\phi_1(y + x) + e^{2x}\phi_2(y - 3x)$
3. $(DD' + aD + bD' + ab)z = 0$ **Ans.** $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$
4. $r + 2s + t + 2p + 2q + z = 0$ **Ans.** $z = e^{-x}\{\phi_1(y - x) + x\,\phi_2(y - x)\}$
5. $(D + 1)(D + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$
6. $(D^2 - D'^2 + D - D')z = 0$ **Ans.** $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$
7. $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y - 6(\partial^2 z/\partial y^2) = 0$ **Ans.** $z = \phi_1(y + 2x) + \phi_2(y - 3x)$
8. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$ **Ans.** $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$
9. $s + p - q - z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-y}\phi_2(x)$
10. $(D^2 - DD' + D' - 1)z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)$
11. $(D^2 + DD' + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$

5.8. Method of finding C.F. of irreducible linear partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$ ... (1)

When the operator $F(D, D')$ in (1) is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to construct solutions which contain as many arbitrary constants as we wish. We now state and prove a theorem which will be used to find C.F. of (1).

Theorem. *To show that* $F(D, D')\, e^{ax+by} = F(a, b)\, e^{ax+by}$

Proof. We know that $F(D, D')$ consists of terms of the form $C_{rs}\, D^r\, D'^s$.

Also $D^r(e^{ax+by}) = a^r e^{ax+by}$ and $D'^s(e^{ax+by}) = b^s\, e^{ax+by}$

so that $(C_{rs}\, D^r D'^s)(e^{ax+by}) = C_{rs} a^r b^s\, e^{ax+by}$

The theorem follows by combining the terms of the operator $F(D, D')$.

We now discuss method of finding C.F. of (1). Consider $F(D, D')\, z = 0$ ... (2)

From the above theorem we see that e^{hx+ky} is a solution of (2) provided $F(h, k) = 0$, so that

$$z = \sum_i A_i\, e^{h_i x + k_i y} \qquad \text{... (3)}$$

in which A_i, h_i, k_i are all constants, is also a solution provided that h_i, k_i are connected by the relation $F(h_i, k_i) = 0$ ... (4)

Thus we can construct solution of (2) containing as many arbitrary constants as we need. The series (3) may not be finite but if it is infinite, it is necessary that it should be uniformly convergent if it has to be a solution of (2).

Remark. We can also present C.F. of irreducible equation (1) in the following manner

$$C.F. = \Sigma A e^{hx+ky}$$

where A, h, k are orbitrary constants such that $F(h, k) = 0$

Working rule for finding C.F. of irreducible non–homogeneous linear partial differential equation with constant coefficients, namely, $F(D, D')z = 0$

Step 1. If necessary, factorize $F(D, D')$ in the form $F_1(D, D')\, F_2(D, D')$, where $F_1(D, D')$ consist of a product of linear factors in D, D' and $F_2(D, D')$ consists of a product of irreducible factors in D, D'.

Step 2. Using Art. 5.6, write down the part of C.F. corresponding to factors of $F_1(D, D')$.

Step 3. Using Art 5.8, write down the part of C.F. corresponding to factors of $F_2(D, D')$.

Step 4. Adding the C.F. corresponding to $F_1(D, D')$ obtained in step 2 and the C.F. corresponding to $F_2(D, D')$ obtained in step 3, we obtain the C.F. of the given equation

$$F(D, D')z = 0, \qquad i.e., \qquad F_1(D, D')F_2(D, D')z = 0.$$

5.9 SOLVED EXAMPLES BASED ON ART. 5.8

Ex. 1. *Solve* $(D - D'^2)z = 0$.

Sol. Here $D - D'^2$ is not a linear factor in D and D'. Let $z = Ae^{hx+ky}$ be a trial solution of the given equation. Then $Dz = Ahe^{hx+ky}$ and $D'^2z = Ak^2e^{hx+ky}$.

Putting these values in the given differential equation, we get

$A(h - k^2)e^{hx+ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

Replacing h by k^2, the most general solution of the given equation is

$$z = \Sigma A e^{k^2x+ky}, \text{ where } A \text{ and } k \text{ are arbitrary constants.}$$

Ex. 2. *Solve* $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.

Sol. $(D - 2D' - 1)$ being linear in D and D', the part of C.F. corresponding to it is $e^x\phi(y + 2x)$, where ϕ is an arbitrary function.

To find C.F. corresponding non–linear factor $D - 2D'^2 - 1$, we now proceed as follows :

Let a trial solution of $\qquad (D - 2D'^2 - 1)z = 0 \qquad$ ...(1)

be $\qquad z = Ae^{hx+ky} \qquad$ ...(2)

$\therefore \quad Dz = Ahe^{hx+ky}$ and $D'^2z = Ak^2e^{hx+ky}$. Hence (1) becomes

$A(h - 2k^2 - 1)e^{hx+ky} = 0$ or $h - 2k^2 - 1 = 0$ or $h = 2k^2 + 1$.

Replacing h by $2k^2 + 1$ in (2), the solution of (1) *i.e.* the part of C.F. corresponding to $(D - 2D'^2 - 1)$ in the given equation is given by

$$\Sigma A e^{(2k^2+1)x+ky}, \ A \text{ and } k \text{ being arbitrary constants.}$$

$\therefore$ The required solution is $\qquad z = e^x\phi(y + 2x) + \Sigma A e^{(2k^2+1)x+ky}$.

Ex. 3. *Solve* $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) = n^2 z$.

Sol. The given equation can be written as $\qquad (D^2 + D'^2 - n^2)z = 0. \qquad$ ...(1)

Let a trial solution of (1) be $\qquad z = Ae^{hx+ky}. \qquad$ ...(2)

$\therefore \quad D^2 z = Ah^2 e^{hx+ky} \quad$ and $\quad D'^2 z = Ak^2 e^{hx+ky}$. Hence (1) gives

$A(h^2 + k^2 - n^2)e^{hx+ky} = 0 \quad$ or $\quad h^2 + k^2 = n^2$. ...(3)

Taking α as parameter, we see that (2) is satisfied if $h = n\cos\alpha$ and $k = n\sin\alpha$.

Putting these values in (2), the required general solution is

$$z = \Sigma A e^{n(x\cos\alpha + y\sin\alpha)}, \text{ } A \text{ and } \alpha \text{ being arbitrary constants.}$$

Ex. 4. *Solve* $(2D^4 - 3D^2D' + D'^2)z = 0$

Sol. Re–writing the given equation, we have $\quad (2D^2 - D')(D^2 - D')z = 0 \quad$ ... (1)

Let $z = A\, e^{hx+ky}$ be a solution of $\quad (2D^2 - D')z = 0$

$\therefore (2h^2 - k)Ae^{hx+ky} = 0 \quad$ so that $\quad 2h^2 - k = 0 \quad$ or $\quad k = 2h^2$

Hence C.F. corresponding to $(2D^2 - D')$ is $\quad \Sigma A e^{hx+2h^2 y} \quad$ ... (2)

Again, let $z = A' e^{h'x+k'y}$ be a solution of $\quad (D^2 - D')z = 0$

$\therefore \quad (h'^2 - k')A' e^{h'x+k'y} = 0 \quad$ so that $\quad h'^2 - k' = 0 \quad$ or $\quad k' = h'^2$.

Hence, C.F. corresponding to $(D^2 - D')$ is $\quad \Sigma A' e^{h'x+h'^2 y} \quad$ ... (3)

From (2) and (3), the general solution of (1) is given by $\quad z =$ Total C.F., i.e.,

$z = \sum_i A_i e^{h_i x + 2h_i^2 y} + \sum_i A_i' e^{h_i' x + h_i'^2 y}$, where A_i, h_i, A_i' and h_i' are arbitrary constants.

Ex. 5. Solve $(D + 2D' - 3)(D^2 + D')z = 0$

Sol. C.F. corresponding to linear factor $(D + 2D' - 3)$ is $e^{3x}\phi(y - 2x)$.

We now find C.F. corresponding to irreducible factor $(D^2 + D')$.

Let $z = Ae^{hx+ky}$ be a solution of $\quad (D^2 + D')z = 0$.

$\therefore \quad (h^2 + k)Ae^{hx+ky} = 0 \quad$ so that $\quad h^2 + k = 0 \quad$ or $\quad k = -h^2$.

Hence C.F. corresponding to $(D^2 + D')$ is $\quad \Sigma A e^{hx-h^2 y}$.

Therefore, the general solution of given equation is $\quad z = e^{3x}\phi(y - 2x) + \sum_i A_i e^{h_i x - h_i^2 y}$,

where ϕ is an arbitrary function and A_i, h_i are arbitrary constants.

Ex. 6. *Solve* $(D^2 - D')\, z = 0$. **[Delhi Maths (H) 2009]**

Sol. Given equation is $\quad F(D, D')z = 0,$ where $F(D, D') = D^2 - D'$.

Let $z = e^{hx+ky}$ be a trial solution of the given equation. Then, the required solution is

$z = \sum_i A_i\, e^{h_i x + k_i y}$, where $\quad F(h_i, k_i) = h_i^2 - k_i = 0 \quad$ so that $\quad k_i = h_i^2$

Hence the required solution is $z = \sum_i A_i\, e^{h_i x + h_i^2 y}$, A_i, h_i being arbitrary constants.

Ex. 7. *Show that* $\partial^2 z/\partial x^2 = (1/k)\times(\partial z/\partial t)$ *possesses solutions of the form*

$$\sum_{n=0}^{\infty} C_n \cos(nx+\epsilon_n)\, e^{-kn^2 t}.$$

Sol. Re–writing giving equation, we get $\{D^2 - (1/k)\times D'\}\, z = 0$, ... (1)

where $D = \partial/\partial x$, $D' = \partial/\partial t$. Note that here we have t in place of usual independent variable y. Let $z = e^{ax+bt}$ be a trial solution of (1). Then, $a^2 - (1/k)\, b = 0$ so that $a^2 = b/k$. This relation is satisfied if we take $a = \pm in$ and $b = -kn^2$. Then solution of (1) will be of the form

$z = \sum_{n=0}^{\infty} C_n\, e^{\pm inx - kn^2 t}$, which can be re-written as $z = \sum_{n=0}^{\infty} C_n \cos(nx+\epsilon_n) e^{-kn^2 t}$

Ex. 8. *Write the form of solution possessed by the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2\,(\partial^2 y/\partial x^2)$ **[Delhi B.Sc (H) 2002]**

Another form. *Show that the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2(\partial^2 y/\partial x^2)$ *possesses solutions of the form* $\sum_{r=0}^{\infty} C_r\, e^{-kt} \cos(w_r t + \delta_r)\cos(\alpha_r x + \epsilon_r)$,

where C_r, α_r, δ_r, ϵ_r are constants and $w_r^2 = \alpha_r^2 c^2 - k^2$.

Sol. Re–writing the given equation, we get $(D'^2 + 2k\, D' - c^2 D^2)\, y = 0$ ... (1)

where $D' \equiv \partial/\partial t$, $D \equiv \partial/\partial x$. Here t is independent variable and y is dependent variable.

Let $z = e^{ax+bt}$ be a solution of (1) Then, $b^2 + 2kb - c^2 a^2 = 0$

so that $b = \{-2k \pm (4k^2 + 4c^2 a^2)\}^{1/2}/2 = -k \pm (k^2 - c^2\alpha_r^2)^{1/2}$, where $a^2 = -\alpha_r^2$

or $b = -k \pm i w_r$, where $w_r^2 = c^2\alpha_r^2 - k^2$ ... (2)

Hence the solution of (1) takes the form

$$y = \sum_{r=0}^{\infty} C_r\, e^{\pm i\alpha_r x + (-k \pm i w_r)t} = \sum_{r=0}^{\infty} C_r\, e^{(-k \pm i w_r)t}\, e^{\pm i\alpha_r x}$$

which can also be re–written as $y = \sum_{r=0}^{\infty} C_r\, e^{-kt}\cos(w_r t + \delta_r)\cos(\alpha_r x + \epsilon_r)$,

where C_r, α_r, δ_r, ϵ_r are constants. Also, by (2), $\omega_r^2 = \alpha_r^2 c^2 - k^2$.

EXERCISE 5(B)

Solve the following partial differential equations:

1. $(D^2 + D + D')z = 0$ **Ans.** $z = \sum_i A_i e^{h_i - (h_i^2 + h_i)y}$, where A_i and h_i are arbitrary constants.

2. $(2D^2 - D'^2 + D)z = 0$

Ans. $z = \sum_i A_i e^{h_i x + k_i y}$, where $2h_i^2 - k_i^2 + h_i = 0$; A_i, h_i, k_i being arbitrary constants.

3. $(D' + 3D)^2(D^2 + 5D + D')z = 0$ **Ans.** $z = \phi_1(3y - x) + x\,\phi_2(3y - x)$ $+ \sum_i A_i\, e^{h_i x - (h_i^2 + 5h_i)y}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

4. $(2D-3D'+7)^2\,(D^2+3D')z=0$. **Ans.** $z=e^{-(7x/2)}\{\phi_1(2y+3x)+x\,\phi_2(2y+x)\}$

$+\sum_i A_i\, e^{h_i x-(h_i^2 y)/3}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

5.10. General solution of non–homogeneous linear partial differential equation with constant coefficients.

Let
$$F(D, D')z = f(x, y) \qquad \text{... (1)}$$
be a non-homogeneous linear partial differential equation with constant coefficients. Let u be the C.F. of (1). Then, by definition u in a solution of $F(D, D')\,z=0$ so that
$$F(D, D')u = 0 \qquad \text{... (2)}$$
Let z' be a particular integral (P.I.) of (1). Hence
$$F(D, D')z' = f(x, y) \qquad \text{... (3)}$$
Now, $F(D, D')\,(u+z') = F(D, D')u + F(D, D')z' = f(x,y)$, by (2) and (3),

Thus $u+z'$ is a solution of (1). Hence, a solution of (1) is z = C.F. + P.I.

5.11. Particular integral of non–homogeneous linear partial differential equation

$$\boldsymbol{F(D, D')y = f(x, y)} \qquad \text{... (1)}$$

The inverse operator $1/F(D, D')$ of the operator $F(D, D')$ is defined by the following indentity:
$$F(D, D')\left(\frac{1}{F(D, D')}f(x, y)\right) = f(x, y)$$
$$\Rightarrow \qquad \text{Particular integral (P.I.)} = \frac{1}{F(D, D')}f(x, y)$$

5.12. Determination particular integral of non–homogeneous linear partial differential equations (reducible or irreducible), namely,

$$\mathbf{F(D, D')z = f(x, y).} \qquad \text{...(1)}$$

The methods of finding particular integrals of non–homogeneous partial differential equations are very similar to those of ordinary linear differential equation with constant coefficients. We now give a list of some cases of finding P.I. of (1).

Case I. When f(x, y) = $e^{ax + by}$ and F(a, b) ≠ 0.

Then, $$\text{P.I.} = \frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(a, b)}e^{ax+by}.$$

Thus in this case we replace D by a and D' by b.

Case II. When f(x, y) = sin (ax + by) or cos (ax + by).

Then, $$\text{P.I.} = \frac{1}{F(D, D')}\sin(ax+by) \qquad \text{or} \qquad \text{P.I.} = \frac{1}{F(D, D')}\cos(ax+by)$$

which is evaluated by putting $D^2=-a^2$, $D'^2=-b^2$, $DD'=-ab$, provided the denominator is non–zero.

Case III. When f(x, y) = $x^m y^n$

Then, $$\text{P.I.} = \frac{1}{F(D, D')}x^m y^n = [F(D, D')]^{-1}\,x^m y^n,$$

which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D'/D or D/D' or D or D' as the case may be. In practice, we shall expand in ascending powers of D'/D. However note that if we expand in ascending powers of D/D', we shall get a P.I. of apparently different form. In this connection remember that both forms of P.I. are correct because the two could be transformed into each other with the help of C.F. of the given equation.

Case IV. When f(x, y) = Ve$^{ax + by}$, when V is a function of x and y.

Then
$$\text{P.I.} = \frac{1}{F(D, D')} Ve^{ax+by} = e^{ax+by} \frac{1}{F(D+a, D'+b)} e^{ax+by}$$

Remark. If $F(a, b) = 0$ and $f(x, y) = e^{ax + by}$. Then, we have

$$\text{P.I.} = \frac{1}{F(D, D')} e^{ax+by},$$

in which case I fails. However, by treating $e^{ax + by}$ as product of $e^{ax + by}$ with '1' and applying the result of case IV, we can evaluate P.I. as follows :

$$\text{P.I.} = \frac{1}{F(D, D')} e^{ax+by} \cdot 1 = e^{ax+by} \frac{1}{F(D+a, D'+b)} \cdot 1,$$

which can be evaluated as explained in case III by treating $1 = x^0 y^0$.

5.13 SOLVED EXAMPLES BASED ON ARTICLES 5.6, 5.8, AND 5.12.

Type 1 : Examples based on case I of Art. 5.12.

Ex. 1. *Solve* $(DD' + aD + bD' + ab)z = e^{mx + ny}$.

Sol. The given equation can be re–written as $\quad (D + b)(D' + a)z = e^{mx + ny}$

$\therefore$ C.F. $= e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

and
$$\text{P.I.} = \frac{1}{(D+b)(D'+a)} e^{mx+xy} = \frac{1}{(m+b)(n+a)} e^{mx+ny}.$$

Hence the required general solution is $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) + [(m + b)(n + a)]^{-1}e^{mx + ny}$.

Ex. 2. *Solve* $(D^2 - D'^2 + D - D')z = e^{2x + 3y}$. **[Ravishankar 2005]**

Sol. The given equation can be re–written as

$[(D - D')(D + D') + (D - D')]z = e^{2x + 3y}$ or $(D - D')(D + D' + 1)z = e^{2x + 3y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

and
$$\text{P.I.} = \frac{1}{(D-D')(D+D'+1)} e^{2x+3y} = \frac{1}{(2-3)(2+3+1)} e^{2x+3y} = -\frac{1}{6} e^{2x+3y}$$

Hence the required general solution is z. $\quad z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - (1/6) \times e^{2x + 3y}$.

Ex. 3. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x - y}$.

Sol. Try yourself. **Ans.** $z = e^{x}\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/2) \times e^{2x - y}$

Ex. 4. *Solve* (*a*) $(D^2 - 4DD' + D - 1)z = e^{3x - 2y}$.

(*b*) $(D^3 - 3DD' + D + 1)z = e^{2x + 3y}$. **[Kanpur 2006]**

Sol. (*a*) Here $(D^2 - 4DD' + D - 1)$ cannot be resolved into linear factors in D and D'. Hence for finding C.F., consider the equation $\quad (D^2 - 4DD' + D - 1)z = 0. \quad$ ...(1)

Let a trial solution of (1) be $\quad z = Ae^{hx + ky}. \quad$ ...(2)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$, $\quad DD' = Ahke^{hx + ky}$, $\quad Dz = Ahe^{hx + ky}$ and so (1) gives

$A(h^2 - 4hk + h - 1)e^{hx + ky} = 0$ so that $\quad h^2 - 4hk + h - 1 = 0$

giving $\quad k = (h^2 + h - 1)/4h. \quad$ ...(3)

$\therefore$ C.F. $= \Sigma Ae^{hx + ky}$, when k is given by (3).

$$\text{Again, P.I.} = \frac{1}{D^2 - 4DD' + D - 1} e^{3x-2y} = \frac{1}{3^2 - 4 \times 3 \times (-2) + 3 - 1} e^{3x-2y} = \frac{1}{35} e^{3x-2y}$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ *i.e.* $z = \Sigma A e^{hx+ky} + (1/35) \times e^{3x-2y}$, where A, h and k are constants and k and h are related by (3).

(b) Proceed as in part (a). **Ans.** $z = \Sigma A e^{hx+ky} - (1/7) \times e^{2x+3y}$, where A, h, k are arbitrary constants and k is given by $k = (h^3 + h + 1)/3h$

Ex. 5. *Solve* $(\partial^2 y/\partial x^2) - (\partial^2 y/\partial z^2) = y + e^{x+z}$.

Sol. Re–writing, $(D^2 - D'^2 - 1)y = e^{x+z}$, where $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial z$.

$$\text{C.F.} = \sum A\, e^{hx+kz}, \qquad \text{where} \qquad h^2 - k^2 - 1 = 0 \qquad \dots (1)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2 - 1} e^{x+z} = \frac{1}{1-1-1} e^{x+z} = -e^{x+z}$$

$\therefore$ The required solution is $z = \Sigma A\, e^{hx+ky} - e^{x+z}$, where A, h and k are arbitrary constants and h and k are connected by relation (1).

EXERCISE 5(C)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D)z = e^{2x+y}$. **Ans.** $z = \phi_1(y) + e^{2x}\phi_2(y+x) - (1/2) \times e^{2x+y}$

2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$.

   Ans. $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10) \times e^{2x+3y}$

3. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$. **Ans.** $z = e^{-2x}\phi_1(y+x) + e^{x}\phi_2(y-x) - (1/4) \times e^{x-y}$

4. $(D^2 + D' + 4)z = e^{4x-y}$.

   Ans. $z = \sum_i A_i\, e^{a_i x - (a_i^2 + 4)y} + (1/19) \times e^{4x-y}$, where A_i and a_i are arbitrary constants.

5. $(D^2 - D'^2 - 3D')z = e^{x+2y}$ **(Purvanchal 2007)**

   Ans. $z = \Sigma A e^{(k^2+3k)^{1/2} x + ky} - (1/9) \times e^{x+2y}$

Type 2 : Examples based on case II of Art. 5.12.

Ex. 1. *Solve* $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$.

[Bilaspur 2003; Bhopal 1998; Jiwaji 1997; Ravishankar 2004]

Sol. The given equation can be re–written as $(D + 1)(D + D' - 1)z = \sin(x + 2y)$.

$\therefore$ C.F. $= e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions

and $$\text{P.I.} = \frac{1}{D^2 + DD' + D' - 1}\sin(x+2y) = \frac{1}{-1^2 - (1\cdot 2) + D' - 1}\sin(x+2y)$$

$$= \frac{1}{D' - 4}\sin(x+2y) = (D'+4)\frac{1}{D'^2 - 16}\sin(x+2y) = (D'+4)\frac{1}{-2^2 - 16}\sin(x+2y)$$

$$= -(1/20) \times (D' + 4)\sin(x+2y) = -(1/20) \times [D'\sin(x+2y) + 4\sin(x+2y)]$$

$$= -(1/20) \times [2\cos(x+2y) + 4\sin(x+2y)].$$

$\therefore$ Solution is $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x) - (1/10) \times [\cos(x + 2y) + 2\sin(x + 2y)]$.

Ex. 2. *Solve* $(\partial^2 z/\partial x^2) - (\partial z/\partial x\partial y) + (\partial z/\partial y) - z = \cos(x + 2y)$.

[Delhi Maths (H) 2001; M.D.U Rohtak 2004]

Sol. The given equation can be re–written as

$(D^2 - DD' + D' - 1)z = \cos(x + 2y)$ or $(D - 1)(D - D' + 1)z = \cos(x + 2y)$.

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y) = \frac{1}{-1^2 + (1\cdot 2) + D' - 1}\cos(x + 2y) = \frac{1}{D'}\cos(x + 2y)$$

$= (1/2) \times \sin(x + 2y)$, as $1/D'$ stands for integration w.r.t. y keeping x as constant

Hence the required solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2) \times \sin(x + 2y)$.

Ex. 3. *Solve* $2(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) - 3(\partial z/\partial y) = 5\cos(3x - 2y)$.

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (1/10) \times [4\cos(3x - 2y) + 3\sin(3x - 2y)]$.

Ex. 4. *Solve* $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. **[KU Kurukshetra 2005]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and
$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)}\sin(2x + 3y) = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-2^2 + 2\times(2\times 3) - 3^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1}\sin(2x + 3y) = D\frac{1}{-3D^2 + 3DD' + D}\sin(2x + 3y)$$

$$= D\frac{1}{-3\times(-2^2) + 3\times(2\times 3) + D}\sin(2x + 3y) = D\frac{1}{D - 6}\sin(2x + 3y)$$

$$= D(D + 6)\frac{1}{D^2 - 36}\sin(2x + 3y) = (D^2 + 6D)\frac{1}{-2^2 - 36}\sin(3x + 2y)$$

$= -(1/40) \times [D^2\sin(2x + 3y) + 6D\sin(2x + 3y)] = -(1/40) \times [-4\sin(2x + 3y) + 12\cos(2x + 3y)]$

$\therefore$ Solution is $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/10) \times [\sin(2x + 3y) - 3\cos(2x + 3y)]$.

Ex. 5. *Solve* (*a*) $(D - D'^2)z = \cos(x - 3y)$ **[Delhi Maths (Hons.) 1998, 2007, 2009, 2011]**

(*b*) $(D^2 - D')z = \cos(3x - y)$.

Sol. (*a*) Here $(D - D'^2)$ cannot be resolved into linear factors in D and D'. Hence in order to find C.F. of the given equation, consider the equation

$$(D - D'^2)z = 0. \qquad ...(1)$$

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2 z = Ak^2e^{hx + ky}$. Then (1) gives

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

$\therefore$ C.F. $= \Sigma\, Ae^{k(kx+y)}$, where A, k are arbitrary constants.

Now,
$$\text{P.I.} = \frac{1}{D - D'^2}\cos(x - 3y) = \frac{1}{D - (-3^2)}\cos(x - 3y)$$

$$= (D - 9)\frac{1}{(D + 9)(D - 9)}\cos(x - 3y) = (D - 9)\frac{1}{D^2 - 81}\cos(x - 3y) = \frac{(D - 9)}{-1^2 - 81}\cos(x - 3y)$$

$= -(1/82) \times [D\cos(x - 3y) - 9\cos(x - 3y)] = -(1/82) \times [-\sin(x - 3y) - 9\cos(x - 3y)]$.

$\therefore$ General solution is $z = \Sigma\, Ae^{k(kx + y)} + (1/82) \times [\sin(x - 3y) + 9\cos(x - 3y)]$

(*b*) **Ans.** $z = \Sigma A e^{h(x+hy)} - (1/82) \times [9 \cos(3x - y) - \sin(3x - y)]$, where A and h are arbitrary constants.

Ex. 5. (*c*) *Solve* $(D^2 - D')z = A \cos(lx + my)$, where A, l, m *are constants*.

Sol. Proceed as in Ex. 5(*a*). **Ans.** $z = \Sigma A' e^{hx + h^2 y} - \{A/(m^2 + l^4)\} \times \{m \sin(lx + my)$ $+ \; l^2 \cos(lx + my)\}$, where A' and h are arbitrary constants.

Ex. 6. *Solve* $(D^2 - DD' - 2D)z = \sin(3x + 4y)$. **[Delhi Maths (Hons.) 1997]**

Sol. The given equation can be re–written as $D(D - D' - 2)z = \sin(3x + 4y)$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $$= \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \frac{1}{-3^2 + (3\times 4) - 2D}\sin(3x+4y) = \frac{1}{3-2D}\sin(3x+4y)$$

$$= (3+2D)\frac{1}{9-4D^2} = \frac{3+2D}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y) = \frac{1}{45}[3\sin(3x+4y) + 2D\sin(3x+4y)]$$

$$= (1/45) \times [3\sin(3x + 4y) + 6\cos(3x + 4y)] = (1/15) \times [\sin(3x + 4y) + 2\cos(3x + 4y)].$$

$\therefore$ Solution is $z = \phi_1(y) + e^{2x}\phi_2(y + x) + (1/15) \times [\sin(3x + 4y) + 2\cos(3x + 4y)]$

Ex. 7. *Solve* $(3DD' - 2D'^2 - D')z = \sin(2x + 3y)$. **[Delhi Maths (H) 2001]**

Sol. Re–writing, given equation becomes $D'(3D - 2D' - 1)\, z = \sin(2x + 3y)$

$\therefore$ C.F. $= \phi_1(x) + e^{x/3}\phi_2(2x + 3y)$, ϕ_1, ϕ_2 are arbitrary functions

P.I. $$= \frac{1}{3DD' - 2D'^2 - D'}\sin(2x+3y) = \frac{1}{-3\times(2\times 3) - 2\times(-3^2) - D'}\sin(2x+3y)$$

$$= -(1/D')\sin(2x + 3y) = (1/3) \times \cos(2x + 3y)$$

$\therefore$ General solution is $z = \phi_1(x) + e^{x/3}\phi_2(2x + 3y) + (1/3) \times \cos(2x + 3y)$.

Ex. 8. $(D + D')\,(D + D' - 2)z = \sin(x + 2y)$ **[Delhi Maths (H) 2000]**

Sol. Here C.F. $= \phi_1(y - x) + e^{2x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary function

P.I. $$= \frac{1}{(D+D')(D+D'-2)}\sin(x+2y) = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}\sin(x+2y)$$

$$= \frac{1}{-1^2 - 2\times(1\times 2) - 2^2 - 2D - 2D'}\sin(x+2y)$$

$$= -\frac{1}{9 + 2(D+D')}\sin(x+2y) = -\{9 - 2(D+D')\}\cdot\frac{1}{81 - 4(D+D')^2}\sin(x+2y)$$

$$= \frac{-(9 - 2D - 2D')}{81 - 4D^2 - 4D'^2 - 8DD'}\sin(x+2y) = -\frac{9 - 2D - 2D'}{81 - 4(-1^2) - 4(-2^2) + 8\times(1\times 2)}\sin(x+2y)$$

$$= (1/117)\times\{-9\sin(x+2y) + 2\cos(x+2y) + 4\cos(x+2y)\} = (3/117)\times\{2\cos(x+2y) - 3\sin(x+2y)\}$$

$\therefore$ Solution is $z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + (3/117) \times \{2\cos(x + 2y) - 3\sin(x + 2y)\}$

Ex. 9. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$

Sol. Re–writing the given equation, we get $(D + D')\,(D - 2D' + 2)\,z = \sin\,(2x + y)$.

$\therefore$ C.F. $= \phi_1\,(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin\,(2x + y) = \frac{1}{-2^2 + (2\times 1) - 2\times(-1^2) + 2D + 2D'}\sin\,(2x + y)$$

$$= \frac{1}{2(D + D')}\sin(2x + y) = \frac{D - D'}{2}\,\frac{1}{D^2 - D'^2}\sin(2x + y) = \frac{D - D'}{2}\,\frac{1}{-2^2 - (-1^2)}\sin(2x + y)$$

$$= -(1/6)\times(D - D')\sin(2x + y) = -(1/6)\times\{2\cos\,(2x + y) - \cos\,(2x + y)\}$$

$\therefore$ General solution is $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/6)\times\cos(2x + y)$

EXERCISE 5 (D)

Solve the following partial differential equations:

1. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$. **[MDU Rohtak 2005]**

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (3/50)\times\{4\cos(3x - 2y) + 3\sin(3x - 2y)\}$

2. $(D^2 + D')\,(D - D - D'^2)z = \sin(2x + y)$ **Ans.** $z = \sum_i A_i\, e^{a_i x - a_i^2 y} + \sum_i B_i\, e^{(b_i + b_i^2)x + b_i y}$

$-(1/34)\times\{5\sin\,(2x + y) - 3\cos\,(2x + y)\}$, which A_i, a_i, B_i, b_i are arbitrary constants

Type 3. Examples based on case III of Art. 5.12.

Ex. 1. *Solve* $s + p - q = z + xy$. **[Delhi Maths (Hons.) 1995, I.A.S. 1991]**

Sol. The given equation can be rewritten as $(\partial^2 z/\partial x\partial y) + (\partial z/\partial x) - (\partial z/\partial y) - z = xy$

or $(DD' + D - D' - 1)z = xy$ or $(D - 1)(D' + 1)z = xy$. ...(1)

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-y}\phi_2(x)$, ϕ_1 and ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 1)(D' + 1)}xy = -\frac{1}{(1 - D)(1 + D')}xy = -(1 - D)^{-1}(1 + D')^{-1}xy$$

$$= -(1 + D + \ldots)(1 - D' + \ldots)xy = -(1 + D - D' - DD' + \ldots)xy = -xy - y + x + 1.$$

$\therefore$ The required solution is $z = e^x\phi_1(y) + e^{-y}\phi_2(x) - xy - y + x + 1$.

Ex. 2. *Solve* (*a*) $r - s + 2q - z = x^2y^2$. **[I.A.S. 1993]**

(*b*) $(D^2 - D' - 1)z = x^2y$.

Sol. (*a*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) + 2(\partial z/\partial y) - z = x^2y^2$ or $(D^2 - DD' + 2D' - 1)z = x^2y^2$. ...(1)

Since $(D^2 - DD' + 2D' - 1)$ cannot be resolved into linear factors in D and D', hence C.F. of (1) is obtained by considering the equation $(D^2 - DD' + 2D' - 1)z = 0$. ...(2)

Let a trial solution of (2) be $z = Ae^{hx + ky}$ ...(3)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$, $DD'z = Ahke^{hx + ky}$, $D'z = Ake^{hx + ky}$. Then (2) gives

$A(h^2 - hk + 2k - 1)e^{hx + ky} = 0$ or $h^2 - hk + 2k - 1 = 0$

so that $k = (1 - h^2)/(2 - h)$. ...(4)

$\therefore$ C.F. $= \Sigma Ae^{hx + ky}$, where A, h, k are arbitrary constants; h, k being related by (4).

$$\text{P.I.} = \frac{1}{D^2 - DD' + 2D' - 1} x^2y^2 = -\frac{1}{1-(D^2 - DD' + 2D')} x^2y^2 = -[1 - (D^2 - DD' + 2D')]^{-1} x^2y^2$$

$= -[1 + (D^2 - DD' + 2D') + (D^2 - DD' + 2D')^2 + \{D^2 + D'(2 - D)\}^3 + ...]x^2y^2$

$= -[1 + (D^2 - DD' + 2D') + (D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + ...) + 3D^2D'^2(2 - D)^2 + ...]x^2y^2$

$= -(1 + D^2 - DD' + 2D' + D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + 12D^2D'^2 + ...)x^2y^2$

$= -x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$

$\therefore$ Solution is $z = \Sigma Ae^{hx+ky} - x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$

(*b*) **Ans.** $z = \Sigma Ae^{hx+(h^2-1)y} + x^2 - x^2y - 2y + 4$, A and h being arbitrary constants.

Ex. 3. *Solve* (*a*) $(D^2 - D')z = 2y - x^2$. **[Delhi Maths (H.) 2004, 10; Agra 2005]**

(*b*) $(2D^2 - D'^2 + D)z = x^2 - y$.

Sol. (*a*) Here $D^2 - D'$ cannot be resolved into linear factors in D and D'. Hence to find C.F., we consider the equation $(D^2 - D')z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ...(2)

So $D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) gives

$A(h^2 - k)e^{hx+ky} = 0$ or $h^2 - k = 0$ so that $k = h^2$.

$\therefore$ C.F. $= \Sigma Ae^{hx+ky} = \Sigma Ae^{hx+h^2y}$, A, h being arbitrary constants.

$$\text{Now, P.I.} = \frac{1}{D^2 - D'}(2y - x^2) = \frac{1}{D^2(1 - D'/D^2)}(2y - x^2) = \frac{1}{D^2}\left(1 - \frac{D'}{D^2}\right)^{-1}(2y - x^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'}{D^2} + ...\right)(2y - x^2) = \frac{1}{D^2}\left\{(2y - x^2) + \frac{1}{D^2}D'(2y - x^2)\right\} = \frac{1}{D^2}\left(2y - x^2 + \frac{1}{D^2}2\right)$$

$$= \frac{1}{D^2}\left(2y - x^2 + 2 \times \frac{x^2}{2}\right) = \frac{1}{D^2}(2y) = 2y \times \frac{x^2}{2} = x^2y.$$

General solution is $z = \Sigma Ae^{hx+h^2y} + x^2y$, A and h being arbitrary constants.

(*b*) **Ans.** $z = \Sigma Ae^{hx+ky} - (1/2) \times x^2y^2 + (1/6) \times y^3 - (1/12) \times xy^4 - (1/6) \times y^4 - (1/360) \times y^6$, where h and k are connected by the relation $2h^2 - k^2 + h = 0$

Ex. 4. *Solve* $r - s + p = 1$. **[Meerut 1993; Sagar 2003; Vikram 2004]**

Sol. The given equation can be re–written as $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) + (\partial z/\partial x) = 1$

or $(D^2 - DD' + D)z = 1$ or $D(D - D' + 1)z = 1$. ...(1)

$\therefore$ C.F. $= \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{D(1 + D - D')}1 = \frac{1}{D}(1 + D - D')^{-1}\,1 = \frac{1}{D}[1 - (D - D') + ...]\,1 = \frac{1}{D}1 = x$$

So the required general solution is $z = \phi_1(y) + e^{-x}\phi_2(y + x) + x.$

Ex. 5. *Solve* (*a*) $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

(*b*) $(D + D' - 1)(D + 2D' - 3)z = 2x + 3y$. **[Bhopal 2000, 03, 04]**

Sol. (*a*) Here C.F. $= \phi_1(y) + e^x\phi_2(y - x) + e^{2x}\phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

$$\text{P.I.} = \frac{1}{D(D + D' - 1)(D + 3D' - 2)}(x^2 - 4xy + 2y^2) = \frac{1}{2D}\{1 - (D + D')\}^{-1}\left\{1 - \frac{D + 3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\{1+(D+D')+(D+D')^2+\ldots\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\ldots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2\right.$$

$$\left.+\frac{(D+D')(D+3D')}{2}+\ldots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left(1+\frac{3D}{2}+\frac{5D'}{2}+\frac{7D^2}{4}+\frac{19D'^2}{4}+\frac{11DD'}{2}+\ldots\right)(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{(x^2-4xy+2y^2)+3(x-2y)+5(2y-2x)+\frac{7}{2}+19-22\right\}$$

$$= \frac{1}{2D}\left(x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4xy+\frac{x}{2}\right)$$

Hence the required general solution is z = C.F. + P.I., *i.e.*,

$z = \phi_1(y) + e^x\phi_2(y-x) + e^{2x}\phi_3(y-3x) - (x^3/6) + x^2y + y^2x - (7x^2/4) + 2xy + x/4.$

(*b*) Try yourself. **Ans.** $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + 2x/3 + y + 23/9$

Ex. 6. *Solve* $(D-1)(D-D'+1)z = 1+xy.$ **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, where ϕ_1 and ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D-1)(D-D'+1)}(1+xy) = -(1-D)^{-1}\{1+(D-D')\}^{-1}(1+xy)$$

$$= -(1+D+\ldots)\{1-(D-D')+(D-D')^2\ldots\}(1+xy) = -(1+D+\ldots)(1-D+D'-2DD'+\ldots)(1+xy)$$

$$= -(1+D'-DD'+\ldots)(1+xy) = -(1+xy+x-1) = -xy-x$$

$\therefore$ The required solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) - xy - x$

Ex. 7. *Solve* $(D^2 - D'^2 - 3D + 3D')z = xy.$

Sol. Re–writing, given equation is $(D-D')(D+D'-3)z = xy.$

Its C.F. $= \phi_1(y+x) + e^{3x}\phi_2(x-y)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left\{1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+\ldots\right\}xy = -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\ldots\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\frac{D'}{D}+\frac{1}{3}D'+\ldots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right) = -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right)$$

$\therefore$ solution is $z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - (1/6)\times x^2y - (x^2/9) - (2x/27) - (x^3/18)$

Ex. 8. Solve $(D+D'-1)(D+2D'-3)z = 4+3x+6y$.

Sol. Here C.F. $= e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+2D'-1)}(4+3x+6y) = \frac{1}{3}\{1-(D+D')\}^{-1}\left\{1-\frac{D+2D'}{3}\right\}^{-1}(4+3x+6y)$$

$$= \frac{1}{3}(1+D+D'+...)\left(1+\frac{D}{3}+\frac{2D'}{3}+...\right)(4+3x+6y) = \frac{1}{3}\left(1+\frac{4D}{3}+\frac{5D'}{3}+...\right)(4+3x+6y)$$

$$= (1/3)\times(4+3x+6y+4+10) = 6+x+2y.$$

$\therefore$ General solution is $z = e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)+6+x+2y.$

EXERCISE 5(E)

Solve the following partial differential equations:

1. $(D-D'-1)(D-D'-2)z = x$. **Ans.** $z = e^x\phi_1(y+x)+e^{2x}\phi_2(y+x)+(2x+3)/4$

2. $(D^2-DD'-2D'^2+2D+2D')z = xy$. **Ans.** $z = \phi_1(y-x)+e^{-2x}\phi_2(y+2x)+(1/4)\times(x^2y-xy-2x)$ $+(3x^2)/8-(x^3/12)$, ϕ_1, ϕ_2 being arbitrary functions

3. $(D^2-D'^2+D+3D'-2)z = x^2y$ **Ans.** $z = e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)-(4x^2y+4xy+6x^2+6y$ $+12x+21)/8$, ϕ_1, ϕ_2 being arbitrary functions.

Type 4. Examples based on case IV of Art. 5.12.

Ex. 1. *Solve* $(D^2 - D')z = xe^{ax+a^2y}$.

Sol. Since $(D^2 - D')$ cannot be resolved into linear factors in D and D', C.F. is obtained by considering the equation $(D^2 - D')z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ...(2)

$\therefore$ $D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) becomes

$A(h^2-k)e^{hx+ky} = 0$ so that $h^2-k=0$ or $k = h^2$.

$\therefore$ From (2), C.F. $= \Sigma A e^{hx+h^2y}$, A, h being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-D'}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{(D+a)^2-(D'+a^2)}x = e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}x = e^{ax+a^2y}\frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+...\right\}x = e^{ax+a^2y}\frac{1}{2aD}\left(x-\frac{1}{2a}\right)$$

$$= e^{ax+a^2y}\left\{(x^2/4a)-(x/4a^2)\right\}.$$

$\therefore$ General solution is $z = \sum Ae^{hx+h^2y} + e^{ax+a^2y}\{(x^2/4a)-(x/4a^2)\}$.

Ex. 2. *Solve* $(D - 3D' - 2)^2z = 2e^{2x}\sin(y + 3x)$ **[I.A.S. 2005]**

Sol. Here C.F. $= e^{2x}[\phi_1(y+3x)+x\phi_2(y+3x)]$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\sin(y+3x) = 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\sin(y+3x)$$

$$= 2e^{2x} \frac{1}{(D' - 3D')^2} \sin(y + 3x) = 2e^{2x} \frac{x^2}{1^2\, 2!} \sin(y + 3x),$$ using formula (*ii*) of Art. 4.8

$\therefore$ Required solution is $\quad z = e^{2x}[\phi_1(y + 3x) + x\phi_2(y + 3x)] + x^2e^{2x} \sin(y + 3x).$

Ex. 3. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = e^{x+y}.$

Sol. Given $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$ or $(D - 2D')(D - 2D' + 1)z = e^{x+y}$.

$\therefore$ C.F. $= \phi_1(y + 2x) + e^{-x}\phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 2D' + 1)}\left[\frac{1}{D - 2D'} e^{x+y}\right] = \frac{1}{D - 2D' - 1}\, \frac{1}{1 - 2} e^{x+y},$$ by case I of Art. 5.12

$$= -e^{x+y} \frac{1}{(D+1) - 2(D'+1) + 1}.1,$$ using result of case IV of Art. 5.12

$$= -e^{x+y} \frac{1}{D - 2D'} 1 = -e^{x+y} \frac{1}{D}\left(1 - \frac{2D'}{D}\right)^{-1} 1 = -e^{x+y} \frac{1}{D} 1 = -xe^{x+y}$$

$\therefore$ The required solution is $\quad z = \phi_1(y + 2x) + e^{-x}\phi_2(y + 2x) - xe^{x+y}.$

Ex. 4. *Solve* $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y} \cos(x + y)$. **[Delhi Maths (H) 1999, 2008]**

Sol. Since $(3D^2 - 2D'^2 + D - 1)$ cannot be resolved into linear factors in D and D', hence

C.F. $= \sum Ae^{hx+ky}$, where A, h are arbitrary constants connected by $3h^2 - 2k^2 + h - 1 = 0$.

$$\text{P.I.} = \frac{1}{3D^2 - 2D'^2 + D - 1} 4e^{x+y} \cos(x + y) = 4e^{x+y} \frac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1} \cos(x + y)$$

$$= 4e^{x+y} \frac{1}{3D^2 + 7D - 2D'^2 - 4D' + 1} \cos(x + y) = 4e^{x+y} \frac{1}{3(-1^2) + 7D - 2(-1^2) - 4D' + 1} \cos(x + y)$$

$$= 4e^{x+y} \frac{1}{7D - 4D'} \cos(x + y) = 4e^{x+y} (7D + 4D') \frac{1}{49D^2 - 16D'^2} \cos(x + y)$$

$$= 4e^{x+y} \frac{7D + 4D'}{49(-1^2) - 16(-1^2)} \cos(x + y)$$

$$= -(4/33) \times e^{x+y} (7D + 4D') \cos(x + y) = -(4/33)e^{x+y} \times [7D \cos(x + y) + 4D' \cos(x + y)]$$

$$= -(4/33) \times e^{x+y} [-7 \sin(x + y) - 4 \sin(x + y)] = (4/3) \times e^{x+y} \sin(x + y).$$

Hence general solution is $\quad z = \Sigma Ae^{hx+ky} + (4/3) \times e^{x+y} \sin(x + y).$

Ex. 5. *Solve* $(D - 1)(D - D' + 1)z = e^y$. **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-1)(D-D'+1)} e^y = \frac{1}{D - D' + 1}\, \frac{1}{D-1}\, e^{0\cdot x + 1\cdot y} = \frac{1}{D - D' + 1}\, \frac{1}{0 - 1}\, e^{0\cdot x + 1\cdot y}\, 1$$

$$= -e^{0\cdot x+1\cdot y} \frac{1}{(D+0) - (D'+1) + 1} 1 = -e^y \frac{1}{D}\left(1 - \frac{D'}{D}\right)^{-1} 1 = -e^y \frac{1}{D}\left(1 + \frac{D'}{D} + \dots\right) = -xe^y$$

$\therefore$ General solution is $\quad z = e^x \phi_1(y) + e^{-x}\phi_2(y + x) - x\, e^y$

Ex. 6. *Solve* $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$. **[Delhi Maths (H) 2004]**

Sol. Here C.F. = $e^{2x}\{\phi_1(y+3x)+x\phi_2(y+3x)\}$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}\,2e^{2x}\tan(y+3x) = 2\frac{1}{(D-3D'-2)^2}\,e^{2x+0\cdot y}\tan(y+3x)$$

$$= 2e^{2x+0\cdot y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\tan(y+3x) = 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$= 2e^{2x}\times(x^2/2!)\tan(y+3x)$, refer formula (*ii*), of Art. 4.12 of chapter 4

$\therefore$ General solution is $\quad z = e^{2x}\{\phi_1(y+3x)+x\,\phi_2(y+3x)\}+x^2e^{2x}\tan(y+3x)$

Ex. 7. *Solve* $(D^2-D'^2-3D+3D')z = e^{x+2y}$. **[Delhi Maths (H) 2005]**

Sol. Re–writing the given equation, we get $\quad (D-D')(D+D'-3)z = e^{x+2y}$.

Its C.F. $=\phi_1(y+x)+e^{3x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D+D'-3}\left\{\frac{1}{D-D'}\,e^{x+2y}\right\} = \frac{1}{D+D'-3}\,\frac{1}{(1-2)}\,e^{x+2y}\cdot 1$$

$$= -e^{x+2y}\frac{1}{D+1+D'+2-3}1 = -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}1$$

$$= -e^{x+2y}(1/D)\,(1-D'/D+\ldots..)\,1 = -e^{x+2y}x$$

$\therefore$ General solution is $\quad z = \phi_1(y+x)+e^{3x}\phi_2(y-x)-x\,e^{x+2y}$

Ex. 8. *Solve* (*i*) $(D^2-D')z = e^{x+y}$

(*ii*) $(D^2-D')z = e^{2x+y}$.

Sol. (*i*) C.F. = $\sum A\,e^{hx+ky}$, where $h^2-k=0$ so that $k=h^2$

$\therefore$ $\quad$ C.F. $= \sum A\,e^{hx+h^2y} = \sum A\,e^{h(x+hy)}$

$$\text{P.I.} = \frac{1}{D^2-D'}e^{x+y}\cdot 1 = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}1 = e^{x+y}\frac{1}{D^2+2D-D'}1$$

$$= -e^{x+y}\frac{1}{D'}\left(1-\frac{D^2+2D}{D'}\right)^{-1}1 = -e^{x+y}\frac{1}{D'}\left(1+\frac{D^2+2D}{D'}+\ldots.\right)1 = -e^{x+y}\frac{1}{D'}1 = -e^{x+y}y$$

$\therefore$ The required solution is $z = \sum A e^{h(x+hy)} - y e^{x+y}$, where A and h are arbitrary constants

(*ii*) C.F. is same as in part (*i*). Its P.I. is given by

$$\text{P.I.} = \frac{1}{D^2-D'}\,e^{2x+y} = \frac{1}{2^2-1}\,e^{2x+y} = \frac{1}{3}\,e^{2x+y}$$

$\therefore$ General solution is $z = \sum A\,e^{h(x+hy)} - (1/3)\times e^{2x+y}$, A, h being arbitrary constants.

Ex. 9. *Solve* $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y}\sin(2x+y)$

Sol. C.F. = $e^x\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+D'-3)(D+D')}\, e^{x+y}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+1+D'+1-1)(D+1+D'+1-3)(D+1+D'+1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+1)(D+D'-1)(D+D'+2)}\sin(2x+y) = e^{x+y}\frac{1}{(D+D'+2)(D^2+2DD'+D'^2-1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{D+D'+2}\,\frac{1}{-2^2-2\times(2\times 1)-1^2-1}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{(D+D')^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{D^2+2DD'+D'^2-4}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{-2^2-2\times(2\times 1)-1^2-4}\sin(2x+y)$$

$$= (1/130)\times e^{x+y}(D+D'-2)\sin(2x+y) = (1/130)\times e^{x+y}\{2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)\}$$

$\therefore$ Solution is $z = e^x\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)+(1/130)\times e^{x+y}\{3\cos(2x+y)-2\sin(2x+y)\}$

Ex. 10. *Solve* $r-3s+2t-p+2q = (2+4x)e^{-y}$

Sol. Re–writing the given equation $\qquad (D^2-3DD'+2D'^2-D+2D')z = (2+4x)\,e^{-y}$

or
$$(D-2D')(D-D'-1)z = (2+4x)\,e^{-y}$$

$\therefore$ C.F. = $\phi_1(y+2x)+e^x\phi_2(y+x)$, where ϕ_1, ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')(D-D'-1)}2\,e^{0\cdot x-y}(1+2x) = 2\,e^{0\cdot x-y}\frac{1}{\{D+0-2(D'-1)\}\{D+0-(D'-1)-1\}}(1+2x)$$

$$= 2e^{-y}\frac{1}{(D-2D'+2)(D-D')}(1+2x) = 2e^{-y}\frac{1}{2D}\left(1+\frac{D-2D'}{2}\right)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1+2x)$$

$$= e^{-y}\frac{1}{D}\left\{1-\frac{1}{2}(D-2D')+\ldots\right\}\left(1+\frac{D'}{D}+\ldots\right)(1+2x) = e^{-y}\frac{1}{D}\left(1-\frac{D}{2}+\ldots\right)(1+2x)$$

$$= e^{-y}(1/D)\,(1+2x-1) = x^2e^{-y}$$

$\therefore$ The required solution is $\qquad z = \phi_1(y+2x)+e^x\phi_2(y+x)+x^2e^{-y}$.

Ex. 11. *Solve* $(D^2-D')z = xe^{x+y}$

Sol. As usual C.F. = $\sum A\,e^{hx+ky}$, when $h^2-k=0$ or $k=h^2$. So C.F. $=\Sigma Ae^{hx+h^2y}$

$$\text{P.I.} = \frac{1}{D^2-D'}x\,e^{x+y} = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}x = e^{x+y}\frac{1}{D^2+2D-D'}x$$

$$= e^{x+y}\frac{1}{2D}\left(1+\frac{D^2-D'}{2D}\right)^{-1}x = e^{x+y}\frac{1}{2D}\left(1-\frac{D^2-D'}{2D}+\ldots\right)x = e^{x+y}\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D'}{2D}+\ldots\right)x$$

$$= e^{x+y}(1/2D)(x-1/2) = e^{x+y}(x^2/4-x/4).$$

$\therefore$ Solution is $z = \sum A\,e^{hx+h^2y}+(x/4)\times(x-1)\,e^{x+y}$, A, h being arbitrary constants.

EXERCISE 5(F)

Solve the following partial differential equations:

1. $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$. **Ans.** $z = \phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)$

$-(1/81)\times(9x^2+36y^2-18x-72y+76)e^{x+2y}$ ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

2. $(D^2+DD'+D+D'-1)z = e^{-2x}(x^2+y^2)$.

Ans. $z = \Sigma Ae^{hx+ky}+(1/27)\times e^{-2x}(9x^2+9y^2+18x+6y+14)$, where $h^2+hk+h+k+1=0$.

3. $(D^2D'+D'^2-2)z = e^{2y}\sin 3x - e^x\cos y$. **Ans.** $z = \Sigma Ae^{hx+ky}-(1/16)\times e^{2y}\sin 3x$

$+(1/20)\times e^x(3\cos 2y-\sin 2y)$, where h and k are related by $h^2k+k^2-2=0$.

4. $(D^2-DD'+D'-1)z = e^y$. **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-xe^y$

5. $(D^2-DD'+D'-1)z = e^x$ **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-(1/2)\times xe^x$

MISCELLANEOUS EXAMPLES ON ART. 5.12.

Ex. 1. *Solve* (*a*) $(D^2-DD'+D'-1)z = \cos(x+2y)+e^y$. **[Jabalpur 2004; I.A.S. 1992]**

(*b*) $(D^2-DD'+D'-1)z = \cos(x+2y)+e^x$.

Sol. (*a*) From given equation $(D-1)(D-D'+1)z = \cos(x+2y)+e^y$. ...(1)

$\therefore$ C.F. $= e^x\phi_1(y)+e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x+2y)$

$$= \frac{1}{D^2-DD'+D'-1}\cos(x+2y) = \frac{1}{-1^2+(1\times 2)+D'-1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\times\sin(x+2y).$$

P.I. corresponding to e^y

$$= \frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{D-D'+1}\cdot\frac{1}{D-1}e^{0\cdot x+1\cdot y} = \frac{1}{D-D'+1}\cdot\frac{1}{0-1}e^{0\cdot x+1\cdot y}$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D(1-D'/D)}1 = -e^y\frac{1}{D}\left(1-\frac{D'}{D}+\ldots\right)^{-1}1$$

$$= -e^y(1/D)(1+\ldots).1 = -e^y x.$$

$\therefore$ The general solution is $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)+(1/2)\times\sin(x+2y)-xe^y$.

(*b*) As in part (*a*), C.F. $= e^x\phi_1(y)+e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary function.

and P.I. corresponding to $\cos(x+2y) = (1/2)\times\sin(x+2y)$.

Now, P.I. corresponding to e^x

$$= \frac{1}{(D-1)(D-D'+1)}e^x = \frac{1}{(D-1)(D-D'+1)}e^{1\cdot x+0\cdot y} = \frac{1}{(D-1)}\frac{1}{(1-0+1)}e^{1\cdot x+0\cdot y}$$

$$= \frac{1}{2}\frac{1}{D-1}e^{1\cdot x+0\cdot y}1 = \frac{1}{2}e^{1\cdot x+0\cdot y}\frac{1}{(D+1)-1}1 = \frac{1}{2}e^x\frac{1}{D}1 = \frac{xe^x}{2}.$$

$\therefore$ The general solution is $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)+(1/2)\times\sin(x+2y)+(x/2)\times e^x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x + y)$. **[Delhi 2008]**

(*b*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy$.

(*c*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x + y)$.

Sol. (*a*) The given equation can be rewritten as

$$(D + D')(D - 2D' + 2)z = e^{2x+3y} + xy + \sin(2x + y). \qquad ...(1)$$

$\therefore$ C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D + D')(D - 2D' + 2)} e^{2x+3y} = \frac{1}{(2+3)(2-6+2)} e^{2x+3y} = -\frac{1}{10} e^{2x+3y}.$$

P.I. corresponding to xy

$$= \frac{1}{(D + D')(D - 2D' + 2)} xy = \frac{1}{D(1 + D'/D) \times 2\{1 + (D/2 - D')\}} xy$$

$$= \frac{1}{2D}\left(1 + \frac{D'}{D}\right)^{-1}\left\{1 + \left(\frac{D}{2} - D'\right)\right\}^{-1} xy = \frac{1}{2D}\left(1 - \frac{D'}{D} + \dots\right)\left\{1 - \left(\frac{D}{2} - D'\right) + \left(\frac{D}{2} - D'\right)^2 + \dots\right\} xy$$

$$= \frac{1}{2D}\left(1 - \frac{D'}{D} + \dots\right)\left(1 - \frac{D}{2} + D' - DD' + \dots\right) xy = \frac{1}{2D}\left(1 - \frac{D'}{D} + \dots\right)\left(xy - \frac{y}{2} + x - 1\right)$$

$$= \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{1}{D}\left(x - \frac{1}{2}\right)\right] = \frac{1}{2D}\left[xy - \frac{y}{2} + x - 1 - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2 y}{2} - \frac{xy}{2} + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{4}\right] = \frac{x^2 y}{4} + \frac{3x^2}{8} - \frac{xy}{4} - \frac{x}{2} - \frac{x^3}{12}.$$

P.I. corresponding to $\sin(2x + y)$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x + y) = \frac{1}{-2^2 + (2 \times 1) - 2 \times (-1^2) + 2D + 2D'} \sin(2x + y)$$

$$= \frac{1}{2(D + D')} \sin(2x + y) \quad = \frac{1}{2}(D - D')\frac{1}{(D^2 - D'^2)} \sin(2x + y)$$

$$= \frac{1}{2}\frac{1}{-2^2 - (-1^2)}(D - D')\sin(2x + y)$$

$$= -(1/6) \times (D - D')\sin(2x + y) = -(1/6) \times [D\sin(2x + y) - D'\sin(2x + y)]$$

$$= -(1/6) \times [2\cos(2x + y) - \cos(2x + y)] = -(1/6) \times \cos(2x + y).$$

The required solution is $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10) \times e^{2x+3y} + (1/4) \times x^2 y$
$+ (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12) - (1/6) \times \cos(2x + y)$

(*b*) As in part (*a*), C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

P.I. corresponding to $e^{2x+3y} = -(1/10) \times e^{2x+3y}$

P.I. corresponding to $xy = (1/4) \times x^2 y + (3/8) \times x^2 - (1/4) \times xy - (1/2) \times x - (1/12) \times x^3$.

$\therefore$ The required general solution is z = C.F. + P.I., *i.e.*

$$z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10) \times e^{2x+3y} + (1/4) \times x^2 y + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12).$$

(*c*) As in part (*a*), C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

P.I. corresponding to $xy = (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12)$
and P.I. corresponding to $\sin(2x + y) = -(1/6) \times \cos(2x + y)$.

$\therefore$ The required solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*, $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$ $+ (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12) \times - (1/6) \times \cos(2x + y)$.

Ex. 3. *Find a particular integral of the differential equation* : $(D^2 - D')z = e^{x+y} + 5\cos(x + 2y)$.

Sol. P.I. corresponding to e^{x+y}

$$= \frac{1}{D^2 - D'} e^{x+y} = e^{x+y} \frac{1}{(D+1)^2 - (D'+1)} 1 = e^{x+y} \frac{1}{D^2 + 2D - D'} 1$$

$$= e^{x+y} \frac{1}{2D}\left[1 + \left(\frac{D}{2} - \frac{D'}{2D}\right)\right]^{-1} 1 = e^{x+y} \frac{1}{2D}\{1 + \ldots\}1 = \frac{1}{2} x e^{x+y}$$

P.I. corresponding to $5\cos(x + 2y)$

$$= 5\frac{1}{D^2 - D'} \cos(x + 2y) = 5\frac{1}{-1^2 - D'} \cos(x + 2y) = -\frac{5}{D' + 1} \cos(x + 2y)$$

$$= -5(D' - 1)\frac{1}{D'^2 - 1} \cos(x + 2y) = -5\frac{1}{-2^2 - 1} (D' - 1) \cos(x + 2y)$$

$$= (D' - 1)\cos(x + 2y) = D'\cos(x + 2y) - \cos(x + 2y) = -2\sin(x + 2y) - \cos(x + 2y)$$

$\therefore$ Required P.I. $= (x/2) \times e^{x+y} - 2\sin(x + 2y) - \cos(x + 2y)$.

Ex. 4. Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$. **[Delhi Maths (Prog) 2007; Delhi Maths (H) 2007; Meerut 1998; Bhopal 1995; Indore 1998; KU Kurukshetra 2004]**

Sol. The given equation can be re–written as $(D - D')(D + D' - 3)z = xy + e^{x+2y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{3x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $xy = \dfrac{1}{(D - D')(D + D' - 3)} xy = -\dfrac{1}{3D}\left(1 - \dfrac{D'}{D}\right)^{-1}\left(1 - \dfrac{D + D'}{3}\right)^{-1} xy$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left[1 + \frac{D + D'}{3} + \left(\frac{D + D'}{3}\right)^2 + \ldots\right]xy$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left(1 + \frac{D + D'}{3} + \frac{2DD'}{9} + \ldots\right)xy = -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)xy$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{2x}{3} + \frac{1}{D}x + \frac{2}{9}\right) = -\frac{1}{9}\left(\frac{x^2 y}{2} + \frac{xy}{2} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D + D' - 3)} \frac{1}{D - D'} e^{x+2y} = \frac{1}{D + D' - 3} \frac{1}{(1 - 2)} e^{x+2y}$$

$$= -\frac{1}{D + D' - 3} e^{1\cdot x + 2y}\, 1 = -e^{1\cdot x + 2y} \frac{1}{(D + 1) + (D' + 2) - 3} 1 = -e^{x+2y} \frac{1}{D + D'} 1$$

$$= -e^{x+2y} \frac{1}{D}\left(1 + \frac{D'}{D}\right)^{-1} 1 = -e^{x+2y} \frac{1}{D}(1 + \ldots)\, 1 = -x\, e^{x+2y}.$$

Hence the required general solution is $z = \text{C.F} + \text{P.I.}$, *i.e.*

$z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - (x^2y/6) - (xy/6) - (x^2/9) - (x^3/18) - (2x/27) - xe^{x+2y}$.

Ex. 5. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$. **[Meerut 2008]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, P.I. corresponding to e^{2x-y}

$$= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y} = \frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}} e^{2x-y} = \frac{1}{2} e^{2x-y}$$

and P.I. corresponding to x

$$= \frac{1}{(D-D'-1)(D-D'-2)} x = \frac{1}{2\{1-(D-D')\}\{1-(D-D')/2\}} x$$

$$= \frac{1}{2}[1-(D-D')]^{-1}\left\{1-\frac{D-D'}{2}\right\}^{-1} x = \frac{1}{2}[1+(D-D')+\ldots]\left\{1+\frac{D-D'}{2}+\ldots\right\}x$$

$$= \frac{1}{2}\left\{1+(D-D')+\frac{D-D'}{2}+\ldots\right\}x = \frac{1}{2}\left\{1+\frac{3}{2}D+\ldots\right\}x = \frac{1}{2}\left(x+\frac{3}{2}\right).$$

$\therefore$ General solution is $\quad z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + (1/2) \times e^{2x-y} + x/2 + 3/4.$

Ex. 6. *Solve* (*a*) $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$. **[Meerut 1995]**

(*b*) $(D^2 - DD' - 2D)z = \sin(3x+4y) + x^2y$ **[Agra 2009, 10]**

Sol. (*a*) The given equation can be re–written as $D(D-D'-2)z = \sin(3x+4y) - e^{2x+y}$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y)$

$$= \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \frac{1}{-3^2 + (3\times 4) - 2D}\sin(3x+4y)$$

$$= \frac{1}{3-2D}\sin(3x+4y) = (3+2D)\frac{1}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y)$$

$= (1/45) \times [3\sin(3x+4y) + 2D\sin(3x+4y)] = (1/45)\times[3\sin(3x+4y) + 6\cos(3x+4y)]$

and P.I. corresponding to $(-e^{2x+y})$

$$= -\frac{1}{D(D-D'-2)}e^{2x+y} = -\frac{1}{2(2-1-2)}e^{2x+y} = \frac{1}{2}e^{2x+y}$$

Hence the required general solution is $\quad z = \text{C.F.} + \text{P.I.}$, *i.e.*

$z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)] + (1/2)\times e^{2x+y}$

(*b*) As in part (*a*), $\quad$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y) = (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)]$.

P.I. corresponding to $x^2y = \dfrac{1}{D(D-D'-2)}x^2y = -\dfrac{1}{2D}\left\{1-\left(\dfrac{D-D'}{2}\right)\right\}^{-1}x^2y$

$$= -\frac{1}{2D}\left\{1+\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2+\left(\frac{D-D'}{2}\right)^3+\ldots\right\}x^2y = -\frac{1}{2D}\left(1+\frac{D}{2}-\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}-\frac{3D^2D'}{8}+\ldots\right)x^2y$$

$$= -\frac{1}{2D}\left(x^2y+xy-\frac{x^2}{2}+\frac{y}{2}-x-\frac{3}{4}\right) = -\frac{1}{2}\left(\frac{x^3y}{3}+\frac{x^2y}{2}-\frac{x^3}{6}+\frac{xy}{2}-\frac{x^2}{2}-\frac{3x}{4}\right)$$

Hence the solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.* $z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y)$
$+ 2\cos(3x+4y)] - (1/6)\times x^3y - (1/4)\times x^2y + (1/12)\times x^3 - (1/4)\times xy - (x^2/4) + 3x/8$

Ex. 7. Solve $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) + (\partial z/\partial x) + 3(\partial z/\partial y) - 2z = e^{x-y} - x^2y.$ **[Rewa 1999]**

Sol. The given equation can be re–written as $\quad (D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$

or $\quad \{(D-D')(D+D') + 2(D+D') - (D-D'+2)\}z = e^{x-y} - x^2y$

or $\{(D+D')(D-D'+2) - (D-D'+2)\}z = e^{x-y} - x^2y$ or $(D-D'+2)(D+D'-1)z = e^{x-y} - x^2y.$

$\therefore$ C.F. $= e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{x-y}

$$= \frac{1}{(D-D'+2)(D+D'-1)}e^{x-y} = \frac{1}{\{1-(-1)+2\}(1-1-1)}e^{x-y} = -\frac{1}{4}e^{x-y}$$

and P.I. corresponding to $(-x^2y)$

$$= \frac{1}{(D-D'+2)(D+D'-1)}(-x^2y) = \frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\{1-(D+D')\}^{-1}x^2y$$

$$= \frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\left(\frac{D-D'}{2}\right)^3+\ldots\right\}\times\{1+(D+D')+(D+D')^2+(D+D')^3+\ldots\}x^2y$$

$$= \frac{1}{2}\left(1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+\ldots\right)\times(1+D+D'+D^2+2DD'+3D^2D'+\ldots)x^2y$$

$= (1/2)\times[1+(1/2)\times D+(3/2)\times D'+(3/4)\times D^2+(3/2)\times DD'+(21/8)\times D^2D'+\cdots]x^2y$
$= (1/2)\times[x^2y+xy+(3x^2/2)+(3y/2)+3x+21/4]$.

Hence general solution is z = C.F. + P.I., *i.e.* $z = e^{-2x}\phi_1(y+x)+e^x\phi_2(y-x)$
$-(1/4)\times e^{x-y}+(1/2)\times x^2y+(1/2)\times xy+(3/4)\times x^2+(3/4)\times y+(3/2)\times x+21/8.$

Ex. 8. *Solve* $(D^2-D')(D-2D')z = e^{2x+y}+xy$.

Sol. C.F. corresponding to linear factor $(D-2D')$ is $\phi(y+2x)$. Now, (D^2-D') cannot be resolved into linear factor in D and D'. To find C.F. corresponding to it, we consider the equation

$$(D^2-D')z = 0. \qquad ...(1)$$

Let a trial solution of (1) be $\quad z = Ae^{hx+ky}$. ...(2)

$\therefore\ D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) becomes
$A(h^2-k)e^{hx+ky} = 0$ so that $h^2-k=0$ or $k = h^2$.

So from (2), C.F. corresponding to (D^2-D') is $\sum Ae^{hx+h^2y}$.

Now, P.I. corresponding to e^{2x+y}

$$= \frac{1}{D-2D'}\cdot\frac{1}{D^2-D'}e^{2x+y} = \frac{1}{D-2D'}\frac{1}{2^2-1}e^{2x+y} = \frac{1}{3}\frac{1}{D-2D'}e^{2x+y}\cdot 1$$

$$= \frac{1}{3}e^{2x+y}\frac{1}{(D+2)-2(D'+1)}1 = \frac{1}{3}e^{2x+y}\frac{1}{D(1-2D'/D)}1 = \frac{1}{3}e^{2x+y}\frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}1$$

$= (1/3)\times e^{2x+y}\times(1/D)(1+\ldots)\,1 = (x/3)\times e^{2x+y}$

and P.I. corresponding to xy

$$= \frac{1}{(D-2D')(D^2-D')}xy = \frac{1}{(-2D')(1-D/2D')(-D')(1-D^2/D')}xy$$

$$= \frac{1}{2D'^2}\left(1-\frac{D}{2D'}\right)^{-1}\left(1-\frac{D^2}{D'}\right)^{-1}xy = \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\ldots\right)(1+\ldots)xy$$

$$= \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\ldots\right)xy = \frac{1}{2D'^2}\left(xy+\frac{1}{2D'}y\right) = \frac{1}{2D'^2}\left(xy+\frac{y^2}{4}\right) = \frac{1}{2}\left(\frac{xy^3}{6}+\frac{y^4}{3\times4\times4}\right).$$

$$\therefore \text{ General solution } z = \phi(y+2x)+\sum Ae^{hx+h^2y}+(x/3)\times e^{2x+y}+(xy^3)/12+y^4/96,$$

where ϕ is an arbitrary function and A and h are arbitrary constants.

Ex. 9. *Solve* $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y + xy + 1$.

Sol. The given equation can be re–written as

$$(D-1)(D-D'+1)z = \cos(x+2y) + e^y + xy + 1. \qquad ...(1)$$

Its C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\times\sin(x+2y),$$

P.I. corresponding to e^y *i.e.* $e^{0\cdot x + 1\cdot y}$

$$= \frac{1}{D^2 - DD' + D' - 1}e^{0\cdot x + 1\cdot y}\cdot 1 = e^{0\cdot x+1\cdot y}\frac{1}{(D+0)^2 - (D+0)(D'+1) + (D'+1) - 1}1$$

$$= e^y\frac{1}{D^2 - DD' + D'}1 = e^y\frac{1}{D'}\left\{1 + \left(\frac{D^2}{D'} - D\right)\right\}^{-1}1 = e^y\frac{1}{D'}\{1+\ldots\}1 = e^y y$$

and P.I. corresponding to $(xy + 1)$

$$= \frac{1}{(D-1)(D-D'+1)}(xy+1) = -(1-D)^{-1}\{1+(D-D')\}^{-1}(xy+1)$$

$$= -\{1+D+\ldots\}\{1-(D-D')+(D-D')^2-\ldots\}(xy+1) = -(1+D+\ldots)(1-D+D'-2DD'+\ldots)(xy+1)$$

$$= -(1+D+\ldots)(xy+1-y+x-2) = -(1+D+\ldots)(xy-y+x-1) = -(xy-y+x-1+y+1) = -(xy+x)$$

$\therefore$ Solution is $\quad z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + (1/2)\times\sin(x+2y) + ye^y - (xy+x)$.

EXERCISE 5(G)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x+y)$ **[KU Kurukshetra 2004]**

Ans. $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10)\times e^{2x+3y} - (1/6)\times\cos(2x+y)$.

2. $(D^2 - DD' + D' - 1)\,z = e^y + xy.$ **[Delhi Maths (H) 2005]**

Hint : Do like solved Ex. 9, Art. 5.13 **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + ye^y - xy - x + 1$

5.14. General method of finding particular integral for only reducible non–homogeneous linear partial differential equation, namely,

$$\boldsymbol{F(D, D')z = f(x, y)}$$

Let $\quad F(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_n)$

$\therefore$ P.I. of the given equation $= \dfrac{1}{F(D, D')}f(x, y)$

or

$$\text{P.I.} = \frac{1}{(a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_2)}f(x, y) \qquad \ldots(1)$$

In order to evaluate P.I. given by (1), we consider a solution of the following equation (assuming that $a \neq 0$)

$(aD + bD + c)z = f(x, y)$ or $\quad a\,p + b\,q = f(x, y) - c\,z \qquad \ldots(2)$

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{f(x,y) - cz}$$

Taking the first two fraction of (3), $ady - bdx = 0$ ... (3)

Integrating, $ay - bx = d$, d being an arbitrary constant ... (4)

From (4), we have $y = (d + bx)/a$, if $a \neq 0$ ... (5)

Taking the first and last fractions of (3) and using (5), we get

$$\frac{dz}{dx} = \frac{f(x,y) - cz}{a} = -\frac{cz}{a} + \frac{1}{a} f\left(x, \frac{d+bx}{a}\right) \quad \text{or} \quad \frac{dz}{dx} + \frac{c}{a} z = \frac{1}{a} f\left(x, \frac{d+bx}{a}\right) \quad \text{... (6)}$$

which is linear differential equation whose I.F. $= e^{\int (c/a)dx} = e^{cx/a}$

$\therefore$ Solution of (6) is $$z e^{cx/a} = \frac{1}{a} \int f\left(x, \frac{d+bx}{a}\right) dx$$

so that $$z = \frac{e^{-(cx/a)}}{a} \int f\left(x, \frac{d+bx}{a}\right) dx, \qquad a \neq 0 \quad \text{and} \quad d = ay - bx.$$

$\therefore$ From (2), $$\frac{1}{(aD + bD' + c)} f(x,y) = \frac{e^{-(cx/a)}}{a} \int f\left(x, \frac{d+bx}{a}\right) dx \quad \text{... (7)}$$

where $ay - bx = d$ and $a \neq 0$ d being arbitrary constant.

Similarly, if $b \neq 0$, we can show that

$$\frac{1}{(aD + bD' + c)} f(x,y) = \frac{e^{-(cy/b)}}{b} \int e^{cy/b} f\left(\frac{d+ay}{b}\right) dy, \quad \text{... (8)}$$

where $bx - ay = d$ and $b \neq 0$, d being an arbitrary constant.

Results (7) and (8) will be used to evaluate P.I. given by (1).

5.15. Working rule for finding P.I. of any reducible linear partial differential equation (homogeneous or non–homogeneous), namely,

$$F(D, D')z = f(x, y) \quad \text{... (1)}$$

Rule I. $\dfrac{1}{aD + bD' + c} f(x,y) = \dfrac{e^{-(cx/a)}}{a} \displaystyle\int e^{cx/a} f\left(x, \frac{d+bx}{a}\right) dx$, $a \neq 0$ where $ay - bx = d$

Note that constant d must be replaced by $ay - bx$ after integration is performed.

Rule II. $\dfrac{1}{aD + bD' + c} f(x,y) = \dfrac{e^{-(cy/b)}}{b} \displaystyle\int e^{cy/b} f\left(\frac{d+ay}{b}\right) dy$, $b \neq 0$, where $bx - ay = d$

Note that constant d must be replaced by $bx - ay$ after integration is performed.

We now consider some special cases of the above rules.

Rule III. $\dfrac{1}{aD + c} f(x,y) = \dfrac{e^{-(cx/a)}}{a} \displaystyle\int e^{cx/a} f(x, d/a)\, dx$, where $ay = d$

Rule IV. $\dfrac{1}{D - mD'} f(x,y) = \displaystyle\int f\,(x,\, d - mx)\, dx$, where $y + mx = d$

Rule V. $\dfrac{1}{bD' + c} f\,(x,y) = \dfrac{e^{-(cy/b)}}{b} \displaystyle\int e^{cy/b} f(d/b,\, y)\; dy$, where $bx = d$

Rule VI. $\dfrac{1}{D'-mD}f(x,y) = \int f(d-my,\ y)\ dy$, where $x+my=d$

Note 1. Results IV and VI have already been obtained in Art. 4.12 of chapter 4.

Note 2. Suppose $F(D,\ D')$ can be factored as $\prod_{r=1}^{n}(a_r D + b_r D' + c_r)$, then

P.I. for (1) $= \dfrac{1}{F(D,D')}f(x,y) = \dfrac{1}{(a_1D+b_1D'+c_1)(a_2D+b_2D'+C_2)....(a_nD+b_n\ D'+c_n)}f(x,y)$,

which is evaluated by using the above six rules for each factor, in succession, from right to the left.

5.16 SOLVED EXAMPLES BASED ON ART. 5.15.

Ex. 1. *Solve* $(D+D')\ (D+D'-2)z = \sin(x+2y)$ **[Delhi Maths (H) 2000]**

Sol. Here C.F. = $\phi_1(y-x)+e^{2x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary function

P.I. $= \dfrac{1}{D+D'-2}\left\{\dfrac{1}{D+D'}\sin\ (x+2y)\right\} = \dfrac{1}{D+D'-2}\int \sin\ (3x+2d)\,dx$, where, $y-x=d$,

[using rule IV of Art. 5.15]

$$= \frac{1}{D+D'-2}\left\{-\frac{\cos(3x+2d)}{3}\right\} = -\frac{1}{3}\ \frac{1}{D+D'-2}\cos\ (2x+y)$$

$= -\dfrac{1}{3}\ e^{2x}\int e^{-2x}\cos(3x+2d)dx$, where $y-x=d$ [using rule I of Art. 5.15]

$$= -\frac{1}{3}e^{2x}\ \frac{1}{(-2)^2+3^2}e^{-2x}\{-2\cos(3x+2d)+3\sin(3x+2d)\} = \frac{2}{39}\cos(x+2y)-\frac{1}{13}\sin(x+2y)$$

$\therefore$ Solution is $\quad z = \phi_1(y-x)+e^{2x}\phi_2(y-x)+(2/39)\times\cos(x+2y)-(1/13)\times\sin(x+2y)$

Ex. 2. *Solve* $(D^3 - DD'^2 - D^2 + DD')z = (x+2)/x^3$

Sol. Re–writing, the given equation $\quad D(D-D')\ (D+D'-1)z = (x+2)/x^3$

Its C.F. = $\phi_1(y)+\phi_2(y+x)+e^{x}\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

P.I. $= \dfrac{1}{(D-D')(D+D'-1)}\dfrac{1}{D}\left(\dfrac{1}{x^2}+\dfrac{2}{x^3}\right) = \dfrac{1}{(D+D'-1)\ (D-D)}\left(-\dfrac{1}{x}-\dfrac{1}{x^2}\right)$

$$= \frac{1}{D+D'-1}\int\left(-\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{1}{D+D'-1}\left(-\log x+\frac{1}{x}\right) = e^x\int e^{-x}\left(-\log x+\frac{1}{x}\right)dx$$

$$= -e^x\int e^{-x}\log x\,dx + e^x\int e^{-x}\frac{1}{x}dx = -e^x\left[(-e^{-x})\log x - \int(-e^{-x})\frac{1}{x}dx\right] + e^x\int e^{-x}\frac{1}{x}dx = \log x$$

[on integration by parts first integral only]

$\therefore$ General solution is $\quad z = \phi_1(y)+\phi_2(y+x)+e^{x}\phi_3(y-x) + \log x$.

Ex. 3. *Solve* $(D^2+DD'+D'-1)\ z = 4\sinh x$.

Sol. Re–writing, the given equation $\quad (D+1)\ (D+D'-1)\ z = 2(e^x - e^{-x})$.

C.F. = $e^{-x}\phi_1(y)+e^{x}\phi_2(y-x)$, where ϕ_1, ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D+1)(D+D'-1)} 2(e^x - e^{-x}) = \frac{1}{(D+1)} 2e^x \int e^{-x}(e^x - e^{-x})dx$$

$$= \frac{1}{(D+1)} 2e^x \left(x + \frac{1}{2} e^{-2x} \right) = \frac{1}{D+1}(2xe^x + e^{-x}) = e^{-x} \int e^x (2x\, e^x + e^{-x})dx$$

[using rule III of Art. 5.15]

$$= 2e^{-x} \int x\, e^{2x} dx + e^{-x} x = 2e^{-x} \left[x \times (e^{2x}/2) - \int 1 \cdot (e^{2x}/2) dx \right] + xe^{-x}$$

$$= x\, e^x - e^{-x} \int e^{2x} dx + xe^{-x} = xe^x - e^{-x} \times (1/2) \times e^{2x} + xe^{-x} = (x - 1/2)e^x + xe^{-x}$$

$\therefore$ General solution is $\quad z = e^{-x}\phi_1(y) + e^x \phi_2(y - x) + (x - 1/2)e^x + xe^{-x}$

Ex. 4. *Solve* $(D^2 - DD' + D' - 1)z = 1 + xy + e^y + \cos(x + 2y)$ **[Delhi Maths (H) 2001]**

Sol. Re–writing the given equation $\quad (D-1)\,(D - D' + 1)z = 1 + xy + e^y + \cos(x + 2y)$

C.F. $= e^x \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-1)\,(D - D' + 1)} \{1 + xy + e^y + \cos\,(x + 2y)\}$$

$$= \frac{1}{D-1} e^{-x} \int e^x \{1 + x\,(d - x) + e^{d-x} + \cos(2d - x)\} dx, \quad \text{where } d = y + x$$

$$= \frac{1}{D-1}\, e^{-x} \left\{ \int (1 + d\,x - x^2) e^x dx + \int e^a dx + \int e^x \cos(x - 2d) dx \right\}$$

$$= \frac{1}{D-1} e^{-x} [(1 + dx - x^2)e^x - (d - 2x)\, e^x + (-2)e^x + e^d x + \frac{e^x}{1^2 + 1^2} \{\cos\,(x - 2d) + \sin\,(x - 2d)\}]$$

$$= \frac{1}{D-1} [(1 + d\,x - x^2 - d + 2x - 2) + e^{d-x} x + (1/2) \times \{\cos(x - 2d) + \sin\,(x - 2d)\}]$$

$$= \frac{1}{D-1} [-1 + x(y + x) - x^2 - (y + x) + 2x + e^y x + (1/2) \times \{\cos(-2y - x) + \sin(-2y - x)\}]$$

$$= \frac{1}{D-1} \{xy - y + x - 1 + x\,e^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin\,(2y + x)\}$$

$$= \frac{1}{D-1} \{(x-1)\,(y+1) + xe^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$$

$$= e^x \int e^{-x} \{(x-1)\,(k+1) + xe^k + (1/2) \times \cos(2k + x) - (1/2) \times \sin(2k + x)\}, \text{ where } y = k$$

$$= e^x \left[(k+1) \int e^{-x}(x-1)dx + e^k \int xe^{-x} dx + \frac{1}{2} \int e^{-x} \cos(2k + x)]\, dx - \frac{1}{2} \int e^{-x} \sin(2k + x) dx \right]$$

$$= e^x (k+1)\, \{(-e^{-x})\,(x+1) - (e^{-x})\,(1)\} + e^x e^k \{(-e^{-x})(x) - (e^{-x})(1)\}]$$

$$+ \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\cos(2k + x) + \sin(2k + x)\} - \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\sin(2k + x) - \cos(2k + x)]$$

$$= -(k + 1)\,x - e^k\,(x + 1) + (1/2) \times \sin\,(2k + x)] = -(y + 1)x - e^y (x + 1) + (1/2) \sin\,(2y + x)$$

$\therefore$ solution is $\quad z = e^x \phi_1(y) + e^{-x}\phi_2(y + x) - x(y + 1) - (x + 1)e^y + (1/2) \times \sin(2y + x))$

Ex 5. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x + y)$

Ans. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) + (x/24) \times (6xy - 6y + 9x - 2x^2 - 12) - (1/6) \times \cos(2x + y).$

5.17. Solutions under given geometrical conditions

We have seen that solutions of non–homogeneous linear partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Example. *Find a surface satisfying* $r + s = 0$, *i.e.*, $(D^2 + DD')z = 0$ *and touching the elliptic paraboloid* $z = 4x^2 + y^2$ *along its section by the plane* $y = 2x + 1$. **[I.A.S. 1994]**

Sol. Given $(D^2 + DD')z = 0.$ or $D(D + D') = 0$ ...(1)

$\therefore$ Solution of (1) is $z = \text{C.F.} = \phi_1(y) + \phi_2(y - x),$ ...(2)

where ϕ_1 and ϕ_2 are arbitrary functions.

Since (2) touches the curve given by $z = 4x^2 + y^2$ ...(3)

and $y = 2x + 1,$ ...(4)

values of p $(= \partial z/\partial x)$ and q $(= \partial z/\partial y)$ obtained from (2) and (3) must be equal for any point on (4).

$\therefore$ $-\phi_2'(y - x) = 8x$ for $y = 2x + 1$ or $\phi_2'(x + 1) = -8x.$ ...(5)

and $\phi_1'(y) + \phi_2'(y - x) = 2y$ for $y = 2x + 1$ or $\phi_1'(2x + 1) + \phi_2'(x + 1) = 4x + 2$ ...(6)

From (5), $\phi_2'(x) = 8 - 8x$

Integrating it, $\phi_2(x) = 8x - 4x^2 + c_1$, c_1 being an arbitrary constant ...(7)

Subtracting (5) from (6), $\phi_1'(2x + 1) = 12x + 2 = 6(2x + 1) - 4$

so that $\phi_1'(x) = 6x - 4.$

Integrating it, $\phi_1(x) = 3x^2 - 4x + c_2$, c_2 being an arbitrary constant ...(8)

From (8), $\phi_1(y) = 3y^2 - 4y + c_2.$

and from (7), $\phi_2(y - x) = 8(y - x) - 4(y - x)^2 + c_1.$

Putting the above values of $\phi_1(y)$ and $\phi_2(y - x)$ in (2), we get

$$z = 3y^2 - 4y + c_2 + 8(y - x) - 4(y - x)^2 + c_1$$

or $z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $c_3 = c_1 + c_2.$ ...(9)

Equating the values of z from (3) and (9), we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $y = 2x + 1.$

$\therefore$ $c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy = 8x^2 + 2(2x + 1)^2 - 4(2x + 1) + 8x - 8x(2x + 1) = -2$

Hence, from (9), the required surface is $4x^2 - 8xy + y^2 - 4y + z + 2 = 0.$

MISCELLANEOUS PROBLEM ON CHAPTER 5

1. Find the solution of the equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$, which tends to zero as $x \to \infty$ and has the value *cos y* when $x = 0$ **Ans.** $z = (1 - x/2)e^{-x}\cos y$

6

Partial Differential Equations Reducible to Equations with Constant Coefficients

6.1. INTRODUCTION

In chapters 4 and 5, we have discussed methods of solving linear partial differential equations with constant coefficients. In this chapter, we propose to discuss the method of solving the so-called *Euler-Cauchy type partial differential equations* of the form

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \ldots + a_n y^n \frac{\partial^n z}{\partial y^n} + \ldots = f(x, y), \quad \ldots(1)$$

having variable coefficients in particular form (namely the term $\partial^n z / \partial x^n$ is multiplied by x^n, $\partial^n z / \partial y^n$ is multiplied by y^n, $\partial^n z / \partial x^r \partial y^{n-r}$, $r = 1, 2, \ldots, n-1$ is multiplied by $x^r y^{n-r}$ and so on. If $D \equiv \partial / \partial x$ and $D' \equiv \partial / \partial y$, then (1) can be re-written as

$$(a_0 x^n D^n + a_1 x^{n-1} y\, D^{n-1} D' + a_2\, x^{n-2} y^2\, D^{n-2} D'^2 + \ldots + a_n y^n D'^n + \ldots) z = f(x, y) \quad \ldots(2)$$

Examples of such equations are: $(x^2 D^2 - y^2 D'^2) z = xy;\ x^2 D^2 - y^2 D'^2 + xD - yD' = \log x$

6.2. METHOD OF REDUCING EULER-CAUCHY TYPE EQUATION TO A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Consider Euler-Cauchy type equation

$$a_0 x^n D^n + a_1 x^{n-1} y D^{n-1} D' + a_2 x^{n-2} y^2 D^{n-2} D'^2 + \ldots + a_n y^n D'^n + \ldots) z = f(x, y) \quad \ldots(1)$$

where $D \equiv \partial / \partial x$ and $D' \equiv \partial / \partial y.$ Define two new variables u and v by

$$x = e^u \quad \text{and} \quad y = e^v \quad \text{so that} \quad u = \log x \quad \text{and} \quad v = \log y \quad \ldots(2)$$

Let $\quad D_1 \equiv \partial / \partial u \quad$ and $\quad D_1' \equiv \partial / dv \quad \ldots(3)$

Now, $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial u}, \text{ using (2)}$$

$$\therefore \quad x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \quad \text{so that} \quad x \frac{\partial}{\partial x} = \frac{\partial}{\partial u} \quad \text{or} \quad xD = D_1 \quad \ldots(4)$$

Again, $$x \frac{\partial}{\partial x}\left(x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}} \right) = x^n \frac{\partial^n z}{\partial x^n} + (n-1) x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\therefore \quad x^n \frac{\partial^n z}{\partial x^n} = \left(x \frac{\partial}{\partial x} - n + 1 \right) x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

so that $$x^n D^n = (D_1 - n + 1) x^{n-1} D^{n-1}. \quad \ldots(5)$$

Putting $n = 2, 3, \ldots$ in (5), we have

$$x^2 D^2 = (D_1 - 1) xD \quad \text{or} \quad x^2 D^2 = D_1 (D_1 - 1), \text{ using (4)} \quad \ldots(6)$$

$x^3D^3 = (D_1 - 2)x^2D^2$ or $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$, using (6) ...(7)

and so on. Similarly, we have

$yD' = D_1'$, $y^2D'^2 = D_1'(D_1' - 1)$, $y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2)$ ...(8)

and so on. Also, we have $xyDD' = D_1D_1'$. ...(9)

and $x^m y^n D^m D'^n = D_1(D_1 - 1)...(D_1 - m + 1)D_1'(D_1' - 1)...(D_1' - n + 1)$. ...(10)

Using the substitutions (2) and results (4), (5), (6), (7), (8), (9) and (10), the given equation (1) reduces to an equation having constant coefficients and now it can easily be solved by the methods already discussed for homogeneous (refer Chapter 4) and non-homogeneous (refer Chapter 5) linear equations with constant coefficients. Finally, with help of (2), the solution is obtained in terms of old variables x and y.

6.3. WORKING RULE FOR SOLVING EULER-CAUCHY TYPE PARTIAL DIFFERENTIAL EQUATION

$$(a_0x^nD^n + a_1x^{n-1}y\,D^{n-1}D' + a_2x^{n-2}y\,D^{n-2}\,D'^2 + ... + a_ny^nD'^n + ...)z = f(x, y) \qquad ...(1)$$

Step 1 Introduce two new variables u and v :

$x = e^u$ and $y = e^v$, *i.e.*, $u = \log x$ and $v = \log y$ ...(2)

Step 2 We have $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$. Also let $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$

Step 3 Use the following results in (1)

$$\left.\begin{array}{l} xD = D_1, \quad yD' = D_1', \quad x^2D^2 = D_1(D_1 - 1), \quad y^2D'^2 = D_1'(D_1' - 1) \\ x^3D^3 = D_1(D_1 - 1)(D_1 - 2), \quad y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2) \text{ and so on} \end{array}\right\} \qquad ...(3)$$

Step 4 Using (2) and (3) in (1), we obtain the following linear partial differential (homogeneous or non-homogeneous) $(b_0\,D_1^n + b_1D_1^{n-1}D_1' + b_2D_1^{n-2}D_1'^2 + ... + b_nD_1'^n + ...)z = g(u, v)$ ...(4)

Step 5 If (4) is homogeneous linear partial differential equation, then it is solved with help of methods of chapter 4. Again, if (4) is non-homogeneous linear partial differential equation, then it is solved with help of methods of chapter 5.

Step 6 Using $u = \log x$ and $v = \log y$ in the solution obtained in step 5, we finally obtain the required solution in terms of the original variables x and y.

6.4. SOLVED EXAMPLES BASED ON ART. 6.3.

Ex. 1. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) - y(\partial z/\partial y) + x(\partial z/\partial x) = 0.$ **(Jabalpur 1996)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1) - D_1' + D_1]z = 0$$

or $(D_1^2 - D_1'^2)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0$.

Hence the required general solution is $z = \text{C.F.} = \phi_1(v + u) + \phi_2(v - u)$

or $z = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (xy) + \phi_2 \log (y/x)$ or $z = f_1(xy) + f_2(y/x)$, where f_1, f_2 are arbitrary functions.

Ex.2. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 0.$ **[Delhi Maths (H) 1994, CDLU 2004]**

or Solve $x^2r + 2xys + y^2t = 0$ **(Purvanchal 2007)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation can be written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = 0$ which reduces to

$$[D_1(D_1 - 1) + 2DD' + D'(D' - 1)]z = 0$$

$$[(D_1 + D_1')^2 - (D_1 + D_1')]z = 0 \qquad \text{or} \qquad (D_1 + D_1')(D_1 + D_1' - 1)z = 0.$$

Hence the required general solution is $z = \text{C.F.} = \phi_1(v - u) + e^u\phi_2(v - u)$

or $z = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (y/x) + x\phi_2 \log (y/x)$ or $z = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Ex. 3. *Solve* $x^2(\partial^2 z/\partial x^2) - 3xy(\partial^2 z/\partial x\partial y) + 2y^2(\partial^2 z/\partial y^2) + 5y(\partial z/\partial y) - 2z = 0.$

Sol. Let $x = e^u$, $y = e^v$, so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 3xyDD' + 2y^2D'^2 + 5yD' - 2)z = 0$ becomes

$$[D_1(D_1 - 1) - 3D_1D_1' + 2D_1'(D_1' - 1) + 5D_1' - 2]z = 0$$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2 - D_1 + 3D_1' - 2)z = 0$ or $(D_1 - D_1' - 2)(D_1 - 2D_1' + 1)z = 0.$

Hence general solution is $z = \text{C.F.} = e^{2u}\phi_1(v + u) + e^{-u}\phi_2(v + 2u)$

or $z = (e^u)^2\phi_1(\log y + \log x) + (e^u)^{-1}\phi_2(\log y + 2\log x) = x^2\phi_1 \log (xy) + x^{-1}\phi_2 \log (yx^2)$, using (1)

or $z = x^2f_1(xy) + x^{-1}f_2(yx^2)$, f_1, f_2 being arbitrary functions.

Ex. 4. *Solve* $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^my^n$, where $(m + n) \neq 0, 1$.

[Delhi Maths (H) 1994, KU Kurukshetra 2004]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/dv$.

Then the given equation reduces to $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = e^{mu}.e^{nv}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = e^{mu + nv}$ or $(D_1 + D_1')(D_1 + D_1' - 1) = e^{mu + nv}$

Here C.F.$= \phi_1(v - u) + e^u\phi_2(v - u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (y/x) + x\phi_2 \log (y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{(D_1 + D_1')(D_1 + D_1' - 1)}e^{mu + nv} = \frac{1}{(m + n)(m + n - 1)}e^{mu+nv}$$

$$= \frac{1}{(m + n)\,(m + n - 1)}(e^u)^m(e^{v})^n = \frac{1}{(m + n)(m + n - 1)}x^my^n \text{, using (1).}$$

$\therefore$ Required general solution is $z = f_1(y/x) + xf_2(y/x) + [1/\{(m + n)(m + n - 1)\}]x^my^n.$

Ex. 5. *Solve* $x^2(\partial^2 z/\partial x^2) - 4xy(\partial^2 z/\partial x\partial y) + 4y^2(\partial^2 z/\partial y^2) + 6y(\partial z/\partial y) = x^3y^4.$

[Jabalpur 2004; Vikram 2004; Meerut 1999; Delhi Maths (H) 1995]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$ becomes

$$[D_1(D_1 - 1) - 4D_1D_1' + 4D_1'(D_1' - 1) + 6D_1']z = e^{3u}e^{4v}$$

or $$[(D^2{}_1 - 4D_1D_1' + 4D'^2_1) - (D_1 - 2D_1')]z = e^{3u + 4v}$$

or $[(D_1 - 2D_1')^2 - (D_1 - 2D_1')]z = e^{3u + 4v}$ or $(D_1 - 2D_1')(D_1 - 2D_1' - 1)z = e^{3u + 4v}$

Here C.F. $= \phi_1(v + 2u) + e^u\phi_2(v + 2u) = \phi_1(\log y + 2\log x) + x\phi_2(\log y + 2\log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (yx^2) + x\phi_2 \log (yx^2) = f_1(yx^2) + xf_2(yx^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{(D_1 - 2D_1')(D_1 - 2D_1' - 1)} e^{3u+4v} = \dfrac{1}{\{3-(2\times 4)\}\times\{3-(2\times 4)-1\}} e^{3u+4v}$

$= (1/30)\times(e^u)^3(e^v)^4 = (1/30)\times x^3y^4$, using (1)

$\therefore$ The required general solution is $z =$ C.F. + P.I. or $z = f_1(yx^2) + xf_2(yx^2) + (1/30)\times x^3y^4$.

Ex. 6. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) - x(\partial z/\partial x) = x^3/y^2$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 + 2xyDD' - xD)z = x^3y^{-2}$ becomes

$$[D_1(D_1-1) + 2D_1D_1' - D_1]z = (e^u)^3(e^v)^{-2}$$

or $(D_1^2 + 2D_1D_1' - 2D_1)z = e^{3u-2v}$ or $D_1(D_1 + 2D_1' - 2)z = e^{3u-2v}$

$\therefore$ C.F. $= \phi_1(v) + e^{2u}\phi_2(v-2u) = \phi_1(v) + (e^u)^2\phi_2(v-2u) = \phi_1(\log y) + x^2\,\phi_2(\log y - 2\log x)$, using (1)

$= \phi_1(\log y) + x^2\phi_2(\log(y/x^2)) = f_1(y) + x^2f_2(y/x^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{D_1(D_1 + 2D_1' - 2)} e^{3u-2v} = \dfrac{1}{3(3-4-2)}(e^u)^3(e^v)^{-2} = -\dfrac{x^3y^{-2}}{9} = -\dfrac{x^3}{9y^2}$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(y) + x^2f_2(y/x^2) - (x^3/9y^2)$.

Ex. 7. *Solve* $x^2r - 3xys + 2y^2t + px + 2qy = x + 2y$.

Sol. The given equation can be re-written as

$$x^2(\partial^2 z/\partial x^2) - 3xy(\partial^2 z/\partial x\partial y) + 2y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) + 2y(\partial z/\partial y) = x + 2y$$

or $(x^2D^2 - 3xyDD' + 2y^2D'^2 + xD + 2yD')z = x + 2y$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

$\therefore$ (1) becomes $[D_1(D_1-1) - 3D_1D_1' + 2D_1'(D_1'-1) + D_1 + 2D_1']z = e^u + 2e^v$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2)z = e^u + 2e^v$ or $(D_1 - D_1')(D_1 - 2D_1')z = e^u + 2e^v$.

$\therefore$ C.F. $= \phi_1(v+u) + \phi_2(v+2u) = \phi_1(\log y + \log x) + \phi_2(\log y + 2\log x)$

or C.F. $= \phi_1\log(xy) + \phi_2\log(x^2y) = f_1(xy) + f_2(x^2y)$, where f_1 and f_2 are arbitrary functions.

Also, P.I. =

$$\frac{1}{(D_1 - D_1')(D_1 - 2D_1')}(e^u + 2e^v) = \frac{1}{(D_1 - D_1')(D_1 - 2D_1')}e^{1\cdot u + 0\cdot v} + 2\frac{1}{(D_1 - D_1')(D_1 - 2D_1')}e^{0\cdot u + 1\cdot v}$$

$$= \frac{1}{(1-0)(1-0)}e^u + 2\frac{1}{(0-1)(0-2)}e^v = x + y, \text{ using (2)}$$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(xy) + f_2(x^2y) + x + y$.

Ex. 8. *Find the general solution of* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) + nz = n\{x(\partial z/\partial x) + y(\partial z/\partial y)\} + x^2 + y^2 + x^3$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation reduces to

$$[x^2D^2 + 2xyDD' + y^2D'^2 - n(xD + yD') + n]z = x^2 + y^2 + x^3$$

or $[D_1(D_1-1) + 2D_1D_1' + D_1'(D_1'-1) - n(D_1 + D_1') + n]z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')^2 - (D_1 + D_1') - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')(D_1 + D_1' - 1) - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $(D_1 + D_1' - 1)(D_1 + D_1' - n)z = e^{2u} + e^{2v} + e^{3u}$.

$\therefore$ C.F. $= e^u\phi_1(v-u) + e^{nu}\phi_2(v-u) = e^u\phi_1(v-u) + (e^u)^n\phi_2(v-u)$

$= x\phi_1(\log y - \log x) + x^n\phi_2(\log y - \log x) = x\phi_1\log(y/x) + x^n\phi_2\log(y/x)$, using (1)

$= xf_1(y/x) + x^n f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}(e^{2u} + e^{2v} + e^{3u}) = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{2u + 0\cdot v}$$

$$+ \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{0\cdot u + 2v} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{3u + 0\cdot v}$$

$$= \frac{(e^u)^2}{(2+0-1)(2+0-n)} + \frac{(e^v)^2}{(0+2-1)(0+2-n)} + \frac{(e^u)^3}{(3+0-1)(3+0-n)} = \frac{x^2+y^2}{2-n} + \frac{x^3}{2(3-n)}$$

Hence general solution is $z = xf_1(y/x) + x^n f_2(y/x) + (x^2 + y^2)/(2 - n) + x^3/2(3 - n)$.

Ex. 9. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) = xy$ or $(x^2D^2 - y^2D'^2)z = xy$.

(Bilaspur 1999, Jabalpur 2003, Jiwaji 2003, 04, Vikram 2004, Ravishankar 2010, I.A.S. 1987, Rohilkhand 1995, Delhi Maths (H) 2004, 06)

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2)z = xy$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1)]z = e^u e^v \quad \text{or} \quad (D_1^2 - D_1'^2 - D_1 + D_1')z = e^{u+v}$$

or $[(D_1 - D_1')(D_1 + D_1') - (D_1 - D_1')]z = e^{u+v}$ or $(D_1 - D_1')(D_1 + D_1' - 1)z = e^{u+v}$.

$\therefore$ C.F. $= \phi_1(v + u) + e^u\phi_2(v - u) = \phi_1(\log y + \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log (xy) + x\phi_2 \log (y/x) = f_1(xy) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1 - D_1')(D_1 + D_1' - 1)} e^{u+v} = \frac{1}{D_1 - D_1'}\frac{1}{(1+1-1)} e^{u+v} = \frac{u}{1!} e^{u+v} = ue^u e^v = xy \log x$$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(xy) + xf_2(y/x) + xy \log x$.

Ex. 10. *Solve* $yt - q = xy$.

Sol. The given equation can be rewritten as $y(\partial^2 z/\partial y^2) - (\partial z/\partial y) = xy$

or $y^2(\partial^2 z/\partial y^2) - y\,(\partial z/\partial y) = xy^2$ or $(y^2D'^2 - yD')z = xy^2$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ... (2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then (1) becomes $[D_1'(D_1' - 1) - D_1']z = e^u e^{2v}$ or $D_1'(D_1' - 2)z = e^{u+2v}$

$\therefore$ C.F. $= \phi_1(u) + e^{2v}\phi_2(u) = \phi_1(\log x) + y^2\phi_2(\log x)$, by (2)

$= f_1(x) + y^2 f_2(x)$, f_1 and f_2 being arbitrary functions.

$$\text{Also, P.I.} = \frac{1}{(D_1' - 2)D_1'} e^{u+2v} = \frac{1}{D_1 - 2}\,\frac{1}{2} e^{u+2v} = \frac{1}{2} e^{u+2v} \frac{1}{D' + 2 + 2} \cdot 1$$

$$= \frac{1}{2} e^{u+2v} \frac{1}{D'} 1 = \frac{1}{2}\, e^u \times (e^v)^2 \times v = \frac{xy^2}{2} \log y.$$

Hence the required general solution is z = C.F. + P.I. or $z = f_1(x) + y^2 f_2(x) + (1/2) \times xy^2 \log y$.

Ex. 11. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = (x^2 + y^2)^{n/2}$. **[Delhi Maths 1999]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation can be re-written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = (x^2 + y^2)^{n/2}$

or $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = (e^{2u} + e^{2v})^{n/2}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = (e^{2u} + e^{2v})^{n/2}$ or $(D_1 + D_1')(D_1 + D_1' - 1)z = (e^{2u} + e^{2v})^{n/2}$.

$\therefore$ C.F. $= \phi_1(v-u) + e^u\phi_2(v-u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log(y/x) + x\phi_2 \log(y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}(e^{2u}+e^{2v})^{n/2} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\left\{1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right\}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu+0\cdot v}+\frac{n}{2}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-2)u+2v}$$
$$+\frac{(n/2)\{(n/2)-1\}}{2!}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-4)u+4v}+\ldots$$

$$= \frac{1}{(n+0)(n+0-1)}e^{nu}+\frac{n}{2}\frac{1}{\{(n-2)+2\}\{(n-2)+2-1\}}e^{(n-2)u+2v}$$
$$+\frac{(n/2)[(n/2)-1]}{2!}\frac{1}{\{(n-4)+4\}\{(n-4)+4-1\}}e^{(n-4)u+4v}+\ldots$$

$$= \frac{e^{nu}}{n^2-n}\left[1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right] = \frac{e^{nu}}{n^2-n}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{n^2-n}\{e^{2u}+e^{2u}e^{2(v-u)}\}^{n/2} = \frac{1}{n^2-n}(e^{2u}+e^{2v})^{n/2} = \frac{1}{n^2-n}(x^2+y^2)^{n/2}, \text{ using (1)}$$

Hence the required general solution is $z = f_1(y/x) + xf_2(y/x) + \{1/(n^2-n)\}(x^2+y^2)^{n/2}$.

Ex.12. *Solve* $x^2r - y^2t + px - qy = \log x$. **[KU Kurukshatra 2004; Meerut 2008]**

Or $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$. **[Delhi Maths (G) 2004; I.A.S. 1997]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$ becomes

$[D_1(D_1-1) - D_1'(D_1'-1) + D_1 - D_1']z = u$ or $(D_1^2 - D_1'^2)z = u$

or $$(D_1 + D_1')(D_1 - D_1')z = u.$$

$\therefore$ C.F. $= \phi_1(v-u) + \phi_2(v+u) = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1 \log(xy) + \phi_2 \log(y/x) = f_1(xy) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

and $$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}u = \frac{1}{D_1^2(1-D_1'^2/D_1^2)}u = \frac{1}{D_1^2}\left(1-\frac{D_1'^2}{D_1^2}\right)^{-1}u = \frac{1}{D_1^2}\left(1+\frac{D_1'^2}{D_1^2}+\ldots\right)u$$

$= u^3/6 = (\log x)^3/6$, using (1)

$\therefore$ Required solution is $z = f_1(xy) + f_2(y/x) + (1/6) \times (\log x)^3$, f_1, f_2 being arbitrary functions.

Ex. 13. *Solve* $(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x) - (1/2)$.

[Delhi B.Sc. (Hons) III 2011]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$[D_1(D_1-1) - D_1D_1' - 2D_1'(D_1'-1) + D_1 - 2D_1']z = \log y - \log x - (1/2)$$

or $(D_1^2 - D_1D_1' - 2D_1'^2)z = v - u - (1/2)$ or $(D_1 - 2D_1')(D_1 + D_1') = v - u - (1/2)$.

$\therefore$ C.F. $= \phi_1(v+2u) + \phi_2(v-u) = \phi_1(\log y + 2\log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1(\log(yx^2)) + \phi_2(\log(y/x)) = f_1(yx^2) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D_1^2 - D_1 D_1' - 2D_1'^2}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2\,(1 - D_1'/D_1 - 2D_1'^2/D_1^2)}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{1 - \left(\frac{D_1'}{D_1} + \frac{2D_1'^2}{D_1^2}\right)\right\}^{-1}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2}\left(1 + \frac{D_1'}{D_1} + \ldots\right)\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{v - u - \frac{1}{2} + \frac{1}{D_1}D_1'\left(v - u - \frac{1}{2}\right)\right\} = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + \frac{1}{D_1}\cdot 1\right) = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + u\right)$$

$$= \frac{1}{D_1^2}\left(v - \frac{1}{2}\right) = \left(v - \frac{1}{2}\right)\frac{u^2}{2} = \frac{1}{2}u^2 v - \frac{1}{4}u^2 = \frac{(\log x)^2 \log y}{2} - \frac{(\log x)^2}{4}, \text{ by (1)}$$

$\therefore$ Required solution is $\quad z = f_1(yx^2) + f_2(y/x) + (1/2) \times (\log x)^2 \log y - (1/4) \times (\log x)^2.$

Ex. 14. *Solve* $(x^2D^2 - 4y^2D'^2 - 4yD' - 1)z = x^2y^2 \log y.$ **[Delhi Maths (H) 2006]**

Sol. Let $\quad x = e^u, \quad y = e^v \quad$ so that $\quad u = \log x, \quad v = \log y.$ ...(1)

Also, let $\quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' \equiv \partial/\partial v.$

Then the given equation reduces to $[D_1(D_1 - 1) - 4D_1'(D_1' - 1) - 4D_1' - 1]z = e^{2u}e^{2v}v$

or
$$(D_1^2 - D_1 - 4D_1'^2 - 1)z = e^{2u+2v}v. \qquad \text{...(2)}$$

Here $(D_1^2 - D_1 - 4D_1'^2 - 1)$ cannot be resolved into linear factors in D_1 and D_1'. To find C.F. corresponding to it, we consider the equation.

$$(D_1^2 - D_1 - 4D_1'^2 - 1)z = 0. \qquad \text{...(3)}$$

Let a trial solution of (3) be $\quad z = Ae^{hu+kv}$ ...(4)

$\therefore \; D_1^2 z = Ah^2e^{hu+kv}, \quad D_1 z = Ahe^{hu+kv}, \quad$ and $\quad D_1'^2 z = Ak^2e^{hu+kv}.$

Then, $\quad (3) \Rightarrow A(h^2 - h - 4k^2 - 1)e^{hu+hv} = 0 \Rightarrow h^2 - h - 4k^2 - 1 = 0.$ ...(5)

$\therefore \quad$ C.F. of (2) $= \Sigma Ae^{hu+kv} = \Sigma A(e^u)^h (e^v)^k = \Sigma Ax^h y^k$

$$\text{P.I. of (2)} = \frac{1}{D_1^2 - D_1 - 4D_1'^2 - 1}e^{2u+2v}v = e^{2u+2v}\frac{1}{(D_1+2)^2 - (D_1+2) - 4(D_1'+2)^2 - 1}v$$

$$= e^{2u+2v}\frac{1}{D_1^2 + 3D_1 - 4D_1'^2 - 16D_1' - 15}v$$

$$= e^{2u+2v}\frac{1}{(-15)\times[1 + (16/15)\times D_1' + (4/15)\times D_1'^2 - (1/5)\times D_1 - (1/15)\times D_1^2]}v$$

$$= e^{2u+2v}\frac{1}{(-15)}\left[1 + \left\{\frac{16}{15}D_1' + \frac{4}{15}D_1'^2 - \frac{1}{5}D_1 - \frac{1}{15}D_1^2\right\}\right]^{-1}v$$

$$= \frac{e^{2u+2v}}{(-15)}\left(1 - \frac{16}{15}D_1' + \ldots\right)v = \frac{e^{2u+2v}}{(-15)}\left(v - \frac{16}{15}\right) = \frac{(e^u)^2 \times (e^v)^2(16 - 15v)}{225}$$

$= (1/225) \times x^2y^2\,(16 - 15\log y)$, using (1).

The required general solution is $\quad z = \Sigma Ax^hy^k + (1/225) \times x^2y^2(16 - 15\log y)$
where $h^2 - h - 4k^2 - 1 = 0$, and A, h and k are arbitrary constants.

Ex. 15. *Solve* $(x^2D^2 + 2xy\,DD' + y^2D'^2)\,z = x^2y^2$ **[Delhi Maths (H) 2007]**

Sol. Let $\quad x = e^u, \quad y = e^v \quad$ so that $\quad u = \log x, \quad v = \log y$ ... (1)

Here $D \equiv \partial/\partial x, \quad D' \equiv \partial/\partial y \quad$ let $\quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' \equiv \partial/\partial v$

$\therefore$ Given equation reduces to

$\{D_1(D_1-1)+2D_1D_1'+D_1'(D_1'-1)\}z=e^{2u}\ e^{2v}$

or $$(D_1+D_1')\ (D_1+D_1'-1)\ z=e^{2u+2v}$$

Its C.F. = $\phi_1(v-u)+e^u\phi_2(v-u)=\phi_1(\log y-\log x)+x\ \phi_2(\log y-\log x)$, by (1)

= $\phi_1\{\log(y/x)\}+x\ \phi_2\{\log(y/x)\}=\psi_1(y/x)+x\psi_2(y/x)]$. ψ_1, ψ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1+D_1')\ (D_1+D_1'-1)}e^{2u+2v}=\frac{1}{(2+2)(2+2-1)}e^{2u+2v}=\frac{(e^u)^2(e^v)^2}{12}=\frac{x^2y^2}{12}$$

Hence the required solution is given by $z=\psi_1(y/x)+x\ \psi_2\ (y/x)+(1/12)\times x^2y^2$.

Ex. 16. *Solve* $(x^2D^2-4xyDD'+4y^2D'^2+4yD'+xD)z=x^2y$

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D\equiv\partial/\partial x$, $D'\equiv\partial/\partial x$, . Let $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$

Then given equation reduces to $\{D_1\ (D_1-1)-4\ D_1D_1'+4D_1'(D_1'-1)+4D_1'+D_1)\}z=e^{2u}e^v$

or $(D_1^2-4D_1D_1'+4D_1'^2)z=e^{2u+v}$ or $(D_1-2D_1')^2z=e^{2u+v}$.

C.F. = $\phi_1\ (v+2u)+u\ \phi_2(v+2u)\ =\phi_1\ (\log y+2\log x)+\log x\ \phi_2(\log y+2\log x)$, by (1)

= $\phi_1(\log yx^2)+\log x\ \phi_2(\log yx^2)=\psi_1(yx^2)+\log x\ \psi_2(yx^2)$, ψ_1, ψ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1-2D_1')^2}\ e^{2u+v}=\frac{u^2}{2!}\ e^{2u+v}=\frac{1}{2}(\log x)^2x^2y$$

$\therefore$ General solution is $z=\psi_1\ (yx^2)+(\log x)\ \psi_2(yx^2)+(1/2)\times x^2y\ (\log x)^2$.

Ex. 17. *Solve* $(x^2D-2xy\ DD'+y^2D'^2-xD+3yD')z=8y/x$ **[Delhi Maths (H) 2005]**

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D\equiv\partial/\partial x$, $D'\equiv\partial/\partial y$. Let $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$.

Then the given equation reduces to $\{D_1(D_1-1)-2D_1D_1'+D_1'(D_1'-1)-D_1+3D_1'\}z=8e^v/e^u$

or $\{(D_1^2-2D_1\ D_1'+D_1'^2)-2\ (D_1-D_1')\}z=8e^{v-u}$ or $(D_1-D_1')\ (D_1-D_1'-2)z=8\ e^{v-u}$

C.F. = $\phi_1\ (v+u)+e^{2u}\phi_2\ (v+u)\ =\ \phi_1\ (\log y+\log x)+(e^u)^2\phi_2(\log y+\log x)$, using (1)

= $\phi_1(\log xy)+x^2\phi_2(\log xy)=\psi_1(xy)+x^2\psi_2(xy)$, ψ_1 ψ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D_1-D_1')\ (D_1-D_1'-2)}8\ e^{-u+v}=8\frac{1}{(-1-1)\ (-1-1-2)}e^{-u+v}=\frac{e^v}{e^u}=\frac{y}{x}$$

$\therefore$ The required solution is $z=\psi_1(xy)+x^2\psi_2(xy)+y/x$.

Ex. 18. $(x^2D^2-2xy\ DD'-3y^2D'^2+xD-3yD')z\ =\ x^2y\cos\ (\log x^2)$

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$\{D_1(D_1-1)-2D_1\ D_1'-3D_1'\ (D_1'-1)\ +\ D_1-3\ D_1'\}z\ =\ e^{2u}\ e^{v}\cos 2u\,.$$

or $\quad (D_1^2-2D_1D_1'-3D_1'^2)z=e^{2u+v}\cos 2u \quad$ or $\quad (D_1-3D_1')\ (D+D_1')z=e^{2u+v}\cos 2u$

Its C.F. $=\ \phi_1\ (u+3u)+\phi_2(v-u)\ =\ \phi_1(\log y+3\log x)+\phi_2(\log y-\log x)$, using (1)

$=\ \phi_1(\log yx^3)+\phi_2(\log(y/x))=\psi_1(x^3y)+\psi_2(y/x)$, $\psi_1\ \psi_2$ being arbitrary functions.

$$\text{P.I.}\ =\ \frac{1}{(D_1-3D_1')\ (D_1+D_1')}\ e^{2u+v}\cos 2u=\ e^{2u+v}\ \frac{1}{\{(D_1+2)-3(D_1'+1)\}\ (D_1+2+D_1'+1)}\cos 2u$$

$$=\frac{(e^u)^2(e^v)}{(D_1-3D_1'-1)\ (D_1+D_1'+3)}\cos 2u\ =\ \frac{x^2y}{D_1^2-2D_1D_1'-3D_1'^2+2D_1-10D_1'-3}\cos(2u+0\cdot v)$$

$$=\ \frac{x^2y}{-2^2-2(-2\cdot 0)-3\cdot 0^2+2D_1-10D_1'-3}\cos 2u=\ \frac{x^2y}{2D_1-10D_1'-7}\cos 2u=x^2y\frac{2D_1-10D_1'+7}{(2D_1-10D_1')^2-49}\cos 2u$$

$$=\ x^2y\frac{2D_1-10D_1'+7}{4D_1^2-40D_1D_1'+100\,D_1'^2-49}\cos(2u+0.v)=\ x^2y\frac{2D_1-10D_1'+7}{4(-2^2)-40(-2\cdot 0)+100(-0^2)-49}\cos(2u+0\cdot v)$$

$= -\ (1/65)\ x^2y\ (-\ 4\ \sin\ 2u + 7\ \cos\ 2u) = (1/65)\ x^2y\ \{4\ \sin\ (2\ \log\ x) - 7\ \cos\ (2\ \log\ x)\}$

$\therefore$ Required solution is $z=\psi_1(x^3y)+\psi_2(y/x)+(1/65)x^2y\ \ \{4\sin(\log x^2)-7\cos(\log x^2)\}$.

Ex. 19. *Solve* $x^2(\partial^2z/\partial x^2)-y^2(\partial^2z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=x^2y^4$ *by reducing it to the equation with constant coefficients.* **[I.A.S. 2001]**

Sol. Re-writing, the given equation $\quad (x^2D^2-y^2D'^2+xD-yD')z=x^2y^4 \quad$ ... (1)

Let $x=e^u$ and $y=e^v$ so that $u=\log x$ and $v=\log y$ ... (2)

Here $D\equiv\partial/\partial x$, $D'\equiv\partial/\partial y$. Let $D_1\equiv\partial/\partial u$, $D_1'\equiv\partial/\partial v$. Then (1) becomes

$\{D_1\ (D_1-1)-D_1'(D_1'-1)+D_1-D_1'\}z\ =\ e^{2u}e^{4v} \quad$ or $\quad (D_1^2-D_1'^2)z=e^{2u+4v}$

or $$(D_1-D_1')\ (D_1+D_1')z=e^{2u+4v} \qquad \text{... (3)}$$

C.F. $=\ \phi_1(v+u)+\phi_2\ (v-u)=\phi_1(\log y+\log x)+\phi_2(\log y-\log x)\ =\ \phi_1(\log xy)+\phi_2(\log(y/x)\}$

or $\quad$ C.F. $=\ \psi_1(xy)+\psi_2(y/x)$, ψ_1,ψ_2 being arbitrary functions

$$\text{P.I.}\ =\ \frac{1}{D_1^2-D_1'^2}\ e^{2u+4v}=\frac{1}{(2^2-4^2)}e^{2u+4v}=-\frac{1}{12}\ (e^u)^2\ (e^v)^4\ =-\frac{1}{12}x^2y^4$$

$\therefore\ \ z=\psi_1\ (xy)+\psi_2(y/x)-(1/12)\times x^2y^4$ is the required solution.

Remark. Sometimes typical substitutions are employed to reduce a given equation into a partial differential equation with constant coefficients as shown in the following Ex. 20.

Ex. 20. *Solve* $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2}-\dfrac{1}{x^3}\dfrac{\partial z}{\partial x}=\dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2}-\dfrac{1}{y^3}\dfrac{\partial z}{\partial y}$.

Sol. Let $x^2/2=u$, $y^2/2=v$ so that $dx/du=1/x$, $dy/dv=1/y$. ...(1)

Now,
$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{dx}{du}=\frac{1}{x}\frac{\partial z}{\partial x}, \text{ using (1).} \qquad ...(2)$$

and
$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{dx}{du}=\left(\frac{1}{x}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^2}\frac{\partial z}{\partial x}\right)\frac{1}{x}, \text{ using (1)}$$

$\therefore$
$$\frac{\partial^2 z}{\partial u^2}=\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}. \qquad ...(3)$$

Similarly,
$$\frac{\partial^2 z}{\partial v^2}=\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}. \qquad ...(4)$$

Using (3) and (4), the given equation reduces to

$\partial^2 z/\partial u^2=\partial^2 z/\partial v^2$ or $(D_1^2-D_1'^2)z=0$ or $(D_1-D_1')(D_1+D_1')z=0$, ...(5)

where $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$. Hence solution of (5) is

$$z=\phi_1(v+u)+\phi_2(v-u)=\phi_1\{(1/2)\times(x^2+y^2)\}+\phi_2\{(1/2)\times(y^2-x^2)\}$$

or $\quad z=f_1(y^2+x^2)+f_2(y^2-x^2)$, f_1, f_2 being arbitrary functions.

EXERCISE 6

Solve the following partial differential equations:

1. $x^2(\partial^2 y/\partial x^2)+2xy(\partial^2 z/\partial x\partial y)+y^2(\partial^2 z/\partial y^2)+x(\partial z/\partial x)+y(\partial z/\partial y)-z=0$

Ans. $z=xf_1(y/x)+x^{-1}f_2(y/x)$, f_1, f_2 being arbitrary functions

2. $(x^2D^2-y^2D'^2)z=x^2y.$ **Ans.** $z=f_1(xy)+x\,f_2(y/x)+(1/2)\times x^2y$

3. $(x^2D^2-xy\,DD'-2y^2D'^2+xD-2yD')z=\log(y/x)$ **(Delhi Maths (H) 2005)**

Ans. $z=f_1(yx^2)+f_2(y/x)+(1/2)\times(\log x)^2\log y$, f_1, f_2 being arbitrary functions

4. $(x^2D^2-2xy\,DD'-3y^2D'^2+xD-3yD')z=x^2y\sin(\log x^2)$ **[Nagpur 2010]**

Ans. $z=f_1(x^3y)+f_2(y/x)-(1/65)\times\{4\cos(\log x^2)+7\sin(\log x^2)\}$

6.5. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS

We have seen that solution of Euler-Cauchy type partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Illustrative example. *Find a surface satisfying equation* $2x^2r-5xys+2y^2t+2(px+qy)=0$ *and touching the hyperbolic paraboloid* $z=x^2-y^2$ *along its section by the plane* $y=1$.

[Meerut 1998]

Sol. Re–writing given equation, $\quad 2x^2\dfrac{\partial^2 z}{\partial x^2}-5xy\dfrac{\partial^2 z}{\partial x\partial y}+2y^2\dfrac{\partial^2 z}{\partial y^2}+2\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=0.$

or
$$\{2x^2D^2-5xyDD'+2y^2D'^2+2(xD+yD')\}z=0 \qquad ...(1)$$

Put $x=e^u$, $y=e^v$ so that $u=\log x$ and $v=\log y$.

If $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$, then (1) reduces to

$$[2D_1(D_1 - 1) - 5D_1D_1' + 2D_1'(D_1' - 1) + 2(D_1 + D_1')]z = 0$$

or $\quad (2D_1^2 - 5D_1D_1' + 2D_1'^2) = 0 \quad$ or $\quad (2D_1 - D_1')(D_1 - 2D_1') = 0.$

$\therefore$ solution is $z = \text{C.F.} = \phi_1(2v + u) + \phi_2(u + 2v)$, ϕ_1, ϕ_2 being arbitrary function

or $\quad z = \phi_1(2 \log y + \log x) + \phi_2(\log y + 2 \log x) = \phi_1(\log y^2x) + \phi_2(\log yx^2)$

or $\quad z = f_1(y^2x) + f_2(yx^2)$, f_1 and f_2 being arbitrary functions ...(2)

The given surface is $\quad z = x^2 - y^2.$ ...(3)

Now (2) and (3) are to touch each other along the section by the plane

$$y = 1. \qquad \text{...(4)}$$

Therefore the values of p and q for (2) and (3) must be equal at $y = 1$. Equating values of p and q from (2) and (3), we get

$$y^2f_1'(y^2x) + 2xyf_2'(x^2y) = 2x \qquad \text{...(5)}$$

and $\quad 2xyf_1'(y^2x) + x^2f_2'(x^2y) = -2y.$ ...(6)

Putting $y = 1$, (5) and (6) reduce to

$f_1'(x) + 2xf_2'(x^2) = 2x \quad$ and $\quad 2xf_1'(x) + x^2f_2'(x^2) = -2.$

Solving these, $\quad f_1'(x) = -(2/3) \times x - (4/3) \times x^{-1}$ ...(7)

and $\quad f_2'(x^2) = (2/3) \times x^{-2} + (4/3)$ ...(8)

Integrating (7), $\quad f_1(x) = -(1/3) \times x^2 - (4/3) \times \log x + c_1$

which gives $\quad f_1(y^2x) = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + c_1.$ ...(9)

Writing X for x^2 in (8), $\quad f_2'(X) = (2/3) \times (1/X) + (4/3)$

Integrating it, $\quad f_2(X) = (2/3) \times \log X + (4/3) \times X + c_2$

which gives $\quad f_2(yx^2) = (2/3) \times \log (yx^2) + (4/3) \times (yx^2) + c_2$ ...(10)

Putting the values of $f_1(y^2x)$ and $f_2(yx^2)$ from (9) and (10) in (2) and writing $c_1 + c_2 = c/3$, the complete solution is

$$z = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + (2/3) \times \log(yx^2) + (4/3) \times (yx^2) + c/3$$

or $\quad 3z = -y^4x^2 - 4(\log x + 2 \log y) + 2(\log y + 2 \log x) + 4yx^2 + c$

or $\quad 3z = -y^4x^2 - 6 \log y + 4yx^2 + c.$

Now equating values of z from (3) and (11) and putting $y = 1$, we have

$$x^2 - 1 = (1/3)[-x^2 - 6 \log 1 + 4x^2 + c], \text{ giving } c = -3.$$

So the required surface is $\quad 3z = 4yx^2 - y^4x^2 - 6 \log y - 3.$

MISCELLANEOUS PROBLEMS ON CHAPTER 6

1. Show that a linear partial differential equation of the type

$$\sum C_{qs} x^q y^s \frac{\partial^{q+s} z}{\partial x^q \partial y^s} = f(x, y)$$

may be reduced to one with constant coefficients by the substitutions $\log x = \xi$, $\log y = \eta$.

(Meerut 2008)

2. Find the general solution of $x^2(\partial^2 z/\partial x^2) + y^2(\partial^2 z/\partial y^2) = z$ **[Pune 2010]**

Sol. Let $\quad x = e^u \quad$ and $\quad y = e^v \quad$ so that $\quad u = \log x \quad$ and $\quad v = \log y$ ... (1)

Also, let $\quad D \equiv \partial/\partial x, \quad D' \equiv \partial/\partial y, \quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' = \partial/\partial v$ ... (2)

Then, the given equation $(x^2 D^2 + y^2 D'^2 - 1)\, z = 0$ reduce to

$\{D_1(D_1 - 1) + D_1'(D_1' - 1) - 1\}z = 0 \quad$ or $\quad (D_1^2 + D_1' - D_1 - D_1' - 1)z = 0$ ... (3)

Let $z = A\, e^{hu + kv}$ be a trial solution of (3). Then, we have

$D_1 z = Ah\, e^{hu+kv}$, $D_1^2 z = Ah^2 e^{hu+kv}$, $D_1' z = Ak\, e^{hu+kv}$ and $D_1' z = Ak^2\, e^{hu+kv}$

Substituting the above values of $D_1\, z, D_1^2 z, D_1' z$ and $D_1'^2 z$ in (3), we have

$A(h^2 + k^2 - h - k - 1)\, e^{hu+kv} = 0$ so that $h^2 + k^2 - h - k - 1 = 0$, taking $A \neq 0$ ... (4)

Hence the required solution is given by

$$z = \sum_i A_i e^{h_i u + k_i v} \quad \text{or} \quad z = \sum_i A_i e^{h_i \log x + k_i \log y}, \text{ using (1)}$$

or $z = \sum A_i x^{h_i} y^{k_i}$, A_i, h_i and k_i being arbitrary constants.

7

Partial Differential Equations of order Two With Variable Coefficients

7.1 INTRODUCTION

In the present chapter, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r\ (= \partial^2 z/\partial x^2)$, $s\ (= \partial^2 z/\partial x\partial y)$, $t\ (= \partial^2 z/\partial y^2)$, but none of higher order ; the quantities p and q may also enter into the equation. Thus, the general form of a second order partial differential equation is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad \text{...(1)}$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad \text{...(2)}$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

In what follows, we shall show how a large class of second order partial differential equations may be solved by using the methods of solving ordinary differential equations.

Note that x and y, being independent variables, are constant with respect to each other in differentiation and integration. To understand this, note the solution of the following equation.

$$s = 2x + 2y. \qquad \text{...(3)}$$

From (3), $\dfrac{\partial^2 z}{\partial x\partial y} = 2x + 2y$ or $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = 2x + 2y.$ ...(4)

Integrating (4) w.r.t. 'x', $(\partial z/\partial y) = x^2 + 2xy + f(y)$, where $f(y)$ in an arbitrary functions of y. ...(5)

Integrating (5) w.r.t. 'y', $z = x^2y + xy^2 + F(y) + g(x)$,

where F *and* g are arbitrary functions and $F(y)$ is given by $F(y) = \int f(y)\,dy$.

In what follows we shall use the following results.

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y} \quad \text{and} \quad s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

We shall now consider some special types of equations based on (2).

7.2. Type I. Under this type, we consider equations of the form

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x, y), \qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_2(x, y), \qquad s = \frac{\partial^2 z}{\partial x\partial y} = \frac{F}{S} = f_3(x, y).$$

These are homogeneons linear partial differential equations with constant coefficients and can be solved by methods discussed in chapter 4. However a more direct method of solving such equation will be used in practice.

7.3 SOLVED EXAMPLES BASED ON ART 7.2.

Ex 1. *Solve the following partial differential equations:*

(*i*) $r = 6x$. **[Agra 2009; Bhopal 2010]** (*ii*) $ar = xy$ **[Meerut 2001; Vikram 2003]**

(*iii*) $r = x^2 e^y$ **[Indore 2004]** (*iv*) $r = 2y^2$

(*v*) $r = \sin(xy)$

Sol. (*i*) Given equation can be written as $\partial^2 z/\partial x^2 = 6x$. ...(1)

Integrating (1) with respect to 'x', $\partial z/\partial x = 3x^2 + \phi_1(y)$, ...(2)

where $\phi_1(y)$ is an arbitrary function of y.

Integrating (2) with respect to 'x', $z = x^3 + x\phi_1(y) + \phi_2(y)$,

where $\phi_2(y)$ is an arbitrary function of y.

(*ii*) Given equation can be written as $\partial^2 z/\partial x^2 = (1/a) \times xy$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial x = (y/a) \times (x^2/2) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = (y/6a) \times x^3 + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

(*iii*) Try yourself. **Ans.** $z = (e^y/12) \times x^4 + x\phi_1(y) + \phi_2(y)$.

(*iv*) Try yourself. **Ans.** $z = x^2y^2 + x\phi_1(y) + \phi_2(y)$.

(v) Given equation can be written as $\partial^2 z/\partial x^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'x'. $\partial z/\partial x = -(1/y) \times \cos(xy) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = -(1/y^2) \times \sin(xy) + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t = \sin(xy)$ **(Meerut 2008)** (*ii*) $t = x^2 \cos(xy)$.

Sol.(*i*) Given equation can be written as $\partial^2 z/\partial y^2 = \sin(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = -(1/x) \times \cos(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t., 'y', $z = -(1/x^2) \times \sin(xy) + y\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) Given equation can be written as $\partial^2 z/\partial y^2 = x^2 \cos(xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = x\sin(xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -\cos(xy) + y\,\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 3. *Solve the following partial differential equations:*

(*i*) $xys = 1$ **[Agra 2007; Rewa 2004, Vikram 2005]** (*ii*) $xy^2s = 1 - 2x^2y$

(*iii*) $\log s = x + y$ (*iv*) $s = x - y$

(*v*) $s = x^2 - y^2$ (*vi*) $x^2s = \sin y$

(*vii*) $s = (x/y) + a$ (*viii*) $s = 0$.

Sol. (*i*) Re-written the given equation, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y) \times \log x + \phi_1(y)$.

Integrating (2) w.r.t. 'y', $z = \log x \log y + \int \phi_1(y)dy + \psi_2(x)$

or $z = \log x \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y)dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) Given equation is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy^2} - \dfrac{2x}{y}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y^2) \times \log x - (x^2/y) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -(1/y) \times \log x - x^2 \log y + \int \phi_1(y)dy + \psi_2(x)$

or $z = -(1/y) \times \log x - x^2 \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y)dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*iii*) The given equation $\log s = x + y$ can be rewritten as

$$s = e^{x+y} \qquad \text{or} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^x \, . \, e^y. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $\partial z/\partial y = e^x e^y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = e^x e^y + \int \phi_1(y)dy + \psi_2(x)$

or $z = e^{x+y} + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*iv*) Try yourself. **Ans.** $z = (1/2) \times (x^2y - xy^2) + \psi_1(y) + \psi_2(x)$.

(*v*) Try yourself. **Ans.** $z = (1/3) \times (x^3y - xy^3) + \psi_1(y) + \psi_2(x)$.

(*vi*)Given equation can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\sin y}{x^2}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = -(1/x) \times \sin y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = (1/x) \cos y + \int \phi_1(y)dy + \psi_2(x)$

or $z = (1/x) \cos y + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*vii*) Try yourself. **Ans.** $z = (1/2) \times x^2 \log y + axy + \psi_1(y) + \psi_2(x)$.

(*viii*) Try yourself. **Ans.** $z = \psi_1(y) + \psi_2(x)$.

Ex. 4. *Solve* (*i*) $xr = p$ **[Agra 2007]** (*ii*) $rx = (n-1)p$.

Sol.(*i*) Given equation can be rewritten as

$$x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} \qquad \text{or} \qquad \frac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \frac{1}{x}.$$

Integrating, $\log(\partial z/\partial x) = \log x + \log \phi_1(y)$ or $\partial z/\partial x = x\phi_1(y)$.

Integrating it w.r.t. x, $z = (x^2/2) \times \phi_1(y) + \phi_2(y)$, where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

(*ii*) Given $x\dfrac{\partial^2 z}{\partial x^2} = (n-1)\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \dfrac{n-1}{x}$.

Integrating, $\log(\partial z/\partial x) = (n-1)\log x + \log \phi_1(y)$ or $\partial z/\partial x = x^{n-1}\phi_1(y)$.

Integrating it, $z = (x^n/n) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 5. *Solve* (*i*) $xr + 2p = 0$ (*ii*) $2yq + y^2t = 1$.

Sol.(*i*) The given equation can be rewritten as

$$x\frac{\partial p}{\partial x} + 2p = 0 \qquad \text{or} \qquad x^2\frac{\partial p}{\partial x} + 2xp = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x}(x^2 p) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $x^2 p = \phi_1(y)$ or $p = \partial z/\partial x = (1/x^2) \times \phi_1(y)$.

Integrating it w.r.t. 'x', $z = -(1/x) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as

$$2yq + y^2\frac{\partial q}{\partial y} = 1 \qquad \text{or} \qquad \frac{\partial}{\partial y}(y^2 q) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'y', $y^2 q = \phi_1(x)$ or $q = \partial z/\partial y = (1/y^2) \times \phi_1(x)$.

Integrating it, $z = -(1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 6. *Solve* $xs + q = 4x + 2y + 2$.

Sol. The given equation can be re-written as

$$x\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y} = 4x + 2y + 2 \qquad \text{or} \qquad \frac{\partial}{\partial y}(xp + z) = 4x + 2y + 2.$$

Integrating it w.r.t. 'y', $\quad xp + z = 4xy + y^2 + 2y + \phi_1(x)$

or $$x\frac{\partial z}{\partial x} + z = 4xy + y^2 + 2y + \phi_1(x) \qquad \text{or} \qquad \frac{\partial}{\partial x}(xz) = 4xy + y^2 + 2y + \phi_1(x).$$

Integrating it w.r.t. 'x', $\quad xz = 2x^2y + xy^2 + 2xy + \int \phi_1(x)dx + \psi_2(y)$

or $\therefore$ Required solution is $\quad xz = 2x^2y + xy^2 + 2xy + \psi_1(x) + \psi_2(y)$, where $\psi_1(x) = \int \phi_1(x)dx$.

Ex. 7. *Solve* $\quad ys + p = \cos(x + y) - y \sin(x + y)$. **[Meerut 1995]**

Sol. The given equation can be rewritten as

$$y\frac{\partial q}{\partial x}+\frac{\partial z}{\partial x} = \cos(x+y) - y\sin(x+y) \qquad \text{or} \qquad \frac{\partial}{\partial x}(yq+z) = \cos(x+y) - y\sin(x+y).$$

Integrating it w.r.t. 'x', $\quad yq + z = \sin(x + y) + y\cos(x + y) + \phi_1(y)$.

or $$y\frac{\partial z}{\partial y} + z = \sin(x+y) + y\cos(x+y) + \phi_1(y) \qquad \text{or} \qquad \frac{\partial(yz)}{\partial y} = \sin(x+y) + y\cos(x+y) + \phi_1(y).$$

Integrating it w.r.t. 'y', $\quad yz = \int \sin(x+y)dy + \int y\cos(x+y)\,dy + \int \phi_1(y)\,dy + \psi_2(y)$

or $$yz = \int \sin(x+y)dy + y\sin(x+y) - \int \sin(x+y)\,dy + \psi_1(y) + \psi_2(y)$$

[Integrating by parts and taking $\psi_1(y) = \int \phi_1(y)\,dy$]

Required solution is $\quad yz = y\sin(x + y) + \psi_1(y) + \psi_2(y)$, ψ_1, ψ_2 being arbitrary functions.

7.4. Type II. Under this type, we consider equations of the form:

$$Rr + Pp = F, \quad i.e., \quad R\frac{\partial p}{\partial x} + Pp = F; \qquad Ss + Pp = F, \quad i.e., \quad S\frac{\partial p}{\partial y} + Pp = F,$$

$$Ss + Qq = F, \quad i.e., \quad S\frac{\partial q}{\partial x} + Qq = F; \qquad Tt + Qq = F, \quad i.e., \quad T\frac{\partial q}{\partial y} + Qq = F.$$

These will be treated as ordinary linear differential equations of order one in which p (or q) is the dependent variable.

7.5 SOLVED EXAMPLES BASED ON ART 7.4

Ex. 1. *Solve* (*i*) $t - xq = x^2$. **[Ravishanker 2010; Nagpur 1996]**

(*ii*) $yt - q = xy$. **[Meerut 1997]**

Sol. (*i*) The given equation can be rewritten as $\quad (\partial q/\partial y) - xq = x^2$, ...(1)

which is linear differential equation in variables q and y, regarding x as constant.

Integrating factor (I.F.) of (1) $= e^{\int(-x)\,dy} = e^{-xy}$ and solution of (1) is

$$q(I.F.) = \int (x^2)(I.F.)dy + \phi_1(x) \qquad \text{or} \qquad qe^{-xy} = \int x^2 e^{-xy}\,dy + \phi_1(x)$$

or $$qe^{-xy} = x^2 \times (-1/x) \times e^{-xy} + \phi_1(x) \qquad \text{or} \qquad q = \partial z/\partial y = -x + e^{xy}\phi_1(x).$$

Integrating it w.r.t. 'y', $\quad z = -xy + (1/x) \times \phi_1(x)\,e^{xy} + \psi_2(x)$

or $\quad z = -xy + \psi_1(x)e^{xy} + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

It is the required solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as $\quad y(\partial q/\partial y) - q = xy \quad$ or $\quad (\partial q/\partial y) - (1/y)\times q = x$, which is differential equation linear in variables q and y, regarding x as constant.

I.F. of (1) $= e^{\int(-1/y)dy} = e^{-\log y} = 1/y$ and solution of (1) is

$q \times \dfrac{1}{y} = \int \left(x \times \dfrac{1}{y} \right) dy + \phi_1(x)$ or $\dfrac{q}{y} = x \log y + \phi_1(x)$

or $q = xy \log y + y\phi_1(x)$ or $\partial z/\partial y = xy \log y + y\phi_1(x)$.

Integrating it, $z = x\left[(y^2/2) \times \log y - \int \{ (y^2/2) \times (1/y) \} \, dy \right] + (y^2/2) \times \phi_1(x) + \phi_2(x)$

or $z = (1/2) \times xy^2 \log y - (1/4) \times xy^2 + (1/2) \times y^2 \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 2. *Solve* $xs + q = 4x + 2y + 2$.

Sol. Re-writing $x\dfrac{\partial q}{\partial x} + q = 4x + 2y + 2$ or $\dfrac{\partial q}{\partial x} + \dfrac{1}{x}q = 4 + \dfrac{2y}{x} + \dfrac{2}{x}$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence its solution is

$$qx = \int x\left(4 + \frac{2y}{x} + \frac{2}{x}\right) dx + \phi_1(y) = 2x^2 + 2xy + 2x + \phi_1(y)$$

or $$q = \partial z/\partial y = 2x + 2y + 2 + (1/x) \times \phi_1(y).$$

Integrating, $$z = 2xy + y^2 + 2y + (1/x) \times \int \phi_1(y)\, dy + \psi_2(x)$$

or $$z = 2xy + y^2 + 2y + (1/x) \times \psi_1(y) + \psi_2(x), \text{ where } \psi_1(y) = \int \phi_1(y)\, dy.$$

Ex. 3. *Solve* $xr + p = 9x^2y^3$. **[Ranchi 2010]**

Sol. The given equation can be re-written as $x\dfrac{\partial p}{\partial x} + p = 9x^2y^3$ or $\dfrac{\partial p}{\partial x} + \dfrac{1}{x}p = 9xy^3$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence solution is

$px = \int \{x \times (9xy^3)\} dx + \phi_1(y)$ or $px = 3x^3y^3 + \phi_1(y)$

or $p = 3x^2y^3 + (1/x) \times \phi_1(y)$ or $(\partial z/\partial x) = 3x^2y^3 + (1/x) \times \phi_1(y)$.

Integrating, $$z = x^3y^3 + \phi_1(y) \log x + \phi_2(y),$$

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 4. *Solve* $ys - p = xy^2 \cos(xy)$.

Sol. Re-writing given equation, $y\dfrac{\partial p}{\partial y} - p = xy^2 \cos(xy)$ or $\dfrac{\partial p}{\partial y} - \dfrac{1}{y}p = xy \cos(xy)$.

Its I.F. $= e^{\int (-1/y) dy} = e^{-\log y} = 1/y$ and so its solution is

$$p \times (1/y) = \int (1/y) \times [xy \cos(xy)] dx + \phi_1(x) = \sin(xy) + \phi_1(x)$$

or $p = y \sin(xy) + y\,\phi_1(x)$ or $\partial z/\partial x = y \sin(xy) + y\phi_1(x)$.

Integrating, $$z = -\cos(xy) + y\int \phi_1(x)\, dx + \psi_2(y)$$

or $$z = -\cos(xy) + y\psi_1(x) + \psi_2(y), \text{ where } \psi_1(x) = \int \phi_1(x)\, dx.$$

Ex. 5. *Solve* $t - xq = -\sin y - x \cos y$.

Sol. Re-writting, $(\partial q/\partial y) - xq = -\sin y - x \cos y$, which is linear differential equation in q and y.

Its I.F. $= e^{\int (-x) dy} = e^{-xy}$ and so its soltion is

$$qe^{-xy} = -\int e^{-xy} (\sin y + x \cos y) dy + \phi_1(x) = -\int e^{-xy} \sin y\, dy - x\int e^{-xy} \cos y\, dy + \phi_1(x)$$

$$= -\int e^{-xy} \sin y\, dy - x\left[-\frac{1}{x}e^{-xy} \cos y - \int \left(\frac{1}{x}e^{-xy} \sin y \right) dy \right] + \phi_1(x)$$

$$= -\int e^{-xy} \sin y \, dy + e^{-xy} \cos y + \int e^{-xy} \sin y \, dy + \phi_1(x)$$

or $\quad (\partial z/\partial y)\, e^{-xy} = e^{-xy} \cos y + \phi_1(x) \quad$ or $\quad (\partial z/\partial y) = \cos y + e^{xy}\, \phi_1(x).$

Integrating, $\quad z = \sin y + (1/x) \times e^{xy}\, \phi_1(x) + \psi_2(x)$

or $\quad z = \sin y + e^{xy}\, \psi_1(x) + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

Ex. 6. *Solve* $xys - qy = x^2$. **[Delhi Maths Hons. 1992, 93]**

Sol. Re-writing the given equation, we have

$$xy \frac{\partial q}{\partial x} - qy = x^2 \quad \text{or} \quad \frac{\partial q}{\partial x} - \frac{1}{x} q = \frac{x}{y}. \qquad \text{...(1)}$$

which is linear differential equation in variables q and x.

Integrating factor of (1) $= e^{-\int (1/x)\, dx} = e^{-\log x} = (1/x)$. Hence solution of (1) is given by

$$q \times \frac{1}{x} = \int \{(x/y) \times (1/x)\}\, dx = \frac{x}{y} + f(y) \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{x^2}{y} + x\, f(y). \qquad \text{...(2)}$$

Integrating (2), $\quad z = x^2 \log y + x\phi_1(y) + \phi_2(x)$, where $\phi_1(y)$ and $\phi_2(x)$ are arbitrary functions.

Ex. 7. *Solve* $xs + q - xp - z = (1 - y)(1 + \log x)$.

Sol. Re-writing the given equations $\quad \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{1}{x}\dfrac{\partial z}{\partial y} - \left(\dfrac{\partial z}{\partial x} + \dfrac{z}{x}\right) = \dfrac{1-y}{x}\,(1 + \log x)$

or
$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) - \left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) = \frac{1-y}{x}\,(1 + \log x). \qquad \text{...(1)}$$

Let $\quad u = (\partial z/\partial x) + (z/x). \qquad$ ...(2)

$\therefore$ (1) $\Rightarrow \dfrac{\partial u}{\partial y} - u = \dfrac{1-y}{x}\,(1 + \log x)$, which is linear differential equation $\qquad$ ...(3)

Integrating factor of (3) $= e^{-\int dy} = e^{-y}$ and so solution of (3) is

$$ue^{-y} = \int \frac{1-y}{x}\,(1 + \log x) e^{-y}\, dy = \frac{1+\log x}{x} \int (1-y) e^{-y} dy$$

$$= \frac{1+\log x}{x}\left[(1-y)(-e^{-y}) - \int (-1)(e^{-y}) dy\right] = \frac{1+\log x}{x}\,[-e^{-y} + y\, e^{-y} + e^{-y}] + \phi(x)$$

or $\quad u = (y/x) \times (1 + \log x) + e^y\, \phi(x). \qquad$ ...(4)

Then, using (2), $\dfrac{\partial z}{\partial x} + \dfrac{z}{x} = \dfrac{y}{x}(1 + \log x) + e^y \phi(x)$, which is linear differential equation $\quad$ ...(5)

Integrating factor of (5) $= e^{\int (1/x) dx} = e^{\log x} = x$ and solution of (5) is

$$zx = \int x\left[\frac{y}{x}(1 + \log x) + e^y \phi(x)\right] dx + \phi_2(y) = y \int (1 + \log x)\, dx + e^y \int x\, \phi(x) dx + \phi_2(y)$$

or $\quad zx = y\left[(1 + \log x) \times x - \int \{(1/x) \times x\}\, dx\right] + e^y \phi_1(x) + \phi_2(y) \quad$ or $\quad zx = xy \log x + e^y \phi_1(x) + \phi_2(y).$

Ex. 8. *Solve* $ys + p = \cos(x + y) - y \sin(x + y)$. **[Meerut 1995]**

Sol. Re-writing given equation, $\quad \dfrac{\partial p}{\partial y} + \dfrac{1}{y} p = \dfrac{1}{y} \cos(x + y) - \sin(x + y), \qquad$ ...(1)

which is linear differential equation whose I.F. $= e^{\int (1/y) dy} = y$ and so solution of (1) is

$$py = \int y\left[\frac{1}{y}\cos(x+y) - \sin(x+y)\right] dy = \sin(x + y) - \int y \sin(x+y)\, dy$$

$$= \sin(x + y) - \left[-y \cos(z + y) - \int \{-\cos(x+y)\}\, dy\right]$$

$$= \sin(x + y) + y \cos(x + y) - \sin(x + y) + F(x)$$

$\therefore \quad y(\partial z/\partial x) = y \cos(x + y) + F(x) \quad$ or $\quad (\partial z/\partial x) = \cos(x + y) + (1/y) \times F(x).$

Integrating, $z = \sin(x+y) + (1/y) \times \phi_1(x) + (1/y) \times \phi_2(y)$, where $\phi_1(x) = \int F(x)\, dx$

or $yz = y \sin(x+y) + \phi_1(x) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 9. *Solve* $yt + 2q = (9y+6)e^{2x+3y}$.

Sol. Re-writing, $\dfrac{\partial q}{\partial y} + \dfrac{2}{y} q = \left(9 + \dfrac{6}{y}\right) e^{2x+3y}$, which is linear differential equations ...(1)

whose integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and so solution of (1) is

$$qy^2 = \int y^2 \left(9 + \frac{6}{y}\right) e^{2x+3y} dy = e^{2x} \int (9y^2 + 6y) e^{3y}\, dy$$

or $$qy^2 = e^{2x}\left[(9y^2+6y)\left(\frac{1}{3}e^{3y}\right) - (18y+6)\left(\frac{1}{9}e^{3y}\right) + 18\left(\frac{1}{27}e^{3y}\right)\right] + \phi_1(x)$$

[using chain rule of integrating by parts]

or $qy^2 = 3y^2 e^{2x+3y} + \phi_1(x)$ or $y^2(\partial z/\partial y) = 3y^2 e^{2x+3y} + \phi_1(x)$.

$\therefore$ $(\partial z/\partial y) = 3e^{2x+3y} + (1/y^2) \times \phi_1(x)$.

Integrating, $z = e^{2x+3y} - (1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 10. *Solve* (*i*) $2yq + y^2 t = 1$ (*ii*) $xr + p = 9x^2 y^2$.

Sol. (*i*) Re-writing given equation $2yq + y^2(\partial q/\partial y) = 1$ or $\partial q/\partial y + (2/y)q = 1/y^2$, which is linear differential equation in variables q and y, regarding x as constant.

Its I.F. $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and solution is $qy^2 = \int (1/y^2)\, dy + \phi_1(x)$

or $qy^2 = y + \phi_1(x)$ or $\partial z/\partial y = (1/y) + (1/y^2) \times \phi_1(x)$.

Integrating it w.r.t. 'y', $z = \log y - (1/y) \times \phi_1(x) + \phi_2(x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) Do as in Ex. 3 of Art 7.5. **Ans.** $z = x^3 y^2 + \log x\ \phi_1(y) + \phi_2(y)$.

7. 6. Type III. Under this type, we consider equations of the form

$Rr + Ss + Pp = F$ or $R(\partial p/\partial x) + S(\partial p/\partial y) = F - Pp$

and $Ss + Tt + Qq = F$ or $S(\partial q/\partial x) + T(\partial q/\partial y) = F - Qq$.

These are linear partial differential equations of order one with p (or q) as dependent variable and x, y, as independent variables. In such situations we shall apply well known Lagrange's method (for more details refer chapter 2).

Recall that $Pp + Qq = R$ is solved by considering its auxiliary equatrions $dx/P = dy/Q = dz/R$. Sometimes the given equation can be reduced to $Pp + Qq = R$ with help of integration of the given equation.

7.7 SOLVED EXAMPLES BASED ON ART 7.6

Ex. 1. *Solve* $t + s + q = 0$. **[Meerut 1994]**

Sol. Re-writing the given equation, $(\partial q/\partial y) + (\partial p/\partial y) + (\partial z/\partial y) = 0$.

Integrating w.r.t. 'y', $q + p + z = f(x)$ or $p + q = f(x) - z$, ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}. \qquad ...(2)$$

From first and second fractions of (2), $dx - dy = 0$

Integrating, $x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From first and third fraction of (2), $(dz/dx) = f(x) - z$ or $(dz/dx) + z = f(x)$.

Its I.F. $= e^{\int dx} = e^x$ and hence its solution is $\quad ze^x = \int e^x f(x)\, dx + c_2$

or $\quad ze^x - \phi(x) = c_2$, where $\phi(x) = \int e^x (x)\, dx$ and c_2 is an arbitrary constant ...(4)

From (3) and (4), the required general solution is

$ze^x - \phi(x) = \psi(x-y)$ or $ze^x = \phi(x) + \psi(x-y)$, where ϕ and ψ are arbitrary functions.

Ex. 2. *Solve* $\quad p + r + s = 1.$ **[Kanpur 2004; Meerut 2005, 10]**

Sol. Re-writing the given equation $\quad (\partial z/\partial x) + (\partial p/\partial x) + (\partial q/\partial x) = 1$

Integrating w.r.t. 'x', $\quad z + p + q = x + f(y) \quad$ or $\quad p + q = x + f(x) - z, \quad$ ...(1)

which is in Langrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + f(y) - z}. \qquad \text{... (2)}$$

From first and second fractions of (2), $\quad dx - dy = 0.$

Integrating, $\quad x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From second and third fractions of (2), $(dz/dy) = x + f(y) - z \quad$ or $\quad (dz/dy) + z = x + f(y).$

Its I.F. $= e^{\int dy} = e^y$ and hence its solution is

$$ze^y = \int \{x + f(y)\} e^y dy + c_2 = x \int e^y\, dy + \int e^y f(y)\, dy + c_2$$

or $$ze^y - xe^y - \phi(y) = c_2, \text{ where } \phi(y) = \int e^y f(y)\, dy. \qquad \text{...(4)}$$

From (3) and (4), the required general solution is $\quad ze^y - xe^y - \phi(y) = \psi(x-y)$

or $\quad (z - x)e^y = \phi(y) + \psi(x - y)$, where ϕ and ψ are arbitrary functions.

Ex. 3. *Solve* $s - t = x/y^2$. **[Ravishankar 2005; I.A.S. 1988]**

Sol. The given equation can be re-written as $\quad \dfrac{\partial p}{\partial y} - \dfrac{\partial q}{\partial y} = \dfrac{x}{y^2} \quad$ or $\quad \dfrac{\partial}{\partial y}(p - q) = xy^{-2}.$

Integrating it w.r.t. 'y', $\quad p - q = -(x/y) + f(x),$ ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-(x/y) + f(x)}. \qquad \text{... (2)}$$

Taking first two fractions of (2), $\quad dx + dy = 0 \quad$ so that $\quad x + y = c_1.$ ...(3)

Taking first and third fractions of (2), $\quad dz = [-(x/y) + f(x)]\, dx$

or $$dz = \left[-\{x/(c_1 - x)\} + f(x)\right] dx, \text{ since from (3), } y = c_1 - x$$

or $$dz = [1 - \{c_1/(c_1 - x)\} + f(x)]\, dx.$$

Integrating, $\quad z = x + c_1 \log(c_1 - x) + \phi(x) + c_2$, where $\phi(x) = \int f(x)\, dx$

or $$z - x - (x + y)\log y - \phi(x) = c_2, \text{ using (3).} \qquad \text{...(4)}$$

From (3) and (4), the required general solution is $\quad z - x - (x+y)\log y - \phi(x) = \psi(x+y)$

or $\quad z = x + (x + y)\log y + \phi(x) + \psi(x + y)$, where ϕ and ψ are arbitrary functions.

Ex. 4. *Solve* $xyr + x^2 s - yp = x^3 e^y$.

Sol. Re-writing the given equations, $\quad xy(\partial p/\partial x) + x^2(\partial p/\partial y) = yp + x^3 e^y.$ ...(1)

Here Lagrange's auxiliary equations for (1) are $\quad \dfrac{dx}{xy} = \dfrac{dy}{x^2} = \dfrac{dp}{yp + x^3 e^y}.$ ...(2)

From the first two fractions of (2), $\quad 2xdx - 2ydy = 0 \quad$ so that $\quad x^2 - y^2 = c_1.$ ... (3)

From second and third fractions of (2), $\quad dp/dy = (yp + x^3 e^y)/x^2$

or $\quad \dfrac{dp}{dy} - \dfrac{yp}{x^2} = xe^y \qquad$ or $\qquad \dfrac{dp}{dy} - \dfrac{y}{y^2+c_1}p = (y^2+c_1)^{1/2}e^y,$

[$\because$ from (3), $x^2 = y^2 + c_1$ so that $x = (y^2+c_1)^{1/2}$]

Its I.F. $= e^{-\int\{y/(y^2+c_1)\}dy} = e^{-(1/2)\times\log(y^2+c_1)} = (y^2+c_1)^{-1/2}$ and solution of above equation is

$$p\,(y^2+c_1)^{-1/2} = \int\{(y^2+c_1)^{-1/2}\cdot e^y(y^2+c_1)^{1/2}\}\,dy + c_2 = \int e^y dy + c_2 = e^y + c_2$$

or $\quad px^{-1} = e^y + c_2,\;$ as from (3) $\;y^2 + c_1 = x^2. \qquad$ ... (4)

From (3) and (4), the general solution of (1) is $\qquad (p/x) - e^y = f(x^2 - y^2)$

or $\quad p = x\,e^y + x\,f(x^2-y^2) \qquad$ or $\qquad \partial z/\partial x = x\,e^y + x\,f(x^2-y^2)$, f being an arbitrary function.

Integrating the above equation w.r.t. 'x', $\qquad z = e^y\int x\,dx + \int x\,f(x^2-y^2)dx + \phi(y)$

or $\quad z = (1/2)\times x^2e^y + \psi(x^2-y^2) + \phi(y)$, where $\psi(x^2-y^2) = \int x\,f(x^2-y^2)dx$

which is the required solution, ϕ and ψ being arbitrary functions.

Ex. 5. (*i*) Solve $xr + ys + p = 10xy^3$ **[Delhi Maths Hons. 1993]**

(*ii*) $xs + yt + q = 10x^3y$.

Sol. Re-writing the given equation $\qquad x(\partial p/\partial x) + y(\partial p/\partial y) = 10xy^3 - p. \qquad$ ...(1)

Its Lagrange's auxiliary equations are $\qquad \dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dp}{10xy^3 - p}. \qquad$ ...(2)

Taking the first two ratios (2), $\qquad (1/x)dx = (1/y)dy \qquad$ so that $\qquad x/y = c_1. \qquad$ ...(3)

Taking the second and third ratios of (2), we have

$\dfrac{dp}{dy} = \dfrac{10xy^3 - p}{y} \qquad$ or $\qquad \dfrac{dp}{dy} + \dfrac{1}{y}p = 10xy^2 \qquad$ or $\qquad \dfrac{dp}{dy} + \dfrac{1}{y}p = 10c_1y^3$, using (3). $\qquad$ ...(4)

I.F. of (4) $= e^{\int(1/y)dy} = e^{\log y} = y$ and so solution is

$py = \int y(10c_1y^3)\,dy = 2c_1y^5 + c_2 \qquad$ or $\qquad py - 2xy^4 = c_2$, using (3) $\qquad$ ...(5)

From (3) and (5), the general solution of (1) is

$py - 2xy^4 = \phi(x/y) \qquad$ or $\qquad (\partial z/\partial x) = 2xy^3 + (1/y)\times\phi(x/y).$

Integrating w.r.t. 'x', $\qquad z = x^2y^3 + \phi_1\!\left(\dfrac{x}{y}\right) + \phi_2(y), \qquad$ where $\qquad \dfrac{\partial}{\partial x}\phi_1\!\left(\dfrac{x}{y}\right) = \dfrac{1}{y}\phi\!\left(\dfrac{x}{y}\right).$

(*ii*) Do yourself. **Ans.** $zx = x^3y^2 + \phi_1(y/x) + \phi_2(x)$.

Ex. 6. *Solve* $\quad sy - 2xr - 2p = 6xy$.

Sol. Re-writing the given equation, $\qquad 2x(\partial p/\partial x) - y(\partial p/\partial y) = -(6xy + 2p). \qquad$ ...(1)

Lagrange's auxiliary equations are $\qquad \dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dp}{-6xy - 2p}. \qquad$ ...(2)

Taking the first and second ratios in (2), we get

$(1/x)dx + (2/y)dy = 0 \qquad$ so that $\qquad xy^2 = c_1. \qquad$ ...(3)

Now, $\qquad$ each ratio of (2) $= \dfrac{2y^3dx - (2yp + 2xy^2)\,dy + y^2dp}{0}$

$\Rightarrow\; (2yp + 2xy^2)dy - 2y^3dx - y^2dp = 0 \;\Rightarrow\; 2y\,(p + 2xy)dy - y^2(dp + 2xdy + 2ydx) = 0.$

$\Rightarrow\; 2y\,(p + 2xy)dy - y^2d\,(p + 2xy) = 0 \qquad \Rightarrow \qquad -\dfrac{2dy}{y} + \dfrac{d(p+2xy)}{p+2xy} = 0$

Integrating it, $\quad -2\log y + \log(p + 2xy) = \log c_1$, being an arbitrary constant

or $\quad \log\{(p + 2xy)/y^2\} = \log c_1 \qquad$ or $\qquad (p + 2xy)/y^2 = c_1. \qquad$ ...(4)

From (3) and (4), the general solution of (1) is

$(p + 2xy)/y^2 = \phi(xy^2)$ or $(\partial z/\partial x) = -2xy + y^2\phi(xy^2)$. ... (5)

Integrating (5) w.r.t. 'x', $z = -x^2y + \phi_1(xy^2) + \phi_2(y)$, where $\dfrac{\partial}{\partial x}\phi_1(xy^2) = y^2\phi(xy^2)$.

Ex. 7. *Solve* $r + (y/x)s = 15xy^2$.

Sol. Re-writing the given equation, $(\partial p/\partial x) + (y/x)(\partial p/\partial y) = 15xy^2$. ...(1)

So Lagrange's auxiliary equations are $$\frac{dx}{1} = \frac{dy}{y/x} = \frac{dp}{15xy^2}. \quad ...(2)$$

From (2), $(1/y)\,dy = (1/x)\,dx \Rightarrow \log y - \log x = \log c_1 \Rightarrow y/x = c_1$. ...(3)

Taking the first and third ratios of (2), $dp = 15xy^2dx = 15c_1^{\,2}x^3dx$, by (3)

Integrating, $p = (15/4) \times c_1^{\,2}x^4 + c_2 \Rightarrow p - (15/4) \times (y/x)^2 \times x^4 = c_2$, by (3)

or $p - (15/4) \times x^2y^2 = c_2$, c_2 being an arbitrary constant ...(4)

Using (3) and (4), the general solution of (1) is

$$p - \frac{15}{4}x^2y^2 = \phi\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{15}{4}x^2y^2 + \phi\left(\frac{y}{x}\right). \quad ...(5)$$

Integrating (5) w.r.t. 'x', $$z = \frac{5}{4}x^3y^2 + y\int \frac{1}{(-y^2/x^2)}\phi\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + \phi_2(y)$$

or $z = (5/4) \times x^3y^2 + y\phi_1(y/x) + \phi_2(y)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 8. *Solve the following partial differential equations* :

(*i*) $2xr - ys + 2p = xy^2$. (*ii*) $2yt - xs + 2q = 4yx^2$.

Ans. (*i*) $z = \phi_1(xy^2) + \phi_2(y) + (1/4) \times x^2y^2$. (*ii*) $z = \phi_1(x^2y) + \phi_2(x) + x^2y^2$.

Ex. 9. *Solve* $s + r = x\cos(x + y)$

Sol. Re-writing the given equation, $\partial p/\partial x + \partial p/\partial y = x\cos(x + y)$ ... (1)

Its Lagrange's auxiliary equations are $$\frac{dx}{1} = \frac{dy}{1} = \frac{dp}{x\cos(x+y)} \quad .... (2)$$

Taking the first two ratios, $dx - dy = 0$ so that $x - y = c_1$ ... (3)

Taking the first and the last fractions of (2) and using (3), we get

$dp = x\cos(x + x - c_1)$ or $dp = x\cos(2x - c_1)$

Integrating, $$p = (1/2) \times x\sin(2x - c_1) - (1/2) \times \int \sin(2x - c_1)dx + c_2$$

or $$p - (1/2) \times x\sin(2x - c_1) - (1/4) \times \cos(2x - c_1) = c_2$$

or $p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = c_2$, using (2). ... (4)

From (3) and (4), the general solution of (1) is given by

$$p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = f(x - y).$$

or $$\partial z/\partial x = (1/2) \times x\sin(x + y) + (1/4) \times \cos(x + y) + f(x - y) \quad ... (5)$$

Integrting (5) w.r.t. 'x', $z = (1/2) \times [-x\cos(x + y) + \int \cos(x + y)\,dx]$

$+ (1/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$, where $\phi_1(x - y) = \int f(x - y)\,dx$

or $z = -(x/2) \times \cos(x + y) + (3/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$.

Ex. 10. *Solve* $yt + xs + q = 8yx^2 + 9y^2$.

Sol. Re-writing the given equation, $x(\partial q/\partial x) + y(\partial q/\partial y) = 8yx^2 + 9y^2 - q$ ... (1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dq}{8yx^2+9y^2-q}$ ... (2)

Taking the first two ratios of (2), $\log y - \log x = \log c_1$ or $y/x = c_1$ ... (3)
Taking the last two ratios of (2), we get

$$\frac{dq}{dy}=9y+8x^2-\frac{q}{y} \quad \text{or} \quad \frac{dq}{dy}+\frac{1}{y}q=9y+8x^2=9y+\frac{8y^2}{c_1^2}, \text{ by (3)}$$

which is linear differential equation. Its I.F. $= e^{\int (1/y)\,dy}=e^{\log y}=y$ and solution is

$$qy=\int y(9y+8y^2/c_1^2)\,dy+c_2=3y^3+(2y^4/c_1^2)+c_2$$

or $\quad qy - 3y^3 - (2y^4/c_1^2) = c_2$ or $qy - 3y^3 - 2y^2x^2 = c_2$, by (3) ... (4)
From (3) and (4), the general solution of (1) is

$qy - 3y^3 - 2y^2x^2 = f(y/x)$ or $\partial z/\partial y = 3y^2+2x^2y+(1/y)\times f(y/x)$
Integrating it w.r.t. 'y' while treating x as constant, we get

$$z=y^3+x^2y^2+\int\frac{1}{y}f(y/x)\,dy+\phi_1(x) \quad \text{or} \quad z=y^3+x^2y^2+\int\frac{f(y/x)}{(y/x)}d\left(\frac{y}{x}\right)+\phi_1(x)$$

or $\quad z= y^3+x^2y^2+\phi_2(y/x)+\phi_1(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 11. *Solve* $xyr+x^2s-yp=x^3e^y$

Sol. Re-writing the given equation, $xy\,(\partial p/\partial x)+x^2\,(\partial p/\partial y)=yp+x^3e^y$ ... (1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{xy}=\dfrac{dy}{x^2}=\dfrac{dp}{yp+x^3e^y}$ ... (2)

Taking the first two ratios of (2), $2xdx-2ydy=0$ so that $x^2-y^2=c_1$ ... (3)
Taking the first and the last ratios of (2), we get

$$\frac{dp}{dx}=\frac{yp+x^3e^y}{xy}=\frac{p}{x}+\frac{x^2e^y}{y} \quad \text{or} \quad \frac{dp}{dx}-\frac{1}{x}p=\frac{x^2}{(x^2-c_1)^{1/2}}e^{(x^2-c_1)^{1/2}}, \text{ by (3)}$$

Its I.F. $= e^{\int(-1/x)\,dx}=e^{-\log x}=1/x$ and solution is

$$p\times\frac{1}{x}=\int\frac{1}{x}\frac{x^2}{(x^2-c_1)^{1/2}}e^{(x^2-c_1)^{1/2}}dx=\int e^t\,dt=e^t+c_2$$

[on putting $(x^2-c_1)^{1/2}=$ t and $\{x/(x^2-c_1^2)^{1/2}\}dx=dt$]

or $\quad (p/x) - e^{(x^2-c_1)^{1/2}} = c_2$ or $(p/x) - e^y = c_2$, using (3) ... (4)
From (3) and (4), the general solution of (1) is

$(p/x) - e^y = f(x^2-y^2)$ or $\partial z/\partial x = x\,e^y+xf(x^2-y^2)$ ... (5)
Integrating (5) w.r.t. 'x' (while treating y as constant), we get

$z=(1/2)\times x^2e^y+\phi_1(x^2-y^2)+\phi_2(y)$, where $\phi_1(x^2-y^2)=\int x\,f(x^2-y^2)\,dx.$

7.8. Type IV. Under this type, we consider equations of the form

$$Rr+Pp+Zz=F \quad \text{or} \quad R\frac{\partial^2 z}{\partial x^2}+P\frac{\partial z}{\partial x}+Zz=F \qquad ...(1)$$

and $\quad Tt + Qq + Zz = F \quad$ or $\quad T\dfrac{\partial^2 z}{\partial y^2} + Q\dfrac{\partial z}{\partial y} + Zz = F, \qquad ...(2)$

which are linear ordinary differential equations of order two with x as independent variable in (1) and y as independent variable in (2).

7.9 SOLVED EXAMPLES BASED ON ART 7.8

Ex. 1. *Solve* $t - 2xq + x^2 z = (x-2)e^{3x+2y}$. **[Delhi Math (G) 1999; Poona 1996]**

Sol. Taking $D' \equiv \partial/\partial y$, the given equation becomes

$$(D'^2 - 2x\,D' + x^2)z = (x-2)e^{3x+2y} \quad \text{or} \quad (D'-x)^2 = (x-2)\,e^{3x+2y} \qquad ...(1)$$

$\therefore$ Complementary function of (1) $= e^{xy}\,\{\phi_1(x) + x\phi_2(x)\}$.

and particular integral of (1) $= \dfrac{1}{(D'-x)^2}(x-2)e^{3x+2y} = \dfrac{(x-2)\,e^{3x+2y}}{(2-x)^2} = \dfrac{e^{3x+2y}}{x-2}$.

$\therefore$ Required solution is $z = e^{xy}\,\{\phi_1(x) + x\phi_2(x)\} + \dfrac{e^{3x+2y}}{x-2}$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t - q - (1/x)\,\{(1/x) - 1\}z = xy^2 - x^2y^2 + 2x^3y - 2x^3$. **[Calicut 1999]**

(*ii*) $r - p - (1/y)\,\{(1/y) - 1\}z = x^2y - x^2y^2 + 2xy^3 - 2y^3$.

Sol. (*i*) Let $D' \equiv \partial/\partial y$. Then given equation can be re-written as

$$[D'^2 - D' - (1/x)\,\{(1/x) - 1\}]\,z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \qquad ...(1)$$

or
$$\left(D' - \frac{1}{x}\right)\left\{D' + \left(\frac{1}{x} - 1\right)\right\}z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \qquad ...(1)'$$

So $\quad$ C.F. $= e^{y/x}\,\phi_1(x) + e^{y-(y/x)}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

In order to determine a particular integral of (1), we assume that

$$z = F_1y^2 + F_2y + F_3, \text{ where } F_1, F_2, F_3 \text{ are functions of } x \text{ or constants.} \qquad ...(2)$$

$$(2) \Rightarrow \partial z/\partial y = 2F_1y + F_2 \quad \Rightarrow \quad \partial^2 z/\partial y^2 = 2F_1.$$

so that $\quad q = 2F_1y + F_2 \quad$ and $\quad t = 2F_1. \qquad ...(3)$

Using (2) and (3), given equation reduces to

$$2F_1 - (2F_1y + F_2) - \frac{1}{x}\left(\frac{1}{x} - 1\right)(F_1y^2 + F_2y + F_3) = xy^2 - x^2y^2 + 2x^3y - 2x^3.$$

Equation coefficients of various powers of y in the above identity, we obtain

$$-\{(1/x^2) - (1/x)\}F_1 = x\,(1-x), \qquad ...(4)$$

$$-2F_1 - \{(1/x^2) - (1/x)\}F_2 = 2x^3 \qquad ...(5)$$

and
$$2F_1 - F_2 - \{(1/x^2) - (1/x)\}F_3 = -2x^3. \qquad ...(6)$$

From (4), $F_1 = -x^3$. Then, from (5), $F_2 = 0$. So (6) $\Rightarrow$ $F_3 = 0$.

$\therefore$ from (2), P.I. $= -x^3y^2$ and so the required solution is

$$z = e^{y/x}\phi_1(x) + e^{y-(y/x)}\phi_2(x) - x^3y^2.$$

(*ii*) Do your as in part (*i*). **Ans.** $z = e^{(x/y)}\phi_1(y) + e^{x-(x/y)}\phi_2(y) - x^2y^3$.

7.10. SOLUTIONS OF EQUATIONS UNDER GIVEN GEOMETRICAL CONDITIONS.

Working rule. As explained in this chapter, we first find the solution of the given equation containing some arbitrary functions of x and y, which are determined with help of the given geometrical conditions. Substituting the values of arbitrary functions in the general solution, we shall obtain surfaces which satisfy the given geometrical conditions.

7.11 SOLVED EXAMPLES BASED ON ART 7.10

Ex. 1.(a) *Find the surface satisfying* $t = 6x^2y$ *containing two lines* $y = 0 = z$ *and* $y = 2 = z$. **[Kanpur 2001; Sagar 2004]**

Sol. Re-writing the given equation, we get $\partial q/\partial y = 6x^2y$.

Integrating it w.r.t. 'y', $q = 3x^2y^2 + f(x)$ or $\partial z/\partial y = 3x^2y^2 + f(x)$.

Integrating it w.r.t. 'y', $z = x^2y^3 + y\,f(x) + \phi(x)$, ...(1)

which is the general solution, f and ϕ being arbitrary functions.

Since (1) contains the given lines $y = 0 = z$ and $y = 2 = z$, we get $0 = \phi(x)$ ...(2)

and $2 = 8x^2 + 2\,f(x) + \phi(x)$. ...(3)

Using (2), (3) becomes $2 = 8x^2 + 2\,f(x)$ or $f(x) = 1 - 4x^2$.

Putting $\phi(x) = 0$ and $f(x) = 1 - 4x^2$ in (1), the required surface is $z = x^2y^3 + y(1 - 4x^2)$.

Ex. 1.(b) *Find a surface satisfying* $t = 6x^3y$ *and containing the two lines* $y = 0 = z, y = 1 = z$.

Sol. Re-writing the given equation, we get $(\partial q/\partial y) = 6x^3y$

Integrating w.r.t. y, $q = 3x^3y^2 + f(x)$ or $\partial z/\partial y = 3x^3y^2 + f(x)$

Integrating it w.r.t. y $z = x^3y^3 + y\,f(x) + \phi(x)$. ...(1)

Since (1) contains the lines $y = 0 = z$ and $y = 1 = z$, we get $0 = \phi(x)$ ...(2)

and $1 = x^3 + f(x) + \phi(x)$. ...(3)

From (2) and (3), $\phi(x) = 0$ and $f(x) = 1 - x^3$.

Putting these values in (1), the required surface is $z = x^3y^3 + y(1 - x^3)$.

Ex. 2. *Find the surface passing through the parabolas* $z = 0$, $y^2 = 4ax$ and $z = 1, y^2 = -4ax$ *and satisfying the equation* $xr + 2p = 0$. **[Kanpur 2000; Agra 1996 ; Meerut 1993 ; I.A.S. 2006]**

Sol. Re-writing the given differential equation, $x(\partial p/\partial x) + 2p = 0$

or $x^2(\partial p/\partial x) + 2px = 0$ or $\partial(x^2p)/\partial x = 0$

Integrating it w.r.t. x, $x^2p = f(y)$ or $p = f(y)/x^2$ or $(\partial z/\partial x) = (1/x^2) \times f(y)$.

Integrating it w.r.t. x, $z = -(1/x) \times f(y) + \phi(y)$. ...(1)

Since (1) passes through $z = 0$, $y^2 = 4ax$, $0 = -(4a/y^2) \times f(y) + \phi(y)$. ...(2)

Again since (1) passes through $z = 1$, $y^2 = -4a\,x$, $1 = (4a/y^2) \times f(y) + \phi(y)$. ...(3)

Adding (2) and (3), $1 = 2\phi(y)$ so that $\phi(y) = 1/2$. ...(4)

Putting $\phi(y) = 1/2$ in (2), we get $f(y) = y^2/8a$. ...(5)

Putting the values of $\phi(y)$ and $f(y)$ given by (4) and (5) in (1), the desired surface is

$z = -y^2/(8ax) + 1/2$ or $8axy = 4ax - y^2$.

Ex. 3. *Show that a surface satisfying* $r = 6x + 2$ *and touching* $z = x^3 + y^3$ *along its section by the plane* $x + y + 1 = 0$ *is* $z = x^3 + y^3 + (x + y + 1)^2$. **[Agra 1994; KU Kurukshetra 2004]**

Sol. Given $r = 6x + 2$ or $\partial p/\partial x = 6x + 2$. ...(1)

Integrating (1) w.r.t. x, $p = 3x^2 + 2x + f(y)$ or $\partial z/\partial x = 3x^2 + 2x + f(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = x^3 + x^2 + xf(y) + F(y)$, ...(3)

where $f(y)$ and $F(y)$ are arbitrary functions.

The given surface is $z = x^3 + y^3$ ...(4)

and the given plane is $x + y + 1 = 0$. ...(5)

Since (3) and (4) touch each other along their section by (5), the values of p and q at any point on (5) must be equal. Thus we must have

$$3x^2 + 2x + f(y) = 3x^2 \qquad ...(6)$$

and $$xf'(y) + F'(y) = 3y^2. \qquad ...(7)$$

From (5) and (6), $f(y) = -2x = 2(y + 1)$ ...(8)

From (8), $f'(y) = 2$. Using this value, (7) gives

$2x + F'(y) = 3y^2$ or $F'(y) = 3y^2 - 2x$ or $F'(y) = 3y^2 + 2(y+1)$, using (5)

Integrating it, $$F(y) = y^3 + y^2 + 2y + c, \quad \text{...(9)}$$

where c is an arbitrary constant. Using (8) and (9), (3) gives

$$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c \quad \text{...(10)}$$

Now at the point of contact of (4) and (10) values of z must be the same and hence we have

$x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c = x^3 + y^3$ or $x^2 + 2x(y+1) + y^2 + 2y + c = 0$

or $x^2 + 2x(-x) + (x+1)^2 - 2(x+1) + c = 0$, as from (5), $y + 1 = -x$ and $y = -(x+1)$

which gives $c = 1$. Putting $c = 1$ in (10), the required surface is

$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + 1$ or $z = x^3 + y^3 + (x + y + 1)^2$

Ex. 4(a). *Show that a surface passing through the circle* $z = 0$, $x^2 + y^2 = 1$ *and satisfying the differential equation* $s = 8xy$ *is* $z = (x^2 + y^2)^2 - 1$. **[Agra 1993 ; Meerut 1994]**

Sol. Re-writing the given equation, $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 8xy. \quad \text{...(1)}$$

Integrating (1) w.r.t. 'x', $$\partial z/\partial y = 4x^2 y + f(y). \quad \text{...(2)}$$

Integrating (2) w.r.t. 'y', $$z = 2x^2y^2 + \int f(y)\,dy + \phi_1(x)$$

or $$z = 2x^2y^2 + \phi_2(y) + \phi_1(x), \quad \text{...(3)}$$

where $\phi_2(y) = \int f(y)\,dy$ and ϕ_1 ϕ_2 are arbitrary functions.

Given circle is given by $x^2 + y^2 = 1$ and $z = 0$. ...(4)

Putting $z = 0$ in (3), we have $$2x^2y^2 + \phi_2(y) + \phi_1(x) = 0. \quad \text{...(5)}$$

Now, $x^2 + y^2 = 1 \quad \Rightarrow \quad (x^2 + y^2)^2 = 1^2 \quad \Rightarrow \quad 2x^2y^2 + x^4 + y^4 = 1.$ ...(6)

Comparing (5) and (6), $$\phi_2(y) + \phi_1(x) = x^4 + y^4 - 1.$$

Substituting the above value of $\phi_2(y) + \phi_1(x)$ in (3), we have

$z = 2x^2y^2 + x^4 + y^4 - 1$ or $z = (x^2 + y^2)^2 - 1$.

Ex. 4(b). *Find the surface passing through the cirlce* $x^2 + y^2 = a^2$, $z = 0$ *and satisfying the differential equation* $s = 8xy$.

Sol. Proceed as in Ex. 4(a).

Ex. 5. *Show that a surface of revolution satisfying the differential equation* $r = 12x^2 + 4y^2$ *and touching the plane* $z = 0$ *is* $z = (x^2 + y^2)^2$. **[Kanpur 1999; Agra 2000, 02 ; Meerut 1993, 97]**

Sol. The given equation can be re-written as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 12x^2 + y^2 \quad \text{or} \quad \frac{\partial p}{\partial x} = 12x^2 + 4y^2. \quad \text{...(1)}$$

Integrating (1) w.r.t. 'x', $$p = \partial z/\partial x = 4x^3 + 4xy^2 + f(y). \quad \text{...(2)}$$

Integrating (2) w.r.t., 'x', $$z = x^4 + 2x^2y^2 + xf(y) + g(y). \quad \text{...(3)}$$

Given the required surface (3) touches the plane $z = 0$. Now, for $z = 0$, $\partial z/\partial x = 0$ and so (2) reduces to $4x^3 + 4xy^2 + f(y) = 0$ or $-f(y) = 4x^3 + 4xy^2$. ...(4)

Since L.H.S. of (4) is function of y alone and R.H.S. is not a function of y alone, (4) shows that we must take each side of (4) equal to zero. Thus, we take $f(y) = 0$ ...(5)

and $4x^3 + 4xy^2 = 0$ so that $x^2 = -y^2$.

Putting $z = 0$, $x^2 = -y^2$ and $f(y) = 0$ in (3), we have

$0 = y^4 - 2y^4 + 0 + g(y)$ so that $g(y) = y^4$. ...(6)

Putting the values of $f(y)$ and $g(y)$ given by (5) and (6) in (3), desired surface is

$z = x^4 + 2x^2y^2 + y^4$ or $z = (x^2 + y^2)^2$.

8

Classification of P.D.E. Reduction to Canonical or Normal Forms. Riemann Method

8.1. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER.

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \qquad \ldots(1)$$

where R, S and T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then (1) is said to be

(*i*) *Hyperbolic* at a point (x, y) in domain D if $S^2 - 4RT > 0$

(*ii*) *Parabolic* at a point (x, y) in domain D if $S^2 - 4RT = 0$

(*iii*) *Elliptic* at a point (x, y) in domain D if $S^2 - 4RT < 0$.

Observe that the type of (1) is determined solely by its principal part ($Rr + Ss + Tt$, which involves the highest order derivatives of z) and that the type will generally change with position in the xy-plane unless R, S and T are constants

Remark. Some authors use u in place of z. Then, we have

$r = \partial^2 u/\partial x^2$, $s = \partial^2 u/\partial x \partial y$ and $t = \partial^2 u/\partial t^2$. etc.

Examples: (*i*) Consider the one-dimensional wave equation $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *i.e.* $r - t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = -1$.
Hence $S^2 - 4RT = 0 - \{4 \times 1 \times (-1)\} = 4 > 0$ and so the given equation is hyperbolic.

(*ii*) Consider the one-dimensional diffusion equation $\partial^2 z/\partial x^2 = \partial z/\partial y$ *i.e.* $r - q = 0$.
Comparing it with (1), here $R = 1$ and $S = T = 0$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 0) = 0$ and so the given equation is parabolic.

(*iii*) Consider two dimensional Laplace's equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ *i.e.* $r + t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = 1$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 1) = -4 < 0$ and so the given equation is elliptic.

Ex. 2. *Classify the following partial differential equations:*

(*i*) $2(\partial^2 u/\partial x^2) + 4(\partial^2 u/\partial x \partial y) + 3(\partial^2 u/\partial y^2) = 2$ **[Meerut 2006]**

(*ii*) $\partial^2 u/\partial x^2 + 4(\partial^2 u/\partial x \partial y) + 4(\partial^2 u/\partial y^2) = 0$ **[I.F.S. 2005]**

(*iii*) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$ **[Delhi Maths (G) 2006]**

(*iv*) $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$ **[Delhi Maths (Prog) 2007]**

(*v*) $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + xp + yq = 0$ **[Delhi 2008]**

(*vi*) $(x-y)(xr - xs - ys + yt) = (x+y)(p-q)$ **[Delhi BA (Prog) II 2011]**

Sol. (*i*) Re-writing the given equation, we get $2r + 4s + 3t - 2 = 0$ ...(1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 2, S = 4$ and $T = 3.$ So $S^2 - 4RT = (4)^2 - (4 \times 2 \times 3) = -8 < 0,$ showing that the given equation is elliptic at all points .

(*ii*) Re-writing the given equation, we get $r + 4s + 4t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, u, p, q) = 0,$ we get $R = 1, S = 4$ and $T = 4.$ So $S^2 - 4RT = (4)^2 - (4 \times 1 \times 4) = 0,$ showing that the given equation is parabolic at all points.

(*iii*) Given $$xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0 \qquad ...(1)$$

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0,$ we get $R = xy, S = -(x^2 - y^2)$ and $T = -xy.$ So, here $$S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 > 0,$$
showing that the given equation is hyperbolic at all points.

(*iv*) Hyperbolic (*v*) Hyperbolic

(*vi*) Hyperbolic

8.2. CLASSIFICATION OF A PARTIAL DIFFERENTIAL EQUATION IN THREE INDEPENDENT VARIABLES.

A linear partial differential differential equation of the second order in 3 independent variables x_1, x_2, x_3 is given by $$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad ... (1)$$

where a_{ij} $(= a_{ji})$, b_i and c are constants or some functions of the independent variables x_1, x_2, x_3 and u is the dependent variable.

Since $a_{ij} = a_{ji}$, $A = [a_{ij}]_{3 \times 3}$ is a real symmetric matrix of order 3 × 3. The eigen values of matrix A are roots of the characteristic equation of A, namely, $| A - \lambda I | = 0$.

With help of matrix A, (1) is classified as follows:

I. If all the eigenvalues of A are non-zero and have the same sign, except precisely one of them, then (1) is known as *hyperbolic type of equation*.

II. If $|A| = 0$, *i.e.,* any one of the eigenvalues of A is zero, then (1) is known as *parabolic type of equation*

III. If all the eigenvalues of A are non-zero and of the same sign, then (1) is known as *elliptic type of equation*.

Note. the matrix A can be remembered as indicated below:

$$A = \begin{bmatrix} \text{Coeff. of } u_{xx} & \text{Coeff. of } u_{xy} & \text{Coeff of } u_{xz} \\ \text{Coeff. of } u_{yx} & \text{Coeff. of } u_{yy} & \text{Coeff of } u_{yz} \\ \text{Coeff. of } u_{zx} & \text{Coeff. of } u_{zy} & \text{Coeff of } u_{zz} \end{bmatrix}$$

8.2.A SOLVED EXAMPLES BASED ON ART. 8.2

Ex. 1. *Classify* $u_{xx} + u_{yy} = u_{zz}$ **[Delhi Maths (H) 2007; Kanpur 2011]**

The matrix A of the given equation is given by $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvalues of A are given by $| A - \lambda I | = 0$, *i.e.*,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad -(1+\lambda)\,(1-\lambda)^2 = 0.$$

Hence $\lambda = -1$, 1, 1, showing that all the eigenvalues are non-zero and have the same sign except one. Hence the given equation is of hyperbolic type.

Ex. 2. *Classify* $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$.

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + u_{yz} + 0 \cdot u_{zx} + u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $\qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$, using properties of determinents

Since $|A| = 0$, the given equation is of parabolic type.

Ex. 3. *Classify* $u_{xx} + u_{yy} + u_{zz} = 0$ **[Meerut 2007, 08; Kanpur 2011]**

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + 0 \cdot u_{yz} + 0 \cdot u_{zx} + 0 \cdot u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $\qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The eigen values of A are given by $|A - \lambda I| = 0$,

i.e. $\qquad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad$ or $\qquad (1-\lambda)^3 = 0$ giving $\qquad \lambda = 1, 1, 1$.

Since all eigenvalues are non-zero and of the same sign, the given equation is of parabolic type.

Ex. 4. *Classify the following equations:*

(*i*) $u_{xx} + u_{yy} = u_z$ **[Kanpur 2011]** (*ii*) $u_{xx} + 2\,u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$. **[Delhi 2008]**

Sol. Try yourself **Ans.** (*i*) parabolic (*ii*) parabolic

8.3. Cauchy's problem for second order partial differential equation. Characteristic equation and characteristic curves (or simply characteristics) of the second order partial differential equations. (Delhi Maths (H) 2001)

Cauchy' problem. Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{... (1)}$$

in which R, S and T are functions of x and y only. The Cauchy's problem consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed values of z and $\partial z / \partial n$, where n is the distance measured along the normal to the curve.

As an example of Cauchy's problem for the second order partial differential equation, consider the following problem :

To determine solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ with the following data prescribed on the x-axis: $z(x, 0) = f(x)$, $z_y(x, 0) = g(x)$. Observe that y-axis is the normal to the given curve (x-axis here)

Characteristic equations and characteristic curves.

Corresponding to (1), consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{... (2)}$$

where $S^2 - 4RT \geq 0$, (2) has real roots. Then, the ordinary differential equations

$$(dy/dx) + \lambda(x, y) = 0 \qquad \text{... (3)}$$

are called the *characteristic equations*.

The solutions of (3) are knwon as *characteristic curves* or simply the *characteristics* of the second order partial differential equation (1).

Now, consider the following three cases:

Case (*i*) If $S^2 - 4RT > 0$ (*i.e.,* if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations $(dy/dx) + \lambda_1(x, y) = 0$ and $(dy/dx) + \lambda_2(x, y) = 0$.

Solving these we get two distinct families of characteristics.

Case (*ii*). If $S^2 - 4RT = 0$ (*i.e.* (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). Solving it, we get only one family of characteristics.

Case (*iii*) If $S^2 - 4RT < 0$ (*i.e.* (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic

8.4 ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 8.3

Ex. 1. *Find the characteristics of $y^2 r - x^2 t = 0$* **[I.A.S. 2009]**

Sol. Given $$y^2 r - x^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$ and $T = -x^2$. Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolaic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad y^2\lambda^2 - x^2 = 0 \quad$ ... (2)

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots). Corresponding characteristic equations are

$$(dy/dx) + (x/y) = 0 \qquad \text{and} \qquad (dy/dx) - (x/y) = 0$$

or $$x\,dx + y\,dx = 0 \qquad \text{and} \qquad x dx - y\,dy = 0$$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Ex. 2. *Find the characteristics of $x^2 r + 2xys + y^2 t = 0$.*

Sol. Given $$x^2 r + 2xys + y^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$. Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ and hence (1) is parabolic everywhere.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad x^2\lambda^2 + 2xy\lambda + y^2 = 0 \quad$ ... (2)

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots). The characteristic equation is $(dy/dx) - (y/x) = 0$ or $(1/y)\,dy - (1/x)\,dx = 0$ giving $y/x = c_1$ or $y = c_1 x$, which is the required family of characteristics. Here it represents a family of straight lines passing through the origin.

Ex. 3. *Find the characteristics of $4r + 5s + t + p + q - 2 = 0$.*

Sol. Try yourself. **Ans.** $y - x = c_1$ and $y - (x/y) = c_2$.

Ex. 4. *Find the characteristics of $(\sin^2 x)\, r + (2 \cos x)\, s - t = 0$*

Sol. Try yourself **Ans.** $y + \operatorname{cosec} x - \cot x = c_1$, $y + \operatorname{cosec} x + \cot x = c_2$

8.5. Laplace transformation. Reduction to Canonical (or normal) forms.

[Himanchal 2007; Avadh 2001; Delhi Maths (H) 2004, 09]

Consider partial differential equation of the type $\quad Rr + Ss + Tt + f(x, y, z, p, q) = 0, \quad$ ...(1)

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation on (1) consists of changing the independent variables x, y to new set of continuously differentiable independent variables u, v where

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y) \qquad ...(2)$$

are to be chosen so that the resulting equation in independent variables u, v is transformed into one of three canonical forms, which are easily integrable. From (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{and} \qquad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad ...(3)$$

$$(3) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}. \qquad ...(4)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right), \text{by (3) and (4)}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Putting the above values of p, q, r, s, t, in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F(u, v, z, \partial z/\partial u, \partial z/\partial v) = 0, \qquad ...(5)$$

where

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2, \qquad ...(6)$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad ...(7)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad ...(8)$$

and $F(u, v, z, \partial z/\partial u, \partial z/\partial v)$ is the transformed form of $f(x, y, z, p, q)$.

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad ...(9)$$

is everywhere either positive, negative or zero, and now we shall present these three cases separately.

Case I. Let $S^2 - 4RT > 0$. When this condition is satisfied, then the roots λ_1, λ_2 of the equation (9) are real and distinct. The coefficients of $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in the equation (5) will vanish if we choose u and v such that

$$\partial u/\partial x = \lambda_1 (\partial u/\partial y) \qquad ...(10)$$

and

$$\partial v/\partial x = \lambda_2 (\partial v/\partial y). \qquad ...(11)$$

Since λ_1 is a root of (9), we have $R\lambda_1{}^2 + S\lambda_1 + T = 0.$ ...(12)

Using (10), (6) gives $A = (R\lambda_1{}^2 + S\lambda_1 + T)\,(\partial u/\partial y)^2 = 0$, by (12) ...(13)

Again, since λ_2 is a root of (9), we have $R\lambda_2{}^2 + S\lambda_2 + T = 0$ ...(14)

Using (11), (8) gives $C = (R\lambda_2{}^2 + S\lambda_2 + T)\,(\partial v/\partial y)^2 = 0$, by (14) ...(15)

Re-writing (10), we have $(\partial u/\partial x) - \lambda_1(\partial u/\partial y) = 0.$ ...(16)

Lagrange's auxiliary equation for (16) are $dx/1 = dy/(-\lambda_1) = du/0$ ...(17)

Taking third fraction of (17), $du = 0$ so that $u = c_1$, c_1 being an arbitrary constant ...(18)

Taking first and second fractions of (17), we get $(dy/dx) + \lambda_1 = 0$ ...(19)

Let the solution of (19) be $f_1(x, y) = c_2$, c_2 being an arbitrary constant ...(20)

From (18) and (20), the general solution of (16) [*i.e.* (10)] is $u = f_1(x, y)$. ...(21)

Similarly, the general solution of (11) can be taken as $v = f_2(x, y)$. ...(22)

Here f_1 and f_2 are arbitrary function

We can easily verify that
$$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

or
$$B^2 = \frac{1}{4}\,(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \text{ as } A = C = 0. \qquad ...(23)$$

Let the Jacobian J of u and v be non-zero, i.e., let
$$J = \partial(u, v)/\partial(x, y) = (\partial u/\partial x)\,(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) \neq 0$$

Since $S^2 - 4\,RT > 0$, (23) shows that $B^2 > 0$. Hence we may divide both sides of (5) by B^2. Then noting that $A = C = 0$, (5) transforms to the form $\partial^2 z/\partial u\,\partial v = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right)$,
...(24)

which is the canonical form of (1) in this case.

Case II. Let $S^2 - 4RT = 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. We now take u exactly as in case I and take v to be any function of x, y which is independent of u. We have, as in case I, $A = 0$. Also, since $S^2 - 4RT = 0$, (23) shows that $B^2 = 0$ so that $B = 0$.

Moreover in this case $C \neq 0$, otherwise v would be a function of u and consequently v would not be independent of u as already assumed.

Putting $A = 0$, $B = 0$ and dividing by C, (5) transforms to the form
$$\partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v). \qquad ...(25)$$
which is the canonical form of (1) in this case.

Case III. Let $S^2 - 4RT < 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are complex. Hence this case III is formally the same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (24) but that the variables u, v instead of being real are now complex conjugates. To obtain a real canonical form we make further transformation $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that

$\alpha = (u + v)/2$, and $\beta = i\,(v - u)/2$. ...(26)

Now,
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad ...(27)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad ...(28)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{dv}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (27) and (28)}$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right] = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\frac{\partial^2 z}{\partial \beta \partial \alpha} + i\frac{\partial^2 z}{\partial \beta^2}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right), \quad \text{as} \quad \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \qquad ...(29)$$

Putting $u = \alpha + i\beta$, $v = \alpha - i\beta$ and using (27), (28) and (29), (24) reduces to

$$(\partial^2 z/\partial \alpha^2) + (\partial^2 z/\partial \beta^2) = \psi(\alpha, \beta, z, \partial z/\partial \alpha, \partial z/\partial \beta), \qquad ...(30)$$

which is the canonical form of (1) in this case.

8.6 Working rule for reducing a hyperbolic equation to its canonical form

Step 1. Let the given equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ ... (1)
be hyperbolic so that $S^2 - 4RT > 0$.

Step 2. Write λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ ... (2)

Let λ_1 and λ_2 be its two distinct roots of (2).

Step 3. Then corresponding characteristic equations are

$$(dy/dx) + \lambda_1 = 0 \quad \text{and} \quad (dy/dx) + \lambda_2 = 0$$

Solving these, we get $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ ... (3)

Step 4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y)$ ... (4)

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s, t obtained in step 4 in (1) and simplifying we shall get the following canonical form of (1):

$$\partial^2 z/\partial u \partial v = \phi\,(u, v, z, \partial z/\partial u, \partial z/\partial u).$$

8.7. SOLVED EXAMPLES BASED ON ART. 8.6

Ex.1. (*a*) *Write canonical form of* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$. **[Sagar 2004; Delhi Maths (H) 2002]**

(*b*) Reduc $3(\partial^2 z/\partial x^2) + 10\,(\partial^2 z/\partial x \partial y) + 3\,(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it **(Himanchal 2008)**

Sol. (*a*) Re-writing the given equation, we get $r - t = 0$ ... (1)

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ so that $S^2 - 4RT = 4 > 0$, showing that (1) is hyperbolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$

Hence $\lambda = 1, -1$. So $\lambda_1 = 1, \lambda_2 = -1$ (Real and distinct roots).

Then the characteristic equations $dy/dx + \lambda_1 = 0$, $dy/dx + \lambda_2 = 0$ reduces to

$(dy/dx) + 1 = 0$ and $(dy/dx) - 1 = 0$.

Integrating these, $y + x = c_1$ and $y - x = c_2$.

In order to reduce (1) to its canonical form, we choose

$$u = y + x \quad \text{and} \quad v = y - x \qquad ... (2)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \qquad ... (5)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ... (6)$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

or $$t=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)=0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u\partial v}=0.$$

(*b*) $\partial^2 z/\partial u\partial v=0;\ z=f(y-3x)+g(3y-x)$

Ex. 2. *Reduce* $\partial^2 z/\partial x^2=(1+y)^2(\partial^2 z/\partial y^2)$ *to canonical form*

Sol. Re-writing the given equation, $r-(1+y^2)\,t=0$ ... (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=1$, $S=0$, and $T=-(1+y)^2$ so that $S^2-4RT=(1+y^2)>0$ for $y\neq -1$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2-(1+y)^2=0$ so that $\lambda=1+y,\ -(1+y)$. Hence the corresponding characteristic equations are given by

$$(dy/dx)+(1+y)=0 \qquad \text{and} \qquad (dy/dx)-(1+y)=0$$

Integrating these, $\log(1+y)+x=C_1$ and $\log(1+y)-x=C_2$.

In order to reduce (1) to its canonical form, we choose

$$u=\log(1+y)+x \qquad \text{and} \qquad v=\log(1+y)-x \qquad \text{... (2)}$$

$\therefore$ $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

and $$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{1+y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) \qquad \text{... (4)}$$

From (3) $$\partial/\partial x\equiv\partial/\partial u-\partial/\partial v \qquad \text{... (5)}$$

$\therefore$ $$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r=\partial^2 z/\partial u^2-2\,(\partial^2 z/\partial u\,\partial v)+\partial^2 z/\partial v^2 \qquad \text{... (6)}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\} = -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{1}{1+y}\left[\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial u\partial v}\right)\frac{1}{y+1}+\left(\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}\right)\frac{1}{1+y}\right], \text{ by (2)}$$

or $$t=\frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right) \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)=0 \qquad \text{or} \qquad 4\,\frac{\partial^2 z}{\partial u\partial v}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}..$$

Ex. 3. *Reduce the differential equation* $t-s+p-q(1+1/x)+(z/x)=0$ *to canonical form.*

[Delhi Maths (H) 2004]

Sol. Given $$0\cdot r - s + t + p - q\,(1 + 1/x) + (z/x) = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 0$, $S = -1$ and $T = 1$. Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$. Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c$, c being an arbitrary constant

Choose $u = x + y$ and $v = x$, ...(2)

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \;\Rightarrow\;$ u and v are independent functions.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad ...(4)$$

From (4), we have $$\partial/\partial y \equiv \partial/\partial u \qquad ...(5)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)} \qquad ...(6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right), \text{ using (5)}$$

or $$t = \partial^2 z/\partial u^2. \qquad ...(6)$$

Using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\partial^2 z/\partial u \partial v - (\partial z/\partial v) + (1/v) \times (\partial z/\partial u) - (z/v) = 0$, which is the required canonical form.

Ex. 4. *Reduce the equation yr + (x + y) s + xt = 0 to canonical form and hence find its general solution.* **(Delhi Maths (Hons) 2007)**

Sol. Given $$yr + (x + y)\,s + xt = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y$, $S = (x + y)$ and $T = x$ so that $S^2 - 4RT = (x + y)^2 - 4xy = (x - y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x + y)\,\lambda + x = 0$ or $(y\lambda + x)\,(\lambda + 1) = 0$ so that $\lambda = -1$, $-x/y$. Then the corresponding characteristic equations are given by

$$(dy/dx) - 1 = 0 \quad \text{and} \quad (dy/dx) - (x/y) = 0$$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$$u = y - x \quad \text{and} \quad v = y^2/2 - x^2/2 \qquad ...(2)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}, \text{ using (2)} \qquad ...(4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right),\text{ using (3)}$$

$$=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

$$=-\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]-\frac{\partial z}{\partial v}$$

$$=-\left(-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}\right)-x\left(-\frac{\partial^2 z}{\partial u\partial v}-x\frac{\partial^2 z}{\partial v^2}\right)-\frac{\partial z}{\partial v},\text{ using (2)}$$

$$\therefore \qquad r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v} \qquad \text{... (5)}$$

Now, $t=\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right)+\dfrac{\partial}{\partial y}\left(y\dfrac{\partial z}{\partial v}\right)$, using (4)

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\therefore\ t=\frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v} \qquad \text{...(6)}$$

Also, $s=\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right)+\dfrac{\partial}{\partial x}\left(y\dfrac{\partial z}{\partial v}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right)+y\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}=-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}-y\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2},\text{ using (2)}$$

$$\therefore \qquad s=-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (5) (6) and (7) in (1), we get $y\left(\dfrac{\partial^2 z}{\partial u^2}+2x\dfrac{\partial^2 z}{\partial u\partial v}+x^2\dfrac{\partial^2 z}{\partial v^2}-\dfrac{\partial z}{\partial v}\right)$

$$+(x+y)\left\{-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}\right\}+x\left(\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}\right)=0$$

or $\quad \{4xy-(x+y)^2\}\dfrac{\partial^2 z}{\partial u\partial v}-y\dfrac{\partial z}{\partial v}+x\dfrac{\partial z}{\partial v}=0 \qquad$ or $\qquad (y-x)^2\dfrac{\partial^2 z}{\partial u\partial v}+(y-x)\dfrac{\partial z}{\partial v}=0$

or $\quad u^2\dfrac{\partial^2 z}{\partial u\partial v}+u\dfrac{\partial z}{\partial v}=0$, by (2) $\qquad$ or $\qquad u\dfrac{\partial^2 z}{\partial v\partial v}+\dfrac{\partial z}{\partial v}=0$, as $u\neq 0$ ... (8)

(8) is the required canonical form of (1).

Solution of (8). Multiplying both sides of (8) by v, we get

$uv\,(\partial^2 z/\partial u\partial v)+v(\partial z/\partial v)=0 \qquad$ or $\qquad (uv\ DD'+vD')z=0 \qquad$ ... (9)

where $D\equiv\partial/\partial u$ and $D'\equiv\partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows. For details refer Art. 6.3.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ... (10)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (9) reduces to

$$(D_1\ D_1' + D_1')z = 0 \qquad \text{or} \qquad D_1'\ (D_1 + 1)\ z = 0$$

Its general solution is $z = e^{-X}\ \phi_1\ (Y) + \phi_2(X) = u^{-1}\ \phi_1(\log v) + \phi_2(\log u)$ [See Art. 5.6]

or $z = u^{-1}\ \psi_1(v) + \psi_2(u) = (y-x)^{-1}\ \psi_1(y^2 - x^2) + \psi_2(y-x)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex.5. *Reduce the equation* $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$ *to canonical form and hence solve it.* **(Himanchal 2008)**

Sol. Given $$r - (2\sin x)\ s - (\cos^2 x)t - (\cos x)q = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2\sin x$ and $T = -\cos^2 x$ so that $S^2 - 4RT = 4(\sin^2 x + \cos^2 x) = 4 > 0$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (2\sin x)\lambda - \cos^2 x = 0$ so that $\lambda = \sin x + 1,\ \sin x - 1$. Hence the corresponding characteristic equations become

$dy/dx + \sin x + 1 = 0$ and $dy/dx + \sin x - 1 = 0$

Integrating these, $y - \cos x + x = c_1$ and $y - \cos x - x = c_2$

Choose $u = y - \cos x + x$ and $v = y - \cos x - x$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (1+\sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

From (4), we have $$\partial/\partial y = \partial/\partial u + \partial/\partial v \qquad \text{... (5)}$$

$$\therefore \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

Now, $$s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{(1+\sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\}, \text{ by (3)}$$

$$= (\sin x + 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + (\sin x - 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$= (\sin x + 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right\} + (\sin x - 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= (\sin x + 1)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + (\sin x - 1)\left(\frac{\partial^2 z}{\partial u \partial x} + \frac{\partial^2 z}{\partial v^2}\right)$$

or $$s = \sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{(\sin x + 1)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\} = \cos x\frac{\partial z}{\partial u} + (\sin x + 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \cos x\frac{\partial z}{\partial v} + (\sin x - 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}+(\sin x-1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u^2}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}+(\sin x-1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u\partial v}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}$$

$$\therefore \quad r=\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+\sin x)^2\frac{\partial^2 z}{\partial u^2}+(\sin x-1)^2\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v} \qquad \text{... (8)}$$

Using (4) (6), (7) and (8) in (1), we get

$$\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+2\sin x+\sin^2 x)\frac{\partial^2 x}{\partial u^2}+(\sin^2 x+1-2\sin x)\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\,\partial v}$$

$$-2\sin x\left\{\sin x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right\}-\cos^2 x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=0$$

or $\quad (1+2\sin x+\sin^2 x-2\sin^2 x-2\sin x-\cos^2 x)\times(\partial^2 z/\partial u^2)+(\sin^2 x+1-2\sin x-2\sin^2 x$

$$+2\sin x-\cos^2 x)\times(\partial^2 z/\partial v^2)-(2\cos^2 x+4\sin^2 x+2\cos^2 x)\times(\partial^2 z/\partial u\,\partial v)=0$$

or $\quad \partial^2 z/\partial u\,\partial v=0$, on simplification. ... (9)

(9) is the required canonical form of (1).

Solution of (9). Integrating (9) w.r.t. 'u', $\partial z/\partial v=\phi(v), \phi$ being an arbitrary function ... (10)

Integrating (10) w.r.t. 'v', $\quad z=\int \phi(v)\,dv+F(u)=G(v)+F(u)$,

where $G(v)=\int \phi(v)\,dv$, F and G are arbitrary functions.

$\therefore \quad z=G(y-\cos x-x)+F(y-\cos x+x)$ is the required solution.

Ex. 6. *Reduce $\partial^2 z/\partial x^2=x^2(\partial^2 z/\partial y^2)$ to canonical form.*

[Agra 2005; Himanchal 2005; Delhi B.Sc. (Prog) II 2002, 07; Kurukshetra 2004; Ravishankar 2004; Nagpur 2010, Kanpur 2011]

Sol. Re-writing the given equation becomes $\quad r-x^2t=0.$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have $R=1$, $S=0$, $T=-x^2$.

Now, the λ-quadratic $R\lambda^2+S\lambda+T=0$ gives $\lambda^2-x^2=0$ so that $\lambda=\pm x$.

$\therefore$ Here $\lambda_1=x$ and $\lambda_2=-x$ (Real and disticnt roots)

Hence characterstic equations $dy/dx+\lambda_1=0$ and $dy/dx+\lambda_2=0$

become $dy/dx+x=0$ and $dy/dx-x=0$.

Integrating these, $y+(x^2/2)=c_1$ and $y-(x^2/2)=c_2$.

Hence in order to reduce (1) to canonical form, we change x, y, to u, v by taking

$u=y+(x^2/2)$ and $v=y-(x^2/2)$ ...(2)

Now, $\quad p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=x\dfrac{\partial z}{\partial u}-x\dfrac{\partial z}{\partial v}$, using (2) ...(3)

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{...(4)}$$

$$\therefore \quad r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial^2 v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = \dfrac{\partial^2 z}{\partial u^2} + 2\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial^2 z}{\partial v^2}$, using (4)

Putting the above values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right) \quad$ or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, by (2)

which is the required canonical form of the given equation.

Ex. 7. *Reduce the equation* $(n-1)^2 (\partial^2 z/\partial x^2) - y^{2n}(\partial^2 z/\partial y^2) = n y^{2n-1} (\partial z/\partial y)$ *to canonical form, and find its general solution*. **[Delhi Maths. (H) 2000, 01, 05; Himanchal 2004; Ravishankar 2004]**

Sol. Given $\quad (n-1)^2 r - y^{2n}\, t - n\, y^{2n-1}\, q = 0. \quad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = (n-1)^2$, $S = 0$, $T = -y^{2n}$.

Now, the λ—quadratic $R\lambda^2 + S\lambda + T = 0$ gives

$(n-1)^2 \lambda^2 - y^{2n} = 0 \quad$ so that $\quad \lambda = \pm (n-1)^{-1} y^n$.

$\therefore$ Here $\quad \lambda_1 = (n-1)^{-1} y^n \quad$ and $\quad \lambda_2 = -(n-1)^{-1} y^n$.

Hence, characterstic equations $\quad dy/dx + \lambda_1 = 0 \quad$ and $\quad dy/dx + \lambda_2 = 0$

become $\quad dy/dx + (n-1)^{-1} y^n = 0 \quad$ and $\quad dy/dx - (n-1)^{-1} y^n = 0$.

Integrating these, $\quad x - y^{-n+1} = c_1 \quad$ and $\quad x + y^{-n+1} = c_2$.

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking

$u = x - y^{-n+1} \quad$ and $\quad v = x + y^{-n+1}. \quad$ ...(2)

$\therefore\ p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} \quad$ so that $\quad \dfrac{\partial}{\partial x} \equiv \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{(n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n}\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Substituting the above values of r, t, q in (1) and simplifying, we obtain

$$\partial^2 z / \partial u\, \partial v = 0, \qquad \text{...(3)}$$

which is the required canonical form of the given equation

Integrating (3) w.r.t. 'v', $\quad \partial z/\partial u = F(u)$, where $F(u)$ is an arbitrary function of u, $\quad$ ...(4)

Integrating (4) w.r.t. 'u', $\quad z = G(u) + H(v)$,

where $G(u) = \int F(u)\, du$ and $G(u), H(v)$ are arbitrary functions

Using (2), the solution of the given equation is $\quad z = G(x - y^{-n+1}) + H(x + y^{-n+1})$.

Ex. 8. *Reduce the equation* $(y-1)\,r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$ to *canonical form and hence solve it.* **[Delhi B.Sc. (Hons) III 2008; Rohilkhand 1992]**

Sol. Given $\qquad (y-1)\,r-(y^2-1)s+y(y-1)t+p-q-2ye^{2x}(1-y)^3=0. \qquad ...(1)$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=y-1, \qquad S=-(y^2-1) \qquad$ and $\qquad T=y(y-1). \qquad ...(2)$

$\therefore$ The λ–quardratic $R\lambda^2-S\lambda+T=0$ gives

$(y-1)\lambda^2-(y^2-1)\lambda+y(y-1)=0 \quad \Rightarrow \quad \lambda_1=1 \quad$ and $\quad \lambda_2=y \quad$ (real and distinct roots)

Hence characterstic equations $\quad (dy/dx)+\lambda_1=0 \quad$ and $\quad (dy/dx)+\lambda_2=0$ become

$(dy/dx)+1=0 \quad$ and $\quad (dy/dx)+y=0.$

Integrating these, $\qquad x+y=c_1 \quad$ and $\quad y\,e^x=c_2.$

To reduce (1) to canonical form, we change the independent variables x, y, to new independent variables u, v by taking

$u=x+y \qquad$ and $\qquad v=y\,e^x. \qquad ...(3)$

$$\therefore \qquad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}, \text{ by (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}, \text{ by (3)} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v}, \text{ by (4)}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Substituting the above values in (1) and simplifying, we have

$$(1-y)^3e^x\frac{\partial^2 z}{\partial u\partial v}=2y\,e^{2x}(1-y)^3 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u\partial v}=2v, \qquad ...(6)$$

which is the canonical form of (1).

Integrating (6) w.r.t. 'v', $\qquad \partial z/\partial u=v^2+\phi(u)$, $\phi(u)$ being an arbitrary function $\qquad ...(7)$

Integrating (7) w.r.t. 'u', $\qquad z=uv^2+\phi_1(u)+\phi_2(v)$, where $\phi_1(u)=\int\phi(u)\,du$

$\therefore$ Using (3) $\qquad z=(x+y)y^2e^{2x}+\phi_1(x+y)+\phi_2(ye^x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 9. *Solve* $x^2(y-1)\,r-x\,(y^2-1)\,s+y(y-1)\,t+xyp-q=0.$

Sol. Given $\qquad x^2(y-1)\,r-x(y^2-1)\,s+y(y-1)t+xyp-q=0. \qquad ...(1)$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=x^2(y-1), \qquad S=-x\,(y^2-1) \qquad$ and $\qquad T=y\,(y-1).$

$\therefore$ λ–quadratic $R\lambda^2+S\lambda+T=0$ reduces to

$x^2(y-1)\,\lambda^2-x(y^2-1)\,\lambda+y(y-1)=0 \qquad \Rightarrow \qquad \lambda_1=y/x \quad$ and $\quad \lambda_2=1/x \quad$ (real and distinct)

So characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$ become

$$(dy/dx) + (y/x) = 0 \quad \text{and} \quad (dy/dx) + (1/x) = 0$$

Integrating these, $xy = c_1$ and $xe^y = c_2$ so for canonical form, we take

$$u = xy \quad \text{and} \quad v = xe^y. \qquad \text{...(2)}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{...(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] = y^2\frac{\partial^2 z}{\partial u^2} + 2ye^x\frac{\partial^2 z}{\partial u\partial v} + e^{2y}\frac{\partial^2 z}{\partial y^2},$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial z}{\partial v} + xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + xy\frac{\partial^2 z}{\partial u^2} + (yxe^y + e^y x)\frac{\partial^2 z}{\partial u\partial v} + xe^{2y}\frac{\partial^2 z}{\partial v^2}$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + xe^y\frac{\partial z}{\partial v} + x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + xe^y\frac{\partial z}{\partial v} + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2x^2e^y\frac{\partial^2 z}{\partial u\partial v} + x^2e^{2y}\frac{\partial^2 z}{\partial y^2} + xe^y\frac{\partial z}{\partial v}.$$

Substituting the above values in (1) and simplifying, we get $\partial^2 z/\partial u\partial v = 0$, ... (5)
which is canonical form of (1).

Integrating (5) w.r.t. 'u', $\partial z/\partial v = \phi(v)$, $\phi(v)$ being an arbitrary function.

Integrating it w.r.t. 'v', $z = \phi_1(v) + \phi_2(u)$, where $\phi_1(v) = \int \phi(v)\,dv$.

$\therefore$ $z = \phi_1(xe^y) + \phi_2(xy)$, by (2). This is the required solution, ϕ_1, ϕ_2 being arbitrary functions

Ex. 10. *Solve* (*i*) $xyr - (x^2 - y^2)\,s - xyt + py - qx = 2(x^2 - y^2)$. **[Delhi Maths (H) 2006]**
(*ii*) $x(y - x)\,r - (y^2 - x^2)\,s + y\,(y - x)\,t + (y + x)(p - x) = 2x + 2y + 2$.

Sol. (*i*) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ...(1)

Comparing (*i*) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = xy, \quad S = -(x^2 - y^2) \quad \text{and} \quad T = -xy.$$

So λ–quadratic $R\lambda^2 + S\lambda + T = 0$ becomes $xy\lambda^2 - (x^2 - y^2)\lambda - xy = 0$ giving $\lambda = -y/x, x/y$.

$$\therefore \quad \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{y}{x} = 0.$$

Integrating, $y/x = c_1$, and $x^2 + y^2 = c_2$. So, we take

$$u = y/x \quad \text{and} \quad v = x^2 + y^2. \qquad \text{...(2)}$$

$\therefore$ Proceeding as usual, we obtain

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v},\quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v},$$

$$r = \left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial u^2} + 2\times(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v\partial u} + 4x^2\frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$$

$$s = \left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial u^2} + \left\{2y\left(-\frac{y}{x^2}\right) + 2x\times\frac{1}{x}\right\}\frac{\partial^2 z}{\partial u\partial v} + 4xy\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u}$$

and $$t = \left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial u^2} + 2\times\frac{1}{x}\times(2y)\frac{\partial^2 z}{\partial u\partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial v}.$$

Substituting these in (1) we get

$$(x^2+y^2)^2\frac{\partial^2 z}{\partial u\partial v} = (y^2-x^2)x^2 \quad\text{or}\quad \frac{\partial^2 z}{\partial u\partial v} = \frac{(y^2-x^2)x^2}{(x^2+y^2)^2} = \frac{u^2-1}{(u^2+1)^2},\ \text{by (2)} \quad ...(3)$$

Integrating (3) w.r.t. 'u', we have

$$\frac{\partial z}{\partial v} = \int\frac{u^2-1}{(u^2+1)^2}du + \phi(v) = \int\frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2} + \phi(v) \quad ...(4)$$

We have, $$\int 1\cdot\frac{1}{u^2+1}du = u\times\frac{1}{u^2+1} - \int u\times\left(\frac{-2u}{(u^2+1)^2}\right)du,\ \text{integrating by parts}$$

or $$\int\frac{du}{u^2+1} = \frac{u}{u^2+1} + 2\int\frac{(u^2+1)-1}{(u^2+1)^2}du = \frac{u}{u^2+1} + 2\int\frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2}$$

Then, $$\int\frac{du}{u^2+1} - 2\int\frac{du}{(u^2+1)^2} = -\frac{u}{u^2+1} \quad ...(4)$$

Using (5), (4) gives $\partial z/\partial v = -u/(u^2+1) + \phi(v)$, $\phi(v)$ being an arbitrary function ...(6)

Integrating (6) w.r.t. v, $z = -(uv)/(u^2+v^2) + \phi_1(v) + \phi_2(u)$, where $\phi_1(v) = \int\phi(v)\,dv$

$\therefore$ Using (2), $z = -xy + \phi_1(x^2+y^2) + \phi_2(y/x)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) **Hint.** Since $R = x(y-x)$, $S = -(y^2-x^2)$, $T = y(y-x)$, so here $\lambda_1 = y/x$, $\lambda_2 = 1$.

So we get $(dy/dx) + (y/x) = 0$ and $(dy/dx) + 1 = 0$ as characteristic equations

These give $xy = c_1$ and $x + y = c_2$. Hence take

$u = xy$ and $v = x + y$. ...(1)

As usual, $p = y\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$ and $q = x\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}$,

$$r = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2},\quad t = x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2},\quad s = xy\frac{\partial^2 z}{\partial u^2} + (x+y)\frac{\partial^2 z}{\partial v\partial u} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}.$$

$\therefore$ Given equation becomes $$-(y-x)^3\frac{\partial^2 z}{\partial v\partial u} = 2x + 2y + 2 \quad ...(2)$$

or $$\frac{\partial^2 z}{\partial v \partial u} = -\frac{2(x+y+1)}{(y-x)^3} = -\frac{2(x+y+1)}{[(y+x)^2-4xy]^{3/2}} = \frac{2(v+1)}{(v^2-4u)^{3/2}}, \text{ by (1)}$$

Integrating (2) w.r.t. 'u', we get $$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2-4u)}} + \phi(v). \qquad \text{... (3)}$$

Integrating, (3) w.r.t. v, $$z = \sqrt{(v^2-4u)} + \log\,[v + \sqrt{(v^2-4u)}\,] + \phi_1(v) + \phi_2(u)$$

or $z = x - y + \log(2x) + \phi_1(x+y) + \phi_2(xy)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 11. *Solve* (*i*) $y(x+y)(r-s) - xp - yq - z = 0$ **[Delhi Maths (H) 1998]**

(*ii*) $xys - x^2r - px - qy + z = -2xy^2y$.

Sol. (*i*) Given $\quad y(x+y)\,r - y\,(x+y)s - xp - yq - z = 0. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, $R = y\,(x+y)$, $S = -\,y\,(x+y)$, $T = 0$.

So, the λ–quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$y(x+y)\lambda^2 - y(x+y)\lambda = 0$, giving $\lambda = 0, 1$. Thus $\lambda_1 = 1$ and $\lambda_2 = 0$ and so

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $\quad x + y = c_1, \quad$ and $\quad y = c_2$.

So we take $\quad u = x + y \quad$ and $\quad v = y \qquad$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}, \text{ by (3)} \qquad \text{...(5)}$$

$$s = \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ using (3) and (4)} \qquad \text{...(6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), we have

$$y(x+y)\left(-\frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0 \quad \text{or} \quad uv\frac{\partial^2 z}{\partial v \partial u} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0.$$

or $$\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0 \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0. \quad \text{... (7)}$$

Let $\quad \partial z/\partial v + (z/v) = w. \qquad$ ...(8)

Then, the above equation (7) becomes $\quad \partial w/\partial u + w/u = 0$.

Integrating, $\quad wu = \phi(v) \quad$ or $\quad w = (1/u) \times \phi(v)$.

Substituting this value of w in (8), we have $$\frac{\partial z}{\partial v} + \frac{1}{v}z = \frac{1}{u}\phi(v), \qquad \text{...(9)}$$

I.F. of (9) $= e^{\int (1/v)dv} = v$ and solution of (9) is

$$zv = \frac{1}{u}\int \phi(v)dv + \phi_2(u) \quad \text{or} \quad z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u), \text{ where } \phi_1(v) = \int \phi(v)\,dv$$

or $$z = \frac{1}{y(x+y)}\,\phi_1(y) + \frac{1}{y}\phi_2(x+y), \text{ by (2); } \phi_1, \phi_2 \text{ being arbitrary functions}$$

(*ii*) **Hint.** Given $$xys - x^2 r - px - qy + z = -2x^2 y. \qquad ...(1)$$

Here, $R = -x^2$, $S = xy$, $T = 0$ and λ–quadratic is $-x^2\lambda^2 + xy\lambda = 0$

so that $\lambda_1 = y/x$ and $\lambda_2 = 0$. Hence, characterstic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $xy = c_1$, $y = c_2$. So we take $u = xy$ and $v = y$. ...(2)

Then, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}y = v\frac{\partial z}{\partial u}, \text{ by (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = v\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right) = v^2\frac{\partial^2 z}{\partial u^2}, \text{ by (3)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = v\frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = v\left(\frac{1}{v}\frac{\partial z}{\partial u} + \frac{u}{v}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right), \text{ by (3) and (4)}$$

Substituting these values in (1), we have

$$xy\left(\frac{\partial z}{\partial u} + u\frac{\partial^2 z}{\partial u^2} + v\frac{\partial^2 z}{\partial u \partial v}\right) - x^2 v^2\frac{\partial^2 z}{\partial u^2} - v\frac{\partial z}{\partial u}x - y\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + z = -2x^2 y$$

or $$u\frac{\partial z}{\partial u} + u^2\frac{\partial^2 z}{\partial u^2} + uv\frac{\partial^2 z}{\partial u \partial v} - u^2\frac{\partial^2 z}{\partial u^2} - u\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -2(u^2/v^2)v, \text{ by (2)}$$

or $$uv\frac{\partial^2 z}{\partial u \partial v} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -\frac{2u^2}{v} \quad \text{or} \quad \frac{\partial^2 z}{\partial u \partial v} - \frac{1}{v}\frac{\partial z}{\partial u} - \frac{1}{u}\frac{\partial z}{\partial v} + \frac{z}{uv} = -\frac{2u}{v^2}.$$

or $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) - \frac{1}{u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) = -\frac{2u}{v^2}. \qquad ...(5)$$

Let $$\partial z/\partial v - z/v = w. \qquad ...(6)$$

Then (5) becomes $\dfrac{\partial w}{\partial u} - \dfrac{1}{u}w = -\dfrac{2u}{v^2}$, which is linear differential equation ...(7)

I.F. of (7) $= e^{-\int (1/u)du} = e^{-\log u} = e^{\log u^{-1}} = (1/u)$ and so its solution is

$$\frac{w}{u} = -\int\left(\frac{2u}{v^2} \times \frac{1}{u}\right)du = -\frac{2u}{v^2} + \phi(v) \quad \text{or} \quad w = -\frac{2u^2}{v^2} + u\,\phi(v)$$

Substituting this value of w in (6), we get $$\frac{\partial z}{\partial v} - \frac{1}{v}z = -\frac{2u^2}{v^2} + u\phi(u).$$

Its I.F. $= e^{-\int (1/v)dv} = e^{-\log v} = e^{\log v^{-1}} = (1/v)$ and so its solution is

$$\frac{z}{v} = \int \frac{1}{v}\left[-\frac{2u^2}{v^2} + u\,\phi(v)\right]dv = \frac{u^2}{v^2} + u\psi(v) + \phi_2(u)$$

or $z = (u^2/v) + uv\psi(v) + v\phi_2(u) = (u^2/v) + u\phi_1(v) + v\phi_2(u)$ or $z = x^2 y + xy\phi_1(y) + y\phi_2(xy)$, by (2).

Ex. 12. *Solve* $x^2 r - y^2 t + px - qy = x^2$. **[Kurukshetra 2003; Delhi Maths (H) 1998]**

Sol. Given $$x^2 r - y^2 t + (px - qy - x^2) = 0. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$R = x^2$, $S = 0$ and $T = -y^2$. ...(2)

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ and (2) give

$x^2\lambda^2 - y^2 = 0$ so that $\lambda = \pm y/x$. (real and distinct roots)

Take $\lambda_1 = y/x$ and $\lambda_2 = -y/x$.

Hence characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$

become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

or $(1/x)dx + (1/y)dy = 0$ and $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x + \log y = \log c_1$ and $\log x - \log y = \log c_2$

or $xy = c_1$ and $x/y = c_2$.

To reduce (1) to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u=xy \quad \text{and} \quad v = x/y. \qquad ...(3)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad ...(4)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad ...(5)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2}\times y + \frac{\partial^2 z}{\partial v \partial u}\times\frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v \partial u}\times y + \frac{\partial^2 z}{\partial v^2}\times\frac{1}{y}\right), \text{ using (3)}$$

$$\therefore \quad r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}. \qquad ...(6)$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}\times x + \frac{\partial^2 z}{\partial v \partial u}\times\left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u \partial v}\times x + \frac{\partial^2 z}{\partial v^2}\times\left(-\frac{x}{y^2}\right)\right]$$

$$\therefore \quad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}. \qquad ...(7)$$

Substituting the values of r, t, p and q given by (6), (7) (3) and (4) in (1), we obtain

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial u^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)$$

$$+ x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) - y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) - x^2 = 0$$

or $4x^2\dfrac{\partial^2 z}{\partial u \partial v} = x^2$ so that $\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right) = \dfrac{1}{4}$, ...(8)

which is the canonical form of (1).

Now, integrating (8) w.r.t. 'u', $\partial z/\partial u = (u/4) + f(v)$. ...(9)

Integrating (9) w.r.t. 'v', $z = (uv)/4 + \int f(v)\,dx + \phi(u)$

or $z = (uv)/4 + \psi(v) + \phi(u)$, where $\psi(v) = \int f(v)\,dv$

or $z = x^2/4 + \psi(x/y) + \phi(xy)$, which is the required solution, ϕ, ψ being arbitrary functions.

Ex. 13. *(a) Reduce* $x^2\,(\partial^2 z/\partial x^2) - y^2\,(\partial^2 z/\partial y^2) = 0$ *to canonical form and hence solve it.*

(b) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form.*

Sol. (*a*) Re-writing the given equation, $x^2 r - y^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2 y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x,\ -y/x$ and hence the corresponding characteristic equations become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

Integrating these, $xy = c_1$ and $x/y = c_2$

In order to reduce (1) to its canonical form, we choose $u = xy$ and $v = x/y$ ... (2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \quad \text{and} \quad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $$4x^2\frac{\partial^2 z}{\partial u \partial v} - \frac{2x}{y}\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad 2xy\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

or $$2u\,(\partial^2 z/\partial u \partial v) - (\partial z/\partial v) = 0, \text{ using (2).} \qquad \text{... (3)}$$

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\frac{\partial^2 z}{\partial u\,\partial v} - v\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad (2uv\,DD' - vD')z = 0 \qquad \text{... (4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $y = \log v$ ... (5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (4) reduces to

$(2D_1 D_1' - D_1')z = 0$ or $D_1'\,(2D_1 - 1)\,z = 0$

Its general solution is given by (use Art. 5.6 of chapter 5)

$$z = e^{X/2}\,\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u), \text{ using (5)}$$

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\,\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions

(*b*) Try yourself. Choose $u = (y^2 - x^2)/2$, $v = (y^2 + x^2)/2$.

Ans. $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{2(u^2 - v^2)}\left(v\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial v}\right).$$

Ex. 14. *Reduce the equation* $x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0$ *to canonical form and hence solve it.*

Sol. Comparing the given equation with $Rr+Ss+Tt+f(x,y,z,p,q)=0$,

here, $R=x(xy-1)$, $S=-(x^2y^2-1)$, $T=y(xy-1)$. ...(1)

Now, the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ and (1) give

$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0$ or $x\lambda^2-(xy+1)\lambda+y=0$

or $(x\lambda-1)(\lambda-y)=0$ so that $\lambda=1/x,\ y$. Take $\lambda_1=1/x$ and $\lambda_2=y$.

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$

become $(dy/dx)+(1/x)=0$ and $(dy/dx)+y=0$

or $dy+(1/x)dx=0$ and $(1/y)dy+dx=0$. ...(2)

Intergrating (2), $y+\log x=\log c_1$ and $\log y+x=\log c_2$

or $\log e^y+\log x=\log c_1$ and $\log y+\log e^x=\log c_2$

$x\,e^y=c_1$ and $y\,e^x=c_2$.

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$$u=x\,e^y \quad \text{and} \quad v=y\,e^x. \qquad ...(3)$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v},\ \text{using (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},\ \text{using (3).} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+ye^x\frac{\partial z}{\partial v}+y\,e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+y\,e^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+y\,e^x\frac{\partial z}{\partial v}+y\,e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore \quad r=e^{2y}\frac{\partial^2 z}{\partial u^2}+2y\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y^2\,e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v}.$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial z}{\partial v}+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$=x\,e^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y\,e^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial^2 z}{\partial u^2}x\,e^y+\frac{\partial^2 z}{\partial u\partial v}e^x\right]+e^x\left[\frac{\partial^2 z}{\partial u\partial v}x\,e^y+\frac{\partial^2 z}{\partial v^2}e^x\right],$$

$$\therefore \qquad t = x^2 e^{2y} \frac{\partial^2 z}{\partial u^2} + 2x\, e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + x\, e^{y} \frac{\partial z}{\partial u} + e^{2x} \frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0. \qquad ...(6)$$

Integrating (6) w.r.t. 'v', $\quad \partial z/\partial u = f(u)$, f being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $\quad z = \int f(u)\,du + \psi(v)$ or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $\quad z = \phi(xe^y) + \psi(ye^x)$, ϕ and ψ being arbitrary functions.

Ex. 15. (*a*) *Reduce the one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) to canonical form and hence find its general solution.*

(b) Find the D'Alembert's solution of the Cauchy's problem: $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) satisfying z (x, 0) = f (x) and* $z_t(x,0) = g(x)$ *where f (x) and g (x) are given functions representing the initial displacement and initial velocity, respectively. Also,* $z_t = \partial z/\partial t$

Sol. (*a*) Given $\quad \partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$, $c > 0$. ... (1)

To re-write (1), put $\quad y = ct$, ... (2)

Then, (1) reduces to $\quad \partial^2 z/\partial x^2 - (\partial^2 z/\partial y^2) = 0 \quad$ or $\quad r - t = 0$ ... (3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad ...(4)$$

where $u = y + x$, $\quad v = y - x \quad$ or $\quad u = ct + x \quad$ and $\quad v = ct - x$. ... (5)

Integrating (4) w.r.t. 'u', $\quad \partial z/\partial v = f(v)$, where f is an arbitrary function ... (6)

Integrating (6) w.r.t. 'v', $\quad z = \int f\ (v)\ dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)\,dv$

or $$z(x,t) = F(ct - x) + \psi\ (ct + x), \text{ using (5)}$$

or $$z(x,t) = \phi(x - ct) + \psi(x + ct), \qquad ...(7)$$

where we take $\phi\ (x - ct) = F\ (ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

(***b***) We are to solve $\quad \partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$ ... (*i*)

subject to the conditions $\quad z(x,0) = f(x)$ ... (*ii*)

and $\quad (\partial z/\partial t)_{t=0} = g(x)$ ... (*iii*)

Proceed exactly as in part (*a*) and get solution of (*i*) as

$$z(x,t) = \phi(x - ct) + \psi(x + ct) \qquad ...(iv)$$

Differentiating (*iv*) partially w.r.t. 't', we get

$$\partial z/\partial t = -c\ \phi'(x - ct) + c\ \psi'(x + ct) \qquad ...(v)$$

where dash denotes the derivative w.r.t. the argument. Putting $t = 0$ in (*iv*) and (*v*) and using (*ii*) and (*iii*) respectively, we get $\quad \phi(x) + \psi(x) = f(x)$ ... (*vi*)

and $\quad -c\ \phi'(x) + c\,\psi'(x) = g(x)$ ... (*vii*)

Integrating (*vii*), $\quad -c\ \phi(x) + c\ \psi\ (x) = \int_a^x g(u)\ du$, ... (*viii*) where a is an arbitrary constant. Solving (*vi*) and (*viii*) for $\phi(x)$ and $\psi(x)$, we have

$\phi(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_a^x g(x)\,du,$ and $\psi(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_a^x g(u)\,du$

so that
$$\phi\,(x-ct)=\frac{1}{2}f(x-ct)-\frac{1}{2c}\int_a^{x-ct} g(u)\,du \qquad \text{... }(ix)$$

and
$$\psi\,(x+ct)=\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_a^{x+ct} g(u)\,du \qquad \text{... }(x)$$

Using (*ix*) and (*x*) in (*iv*), we get the required so called D'Alembert's solutiion of the Cauchy problem (which represents the vibrations of an infinite string in the present problem)

$$z(x,t)=\frac{1}{2}\left[f(x-ct)+f(x+ct)\right]+\frac{1}{2c}\left[\int_{x-ct}^{a} g(u)\,du+\int_a^{x+ct} g(u)\,dx\right]$$

or
$$z(x,t)=\frac{1}{2}\,\{f\,(x-ct)+f(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du \qquad \text{... }(xi)$$

Particular Case I. If in the above problem, we take $g\,(x) = 0$ so that the initial velocity of the string is zero, then (*xi*) reduces to

$$z(x,t)=\{f(x-ct)+f(x+ct)\}/2\,,$$

where $f\,(x - ct)$ represents a right travelling wave travelling with the speed c (along OX) and $f\,(x + ct)$ represents a left travelling wave travelling with the speed c.

Particular case II. If $f\,(x) = \sin x$ and $g\,(x) = \cos x$ in the above problem, then the corresponding solution (*xi*) reduces to

$$z(x,t)=\frac{1}{2}\{\sin\,(x-ct)+\sin(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct}\cos u\,du$$

or $z(x,t)=\sin x\cos ct+(1/2c)\times\{\sin(x+ct)-\sin(x-ct)\}$ or $z(x,t)=\sin x\cos ct+(1/c)\times\cos x\sin ct$.

Particular case III. If $f(x) = \sin x$ and $g(x) = x^2$, then (*xi*) gives

$z(x,\,t) = \sin x \cos ct + x^2t + (c^3t^3)/3$, on simplification.

8.8 Working rule for reducing a parabolic equation to its canonical form.

Step 1. Let the given equation $Rr+Ss+Tt+f(x,y,z,p,q)=0$ ... (1)

be parabolic so that $S^2-4RT=0$.

Step 2. Write λ-quadratic equation $R\lambda^2+S\lambda+T=0$ ... (2)

Let λ_1, λ_1 be two equal roots of (2)

Step 3. Write the characteristic equation corresponding to $\lambda=\lambda_1$, *i.e.*, $(dy/dx)+\lambda_1=0$

Solving it, we get $f_1(x,y)=C_1$, C_1 being an arbitrary constant ... (3)

Step 4. Choose $u=f_1(x,y)$ and $v=f_2(x,y)$ ... (4)

where $f_2\,(x, y)$ is an arbitrary function of x and y and is independent of $f_1\,(x, y)$. For this verify that Jacobian J of u and v given by (4) is non-zero,

i.e.
$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\neq 0 \qquad \text{... }(5)$$

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s and t obtained in step (1) and simplifying we get the following canonical forms of (1)

$$\partial^2 z/\partial u^2 = \phi\,(u, v, z, \partial z/\partial u, \partial z/\partial v) \qquad \text{or} \qquad \partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v)$$

8.9 SOLVED EXAMPLES BASED ON ART. 8.8

Ex. 1. *Reduce the equation* $\partial^2 z/\partial x^2 + 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = 0$ *to canonical form and hence solve it.* **[Delhi Maths (H) 2000, 06; 08; Jabalpur 2004; Delhi Maths (Prog) II 2008; Delhi B.Sc. (Prog) II 2008, 11; Himanchal 2001; 05 Rajasthan 2003; Lucknow 2010]**

Sol. Re-writing the given equation, we get $\qquad r + 2s + t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$, $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadrate equation reduces to $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda = -1, -1$ (equal roots).

The corresponding characteristic equation is $(dy/dx) - 1 = 0$ or $dx - dy = 0$

Integrating, $x - y = c$, c being an arbitrary constant.

Choose $u = x - y$ and $v = x + y$, ... (2)

where we have chosen $v = x + y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1\cdot 1 + 1\cdot 1 = 2 \neq 0.$$

Now,
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

From (3) and (4),
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{... (5)}$$

$\therefore$
$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

and
$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (8)}$$

Using (6), (7) and (8) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2}=0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)=0 \qquad \text{... (9)}$$

To find the required solution. Integrating (9) partially w.r.t. 'v', we get

$$\partial z/\partial v=\phi(u),\ \phi \text{ being an arbitrary function.} \qquad \text{... (10)}$$

Integrating (10) partially w.r.t 'v', $\quad z=\int \phi(u)\,dv+\psi(u)=v\phi(u)+\psi(u)$

or $\quad z=(x+y)\,\phi(x-y)+\psi(x-y)$, which is the desired solution, ϕ, ψ being arbitrary functions.

Ex. 2. *Reduce the equation* $y^2(\partial^2 z/\partial x^2)-2xy\,(\partial^2 z/\partial x\,\partial y)+x^2(\partial^2 z/\partial x^2)=(y^2/x)(\partial z/\partial x)+(x^2/y)(\partial z/\partial y)$ *to canonical form and hence solve it.* **[Nagpur 2005; Delhi Maths (H) 2001, 05, 09; Avadh 2001, Himanchal 2009; Delhi B.Sc. (Prog) II 2007; Meerut 2005, 06, 11; G.N.D.U. Amritsar 2005]**

Sol. Re-writing the given equation, $\quad y^2r-2xys+x^2t-(y^2/x)\,p-(x^2/y)q=0 \qquad \text{... (1)}$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=y^2$, $S=-2xy$, $T=x^2$ so that $S^2-4RT=0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$$y^2\lambda^2-2xy\,\lambda+x^2=0 \qquad \text{or} \qquad (y\lambda-x)^2=0 \qquad \text{so that} \qquad \lambda=x/y,\ x/y.$$

The corresponding characteristic equation is $\quad dy/dx+x/y=0$

or $\quad x\,dx+y\,dy=0 \quad$ so that $\quad x^2/2+y^2/2=C_1$

Choose $\quad u=x^2/2+y^2/2 \quad$ and $\quad v=x^2/2-y^2/2, \qquad \text{... (2)}$

where we have chosen $v=x^2/2-y^2/2$ in such a manner that u and v are independent functions as verified below

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=-2xy\neq 0.$$

Now,
$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (3)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \text{... (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}+x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}+x^2\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right), \text{ using (2)} \qquad \text{... (5)}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]=\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{... (6)}$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

or $$s = xy\,(\partial^2 z/\partial u^2 - \partial^2 z/\partial v^2) \quad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\,(\partial^2 z/\partial v^2) = 0$ so that $\partial^2 z/\partial v^2 = 0$, ... (8)

which is the required canonical form.

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v = \phi(u)$, ϕ being arbitrary function. ... (9)

Integrating (9) partially w.r.t. 'v', $z = v\,\phi(u) + \psi(u)$, ψ being arbitrary function.

or $$z = [(x^2 - y^2)/2]\,\phi\{(x^2 + y^2)/2\} + \psi\,\{(x^2 + y^2)/2\}, \text{ using (2)}$$

or $$z = (x^2 - y^2)\,F(x^2 + y^2) + G\,(x^2 + y^2), \text{ F, G being arbitrary functions}$$

Ex. 3. *(a) Reduce $r + 2xs + x^2 t = 0$ to canonical form*

(b) Reduce $r - 6s + 9t + 2p + 3q - z = 0$ to canonical form

(c) Reduce $r - 2s + t + p - q = 0$ to canonical form and hence solve it.

Sol. (*a*) Given $$r + 2xs + x^2 t = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2x$ and $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda x + x^2 = 0$ or $(\lambda + x^2) = 0$ so that $\lambda = -x, -x$.

The corresponding characteristic equation is $(dy/dx) - x = 0$ or $dy - x\,dx = 0$

Integrating, $y - x^2/2 = c_1$, c_1 being an arbitrary constant. ... (2)

Choose $u = y - x^2/2$ and $v = x$ ... (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial\,(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -1 \neq 0$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= -\frac{\partial z}{\partial u} - x\left(-x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ by (4)} \quad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2}, \text{ by (4)} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally obtain $\partial^2 z/\partial v^2 = \partial z/\partial u$, which is required canonical form.

3. (*b*) Hint. Here $\lambda = 3$, $u = y + 3x$. Choose $v = y$. The canonical form will be

$$\partial^2 z/\partial v^2 = z/9 - (\partial z/\partial u) + (1/3) \times (\partial z/\partial v).$$

3. (c) Hints. Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v$.

Solution is $\quad z = \phi\,(x+y) + e^y \psi\,(x+y)$, ϕ, ψ being arbitrary functions

Ex. 4. *Reduce the following to canonical form and hence solve*

(*a*) $x^2\, r + 2xy\, s + y^2\, t = 0$

(*b*) $r - 4s + 4t = 0$

(*c*) $x^2\, r + 2xys + y^2 t + xyp + y^2 q = 0$

(*d*) $2r - 4s + 2t + 3z = 0$.

Sol. (a) Given $\quad x^2 r + 2xys + y^2 t = 0 \quad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x^2\lambda^2 + 2xy\lambda + y^2 = 0 \quad$ or $\quad (x\lambda + y)^2 = 0 \quad$ giving $\quad \lambda = -y/x,\ -y/x$.

The corresponding characteristic equation is $\quad dy/dx - y/x = 0$

or $\quad (1/y)dy - (1/x)dx = 0 \quad$ so that $\quad \log y - \log x = c_1 \quad$ or $\quad y/x = c_1$

Choose $\quad u = y/x \quad$ and $\quad v = y, \quad$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\frac{y}{x^2} \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{\partial z}{\partial u}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right), \text{ by (3)}$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{2y}{x^3}\frac{\partial z}{\partial u} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} \qquad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}.$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} - \frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 z}{\partial u \partial v} \qquad \text{... (6)}$$

$$t = \frac{\partial^2 y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ using (4)}$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{2}{x}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally get as the canonical form $\quad \partial^2 z/\partial v^2 = 0$ ... (8)

Integrating (8) partially w.r.t. 'v', $\quad \partial z/\partial v = \phi(u)$ ... (9)

Integrating (9) partially w.r.t 'v', $\quad z = v\phi(u) + \psi(u)$

or $\quad z = y\phi(y/x) + \psi(y/x), \phi, \psi$ being arbitrary functions.

(*b*) **Hint.** Here $\lambda = 2$, $u = y + 2x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = 0$ and solution is $z = y\phi(y+2x) + \psi(y+2x)$.

(*c*) **Hint.** Here $\lambda = -y/x$, $u = y/x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(\partial z/\partial v)$ and solution is $z = \phi(y/x) + e^{-y}\psi(y/x)$

(*d*) **Hint.** Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(3z/2)$ and solution is $z = e^{(i\sqrt{3}/2)y}\phi(y+x) + e^{-(i\sqrt{3}/2)}\psi(y+x)$,

Ex. 5. *Reduce the following in canonical form and solve them*

(*a*) $r - 2s + t + p - q = e^x(2y-3) - e^y$

(*b*) $r - 2s + t + p - q = e^{x+y}$

Sol. (*a*) Given $\quad r - 2s + t + p - q - e^x(2y-3) + e^y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2$ and $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 - 2\lambda + 1 = 0 \quad$ or $\quad (\lambda - 1)^2 = 0 \quad$ so that $\quad \lambda = 1, 1$ (equal roots)

So the corresponding characteristic equation is $\quad dy/dx + 1 = 0 \quad$ or $\quad dx + dy = 0$

Integrating it, $\quad x + y = c_1$, c_1 being an arbitrary constant.

Choose $\quad u = x + y \quad$ and $\quad v = y$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} \text{ by (3)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}, \text{ using (4)} \qquad \text{... (7)}$$

Using (2) (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^v$$

or $$\partial^2 z/\partial v^2 - \partial z/\partial v = e^{u-v}(2v-3) - e^v \qquad \text{... (8)}$$

which is the required canonical form of (1) Let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$.

Then (8) can be re-written as $$D'(D'-1)\,z = e^{u-v}(2v-3) - e^v, \qquad \text{... (9)}$$

which is non-honmogeneous linear partial differential equation with constant coefficients. To solve it, we shall use results of chapter 5. Accordingly, we have

$$\text{C.F.} = \phi(u) + e^v\ \psi(u) = \phi(x+y) + e^y\psi(x+y), \text{ by (2)}$$

P.I. corresponding to $e^{u-v}(2v-3)$

$$= \frac{1}{D'(D'-1)}e^{u+(-1)v}(2v-3) = e^{u+(-1)v}\frac{1}{(D'-1)\,(D'-1-1)}(2v-3)$$

$$= (1/2)\times e^{u-v}\ (1-D')^{-1}(1-D'/2)^{-1}(2v-3) = (1/2)\times\ e^{u-v}\ (1+D'+...)(1+D'/2+...)\ (2v-3)$$

$$= (1/2)\times e^{u-v}(1+3D'/2+...)\ (2v-3) = (1/2)\times e^{u-v}\ (2v-3+3) = v\ e^{u-v} = y\ e^{x+y-y} = ye^x, \text{ using(2)}$$

P.I. Corresponding to $(-\ e^v)$

$$= \frac{1}{D'(D'-1)}(-e^v) = -\frac{1}{D'-1}\frac{1}{D'}e^v = -\frac{1}{D'-1}\ (e^v \times 1) = -e^v\frac{1}{D'+1-1}1 = -e^y\frac{1}{D'}1$$

$$= -e^v v = -e^y y, \text{ using (2)}$$

Hence the required general solution is given by $y = \phi\ (x+y) + e^y\ \psi\ (x+y) + y\ e^x - y\ e^y$

or $$y = \phi\ (x+y) + e^y\ \psi\ (x+y) + ye^x - (x+y)\ e^y + x\ e^y$$

or $$y = \phi\ (x+y) + e^y\ \{\phi\ (x+y) + (x+y)\} + y\ e^x + x\ e^y$$

or $$y = \phi\ (x+y) + e^y F(x+y) + y\, e^x + x\, e^y,$$

where ϕ and F are arbitrary functions and $F(x+y) = \phi\ (x+y) + x + y$

(*b*) Hint. Here $\lambda = 1$, $u = x + y$, choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v + e^u$ and solution is $z = \phi\ (x+y) + e^y \psi(x+y) - y\, e^{x+y}$, ϕ, ψ being arbitrary functions.

Ex. 6. *Reduce the equation $x^2 r - 2xys + y^2 t - xp + 3yq = 8y/x$ to canonical form.*

[Delhi B.Sc. (H) 1999]

Sol. Given $$x^2 r - 2xy\, s + y^2 t - xp + 3yq - 8y/x = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, y, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 \qquad \text{or} \qquad (x\lambda - y)^2 = 0 \qquad \text{so that} \qquad \lambda = y/x,\ y/x.$$

The corresponding characteristic equation is

$$dy/dx + y/x = 0 \qquad \text{or} \qquad (1/y)\, dy + (1/x)\, dx = 0 \qquad \text{so that} \qquad xy = C_1$$

Choose $$u = xy \qquad \text{and} \qquad v = x \qquad \text{... (2)}$$

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0\,.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left(y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}, \qquad \text{.... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v} \qquad \text{... (6)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right), \text{ by (4)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2}, \text{ by (2)} \qquad \text{... (7)}$$

Using (2), (3), (4), (5), (6) and (7) in (1), we have

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-2xy\left(\frac{\partial z}{\partial u}+xy\frac{\partial^2 z}{\partial u^2}+x\frac{\partial^2 z}{\partial u\partial v}\right)+y^2x^2\frac{\partial^2 z}{\partial u^2}-x\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+3y\,x\frac{\partial z}{\partial u}-\frac{8y}{x}=0$$

or $\quad x^2\dfrac{\partial^2 z}{\partial v^2}--x\dfrac{\partial z}{\partial v}=\dfrac{8y}{x}\quad$ or $\quad v^2\dfrac{\partial^2 z}{\partial v^2}-v\dfrac{\partial z}{\partial v}=\dfrac{8u}{v^2}$, by (2)

or $\quad (v^2D'^2-vD')z=8u/v^2,\quad$ where $\quad D\equiv\partial/\partial u,\quad D'\equiv\partial/\partial v \quad$ ... (8)

As explained in chapter 6, we shall reduce (8) to linear partial differential equation with constant coefficients and then use methods of chapter 5 to solve the resulting equation.

To solve (8), let $\quad u=e^X \quad$ and $\quad v=e^Y \quad$ so that $\quad X=\log u, \quad Y=\log v \quad$ ... (9)

Then (8) becomes $\quad \{D'(D'-1)-D'\}z=8e^{X-2y} \quad$ or $\quad D'(D'-2)z=8e^{X-2y}$.

C.F. $=\phi(X)+e^{2Y}\psi(X)=\phi(\log u)+v^2\psi(\log u)$, using (9)

$=F(u)+v^2G(u)=F(xy)+x^2G(xy)$, using (2)

P.I. $=\dfrac{1}{D'(D'-1)}8e^{X-2Y}=8e^{X-2Y}\dfrac{1}{(D'-2)(D'-2-2)}\cdot 1$

$$=\frac{8e^X}{(e^Y)^2}\times\frac{1}{8}\left(1-\frac{D'}{2}\right)^{-1}\left(1-\frac{D'}{4}\right)^{-1}\cdot 1=\frac{u}{v^2}\left(1+\frac{D'}{2}+\ldots\right)\left(1+\frac{D'}{4}+\ldots\right)\cdot 1=\frac{u}{v^2}\times 1=\frac{xy}{x^2}=\frac{y}{x}, \text{ by (2)}$$

$\therefore$ Required solution is $\quad z=F(xy)+x^2G(xy)+y/x$, F, G being arbitrary functions.

8.10 Working rule for reducing an elliptic equation to its canonical form.

Step 1. Let the given equation $\quad Rr+Ss+Tt+f(x,y,z,p,q)=0 \quad$ ... (1)

be ellipitic so that $\quad S^2-4RT<0\cdot \quad$ ... (2)

Step 2. Write λ quadratic equation $\quad R\lambda^2+S\lambda+T=0 \quad$ ... (2)

Let roots λ_1, λ_2 of (2) be complex conjugates.

Step 3. Then corresponding charactersitic equations are

$(dy/dx)+\lambda_1=0 \quad$ and $\quad dy/dx+\lambda_2=0$

Solving these, we shall obtain solutions of the form

$f_1(x,y)+i\,f_2(x,y)=c_1 \quad$ and $\quad f_1(x,y)-i\,f_2(x,y)=c_2 \quad$ ... (3)

Step 4. Choose $\quad u=f_1(x,y)+i\,f_2(x,y), \quad v=f_1(x,y)-i\,f_2(x,y)$

Let α and β be two new real independent variables such that $u=\alpha+i\beta$ and $v=\alpha-i\beta$,

so that $\quad \alpha=f_1(x,y) \quad$ and $\quad \beta=f_2(x,y) \quad$ ... (4)

Step 5. Using relations (4), find *p*, *q*, *r*, *s* and *t* in terms of α and β (in place of *u* and *v* as we did in Art 8.6 and 8.8 corresponding to the cases of hyperbolic and parabolic equations).

Step 6. Substituting the values of *p*, *q*, *r*, *s* and *t* and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2=\phi(\alpha,\beta,z,\partial z/\partial\alpha,\ \partial z/\partial\beta).$$

8.11 SOLVED EXAMPLES ON ART 8.10

Ex. 1. *Reduce the following partial differential equations to canonical forms:*

(*a*) $\partial^2 z/\partial x^2+x^2(\partial^2 z/\partial y^2)=0$ *or* $\quad r+x^2t=0$

[Delhi B.Sc. (Prog) II 2010; Delhi B.Sc. (Hons) III 2011; Kanpur 2011]

(*b*) $y^2(\partial^2 z/\partial y^2)+\partial^2 z/\partial x^2 = 0$ **[Delhi Math (Hons.) 1995, 98, 2005]**

Sol. (*a*) Re-writing the given equations, we get $\qquad r + x^2 t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, $T = x^2$ so that $S^2 - 4RT = -4x^2 < 0$, $x \neq 0$, showing that (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + x^2 = 0$ giving $\lambda = ix, -ix$.
The corresponding characteristic equations are given by

$$dy/dx + ix = 0 \quad \text{and} \quad dy/dx - ix = 0$$

Integrating, $\quad y + i\,(x^2/2) = c_1 \quad$ and $\quad y - i\,(x^2/2) = c_2$.

Choose $\quad u = y + i(x^2/2) = \alpha + i\beta \quad$ and $\quad v = y - i(x^2/2) = \alpha - i\beta$,

where $\quad \alpha = y \quad$ and $\quad \beta = x^2/2 \quad$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad .... (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad ... (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} \qquad ... (5)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ by (4)} \qquad ... (6)$$

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}, \text{ as } \beta = \frac{x^2}{2}.$$

(***b***) Do as in part (a). **Ans.** $\partial^2 z/\partial \alpha^2 + \partial^2 z/\partial \beta^2 = -(1/2\alpha)\times(\partial z/\partial \alpha)$, where $\alpha = y^2/2$, $\beta = x$.

Ex. 2. *Reduce* $y^2(\partial^2 z/\partial x^2) + x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form*

Sol. Re-writing the given equation, we get $\qquad y^2 r + x^2 t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$, $T = x^2$ so that $S^2 - 4RT = -4x^2y^2 < 0$ for $x \neq 0$, $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 + x^2 = 0 \quad$ or $\quad \lambda^2 = -x^2/y^2 \quad$ so that $\quad \lambda = ix/y, \quad -ix/y$
The corresponding characteristic equations are

$$dy/dx + ix/y = 0 \quad \text{and} \quad dy/dx - ix/y = 0$$

Integrating, $\quad y^2 + ix^2 = C_1 \quad$ and $\quad y^2 - ix^2 = C_2$

Choose $\quad u = y^2 + ix^2 = \alpha + i\beta \quad$ and $\quad v = y^2 - ix^2 = \alpha - i\beta$, where $\alpha = y^2$ and $\beta = x^2$ ... (2)

are now two new independent variables

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha}, \text{ by (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= 2\frac{\partial z}{\partial \beta} + 2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\} = 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\} = 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \quad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0$$

Ex. 3. *Reudce* $\partial^2 z/\partial x^2 + y^2(\partial^2 z/\partial y^2) = y$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + y^2 t - y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here R = 1, S = 0 and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy, -iy$.

The corresponding characteristic equations are given by

$dy/dx + iy = 0$ and $dy/dx - iy = 0$

Integrating these, $\log y + ix = c_1$ and $\log y - ix = c_2$

Choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$,

where $\alpha = \log y$ and $\beta = x$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \alpha}, \text{ using (2)} \quad \text{.... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}, \text{ by (3)} \quad \text{... (5)}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right)=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$$

$$=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\frac{1}{y}\right)=\frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2}-\frac{\partial z}{\partial \alpha}\right) \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}-\frac{\partial z}{\partial \alpha}-y=0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\frac{\partial z}{\partial \alpha}+e^{\alpha}, \quad \text{using (2)}$$

Ex. 4. *Reduce* $x\,(\partial^2 z/\partial x^2)+\partial^2 z/\partial y^2=x^2$ *($x>0$) to canonical form.* **[Delhi Maths(H) 2007, 11]**

Sol. Re-writing the given equation, we get $\qquad xr+t-x^2=0,\ (x>0)$ ... (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=x$, $S=0$ and $T=1$ so that $S^2-4RT=-4x<0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$$x\lambda^2+1=0 \qquad \text{or} \qquad \lambda^2=-(1/x^2) \qquad \text{so that} \qquad \lambda=i/x^{1/2},\ -i/x^{1/2}$$

The corresponding characteristic equations are given by

$$dy/dx+i\,x^{-1/2}=0 \qquad \text{and} \qquad dy/dx-i\,x^{-1/2}=0.$$

Integrating these, $\qquad y+2i\,x^{1/2}=C_1 \qquad$ and $\qquad y-2i\,x^{1/2}=C_2$

Choose $\qquad u=y+2i\,x^{1/2}=\alpha+i\beta \qquad$ and $\qquad v=y-2i\,x^{1/2}=\alpha-i\beta$,

where $\qquad \alpha=y \qquad$ and $\qquad \beta=2x^{1/2}$ ... (2)

are now two new independent variables.

Now, $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=x^{-1/2}\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{... (3)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right)=-\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta}+x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

or $$r=-\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta}+x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right)=-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta}+\frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}, \text{ using (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta}+\frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right)+\frac{\partial^2 z}{\partial \alpha^2}=x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=x^2+\frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

or $\partial^2 z/\partial\alpha^2 + \partial^2 z/\partial\beta^2 = (\beta^2/4) + (1/\beta)\times(\partial z/\partial\beta)$, as $\beta = 2\,x^{1/2}$.

Ex. 5. *Reduce* $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + 5\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} - 3z = 0$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + 2s + 5t + p - 2q - 3z = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f\,(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$ and $T = 5$ so that $S^2 - 4RT = -16 < 0$, showing (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda + 5 = 0$ so that $\lambda = \{-2 \pm (4-20)^{1/2}\}/2 = -1 \pm 2i$

The corresponding characterisitc equations are given by

$dy/dx + (-1+2i) = 0$ and $dy/dx + (-1-2i)x = 0.$

Integrating these, $y + (-1+2i)x = C_1$ and $y + (-1-2i)x = C_2$

Let $u = y - x + 2ix = \alpha + i\beta$ and $v = y - x - 2ix = \alpha - i\beta$,

where $\alpha = y - x$ and $\beta = 2x$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = -\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ using (2)} \qquad \text{... (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ using (3)} \qquad \text{... (5)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= -\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right\} + 2\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \beta \partial \alpha}\right) + 2\left(-\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\frac{\partial^2 a}{\partial \beta^2}\right), \text{ by (2)}$$

$$\therefore \qquad r = \partial^2 z/\partial\,\alpha^2 + 4\,(\partial^2 z/\partial\beta^2) - 4\,(\partial^2 z/\partial\alpha\partial\beta) \qquad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}$$

or $$s = -(\partial^2 z/\partial\alpha^2) + 2\,(\partial^2 z/\partial\alpha\,\partial\beta), \text{ using (2)} \qquad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial \alpha^2} + 4\frac{\partial^2 z}{\partial \beta^2} - 4\frac{\partial^2 z}{\partial \alpha \partial \beta} + 2\left(-\frac{\partial^2 z}{\partial \alpha^2} + 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right) + 5\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} + 2\frac{\partial z}{\partial \beta} - 2\frac{\partial z}{\partial \alpha} - 3z = 0$$

or $$4\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) = 3z + 3\frac{\partial z}{\partial \alpha} - 2\frac{\partial z}{\partial \beta}$$ or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3z}{4} + \frac{3}{4}\frac{\partial z}{\partial \alpha} - \frac{1}{2}\frac{\partial z}{\partial \beta},$$

which is the required canonical form of given equation (1).

8.12. The solution of linear hyperbolic equations. It what follows we aim at sketching the existence theorems for two types of initial conditions on the linear hyperbolic equation

$$\partial^2 z/\partial x \partial y = f(x, y, z, p, q). \qquad \text{...(1)}$$

For both kinds of initial condition, we assume that the function $f(x, y, z, p, q)$ satisfies the following two conditions :

(*i*) f is continuous at all points of a rectangular region R defined by $\alpha < x < \beta$, $\gamma < y < \delta$ for all values of x, y, z, p, q concerned.

(*ii*) f satisfies the so called Lipschitz condition, namely,

$$|f(x, y, z_2, p_2, q_2) - f(x, y, z_1, p_1, q_1)| \leq M\{|z_2 - z_1| + |p_2 - p_1| + |q_2 - q_1|\}$$

in all bounded subrectangles r of R.

We now state (without proof) two existence theorems.

Theorem 1. Initial conditions of the first kind. *If $F(x)$ and $G(x)$ are defined in the open intervals (α, β), (γ, δ), respectively, and have continuous first derivatives, and if (ξ, η) is a point inside R such that $F(\xi) = G(\eta)$, then (1) has at least one integral $z = \phi(x, y)$ in R such that*

$$\phi(x, y) = \begin{cases} F(x), \text{ when } y = \eta \\ G(y), \text{ when } x = \xi. \end{cases}$$

Theorem II. *Initial conditions of the second kind.* *Let C_1 be a space curve defined by $x = x(\lambda)$, $y = y(\lambda)$, $z = z(\lambda)$ in terms of a single parameter λ and also let C_0 be the projection of C_1 on the xy-plane. If we are given (x, y, z, p, q) along a strip C_1, then (1) has an integral which takes on the given values of z, p, q along the curve C_0. This intergral exists at every point of the region R, which is defined as the smallest rectangle completely enclosing the curve C_0.*

8.13. Riemann method of solution of general linear hypobolic equation of the second order. [Himanchal 2002; Meerut 2005, 07, 08; Delhi Maths (Hons.) 1995, 1999, 2000]

Assume that the given linear hyperbolic equation is reducible to canonical form

$$L(z) = f(x, y), \qquad \text{...(1)}$$

where L denotes the linear operator given by
$$L \equiv \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c, \qquad \text{...(2)}$$

where a, b, c are functions of x and y only.

Let w be another function with continuous derivatives of the first order. Again, let M be another operator defined by the relation

$$Mw = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw. \qquad \text{...(3)}$$

The operator M defined by (4) is called the *adjoint operator* to the operator L.

$$\therefore \quad w\,Lz - z\,Mw = w\left(\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c\right) - z\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw\right)$$

$$= \left(w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y}\right) + \left(wa\frac{\partial z}{\partial x} + z\frac{\partial(aw)}{\partial x}\right) + \left(wb\frac{\partial z}{\partial y} + z\frac{\partial(bw)}{\partial y}\right) = \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right) + \frac{\partial(awz)}{\partial x} + \frac{\partial(bwz)}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(awz - z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(bwz + w\frac{\partial z}{\partial x}\right) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \qquad \text{...(4)}$$

where $U = awz - z(\partial w/\partial y)$ and $V = bwz + w(\partial z/\partial x)$. ...(5)

Now if C' is a closed curve enclosing an area S, then

$$\iint_S (w\,Lz - z\,Mw)\,dx\,dy = \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\,dy = \int_{C'} (U\,dy - V\,dx), \text{ by Green's theorem} \qquad \text{...(6)}$$

Assume that the values of z and $\partial z/\partial x$ (or $\partial z/\partial y$) are prescribed along a curve C in the xy-plane (refer figure 1) and further assume that we are required to determine the solution of (1) at the point P (ξ, η) agreeing with these boundary conditions. Draw PA, PB parallel to x-axis and y-axis and cutting the curve C in the points A and B respectively. The closed circuit $PABP$ can be taken as the closed curve C'. Then (6) reduces to

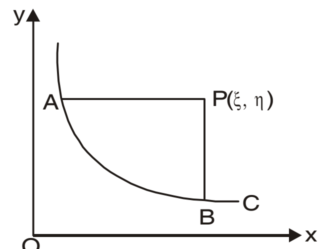


Fig. 1.

$$\iint_S (w\,Lz - z\,Mw)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}(Udy - Vdx) + \int_{PA}(Udy - Vdx)$$
$$= \int_{AB}(Udy - Vdx) + \int_{BP} Udy - \int_{PA} Vdx, \qquad ...(7)$$

where we have used the following facts:

along BP, x = constant so that $dx = 0$ and along PA, y = constant so that $dy = 0$.

Now, $\int_{PA} Vdx = \int_{PA}\left(bwz + w\dfrac{\partial z}{\partial x}\right)dx$, by (5)

$$= \int_{PA} bwzdz + \int_{PA} w\frac{\partial z}{\partial x}\,dx = \int_{PA} bwzdz + [wz]_P^A - \int_{PA} z\frac{\partial w}{\partial x}\,dx, \text{ integrating by parts}$$
$$= [wz]_A - [wz]_P + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx. \qquad ...(8)$$

Using (5) and (8), (7) becomes

$$\iint_S (w\,Lz - z\,Mz)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}\left(awz - z\frac{\partial w}{\partial y}\right)dy - [wz]_A + [wz]_P - \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx.$$

$$\therefore\ [wz]_P = [wz]_A + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx - \int_{BP} z\left(aw - \frac{\partial w}{\partial y}\right)dy$$
$$- \int_{AB}(Udy - Vdx) + \iint_S (w\,Lz - z\,Mw)\,dxdy \quad ...(9)$$

So far we have treated w as an arbitrary function. Now, we choose a function $w(x, y, \xi, \eta)$ which has the following four properties, namely,

(*i*) $Mw = 0$, (*ii*) $w = 1$, when $x = \xi$, $y = \eta$ *i.e.*, at $P(\xi, \eta)$,

(*iii*) $\partial w/\partial x = bw$ when $y = \eta$, (*iv*) $\partial w/\partial y = aw$ when $x = \xi$.

Such a function $w(x, y, \xi, \eta)$ is known as *Green's function* for the problem or sometimes a *Riemann-Green function*. Using the above four properties of w, (9) may be re-written as

$$[z]_P = [wz]_A - \int_{AB}(Udy - Vdx) + \iint_S w\,Lz\,dxdy$$
$$= [wz]_A - \int_{AB}\left(awz - z\frac{\partial w}{\partial y}\right)dy + \int_{AB}\left(bwz + w\frac{\partial z}{\partial x}\right)dx + \iint_S (wf)dxdy, \text{ using (1) and (5)}$$
$$= [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}\,dy + w\frac{\partial z}{\partial x}\,dx\right) + \iint_S wf\,dxdy. \qquad .(10)$$

Equation (10) may be used to determine the value of z at the point P when $\partial z/\partial x$ is prescribed along the curce C.

Suppose, in place of the prescribed value of $\partial z/\partial x$, we are now given a prescribed value of $\partial z/\partial y$. Then, we make use of the following relation

$$\int_{AB} d(wz) = \int_{AB}\left(\frac{\partial(wz)}{\partial x}\,dx + \frac{\partial(wz)}{\partial y}\,dy\right)$$

$$\Rightarrow \qquad 0 = [wz]_B - [wz]_A - \int_{AB}\left(\frac{\partial(wz)}{\partial x}\,dx + \frac{\partial(wz)}{\partial y}\,dy\right). \qquad ...(11)$$

Adding the corresponding sides of (10) and (11), we get

$$[z]_P = [wz]_B - \int_{AB} wz\,(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}\,dy + w\frac{\partial z}{\partial x}\,dx\right) - \int_{AB}\left(\frac{\partial (wz)}{\partial x}\,dx + \frac{\partial (wz)}{\partial y}\,dy\right) + \iint_S (wf)\,dxdy$$

$$= [wz]_B - \int_{AB} wz\,(ady - bdx) - \iint_{AB}\left(z\frac{\partial w}{\partial x}\,dx + w\frac{\partial z}{\partial y}\,dy\right) + \iint_S (wf)\,dxdy \qquad ..(12)$$

Equation (12) may be used to determine the z at the point P when $\partial z/\partial y$ is precribed along the curve C.

Finally, by adding (10) and (12), we get the following symmetrical result which can be used to find value of z at the point P when both $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along the curve C.

$$[z]_P = \frac{1}{2}\{[wz]_A + [wz]_B\} - \int_{AB} wz\,(ady - bdx) + \iint_S (wf)\,dxdy$$

$$-\frac{1}{2}\int_{AB} w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) - \frac{1}{2}\int_{AB} z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right). \qquad ...(13)$$

By means of whichever of the formulas (10), (12) and (13) is suitable, we may determine the solution of (1) at any point in terms of the prescribed values of z, $\partial z/\partial x$ or/and $\partial z/\partial y$ along a given curve C.

We now discuss four particular cases:

Particular Case I *Determine the solution of*

$$\partial^2 z/\partial x\,\partial y + a(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x,y) \qquad ...\ (i)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along curve C in the xy-plane.* **[Delhi Maths (H) 1995, 99, 2000, 06, 08; Meerut 2010]**

Hint. Proceed as in Art. 8.13 upto equation (10), *i.e.,*

$$[z]_P = [wz]_A - \int_{AB} wz\,(a\,dy - b\,dx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wfdxdy \qquad ...\ (ii)$$

Relation (*ii*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular case II *To determine the solution of the equation*

$$\partial^2 z/\partial x\,\partial y = f(x,y) \qquad ...\ (iii)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along a curve C in the xy-plane.* **[Meerut 2010; Delhi. Maths (H) 1995, 99, 2000, 06, 08, 09, 10]**

Hint. First state and prove that above particular case I. Note that (*ii*) is solution of (*i*). Comparing (*iii*) with (*i*), we have $a = b = c = 0$ and hence for the present equation (*iii*), (*ii*) gives

$$[z]_P = [wz]_A + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\ dx\,dy \qquad ...\ (iv)$$

where the Green's function w satisfies the following four properties (refer Art. 8.13 and note that $a = b = c = 0$ for the present case).

(a) $\partial^2 w/\partial x\,\partial y = 0$ at all points of S — (b) $w = 1$ at $P\ (\xi, \eta)$

(c) $\partial w/\partial x = 0$ when $y = \eta$ — (d) $\partial w/\partial y = 0$ when $x = \xi$

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*iv*), the required solution takes the following form

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}\,dx + \iint_S f(x,y)\,dx\,dy \qquad ...\ (v)$$

The relation (*v*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular Case III *Determine the solution of*

$$\partial^2 z/\partial x\partial y + a\,(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x, y) \qquad \text{... } (vi)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane.*

Hint. Proceed as in Art. 8.13 upto equation (12), *i.e.*

$$[z]_P = [wz]_B - \iint_{AB} wz\,(ady - bdx) - \iint_{AB}\left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right) + \iint_S wf\;dx\,dy \qquad \text{... } (vii)$$

Relation (*vii*) may be used to determine the value of z at the point P when z and $\partial z/\partial y$ are prescribed along a curve C in the xy-plane.

Particular Case IV To *determine the solution* of

$$\partial^2 z/\partial x\partial y = f(x, y) \qquad \text{... } (viii)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane.* **[Delhi Maths (H) 1974, 97, 2001]**

Hint. First state and prove the above particular case III. Note that (*vii*) is solution of (*vi*). Comparing (*viii*) with (*vi*), we have a = b = c = 0 and hence for the present equation (*viii*), (*vii*) reduces to

$$[z]_P = [wz]_B - \iint_{AB}\left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right) + \iint_S wf\;dx\,dy \qquad \text{... } (ix)$$

where the Green's function w satisfies the following four properties (refer Art. 8.13 and note that $a = b = c = 0$ for the present case)

(*a*) $\partial^2 w/\partial x\,\partial y = 0$ at all points of S (*b*) $w = 1$ at $P(\xi, \eta)$

(*c*) $\partial w/\partial x = 0$ when $y = \eta$ (*d*) $\partial w/\partial y = 0$ when $x = \xi$.

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*ix*), the required solution takes the following form.

$$[z]_P = [z]_B - \iint_{AB}\frac{\partial z}{\partial y}\,dy + \iint_S f(x, y)\;dx\;dy\,. \qquad \text{... } (x)$$

The relation (*x*) may be used to determine the value of z at the point P when z and $\partial z/\partial y$ are prescribed along a curve.

Note: Relations (10), (12) (13) (*ii*), (*v*), (*vii*) and (*x*) must be remembered and may be used directly in solving problems based on them.

8.14 SOLVED EXAMPLES BASED ON ART 8.13

Ex 1. *Find the solution, valid when x, y>0, xy > 1 of the equation* $\partial^2 z/\partial x\partial y = 1/(x+y)$ *such that z = 0,* $p = (2y)/(x+y)$ *on the hyperbola xy = 1.* **[Meerut 2007 Delhi Maths (H) 1999, 2006 Himanchal 1997, K.U. Kurukshatra 1999]**

Sol. Comparing the given equation with $L(z) = f(x, y),$ we have $a = b = c = 0$ and $f(x, y) = 1/(x + y)$. Hence the adjoint operator M of the operator L is given by $M \equiv \partial^2/\partial x\partial y$.

So Green's function can be taken as $w = 1$. ...(1)

In the present problem, the values of z and $\partial z/\partial x$ $(=p)$ are given by

$$z = 0, \qquad \partial z/\partial x = (2y)/(x+y), \qquad \text{...(2)}$$

along the curve C, which is hyperbola.

$$xy = 1. \qquad \text{...(3)}$$

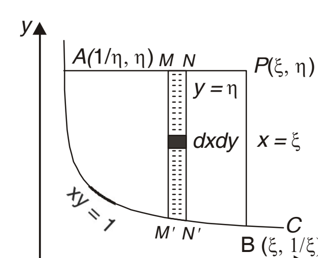


Fig. 2.

Then we wish to find the solution of given equation at the point P (ξ, η) agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $xy = 1$ in the point A and PB parallel to the y-axis and cutting $xy = 1$ in B. Then region enclosed by $xy = 1, x = \xi, y = \eta$ is denoted by S. Now, we know that (refer equation (10) of Art 8.13.)

$$[z]_P = [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\ dxdy$$

or $$[z]_P = \int_{AB}\frac{2y}{x+y}dx + \iint_S \frac{1}{x+y}dxdy, \text{ by (1) and (2).} \qquad ...(4)$$

Now, $$\int_{AB}\frac{2y}{x+y}dx = 2\int_A^B \frac{xy}{x^2+xy}dx = 2\int_{1/\eta}^{\xi}\frac{1}{1+x^2}dx = 2\ \{\tan^{-1}\xi - \tan^{-1}(1/\eta)\} \qquad ...(5)$$

and $$\iint_S \frac{1}{1+x}dxdy = \int_{x=1/\eta}^{\xi}\left\{\int_{y=1/x}^{\eta}\frac{1}{x+y}dy\right\}dx, \qquad ...(6)$$

since to integrate over area bounded by $PABP$, we first integrate along the strip $MNN'M'$ by fixing x and varying y from $y = 1/x$ at M' to $y = \eta$ at M and then integrate from A to P (keeping y fixed) by varying x from $x = 1/\eta$ to $x = \xi$. Evaluating the double integral on R.H.S. of (6) by the usual rule,

$$\iint_S \frac{1}{x+y}dxdy = \int_{1/\eta}^{\xi}[\log(x+y)]_{1/x}^{\eta}dx = \int_{1/\eta}^{\xi}[\log(x+\eta) - \log(x+1/x)]\,dx$$

$$= \int_{1/\eta}^{\xi}\{\log(x+\eta) - \log(1+x^2) + \log x\}\,dx$$

$$= \Big[\{\log(x+\eta) - \log(1+x^2) + \log x\}\,x\Big]_{1/\eta}^{\xi} - \int_{1/\eta}^{\xi} x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right)dx$$

$$= \xi\ \{\log(\xi+\eta) - \log(1+\xi^2) + \log\xi)\} - \frac{1}{\eta}\left\{\log\left(\frac{1}{\eta}+\eta\right) - \log\left(1+\frac{1}{\eta^2}\right) + \log\frac{1}{\eta}\right\}$$

$$-\int_{1/\eta}^{\xi}\left(\frac{2}{1+x^2} - \frac{\eta}{x+\eta}\right)dx, \text{ on re-arranging*}$$

$$= \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - \Big[2\tan^{-1}x - \eta\log(x+\eta)\Big]_{1/\eta}^{\xi} = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - 2\left(\tan^{-1}\xi - \tan^{-1}\frac{1}{\eta}\right) + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2} \qquad ...(7)$$

Using (5) and (7), (4) reduces to $$[z]_P = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2}. \qquad ...(8)$$

Replacing ξ and η by x and y respectively in (8), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by $$z = x\log\frac{x(x+y)}{1+x^2} + y\log\frac{y(x+y)}{1+y^2}.$$

Ex. 2. *Prove that, for the equation* $(\partial^2 z/\partial x\partial y) + (z/4) = 0$, *the Green's function is* $w(x, y\,;\xi, \eta) = J_0\sqrt{(x-\xi)(y-\eta)}$, *where* $J_0(z)$ *denotes Bessel's function of the first kind of order zero.*

[Himanchal 1998, 2004, Kerala 2001; Kurukshetra 2000; Nagpur 2000, 03, 05 Delhi Maths Hons. 2004, 07, 09]

Sol. Here $$L(z) = (\partial^2 z/\partial x\partial y) + (z/4) = 0. \qquad ...(1)$$

$$\therefore\quad L \equiv \frac{\partial^2}{\partial x\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c = \frac{\partial^2}{\partial x\partial y} + \frac{z}{4}. \quad \Rightarrow \quad a = 0, \quad b = 0, \quad c = z/4. \qquad ...(2)$$

$$^*\ x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right) = \frac{x}{x+\eta} - \frac{2x^2}{1+x^2} + 1 = \frac{(x+\eta)-\eta}{x+\eta} - 2\frac{(1+x^2)-1}{1+x^2} + 1 = -\frac{\eta}{x+\eta} + \frac{2}{1+x^2}$$

So the adjoint operator M to the operator L is given by $M \equiv (\partial^2/\partial x\,\partial y)+(1/4)=0.$ ...(3)

Given, $$w = J_0\sqrt{(x-\xi)(y-\eta)}. \quad ...(4)$$

From (4) $$\frac{\partial w}{\partial x} = \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}}J_0' \quad ...(5)$$

From (5), $$\frac{\partial^2 w}{\partial y\partial x} = \frac{1}{4}\frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}} \times \frac{\sqrt{(x-\xi)}}{2\sqrt{(y-\eta)}}J_0''$$

or $$\frac{\partial^2 w}{\partial y\partial x} = \frac{1}{4}\left\{J_0''+\frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0'\right\}. \quad ...(6)$$

So (3) and (6) $\Rightarrow$ $$Mw = \frac{1}{4}\left\{J_0''+\frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + J_0\right\}. \quad ...(7)$$

Now, Bessel's eqution of order zero is given by

$x^2y'' + xy' + x^2y = 0$ or $y'' + (1/x) \times y' + y = 0.$ ...(8)

Since $y = J_0\{\sqrt{(x-\xi)(y-\eta)}\}$ is a solution of (8), we get

$$J_0''+\frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + J_0 = 0 \quad \text{or} \quad Mw = 0, \text{ by (7)} \quad ...(9)$$

Again, (5) $\Rightarrow$ $(\partial w/\partial x) = 0 = bw$ when $y = \eta$, as $b = 0$ ...(10)

Similarly, $(\partial w/\partial y) = 0 = aw$ when $x = \xi$, as $a = 0$ ...(11)

Finally, when $x = \xi$, $y = \eta$, $w = J_0(0) = 1.$ ...(12)

Since w satisfies four properties (9), (10), (11) and (12) of a Green's function, it follows that w must be a Green's function of the given equation (1).

Ex. 3. *Prove that for the equation* $\dfrac{\partial^2 z}{\partial y\partial x}+\dfrac{2}{x+y}\left(\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}\right) = 0$, *the Green's function is*

$$w(x, y\,;\xi, \eta) = \frac{(x+y)\{2xy+(\xi-\eta)(x+y)+2\xi\eta\}}{(\xi+\eta)^3}.$$

Hence find the solution of the differential equation which satisfies the conditions $z = 0$, $\partial z/\partial x = 3x^2$ on $y = x$. **[Bangalore 2003, Himanchel 2001; Kurukshetra 2004; Delhi Maths (H) 2001,05,11]**

Sol. Compare the given equation with $L(z) = f(x, y)$ where $L \equiv \dfrac{\partial^2}{\partial y\partial x}+a\dfrac{\partial}{\partial x}+b\dfrac{\partial}{\partial y}+c$, we find

$a = 2/(x + y)$, $b = 2/(x + y)$, $c = 0$, $f(x, y) = 0.$ ...(1)

So the adjoint operator M to the operator L is given by

$$Mw \equiv \frac{\partial^2 w}{\partial y\partial x} - \frac{\partial}{\partial x}\left(\frac{2}{x+y}w\right) - \frac{\partial}{\partial y}\left(\frac{2}{x+y}w\right). \quad ...(2)$$

Given $$w(x, y\,;\xi, \eta) = \frac{(x+y)\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}. \quad ...(3)$$

(3) $\Rightarrow$ $$\frac{\partial w}{\partial x} = \frac{2xy+(\xi-\eta)(x-y)+2\xi\eta+(x+y)(2y+\xi-\eta)}{(\xi+\eta)^3}. \quad ...(4)$$

(3) $\Rightarrow$ $$\frac{\partial w}{\partial y} = \frac{2xy+(\xi-\eta)(x-y)+2\xi\eta+(x+y)(2y-\xi+\eta)}{(\xi+\eta)^3}. \quad ...(5)$$

(5) $\Rightarrow$ $$\frac{\partial^2 w}{\partial x\partial y} = \frac{2y+\xi-\eta+2x-\xi+\eta+2(x+y)}{(\xi+\eta)^3} = \frac{4(x+y)}{(\xi+\eta)^3}. \quad ..(6)$$

Using (6) and (3), (2) reduces to

$$Mw = \frac{4(x+y)}{(\xi+\eta)^3} - 2\frac{\partial}{\partial x}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\} - 2\frac{\partial}{\partial y}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2(2y+\xi-\eta)+2(2x-\xi+\eta)}{(\xi+\eta)^3} = 0. \qquad ...(7)$$

At $y=\eta$, $$\frac{\partial w}{\partial x} = \frac{2x\eta+(\xi-\eta)(x-\eta)+2\xi\eta+(x+\eta)(2\eta+\xi-\eta)}{(\xi+\eta)^3}, \text{ by (4)}$$

$$= \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(8)$$

From (1) and (3), $$bw = \frac{2\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(9)$$

So at $y=\eta$, (9) reduces to

$$bw = \frac{2\{2x\eta+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3} = \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(10)$$

From (8) and (10), $\partial w/\partial x = bw$ when $y=\eta$. ...(11)

Similarly, $\partial w/\partial y = aw$ when $x=\xi$. ...(12)

From (3), when $x=\xi$, $y=\eta$, we get $$w = \frac{(\xi+\eta)\{2\xi\eta+(\xi-\eta)^2+2\eta\}}{(\xi+\eta)^3} = 1. \qquad ...(13)$$

Since w satisfies four properties (7), (11), (12) and (13) of a Green's function, it follows that w must be a Green's function of the given equation.

To find the solution of the equation. In the present problem, the values of z and $\partial z/\partial x\ (=p)$ are given by

$z = 0$ and $\partial z/\partial x = 3x^2$. ...(14)

along the line AB, $y = x$. ...(15)

Then we wish to find the solution of given equation at the point $P(\xi,\eta)$ agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $y=x$ in the point A and PB parallel to the y-axis and cutting $y=x$ in B. Then triangular region enclosed by straight lines $y=x$, $y=\eta$ and $x=\xi$ is denoted by S. Then we know that (refer equation (10) of Art 8.13).

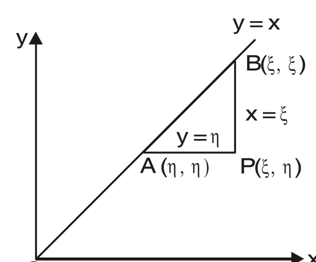


Figure-3

$$[z]_P = [wz]_A - \int_{AB} wz(a\,dy - b\,dx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\,dxdy. \qquad ..(16)$$

Now on line AB, from (3), $$w = \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^2}, \text{ as } y = x. \qquad ...(17)$$

Using (1), (14) and (17), (16) reduces to

$$[z]_P = \int_A^B \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^3}\,3x^2 dx = \frac{12}{(\xi+\eta)^3}\int_\eta^\xi (x^5+\xi\eta x^3)dx$$

$$= \frac{12}{(\xi+\eta)^3}\left[\frac{x^6}{6}+\xi\eta\frac{x^4}{4}\right]_\eta^\xi = \frac{12}{(\xi+\eta)^3}\left[\frac{\xi^6-\eta^6}{6}+\frac{\xi\eta}{4}(\xi^4-\eta^4)\right]$$

$$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)(\xi^3-\eta^3)+3\xi\eta(\xi^2-\eta^2)(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)\,(\xi-\eta)\,(\xi^2+\eta^2+\xi\eta)+3\xi\eta(\xi-\eta)(\xi+\eta)(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)\,(\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)-3\xi\eta(\xi+\eta)(\xi^2+\eta^2)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)\,(\xi^2+\eta^2)+6\xi\eta(\xi+\eta)(\xi^2+\eta^2)\;-3\xi\eta(\xi+\eta)(\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)\right\}$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left[2(\xi^2+\eta^2)\left\{\xi^3+\eta^3+3\xi\eta\,(\xi+\eta)\right\}\;-\xi\eta\left\{3(\xi+\eta)\,(\xi^2+\eta^2)-2\,(\xi^3+\eta^3)\right\}\right]$$

$$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^2+\eta^2)(\xi+\eta)^3-\xi\eta(\xi+\eta)^3\right\} = (\xi-\eta)(2\xi^2+2\eta^2-\xi\eta)$$

or $$[z]_P = 2\xi^3+3\xi\eta^2-3\xi^2\eta-2\eta^3. \qquad ....(18)$$

Replacing ξ and η by x and y respectively in (18), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by

$$z = 2x^3 + 3xy^2 - 3x^2y - 2y^3.$$

Ex.4. *Obtain the solution of* $\partial^2 z/\partial x\partial y = 1/(x+y)$ *such that* $z = 0$, $p = 2y/(x+y)$ *on* $y = x$.

[Delhi Maths (H) 1998]

Sol. Here we are solve $$\partial^2 z/\partial x\partial y = 1/(x+y), \qquad ...(1)$$

where $z = 0$ and $p = \partial z/\partial x = 2y/(x+y)$ on $y = x$ ... (2)

Here the given curve C is straight line $y = x$. Then we wish to find the solution of (1) at $P(\xi, \eta)$ agreeing with boundary conditions (2). Through P we draw PA parallel to the x-axis and cutting $y = x$ at the point A and PB parallel to the y-axis and cutting $y = x$ in B. The triangular-region enclosed by stright lines $y = x$, $y = \eta$, $x = \xi$ is denoted by S (draw figure as shown in figure 3 of solved Ex. 3). Then we know that (refer particular case II of Art. 8.13).

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}dx + \iint_S f(x,y)\; dx\; dy \qquad ...(3)$$

Comparing (1) with $\partial^2 z/\partial x\;\partial y = f(x,y)$, here $f(x,y) = 1/(x+y)$.

Since A lies on given curve AB and it is given that $z = 0$ on AB, hence $[z]_A = 0$.

From (2), $\partial z/\partial x = 2y/(x+y)$ on AB. *i.e.* $y = x$

so that $\partial z/\partial x = 2x/(x+x) = 1$ on $y = x$.

Using the above facts, (3) reduces to

$$[z]_P = \int_\eta^\xi dx + \int_{x=\eta}^{\xi}\left\{\int_{y=\eta}^{x}\frac{1}{x+y}dy\right\}dx = \xi-\eta+\int_\eta^\xi\left[\log(x+y)\right]_{y=\eta}^{x}dx = \xi-\eta+\int_\eta^\xi[\log(2x)-\log(x+\eta)]\,dx$$

$$= \xi-\eta+\left[x\log\frac{2x}{x+\eta}\right]_\eta^\xi - \int_\eta^\xi\left(\frac{1}{x}-\frac{1}{x+\eta}\right)x\;dx = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\log 1-\int_\eta^\xi\frac{\eta}{x+\eta}dx$$

$$= \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\left[\log\,(x+\eta)\right]_\eta^\xi = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}+\eta\log\frac{2\eta}{\xi+\eta}$$

Replacing ξ and η by x and y respectively in the above equation, the value of z (*i.e.,* solution of (1) at any point (x, y) is given by

$$z = x-y+x\log\;\{2x/(x+y)\}+y\log\{2y/(x+y)\}.$$

8.15. Riemann-Volterra method for solving the Cauchy problem for the one-dimensional wave equation

The entire procedure of solution will become clear from the following solved examples.

Ex. 1. *Using Riemann-Volterra method, solve* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$, *when* z, $\partial z/\partial x$, $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane*

Sol. Given $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0 \qquad$ ... (1)

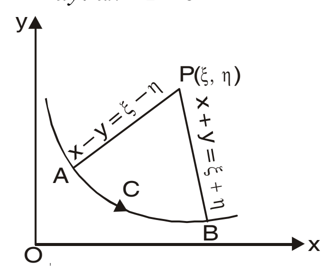
Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, p) = 0$, here R = 1, S = 0 and T = – 1. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$dy/dx + 1 = 0 \qquad$ and $\qquad dy/dx - 1 = 0$

Integrating these,

$x + y = C_1 \qquad$ and $\qquad x - y = C_2, \qquad$ ... (2)

which are characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in xy-plane. We now obtain characteristics of (1) passing through P. So putting $x = \xi$ and $y = \eta$ in (2), we have $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through P are given by



$x + y = \xi + \eta \qquad$ and $\qquad x - y = \xi - \eta, \qquad$ ... (3)

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut the given curve *C* in *A* and *B* respectively. Let C' denote the closed curve *PABP* (which is made up of straight line *PA*, curve *C* (*i.e. AB*) and straight line *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\, dy = 0 \qquad \text{or} \qquad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\ dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) = 0, \text{ by Green's theorem.*}$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) = 0$$

Equation of *BP* is $x + y = \xi + \eta$ and hence $dx = -\,dy$ on *BP*. Similarly, equation of *PA* is $x - y = \xi - \eta$ and hence $dx = dy$ on *PA*. Using these facts in the above equation, we get

$$\int_C \left(\frac{\partial z}{\partial y}\, dx + \frac{\partial z}{\partial x}\, dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx\right) + \int_P^A \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx\right) = 0$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) - \int_B^P dz + \int_P^A dz = 0, \text{ as } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) - (z_P - z_B) + (z_A - z_P) = 0$$

---

* **Green's theorem.** Let C' be a closed curve bounding the region S on *xy*-plane and $u(x, y)$, $v(x, y)$ be differential functions in *S* and continusou on C', then $\oint_{C'} (udx + vdy) = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\ dy$

$$\therefore \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right), \qquad \text{... (4)}$$

which is the required solution of (1) at any point P.

Ex. 2. *Solve one dimensional wave equation by Riemann Volterra method.*

[Kurukshetra 2001; Delhi Maths (H) 1996]

or *Solve homogeneous one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *when* $z, \partial z/\partial x, \partial z/\partial t$ *are prescribed along a curve C.*

Sol. Given $\qquad \partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2) \qquad$ ... (*i*)

Let y be a new variable such that $\qquad y = ct \qquad$ ... (*ii*)

Then (*i*) becomes $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0 \qquad$ ... (*iii*)

for which z, $\partial z/\partial x$ and $\partial z/\partial y$ are now prescribed along C.

Proceed with (*iii*) as we did in solved Ex. 1 upto equation (4).

Ex. 3. *Find z (x, y) such that* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *and* $z = f(x)$ *and* $\partial z/\partial y = g(x)$ *on y = 0.*

[Kanpur 2003; Delhi Maths (H) 2000, 08]

Sol. Given $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0, \qquad$ ... (1)

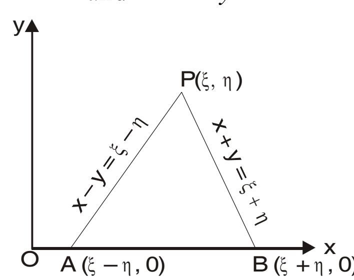
where $\quad z(x,0) = f(x), \quad$ *i.e.*, $\quad z = f(x) \quad$ on $\quad y = 0 \quad$ *i.e.* x – axis ... (2)

and $\quad (\partial z/\partial y)_{y=0} = g(x) \quad$ *i.e.*, $\quad \partial z/\partial y = g(x) \quad$ on $\quad y = 0 \quad$ *i.e.* x – axis ...(3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by $\qquad dy/dx + 1 = 0 \qquad$ and $\qquad dy/dx - 1 = 0$

Integrating these $\quad x + y = C_1 \quad$ and $\quad x - y = C_2 \quad$ ... (4)

which are the characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in *xy*-plane. We now obtain characteristics of (1) passing through *P*. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through *P* are given by



$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \qquad \text{... (5)}$$

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut given curve (here $y = 0$ *i.e.* x-axis) in $A\,(\xi - \eta, 0)$ and $B\,(\xi + \eta, 0)$ respectively. Let C' denote the closed curve *PA BP* (which is made up of straight lines *PA*, *AB* and *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over *S*, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dxdy = 0 \qquad \text{or} \qquad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0, \text{ by Green's theorem}$$

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

On AB (*i.e.,* x-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z/\partial y = g(x)$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -dy$. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, the above equation reduces to

$$\int_A^B g(x)dx + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) = 0$$

or $$\int_A^B g(x)\, dx - \int_B^P dz + \int_P^A dz = 0, \quad \text{as} \quad \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx = dz$$

or $$\int_A^B g(x) - (z_P - z_B) + z_A - z_P = 0 \qquad \text{or} \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B g(x)\, dx \qquad \text{... (6)}$$

From (2), $z = f(x)$ on $y = 0$ (*i.e.,* x-axis). Since x-coordinates of A and B are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = f(\xi - \eta)$ and $z_B = f(\xi + \eta)$. Hence (6) reduces to

$$z_P = \frac{1}{2}\{f(\xi - \eta) + f(\xi + \eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} g(x)\, dx \qquad \text{... (7)}$$

Replacing ξ and η by x and y respectively in (7), the value of z (*i.e.,* solution of (1) at any point $P(x, y)$) is given by

$$z(x, y) = \frac{1}{2}\{f(x - y) + f(x + y)\} + \frac{1}{2}\int_{x-y}^{x+y} g(u)\, du$$

Ex. 4. *Find the solution of one-dimensional non-homogeneous wave equation* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0$ *by Riemann-Vatterra method.*

[A.M.I.E. 2005; Delhi Maths (H) 1998, 2002; Kanpur 1998]

Sol. Given $\quad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0 \quad$ or $\quad r - t + f(x, y) = 0$. ... (1)

Suppose that z, $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along a given curve C. Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations $dy/dx + 1 = 0$ and $dy/dx - 1 = 0$ give, on integration,

$$x + y = C_1 \qquad \text{and} \qquad x - y = C_2 \qquad \text{... (2)}$$

which are characteristics of (1). Draw figure as in solved Ex. 1. Let $P(\xi, \eta)$ be any point in xy-plane. Then characteristics of (1) passing through $P(\xi, \eta)$ are given by

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \qquad \text{... (3)}$$

which have been shown by straight lines PB and PA respectively. Let the characteristics PA and PB cut the given curve C in A and B respectively. Let C' denote the closed curve $PABP$ and let S denote the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\, dy + \iint_S f(x, y)\, dx\, dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x, y)\, dx\, dy = 0, \text{ by Green's theorem}$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x, y)\, dxdy = 0$$

Equation of BP is $x+y=\xi+\eta$ and so $dx = -dy$ on BP. Similarly, equation of PA is $x-y = \xi-\eta$ and so $dx = dy$ on PA. Using these facts, the above equation reduces to

$$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_B^P \left(-\frac{\partial z}{\partial y}dy-\frac{\partial z}{\partial x}dx\right)+\int_P^A \left(\frac{\partial z}{\partial y}dy+\frac{\partial z}{\partial x}dx\right)+\iint_S f(x,y)\,dx\,dy=0$$

or
$$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\int_B^P dz+\int_P^A dz+\iint_S f(x,y)\,dx\,dy=0$$

or
$$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-(z_P-z_B)+z_A-z_P+\iint_S f(x,y)\,dx\,dy=0$$

or
$$z_P=\frac{1}{2}(z_A+z_B)+\frac{1}{2}\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\frac{1}{2}\iint_S f(x,y)\,dx\,dy=0,$$

which is the required solution of (1) at any point P.

Ex. 5. *Solve* $\partial^2 z/\partial x^2-\partial^2 z/\partial y^2=1$, *when* $z(x,0)=\sin x$, $z_y(x,0)=x$.

Sol. Given $\quad \partial^2 z/\partial x^2-\partial^2 z/\partial y^2-1=0 \quad$ or $\quad r-t-1=0 \quad$ ... (1)

where $\quad z(x,0)=\sin x$, *i.e.*, $z=\sin x$ on $y=0$, *i.e.*, x-axis ... (2)

and $\quad z_y(x,0)=x$, *i.e.*, $\partial z/\partial y=x$ on $y=0$, *i.e.*, x-axis ... (3)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=1$, $S=0$ and $T=-1$. Hence the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2-1=0$ so that $\lambda=1,-1$. The corresponding characteristic equations of (1) are

$$dy/dx+1=0 \qquad \text{and} \qquad dy/dx-1=0$$

Integrating these, $\quad x+y=C_1 \quad$ and $\quad x-y=C_2$ ... (4)

which are the characteristics of (1) and these are two families of straight lines Draw a figure as in solved Ex. 3. Let $P(\xi,\eta)$ be any point in xy-plane. Putting $x=\xi$, $y=\eta$ in (4), we get $C_1=\xi+\eta$, $C_2=\xi-\eta$. Hence the characteristics of (1) passing through P are given by

$$x+y=\xi+\eta \qquad \text{and} \qquad x-y=\xi-\eta \qquad \text{... (5)}$$

which have been shown by straight lines PB and PA respectively in the figure. Let the characteristics PA and PB cut the given line $y=0$ *i.e.*, x-axis in $A(\xi-\eta, 0)$ and $B(\xi+\eta, 0)$ respectively. Let C' denote the closed curve $PABP$ and let S denote the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\}-\iint_S dx\,dy=0$$

or
$$\oint_{C'} \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\iint_S dx\,dy=0, \text{ by Green's theorem} \qquad \text{... (5)}$$

Now, $\iint_S dx\,dy$ = area of the triangle SAB = (1/2) × AB × perpendicular distance of P from AB

$$= (1/2)\times\{\xi+\eta-(\xi-\eta)\}\times\eta=\eta^2$$

Hence (5) reduces to

$$\int_{AB} \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_{BP} \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_{PA} \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\eta^2=0 \qquad \text{... (6)}$$

On AB (*i.e.,* x-axis), $y = 0$ so that $dy = 0$. Also from (3), $\partial z/\partial y = x$ on $y = 0$. On BP (*i.e.,* $x + y = \xi+\eta$), $dx = -dy$. Similarly on PA (*i.e.,* $x - y = \xi-\eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_A^B x\,dx + \int_B^P\left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A\left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) - \eta^2 = 0$$

or $$\int_A^B x\,dx - \int_B^P dz + \int_P^A dz - \eta^2 = 0, \text{ as } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

or $$\int_A^B x\,dx - (z_P - z_B) + z_A - z_P - \eta^2 = 0 \quad \text{or} \quad z_P = \frac{1}{2}(z_Z + z_B) + \frac{1}{2}\int_A^B x\,dx - \frac{1}{2}\eta^2 \quad \text{... (7)}$$

From (2), $z = \sin x$ on $y = 0$ (*i.e.,* x-axis). Since x-coordinates of A and B are $\xi-\eta$ and $\xi+\eta$ respectively, it follows that $z_A = \sin(\xi-\eta)$ and $z_B = \sin(\xi+\eta)$. Hence (7) reduces to

$$z_P = \frac{1}{2}\{\sin(\xi-\eta) + \sin(\xi+\eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} x\,dx - \frac{\eta^2}{2} \quad \text{or} \quad z_P = \sin\xi\cos\eta - \frac{1}{4}\left[x^2\right]_{\xi-\eta}^{\xi+\eta} - \frac{\eta^2}{2}$$

or $$z_P = \sin\xi\cos\eta - (1/4) \times \{(\xi+\eta)^2 - (\xi-\eta)^2\} - (1/2)\times\eta^2$$

or $$z_P = \sin\xi\cos\eta - \xi\eta - (\eta^2/2) \quad \text{... (8)}$$

Replacing ξ and η by x and y respectively in (8), the value of z (*i.e.,* solution of (1)) at any point (x, y) is given by $\quad z(x, y) = \sin x \cos y - xy - (y^2/2)$.

Ex. 6. *A function $z(x, y)$ satisfies the non-homogeneous equation $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0$ and the initial conditions $z = \partial z/\partial y = 0$ when $y = 0$. Show that (using Riemann - Volterra method)*

$$z(x, y) = \frac{1}{2}\iint_\Gamma f(u, v)\,du\,dv,$$

where Γ is the triangle cut off from the upper half of uv-plane by two characteristics through the point (x, y). **[Delhi Maths (Hons) 2002, 07, 11; A.M.I.E. 2005; Amaravati 2003; Kanpur 1999; Rohilkhand 2004]**

Sol. Given $\quad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0 \quad$ or $\quad r - t + f(x, y) = 0, \quad$ ... (1)

where $\quad z(x, 0) = 0, \quad$ *i.e.,* $\quad z = 0 \quad$ on $\quad y = 0 \quad$ (x-axis) ... (2)

and $\quad (\partial z/\partial y)_{y=0} = 0, \quad$ *i.e.,* $\quad (\partial z/\partial y) = 0 \quad$ on $\quad y = 0 \quad$ (x-axis) ... (3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations are given by

$$dy/dx + 1 = 0 \quad \text{and} \quad dy/dx - 1 = 0$$

Integrating these, $\quad x + y = c_1 \quad$ and $\quad x - y = c_2, \quad$ ... (4)

which are characteristics of (1). Draw figure as in solved Ex. 3. Let $P(\xi, \eta)$ be any point in xy-plane. We now obtain characteristics of (1) passing through P. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi+\eta$ and $C_2 = \xi-\eta$. Hence the characteristics of (1) passing through P are given by

$$x + y = \xi+\eta \quad \text{and} \quad x - y = \xi-\eta \quad \text{... (5)}$$

which have been shown by straight lines PB and PA respectively in the figure. Let the characteristics PA and PB cut $y = 0$ (*i.e.,* x-axis) at A and B respectively. Let C' denote the closed curve $PA\,BP$ and let Γ denote the triangular region enclosed by C'.

Integrating both sides of (1) over Γ, we have

$$\iint_\Gamma \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right) dx\, dy + \iint_\Gamma f(x,y)\, dx\, dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_\Gamma f(x,y)\, dx\, dy = 0 \text{, using Green's theorem}$$

or $$\int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_\Gamma (x,y)\, dx\, dy = 0 \quad \text{... (6)}$$

On AB (*i.e., x*-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z/\partial y = 0$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -dy$. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx \right) + \int_P^A \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx \right) + \iint_\Gamma (x,y)\, dxdy = 0$$

or $$-\int_B^P dz + \int_P^A dz + \iint_\Gamma f(x,y) dxdy = 0, \text{ as } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $$-(z_P - z_B) + z_A - z_P + \iint_\Gamma f(x,y)\, dx\, dy = 0 \quad \text{... (7)}$$

From (2), $z = 0$ on $y = 0$ (*i.e., x*-axis) and so $z_A = z_B = 0$, because A and B both lie on $y = 0$. Hence (7) becomes

$$2z_P = \iint_\Gamma f(x,y)\, dxdy \qquad \text{or} \qquad z(x,y) = \frac{1}{2}\iint_\Gamma f(u,v)\, du\, dv,$$

which gives the value of z (*i.e.,* solution of (1)) at any point (x, y)

Ex.7. *Find the solution of the non-homogeneous wave equation* $\partial^2 z/\partial x^2 - (1/c^2)(\partial^2 z/\partial t^2) + f(x,t) = 0$ *with initial conditions* $z(x, 0) = f(x)$, $z_t(x, t) = g(x)$.

Sol. Let y be a new variable such that $\qquad y = c\,t \qquad$ ... (1)

Then the given problem may be re-written as

$$\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + F(x,y) = 0 \qquad \text{or} \qquad r - t + F(x,y) = 0, \quad \text{... (2)}$$

with the modified initial conditions given below

$z(x, 0) = f(x)$, *i.e.,* $z = f(x)$ on $y = 0$, *i.e., x*-axis ... (3)

$(\partial z/\partial y)_{y=0} = G(x)$, *i.e.,* $\partial z/\partial y = G(x)$ on $y = 0$, *i.e., x*-axis ... (4)

Here $F(x, y) = f(x, t)$ and $G(x) = (1/c) \times g(x)$ ... (5)

Comparing (2) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$$dy/dx + 1 = 0 \qquad \text{and} \qquad dy/dx - 1 = 0 \quad \text{... (6)}$$

Integrating these, $\quad x + y = C_1 \quad$ and $\quad x - y = C_2$,

which are characteristics of (1). Draw a figure same as in Ex. 3.

Let $P(\xi, \eta)$ be any point in xy-plane. Then characteristics (2) passing through $P(\xi, \eta)$ are

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \quad \text{... (7)}$$

which have been shown by straight lines PB and PA respectively in the figure. Here lines given by (7) cut x-axis (*i.e.,* $y = 0$) in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let C' denote the closed curve $PABP$ and let S be the region enclosed by C'.

Integrating both sides of (2) over S, we have

$$\iint_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \right\} \, dx\,dy + \iint_S F(x,y)\,dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S F(x,y)\,dx\,dy = 0 \text{, using Green's theorem}$$

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S F(x,y)\,dx\,dy = 0 \quad \text{... (8)}$$

On AB (*i.e., x*-axis), $y = 0$ so that $dy = 0$. Also, from (4), $\partial z/\partial y = G(x)$ on $y = 0$. On BP (*i.e.,* $x + y = \xi+\eta$), $dx = -$ dy. Similarly, on PA (*i.e.,* $x - y = \xi-\eta$), $dx = dy$. Using these facts, (8) reduces to

$$\int_A^B G(x)\,dx + \int_B^P \left(-\frac{\partial z}{\partial y}\,dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A\left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) + \iint_S F(x,y)\,dx\,dy = 0$$

or $$\int_A^B G(x)dx - \int_B^P dz + \int_P^A dz + \iint_S F(x,y)\,dx\,dy = 0$$

or $$\int_A^B G(x)\,dx - (z_P - z_B) + z_A - z_P + \iint_S F(x,y)\,dx\,dy = 0$$

or $$z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B G(x)\,dx + \frac{1}{2}\iint_S F(x,y)\,dx\,dy \quad \text{... (9)}$$

From (3), $z = f(x)$ on $y = 0$ (*i.e., x*-axis). Since x-coordinates of A and B are $\xi-\eta$ and $\xi+\eta$ respectively, it follows that $z_A = f(\xi-\eta)$ and $z_B = f(\xi+\eta)$. Hence (9) reduces to

$$z_p = \frac{1}{2}\{f(\xi-\eta) + f(\xi+\eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} G(x)\,dx + \frac{1}{2}\iint_s F(x,y)\,dx\,dy \quad \text{... (10)}$$

Replacing ξ and η by x and y (= ct) respectively and using (5), (10) reduces to

$$z(x,y) = \frac{1}{2}\{f(x-y) + f(x+y)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)du + \frac{1}{2}\iint_s f(x,t)dx\,dt$$

Miscellaneous Problems on chapter 8

Ex. 1. $\partial^2 u/\partial t^2 = c^2(\partial^2 u/\partial x^2)$ is hyperbolic or parabolic. Classify it. **[Agra 2008]**

Hint. See Art 8.1 **Ans.** Hyperbolic

Ex. 2. The equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ is

(a) parabolic (b) hyperbolic (c) elliptic (d) Nonw of these **[Agra 2007]**

Sol. Ans. (b.) See Art 8.1.

Ex. 3. Classify and solve the following equation $\partial^2 z/\partial x^2 = x^2(\partial^2 z/\partial y^2)$. **[Bhopal 2010]**

9

Monge's Methods

9.1 INTRODUCTION

The most general form of partial differential equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

It is only in special cases that (1) can be integrated. Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equations of the form (1). Monge's methods consists in finding one or two first integrals of the form

$$u = \phi(v), \qquad ...(2)$$

where u and v are known functions of x, y, z, p and q and ϕ is an arbitrary function. In other words, Monge's methods consists in obtaining relations of the form (2) such that equation (1) can be derived from (2) by eliminating the arbitrary function. A relation of the form (2) is known as an *intermediate integral* of (1). Every equation of the form (1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (2) as an intermediate integral are of the following forms

$$Rr + Ss + Tt = V \quad \text{and} \quad Rr + Ss + Tt + U(rt - s^2) = V, \qquad ...(3)$$

where R, S, T, U and V are functions of x, y, z, p and q. Even equations (3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (3) exists.

9.2. MONGE'S METHOD OF INTERGRATING Rr + Ss + Tt = V. [Agra 2005; Delhi Maths (Hons) 2000, 02, 08, 09, 11; Garhwal 1994; Patna 2003; Kanpur 1997; Meerut 2000]

Given $$Rr + Ss + Tt = V, \qquad ...(1)$$

where R, S, T and V are functions of x, y, z, p and q.

We know that $p = \partial z/\partial x$, $\qquad q = \partial z/\partial y$,

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}, & t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}, \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \quad \text{and} & s &= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \quad ...(2)$$

Now, $$dp = (\partial p/\partial x)\, dx + (\partial p/\partial y) dy = rdx + sdy, \text{ using (2)} \qquad ...(3)$$

and $$dq = (\partial q/\partial x) dx + (\partial q/\partial y) dy = sdx + tdy, \text{ using (2)} \qquad ...(4)$$

From (3) and (4), $\quad r = (dp - s\,dy)/dx \quad$ and $\quad t = (dq - s\,dx)/dy \qquad ...(5)$

Substituting the values of r and s given by (5) in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V \quad \text{or} \quad R(dp - sdy)dy + Ss\, dxdy + T(dq - sdx)dx = V\, dxdy$$

or $$(Rdpdy + Tdqdx - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2\} = 0. \qquad ...(6)$$

Clearly any relation between x, y, z, p and q which satisfies (6) must also satisfy the following two simultaneous equations

$$Rdpdy + Tdq\,dx - Vdxdy = 0. \qquad ...(7)$$

and
$$(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad ...(8)$$

The equations (7) and (8) are called *Monge's subsidiary equations* and the relations which satisfy these equations are called *intermediate integrals*.

Equation (8) being a quadratic, in general, it can be resolved into two equations, say

$$dy - m_1\,dx = 0 \qquad ...(9)$$

and
$$dy - m_2\,dx = 0. \qquad ...(10)$$

Now the following two cases arise :

Case I. When m_1 and m_2 are distinct in (9) and (10).

In this case (7) and (9), if necessary by using well known result $dz = pdx + qdy$, will give two integrals $u_1 = a$ and $v_1 = b$, where a and b are arbitrary constants. These give

$$u_1 = f_1(v_1), \qquad ...(11).$$

where f_1 is an arbitrary function. It is called an *intermediate integral* of (1).

Next, taking (7) and (10) as before, we get another intermediate integral of (1), say

$$u_2 = f_2(v_2), \text{ where } f_2 \text{ is an arbitrary function.} \qquad ...(12)$$

Thus we have in this case two distinct intermediate integrals (11) and (12). Solving (11) and (12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well known relation $\quad dz = pdx + qdy \quad$ ...(13)

and then integrating (13), we get the required complete integral of (1).

Case II . When $m_1 = m_2$ i.e., (8) is a perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$$Pp + Qq = R. \qquad ...(14)$$

Solving (14) with help of Lagrange's method (refer Art. 2.3, chapter 2), we get the required complete integral of (1).

Remark 1. Usually while dealing with case I, we obtain second intermediate integral directly by using symmetry. However sometimes in absence of any symmetry, we find the complete integral with help of only one indetermediate integral. This is done with help of using Lagrange's method.

Remark 2. While obtaining an intermediate integral, remember to use the relation $dx = pdx + qdy$ as explained below :

(*i*) $pdx + qdy + 2xdx = 0$ can be re–written as $\quad dz + 2xdx = 0 \quad$ so that $\quad z + x^2 = c$.

(*ii*) $xdp + ydq = dx$ can be re–written as $\quad xdp + ydq + pdx + qdy = dx + pdx + qdy$

or $\quad d(xp) + d(yq) = dx + dz \quad$ so that $\quad xp + yq = x + z + c$, on integration

Remark 3. While integrating, we shall use the following types of calculations. In what follows, f and g are arbitrary functions and k and a are a constants.

(*i*) $\int k\,f(t)\,dt = g(t)$ $\qquad$ (*ii*) $\int k\frac{1}{t}f(t)\,dt = g(t)$. $\qquad$ (*iii*) $\int k\frac{1}{t^2}f(t^2)\,d(t^2) = g(t^2)$

(*iv*) $\int k\,f(x+y)\,d(x+y) = g(x + y)$. $\qquad$ (*v*) $\int k t^2 f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = \int \frac{k}{(1/t)^2} f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = g\left(\frac{1}{t}\right)$

(*vi*) $\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{(at^2)} f(at^2)\,d(at^2) = g(at^2)$

Proof of (vi). Putting $at^2 = u$, and $d(at^2) = du$ we have

$$\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{u} f(u)\,d(u) = g(u) = g(at^2), \text{ as } u = at^2.$$

Similarly, other results can be proved. In examination we shall not use substitution as explained above. With good practice, the students will be able to write direct results of integration very easily.

Important Note. For sake of convenience, we have divided all questions based on $Rr + Ss + Tt = V$ in four types. We shall now discuss them one by one.

9.3. Type 1. When the given equation Rr + Ss + Tt = V leads to two distict intermediate intergrals and both of them are used to get the desired solution.

Working rule for solving problems of type 1.

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations:

$Rdpdy + Tdqdx - Vdxdy = 0$ ...(1) $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Using one of the factors obtained in (1), (2) will lead to an intermediate integral. In general, the second intermediate integral can be obtained from the first one by inspection, taking advantage of symmetry. In absence of any symmetry, the second factor obtained in step 3 is used in (2) to arrive at second intermediate integral. You should use remark 2 of Art. 9.2 while finding intermediate integrals.

Step 5. Solve the two intermediate integrals obtained in step 4 and get the values of p and q.

Step 6. Substitute the values of p and q in $dz = pdx + q\,dy$ and integrate to arrive at the required general solution. You should use remark 3 of Art. 9.2 while integrating $dz = pdx + qdy$.

9.4. SOLVED EXAMPLES BASED ON ART. 9.3.

Ex. 1. (a) *Solve* $r = a^2 t$. **[Agra 2008; Lucknow 2010; Patna 2003; Meerut 2008]**

(b) $r = t$. **[Agra 2006]**

(c) Solve one-dimensions wave equation by Monge's method: $\partial^2 y/dx^2 = a^2(\partial^2 y/\partial t^2)$. **[Meerut 2003]**

Sol. (*a*) Given equation is $\qquad r - a^2 t = 0.$

Comparing it with $Rr + Ss + Tt = V$, we have $\quad R = 1, \quad S = 0, \quad T = -a^2, \quad V = 0.$

Hence Monge's subsidiary equations

$Rdpdy + Tdq\,dx - Vdxdy = 0 \qquad$ and $\qquad R(dy)^2 - S\,dxdy + T\,(dx)^2 = 0$

become $\qquad dpdy - a^2 dqdx = 0 \qquad$ ...(1)

and $\qquad (dy)^2 - a^2(dx)^2 = 0. \qquad$ ...(2)

Equation (2) may be factorised as $\qquad (dy - adx)\,(dy + adx) = 0$

Hence two systems of equations to be considered are

$dpdy - a^2\,dqdx = 0, \qquad dy - adx = 0.$ ...(3)

and $\quad dpdy - a^2\,dqdx = 0, \qquad dy + adx = 0.$ ...(4)

Integrating the second equation of (3), we get $\qquad y - ax = c_1.$ ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp - adq = 0 \qquad$ so that $\qquad p - aq = c_2.$ ...(6)

Hence the intermediate integral corresponding to (3) is $\qquad p - aq = \phi_1(y - ax).$ ...(7)

Similarly another intgermediate integral corresponding to (4) is $\quad p + aq = \phi_2(y + ax).$...(8)

Here ϕ_1 and ϕ_2 are arbitrary functions.

Solving (7) and (8) for p and q, we have

$p = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\} \quad$ and $\quad q = (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}.$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\}dx + (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$
$$= (1/2a) \times \phi_2(y + ax)\,(dy + adx) - (1/2a) \times \phi_1(y - ax)(dy - adx)$$

Integrating, $\qquad z = \psi_2(y + ax) + \psi_1(y - ax)$, ψ_1, ψ_2 being arbitrary functions.

(*b*) This is a particular case of part (*a*). Here $a = 1$. **Ans.** $z = \psi_2(y + x) + \psi_1(y - x)$.

(*c*) Refer part (*a*). Note that $\partial^2 y/\partial x^2 = r$ and $\partial^2 y/\partial t^2 = t$

Ex. 2. *Solve* $r + (a + b)s + abt = xy$. **[Vikram 2003]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = a + b$, $T = ab$, $V = xy$. The usual Monge's subsidiary equations

$Rdpdy + Tqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$.

become $$dp\, dy + a\, b\, dq\, dx - xy\, dx\, dy = 0 \qquad ...(1)$$

and $$(dy)^2 - (a + b)\, dxdy + ab\, (dx)^2 = 0. \qquad ...(2)$$

Factorizing, (2) gives $(dy - bdx)\,(dy - adx) = 0$.

Hence two systems to be considered are

$$dp\, dy + ab\, dq\, dx - xy\, dx\, dy = 0, \qquad dy - b\, dx = 0. \qquad ...(3)$$

and $$dp\, dy + ab\, dq\, dx - xy\, dx\, dy = 0, \qquad dy - a\, dx = 0. \qquad ...(4)$$

Integrating the second equation of (3), $y - bx = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp + a\, dq - xy\, dx = 0$ or $dp + a\, dq - x(c_1 + bx)\, dx = 0$, by (5) ...(6)

Integrating (6), $p + aq - (c_1/2)x^2 - (b/3)x^3 = c_2$ or $p + aq - (x^2/2)\,(y - bx) - (b/3)x^3 = c_2$, using (5)

or $$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = c_2. \qquad ...(7)$$

Using (5) and (7), the first intermediate integral corresponding to (3) is

$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = \phi_1(y - bx)$, ϕ_1 being an arbitrary function ...(8)

Similarly, another intermediate integral corresponding to (4) is

$p + bq - (1/2) \times yx^2 + (1/6) \times ax^3 = \phi_2(y - ax)$, ϕ_2 being an arbitrary functionJ ...(9)

Solving (8) and (9) for p and q, we have

$$p = (1/2) \times x^2y - (1/6) \times (a + b)x^3 + (a - b)^{-1}\, [a\phi_2(y - ax) - b\phi_1(y - ax)]$$

and $$q = (1/6) \times x^3 + (a - b)^{-1}\, [\phi_1(y - bx) - \phi_2(y - ax)].$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = (1/2) \times x^2ydx - (1/6) \times (a + b)x^3dx + (a - b)^{-1}\, [\phi_2(y - bx)dx - \phi_1(y - ax)dx]$$
$$+ (1/6) \times x^3dy + (a - b)^{-1}\, [\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times (3x^2ydx + x^3dy) - (1/6) \times (a + b)\, x^3dx - (b - a)^{-1}\, [\phi_2(y - bx)dx$$
$$- \phi_1(y - ax)dx] - (b - a)^{-1}\, [(\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\, x^3dx + (b - a)^{-1}\phi_2(y - ax)\,(dy - adx)$$
$$- (b - a)^{-1}\phi_1(y - bx)\,(dy - bdx)$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\, x^3dx + (b - a)^{-1}\, \phi_2(y - ax)\, d(y - ax)$$
$$- (b - a)^{-1}\, \phi_1(y - bx)\, d(y - bx)$$

Integrating, $$z = (1/6) \times x^3y - (1/24) \times (a + b)x^4 + \psi_2(y - ax) + \psi_1(y - bx),$$

where ψ_1 and ψ_2 are arbitrary functions.

Ex. 3. *Solve* $r - t\cos^2 x + p\tan x = 0$. **[K.U. Kurukshetra 2005; Meerut 1993]**

Sol. Given $$r - t\cos^2 x = - p\tan x \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -p\tan x$. ...(2)

Monge's subsidiary equations are $Rdp\, dy + Tdq\, dx - V\, dx\, dy = 0$ ...(3)

and $$R\,(dy)^2 - S\, dx\, dy + T\,(dx)^2 = 0 \qquad ...(4)$$

Putting the values of R, S, T and V, (3) and (4) become

$$dp\, dy - \cos^2 x\, dq\, dx + p\tan x\, dx\, dy = 0 \qquad ...(5)$$

and $$(dy)^2 - \cos^2 x\,(dx)^2 = 0 \qquad ...(6)$$

Equation (6) may be factoriesed as $(dy - \cos x\, dx)\,(dy + \cos x\, dx) = 0$

$\therefore$ $$dy - \cos x\, dx = 0 \qquad ...(7)$$
or $$dy + \cos x\, dx = 0 \qquad ...(8)$$

Putting the value of dy from (7) in (5), we get

$dp \cos x\, dx - \cos^2 x\, dq\, dx + p \tan x\, dx \cos x\, dx = 0$ or $dp - \cos x\, dq + p \tan x\, dx = 0$

or $\sec x\, dp + p \sec x \tan x\, dx - dq = 0$ or $d\,(p \sec x) - dq = 0.$

Integrating it, $p \sec - q = c_1$, c_1 being an arbitrary constant ....(9)

Integrating (7), $y - \sin x = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p \sec x - q = f(y - \sin x).$ ...(11)

In a similar manner, (8) and (5) give another integral of (1)
$$p \sec x + q = g(y + \sin x). \qquad ...(12)$$

Solving (11) and (12) for p and q, we find

$p = (f + g)/2 \sec x = (1/2) \times (f + g) \cos x$ and $q = (g - f)/2$ ...(13)

Now, $dz = p\, dx + q\, dy$ or $dz = (1/2) \times (f + q) \cos x\, dx + (1/2) \times (g - f)\, dy$, by (13)

or $dz = -(1/2) \times f(y - \sin x)\,(dy - \cos x\, dx) + (1/2) \times g(y + \sin x)\,(dy + \cos x\, dx)$

Integrating, $z = F(y - \sin x) + G(y + \sin x)$, F and G being arbitrary functions.

Ex. 4. *Solve* $t - r \sec^4 y = 2q \tan y$. **[Delhi Maths Hons 1995; Kanpur 1995; Meerut 1995]**

Sol. Given $$t - r \sec^4 y = 2q \tan y. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = -\sec^4 y$, $S = 0$, $T = 1$, $V = 2q \tan y.$ ...(2)

Monge's subsidiary equations are $$R dp\, dy + T\, dq\, dx - V\, dx\, dy = 0 \qquad ...(3)$$
and $$R(dy)^2 - S\, dxdy + T\,(dx)^2 = 0 \qquad ...(4)$$

Putting the values of R, S, T and V, (3) and (4) become
$$-\sec^4 y\, dp\, dy + dq\, dx - 2q \tan y\, dx\, dy = 0 \qquad ...(5)$$
and $$-\sec^4 y\,(dy)^2 + (dx)^2 = 0. \qquad ...(6)$$

Equation (6) may be factorised as $(dx - \sec^2 y\, dy)\,(dx + \sec^2 y\, dy) = 0$ so that
$$dx - \sec^2 y\, dy = 0 \qquad ...(7)$$
or $$dx + \sec^2 y\, dy = 0. \qquad ...(8)$$

Putting the value of dx from (7) in (5), we get

$-\sec^4 y\, dp\, dy + dq \sec^2 y\, dy - 2q \tan y\, dy \times \sec^2 y\, dy = 0$ or $-dp + \cos^2 y\, dq - 2q \sin y \cos y\, dy = 0$

or $dp - (\cos^2 x\, dq - q \times 2 \sin y \cos y\, dy) = 0$ or $dp - d(q \cos^2 y) = 0.$

Integrating it, $p - q \cos^2 y = c_1$, c_1 being an arbitrary constant ...(9)

Integrating (7), $x - \tan y = c_2$, being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p - q \cos^2 y = f(x - \tan y).$ ...(11)

Similarly, from (8) and (5) the other integral of (1) is $p + q \cos^2 y = g(x + \tan y).$ ...(12)

Solving (11) and (12) for p and q, we find

$p = (f + g)/2$ and $q = (g - f)/(2 \cos^2 y) = (1/2) \times (g - f) \times \sec^2 y$ ...(13)

Now, we have $$dz = pdx + qdy$$
or $dz = (1/2) \times (f + g)dx + (1/2) \times (g - f) \times \sec^2 y\, dy$, using (13)

or $dz = (1/2) \times f(x - \tan y)\,(dx - \sec^2 y\, dy) + (1/2) \times g(x + \tan y)\,(dx + \sec^2 y\, dy)$

or $dz = (1/2) \times f(x - \tan y)\, d(x - \tan y) + (1/2) \times g(x + \tan y)\, d(x + \tan y).$

Integrating, $z = F(x - \tan y) + G(x + \tan y)$, F, G being arbitrary functions.

Ex. 5. *Solve* $q(yq + z)r - p\,(2yq + z)s + yp^2 t + p^2 q = 0.$ **[Delhi 2008]**

Sol. As usual, here Monge's subsidiary equations are
$$q(yq + z)dp\, dy + yp^2 dqdx + p^2 qdxdy = 0 \qquad ...\ (1)$$
and $$q(yq + z)(dy)^2 + p(2yq + z)dxdy + yp^2(dx)^2 = 0. \qquad ...\ (2)$$

On factorization, (2) gives $(qdy + pdx)\ \{(yq + z)dy + ypdx\} = 0.$

Hence two systems to be considered are

$$q(yq + z)dpdy + yp^2dqdx + p^2qdxdy = 0, \qquad qdy + pdx = 0 \qquad \ldots (3)$$

and $$q(yq + z)dpdy + yp^2dqdx + p^2q\ dxdy = 0, \qquad (yq + z)dy + ypdx = 0 \ldots (4)$$

Using $dz = pdx + qdy$, the second equation of (3) reduces to

$dz = 0$ so that $z = c_1$. ... (5)

From second equation of (3), $qdy = -pdx$. Hence first equation of (3) reduces to

$(yq + z)dp - ypdq - pqdy = 0$ or $(yq + z)dp - p\ d(yq) = 0$

or $(yq + z)dp - pd(yq + z) = 0,$ as $dz = 0$, by (5)

or $\dfrac{d(yq+z)}{yq+z} - \dfrac{dp}{p} = 0$ so that $\log (yq + z) - \log p = \log c_1$

or $(yq + z)/p = c_2$, c_2 being an arbitrary constant ... (6)

From (5) and (6), the intermediate integral corresponding to (3) is

$(yq + z)/p = \phi_1(z)$ or $yq + z = p\phi_1(z)$, ...(7)

where ϕ_1 is an arbitrary function.

Using $dz = pdx + qdy$, the second equation of (4) becomes

$y(qdy + pdx) + zdy = 0$ or $ydz + zdy = 0$ or $d(yz) = 0.$

Integrating it, $yz = c_3$, c_3 being an arbitrary constant ...(8)

From second equation of (4), $(yq + z)dy = -ypdx.$

Using this fact, first equation of (4) reduces to

$qdp - pdq - (pq/y)dy = 0$ or $-(1/p)dp + (1/q)dq + (1/y)dy = 0.$

Integrating, $-\log p + \log q + \log y = \log c_1$ or $(yq)/p = c_2$ ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$(qy)/p = \phi_2(yz)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have $p = \dfrac{z}{\phi_1(z) - \phi_2(yz)}$, $q = \dfrac{z\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}$.

Substituting these in $dz = pdx + qdy$, $dz = \dfrac{z}{\phi_1(z) - \phi_2(yz)}\{dx + (1/y) \times \phi_2(yz)\ dy\}$

or $\phi_1(z)dz = zdx + \phi_2(yz)\dfrac{zdy + ydz}{y}$ or $\dfrac{\phi_1(z)\,dz}{z} = dx + \dfrac{\phi_1(yz)\,d(yz)}{yz}$.

Integrating, $\psi_1(z) = x + \psi_2(yz)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py.$ **[Delhi Maths 2005, Kurukshetra 2005 (H)]**

Sol. Usual Monge's auxiliary equations are

$$xydpdy - xydqdx - (qx - py)dxdy = 0 \qquad \ldots(1)$$

and $$xy(dy)^2 + (x^2 - y^2)\ dxdy - xy\ (dx)^2 = 0. \qquad \ldots(2)$$

On factorizing, (2)gives $(xdy - ydx)\ (ydx + xdy) = 0.$

Hence, two systems to be considered are

$$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad xdy - ydx = 0 \qquad \ldots(3)$$

and $$xydpdy - xydqdx - (qx - py)\ dxdy = 0, \qquad ydx + xdy = 0. \qquad \ldots(4)$$

Second equation of (3) gives $y/z = c_1$, c_1 being an arbitrary constant ...(5)

Using second equation, first equation of (3) reduces to

$ydp - xdq - qdx + pdy = 0$ or $d(yp - xq) = 0$

Integrating, $yp - xq = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), intermediate integral corresponding to (3) is

$yp - xq = \phi_1(y/x)$, where ϕ_1 is an arbitrary function. ...(7)

Second equation of (4) gives $x^2 + y^2 = c_3$, c_3 being arbitrary constant ...(8)

Using second equation, first equation of (4) reduces to

$xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$

Integrating, $xp + yq = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$xp + yq = \phi_2(x^2 + y^2)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x^2+y^2}\left\{y\phi_1\left(\frac{y}{x}\right)+x\phi_2(x^2+y^2)\right\} \quad \text{and} \quad q = \frac{1}{x^2+y^2}\left\{y\phi_2(x^2+y^2)-x\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = \frac{1}{x^2+y^2}\left[\left\{y\phi_1\left(\frac{y}{x}\right)+x\phi_2(x^2+y^2)\right\}dx+\left\{y\phi_2(x^2+y^2)-x\phi_1\left(\frac{y}{x}\right)dy\right\}\right]$$

or $$dz = \frac{ydx-xdy}{x^2+y^2}\phi_1\left(\frac{y}{x}\right)+\frac{xdx+ydy}{x^2+y^2}\phi_2(x^2+y^2) \quad \text{or} \quad dz = -\frac{\phi_1(y/x)}{1+(y/x)^2}d\left(\frac{y}{x}\right)+\frac{1}{2}\frac{\phi_2(x^2+y^2)}{x^2+y^2}d(x^2+y^2).$$

Integrating, $z = \psi_1(y/x) + \psi_2(x^2 + y^2)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 7. *Solve* $(r - s)\, x = (t - s)\, y$. **(M.D.U Rohtak 2005)**

Sol. Usual Monge's subsidiary equations are $xdpdy - ydqdx = 0$ ...(1)

and $x(dy)^2 + (x - y)\, dxdy - y(dx)^2 = 0.$ ...(2)

Factorising, (2) $\Rightarrow$ $(xdy - ydx)\,(dy + dx) = 0.$

Hence two systems to be considered are

$xdpdy - ydqdx = 0,\quad xdy - ydx = 0$ ...(3)

and $xdpdy - ydqdx = 0,\quad dy + dx = 0.$ ...(4)

Integrating second equation of (3), $y/x = c_1$, c_1 being an arbitrary constant ...(5)

Eliminating dy/dx between equations of (3), we get

$dp - dq = 0$ so that $p - q = c_2$, c_2 being an arbitrary constant ...(6)

Hence the intermediate integral corresponding to (3) is $p - q = \phi_1(y/x)$. ...(7)

Integrating second equation of (4), $x + y = c_3$, c_3 being an arbitrary constant ...(8)

Eliminating dy/dx between equations of (4), we get

$xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0,$ as $dz = pdx + qdy.$

Integrating, $xp + yq - z = c_4$, c_4 being an arbitrary constant ...(9)

Hence the intermediate integral corresponding to (4) is

$xp + yq - z = \phi_2(x + y)$ or $xp + yq = z + \phi_2(x + y)$, ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x+y}\left\{z+\phi_2(x+y)+y\phi_1\left(\frac{y}{x}\right)\right\} \quad \text{and} \quad q = \frac{1}{x+y}\left\{z+\phi_2(x+y)-x\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{1}{x+y}\left[\left\{z+\phi_2(x+y)+y\phi_1\left(\frac{y}{x}\right)\right\}dx+\left\{z+\phi_2(x+y)-x\phi_1\left(\frac{y}{x}\right)\right\}dy\right]$$

$$\Rightarrow \quad \frac{(x+y)\,dx-zdx}{(x+y)^2} = \frac{\phi_2(x+y)\,d\,(x+y)}{(x+y)^2}+\frac{(ydx-xdy)\,\phi_1(y/x)}{(x+y)^2}$$

$$\Rightarrow \qquad d\left(\frac{z}{x+y}\right) = \frac{\phi_2(x+y)}{(x+y)^2}\,d(x+y) - \frac{\phi_1(y/x)}{1+(y/x)^2}\,d\left(\frac{y}{x}\right).$$

Integrating, $z/(x+y) = \psi_2(x+y) + \psi_1(y/x)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 8. *Solve* $r + ka^2t - 2as = 0$.

Sol. Given $$r - 2as + ka^2t = 0. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we have $R = 1$, $S = -2a$, $T = ka^2$, $V = 0$.

Hence the Monge's subsidiary equations

$Rdp\,dy + Tdq\,dx - Vdx\,dy = 0$ and $R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$

become $$dp\,dy + ka^2\,dq\,dx = 0 \qquad ...(2)$$

and $$(dy)^2 + 2a\,dx\,dy + ka^2\,(dx)^2 = 0. \qquad ...(3)$$

From (3), $dy = [-2a\,dx \pm \{4a^2(dx)^2 - 4ka^2\,(dx)^2\}^{1/2}]/2 = -a\,dx \pm a\sqrt{(1-k)}\,dx$

or $dy + a\,\{1 \pm \sqrt{(1-k)}\}dx = 0$ or $dy + a\,(1 \pm l)\,dx = 0$, where $l = \sqrt{(1-k)}$.

Hence (3) reduces to the following two equations :

$$dy + a(1 + l)dx = 0 \qquad ...(4)$$

and $$dy + a(1 - l)dx = 0. \qquad ...(5)$$

From (2) and (4) , eliminating dy, we have

$dp\{-a\,(1 + l)\,dx\} + ka^2\,dqdx = 0$ or $(1 + l)dp - ka\,dq = 0$.

Integrating it, $(1 + l)p - kaq = c_1$, c_1 being an arbitrary constant ...(6)

Again, integrating (4), $y + a(1 + l)x = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), first intermediate integral is

$(1 + l)p - kaq = f_1\{y + a(1 + l)x\}$, where f_1 is an arbitrary function. ...(8)

Similary, from (2) and (5), second intermediate intgegral is given by (replacing l by $-l$ in (8) since (5) differs from (4) in having $-l$ in place of l)

$(1 - l)p - kaq = f_2\{y + a(1 - l)x\}$, where f_2 is an arbitrary function ...(9)

Solving (8) and (9) for p and q, $p = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]$

and $q = (1/2akl) \times [(1 - l)f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx + (1/2akl) \times [(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy$$

or $dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx$

$+ \dfrac{1}{2al(1-l^2)}\,[\,(1 - l)\,f_1\{y + a(1 + l)x\} - (1 + l)\,f_2\{y + a(1 - l)x\}]dy$, as $l = (1 - k)^{1/2} \Rightarrow k = 1 - l^2$

or $dz = (1/2l)\,[dx f_1\,\{y + a(1 + l)x\} - dx f_2\{y + a(1 - l)x\}] + \dfrac{1}{2al}\left[\dfrac{dy}{1+l} f_1\{y + a(1+l)x\} - \dfrac{dy}{1-l} f_2\{y + (1-l)x\}\right]$

$= \dfrac{1}{2al(l+1)} f_1\{y + a(1 + l)x\}\{dy + a(1 + l)dx\} - \dfrac{1}{2al(1-l)}\,f_2\{y + a(1 - l)x\}\,\{dy + a(1 - l)dx\}$

or $dz = \dfrac{1}{2al(l+1)}\,f_1\{y + a(1 + l)x\}\,d\{y + a(1 + l)x\} - \dfrac{1}{2al(1-l)} f_2\{y + a(1 - l)x\}\,d\{y + a\,(1 - l)x\}$.

Integrating, $z = F_1\{y + a(1 + l)x\} + F_2\{y + a\,(1 - l)x\}$, where F_1 and F_2 are arbitrary functions.

Ex. 9. *Solve* $x^{-2}r - y^{-2}t = x^{-3}p - y^{-3}q$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^{-2}$, $S = 0$, $T = y^{-2}$, $V = x^{-3}p - y^{-3}q$. Then Monge's subsidiary equations

$Rdpdy + Tdqdx + Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^{-2}dpdy + y^{-2}dqdx - (x^{-3}p - y^{-3}q)\, dxdy = 0. \qquad ...(1)$$

and $$x^{-2}(dy)^2 - y^{-2}(dx)^2 = 0. \qquad ...(2)$$

Multiplying both sides of (1) by x^3y^3, we get

$$xy^3dpdy - x^3ydqdx - py^3dxdy + qx^3dxdy = 0. \qquad ...(3)$$

Again, (2) $\Rightarrow$ $x^2y^2(y^2dy^2 - x^2dx^2) = 0$ or $x^2y^2(ydy + xdx)\,(ydy - xdx) = 0$

Hence (2) is equivalent to the equations

$ydy + xdx = 0$ *i.e.*, $ydy = -xdx$ ...(4)

and $ydy - xdx = 0$. ...(5)

Integrating (4), $y^2/2 + x^2/2 = c_1/2$ or $x^2 + y^2 = c_1$. ...(6)

From (3), $xy^2dp(ydy) - x^2ydq(xdx) - py^2dx(ydy) + qx^2dy(xdx) = 0$

or $xy^2dp(-xdx) - x^2ydq(xdx) - py^2dx\,(-xdx) + qx^2dy(xdx) = 0$, using (4)

or $-xy^2dp - x^2ydq + py^2dx + qx^2dy = 0$ or $y^2(xdp - pdx) + x^2(ydq - qdy) = 0$

or $$\frac{xdp - pdx}{x^2} + \frac{ydq - qdy}{y^2} = 0 \quad \text{or} \quad d\left(\frac{p}{x}\right) + d\left(\frac{q}{y}\right) = 0.$$

Integrating, $(p/x) + (q/y) = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is

$(1/x)p + (1/y)q = f(x^2 + y^2)$, where f is an arbitrary function. ...(8)

Similarly, from (3) and (5), another intermediate integral is

$(1/x)p - (1/y)q = g(x^2 - y^2)$, where g is an arbitrary function ...(9)

Solving (8) and (9) for p and q, we obtain

$p = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}$ and $q = (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}dx + (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}dy$$

or $$dz = (1/4) \times f(x^2 + y^2)\,(2xdx + 2ydy) + (1/4) \times g(x^2 - y^2)\,(2xdx - 2ydy) \qquad ...(10)$$

Putting $x^2 + y^2 = u$, $x^2 - y^2 = v$ so that $2xdx + 2ydy = du$ and $2xdx - 2ydy = dv$, (10) gives

$$dz = (1/4) \times f(u)\, du + (1/4) \times g(v)dv, \qquad ...(11)$$

Integrating (11), $z = F(u) + G(v) = F(x^2 + y^2) + G(x^2 - y^2)$,

where F and G are arbitrary functions.

Ex. 10. *Solve* $rx^2 - 3s\,xy + 2t\,y^2 + px + 2qy = x + 2y$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = -3xy$, $T = 2y^2$, $V = x + 2y - px - 2qy$.

Hence Monge's subsidiary equations are

$Rdpdy + Tdqdy - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2\, dpdy + 2y^2dqdx - (x + 2y - px - 2qy)\, dxdy = 0 \qquad ...(1)$$

and $$x^2(dy)^2 + 3xy\, dxdy + 2y^2(dx)^2 = 0. \qquad ...(2)$$

Here (2) $\Rightarrow$ $(xdy + 2ydx)\,(xdy + ydx) = 0$.

Hence (2) resolves into the following two equations

$xdy + 2ydx = 0$ *i.e.*, $2ydx = -xdy$ ...(3)

and $xdy + ydx = 0$. ...(4)

Re–writing (3), $(1/y)dy + 2(1/x)dx = 0$

Integrating, $\log y + 2 \log x = \log c_1$ or $yx^2 = c_1$. ...(5)

Re–writing (1), $(xdp)(xdy) + ydq(2ydx) - dx(xdy) - dy(2ydx) + pdx(xdy) + qdy(2ydx) = 0$

or $(xdp)(xdy) + ydq(-xdy) - dx(xdy) - dy(-xdy) + pdx(xdy) + qdy(-xdy) = 0$, using (3)

or $xdp - ydq - dx + dy + pdx - qdy = 0$

or $(xdp + pdx) - (ydq + qdy) - dx + dy = 0$ or $d(xp) - d(yq) - dx + dy = 0$.

Integrating, $xp - yq - x + y = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$xp - yq - x + y = f(x^2y)$, where f is an arbitrary function. ...(7)

Similarly from (1) and (4), another intermediate integral is

$xp - 2yq - x + 2y = g(xy)$, where g is an arbitrary function. ...(8)

Solving (7) and (8) for p and q, we have

$p = (1/x) \times \{x + 2f(x^2y) - g(xy)\}$, and $q = (1/y) \times \{y + f(x^2y) - g(xy)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/x) \times \{x + 2f(x^2y) - g(xy)\}dx + (1/y) \times \{y + f(x^2y) - g(xy)\}]dy$$

or $$dz = dx + dy + f(x^2y)\left(\frac{2}{x}dx + \frac{1}{y}dy\right) - g(xy)\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

or $$dz = dx + dy + f(x^2y)\, d[\log(x^2y)] - g(xy)\, d[\log(xy)].$$

Integrating, $z = x + y + F(x^2y) + G(xy)$, G, and F being arbitrary functions.

Ex. 11. *Find the general solution of the equation $r + 4t = 8xy$, by Monge's method. Find also the particular solution for which $z = y^2$ and $p = 0$, when $x = 0$* **[Delhi Maths (Hons) 2006, 09]**

Sol. Given $r + 4t = 8xy$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 1$, $S = 0$, $T = 4$ and $V = 8xy$. Hence Monge's subsidians equations $Rdp\,dy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - Sdx\,dy + T(dx)^2 = 0$ become

$$dpdy + 4\,dqdx - 8xydxdy = 0 \qquad \text{... (2)}$$

and $$(dy)^2 + 4(dx)^2 = 0 \qquad \text{... (3)}$$

Re–writing (3), $dy^2 - 4i^2dx^2 = 0$ or $(dy - 2idx)(dy + 2idx) = 0$

so that $dy - 2idx = 0$ or $dy = 2idx$ ... (4)

and $dy + 2idx = 0$ or $dy = -2idx$ ... (5)

We first consider (4) and (2). Integrating (4), $y - 2ix = C_1$ ... (6)

Using (4) and (6), (2) gives $dp\,(2i\,dx) + 4\,dq\,dx - 8x(C_1 + 2ix)\,(2i\,dx)\,dx = 0$

or $i\,dp + 2dq - 8xi\,(C_1 + 2ix) = 0$, by (6) or $idp + 2dq - 8C_1ix\,dx + 16x^2dx = 0$

Integrating, $ip + 2q - 4C_1ix^2 + (16/3)x^3 = C_2$, C_2 being an arbitrary constant

or $$ip + 2q - 4ix^2(y - 2ix) + (16/3) \times x^3 = C_2\text{, by (6)} \qquad \text{... (7)}$$

From (6) and (7) first intermediate integral of (1) is $ip + 2q - 4ix^2(y - 2ix) + (16/3)x^3 = f(y - 2ix)$

or $ip + 2q = (8/3) \times x^3 + 4ix^2y + f(y - 2ix)$, f being an abritrary function ... (8)

Similarly considering the pair (5) and (2), the second intermediate integral of (1) is

$$ip - 2q = -(8/3) \times x^3 + 4ix^2 y + g\ (y + 2ix),\ g \text{ being an arbitrary function} \quad \text{... (9)}$$

Solving (8) and (9) for p and q, $\quad p = \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}/2i$

and $\quad q = \{(16/3) \times x^3 + f(y - 2ix) - g(y + 2ix)\}/4$

Putting the above values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2i) \times \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}dx + (1/4) \times \{(16/3) \times x^3 + f(y - 2ix) - g(y - 2ix)\}dy$$

$$= (4/3) \times (3x^2 ydx + x^3 dy) + (1/4) \times f(y - 2ix)d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

$$\therefore \quad dz = (4/3) \times d(x^3 y) + (1/4) \times f(y - 2ix)\ d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

Integrating, $\quad z = (4/3) \times x^3 y + F(y - 2ix) + G\ (y + 2ix), \quad$ ... (10)

which is the general solution of (1) containing F and G as arbitrary functions

To find particular solution of (1) Given conditions are

$z = y^2 \quad$ and $\quad p = \partial z/\partial x = 0 \quad$ when $x = 0 \quad$ ... (11)

From (11), $\quad \partial z/\partial y = 2y \quad$ when $\quad x = 0 \quad$ ... (12)

Differentiating (10) partially w.r.t. 'x' and 'y', we get

$$\partial z/\partial x = 4x^2 y - 2i\ F'(y - 2ix) + 2i\ G'(y + 2ix) \quad \text{... (13)}$$

and $\quad \partial z/\partial y = (4/3) \times x^3 + F'(y - 2ix) + G'(y + 2ix) \quad$ ... (14)

Using (11) and (12), (10), (13) and (14) reduce to

$$F(y) + G(y) = y^2 \quad \text{... (15)}$$

$$F'(y) - G'(y) = 0 \quad \text{... (16)}$$

and $\quad F'(y) + G'(y) = 2y \quad$ ... (17)

From (16) and (17), $\quad F'(y) = y \quad$ and $\quad G'(y) = y$

Integrating these, $\quad F(y) = y^2/2 \quad$ and $\quad G(y) = y^2/2 \quad$ ... (18)

which also satisfy (15).

From (18), $\quad F(y - 2ix) = (y - 2ix)^2/2 \quad$ and $\quad G(y + 2ix) = (y + 2ix)^2/2$

Putting these values in (10), the required particular solution is

$z = (4/3) \times x^3 y + (y - 2ix)^2/2 + (y + 2ix)^2/2 \quad$ or $\quad z = (4/3) \times x^3 y + y^2 - 4x^2.$

9.5. Type 2. When the given equation $Rr + Ss + Tt = V$ leads to two distinct intermediate integrals and only one is employed to get the desired solution.

Working rule for solving problems of type 2.

Step 1. Write the given equation in the standard form $\quad Rr + Ss + Tt = V.$

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0 \quad$ ... (1) $\quad R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Take one of the factors of step 3 and use (2) to get an intermediate integral. Don't find second intermediate integral as we did in type 1. If required use remark 1 of Art. 9.2.

Step 5. Re–write the intermediate integral of the step 4 in the form of Lagrange equation, namely, $Pp + Qq = R$ (refer chapter 2). Using the well known Lagrange's method we arrive at the desired general solution of the given equation.

9.6 SOLVED EXAMPLES BASED ON ART. 9.5.

Ex. 1. *Solve* $(r - s)y + (s - t)x + q - p = 0$.

Sol. The given can be written as $\qquad yr + s(x - y) - tx = p - q. \qquad ...(1)$

Comparing (1) with $Rr + Ss + Tt = V$, $R = y$, $S = x - y$, $T = -x$ and $V = p - q$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become
$$ydpdy - xdqdx + (q - p)dxdy = 0 \qquad ...(1)$$
and
$$y(dy)^2 - (x - y)\, dxdy - x\,(dx)^2 = 0. \qquad ...(2)$$

Re–writing (2), $\qquad (dy + dx)\,(ydy - xdx) = 0.$

so that $\quad dy + dx = 0 \quad$ or $\quad dy = -\,dx \qquad ...(3)$

and $\qquad ydy - xdx = 0. \qquad ...(4)$

Using (3), (1) becomes $\qquad -\,ydpdx - xdqdx + q\,dx(-\,dx) - p\,dxdy = 0$

or $\quad ydp + xdq + qdx + pdy = 0 \quad$ or $\quad (ydp + pdy) + (xdq + qdx) = 0$

or $\quad d(yp) + d(xq) = 0 \quad$ so that $\quad yp + xq = c_1. \qquad ...(5)$

Integrating (3), $\quad x + y = c_2$, c_2 being an arbitrary constant $\qquad ...(6)$

From (5) and (6), one intermediate integral is $\qquad yp + xq = f(x + y), \qquad ...(7)$

which is of the Lagrange's form and so its subsidiary equations are
$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x+y)}. \qquad ...(8)$$

From first and second fractions of (8), $\qquad 2xdx - 2ydy = 0.$

Integrating, $\quad x^2 - y^2 = a$, a being an arbitrary constant $\qquad ...(9)$

Taking first and third fractions of (8), we get
$$\frac{dx}{y} = \frac{dz}{f(x+y)} \quad \text{or} \quad \frac{dx}{(x^2-a)^{1/2}} = \frac{dz}{f\left[x+(x^2-a)^{1/2}\right]}, \text{ as (9)} \quad \Rightarrow y = (x^2 - a)^{1/2}$$

or
$$dz = f[x + (x^2 - a^2)^{1/2}]\,(x^2 - a^2)^{-1/2}dx \qquad ...(10)$$

Put $\quad x + (x^2 - a)^{1/2} = v \quad$ so that $\quad \left[1 + x/(x^2 - a)^{1/2}\right]dx = dv \qquad ...(11)$

or $\quad \dfrac{x+(x^2-a)^{1/2}}{(x^2-a)^{1/2}}\,dx = dv \quad$ or $\quad \dfrac{dx}{(x^2-a)^{1/2}} = \dfrac{dv}{v}$, using (11)

Then, (10) reduces to $\qquad dz - (1/v)\,f(v)dv = 0.$

Integrating, $\quad z - F(v) = b \quad$ or $\quad z - F[x + (x^2 - a)^{1/2}] = b$, by (11)

or $\quad z - F(x + y) = b, \quad$ as $\quad y = (x^2 - a)^{1/2}$, by (9) $\qquad ...(12)$

From (9) and (12), the required general solution is $\qquad z - F(x + y) = G(x^2 - y^2)$

or $\quad z = F(x + y) + G(x^2 - y^2)$, where F and G are arbitrary functions.

Ex. 2. *Solve* : $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0$. **[Meerut 1994; I.A.S. 1974]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we find
$$R = q(1 + q), \quad S = -\,(p + q + 2pq), \quad T = p(1 + p), \quad V = 0 \qquad ...(1)$$

Monge's subsidiary equations are $\qquad Rdpdy + Tdq\,dx - Vdxdy = 0 \qquad ...(2)$

and
$$R(dy)^2 - Sdxdy + T(dx)^2 = 0 \qquad ...(3)$$

Using (1), (2) and (3) become $\qquad (q + q^2)dpdy + (p + p^2)dqdx = 0 \qquad ...(4)$

and
$$(q + q^2)\,(dy)^2 + (p + q + 2pq)dxdy + (p + p^2)\,(dx)^2 = 0. \qquad ...(5)$$

In order to factorise (5), we re–write it as

$$q(1+q)(dy)^2 + (p+pq)dxdy + (q+pq)dxdy + p(1+p)(dx)^2 = 0$$

or
$$q(1+q)(dy)^2 + p(1+q)dxdy + q(1+p)dxdy + p(1+p)(dx)^2 = 0$$

or
$$(1+q)dy(qdy+pdx) + (1+p)dx(qdy+pdx) = 0$$

or
$$(qdy+pdx)\,[(1+q)dy + (1+p)dx] = 0. \qquad \text{... (6)}$$

Then, from (6), we get $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(7)

and
$$(1+q)dy + (1+p)dx = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (4) may be re–written as $(1+q)dp\,(qdy) - (1+p)dq\,(-pdx) = 0$

From (7), qdy and $(-pdx)$ are equivalent. Hence dividing each term of the above equation by qdy, or its equivalent $(-pdx)$, we get

$(1+q)dp - (1+p)dq = 0$ or $dp/(1+p) - dq/(1+q) = 0.$

Integrating it, $\log(1+p) - \log(1+q) = \log c_1$ or $(1+p)/(1+q) = c_1.$ ...(9)

Using $dz = pdx + qdy$, (7) becomes $dz = 0$ so that $z = c_2.$ ...(10)

From (9) and (10), one intermediate integral of (1) is given by

$(1+p)/(1+q) = f(z)$ or $p - f(z)q = f(z) - 1,$ ...(11)

which is of the form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (11) are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{f(z)-1}. \qquad \text{...(12)}$$

Choosing 1, 1, 1 as multipliers, each fraction in (12) $= \dfrac{dx+dy+dz}{1-f(z)+f(z)-1} = \dfrac{dx+dy+dz}{0}$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_2.$ ...(13)

From first and third fractions in (12), we get $dx - [f(z) - 1]^{-1}\,dz = 0.$

Integrating it, $x + F(z) = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$$x + F(z) = G(x+y+z),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 3. *Solve* $(x-y)(xr - xs - ys + yt) = (x+y)(p-q).$

[Delhi Maths (H) 97, 2000; Meerut 1999; Garhwal 1996]

Sol. Given $(x-y)xr - (x^2-y^2)s + (x-y)yt = (x+y)(p-q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = x(x-y),$ $S = -(x^2-y^2),$ $T = y(x-y),$ $V = (x+y)(p-q).$ ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and
$$R(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad \text{...(4)}$$

Using (2), (3) and (4) become

$$x(x-y)dpdy + y(x-y)dqdx - (x+y)(p-q)dxdy = 0 \qquad \text{...(5)}$$

and
$$x(x-y)(dy)^2 + (x^2-y^2)dxdy + y(x-y)(dx)^2 = 0. \qquad \text{...(6)}$$

Since $x^2 - y^2 = (x-y)(x+y)$, dividing (6) by $(x-y)$ gives

$xdy^2 + (x+y)dxdy + ydx^2 = 0$ or $(xdy + ydx)(dx + dy) = 0$

Thus we get $xdy + ydx = 0$ or $xdy = -\,ydx$ ...(7)

and
$$dx + dy = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (5) may be rewritten as

$$(x-y)dp\,(xdy) - (x-y)\,dq\,(-ydx) - (p-q)\,dx\,(xdy) + (p-q)dy(-\,ydx) = 0.$$

From (7), $x\,dy$ and $(-y\,dx)$ are equal. So dividing each term of the above equation by $x\,dy$, or its equivalent $(-y\,dx)$, we get

$(x-y)\,dp - (x-y)dq - (p-q)\,dx + (p-q)dy = 0$ or $(x-y)\,(dp-dq) - (p-q)\,(dx-dy) = 0$

or $\dfrac{dp-dq}{p-q} - \dfrac{dx-dy}{x-y} = 0$ so that $\dfrac{p-q}{x-y} = c_1$ ...(9)

Integrating (7), $xy = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (10) is

$(p - q)/(x - y) = f(xy)$ or $p - q = (x - y)f(xy)$ ...(11)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)\,f(xy)}. \qquad ...(12)$$

Taking the first two fractions of (12), we get

$dx + dy = 0$ so that $x + y = c_3$, c_3 being an arbitrary constant ...(13)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (12) = $\dfrac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{0}$

so that $f(xy) \times (ydx + x\,dy) + dz = 0$ or $f(xy) \times d(xy) + dz = 0$.

Integrating it, $F(xy) + z = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$F(xy) + z = G(x + y)$, where F and G are arbitrary functions.

Ex. 4. $xy\,(t - r) + (x^2 - y^2)\,(s - 2) = py - qx$. **[Delhi Maths (H) 2001]**

Sol. Given $-\,xyr + (x^2 - y^2)\,s + xyt = py - qx + 2(x^2 - y^2)$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = -xy$, $S = x^2 - y^2$, $T = xy$, $V = py - qx + 2(x^2 - y^2)$. ...(2)

Monge's subsidiary equations are $Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$. ...(4)

Using (2), (3) and (4) become

$$-\,xy\,dp\,dy + xydqdx - [py - qx + 2(x^2 - y^2)]dxdy = 0 \qquad ...(5)$$

and $$-\,xy\,(dy)^2 - (x^2 - y^2)dxdy + xy(dx)^2 = 0. \qquad ...(6)$$

From (6), $xy(dy)^2 + x^2dxdy - y^2dxdy - xy(dx)^2 = 0$

or $xdy(ydy + xdx) - ydx\,(ydy + xdx) = 0$ or $(xdy - ydx)\,(ydy + xdx) = 0$.

So, we get $xdx + ydy = 0$, *i.e.*, $xdx = -ydy$ ...(7)

and $xdy - ydx = 0$. ...(8)

Keeping (7) in view, (5) may be re–written as

$$xdp(-ydy) + ydq\,(xdx) + pdx\,(-ydy) + qdy(xdx) - 2xdy(xdx) - 2ydx(-ydy) = 0.$$

From (7), xdx and $(-ydy)$ are equivalent. So dividing each term of the above equation by xdx, or its equivalent $(-ydy)$, we get

$xdp + ydq + pdx + qdy - 2xdy - 2ydx = 0$ or $(xdp + pdx) + (ydq + qdy) - 2(xdy + ydx) = 0$.

Integrating it, $xp + yq - 2xy = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2$. ...(10)

From (9) and (10), one intgegral of (1) is

$xp + qy - 2xy = f(x^2 + y^2)$ or $xp + yq = 2xy + f(x^2 + y^2)$, ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations for (11)are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2xy + f(x^2 + y^2)}. \qquad ..(12)$$

Taking the first two fractions in (12), we get

$\log y - \log x = \log c_3$ or $y/x = c_3$ or $y = xc_3$ ...(13)

Taking the first and the last fractions in (12) and using $y = xc_3$ in it, we get

$dz = (1/x) \times [2c_3x^2 + f(x^2 + x^2c_3^{\,2})]dx$ or $dx = 2c_3xdx + (1/x) \times f\{(1 + c_3^{\,2})x^2\}dx$

or $$dz = 2c_3xdx + (1/2x^2) \times f\{(1 + c_3^{\,2})x^2\}d\,(x^2).$$

Integrating $z - 2c_3(x^2/2) + F\{(1 + c_3^{\,2})x^2\} = c_4$ or $z - (y/x)x^2 + F\{(1 + y^2/x^2)x^2\} = c_4$, by (13)

or $$z - xy + F(x^2 + y^2) = c_4,\ c_4 \text{ being an arbitrary constant} \qquad ...(14)$$

From (13) and (14), the required general solution is

$$z - xy + F(x^2 + y^2) = G(y/x), \text{ where } F \text{ and } G \text{ are arbitrary functions.}$$

Ex. 5. *Solve* $x^2r - y^2t - 2xp + 2z = 0$.

Sol. Given $$x^2r - y^2t = 2xp - 2z. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = x^2$, $S = 0$, $T = -y^2$, $V = 2xp - 2z$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2dpdy - y^2dqdx - (2xp - 2z)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0. \qquad ...(3)$$

On factorizing, $(3) \Rightarrow (xdy - ydx)\,(xdy + ydx) = 0$

Thus, we have $xdy - ydx = 0$ *i.e.*, $xdy = ydx$. ...(4)

and $$xdy + ydx = 0. \qquad ...(5)$$

Re–writing (2), $xdp\,(xdy) - ydq(ydx) - 2(xp - z)\,(xdy)(1/x)dx = 0$

or $xdp(xdy) - ydq(xdy) - 2(xp - z)\,(xdy)\,(1/x)dx = 0$, using (4)

or $$xdp - ydq - 2(xp - z)\,(1/x)dx = 0$$

or $xdp - dz + pdx + qdy - ydq - 2(xp - z)\,(1/x)dx = 0$ as $dz = pdx + qdy \Rightarrow -dz + pdx + qdy = 0$

or $$d(xp - z) - d(yq) + 2qdy - 2(xp - z)\,(1/x)dx = 0$$

or $d(xp - yq - z) + 2qy(1/x)dx - 2(xp - z)\,(1/x)dx = 0$, as from (4), $dy = (y/x)dx$

or $d(xp - yq - z) - 2\,(xp - yq - z)\,(1/x)\,dx = 0$ or $\dfrac{d(xp - yq - z)}{xp - yq - z} - \dfrac{2dx}{x} = 0.$

Integrating, $\log\,(xp - yq - z) - 2\log x = \log c_1$ or $(xp - yq - z)/x^2 = c_1$. ...(6)

From (4), $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = \log c_2$

or $$y/x = c_2,\ c_2 \text{ being an arbitrary constant} \qquad ...(7)$$

From (6) and (7), an intermediate integral is

$(xp - yq - z)\,/x^2 = \phi_1(y/x)$ or $xp - yq = z + x^2\phi_1(y/x)$. ...(8)

Lagrange's auxiliary equations for (8) are $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dz}{z + x^2\phi_1(y/x)}$. ...(9)

From the first two ratios of (9), we get

$(1/x)\,dx + (1/y)\,dy = 0$ so that $xy = c_3$. ...(10)

Taking the second and third ratios of (9), we get

$$\frac{dz}{dy} + \frac{z}{y} = -\frac{x^2}{y}\phi_1\left(\frac{y}{x}\right) = -\frac{c_3^2}{y^3}\phi_1\left(\frac{y^2}{c_3}\right), \text{ by (10)}$$

Its I.F $= e^{\int(1/y)dy} = y$ and so solution is $zy = -\displaystyle\int \frac{c_3^2}{y^2}\phi_1\left(\frac{y^2}{c_3^2}\right)dy + c_4$

or $zy + \dfrac{c_3^{3/2}}{2}\displaystyle\int\left(\frac{c_3}{y^2}\right)\phi_1\left(\frac{y^2}{c_3}\right)\left(\frac{\sqrt{c_3}}{y}\right)d\left(\frac{y^2}{c_3}\right) = c_4$ or $zy + c_3^{\,3/2}\,\psi_1\left(\dfrac{y^2}{c_3}\right) = c_4$

or $$zy + (xy)^{3/2}\,\psi_1(y/x) = c_4, \text{ using (10).} \qquad \text{...(11)}$$

From (10) and (11), the required general solution is

$zy + (xy)^{3/2}\,\psi_1(y/x) = \psi_2(xy)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$.

Sol. Given $$xyr - (x^2 - y^2)s - xyt = qx - py. \qquad \text{...(1)}$$

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$xy\,dpdy - xy\,dqdx - (qx - py)\,dxdy = 0 \qquad \text{...(2)}$$

and $$xy\,(dy)^2 + (x^2 - y^2)\,dxdy - xy\,(dx)^2 = 0. \qquad \text{...(3)}$$

Now, (3) $\Rightarrow$ $(xdx + ydy)\,(xdy - ydx) = 0$

Hence, $xdx + ydy = 0$ *i.e.*, $xdx = -\,ydy$ ...(4)

and $$xdy - ydx = 0 \qquad \text{...(5)}$$

Re–writing (2), $(xdp)\,(ydy) - ydq(xdx) - qdy(xdx) + pdx\,(ydy) = 0$

or $(xdp)\,(ydy) - ydq(-ydy) - qdy(-ydy) + pdx(ydy) = 0$, using (4)

or $xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0.$

Integrating, $xp + yq = c_1$, c_1 being an arbitrary constant ...(6)

Integrating (4) $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2$. ...(7)

From (6) and (7), are intremediate integral is

$$xp + yq = f(x^2 + y^2),\ f \text{ being an arbitrary function.} \qquad \text{...(8)}$$

Lagrange's subsidiary equations for (8) are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x^2+y^2)}. \qquad \text{...(9)}$$

Taking the first and second fractions of (9), $(1/y)dy - (1/x)dx = 0.$

Integrating, $\log y - \log x = \log a$ or $y/x = a$, ...(10)

where a is an arbitrary constant.

Taking the first and third fraction of (9), we get

$$\frac{dx}{x} = \frac{dz}{f(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{f(x^2+a^2x^2)}, \text{ using (10)}$$

or $$dz = (1/x) \times f[x^2\,(1 + a^2)]\,dx = (1/x^2) \times f[x^2\,(1 + a^2)]\,xdx. \qquad \text{...(11)}$$

Putting $x^2(1 + a^2) = v$ and $2x(1+ a^2)dx = dv$, (11) gives

$$dz = \frac{1+a^2}{v}\ f(v) \times \frac{1}{2(1+a^2)}\,dv = \left(\frac{1}{2v}\right) f(v)dv.$$

Integrating, $z = F(v) + b$ or $z - F[x^2\,(1 + a^2)] = b$

or $z - F(x^2 + x^2a^2) = b$ or $z - F(x^2 + y^2) = b$, using (10). ...(12)

Here b is an arbitrary constant. From (10) and (12), general solution of (1) is

$z - F(x^2 + y^2) = G(y/x)$ or $z = F(x^2 + y^2) + G(y/x)$,

where F and G are arbitrary functions.

Ex. 7. *Solve* $2xr - (x + 2y)s + yt = [(x + 2y)\,(2p - q)]/(x - 2y)$

Sol. Comparing the given equation with $Rr + Ss + Tr = V$, we have

$R = 2x$, $S = -(x + 2y)$, $T = y$, $V = [(x - 2y)\,(2p - q)]/(x - 2y)$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$2xdpdy + ydqdx - \frac{x+2y}{x-2y}\,(2p - q)\,dxdy = 0 \qquad \text{...(1)}$$

and $$2x(dy)^2 + (x + 2y)dxdy + y(dx)^2 = 0. \qquad ...(2)$$

The equation (2) can be resolved into the following two equations

$$xdy + ydx = 0 \quad i.e., \quad xdy = -ydx \qquad ...(3)$$

and $$dx + 2ydy = 0. \qquad ...(4)$$

Re–writing (1), $$2dp(xdy) + dq(ydx) - \frac{2p-q}{x-2y}[(xdy)dx + 2(ydx)dy]$$

or $$2dp(-ydx) + dq(ydx) - \frac{2p-q}{x-2y}\{(-ydx)dx + 2(ydx)dy\} = 0 \text{ using (3)}$$

or $$-2dp + dq - \frac{2p-q}{x-2y}(-dx + 2dy) = 0 \quad \text{or} \quad \frac{2dp-dq}{2p-q} - \frac{dx-2dy}{x-2y} = 0.$$

Intergrating, $\log(2p - q) - \log(x - 2y) = \log c_1$ or $(2p - q)/(x - 2y) = c_1$. ...(5)

Re–writing (3), $(1/y)dy + (1/x)dx = 0$ so that $\log x + \log y = \log c_2$

$\therefore$ $xy = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$(2p - q)/(x - 2y) = f(xy)$ or $2p - q = (x - 2y)\, f(xy)$, ...(7)

where f is an arbitrary function. The equation (7) is of Lagrange's form $Pp + Qq = R$. So Lagrange's, subsidiary equation for (7) are

$$\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{(x-2y)\,f(xy)}. \qquad ...(8)$$

Taking the first and second fractions of (8), $dx + 2dy = 0$.

Integrating, $x + 2y = a$, a being an arbitrary constant ...(9)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (8)

$$= \frac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{2y\,f(xy) - x\,f(xy) + (x-2y)\,f(xy)} = \frac{f(xy)(ydx + x\,dy) + dz}{0}$$

This $\Rightarrow$ $f(xy)\, d(xy) + dz$, as $ydx + xdy = d(xy)$

Integrating, $F(xy) + z = b$, b being an arbitrary constant. ...(10)

From (9) and (10), the required complete integral is

$F(xy) + z = G(x + y)$, F and G being arbitrary functions.

Ex. 8. *Solve $xr + (x + y)s + yt + p + q = 0$ by Monge's method.*

Sol. Given $$xr + (x + y)s + yt = -(p + q) \qquad ...(1)$$

Comparing (1) with $Rr + Sr + Tt = V$, here $R = x$, $S = x + y$, $T = y$ and $V = -(p + q)$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$xdpdy + ydqdx + (p + q)dxdy = 0 \qquad ...(2)$$

and $$x(dy)^2 - (x + y)dxdy + y(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $$(xdy - ydx)(dy - dx) = 0$$

so that $$xdy - ydx = 0 \qquad ...(4)$$

and $$dy - dx = 0 \quad i.e., \quad dy = dx \qquad ...(5)$$

For the required solution, we consider relation (5) only.

Integrating (5), $x - y = c_1$, being an arbitrary constant ... (6)

Using (5), (2) becomes $xdpdx + ydqdx + (p + q)(dx)^2 = 0$

or $xdp + ydq + pdx + qdx = 0$, on dividing by dx (as $dx \neq 0$)

or $(xdp + pdx) + (ydq + qdx) = 0$ or $(xdp + pdx) + (ydq + qdy) = 0$ by(5)

or $d(xp) + d(yq) = 0$ so that $xp + yq = c_2$. ...(7)

From (6) and (7), one intermediate integral of (1) is

$$xp + yq = f(x - y), \; f \text{ being an arbitrary function} \qquad ...(8)$$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x-y)} \qquad ...(9)$$

Taking the first two fields of (9), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_3$ or $x/y = c_3$ ...(10)

Now, each fraction of (9) $= \dfrac{dx - dy}{x - y} = \dfrac{d(x-y)}{x-y}$ ...(11)

Combining this fraction with last fraction of (9), we get

$$\frac{dz}{f(x-y)} = \frac{d(x-y)}{x-y} \quad \text{or} \quad dz = \frac{f(x-y)}{x-y} d(x-y) = \frac{f(u)du}{u}, \text{ if } \quad u = x - y$$

Integrating, $z = F(u) + c_4 = F(x-y) + c_4$, where $F(u) = \int \frac{1}{u} f(u)\, du$

or $z - F(x - y) = c_4$, c_4 being an arbitrary constant ...(12)

From (10) and (12), the required solution is

$z - F(x-y) = G(x/y)$ or $z = G(x/y) + F(x-y)$,

where F and G are arbitrary functions.

Ex. 9. *Solve $rq^2 - 2pqs + p^2t = pt - qs$ by Monge's method.* **[Delhi Maths (Hons) 2002]**

Sol. Given $q^2r - q\,(2p - 1)s + p(p - 1)t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -\,q(2p - 1)$, $T = p(p - 1)$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\, dxdy + T(dx)^2 = 0$ become

$$q^2dpdy + p(p - 1)dqdx = 0 \qquad ...(2)$$

and $$q^2(dy)^2 + q(2p - 1)dxdy + p\,(p - 1)\,(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $(qdy + pdx)\;\{qdy + (p - 1)dx\} = 0$

so that $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(4)

and $qdy + (p - 1)\, dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Since $dz = pdx + qdy$, (4) reduces to $dz = 0$ and so $z = c_1$ ...(6)

Re–writing (2), $(qdp)\,(qdy) + (p - 1)\; dq(pdx) = 0$

or $(qdp)\,(-pdx) + (p - 1)\; dq(pdx) = 0$, since from (4), $qdy = -pdx$

or $-qdp + (p - 1)dq = 0$ or $\{1/(p - 1)\}dp - \{1/q\}dq = 0$

Integrating, $\log(p-1) - \log q = \log c_2$ or $(p-1)/q = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(p - 1)/q = f(z)$ or $p - qf(z) = 1$, ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{1} \qquad ...(9)$$

From the first and the last fractions of (9), $dx - dz = 0$ so that $x - z = c_3$ ...(10)

From the last two fractions of (9), $dy - f(z)dz = 0$

Integrating, $y - F(z) = c_4$, where $F(z) = \int f(z)dz$ ...(11)

From (10) and (11), the required solution is $y - F(z) = G(x - z)$

or $y = F(z) + G(x - z)$, where F, G are arbitrary functions.

Ex. 10. *Solve $e^{2y}(r - p) = e^{2x}(t - q)$ by Monge's method.*

Sol. Given $e^{2y}r - e^{2x}t = pe^{2y} - qe^{2x}$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = e^{2y}$, $S = 0$, $T = -e^{2x}$ and $V = pe^{2y} - qe^{2x}$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $e^{2y}dxdy - e^{2x}dqdx - (pe^{2y} - qe^{2x})dxdy = 0$ ...(2)

and $e^{2y}(dy)^2 - e^{2x}(dx)^2 = 0$ ...(3)

From (3), $(e^y dy - e^x dx)(e^y dy + e^x dx) = 0$

so that $e^y dy - e^x dx = 0$, that is, $e^x dx = e^y dy$ ...(4)

and $e^y dy + e^x dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Integrating (4), $e^x - e^y = c_1$, c_1 being arbitrary constant ...(6)

Rewriting (2), $(e^y dp)(e^y dy) - (e^x dq)(e^x dx) - p(e^y dy)(e^y dx) + q(e^x dx)(e^x dy) = 0$

or $(e^y dp)(e^x dx) - (e^x dq)(e^x dx) - p(e^x dx)(e^y dx) + q(x^x dx)(e^y dy) = 0$, by (4)

or $e^y dp - e^x dq - pe^y dx + qe^x dy = 0$ or $\{d(e^y p) - pe^y dy\} - \{d(e^x q) - qe^x dx\} = pe^y dx - qe^x dy$

or $d(e^y p) - d(e^x q) = pe^y(dx + dy) - qe^x(dx + dy)$ or $d(e^y p - e^x q) = (e^y p - e^x q)(dx + dy)$

or $$\frac{d(e^y p - e^x q)}{e^y p - e^x q} = d(x + y)$$

Integrating, $\log(e^y p - e^x q) - \log c_2 = x + y$ or $(e^y p - e^x q)/c_2 = e^{x+y}$

or $(e^y p - e^x q)/e^{x+y} = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one intermediate integral of (1) is

$(e^y p - e^x q)/e^{x+y} = f(e^x - e^y)$ or $e^y p - e^x q = e^{x+y} f(e^x - e^y)$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{e^y} = \frac{dy}{-e^x} = \frac{dz}{e^{x+y} f(e^x - e^y)} \qquad ...(8)$$

From the first two fractions of (8), $e^x dx + e^y dy = 0$ so that $e^x + e^y = c_3$ ...(9)

Taking the first and third fraction of (8) and noting that $e^y = c_3 - e^x$ from (9), we get

$$\frac{dx}{e^y} = \frac{dz}{e^x e^y f(e^x - c_3 + e^x)} \quad \text{or} \quad dz = e^x f(2e^x - c_3)dx$$

or $dz - (1/2) \times f(2e^x - c_3)d(2e^x - c_3) = 0$ or $dz - (1/2) \times f(u)du = 0$, taking $u = 2e^x - c_3$

Integrating, $z - F(u) = c_3$, where $F(u) = \int (1/2) \times f(u)du$

or $z - F(2e^x - c_3) = c_4$ or $z - F(e^x - e^y) = c_4$, by (9) ...(10)

From (9) and (10), the required solution is $z - F(e^x - e^y) = G(e^x + e^y)$

or $z = F(e^x - e^y) + G(e^x + e^y)$, where F, G are arbitrary functions.

Ex. 11. *Solve $x^2r - y^2t = xp - yq$ by Monge's method.*

Sol. Given $$x^2r - y^2t = xp - yq \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2$, $S = 0$, $T = -y^2$ and $V = xp - yq$. Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$x^2dpdy - y^2dqdx - (xp - yq)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $$(xdy - ydx)\,(xdy + ydx) = 0$$

so that $xdy - ydx = 0$ that is, $xdy = ydx$ ...(4)

and $$xdy + ydx = 0 \qquad ...(5)$$

From (4), $(1/y)dy - (1/x)dx = 0$ so that $y/x = c_1$ ...(6)

For the required solution, we consider relation (4) only.

Re–writing (2), $$(xdp)(xdy) - (ydq)(ydx) - (pdx)(xdy) + (qdy)(ydx) = 0$$

or $$(xdp)(ydx) - (ydq)(ydx) - (pdx)(ydx) + (qdy)(ydx) = 0, \text{ by (4)}$$

or $xdp - ydq - pdx + qdy = 0$ or $\{d(xp) - pdx\} - \{d(yq) - qdy\} - pdx + qdy = 0$

or $d(xp - yq) - 2pdx + 2qdy = 0$ or $d(xp - yq) - 2p\,dx + 2(y/x)dx = 0$, by (4)

or $d(xp - yq) - (2/x)(xp - yq)dx = 0$ or $\dfrac{d(xp - yq)}{xp - yq} - \dfrac{2dx}{x} = 0$

Integrating, $\log(xp - yq) - 2\log x = c_2$ or $(xp - yq)/x^2 = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(xp - yq)/x^2 = f(y/x)$ or $xp - yq = x^2 f(x/y)$ ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x^2 f(y/x)} \qquad ...(9)$$

Taking the first two ratios of (9), $(1/x)\,dx + (1/y)\,dy = 0$ so that $\log x + \log y = c_3$

or $$xy = c_3,\ c_3 \text{ being an arbitrary constant} \qquad ...(10)$$

Taking the first and last fractions of (9), we get

$dz = x\,f(y/x)dx$ or $dz = x\,f(c_3/x^2)$, since by (10), $y = c_3/x$

$$\therefore\quad z = \int\left(-\frac{x^4}{2c_3}\right) f\left(\frac{c_3}{x^2}\right)\left(-\frac{2c_3}{x^3}\right)dx = \int\left(-\frac{c_3^2}{2c_3t^2}\right)f(t)dt, \text{ putting } \frac{c_3}{x^2} = t \text{ and } -\frac{2c_3}{x^3}dx = dt$$

or $z = -\dfrac{c_3}{2}\displaystyle\int \frac{f(t)}{t^2}dt + c_4 = c_3F(t) + c_4,$ where $F(t) = -\dfrac{1}{2}\displaystyle\int \frac{f(t)}{t^2}dt$

or $z - c_3F(c_3/x^2) = c_4$ or $z - xy\,F(y/x) = c_4$, by (10) ... (11)

From (10) and (11), the required solution is

$z - xy\ F(y/x) = G(xy)$ or $z = x^2(y/x)\ F\ (y/x) + G(xy)$

or $z = x^2H(y/x) + G(xy)$ where $H(y/x) = (y/x)\ \ F(y/x)$ and H, G are arbitrary functions.

Ex. 12. *Solve* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0.$
and hence find the surface satisfying the above equation and touching the hyperbolic paraboloid $z = x^2 - y^2$ *along its section by the plane* $y = 1$. **[Meerut 2001, I.A.S. 1978, Ranchi 2010]**

Sol. Given $$2x^2r - 5xys + 2y^2t = -2(px + qy). \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = 2x^2$, $S = -5xy$, $T = 2y^2$, $V = -2\,(px + qy)$

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0.$

become $$2x^2dpdy + 2y^2dqdx + 2(px + qy)dxdy = 0. \qquad ...(2)$$
and $$2x^2(dy)^2 + 5xydxdy + 2y^2(dx)^2 = 0. \qquad ...(3)$$

Re–writing (3), $$(xdy + 2ydy)\,(2xdy + ydx) = 0.$$
so that $xdy + 2ydx = 0$, *i.e.*, $xdy = -\,2ydx$ ...(4)

and $$2xdy + ydx = 0. \qquad ...(5)$$

Keeping (4) in view, (2) may be re–written as

$$2xdp(xdy) - ydq\,(-\,2ydx) + 2pdx\,(xdy) - qdy\,(-\,2ydx) = 0.$$
or $2xdp(xdy) - ydq\,(xdy) + 2pdx\,(xdy) - qdy\,(xdy) = 0$, using (4)

or $2xdp - ydq + 2pdx - qdy = 0$ or $2(xdp + pdx) - (ydq + qdy) = 0$

or $2d(xp) - d(yq) = 0$ so that $2xp - yq = c_1$. ...(6)

From (4), $(1/y)dy + 2(1/x)dx = 0$ so that $\log y + 2\log x = \log c_2$

or $\log y + \log x^2 = \log c_2$ or $x^2y = c_2$. ...(7)

From (6) and (7), one intermediate integral is

$$2xp - yq = f(x^2y), \; f \text{ being an arbitrary function.} \qquad ...(8)$$

which is of Lagrange's form. Hence Lagrange's subsidiary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{f(x^2y)}. \qquad ...(9)$$

Taking the first two fractions of (9), $2(1/y)dy + (1/x)dx = 0.$

Integrating, $2\log y + \log x = \log a$ or $y^2x = a$ or $x = a/y^2$. ...(10)

Taking the second and third fractions of (9) and using (10), we get

$$\frac{dy}{-y} = \frac{dz}{f(a^2/y^3)} \quad \text{or} \quad dz + \frac{1}{y} f\left(\frac{a^2}{y^3}\right) dy = 0. \qquad ...(11)$$

Putting $(a^2/y^3) = v$ so that $-\,(3a^2/y^4)\,dy = d\,v$, (11) gives

$$dz + \frac{1}{y} f(v) \times \left(-\frac{y^4}{3a^2}\right) dv = 0 \quad \text{or} \quad dz - \frac{f(v)}{3(a^2/y^3)} dv = 0$$

or $$dz - (1/3\,v) \times f(v)\,dv = 0, \text{ as } v = a^2/y^3.$$

Integrating, $z - F(v) = b$ or $z - F(a^2/y^3) = b$, b being an arbitrary constant.

or $$z - F(x^2y) = b, \quad \text{as} \quad y^2x = a. \qquad ...(12)$$

From (10) and (12), the required complete solution is

$$z - F(x^2y) = G(xy^2), \; F \text{ and } G \text{ being arbitrary fucntions.}$$
or $$z = F(x^2y) + G(xy^2). \qquad ...(13)$$

Second Part. The given surface is $z = x^2 - y^2$. ...(14)

(13) $\Rightarrow$ $p = \partial z/\partial x = 2xy\,F'(x^2y) + y^2G'(xy^2)$ and $q = \partial z/\partial y = x^2F'(x^2y) + 2xyG'(xy^2)$. ...(15)

From (14), $p = \partial z/\partial x = 2x$ and $q = \partial x/\partial y = -2y$. ...(16)

Since (13) and (14) touch each other along their section by the plane $y = 1$, the values of p and q given by (15) and (16) at any point on $y = 1$ must be equal

Thus, $$2xyF'(x^2y) + y^2G'(xy^2) = 2x, \text{ where } y = 1 \qquad \text{...(17)}$$

and $$x^2F'(x^2y) + 2xy\, G'(xy^2) = -2y, \text{ where } y = 1. \qquad \text{...(18)}$$

From (17), $$2xF'(x^2) + G'(x) = 2x. \qquad \text{...(19)}$$

From (18), $$x^2F'(x^2) + 2xG'(x) = -2. \qquad \text{...(20)}$$

Solving (19) and (20) for $F'(x^2)$ and $G'(x)$, we have

$$F'(x^2) = (4/3) + (2/3) \times (1/x^2). \qquad \text{...(21)}$$

and $$G'(x) = -(2/3)\times x - (4/3) \times (1/x). \qquad \text{...(22)}$$

(21) $\Rightarrow$ $F'(u) = (4/3) + (2/3) \times (1/u)$, on putting $x^2 = u$

Integrating, $F(u) = (4/3) \times u + (2/3) \times \log u + c_1$, c_1 being an arbitrary constant

This $\Rightarrow$ $$F(x^2y) = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1. \qquad \text{...(23)}$$

Integrating (22), $G(x) = -(2/3)(x^2/2) - (4/3) \log x + c_2$, being an arbitrary constant

This $\Rightarrow$ $$G(xy^2) = -(1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2. \qquad \text{...(24)}$$

Putting values of $F(x^2y)$ and $G(xy^2)$ given by (23) and (24) in (13), we get

$$z = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1 - (1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2$$

or $$z = (4/3) \times x^2y - (1/3)\times x^2y^4 + (2/3) \times [\log (x^2y) - 2 \log (xy^2)] + c, \text{ taking } c_1 + c_2 = c$$

or $$z = (4/3)\times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log (x^2y) - \log (xy^2)^2]$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log \{(x^2y)/(x^2y^4)\} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times \log y^{-3} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c. \qquad \text{...(25)}$$

Now at the point of contact of (14) and (25), the values of z must be the same and hence

$$x^2 - y^2 = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c, \text{ where } y = 1$$

$\Rightarrow$ $$x^2 - 1 = (4/3) \times x^2 - (1/3) \times x^2 + c, \text{ putting } y = 1$$

$\Rightarrow$ $$x^2 - 1 = x^2 + c \quad \Rightarrow \quad c = -1.$$

Putting $c = -1$ in (25), the required surface is

$$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y - 1 \quad \text{or} \quad 3z = 4x^2y - x^2y^4 - 6 \log y - 3.$$

9.7. Type 3. When the given equation Rr + Ss + Tt = V leads to two identical intermediate intergrals.

Working rule for solving problems of type 3

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rpdy + Tdqdx - Vdx\, dy = 0$ ... (1) $\qquad R(dy)^2 - S\, dxdy + T(dx)^2 = 0$ ... (2)

Step 3. R.H.S. of (2) reduces to a perfect square and hence it gives only one distinct factor in place of two as in type 1 and type 2.

Step 4. Start with the only one factor of step 3 and use (2) to get an intermediate integral.

Step 5. Re–write the intermediate integral of the step 4 in the form of $Pp + Qq = R$ and use Lagrange's method to obtain the required general solution of the given equation.

9.8. Solved examples based on Art 9.7

Ex. 1. *Solve* : $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$

[Meerut 2002, Delhi Maths (H) 1999 2007, 10; Rohailkhand 1997; Kanpur 1994]

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, ...(1)

$R = (1 + q)^2, \quad S = -2(1 + p + q + pq), \quad T = (1 + p)^2, \quad V = 0.$ ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$(1 + q)^2 dpdy + (1 + p)^2 dqdx = 0 \quad ...(5)$$

and $(1 + q)^2(dy)^2 + 2(1 + p + q + pq)dxdy + (1 + p)^2 (dx)^2 = 0.$ ...(6)

Since $1 + p + q + pq = (1 + p)(1 + q)$, (6) becomes $[(1 + q)dy + (1 + p)dx]^2 = 0$

so that $(1 + q)dy + (1 + p)dx = 0$ or $(1 + q)dy = -(1 + p)dx.$ ...(7)

Keeping (7) in view, (5) may be re–written as

$$(1 + q)dp\ \{(1 + q)dy\} - (1 + p)dq\ \{-(1 + p)dx\} = 0. \quad ...(8)$$

Dividing each term of (8) by $(1 + q)dy$, or its equivalent $-(1 + p)dx$, we get

$(1 + q)dp - (1 + p)dq = 0$ or $dp/(1 + p) - dq/(1+ q) = 0.$

Integrating it, $(1+ p)/(1 + q) = c_1$, c_1 being an arbitrary constant ...(9)

From (7), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (1) is

$(1 + p)/(1 + q) = F(x + y + z)$ or $1 + p = (1 + q)\, F(x + y + z)$

or $p - q\, F(x + y + z) = F(x + y + z) - 1,$ ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1} \quad ...(12)$$

Choosing 1, 1, 1 as multipliers, each fraction of (12) $= (dx + dy + dz)/0$

so that $dx + dy + dz = 0$ giving $x + y + z = c_2$ ... (13)

Using (13) and taking the first two fractions of (12), we have

$dx = -\,dy/F(c_2)$ or $dy + F(c_2)dx = 0.$

Integrating it, $y + xF(c_2) = c_3$ or $y + x\,F(x + y + z) = c_3$ ...(14)

From (13) and (14), the required general solution is

$y + x\,F(x + y + z) = G(x + y + z)$, F, G being arbitrary functions.

Ex. 2. *Solve* $y^2r + 2xys + x^2t + px + qy = 0.$ **[Bilaspur 2004]**

Sol. Given $y^2r + 2xys + x^2t = -(px + qy).$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = y^2, \; S = 2xy, \; T = x^2, \; V = -(px + qy).$ ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx + Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$y^2dpdy + x^2dqdx + (px + qy)\, dxdy = 0 \quad ...(5)$$

and $y^2(dy)^2 - 2xydxdy + x^2(dx)^2 = 0.$ ...(6)

From (6), $(xdx - ydy)^2 = 0$ so that $xdx - ydy = 0$ or $xdx = ydy.$ ...(7)

Keeping (7) in view, (5) may be re–written as

$$ydp\,(ydy) + xdq\,(xdx) + pdy\,(xdx) + qdx\,(ydy) = 0. \quad ...(8)$$

Dividing each term of (8) by xdx, or its equivalent ydy, we get

$ydp + xdq + pdy + qdx = 0$ or $(ydp + pdy) + (xdq + qdx) = 0$

Integrating it, $yp + xq = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 - y^2/2 = c_2/2$ or $x^2 - y^2 = c_2.$ ...(10)

From (9) and (10), one intermediate integral of (1) is $\quad yp + xq = F(x^2 - y^2)$, $\quad$ ...(11)
which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{F(x^2 - y^2)}. \qquad ...(12)$$

From the first two fractions of (2), $\quad xdx - ydy = 0 \quad$ so that $\quad x^2 - y^2 = c_2$. $\quad$ ...(13)

Taking the last two fractions and using (13), we get

$$\frac{dy}{(y^2 + c_2)^{1/2}} = \frac{dz}{F(c_2)} \qquad \text{or} \qquad dz - F(c_2)\frac{dy}{(y^2 + c_2)^{1/2}} = 0.$$

Integrating, $\qquad z - F(c_2) \log [y + (y^2 + c_2)^{1/2}] = c_3$

or $\qquad z - F(x^2 - y^2) \log [y + \sqrt{(y^2 + x^2 - y^2)}\,] = c_3$, using (13)

or $\qquad z - F(x^2 - y^2) \log (x + y) = c_3$, c_3 being an arbitrary constant $\quad$ ...(14)

From (13) and (14), the required general solution is

$z - F(x^2 - y^2) \log (x + y) = G(x^2 - y^2)$, F, G being arbitrary functions.

Ex. 3(a). *Obtain the integral of* $q^2r - 2pqs + p^2t = 0$ *in the form* $y + xf(z) = F(z)$.

[Delhi Maths Hons. 1999, 2007; Meerut 1994, 95; Nagpur 2005]

(b) *Show also that this solution represents a surface generated by straight lines that are parallel to a fixed plane.*

Sol. (*a*) Given $\qquad q^2r - 2pqs + p^2t = 0$. $\quad$ ...(1)

As ususal Monge's subsidiary equations are $\quad q^2dpdy + p^2dp\,dx = 0$ $\quad$ ...(2)

and $\quad q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0 \quad$ or $\quad (qdy + pdx)^2 = 0$. $\quad$ ...(3)

From (3), we have $\quad qdy + pdx = 0 \quad$ or $\quad qdy = -pdx$. $\quad$ ...(4)

In view of (4), (2) may be re–written as $\quad qdp\,(qdy) - pdq\,(-pdx) = 0$. $\quad$ ...(5)

Dividing each term of (5) by qdy, or its equivalent $(-pdx)$, we find

$qdp - pdq = 0 \qquad$ or $\qquad (1/p)dp - (1/q)\,dp = 0$.

Integrating it, $\quad p/q = c_1$, c_1 being an arbitrary constant $\quad$ ...(6)

From (4), $\quad dz = 0$, $\quad$ (as $dz = pdx + qdy$) $\quad$ so that $\quad z = c_2$. $\quad$ ...(7)

From (6) and (7), one integral of (1) is $\quad p/q = f(z) \quad$ or $\quad p - f(z)q = 0$, $\quad$ ...(8)
which is of the form $Pp + Qq = R$. Here f is an arbitrary function. Its Lagrange's auxiliary equations

are $$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}. \qquad ...(9)$$

The last fraction in (9) gives $\quad dz = 0 \quad$ so that $\quad z = c_2$ $\quad$ ...(10)

From the first two fractions in (9) and (10), we find

$$\frac{dx}{1} = \frac{dy}{-f(c_2)} \qquad \text{or} \qquad dy + f(c_2)dx = 0.$$

Integrating, $\quad y + xf(c_2) = c_3 \quad$ or $\quad y + xf(z) = c_3$, by (10). $\quad$ ...(11)

From (10) and (11), the required integral is $\quad y + xf(z) = F(z)$. $\quad$ ...(12)

Part (*b*). Let $z = k$, k being an arbitrary constant. Then (12) is the locus of the straight lines given by the intersection of the planes

$$z = k \qquad \text{and} \qquad y + xf(k) - F(k) = 0. \qquad ...(13)$$

Clearly the lines are parallel to the plane $z = 0$ (which is a fixed plane) because these lie on the plane $z = k$ for different values of k.

Ex. 4. *Solve* $y^2r - 2ys + t = p + 6y$. $\quad$ **[Agra 1993; Bhopal 2004; Vikram 2004; Meerut 2009; Delhi Maths Hons 1994, 98, 2006, 09, 10]**

Sol. As usual Monge's subsidiary equations are

$$y^2dpdy + dqdx - (p + 6y)dxdy = 0 \qquad ...(1)$$

and $y^2(dy)^2 + 2ydydx + (dx)^2 = 0$ or $(ydy + dx)^2 = 0$. ...(2)

From (2), $ydy + dx = 0$ or $dx = -ydy$. ...(3)

Putting the value of dx from (3) in (1), we find

$$y^2dpdy + dq(-ydy) - (p + 6y)\,dy\,(-ydy) = 0$$

or $ydp - dq + (p + 6y)\,dy = 0$ or $(ydp + pdy) - dq + 6ydy = 0$.

Integrating it, $yp - q + 3y^2 = c_1$, c_1 being an arbitrary constant ...(4)

Integrating (4), $y^2/2 + x = c_2/2$ or $y^2 + 2x = c_2$. ...(6)

From (5) and (6), one integral of (1) is

$yp - q + 3y^2 = F(y^2 + 2x)$ or $yp - q = F(y^2 + 2x) - 3y^2$, ...(7)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}. \qquad ...(8)$$

From the first two fractions of (8), $2ydy + 2dx = 0$ so that $y^2 + 2x = c_2$. ...(9)

Taking the last two fractions of (8) and using (9), $dz + [F(c_2) - 3y^2]dy = 0$.

Integrating, $z + yF(c_2) - y^3 = c_2$ or $z + yF(y^2 + 2x) - y^3 = c_3$. ...(10)

From (9) and (10), the required general solution is

$$z + yF(y^2 + 2x) - y^3 = G(y^2 + 2x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 5. *Solve* $(b + cq)^2r - 2(b + cq)(a + cp)s + (a + cp)^2t = 0$

Sol. Usual Monge's subsidiary equations are $(b + cq)^2\,dpdy + (a + cp)^2\,dqdx = 0$. ...(1)

and $(b + cq)^2(dy)^2 + 2(b + cq)(a + cp)\,dxdy + (a + cp)^2(dx)^2 = 0$. ...(2)

(2) $\Rightarrow$ $\{(b + cq)dy + (a + cp)dx\}^2 = 0$ ...(3)

or $(b + cq)dy + (a + cp)dx = 0$ or $adx + bdy + c(pdx + qdy) = 0$

or $adx + bdy + cdz = 0$, as $dz = pdx + qdy$.

Integrating, $ax + by + cz = c_1$, c_3 being an arbitrary constant ...(4)

From (3), $(b + cq)dy = -(a + cp)dx$. So (1) reduces to $(b + cq)dp - (a + cp)dq = 0$

or $\dfrac{dp}{a+cp} - \dfrac{dq}{b+cq} = 0$ so that $\dfrac{a+cp}{b+cq} = c_2$ ...(5)

So the intermediate integral of the given equation is $(a + cp)/(b + cq) = \phi_1(ax + by + cz)$

or $cp - c\phi_1(ax + by + cz)q = -a + b\,\phi_1(ax + by + cz)$. ...(6)

Lagrange's auxiliary equations are

$$\frac{dx}{c} = \frac{dy}{-c\,\phi_1(ax+by+cz)} = \frac{dz}{-a+b\,\phi_1(ax+by+cz)}. \qquad ...(7)$$

Using a, b, c as multipliers, each fraction of (7) = $(adx + bdy + cdz)/0$

$\therefore$ $adx + bdy + cdz = 0$ so that $ax + by + cz = c_3$. ...(8)

Using (8) and taking the first two ratios of (7), we get

$dx = -\,dy/\phi_1(c_3)$ or $dy + \phi_1(c_3)dx = 0$.

Integrating, $y + x\phi_1(c_3) = c_4$ or $y + x\phi_1(ax + by + cz) = c_4$. ...(9)

From (8) and (9), the required solution is

$$y + x\phi_1(ax + by + cz) = \phi_2(ax + by + cz),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

Ex. 6. *Solve* $x^2r - 2xs + t + q = 0$. **[K.U. Kurukshetra 2004; Ravishankar 2005]**

Sol. Usual Monge's subsidiary equations are $x^2dpdy + dqdx + qdxdy = 0$ ...(1)

and $x^2(dy)^2 + 2xdxdy + (dx)^2 = 0$. ...(2)

Now, $(2) \Rightarrow \quad (xdy + dx)^2 = 0 \quad \Rightarrow \quad xdy + dx = 0$ ...(3)

$(3) \Rightarrow \quad (dx)/x + dy = 0 \quad \Rightarrow \quad y + \log x = c_1.$ ...(4)

Using (3), (1) reduces to $\quad x^2 dpdy + dq\,(-x\,dy) + q(-x\,dy)dy = 0$

or $\quad dp - \left(\dfrac{dq}{x} - \dfrac{qdx}{x^2}\right) = 0 \quad$ or $\quad d\left(p - \dfrac{q}{x}\right) = 0.$

Integrating, $\quad p - (q/x) = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$p - (q/x) - \phi_1(y + \log x) \quad$ or $\quad xp - q = x\phi_1(y + \log x).$ ...(6)

Lagrange's auxiliary equations for (6) are $\quad \dfrac{dx}{x} = \dfrac{dy}{-1} = \dfrac{dz}{x\,\phi_1(y + \log x)}.$ ...(7)

Taking the first two fractions of (7), $\quad (1/x)dx + dy = 0 \Rightarrow y + \log x = c_3.$ ...(8)

Using (8), first and third fractions of (7) give $\quad \dfrac{dx}{x} = \dfrac{dz}{x\,\phi_1(c_3)} = \Rightarrow z - x\phi_1(c_3) = c_4$

or $\quad z - x\phi_1(y + \log x) = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9) the required solution is

$z - x\phi_1(y + \log x) = \phi_2(y + \log x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 7. *Solve* $(y - x)\,(q^2 r - 2pqs + p^2 t) = (p + q)^2\,(p - q)$.

Sol. The usual Monge's subsidiary equations are

$(y - x)\,(q^2 dpdy + p^2 dqdx) - (p + q)^2\,(p - q)dxdy = 0$ ...(1)

and $\quad q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0.$ ...(2)

$(2) \Rightarrow \quad (qdy + pdx)^2 = 0 \quad$ or $\quad qdy + pdx = 0.$ ...(3)

$dz = pdx + qdy$ and $(3) \quad \Rightarrow \quad dz = 0 \quad \Rightarrow \quad z = c_1.$ ...(4)

Using (3), (1) reduces to $\quad (y - x)\,(qdp - pdq) - (p^2 - q^2)\,(dx - dy) = 0$

or $\quad q^2 d\left(\dfrac{p}{q}\right) - (p^2 - q^2)\,\dfrac{d(x-y)}{y-x} = 0 \quad$ or $\quad \dfrac{d(x-y)}{x-y} + \dfrac{d(p/q)}{(p/q)^2 - 1} = 0$

Integrating, $\quad \log(x - y) + \dfrac{1}{2}\log\dfrac{(p/q)-1}{(p/q)+1} = \dfrac{1}{2}\log c_2 \quad$ or $\quad (x - y)^2\dfrac{p-q}{p+q} = c_2.$ ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(x - y)^2\dfrac{p-q}{p+q} = \phi_1(z) \quad$ or $\quad (x - y)^2(p - q) = (p + q)\phi_1(z)$

or $\quad p\{(x - y)^2 - \phi_1(z)\} - q\{(x - y)^2 + \phi_1(z)\} = 0.$ ...(6)

Here Lagrange's subsidiary equation for (6) are

$$\frac{dx}{(x-y)^2 - \phi_1(z)} = \frac{dy}{-\{(x-y)^2 + \phi_1(z)\}} = \frac{dz}{0}. \qquad ...(7)$$

Now, the third fraction of (7) $\Rightarrow \quad dz = 0 \quad$ so that $\quad z = a,$ ...(8)

where 'a' is an arbitrary constant.

Now, each fraction of (7) $= \dfrac{dx + dy}{-2\phi_1(z)} = \dfrac{dx - dy}{2(x-y)^2} \quad \Rightarrow \quad d(x + y) = -\,\phi_1(a)\dfrac{d(x-y)}{(x-y)^2}$, by (8).

Integrating it, $\; x + y - \phi_1(a)\,(x - y)^{-1} = b \;$ or $\; x + y - \phi_1(z)\,(x - y)^{-1} = b$, using (8). ...(9)

From (8) and (9), the required general solution is

$x + y - (x - y)^{-1}\,\phi_1(z) = \phi_2(z)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 8. *Solve* $x^2r + 2xys + y^2t = 0$. **[Meerut 2003, Garhwal 1993; Delhi Maths (H) 2001]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = 2xy$, $T = y^2$. Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$

become $$x^2dpdy + y^2dqdx = 0 \qquad ...(1)$$

and $$x^2(dy)^2 - 2xydxdy + y^2(dx)^2 = 0. \qquad ...(2)$$

Now, (2) gives $(xdy - ydx)^2 = 0$ so that $xdy - ydx = 0$. ...(3)

Re–writing (1), $(xdp)\,(xdy) + (ydx)\,(ydq) = 0$

or $(xdp)(xdy) + (xdy)\,(ydq) = 0$ [$\because$ from (3), $ydx = xdy$]

or $xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0$, as $dz = pdx + qdy$.

Integrating (1) $xp + yq - z = c_1$, c_1 being an arbitrary constant ...(4)

Now (3) gives $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log c_2$ or $y/x = c_2$. ...(5)

From (4) and (5), the intermediate integral of the given equation is

$xp + yq - z = f(y/x)$ or $xp + yq = z + f(y/x)$, ...(6)

where f is an arbitrary function. Lagrange's subsidiary equation for (6) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(y/x)}. \qquad ...(7)$$

Taking the first two fractions of (7), $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log a$ so that $y/x = a$. ...(8)

Taking the last two fractions of (7) and using (8), we get $\dfrac{dz}{z + f(a)} - \dfrac{dy}{y} = 0$.

Integrating it, $\log\,[z + f(a)] - \log y = \log b$, b being an arbitrary constant

so that $[z + f(a)]/y = b$ or $[z + f(y/x)]/y = b$, using (8) ...(9)

From (8) and (9), the required solution is

$[z + f(y/x)]/y = g(y/x)$ or $z = yg(y/x) - f(y/x)$, where f and g are arbitrary functions.

Ex. 9. *Solve* $r - 2s + t = \sin(2x + 3y)$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = -2$, $T = 1$, $V = \sin(2x + 3y)$. So Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$dpdy + dqdx - \sin(2x + 3y)dxdy = 0. \qquad ...(1)$$

and $$(dy)^2 + 2\,dxdy + (dx)^2 = 0. \qquad ...(2)$$

Now, (2) gives $(dy + dx)^2 = 0$ so that $dy + dx = 0$. ...(3)

From (3), $dy = -\,dx$. Then, (1) becomes $-\,dpdx + dqdx + \sin(2x + 3y)dxdy = 0$

or $$dp - dq + \sin(2x + 3y)dy = 0, \text{ as } dx \neq 0. \qquad ...(4)$$

Now, integrating (3), $x + y = c_1$, c_1 being an arbitrary constant ...(5)

From (4), $dp - dq + \sin[2(x + y) + y]dy = 0$ or $dp - dq + \sin(2c_1 + y)dy = 0$, using (5).

Integrating, $p - q - \cos(2c_1 + y) = c_2$

or $$p - q - \cos(2x + 3y) = c_2, \text{ as } c_1 = x + y \qquad ...(6)$$

From (5) and (6), an intermediate integral is

$p - q - \cos(2x + 3y) = f(x + y)$ or $p - q = \cos(2x + 3y) + f(x + y)$, ...(7)

where f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\cos(2x+3y)+f(x+y)}. \qquad ...(8)$$

Taking the first two fractions of (8), $dx + dy = 0$ so that $x + y = a$ ... (9)

Taking the last two fractions of (8) and using (9), we get

$$\frac{dy}{-1} = \frac{dz}{\cos(2a+y)+f(a)} \qquad \text{or} \qquad dz + [\cos(2a+y) + f(a)]dy = 0.$$

Integrating it, $z + \sin(2a + y) + y f(a) = b$, b being an arbitrary constant

or $z + \sin(2x + 3y) + y f(x + y) = b$, using (9). ...(10)

From (9) and (10) the required complete integral is

$z + \sin(2x + 3y) + y f(x + y) = g(x + y)$, f and g being an arbitrary functions.

Ex. 10. *Solve* $q^2r - 2pqs + p^2t = pq^2$. **[I.A.S. 1986]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = q^2$, $S = -2pq$, $T = p^2$, $V = pq^2$. The Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q^2dpdy + p^2dqdx - pq^2dxdy = 0$ ...(1)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0.$ ...(2)

Re–writing (2), $(qdy + pdx)^2 = 0$ so that $pdx + qdy = 0.$ ...(3)

Since $dz = pdx + qdy$, (3) $\Rightarrow$ $dz = 0$ so that $z = c_1$. ...(4)

Re–writing (1), $(qdy)(qdp) + (pdx)(pdq) - (qdy)(pqdx) = 0$

or $(qdy)(qdp) - (qdy)(pdq) - (qdy)(pqdx) = 0$, as from (3), $pdx = -qdy$

or $qdp - pdq - pqdx = 0$ or $(1/p)dp - (1/q)dq = dx.$

Integrating, $\log p - \log q - \log c_2 = x$ or $p/(c_2q) = e^x$

or $(p/q)e^{-x} = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(p/q)e^{-x} = f(z)$ or $px^{-x} - f(z)q = 0.$ ...(6)

Lagrange's auxiliary equations for (6) are $\dfrac{dx}{e^{-x}} = \dfrac{dy}{-f(z)} = \dfrac{dz}{0}.$ ...(7)

The last fraction of (7) $\Rightarrow$ $dz = 0$ so that $z = a.$ ...(8)

Taking the first fractions of (7) and using (8), we get

$$\frac{dx}{e^{-x}} = \frac{dy}{-f(a)} \qquad \text{or} \qquad e^x f(a)dx + dy = 0.$$

Integrating, $e^x f(a) + y = b$ or $e^x f(z) + y = b$, as from (8), $a = z$ ...(9)

From (8) and (9), the required complete integral is

$e^x f(z) + y = g(z)$, where f and g are arbitrary functions.

Ex. 11. *Solve* $q^2r - 2q(1+p)s + (1+p)^2t = 0$ *by Monge's method.*

Sol. Given $q^2r - 2q(1+p)s + (1+p)^2t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -2q(1+p)$ and $T = (1+p)^2$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$q^2 dpdy + (1+p)^2\ dqdx = 0 \qquad \text{... (2)}$$

and
$$q^2 (dy)^2 + 2q(1+p)dxdy + (1+p)^2\ (dx)^2 = 0 \qquad \text{... (3)}$$

Rewriting (3), $\{qdy + (1+p)dx\}^2 = 0$ or $qdy + (1+p)\,dx = 0$ ... (4)

From (4), $dx + (pdx + qdy) = 0$ or $dx + dz = 0$, as $dz = pdx + qdy$

Integrating, $x + z = C_1$, C_1 being an arbitrary constant ... (5)

Re–writing (2), $(qdy)\ (qdp) + [(1+p)\ dx\} \times \{(1+p)dq\} = 0$

or $(qdy)\ (qdp) + (-qdy)\ [(1+p)\ dq] = 0$, using (4)

or $qdp - (1+p)\,dq = 0$ or $\{1/(1+p)\}dp - (1/q)dq = 0$

Integrating, $\log(1+p) - \log q = \log C_2$ or $(1+p)/q = C_2$ ... (6)

From (5) and (6), the intermediate integral of (1) is

$(1+p)/q = f(x+z)$ or $p - q\,f(x+z) = -1$ ... (7)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+z)} = \frac{dz}{-1} \qquad \text{... (8)}$$

Taking the first and last ratios, $dx + dz = 0$ $\Rightarrow$ $x + z = C_3$ ... (9)

Using (9) and taking the first two ratios of (8), we get

$dy + f(C_3)dx = 0$ so that $y + xF(C_3) = C_4$

or $y + x\ f(x+z) = C_4$, using (9) ... (10)

From (9) and (10), the required general solution is

$$y + xf(x+z) = g(x+z),\ f,\ g \text{ are arbitrary functions}$$

Ex. 12. *Solve* $(x-y)\ (x^2 - 2xys + y^2 t) = 2xy\ (p-q)$. **[Delhi B.Sc. (Hons) 2011]**

Sol. Given $x^2(x-y)r - 2xy(x-y)s + y^2(x-y)t = 2xy(p-q)$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2(x-y)$, $S = -2xy\ (x-y)$, $T = y^2\ (x-y)$ and $V = 2xy\ (p-q)$. Hence Monge's subsidiars equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dyxy + T(dx)^2 = 0$ become

$$x^2(x-y)\ dpdy - 2xy\ (p-q)\ dxdy + y^2(x-y)dqdx = 0 \qquad \text{... (2)}$$

and
$$(x-y)\{x^2(dy)^2 + 2xy\ dx\ dy + y^2(dy)^2\} = 0 \qquad \text{... (3)}$$

Since $x \neq y$, (3) gives $(xdy + ydx)^2 = 0$ so that $ydx = -xdy$ ... (4)

From (4), $(1/x)dx + (1/y)dy = 0$ so that $xy = C_1$ ... (5)

Re–writing (2), $x(x-y)\ dp\ (xdy) - 2(p-q)(xdy)\ (ydx) + y(x-y)\,dq(ydx) = 0$

or $x(x-y)\,dp(xdy) - 2(p-q)\ (xdy)\ (ydx) + y(x-y)dq(-xdy), by (4)$

or $x(x-y)dp - 2\ (p-q)\ (ydx) - y(x-y)\ dq = 0$

or $\quad (x-y)(xdp-ydq)=2y(p-q)dx \quad$ or $\quad xdp-ydq=\{2y(p-q)dx\}/(x-y)$

or $\quad (xdp+pdx)-(ydq+qdy)=\{2y(p-q)\,dx\}/(x-y)+pdx-qdy$

or $\quad d(xp)-d(yq)=\{2(p-q)\,ydx+(x-y)p\,dx-(x-y)\,qdy\}/(x-y)$

or $\quad (x-y)d(xp-yq)=2pydx-2qydx+xpdx-ypdx-xqdy+yqdy$

$$= pydx-2qydx+xpdx+qydx+yqdy \;=\; -pxdy-qydx+xpdx+yqdy,\; by(4)$$

$\therefore \quad (x-y)\,d(xp-yq)=xp(dx-dy)-yq(dx-dy)=(xp-yq)(dx-dy)$

or $\quad \dfrac{d(xp-yq)}{xp-yq}=\dfrac{dx-dy}{x-y} \quad$ or $\quad \dfrac{d(xp-yq)}{xp-yq}-\dfrac{d(x-y)}{x-y}=0.$

Integrating, $\quad \log(xp-yq)-\log(x-y)=\log C_2 \quad$ or $\quad (xp-yq)/(x-y)=C_2 \quad$ ... (6)

From (5) and (6), the intermediate integral of the given equation is

$(xp-yq)/(x-y)=f(xy) \quad$ or $\quad xp-yq=(x-y)f(xy), \quad$ ... (7)

which is of Lagrange's form. Its auxiliary equations are $\quad \dfrac{dx}{x}=\dfrac{dy}{-y}=\dfrac{dz}{(x-y)f(xy)} \quad$ ... (8)

Taking the first two fractions, $\quad (1/x)dx+(1/y)dy=0 \quad$ so that $\quad xy=C_3 \quad$ ... (9)

Now, $\quad$ each fraction of (8) $=\dfrac{dx+dy}{x-y}=\dfrac{dz}{(x-y)f(xy)}$

or $\quad dz=f(xy)\,d(x+y) \quad$ or $\quad dz=f(C_3)\,d(x+y)$, by (9)

Integrating, $\quad z-(x+y)\,f(C_3)=C_4 \quad$ or $\quad z-(x+y)\,f(xy)=C_4 \quad$ ... (10)

From (9) and (10), the required solution is $\quad z-(x+y)f(xy)=g(xy)$

or $\quad z=(x+y)f(xy)+g(xy)$, f and g being arbitrary functions.

9.9 Type 4. When the given equation Rr + Ss + Tt = V fails to yield an intermediate integral as in cases 1, 2 and 3.

Working rule for solving problems of type 4.

Suppose the R.H.S. of $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ neither gives two factors nor a perfect square (as in Types 1, 2 and 3 above). In such cases factors dx, dy, p, $1+p$ etc. are cancelled as the case may be and an integral of given equation is obtained as usual. This integral is then integrated by methods explained in chapter 7.

9.10 SOLVED EXAMPLES BASED ON ART 9.9

Ex. 1. *Solve* $(q+1)s=(p+1)t$. **[Agra 2009]**

Sol. Given $\quad (q+1)s-(p+1)t=0. \quad$ ...(1)

Comparing (1) with $Rr+Ss+Tt=V$, we find $R=0$, $S=(q+1)$. $T=-(p+1)$, $V=0$....(2)

Monge's subsidiary equations are $\quad Rdpdy+Tdqdx-Vdxdy=0. \quad$ ...(3)

and $\quad R(dy)^2-Sdxdy+T(dx)^2=0. \quad$ ...(4)

Using (2), (3) and (4) become $\quad -(p+1)dqdx=0 \quad$ ...(5)

and $\quad -(q+1)dxdy-(p+1)(dx)^2=0. \quad$ ...(6)

Dividing (5) by $-(p+1)dx$, we obtain $\quad dq=0. \quad$ ...(7)

and dividing (6) by $-dx$ we get $\quad (q+1)+(p+1)dx=0. \quad$ ...(8)

From (8), $\quad dx+dy+pdx+qdy=0 \quad$ or $\quad dx+dy+dz=0, \quad$ as $\quad dz=pdx+qdy$

Integrating it, $x + y + z = c_1$, being an arbitrary constant ...(9)

Integrating (7), $q = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), an integral of (1) is

$q = f(x + y + z)$ or $\partial z/\partial y = f(x + y + z)$ ...(11)

Integrating (11) partially w.r.t. y (treating x as constant), we find

$$z = F(x + y + z) + G(x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 2. *Solve* $pq = x(ps - qr)$. **[Delhi. Maths (H) 2002, 08]**

Sol. Given $xqr - xps + 0.t = -pq$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = xq$, $S = xp$, $T = o$ and $V = -pq$

Monge's subsidiary equations $Rdp\,dy + T\,dq\,dx - V\,dx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $xqdpdy + pqdxdy = 0$. ...(2)

and $xq(dy)^2 + xpdxdy = 0$. ...(3)

Dividing (2) by qdy we get $xdp + pdx = 0$ ...(4)

and dividing (3) by xdy, we get $qdy + pdx = 0$. ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = c_1$ ...(6)

Integrating (4), $xp = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one integral of (1) is

$$xp = f(z) \quad \text{or} \quad x\frac{\partial z}{\partial x} = f(z) \quad \text{or} \quad \frac{1}{f(z)}\frac{\partial z}{\partial x} = \frac{1}{x}.$$

Integrating it partially w.r.t. x, $F(z) = \log x + G(y)$, F, G being arbitrary functions.

Ex. 3. *Solve* $pt - sqs = q^3$ **[MDU Rohtak 2004; Ravishankar 2004; Delhi Maths (H) 2005; Meerut 2005; 06 ; Rohilkhand 1994]**

Sol. Given $pt - qs = q^3$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = -q$, $T = p$, $V = q^3$.

$\therefore$ Monge's subsidiary equations $Rdpdy + Tdqdx - Vdxdy = 0$, $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $pdqdx - q^3dxdy = 0$ ...(2)

and $qdxdy + p(dx)^2 = 0$. ...(3)

Dividing (2) by dx, we get $pdq - q^3dy = 0$ ...(4)

and dividing (3) by dx, we get $pdx + qdy = 0$. ...(5)

From (5), $dy = -(pdx)/q$. Putting this value of dy into (4) gives

$pdq - q^3(pdx/q) = 0$ or $(1/q^2)dq + dx = 0$.

Integrating it, $-1/q + x = C_1$, C_1 being an arbitrary constant ...(6)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_2$. ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + x = f(z) \quad \text{or} \quad \frac{\partial y}{\partial z} = x - f(z), \text{ as } q = \frac{\partial z}{\partial y},$$

Integrating with respect to z partially (treat x as constant), we obtain

$$y = xz - F(z) + G(x),\ F,\ G \text{ being arbitrary functions, where } F(z) = \int f(z)\,dz.$$

Ex. 4. *Solve* $z(qs - pt) = pq^2$. **[Delhi Maths (H) 1998; 2004, 11]**

Sol. Given $zqs - zpt = pq^2$. ...(1)

The usual Monge's subsidiary equations are $-zpdqdx - pq^2dxdy = 0$ ...(2)

and $-zqdxdy - zp(dx)^2 = 0$. ...(3)

Dividing (2) by $-pdx$, we get $zdq + q^2dy = 0$ ...(4)

and dividing (3) by $-z\,dx$ we get $qdy + pdx = 0.$ ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_1$. ...(6)

Using (6) in (4), $C_1dq + q^2dy = 0$ or $(1/q^2)dq + (1/C_1)dy = 0.$

Integrating it, $-1/q + y/C_1 = C_2$ or $-1/q + y/z = C_2$, by (6) ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q}+\frac{y}{z}=f(z) \quad \text{or} \quad \frac{\partial y}{\partial z}-\frac{1}{z}y=-f(z), \quad \text{as } q=\frac{\partial y}{\partial z}$$

which is linear in variables y and z (treating x as constant).

Its integrating factor (I.F.) $= e^{-\int(1/z)dz} = e^{-\log z} = z^{-1}$ and so its solution is

$$yz^{-1}=-\int z^{-1}f(z)\,dz+G(x) \quad \text{or} \quad yz^{-1}=F(z)+G(x), \quad \text{where} \quad F(z)=\int f(z)\,dz$$

or $y = zF(z) + zG(x)$ or $y = H(z) + zG(x)$,

where $H(z)[= zF(z)]$ and $G(x)$ are arbitrary functions.

Ex. 5. *Solve* $2yq + y^2t = 1$.

Sol. Given equation is $0.r + 0.s + y^2.t = 1 - 2yq.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = 0$, $T = y^2$, $V = 1 - 2yq$.

Hence the usual subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $y^2dqdx - (1 - 2yq)dxdy = 0$ ...(2)

and $y^2(dx)^2 = 0.$ ...(3)

From (3), $dx = 0$ so that $x = c_1$. ...(4)

From (2), $y^2dq + 2yq\,dy - dy = 0$ or $d(y^2q) - dy = 0.$

Integrating it, $y^2q - y = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), an intermediate integral is

$y^2q - y = f(x)$ or $y^2(\partial z/\partial y) - y = f(x)$

or $$\partial z/\partial y = 1/y + (1/y^2)\times f(x)$$ ...(6)

Integrating (6) w.r. t. y, treating x as constant, we get

$z = \log y - (1/y) f(x) + g(x)$ or $yz = y \log y - f(x) + y\,g(x)$,

where f and g being arbitrary functions.

Ex. 6. *Solve* $(e^x - 1)(qr - ps) = pqe^x$.

Sol. Given $q(e^x - 1)r - p(e^x - 1)s = pqe^x.$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = q(e^x - 1)$, $S = -p(e^x - 1)$, $T = 0$, $V = pqe^x$.

Then the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q(e^x - 1)dpdy - pqe^xdxdy = 0$ ...(2)

and $q(e^x - 1)(dy)^2 + p(e^x - 1)dxdy = 0.$ ...(3)

Now, (3) $\Rightarrow$ $qdy + pdx = 0$ $\Rightarrow$ $dz = 0$, as $dz = pdx + qdy$.

Integrating, $z = c_1$, c_1 being an arbitrary constant ...(4)

Again, from (2), $(e^x - 1)dp - pe^xdx = 0$ or $$\frac{dp}{p}-\frac{e^x}{e^x-1}dx=0$$

Integrating, $\log p - \log(e^x - 1) = \log c_2$ or $p/(e^x - 1) = c_2$. ...(5)

From (4) and (5), an intermediate integral is $p/(e^x - 1) = f(z)$, f being an arbitrary function or $\quad \dfrac{\partial z}{\partial x} = (e^x - 1)f(z),$ $\quad$ or $\quad \dfrac{1}{f(z)}\dfrac{\partial z}{\partial x} = e^x - 1.$

Integrating w.r.t. 'x', treating y as constant, we get

$F(z) = e^x - x + G(y)$ $\quad$ or $\quad x = e^x + G(y) - F(z),$

F and G being arbitrary functions, where $\int (1/f(z))dz = F(z)$.

Miscellaneous problems based on types 1, 2, 3 and 4

Solve the following partial differential equations by using Monge's method:

1. $x^2r - y^2t = xy.$ **Ans.** $z = xy \log x + x F(y/x) + G(xy)$

2. $(1 + pq + q^2)r + s(q^2 - p^2) - (1 + pq + p^2)t = 0$ **Ans.** $z\{2 + (x + y)\}^{1/2} = F(x + y) + G(x - y)$

3. $q(1 + q)r - (1 + 2q)(1 + p)s + (1 + p)^2t = 0$ **Ans.** $x = F(x + y + z) + G(x + z)$

4. $x^2r - y^2t - xp + yq = xy.$ **Ans.** $z = (xy/4) \times \{(\log x)^2 - (\log y)^2\} + xyF(x/y) + G(xy)$

9.11. Monge's Method of integrating the equation Rr + Ss + Tt + U(rt – s²) = V, where r, s, t have their usual meaning and R, S, T, U, V are functions of x, y, z.

Given $\quad Rr + Ss + Tt + U(rt - s^2) = V.$ ...(1)

We have $\quad dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy = rdx + sdy$

and $\quad dq = (\partial q/\partial x)\,dx + (\partial q/\partial y)\,dy = sdx + tdy$

which give $\quad r = (dp - sdy)/dx \quad$ and $\quad t = (dq - sdx)/dy.$

Putting these values in (1) and simplifying, we get

$(Rdpdy + Tdqdx - Udpdq - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy\} = 0.$

Hence the usual Monge's subsidiary equations are

$$L \equiv Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \qquad ...(2)$$

and
$$M \equiv R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy = 0. \qquad ...(3)$$

We cannot factorise M as we did before (see Art 9.1), on account of the presence of the additional terms, $Udpdx + Udqdy$. Hence let us factorise $M + \lambda L$, where λ is some multiplier to be determined later. Now, we have

$$M + \lambda L \equiv R(dy)^2 + T(dx)^2 - (S + \lambda V)dxdy + Udpdx + Udqdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq = 0. \quad ...(4)$$

Factorising L.H.S. of (4), let k and m be constants such that

$$M + \lambda L \equiv (Rdy + mTdx + kUdp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0. \qquad ...(5)$$

Comparing coefficients in (4) and (5), we get $\quad R/m + mT = -(S + \lambda V),$ ...(6)

$k = m \quad$ and $\quad R\lambda/k = U.$ ...(7)

Now, the two relations of (7) give $\quad m = R\lambda u$

Putting this valus of m in (6) and simplifying, we get $\quad \lambda^2(UV + RT) + \lambda US + U^2 = 0,$ ...(8)

which is quadratic in λ. Let λ_1 and λ_2 be its roots.

When $\lambda = \lambda_1$, (7) $\Rightarrow \quad R\lambda_1/k = U \quad \Rightarrow \quad k = R\lambda_1/U \quad \Rightarrow \quad m = R\lambda_1/U$

Hence (5) gives
$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

or
$$(Udy + \lambda_1 Tdx + \lambda_1 Udp)\,(Udx + \lambda_1 Rdy + \lambda_1 Udq) = 0. \qquad ...(9)$$

Similarly for $\lambda = \lambda_2$, (5) gives

$$(Udy + \lambda_2 Tdx + \lambda_2 Udp)\,(Udx + \lambda_2 Rdy + \lambda_2 Udq) = 0. \qquad ...(10)$$

Now one factor of (9) is combined with one factor of (10) to give an intermediate integral. Exactly similarly, the other pair will give rise to another intermediate integral. In this connection remember that we must combine first factor of (9) with the second factor of (10) and similarly the second factor of (9) with the first factor of (10). Thus for the desired solution the proper method is to combine the factors in the following manner :

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad ...(11)$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0, \qquad Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \qquad ...(12)$$

Let equations (11) give two integrals $u_1 = c$ and $v_1 = d_1$ so that one intermediate integral is

$$u_1 = f_1(v_1), f_1 \text{ being an arbitrary function} \qquad ...(13)$$

Similarly, (12) gives second intermediate integral $\qquad u_2 = f_2(v_2), \qquad ...(14)$

where f_2 is an arbitrary function

We now solve (13) and (14) for p and q and substitute in $dz = pdx + qdy$, which after integration gives the desired general solution.

Remark 1. There are in all four ways of combining factors of (9) and (10). By combining the first factors in these equations, we would get $u\, dy = 0$ on substraction (after dividing equations by λ_1 and λ_2 respectively) and this would not produce any solution. Similarly, combining the second factors in these equations would give $u\, dx = 0$ and hence would produce no solution. Hence for getting integrals of the given equation we must proceed as explained in (11) and (12).

Remark 2. In what follows we shall use the following two results of equation $a\lambda^2 + b\lambda + c = 0$

(*i*) $a = b = 0$, *i.e.*, the coefficients of λ^2 and λ both equal to zero imply that both roots of the equatin are equal to ∞

(*ii*) $a = 0$ but $b \neq 0$, *i.e.*, the coefficient of λ^2 is zero but that of λ is non–zero imply that one root of the equation is ∞ and the other is $-c/b$.

Remark 3. When the two values of λ are equal, we shall have only one intermediate integral $u_1 = f(v_1)$ and proceed as explained in solved examples of type 1 based on $Rr + Ss + Tt + U(rt - s^2) = V$ given below.

An integral of *a* more general form can be obtained by taking the arbitrary function occuring in the intermediate integral to be linear.

Let $u_1 = mv_1 + n$, where m and n are some constants. Then integrating it by Lagrange's method we find the solution of the given equation.

9.12. Type 1: When the roots of λ –quadratic (8) of Art 9.11 are identical.

Solved examples of type 1 based on Rr + Ss + Tt + U(rt – s²) = V

Ex. 1. *Solve* $5r + 6s + 3t + 2(rt - s^2) + 3 = 0$. **[I.A.S. 1973 ; Meerut 1998]**

Sol. Given equation $\qquad 5r + 6s + 3t + 2(rt - s^2) = -3. \qquad ...(1)$

Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = 5$, $S = 6$, $T = 3$, $U = 2$ and $V = -3$. Hence the λ–quadratic $\qquad \lambda^2(UV + RT) + \lambda SU + U^2 = 0$

becomes $\quad 9\lambda^2 + 12\lambda + 4 = 0 \quad$ or $\quad (3\lambda + 2)^2 = 0 \quad$ so that $\quad \lambda_1 = \lambda_2 = -2/3$.

There is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \quad \text{and} \quad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $\quad 2dy + (-2/3) \times 3dx + (-2/3) \times 2dp = 0 \quad$ and $\quad 2dx + (-2/3) \times 5dy + (-2/3) \times 2dq = 0$

or $\quad 3dy - 3dx - 2dp = 0 \quad$ and $\quad 3dx - 5dy - 2dq = 0.$

Integrating, $\quad 3y - 3x - 2p = c_1 \quad$ and $\quad 3x - 5y - 2q = c_2. \qquad ...(2)$

Hence here the only intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q), \text{ where } f \text{ is an arbitrary function.} \qquad ...(3)$$

Solving the two equations of (2) for p and q, we have

$$p = (1/2) \times (3y - 3x - c_1) \quad \text{and} \quad q = (1/2) \times (3x - 5y - c_2).$$

Putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = (1/2) \times (3y - 3x - c_1)dx + (1/2) \times (3x - 5y - c_2)dy$$

or $$2dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1dx - c_2dy.$$

Integrating, $$2z = 3xy - (3x^2/2) - (5y^2/2) - c_1x - c_2y + c_3,$$

which is the required complete integral, c_1, c_2 and c_3 being arbitrary constants.

Alternative solution. An integral of a more general form can be obtained by supposing the arbitrary function f occuring in the intermediate integral (3) to be linear, giving

$$3y - 3x - 2p = m(3x - 5y - 2q) + n, \text{ where } m \text{ and } n \text{ are arbitrary constants.} \quad ...(4)$$

Re–writing (4), $$2p - 2mq = 3y - 3x + 5my - 3mx - n. \quad ...(5)$$

Lagrange's auxiliary equations for (5) are $$\frac{dx}{2} = \frac{dy}{-2m} = \frac{dz}{3y-3x+5my-3mx-n}. \quad ...(6)$$

Taking the first two fractions of (6), we have

$$dy + mdx = 0 \qquad \text{so that} \qquad y + mx = a. \quad ...(7)$$

Now, $$\text{each fraction of (6)} = \frac{3xdx+5ydy+2dz}{6x-10my+6y-6x+10my-6mx-2n} \quad ...(8)$$

Hence taking first fraction of (6) and fraction (8), we have

$$\frac{dx}{2} = \frac{3xdx+5ydy+2dz}{6y-6mx-2n} \qquad \text{or} \qquad dx = \frac{3xdx+5ydy+2dz}{3y-3mx-n}$$

or $$3xdx + 5ydy + 2dz = (3y - 3mx - n)dx$$

or $$2dz + 3xdx + 5ydy = \{3(a - mx) - 3mx - n\}dx, \text{ using (7)}$$

or $$2dz + 3xdx + 5ydy = (3a - 6mx - n)dx\ .$$

Integrating, $$2z + (3x^2/2) + (5y^2/2) = 3ax - 3mx^2 - nx + b/2$$

or $$4z + 3x^2 + 5y^2 = 6x(y + mx) - 6mx^2 - 2xn + b, \text{ using (7)}$$

or $$4z - 6xy + 3x^2 + 5y^2 + 2nx = b. \quad ...(9)$$

From (7) and (9), the required general solution is $4z - 6xy + 3x^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 2. *Solve* $3r + 4s + t + (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 3$, $S = 4$, $T = 1$, $U = 1$, $V = 1$. Then, λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $4\lambda^2 + 4\lambda + 1 = 0$ or $(2\lambda + 1)^2 = 0$ so that $\lambda_1 = \lambda_2 = -1/2$.

There is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \qquad \text{and} \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $$dy + (-1/2) \times dx + (-1/2) \times dp = 0 \qquad \text{and} \qquad dx + (-1/2) \times 3dy + (-1/2) \times dq = 0$$

or $$-2dy + dx + dp = 0 \qquad \text{and} \qquad 3dy - 2dx + dq = 0. \quad ...(1)$$

Integrating, $$-2y + x + p = c_1 \qquad \text{and} \qquad 3y - 2x + q = c_2. \quad ...(2)$$

Hence the only intermediate integral is

$$-2y + x + p = f(3y - 2x + q), \text{ where } f \text{ is an arbitrary function.} \quad ...(3)$$

Solving (2) for p and q, $p = 2y - x + c_1$ and $q = -3y + 2x + c_2$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

or $$dz = 2(ydx + xdy) - xdx - 3ydy + c_1dx + c_2dy.$$

Integrating, $$z = 2xy - (x^2/2) - (3y^2/2) + c_1x + c_2y + c_3,$$

which is the required complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. In order to get the more general solution, we assume the arbitrary function ϕ in (3) to be linear. Thus, we take

$$-2y + x + p = m(3y - 2x + q) + n, \ m, n \text{ being arbitrary constants}$$

or $$p - mq = 2y - x + 3my - 2mx + n. \qquad ...(4)$$

Lagrange's auxiliary equations for (4) are $$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{2y-x+3my-2mx+n}. \qquad ...(5)$$

Taking the first two fractions of (5), $dy + mdx = 0$ so that $y + mx = a$. ...(6)

Now, $$\text{each fraction of (5)} = \frac{xdx+3ydy+dz}{x-3my+2y-x+3my-2mx+n} \qquad ...(7)$$

Taking the first fraction of (5) and the fraction (7), we have $$\frac{dx}{1} = \frac{xdx+3ydy+dz}{2y-2mx+n}$$

or $$xdx + 3ydy + dz = (2y - 2mx + n)dx$$

or $$xdx + 3ydy + dz = 2(a - mx)dx - 2mxdx + ndx, \text{ using (6)}$$

Integrating, $$(x^2/2) + (3y^2/2) + z = 2ax - mx^2 - mx^2 + nx + b/2$$

or $$x^2 + 3y^2 + 2z - 2x(y + mx) + 2mx^2 - nx = b, \text{ using (6)} \qquad ...(8)$$

From (6) and (8), the required general solution is $x^2 + 3y^2 + 2z - 2xy - nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 3. *Solve* $(q^2 - 1)zr - 2pqzs + (p^2 - 1)zt + z^2(rt - s^2) = p^2 + q^2 - 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = z(q^2 - 1)$, $S = -2pqz$, $T = z(p^2 - 1)$, $U = z^2$ and $V = p^2 + q^2 - 1$.

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes

$p^2q^2\lambda^2 - 2pqz + z^2 = 0$ or $(pq\lambda - z)^2 = 0$ so that $\lambda_1 = \lambda_2 = z/pq$.

There is only one intermediate integral given by equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \quad \text{and} \quad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $$z^2dy + \frac{z^2(p^2-1)}{pq}dx + \frac{z^3}{pq}dp = 0 \quad \text{and} \quad z^2dx + \frac{z^2(q^2-1)}{pq}dy + \frac{z^3}{pq}dq = 0$$

or $$pqdy + (p^2 - 1)dx + zdp = 0 \quad \text{and} \quad pqdx + (q^2 - 1)dy + zdq = 0$$

or $$p(qdy + pdx) - dx + zdp = 0 \quad \text{and} \quad q(pdx + qdy) - dy + zdq = 0$$

or $$pdz + zdp - dx = 0 \quad \text{and} \quad qdz + zdq - dy = 0, \text{ as } dz = pdx + qdy$$

or $$d(pz) - dx = 0 \quad \text{and} \quad d(qz) - dy = 0.$$

Integrating, $$pz - x = c_1 \quad \text{and} \quad qz - y = c_2. \qquad ...(1)$$

Hence the only intermediate integral is $pz - x = f(qz - y)$, f being an arbitrary function. ...(2)

Solving (1) for p and q, $p = (c_1 + x)/z$ and $q = (c_2 + y)/z$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (1/z) \times (c_1 + x)dx + (1/z) \times (c_2 + y)dy$ or $zdz = (c_1 + x)dx + (c_2 + y)dy$.

Integrating, $$(1/2) \times z^2 = (1/2) \times (c_1 + x)^2 + (1/2) \times (c_2 + y)^2 + (1/2) \times c_3'.$$

or $$z^2 = x^2 + y^2 + 2c_1x + zc_2y + c_3, \text{ where } c_3 = c_1^2 + c_2^2 + c_3'$$

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. To find the more general solution, we take the arbitrary function f in (2) to be linear. So, let $pz - x = m(qz - y) + n$, m, n being arbitrary constants.

or $$pz - mqz = x - my + n. \qquad ...(3)$$

Lagrange's auxiliary equation for (3) are $$\frac{dx}{z} = \frac{dy}{-mz} = \frac{dz}{x-my+n}. \qquad ...(4)$$

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = a$. ...(5)

Now, each fraction of (4) $= \dfrac{(-x/z)dx-(y/z)dy+dz}{z\times(-x/z)-mz\times(-y/z)+x-my+n}$. ...(6)

Taking the first fraction of (4) and fraction (6), $\dfrac{dx}{z} = \dfrac{-(x/z)dx-(y/z)dy+dz}{n}$

or $-xdx - ydy + zdz = ndx$ or $-2zdz + 2xdx + 2ydy + 2ndx = 0$.

Integrating, $-z^2 + x^2 + y^2 + 2nx = b$, b being an arbitrary constant ...(7)

From (5) and (7), the required general solution is $-z^2 + x^2 + y^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m, n are arbitrary constants.

Ex. 4. *Solve* $2s + (rt - s^2) = 1$. **[Garwhal 1995; Meerut 2000]**

Sol. Comparing the given equation with the equation $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 0$, $S = 2$, $T = 0$, $U = 1$, $V = 1$, so λ–quardratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ becomes $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda_1 = \lambda_2 = -1$.

Since we have equal values of l, there would be only one intermediate integral given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $dy - dp = 0$ and $dx - dq = 0$, using (1)

which give $y - p = c_1$, and $x - q = c_2$.

Solving these for p and q, $p = y - c_1$ and $q = x - c_2$.

$\therefore$ $dz = pdx + qdy = (y - c_1)dx + (x - c_2)dy = (ydx + xdy) - c_1dx - c_2dy$,

or $dz = d(xy) - c_1dx - c_2dy$.

Integrating, $z = xy - c_1x - c_2y + c_3$, which is solution, c_1, c_2, c_3 being arbitrary constants.

Ex. 5. $z(1 + q^2)r - 2pqzs + z(1 + p^2)t + z^2(s^2 - rt) + 1 + p^2 + q^2 = 0$.

Sol. Comparing the give equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = z(1 + q^2)$, $S = -2pqz$, $T = z(1 + p^2)$, $U = z^2$ and $V = -(1 + p^2 + q^2)$. ... (1)

Hence λ–quadratic *i.e.* $\lambda^2(RT + UV) + \lambda US + U^2 = 0$ gives

$\lambda^2(p^2q^2) - 2\lambda zpq + z^2 = 0$ or $(\lambda pq - z)^2 = 0$.

Thus here we obtain $\lambda_1 = \lambda_2 = z/pq$. Hence there would be only one intermediate integral which is given by

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0. \qquad ...(2)$$

and

$$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad ...(3)$$

Using (1), (2) becomes $pq\,dy + (1 + p^2)dx + zdp = 0$ ...(4)

Using (1), (3) becomes $pqdx + (1 + q^2)dy + zdq = 0$ ...(5)

Now from (4), $p(pdx + qdy) + dx + zdp = 0$ or $pdz + dx + zdp = 0$, as $dz = pdx + qdy$

or $d(zp) + dx = 0$ so that $zp + x = c_1$. ...(6)

Similarly (5) gives $zq + y = c_2$, c_2 being an arbitrary constant ...(7)

Solving (6) and (7), we get $p = (c_1 - x)/z$ and $q = (c_2 - y)/z$.

$\therefore$ $dz = pdx + qdy = \{(c_1 - x)/z\}dx + \{(c_2 - y)/z\}dy$ or $zdz = c_1dx + c_2dy - (xdx + ydy)$.

Integrating, $(1/2) \times z^2 = c_1x + c_2y - (x_2 + y^2)/2 + c_3/2$ or $z^2 = 2c_1x + 2c_2y - x^2 - y^2 + c_3$, which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 6. *Solve* $2r + te^x - (rt - s^2) = 2e^x$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 2$, $S = 0$, $T = e^x$, $U = -1$ and $V = 2e^x$. ... (1)

Hence the λ–quadratic $\lambda^2(UV+RT)+\lambda SU+U^2=0$ gives $\lambda^2(2e^x-2e^x)+(\lambda\times 0)+1=0.$

Since the coefficient of λ^2 and λ in the above quadratic vanish, it follows from the theory of equations that its both the roots must be infinite. Thus $\lambda_1=\lambda_2=\infty$. Since the two roots are equal there would be only one intermediate integral which is given by

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad Udx+\lambda_2 Rdy+\lambda_2 Udq=0,$$

i.e., by $\quad (U/\lambda_1)dy+Tdx+Udq=0 \quad$ and $\quad (U/\lambda_2)dx+Rdy+Udq=0,$

i.e., by $\quad e^x dx-dp=0$ using (1) $\quad$ and $\quad 2dy-dq=0$, using (1)

Integrating these $\quad e^x-p=c_1 \quad$ and $\quad 2y-q=c_2.$

Solving these, $\quad p=e^x-c_1 \quad$ and $\quad q=2y-c_2.$

Now, $\quad dz=pdx+qdy=(e^x-c_1)dx+(2y-c_2)dy.$

Integrating, $\quad z=e^x-c_1x+y^2-c_2y+c_3,$

which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 7. *Solve* $r+t-(rt-s^2)=1.$

Sol. Comparing the given equation with $\quad Rr+Ss+Tt+U(rt-s^2)=V,$

$$R=1, \qquad S=0, \qquad T=1, \qquad U=-1, \qquad V=1. \qquad \text{...(1)}$$

So λ–quadratic $\lambda^2(UV+RT)+\lambda US+U^2=0$ becomes $(0\times\lambda^2)+(0\times\lambda)+1=0$. Since the coeffieicnts of both λ^2 and λ are zero, so both roots of this quadratic are equal to ∞. So $\lambda_1=\lambda_2=\infty$

Now, the only one intermediate integral is given by equations

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad \lambda_1 Rdy+Udx+\lambda_1 Udq=0$$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

or $\quad (1/\lambda_1)\times Udy+Tdx+Udp=0 \quad$ and $\quad Rdy+(1/\lambda_1)\times Udx+Udq=0$

or $\quad Tdx+Udp=0$, as $\lambda_1=\infty \quad$ and $\quad Rdy+Udq=0$, as $\lambda_1=\infty$

or $\quad dx-dp=0 \quad$ and $\quad dy-dq=0$, using (1)

Integrating, $\quad p-x=c_1 \quad$ and $\quad q-y=c_2. \qquad$ ...(2)

Solving (2) for p and q, $\quad p=x+c_1 \quad$ and $\quad q=y+c_2.$

Putting these values of p and q in $dz=pdx+qdy$, we get $\quad dz=(x+c_1)dx+(y+c_2)dy$

Integrating, $\quad z=x^2/2+c_1x+y^2/2+c_2y+c_3,$

which is the required integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 8. *Solve* $2pr+2qt-4pq(rt-s^2)=1.$

Sol. Comparing the given equation with $Rr+Ss+Tt+U(rt-s^2)=V$, we have

$$R=2p, \qquad S=0, \qquad T=2q, \qquad U=-4pq, \qquad V=1. \qquad \text{...(1)}$$

Then the λ–quadratic $\lambda^2(UV+RT)+\lambda SU+U^2=0$ becomes $(0\times\lambda^2)+(0\times\lambda)+4p^2q^2=0$. Since the coefficients of both λ^2 and λ are zero, so both roots of the λ– quadratic are equal to ∞. So $\lambda_1=\lambda_2=\infty$.

Now the only intermediate integral is given by the equation

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \quad \text{and} \quad \lambda_1 Rdy+Udx+\lambda_1 Udq=0$$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

$$(1/\lambda_1)\times Udy+Tdx+Udp=0 \quad \text{and} \quad Rdy+(1/\lambda_1)\times Udx+Udq=0$$

or $\quad 2qdx-4pqdp=0 \quad$ and $\quad 2pdy-4pqdq=0$, using (1)

or $\quad 2pdp-dx=0 \quad$ and $\quad 2qdq-dy=0.$

Integrating, $p^2 - x = c_1$ and $q^2 - y = c_2$.

Hence $p = \pm (c_1 + x)^{1/2}$ and $q = \pm (c_2 + y)^{1/2}$

Putting values of p and q in $dz = pdx + qdy$ gives $dz = \pm (c_1 + x)^{1/2}dx \pm (c_2 + y)^{1/2}dy$.

Integrating, $z = \pm (2/3) \times (c_1 + x)^{3/2} \pm (2/3) \times (c_2 + y)^{3/2} + c_3/2$

or $3z = \pm 2(c_1 + x)^{3/2} \pm 2(c_2 + y)^{3/2} + c_3$,

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 9. *Solve* $(1 + q^2)r - 2pqs + (1 + p^2)t + (1 + p^2 + q^2)^{-1/2} (rt - s^2) = -(1 + p^2 + q^2)^{3/2}$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 1 + q^2$, $S = -2pq$, $T = 1 + p^2$, $U = (1 + p^2 + q^2)^{-1/2}$, $V = -(1 + p^2 + q^2)^{3/2}$ ...(1)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes

$$\lambda^2 \{-(1 + p^2 + q^2) + (1 + q^2)(1 + p^2)\} - 2pq (1 + p^2 + q^2)^{-1/2}\lambda + (1 + p^2 + q^2)^{-1} = 0$$

or $$p^2q^2(1 + p^2 + q^2)\lambda^2 - 2pq(1 + p^2 + q^2)^{1/2} \lambda + 1 = 0$$

or $\{pq(1 + p^2 + q^2)^{1/2}\lambda - 1\}^2 = 0$ so that $\lambda_1 = \lambda_2 = 1/pq(1 + p^2 + q^2)^{1/2}$.

Here there is only intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $$\frac{1}{(1+p^2+q^2)^{1/2}}dy + \frac{1+p^2}{pq\,(1+p^2+q^2)^{1/2}}dx + \frac{dp}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

and $$\frac{1}{(1+p^2+q^2)^{1/2}}dx + \frac{1+q^2}{pq\,(1+p^2+q^2)^{1/2}}dy + \frac{dq}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

or $$pqdy + (1 + p^2)dx + [1/(1 + p^2 + q^2)^{1/2}]dp = 0 \qquad ...(2)$$

and $$pqdx + (1 + q^2)dy + \{1/(1 + p^2 + q^2)^{1/2}\}dq = 0. \qquad ...(3)$$

Eliminating dy between (2) and (3), $\{(1 + p^2)(1 + q^2) - p^2q^2\}dx + \dfrac{(1+q^2)dp - pqdq}{(1+p^2+q^2)^{1/2}} = 0$

or $$(1 + p^2 + q^2)dx + \frac{(1 + p^2 + q^2)dp - (p^2dp + pqdq)}{(1 + p^2 + q^2)^{1/2}} = 0$$

or $$dx + \frac{dp}{(1+p^2+q^2)^{1/2}} - \frac{p}{2}\frac{2pdp+2qdq}{(1+p^2+q^2)^{3/2}} = 0 \qquad \text{or} \qquad dx + d\left\{\frac{p}{(1+p^2+q^2)^{1/2}}\right\} = 0$$

Integrating, $x + p(1 + p^2 + q^2)^{-1/2} = a$, where a is an arbitrary constant. ...(4)

Similarly, eliminating dx between (2) and (3), we have

$y + q(1 + p^2 + q^2)^{-1/2} = b$, where b in an arbitrary constant. ...(5)

From (4) and (5), $x - a = -p(1 + p^2 + q^2)^{-1/2}$, $y - b = -q(1 + p^2 + q^2)^{-1/2}$.

$$\therefore \quad \frac{x-a}{y-b} = \frac{p}{q} \quad \text{so that} \quad p = \frac{x-a}{y-b}q. \qquad ...(6)$$

Putting the above value of p in (4), we have

$$x + q\frac{x-a}{y-b}\left\{1+q^2\frac{(x-a)^2}{(y-b)^2}+q^2\right\}^{-1/2} = a \quad \text{or} \quad (x-a) + \frac{x-a}{y-b}q\left[1+\frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2\right]^{-1/2} = 0$$

or $$1 + \frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2 = \frac{q^2}{(y-b)^2} \quad \text{or} \quad (y-b)^2 = q^2[1 - \{(x-a)^2 + (y-b)^2\}].$$

Thus, $$q = (y-b)/[1-\{(x-a)^2+(y-b)^2\}]^{1/2}. \qquad \text{... (7)}$$

Now, (6) and (7) $\Rightarrow$ $$p = \frac{x-a}{y-b}q = \frac{x-a}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}. \qquad \text{... (8)}$$

$$\therefore\ dz = pdx + qdy = \frac{(x-a)\,dx + (y-b)\,dy}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}, \text{ by (7) and (8)}$$

Integrating, $z = [1 - \{(x-a)^2 + (y-b)^2\}]^{1/2} + c$ or $(z-c)^2 = 1 - \{(x-a)^2 + (y-b)^2\}$

$\therefore$ $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ is the complete integral, a, b, c being arbitrary constants.

9.13 Type 2. When the roots of λ –quadratic (8) of Art 9.11 are distinct.

Solved Examples of Type –2 based on Rr + Ss + Tt + U(rt – s²) = V

Ex. 1. *Solve* $3s + rt - s^2 = 2$.

Sol. Given $$3s + (rt - s^2) = 2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0, S = 3, V = 0, U = 1, V = 2$. ...(2)

λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \text{...(3)}$$

Using (2), (3) reduces to $2\lambda^2 + 3\lambda + 1 = 0$ so $\lambda_1 = -1, \lambda_2 = -(1/2)$. ... (4)

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0. \end{aligned}\right\} \qquad \text{... (5)}$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0. \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\begin{aligned} dy - dp = 0 &\quad \text{or} \quad dp - dy = 0 \\ dx - (1/2)dq = 0 &\quad \text{or} \quad dq - 2dx = 0 \end{aligned} \qquad \text{... (5A)}$$

and

$$\begin{aligned} dy - (1/2)dp = 0 &\quad \text{or} \quad dp - 2dy = 0 \\ dx - dq = 0 &\quad \text{or} \quad dq - dx = 0. \end{aligned} \qquad \text{... (6A)}$$

Integration of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q - 2x = c_2 \qquad \text{...(5B)}$$

and $$p - 2y = c_3, \qquad q - x = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$$p - y = f(q - 2x) \quad \text{and} \quad p - 2y = F(q - x), \qquad \text{...(7)}$$

where f and F are arbitrary functions.

Let $$q - 2x = \alpha, \qquad \text{...(8)}$$

and $$q - x = \beta. \qquad \text{...(9)}$$

Then from (7) $$p - y = f(\alpha), \qquad \text{...(10)}$$

and $$p - 2y = F(\beta). \qquad \text{...(11)}$$

[If we treat α and β as constants, then solution of four simultaneous equation (8), (9), (10) and (11) would show that x, y, p and q are all constants which is absurd. Hence α and β will be regarded as variables (parameters) and we will get the general solution in parametric form involving α and β as parameters].

Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10) $$p = y + f(\alpha). \qquad \text{...(14)}$$

From (9) $$q = x + \beta. \qquad ...(15)$$

From (12) and (13), $dx = d\beta - d\alpha$, and $dy = f'(\alpha)d\alpha - F'(\beta)d\beta$. ...(16)

$\therefore$ $dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy$, using (14) and (15)

or $dz = ydx + xdy + f(\alpha)dx + \beta dy = d(xy) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

Thus, $$dz = d(xy) + [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$$

or $$dz = d(xy) + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get $$z = xy + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)d\beta]$$

or $$z = xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)d\beta. \qquad ...(17)$$

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$ ...(19)

Using (18) and (19), (12), (13) and (17) give

$x = \beta - \alpha$, $\quad y = \phi'(\alpha) - \psi'(\beta)$ $\quad z = xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, ϕ and ψ being arbitrary functions and α and β being parameters.

Ex. 2. *Solve* $r + 4s + t + rt - s^2 = 2$. **[I.A.S. 1979]**

Sol. Given $$r + 4s + t + (rt - s^2) = 2. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1$, $S = 4$, $T = 1$, $U = 1$, $V = 2$. ...(2)

λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0. \qquad ...(3)$$

Using (2), (3) reduces to $3\lambda^2 + 4\lambda + 1 = 0$ so $\lambda_1 = -1$, $\lambda_2 = -(1/3)$.

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad ...(5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad ...(6)$$

Using (2) and (4), (5) and (6) respectively gives

$$\begin{aligned} &dy - dx - dp = 0 && \text{or} && \left.\begin{aligned} dp + dx - dy &= 0 \\ dq + dy - 3dx &= 0 \end{aligned}\right\} \quad ...(5A) \\ &dx - (1/3) \times dy - (1/3) \times dq = 0 && \text{or} \\ &dy - (1/3) \times dx - (1/3) \times dp = 0 && \text{or} && \left.\begin{aligned} dp + dx - 3dy &= 0 \\ dq + dy - dx &= 0 \end{aligned}\right\} \quad ...(6A) \\ &dx - dy - dq = 0 && \text{or} \end{aligned}$$

Integration of (5A) and (6A) respectively gives

$$p + x - y = c_1, \qquad q + y - 3x = c_2 \qquad ...(5B)$$

and $$p + x - 3y = c_3, \qquad q + y - x = c_4, \qquad ...(6B)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$$p + x - y = f(q + y - 3x) \quad \text{and} \quad p + x - 3y = F(q + y - x). \qquad ...(7)$$

Let $$q + y - 3x = \alpha, \qquad ...(8)$$

and $$q + y - x = \beta. \qquad ...(9)$$

Then from (7), $$p + x - y = f(\alpha), \qquad ...(10)$$

and $$p + x - 3y = F(\beta). \qquad ...(11)$$

Here α and β are treated as parameters. Solving (8) and (9) for x and (10) and (11) for y gives

$$x = (\beta - \alpha)/2 \qquad \text{...(12)}$$

and
$$y = [f(\alpha) - F(\beta)]/2 \qquad \text{...(13)}$$

From (10), $\quad p = y - x + f(\alpha) \qquad$ ...(14)

From (9), $\quad q = x - y + \beta \qquad$ ...(15)

From (12) and (13), $\quad dx = (1/2) \times (d\beta - d\alpha), \quad dy = (1/2) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]. \quad$ ...(16)

$\therefore \quad dz = pdx + qdy = [y - x + f(\alpha)]dx + (x - y + \beta)dy$, by (14) and (15)

$= ydx + xdy - xdx - ydy + f(\alpha)dx + \beta dy$

$= d(xy) - xdx - ydy + f(\alpha) \times (1/2)\times(d\beta - d\alpha) + \beta \times (1/2)\times[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

$= d(xy) - xdx - ydy + (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - (1/2) \times f(\alpha)d\alpha - (1/2) \times \beta F'(\beta)d\beta$

or
$$2dz = 2d(xy) - 2xdx - 2ydy + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get
$$2z = 2xy - x^2 - y^2 + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)\, d\beta]$$

or
$$2z = 2xy - x^2 - y^2 + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)\, d\beta. \qquad \text{...(17)}$$

Let $\quad \int f(\alpha)d\alpha = \phi(\alpha) \quad$ and $\quad \int F(\beta)\, d\beta = \psi(\beta) \qquad$ ...(18)

so that $\quad f(\alpha) = \phi'(\alpha) \quad$ and $\quad F(\beta) = \psi'(\beta). \qquad$ ...(19)

Using (18) and (19), (12), (13) and (17) give

$2x = \beta - \alpha, \quad 2y = \phi'(\alpha) - \psi'(\beta), \quad 2z = 2xy - x^2 - y^2 + \beta[\phi'(a) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 3. *Solve* $rt - s^2 + 1 = 0$

Sol. Given that $\quad 0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -1. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$ and $V = -1$. ...(2)

Here λ–quadratic $\quad \lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad$ ...(3)

becomes $\quad \lambda^2 - 1 = 0 \quad$ so that $\quad \lambda_1 = -1 \quad$ and $\quad \lambda_2 = 1. \quad$ ...(4)

Since the two values of λ are distinct, we shall get two intermediate integrals which are given by the following sets of equations

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad \text{... (5A)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad \text{... (5B)}$$

Using (2) and (4), equations (5) and (6) reduces to

$$\left.\begin{aligned} dy - dp = 0 \quad & i.e., \quad dp - dy = 0 \\ dx + dq = 0 \quad & i.e., \quad dq + dx = 0 \end{aligned}\right\} \qquad \text{... (5A)}$$

$$\left.\begin{aligned} dy + dp = 0 \quad & i.e., \quad dp + dy = 0 \\ dx - dq = 0 \quad & i.e., \quad dq - dx = 0 \end{aligned}\right\} \qquad \text{... (6A)}$$

Integrating of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q + x = c_2. \qquad \text{...(5B)}$$

and
$$p + y = c_3, \qquad q - x = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 are c_4 are arbitrary constants

From (5B) and (6B), two intermediate integrals are given by

$$p - y = f(q + x) \qquad \text{and} \qquad p + y = F(q - x), \qquad \text{...(7)}$$

where f and F are arbitrary functions.

Let $$q + x = \alpha \qquad ...(8)$$

and $$q - x = \beta. \qquad ...(9)$$

Then, from (7), $$p - y = f(\alpha) \qquad ...(10)$$

and $$p + y = F(\beta). \qquad ...(11)$$

In what follows α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\alpha - \beta)/2 \qquad ...(12)$$

and $$y = [F(\beta) - f(\alpha)]/2 \qquad ...(13)$$

From (10), $$p = y + f(\alpha) \qquad ...(14)$$

From (9), $$q = x + \beta. \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\alpha - d\beta)$, $\quad dy = (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha]$. ...(16)

$$\therefore \quad dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy, \text{ using (14) and (15)}$$
$$= (ydx + xdy) + f(\alpha)dx + \beta dy$$
$$= d(xy) + f(\alpha) \times (1/2) \times (d\alpha - d\beta) + \beta \times (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha], \text{ by (16)}$$
$$= d(xy) + (1/2) \times f(\alpha)d\alpha - (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] + (1/2) \times \beta F'(\beta)d\beta$$

or $$2dz = 2d(xy) + f(\alpha)d\alpha - d[\beta f(\alpha)] + \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$2z = 2xy + \int f(\alpha)d\alpha + \beta f(\alpha) + \beta F(\beta) - \int F(\beta)\, d\beta. \qquad ...(17)$$

Let $\int f(\alpha)\, d\alpha = \phi(\alpha)$ and $\int F(\beta)\, d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) may be re–written as

$$2x = (\alpha - \beta), \qquad 2y = \psi'(\beta) - \phi'(\alpha), \qquad 2z = 2xy - \phi(\alpha) + \beta\{\phi'(\alpha) + \psi'(\beta)\} - \psi(\beta)$$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 4. *Solve* $r + 3s + t + (rt - s^2) = 1$. **[Rohilkhand 1995]**

Sol. Given $$r + 3s + t + (rt + s^2) = 1 \qquad ... (1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $\quad R = 1, S = 3, T = 1, U = 1, V = 1$. ...(2)

Now, λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

or $2\lambda^2 + 3\lambda + 1 = 0$ so that $\lambda = -1, \; -1/2$. Here $\lambda_1 = -1$, $\quad \lambda_2 = -1/2$. ...(4)

Two intermediate integrals of (1) are giving by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad ... (5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad ... (6)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\begin{aligned} dy - dx - dp = 0 \qquad & i.e., \qquad \left.\begin{aligned} dp + dx - dy = 0 \\ dq - 2dx + dy = 0 \end{aligned}\right\} \quad ... 5(A) \\ dx - (1/2) \times dy - (1/2) \times dq = 0 \qquad & i.e., \end{aligned}$$

and
$$\begin{aligned} dy - (1/2) \times dx - (1/2) \times dp = 0 \qquad & i.e., \qquad \left.\begin{aligned} dp + dx - 2dy = 0 \\ dq - dx + dy = 0 \end{aligned}\right\} \quad ...(6A) \\ dx - dy - dq = 0 \qquad & i.e., \end{aligned}$$

Integrating of (5*A*) and (6*A*) respectively gives

$$p + x - y = c_1, \qquad q - 2x + y = c_2 \qquad ...(5B)$$

and

$$p + x - 2y = c_3, \qquad q - x + y = c_4, \qquad ...(6B)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + x - y = f(q - 2x + y) \quad \text{and} \quad p + x - 2y = F(q - x + y), \qquad ...(7)$$

where f and F are arbitrary functions

Let $$q - 2x + y = \alpha \qquad ...(8)$$

and $$q - x + y = \beta. \qquad ...(9)$$

Then, from (7) $$p + x - y = f(\alpha) \qquad ...(10)$$

and $$p + x - 2y = F(\beta). \qquad ...(11)$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad ...(12)$$

and $$y = f(\alpha) - F(\beta). \qquad ...(13)$$

From (10), $$p = y - x + f(\alpha) \qquad ...(14)$$

From (9), $$q = x - y + \beta. \qquad ...(15)$$

From (12) and (13), $$dx = d\beta - d\alpha, \qquad dy = f'(\alpha)d\alpha - F'(\beta)d\beta. \qquad ...(16)$$

$\therefore$ $$dz = pdx + qdy = [y - x + f(\alpha)]dx + [x - y + \beta]dy, \text{ using (14) and (15)}$$

$$= -(x - y)(dx - dy) + f(\alpha)dx + \beta dy$$

$$= -(x - y)\, d(x - y) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)dx - F'(\beta)d\beta], \text{ by (16)}$$

$$= -(x - y)\, d(x - y) - f(\alpha)\, d\alpha + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - \beta F'(\beta)d\beta$$

or $$dz = -(x - y)\, d(x - y) - f(\alpha)\, d\alpha + d[\beta f(\alpha)] - \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$z = -(1/2) \times (x - y)^2 - \int f(\alpha)\, d\alpha + \beta f(\alpha) - \left[\beta F(\beta) - \int F(\beta)\, d\beta\right]. \qquad ...(17)$$

Let $$\int f(\alpha)\, d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad ...(18)$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta). \qquad ...(19)$$

Using (18) and (19), (12), (13) and (17) may be written as

$$x = \beta - \alpha, \qquad y = \phi'(\alpha) - \psi'(\beta), \qquad z = -(1/2)\times (x - y)^2 - \phi(\alpha) + \psi(\beta) + \beta[\phi'(\alpha) - \psi'(\beta)]$$

which is the required solution in parametric form, α and β being parameters, and ϕ and ψ being arbitrary functions.

Ex. 5. *Solve* $rt - s^2 + a^2 = 0$. **[Rohilkhand 1993]**

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -a^2. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$, $V = -a^2$. ...(2)

Then, the λ–quadratic $$\lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad ...(3)$$

becomes $-\lambda^2 a^2 + 1 = 0$ or $\lambda = \pm 1/a$. So $\lambda_1 = 1/a$, $\lambda_2 = -1/a$. ...(4)

Two intermediate integrals of (1) are given by the following two sets

$$\left.\begin{array}{l} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{array}\right\} \qquad ... (5)$$

and

$$\left.\begin{array}{l} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{array}\right\} \qquad ... (6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + (1/a) \times dp = 0 \quad & i.e., \quad dp + ady = 0 \\ dx - (1/a) \times dq = 0 \quad & i.e., \quad dq - adx = 0 \end{aligned}\right\} \quad \ldots (5A)$$

and

$$\left.\begin{aligned} dy - (1/a) \times dp = 0 \quad & i.e., \quad dp - ady = 0 \\ dx + (1/a) \times dq = 0 \quad & i.e., \quad dq + adx = 0 \end{aligned}\right\} \quad \ldots (6A)$$

Integration of (5A) and (6A) respectively gives

$$p + ay = c_1, \qquad q - ax = c_2 \qquad \ldots(5B)$$

and

$$p - ay = c_3, \qquad q + ax = c_4. \qquad \ldots(6B)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + ay = f(q - ax) \qquad \text{and} \qquad p - ay = F(q + ax). \qquad \ldots(7)$$

where f and F are arbitrary functions

Let $$q - ax = \alpha \qquad \ldots(8)$$

and $$q + ax = \beta. \qquad \ldots(9)$$

Then, from (7) $$p + ay = f(\alpha) \qquad \ldots(10)$$

and $$p - ay = F(\beta). \qquad \ldots(11)$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (1/2a) \times (\beta - \alpha) \qquad \ldots(12)$$

and $$y = (1/2a) \times [f(\alpha) - F(\beta)]. \qquad \ldots(13)$$

From (10), $$p = f(\alpha) - ay. \qquad \ldots(14)$$

From (9), $$q = \beta - ax. \qquad \ldots(15)$$

From (12) and (13), $dx = (1/2a) \times (d\beta - d\alpha)$, $dy = (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$ ...(16)

$\therefore$ $dz = pdx + qdy = [f(\alpha) - ay]dx + (\beta - ay)dy$, using (14) and (15)

$$= f(\alpha)dx + \beta dy - a(ydx + xdy)$$

$$= f(\alpha) \times (1/2a) \times (d\beta - d\alpha) + \beta \times (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta] - ad(xy), \text{ by (16)}$$

or $$2adz = \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - 2a^2d(xy) - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integration by parts in the last term on R.H.S., we have

$$2az = \beta f(\alpha) - \int f(\alpha)d\alpha - 2a^2xy - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \ldots(17)$$

Let $$\int f(\alpha)d\alpha = \phi(\alpha) \qquad \text{and} \qquad \int F(\beta)\,d\beta = \psi(\beta) \quad \ldots(18)$$

so that $$f(\alpha) = \phi'(\alpha) \qquad \text{and} \qquad F(\beta) = \psi'(\beta). \quad \ldots(19)$$

Using (18) and (19), (12), (13) and (17) reduces to

$$2ax = \beta - \alpha, \qquad 2ay = \phi'(\alpha) - \psi'(\beta), \qquad 2az = \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) - 2a^2xy + \psi(\beta).$$

which is the required solution in parametric form, α, β, being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being arbitrary functions.

Ex. 6. *Solve* $7r - 8s - 3t + (rt - s^2) = 36$.

Sol. Given that $$7r - 8s - 3t + (rt - s^2) = 36. \qquad \ldots(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 7$, $S = -8$, $T = -3$, $U = 1$, $V = 36$. ...(2)

The λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \ldots(3)$$

becomes $15\lambda^2 - 18\lambda + 1 = 0$ or $(5\lambda - 1)(3\lambda - 1) = 0$. So $\lambda_1 = 1/5$, $\lambda_2 = 1/3$. ...(4)

Two intermediate integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \quad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \quad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$dy + (1/5) \times (-3)dx + (1/5) \times dp = 0 \quad i.e., \quad dp - 3dx + 5dy = 0$$
$$dx + (1/3) \times 7\, dy + (1/3) \times dq = 0 \quad i.e., \quad dq + 7dy + 3dx = 0 \quad \text{... (5A)}$$
$$dy + (1/3) \times (-3)dx + (1/3) \times dp = 0 \quad i.e., \quad dp - 3dx + 3dy = 0$$
$$dx + (1/5) \times 7dy + (1/5) \times dq = 0 \quad i.e., \quad dq + 7dy + 5dx = 0 \quad \text{... (6A)}$$

Integrating of (5*A*) and (6*A*) respectively, gives

$$p - 3x + 5y = c_1, \qquad q + 7y + 3x = c_2 \qquad \text{...(5B)}$$

and $$p - 3x + 3y = c_3, \qquad q + 7y + 5x = c_4, \qquad \text{...(6B)}$$

where c_1, c_2 c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediatre integrals are given by

$$p - 3x + 5y = f(q + 7y + 3x) \quad \text{and} \quad p - 3x + 3y = F(q + 7y + 5x) \quad \text{...(7)}$$

where *f* and *F* are arbitrary functions

Let $$q + 7y + 3x = \alpha \qquad \text{...(8)}$$

and $$q + 7y + 5x = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p - 3x + 5y = f(\alpha) \qquad \text{...(10)}$$

and $$p - 3x + 3y = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\beta - \alpha)/2 \qquad \text{...(12)}$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad \text{...(13)}$$

From (10), $$p = f(\alpha) + 3x - 5y. \qquad \text{...(14)}$$

From (9), $$q = \beta - 7y - 5x. \qquad \text{...(15)}$$

From (12) and (13), $$dx = (1/2) \times (d\beta - d\alpha), \; dy = (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)\, d\beta\}. \qquad \text{...(16)}$$

$\therefore$ $dz = pdx + qdy = \{f(\alpha) + 3x - 5y\}dx + \{\beta - 7y - 5x)dy$, using (14) and (15)

$$= 3xdx - 7ydy - 5(ydx + xdy) + f(\alpha)dx + \beta dy$$
$$= 3xdx - 7ydy - 5d(xy) + f(\alpha) \times (1/2) \times (d\beta - d\alpha) + \beta \times (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)d\beta\}$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + d\{\beta f(\alpha)\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integrating by parts in the last term on R.H.S., we have

$$2z = 3x^2 - 7y^2 - 10xy + \beta f(\alpha) - \int f(\alpha)\, d\alpha - [\beta F(\beta) - \int F(\beta)\, d\beta]$$

or $$2z = 3x^2 - 7y^2 - 10xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)\, d\alpha + \int F(\beta)\, d\beta. \qquad \text{...(17)}$$

Let $$\int f(\alpha)\, d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta) \qquad \text{...(19)}$$

Using (18) and (19), relation (12), (13) and (17) become

$x = (1/2) \times (\beta - \alpha)$, $y = (1/2) \times [\phi'(\alpha) - \psi'(\beta)]$, $2z = 3x^2 - 7y^2 - 10xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$.
which is required solution in parametric form, α and β being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being

arbitrary functions.

9.14 Miscellaneous examples on Rr + Ss + Tt + U(rt – s²) = V.

In some problems only one intermediate integral is possible. Sometimes even after getting two intermediate integrals, it may not be possible to get p and q from those intermediate integrals. In such problems, final solution is obtained by integrating only one intermediate integral by the methods of solution of first order equation, for example, Charpit's method. Again, we can avoid Charpit's method by taking $u_1 = \phi_1(v_1)$ and u_2 = constant = λ (say) to obtain final solution. [Here we have assumed that $u_1 = \phi_1(v_1)$ and $u_2 = \phi_2(v_2)$ are two intermediate integrals]. Since an arbitrary constant can be regarded as a particular case of an arbitrary function, the values of p and q derived from $u_1 = \phi_1(v_1)$ and $u_2 = \lambda$ will make $dz = pdx + qdy$ integrable. The complete integral so obtained will involve one arbitrary funtion ϕ_1 and two arbitrary constants, namely, λ and the constant of integration. To obtain the general integral, express one of the arbitrary constant as an arbitrary function of the other and eliminate this remaining constant between the equation so obtained and that deduced from it by differentiation with respect to that constant.

Ex. 1. *Obtain the intermediate integral of* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$. **[Rohilkhand 1992]**

Sol. Given $$2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$$R = 2y, \quad S = px + qy, \quad T = x, \quad U = -xy, \quad V = 2 - pq. \qquad ...(2)$$

Now, λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

reduces to $$\lambda^2[2yx - xy(2 - pq)] + \lambda[-xy(px + qy)\} + x^2y^2 = 0$$

or $\lambda^2 pq - \lambda(px + qy) + xy = 0$ or $(\lambda p - y)(\lambda q - x) = 0$

or $\lambda = y/p,\ x/p$ so that $\lambda_1 = y/p$ and $\lambda_2 = x/q$. ...(4)

Two intermediate integrals are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad ...(5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad ...(6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} -xydy + (y/p)xdx + (y/p)(-xy)dp = 0 \quad & i.e., \quad (pdy + ydp) - dx = 0 \\ \text{and} \quad -xydx + (x/q)(2y)dy + (x/q)(-xy)dq = 0 \quad & i.e., \quad (qdx + xdq) - 2dy = 0. \end{aligned}\right\} \qquad ...(5A)$$

$$\left.\begin{aligned} -xydy + (x/q)xdx + (x/q)(-xy)dp = 0 \quad & i.e., \quad -qydy + xdx - xydp = 0 \\ \text{and} \quad -xydx + (y/p)(2y)dy + (y/p)(-xy)dq = 0 \quad & i.e., \quad -pxdx + 2ydy - xydq = 0 \end{aligned}\right\} \qquad ...(6A)$$

Integrating (5A), $py - x = c_1$ and $qx - 2y = c_2$

Hence one intermediate integral is $$py - x = \phi(qx - 2y). \qquad ...(7)$$

Note, the equation (6A) cannot be integrated. Hence in this problem we can obtain only one intermediate integral, *i.e.*, (7). Here ϕ is an arbitrary function.

Ex. 2. *Solve* $qr + (p + x)s + yt + y(rt - s^2) + q = 0$. **[Rohilkhand 1992]**

Sol. Given $$qr + (p + x)s + yt + y(rt - s^2) = -q. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$$R = q, \quad S = p + x, \quad T = y, \quad U = y \quad \text{and} \quad \mathrm{T} = -q. \qquad ...(2)$$

Now, the λ–quadratic $$\lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad ...(3)$$

reduces to $(0 \times \lambda^2) + \lambda y(p + x) + y^2 = 0$. Since coefficient of λ^2 is zero, it follows that its one root is ∞. The other root is $-y/(p + x)$.

Let $\lambda_1 = -y/(p + x)$ and $\lambda_2 = \infty$. ...(4)

One intermediate integral is given by the following sets

$$\left.\begin{array}{lll} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 & ,i.e., & Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 & & (1/\lambda_2)\times Udx + Rdy + Udq = 0. \end{array}\right\} \quad ...(5)$$

Using (2) and (4), equations of (5) reduce to

$$\left.\begin{array}{lll} ydy - \dfrac{y^2}{p+x}dx - \dfrac{y^2}{p+x}dp = 0 & i.e. & \dfrac{dp+dx}{p+x} - \dfrac{dy}{y} = 0 \\ \text{and} \quad qdy + ydq = 0 & i.e., & d(yq) = 0 \end{array}\right\} \quad ...(5A)$$

Integrating (5A), $\log(p + x) - \log y = \log c_1$ or $(p + x)/y = c_1$ ...(6)

and $yq = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is given by $qy = f\left(\dfrac{p+x}{y}\right)$. ...(8)

Charpit's auxiliary equations for (8) are $\dfrac{dx}{-\frac{1}{y}f'\left(\frac{p+x}{y}\right)} = \dfrac{dy}{y} = \dfrac{dp}{\frac{1}{y}f'\left(\frac{p+x}{y}\right)}$. ...(9)

Taking the first and third fractions of (9), we have

$dx + dp = 0$ so that $x + p = c$, where c is an arbitrary constant. ...(10)

Solving (8) and (10) for p and q, $p = c - x$ and $q = (1/y) \times f(c/y)$.

Putting these values of p and q in $dz = pdx + qdy$, $dz = (c - x)dx + (1/y) \times f(c/y)dy$.

Integrating, $z = cx - x^2/2 + F(c/y) + G(\lambda)$,

which is the complete integral, F and G being arbitrary functions.

Ex. 3. *Solve* $qxr + (x + y)s + pyt + xy(xt - s^2) = 1 - pq$.

Sol. Given $qxr + (x + y)s + pyt + xy(rt - s^2) = 1 - pq$. ...(1)

Comparing (1) with $Rs + Ss + Tt + U(rt - s^2) = V$, we have

$R = qx$, $S = x+y$, $T = py$, $U = xy$ and $V = 1 - pq$ ...(2)

Now, the λ–qaudratic $\lambda^2(UV+RT) + \lambda US + U^2 = 0$ ...(3)

reduces to $\lambda^2[qxpy+xy(1 - pq)] + \lambda xy(x + y) + x^2y^2 = 0$

or $\lambda^2 + (x + y)\lambda + xy = 0$ or $(\lambda + x)(\lambda + y) = 0$ so that $\lambda = -x, -y$.

Let $\lambda_1 = -x$ and $\lambda_2 = -y$. ...(4)

Two intermediate integrals are given by the following two sets :

$$\left.\begin{array}{l} Udy + \lambda_1 T\,dx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{array}\right\} \quad ...(5)$$

and

$$\left.\begin{array}{l} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0. \end{array}\right\} \quad ...(6)$$

Using (2) and (4), equation (5) and (6) reduce to

$$\left.\begin{array}{lll} xydy - xpydx - x^2\,ydp = 0 & i.e., & (xdp + pdx) - dy = 0 \\ \text{and} \quad xydx - yqxdy - yxydq = 0 & i.e., & (ydq + qdy) - dx = 0 \end{array}\right\} \quad ...(5A)$$

$$\left.\begin{array}{lll} xydy - ypydx - y\,xy\,dp = 0 & i.e., & xdy - pydx - xydp = 0 \\ \text{and} \quad xydx - xyqdy - x^2\,ydq = 0 & i.e., & ydx - qxdy - xydq = 0 \end{array}\right\} \quad ...(6A)$$

We observe that (5*A*) can be integrated whereas (6*A*) cannot the integrated. So we shall obtain only one intermediate integral with help of (5*A*) :

Integrating (5*A*), $px - y = c_1$ and $qy - x = c_2$

Hence the only intermediate integral of (1) is given by

$px - y = f(qy - x)$, where f is an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the intermediate integral (7) to be linear, giving

$px - y = m(qy - x) + n,$ where m and n are arbitrary constants

or $xp - myq = y - mx + n$, which is in Lagrange's form

Hence here Lagrange axuiliary equations are $$\frac{dx}{x} = \frac{dy}{-my} = \frac{dz}{y - mx + n} \qquad ...(8)$$

From first and second fractions of (8), $m(1/x)dx + (1/y)dy = 0$

Integrating, $m \log x + \log y = \log a$ or $x^m y = a$. ...(9)

Chossing $m,\ 1/m,\ 1$ as multipliers, each fraction of (8)

$$= \frac{mdx + (1/m)dy + dz}{mx + (1/m)(-my) + y - mx + n} = \frac{mdx + (1/m)dy + dz}{n} \qquad ...(10)$$

Taking first fraction of (8) and (10), we have

$$\frac{dx}{x} = \frac{mdx + (1/m)dy + dz}{n} \qquad \text{or} \qquad dz + \frac{1}{m}dy + mdx - \frac{n}{x}dx = 0.$$

Integrating, $z + (1/m)y + mx - n \log x = b$, b being an arbitrary constant ...(11)

From (9) and (11) the required general solution in

$z + (1/m)y + mx - \log x^n = \psi(x^m y)$, ψ being an arbitrary function

Ex. 4. *Solve* $(rt - s^2) - s(\sin x + \sin y) = \sin x \sin y$. **[Meerut 1999]**

Sol. Given $0 \cdot r - s(\sin x + \sin y) + 0 \cdot t + (rt - s^2) = \sin x \sin y$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 0, \quad S = -(\sin x + \sin y), \quad T = 0, \quad U = 1, \quad V = \sin x \sin y$. ...(2)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$. ...(3)

reduces to $\sin x \sin y\, \lambda^2 - (\sin x + \sin y)\, \lambda + 1 = 0$

or $(\lambda \sin x - 1)(\lambda \sin y - 1) = 0$ so that $\lambda = \operatorname{cosec} x$ or $\operatorname{cosec} y$.

Let $\lambda_1 = \operatorname{cosec} x$ and $\lambda_2 = \operatorname{cosec} y$. ...(4)

Two intermediate integral are given by the following two sets :

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad ...(5)$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad ...(6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + \operatorname{cosec} x\, dp &= 0 \\ dx + \operatorname{cosec} y\, dq &= 0 \end{aligned}\right\} \qquad ...(5A)$$

$dy + \operatorname{cosec} y\, dp = 0$ *i.e.*, $dp + \sin y\, dy = 0$

and $dx + \operatorname{cosec} x\, dq = 0$ *i.e.*, $dq + \sin x\, dx = 0$ ...(6A)

We observe that (5A) cannot be integrated where as (6A) can be integrated. So we shall obtain only one intermediate integral with help of (6A).

Integrating (6A), $p - \cos y = c_1$ and $q - \cos x = c_2$.

Hence the only intermediate integral of (1) is given by

$p - \cos y = f(q - \cos x)$, f being an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the the intermediate integral (7) to be linear, giving

$p - \cos y = m[q - \cos x] + n$, m, n being arbitrary constants

or $p - mq = \cos y - m \cos x + n$, which is in Lagrange's from.

Its Lagrange's auxiliary equations are $$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\cos y - m \cos x + n}. \qquad ...(8)$$

From the first two fractions of (8), $dy + mdx = 0$ so that $y + mx = a$. ...(9)

Again, taking the first and third fractions of (8), we have

$dz = (\cos y - m \cos x + n)dx = [\cos (a - mx) - m \cos x + n]dx$, as from (9), $y = a - mx$

Intergrating, $z = -(1/m) \times \sin (a - mx) - m \sin x + nx + (1/m) \times b$

or $mz + \sin y + m^2 \sin x - mnx = b$, b being an arbitrary constants ...(10)

From (9) and (10), the required general solution of (1) is

$$mz + \sin y + m^2 \sin x - mnx = \phi(y + mx), \phi \text{ being an arbitrary function.}$$

Ex. 5. *Solve* $xqr + (p + q)s + ypt + (xy - 1)(rt - s^2) + pq = 0$.

Ans. $z - \log (x - m)^n = \phi\{(x - m)^n (1 - my)$

Ex. 6. *Solve* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$.

Ans. $z + (1/m) \times (a^2 - mx^2)^{1/2} + (x/\sqrt{m}) \times \sin^{-1}(x\sqrt{m}/a) + 2mx = \phi(mx^2 + y^2)$

Ex. 7. *Solve* $ar + bs + ct + e(rt - s^2) = h$, *where a, b, c, e and h are constants.*

Sol. Comparing with $Rr + Ss + Tt + U(rt - s^2) = V$, here $R = a$, $S = b$, $T = c$, $U = e$, $V = h$.

The λ–quadratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ gives $(ac + eh)\lambda^2 + \lambda be + e^2 = 0$. ...(1)

Let $\lambda = -e/m$. ...(2)

$\therefore$ (1) reduces to $m^2 - bm + (ac + eh) = 0$. ...(3)

Let m_1 and m_2 be the roots of (3). The first intermediate integral is given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$, where $\lambda_1 = -e/m_1$

and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$, where $\lambda_2 = -e/m_2$

i.e., $e\,dy - (e/m_1) \times c\,dx - (e/m_1) \times e\,dp$ and $ex - (e/m_2) \times a\,dy - (e/m_2) \times e\,dq = 0$

i.e., $c\,dx + edp - m_1\,dy = 0$ and $ady + edq - m_2 dx = 0$.

Integrating, $cx + ep - m_1 y = c_1$ and $ay + eq - m_2 x = c_2$.

So the first intermediate integral is $cx + ep - m_1 y = \phi_1(ay + eq - m_2 x)$. ...(4)

Proceding as before, the second intermediate integral is

$$cx + cp - m_2 y = \phi_2(ay + eq - m_1 x). \quad ...(5)$$

Notice that p and q cannot be determined from (4) and (5). Hence we proced as follows :

We also have, $cx + ep - m_2 y = c_3$ ...(6)

From (4) and (6), $(m_2 - m_1)y = \phi_1(ay + eq - m_2 x) - c_3$

$\therefore$ $ay + eq - m_2 x = \psi_1\{(m_2 - m_1)y + c_3\}]$ ...(7)

where ψ_1 is the inverse function of ϕ_1.

From (7). $q = (1/e) \times [-ay + m_2 x + \phi_1\{(m_2 - m_1)y + c_3\}]$

and from (6) $p = (-cx + m_2 y + c_3)/e$.

Putting these values in $dz = pdx + qdy$, we get

$$edz = -(xdx - aydy + m_2(xdy + ydx) + c_3 dx + \phi_1\{(m_2 - m_1)y + c_3\}dy$$

Integrating, $ez = -(1/2) \times cx^2 - (1/2) \times ay^2 + m_2 xy + c_3 x + F\{(m_2 - m_1)y + c_3\} + k$.

EXERCISE

Solve the following partial differential equation:

1. $3r + s + t + (rt - s^2) = -9$ **[K.U. Kurukshetra 2004]**

2. $3s - 2(rt - s^2) = 2$ **3.** $2r - 6s + 2t + (rt - s^2) = 4$

Objective problems

1. The equations $R\,dpdy + T\,dqdx - V\,dxdy = 0$ and $Rdy^2 - S\,dxdy + T\,dx^2 = 0$ are called

Sol. Ans. Monge's subsidiary equations. Refer Art. 9.2. **[Meerut 2003].**

2. Monge's method is used to solve a partial differential equation of

(*a*) nth order (*b*) first order (*c*) second order (*d*) none of these **[Agra 2007]**

Sol. Ans. (**c**). Refer Art 9.2

10

Transport Equation

10.1 INTRODUCTION

The hyperbolic character of a system of first order differential equations exhibits in the fact that it is possible to have solutions whose derivatives are discontinuous and these discontinuities propagate along the characteristic curves. In this chapter we propose to use the above fact and derive a system of linear homogeneous ordinary differential equations known as *transport equation.*

10.2 An IMPORTANT THEOREM

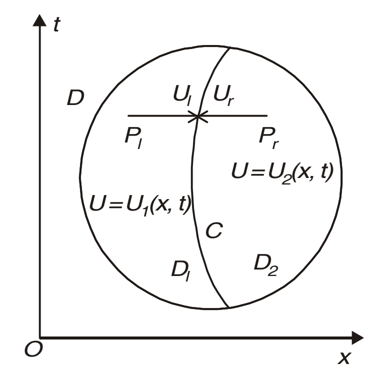
If the first order partial derivatives of a continuous function U (x, t), satisfying a system of quasi-linear equations of first order on both sides of a curve C in xt-plane, are discontinuous across a curve C, then the curve C must be a characteristic curve of the system of equations.

Proof : Suppose that the given first-order system of n quasi-linear partial differential differential equations be given by

$$\left.\begin{aligned} A_{ij}\,(\partial u_j/\partial t) + B_{ij}\,(\partial u_j/\partial x) + C_i = 0, \quad i = 1, 2, ..., n; \quad j = 1, 2, ..., n \\ \text{or} \qquad A(\partial U/\partial t) + B\,(\partial U/\partial t) + C = 0, \end{aligned}\right\} \qquad ...(1)$$

where the n components $u_1, u_2, ..., u_n$ of the column vector U are dependent variables, A and B are $n \times n$ matrices and C is a $n \times 1$ column vector.

Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C such that D_1 is on the left and D_2 on the right of C as shown in the adjoining figure. Let U_1 be the genuine solution of (1) in the domain D_1 and U_2 that in the domain D_2. Suppose that the limiting value U_l *of* U_1 as we approach a point P on C from the domain D_1 and the limiting value U_r as we approach P from the domain D_2 exist and are such that $U_l = U_r$ at every point of the curve C. Let a function U be defined in the domain D such that



$$U = \begin{cases} U_1 & \text{in } D_1 \\ U_2 & \text{in } D_2 \end{cases} \qquad ...(2)$$

The function U given by (2) is a genuine solution of (1) in D_1 and D_2 respectively. The function U is continuous in D but its derivatives may be discontinuous across the curve C. Suppose that the limiting values of the derivatives of U as we approach P on the curve C from the two domains D_1 and D_2 exist. Also assume that these derivative, if discontinuous across the curve C, have only a finite jump across the curve C.

Let the equation of the curve C be $\phi(x, t) = 0$ and let $\eta(x,t)$ be any other function independent of ϕ such that ϕ and η are sufficiently smooth and the Jacobian $\partial(\phi\,,\eta)/\partial(x,t) \neq 0$ in the domain D. Therefore, if we can introduce a new set of independent variables (ϕ, η) in place of (x, t), then U_ϕ represents an exterior derivative and U_η is a tangential derivative along the curve C.

We have
$$\left.\begin{aligned} U_x &= \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = U_\phi\, \phi_x + U_\eta\, \eta_x \\ \text{and} \qquad U_t &= \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = U_\phi \phi_t + U_\eta \eta_t \end{aligned}\right\} \qquad ...(3)$$

Let us now assume that the first order partial derivatives U_x and U_t are discontinuous across the curve C. Since the function U is continuous across C, its tangential derivative U_η is also continuous across C. Hence from the above two relations (3), it follows that the exterior derivative U_ϕ must be discontinuous across C. Let $(U_\phi)_r$ and $(U_\phi)_l$ be the limiting values of U_ϕ across C, as we approach a point P on C from the domains D_1 and D_2 respectively. Then, the jump $[U_\phi]$ in U_ϕ across C is given by

$$[U_\phi] = (U_\phi)_r - (U_\phi)_l \qquad ....(4)$$

Since U_η is continuous across the curve C, hence from (3), the jumps in the first order derivatives U_x and U_t are related to $[U_\phi]$ by the relations

$$[U_x] = [U_\phi]\phi_x \qquad \text{and} \qquad [U_t] = [U_\phi]\phi_t \qquad ...(5)$$

The quasi-linear equation (1) is valid everywhere in D except at the points on the curve C. Since all the terms appearing in it other than the first order derivatives are continuous across the curve C, hence taking the limit of (1) as we move from the region D_1 to P and again as we more from the region D_2 to P and then subtracting the equations so obtained, we obtain

$$(A\phi_t + B\phi_x)_{at\,P}\,[U_\phi] = 0 \qquad ...(6)$$

Since $[U_\phi]$ is not a zero vector, the matrix $A\phi_t + B\phi_x$ must be singular on the curve C, i.e., at every point of the curve C, we have

$$\left.\begin{aligned} &\det\,(A\phi_t + B\phi_x) = 0 \\ \text{or} \qquad &\det(-\lambda\, A + B) = 0, \text{ where } \lambda = -\phi_t/\phi_x \end{aligned}\right\} \qquad ...(6)$$

Hence, it follows that C is a characteristic curve.

Note 1. In (6), det X stands for determinant of the matrix X.

Note 2. If the derivatives of a solution U of system (1) upto order $(r \geq 1)$ are continuous across a curve C and the $(r + 1)$ th derivatives are discontinuous across C, then differentiating (1) r times and then proceeding as discussed above, we can show that C is necessarily a characteristic curve.

10.3 GENERALISED OR WEAK SOLUTION

[Allahabad 2003; G.N.D.U. Amritsar 2004; Kanpur 2003, 05)

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x, t)U_t + B(x, t)\,U_x + H(x, t)\,J\,(x, t) = 0, \qquad ...(1)$$

where the elements of matrices A, B and H (each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t = \partial U/\partial t$ and $U_x = \partial U/\partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C.

Suppose we have a solution U which satisfies (1) in D_1 and D_2 separately but is itself discontinuous across C. It has been established that discontinuities in the solution cannot be discussed for every function U satisfying (1) in D_1 and D_2. However, such discontinuities can be discussed for a "*generalised*" or '*weak*" solution.

To end, we first reduce the system (1) to the characteristic canonical form and note that $(\partial/\partial t) + \lambda_M(x, t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M *to a point* (ξ, τ), *both lying on a* characteristic of the Mth field, we obtain.

$$W_M(\xi, \tau) = -\int_{P_M}^{P} [H_{M_i}\{x_M(t, \xi, \tau), t\} W_i\{x_M(t, \xi, \tau), t\}]dt$$

$$+ (W_M)_{at\, P_M} \int_{P_M}^{P} J_M\{x_M(t, \xi, \tau), t\}dt, \text{ for } M = 1, 2, \ldots n \qquad ...(2)$$

where $$H_{M_i} = [l^{(M)} A\{\partial r^{(i)}/\partial t + \lambda_M(\partial r^{(i)}/\partial x)\} + (l^{(M)}\, H\, r^{(i)})]/(l^{(M)} A\, r^{(M)}) \qquad ...(3)$$

and $$J_M = (l^{(M)} J)/(l^{(M)}\; A\, r^{(M)}) \qquad ...(4)$$

with no sum over M in these expressions and $x = x_M(t, \xi, \tau)$ is the characteristic of Mth field through the point P.

We can rewrite in compact form the expression for H_{M_i} in terms of the operator $\tau \equiv A(\partial/\partial t) + B(\partial/\partial x) + H$. Thus, (3) takes the form

$$H_{M_i} = [l^{(M)} \tau\, r^{(i)}]/[l^{(M)}\; A\; r^{(M)}] \qquad ...(5)$$

We now define a generalised or weak solution of (1) to be a function $U(x, t)$ obtained from $W_1, W_2, \ldots, W_n$ which satisfy the system of equations (2).

10.4 TRANSPORT EQUATION FOR A LINEAR-HYPERBOLIC SYSTEM

[Calicut 2003 G.N.D.U. (Amritsar 2005; Kanpur 2004; Meerut 2005, 06, 10, 11]

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x, t)\, U_t + B(x, t)\, U_x + H(x, t)\, J\,(x, t) = 0, \qquad ...(1)$$

where the elements of matrices A, B and H(each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t = \partial u/\partial t$ and $U_x = \partial U/\partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C. Suppose $U(x, t)$ is a weak solution of (1) which is continuous in the domain D except on the curve C and is a genuine solution of (1) in the domains D_1 and D_2. We also suppose that the function U has a jump discontinuity across C.

We now reduce the system (1) to the characteristic canonical form and note that $(\partial/dt) + \lambda_M(x, t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M to a point (ξ, τ), both lying on a characteristic of the Mth field, we have

$$W_M(\xi, \tau) = -\int_{P_M}^{P} [H_{M_i}\{x_M(t, \xi, \tau), t\} W_i\{x_M(t, \xi, \tau), t] dt$$

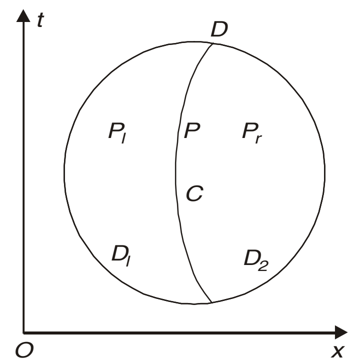
$$+ (W_M)_{at\, P_M} - \int_{P_M}^{P} [J_M\{x_M(t, \xi, \tau), t\}\, dt, \text{ for } M = 1, 2, \ldots n \qquad ...(2)$$

where $$H_{M_i} = [l^{(M)} A\{\partial r^{(i)} / \partial t + \lambda_M (\partial r^{(i)} / \partial x)\} + l^{(M)}\ Hr^{(i)}]/[l^{(M)} A\, r^{(M)}] \qquad ...(3)$$

and $$J_M = (l^{(M)} J)/[l^{(M)} A\ r^{(M)}] \qquad ...(4)$$

with no sum over M in these expressions and $x = x_M(t, \xi, \tau)$ is the characteristic of Mth field through the point P.

In the case when the function U has a jump discontinuity across the curve C, the integrands on the right hand side of (2) are continuous functions of t except for a finite jump across C. On performing the integration in (2) along a characteristic of the Mth family, we find that the characteristic variable W_M is given by a continuous function of this curve. If a curve C is not tangential to a characteristic of the Mth family, W_M must be continuous across the curve C. However, according to our assumption at least one of W_1, W_2, ..., W_n must be discontinuous across the curve C. Therefore, it follows that C, the curve of discontinuity, must be a characteristic curve of jth family (say), and the jump in all characteristic variables W_i, $i \neq j$, must be zero across the carve C.



Now, suppose that the curve of discontinuity C is a characteristic curve of the jth family then, the jump $[W_i]$ in W_i satisfies

$$[W_i] = 0, \text{ for } \quad i = j \qquad \text{and} \qquad [W_j] = 0 \qquad ...(5)$$

Again, we have $$[U] = r^{(j)}[W_j], \text{ on sum over } j \qquad ...(6)$$

Re-writing the equation, we have

$$(\partial W_M / \partial t) + \lambda_M (\partial W_M / \partial x) + C_1\ M_i W_i + J_M = 0, \qquad M = 1, 2, ..., n \qquad ...(7)$$

Consider two points P_l and P_r on the two sides of C in the regions D_1 and D_2 respectively as shown in the figure. Taking limit as both these points tend to P on the curve C and substracting the results so obtained we obtain

$$\frac{d}{dt}\left[W_j\right] = -H_{jj}[W_j], \text{ no sum over } j \qquad ...(8)$$

where $$d / dt \equiv (\partial / \partial t) + (\lambda_j)(\partial / \partial x) \qquad ...(9)$$

The above equation (8) is known as the *transport equation*.

Along a given characteristic curve $x = x_j(t)$ of the j the family, the function $H_{jj}(x, t) = H_{jj}(x_j(t), t)$ is a function of t only.

Thus, the transport equation (8) is a linear homogeneous ordinary differential equation of first order and determines the vibration of $[W_j]$, jump in W_j along a characteristic curve of the jth family. From the properties of solutions of linear homogeneous ordinary differential equations, it follows that if there is a discontinuity in U at some point of a characteristic curve C, the discontinuity in U remains non-zero at every point on the curve.

Note. In order to obtain the transport equation for the discontinuities in the derivatives of U of order $n(n \geq 1)$, differentiate (7) $n(n \geq 1)$ times and proceed in exactly same manner as above.

EXERCISE

1. Write short note on the transport equation for a linear hyperbolic system of first order equations. **(Calicut 2003; Meerut 2005; 06)**

2. When all the characteristic velocities λ_i are different from zero, prove that the first order quasi-linear hyperbolic system

$$A(x, t, U)\ (\partial U / \partial t) + B(x, t, U)\ (\partial U / \partial x) + C(x, t, U) = 0$$

can be reduced to a diagonal canonical system of $2n$ equations $(\partial U / \partial t) - RW = 0$ and $(\partial U / \partial t) + A(\partial W / \partial x) + F = 0,$ where the coefficients A, R and F are functions of x, t, U and W

3. Consider the hyperbolic system $\qquad u_t + (x+t)v_x = 0, \qquad (x+t)v_t + u_x = 0$

Show that the variation in jump $[v]$ along the characteristic curve $(x-t) = C$ (C = constant) is given by

$$[v] = A/(2t+c)^{1/2}, \text{ A being a constant.}$$

Derive also the transport equation of discontinuities in the first order partial derivatives of u and v.

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *The solution of* $xu_x + yu_y = 0$ *is of the form*

(*a*) $f(y/x)$ (*b*) $f(y+x)$ (*c*) $f(x-y)$ (*d*) $f(xy)$ **[GATE 2008]**

Sol. Ans. (*a*). Given $xu_x + yu_y = 0$ ...(1)

which is in the form of Lagrange equation $Pp + Qq = R$, with u in place z. Hence, the Lagrange's auxiliary equations for (1) are given by

$$(dx)/x = (dy)/y = du/0 \qquad \text{...(2)}$$

Taking the first two fractions of (2), $(1/y)\, dy - (1/x)\, dx = 0$

Integrating, $\log y - \log x = \log c_1$ or $y/x = c_1$ ...(3)

Again, the last fraction of (2) yields $du = 0$ so that $u = c_2$ ...(4)

From (3) and (4), the required solution is $u = f(y/x)$.

Ex. 2. *Solve* $x(y^2 + z)p + y(z + x^2)q = z(x^2 - y^2)$ **[Madurai Kamraj 2008]**

Sol. Do like Ex. 6, page 2.10. Here Lagrange's auxiliary equations are

or
$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(z+x^2)} = \frac{dz}{z(x^2-y^2)} \qquad \text{... (1)}$$

Choosing $1/x$, $-(1/y)$, $1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy + (1/z)dz}{y^2 + z - (z + x^2) + x^2 - y^2} = \frac{(1/x)dx - (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow$ $(1/x)\, dx - (1/y)\, dy + (1/z)\, dz = 0$ so that $\log x - \log y + \log z = \log c_1$

or $\log (xz/y) = \log c_1$ or $(xz)/y = c_1$ ...(2)

Choosing $x, -y, -1$ as multipliers each fraction of (1)

$$= \frac{xdx - ydy - dz}{x^2(y^2+z) - y^2(z+x^2) - z(x^2-y^2)} = \frac{xdx - ydy - dz}{0}$$

$\Rightarrow$ $xdx - ydy - dz = 0$ or $2xdx - 2ydy - 2dz = 0$

Integrating, $x^2 - y^2 - 2z = c_2$, c_2 being an arbitrary constant ...(3)

From (2) and (3), the required solution is given by

$(xz)/y = \phi\,(x^2 - y^2 - 2z)$, ϕ being an arbitrary function

Ex. 3. *If the partial differential* $(x-1)^2 u_{xx} - (y-2)^2 u_{yy} + 2xu_x + 2yu_y + 2xyu = 0$ *is parabolic in* $S \subseteq R^2$ *but not in* $R^2 \setminus S$, *then S is*

(*a*) $\left\{(x, y) \in R^2 : x = 0 \text{ or } y = 2\right\}$ (*b*) $\left\{(x, y) \in R^2 : x = 1 \text{ and } y = 2\right\}$

(*c*) $\left\{(x, y) \in R^2 : x = 1\right\}$ (*d*) $\left\{(x, y) \in R^2 : y = 2\right\}$ **[GATE 2008]**

Sol. Ans. (*a*). Refer Art. 8.1. Here R is the set of all real numbers

Ex. 4. *Find the complete integral of* $xp + 3yq = 2(z - x^2q^2)$ **[Delhi Maths (H) 2008]**

Sol. Here given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$ ... (1)

Charpit's auxiliary equations are
$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

i.e., $$\frac{dp}{p+4xq^2-2p}=\frac{dq}{3q-2q}=\frac{dz}{-px-q(3y+4x^2q)}=\frac{dx}{-x}=\frac{dy}{-(3y+4x^2q)}, \text{ using (1)} \qquad \ldots(2)$$

Taking the second and fourth fractions of (2), we have

$(1/q)\, dq + (1/x)\, dx = 0$ so that $\log q + \log x = \log a$, giving

$qx = a$ so that $q = a/x$, a being an arbitrary constant ...(3)

Substituting the value of q given by (3) in (1), we have

$xp + (3ya)/x - 2z + 2a^2 = 0$ or $xp = 2(z - a^2) - (3ya)/x$

Thus, $p = 2(z - a^2)/x - (3ya)/x^2$. ...(4)

Substituting values of q and p given by (3) and (4) in $dz = pdx + qdy$, we get

$dz = \{2(z - a^2)/x - (3ya)/x^2\}dx + (a/x)\, dy$ or $x^2dz = 2x\,(z - a^2)\, dx - 3yadx + axdy$

or $x^2dz - 2x(z - a^2)\, dx = axdy - 3yadx$ or $[x^2dz - 2x\,(z - a^2)dx]/x^4 = ax^{-3}dy - 3ayx^{-4}\, dx$

or $d\{(z - a^2)/x^2\} = d\,(ayx^{-3})$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$, b being an arbitrary constant

or $z = a(a + y/x) + bx^2$, which is the required solution.

Ex.5. *Find the general integral of the partial differential equation $px(z - 2y^2) = (z - qy)\,(z - y^2 - 2x^3)$. Also, find the particular integral which passes through the straight line $x = -1$, $z = 1$.* **(Delhi B.A. (Prog.) 2009)**

Sol. Re-writing the given partial differential equation, we have

$$px\,(z - 2y^2) + qy(z - y^2 - 2x^3) = z\,(z - y^2 - 2x^3) \qquad \ldots(1)$$

Hence the usual Lagrange's, subsidiary equations are given by

$$\frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}=\frac{dz}{z(z-y^2-2x^3)} \qquad \ldots(2)$$

Taking the last two fractions of (1), we get $(1/y)dy = (1/z)\mathrm{d}z$

Integrating, $\log y = \log z + \log c_1$, c_1 being an arbitrary constant

Thus, $y = c_1 z$. ...(3)

Next, taking the first and third fractions of (2) and using (3), we obtain

$$\frac{dx}{x(z-2c_1^2z^2)}=\frac{dz}{z(z-c_1^2z^2-2x^3)} \quad \text{or} \quad \frac{dz}{dx}=\frac{z-c_1^2z^2-2x^3}{x(1-2c_1^2z)}$$

Re-writing it, $(1-2c_1^2z)\,(dz/dx) - (z-c_1^2z^2)\times(1/x) = -2x^2$ ...(4)

Putting $z-c_1^2z^2 = v$ so that $(1-2c_1^2z)(dz/dx) = dv/dx$, (4) reduces to

$$(dv/dx)-(1/x)v=-2x^2 \qquad \ldots(5)$$

whose integrating factor is $e^{\int(-1/x)dx} = e^{-\log x} = x^{-1}$ and hence solution of (5) is

$v \times x^{-1} = \int\{(-2x^2)\times x^{-1}\}\,dx + c_2 = -x^2 + c_2$, c_2 being an arbitrary constant

or $(z-c_1^2z^2)/x + x^2 = c_2$ or $(z - y^2)/x + x^2 = c_2$, using (3) ...(6)

The required general integral is given by (3) and (6).

We now find the required particular integral. To this end, replacing x by -1 and z by 1 in (3) and (6), we obtain

$y = c_1$ and $-(1 - y^2) + 1 = c_2$ ...(7)

Eliminating y between two relations of (7), we have $\qquad c_1^2 = c_2 \qquad$ ...(8)

Substituting the values of c_1 and c_2 given by (3) and (6) in (8), the required particular integral is given by

$$y^2/z^2 = (z - y^2) / x + x^2 \qquad \text{or} \qquad y^2x = z^2(z - y^2 + x^3)$$

Ex.6. *Solve the partial differential equation $z = px + qy + 3p - 2q$ by Lagrange's method as well as Charpit's method. Hence or otherwise give two different solutions of the above partial differential equation passing through (–3, 2, 0).* **(Delhi B.A. (Prog). II 2009)**

Sol. Solution of the given equation by Lagrange's method:

Re-writing the given equation, $\qquad (x + 3)p + (y - 2)q = z \qquad$ ...(1)

Here the usual Lagrange's subsidiary equations are given by

$$(dx)/(x + 3) = (dy)/(y - 2) = (dz) / z \qquad \text{...(2)}$$

Taking the first two fractions of (2), $\qquad (dx)/(x + 3) = (dy)/(y - 2)$

Integrating, $\quad \log(x + 3) = \log(y - 2) + \log a \quad$ or $\quad (x + 3) = a\,(y - 2) \qquad$ ...(3)

Next, taking the last two fractions of (2), $\qquad (dy)/(y - 2) = (dz)/z$

Integrating, $\quad \log\,(y - 2) + \log b = \log z \quad$ or $\quad b\,(y - 2) = z \qquad$ ...(4)

From (3) and (4), $\quad (x + 3)/a = (y - 2)/1 = (z - 0)/b$, a and b being arbitrary constants ...(5)
which is the required solution of the given equation passing through (–3, 2, 0).

Solution of the given equation by Charpit's method:

Let $\qquad f(x, y, z, p, q) = (x + 3)p + (y - 2)q - z = 0 \qquad$ ...(6)

Here Charpit's auxiliary equations $\quad \dfrac{dp}{f_x + pf_z} = \dfrac{dq}{f_y + qf_z} = \dfrac{dz}{-pf_p - qf_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

yield $$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x+3) - q(y-2)} = \frac{dx}{-(x+3)} = \frac{dy}{-(y-2)}$$

Hence, $\quad dp = 0 \quad$ so that $\quad p = c_1$, c_1 being an arbitrary constant $\qquad$ ...(7)

From (6) and (7), $(x + 3)c_1 + (y - 2)q - z = 0 \quad \Rightarrow \quad q = \{z - (x + 3)c_1\}/(y - 2) \qquad$ ...(8)

Substituting the values of p and q given by (7) and (8), we have

$$dz = pdx + qdy = c_1dx + [\{z - (x + 3)c_1\}/(y - 2)]dy$$

or $\quad dz - \dfrac{z}{y-2}dy = c_1dx - \dfrac{(x+3)c_1dy}{y-2} \quad$ or $\quad \dfrac{(y-2)\,dz - z\,dy}{(y-2)^2} = \dfrac{c_1(y-2)\,dx + c_1(x+3)\,dy}{(y-2)^2}$

or $\quad d\left(\dfrac{z}{y-2}\right) = d\left(\dfrac{c_1(x+3)}{y-2}\right) \quad$ giving $\quad \dfrac{z}{y-2} = \dfrac{c_1(x+3)}{y-2} + c_2$

Thus, $\qquad z = c_1(x + 3) + c_2\,(y - 2),$
which is the required second solution of the given equation passing through (–3, 2, 0).

Ex. 7. *Define the singular integral of first order partial differential equation. Is it true that singular integral always exists? Justify your answer.* **[Delhi Math (Hons.) 2009]**

Hint: Refer Art. 3.1.

Ex. 8. *Write the form of the solution of the equation $F(D, D') = 0$, where $F(D, D')$ is not reducible.* **[Delhi Math (Hons). 2009]**

Sol. Let $z = e^{hx + ky}$ be a trial solution of the given equation. Then, the required solution is

$$z = \sum_i A_i\, e^{h_i x + k_i y}, \text{ where } A_i,\ h_i \text{ and } k_i \text{ are arbitrary constants,}$$

and h_i, k_i are connected by the relation $F(h_i, k_i) = 0$.

Ex. 9. *Solve $z = px + qy + p^2 + q^2$.* **[Kanpur 2009]**

Sol. Refer Art 3.12. The complete integral is $z = ax + by + a^2 + b^2$ … (1)

Singular integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$0 = x + 2a \quad \text{and} \quad 0 = y + 2b \qquad \ldots(2)$$

From (2), $a = -(x/2)$ and $b = -(y/2)$. Substituting these values of a and b in (1), we get

$$z = -(x^2/2) - (y^2/2) + x^2/4 + y^2/4 \quad \text{or} \quad 4z + x^2 + y^2 = 0.$$

General integral Take $b = \phi(a)$, where ϕ is an arbitrary function

Then, (1) yields $z = ax + y\,\phi(a) + a^2 + [\phi(a)]^2$ …(3)

Differentiating (3) partially w.r.t. 'a', $0 = x + y\phi'(a) + 2a + 2\phi(a)\phi'(a)$ …(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 10. *Classify the following partial differential equation into elliptic, parabolic or hyperbolic and find its degree and order* $x(y-x)r - (y^2 - x^2)s + y(y-x)t + (y+x)(p-q) = 0$.

Hint. Use Art 1.3, 1.4 and 8.1. **[Delhi BA (Prog.) II 2009]**

Ans. The given equation is hyperbolic, its degree is one and its order is two.

Ex. 11. *Find the characteristic strips of the equation $xp + yq - pq = 0$ and then find the equation of the integral surface through the curve $z = x/2$, $y = 0$* **[Meerut 2011]**

Sol. Given equation is $xp + yq - pq = 0$ ...(1)

We are to find its integral surface passing through the given curve, namely

$$z = x/2, \qquad y = 0 \qquad \ldots(2)$$

Re-writing (2) in parametric form, we have

$$x = \lambda, \quad y = 0, \quad z = \lambda/2; \quad \lambda \text{ being a parameter} \qquad \ldots(3)$$

Let the initial values of x_0, y_0, z_0, p_0 and q_0 of x, y, z, p and q be taken as

$$x_0 = x_0(\lambda) = \lambda \qquad y_0 = y_0(\lambda) = 0, \qquad z_0 = z_0(\lambda) = \lambda/2 \qquad \ldots(4\text{ A})$$

Let p_0 and q_0 be the initial values of p and q corresponding to the initial values of x_0, y_0, z_0. Since the initial values x_0, y_0, z_0, p_0 and q_0 satisfy (1), we have

$$x_0p_0 + y_0q_0 - p_0q_0 = 0 \quad \text{or} \quad \lambda p_0 - p_0q_0 = 0 \quad \text{or} \quad q_0 = \lambda, \text{ using (4A)} \qquad \ldots(5)$$

Also, we have $z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$

so that $1/2 = p_0 + 0$ giving $p_0 = 1/2$, using (4A) …(6)

Thus, from (5) and (6), $p_0 = 1/2$ and $q_0 = \lambda$ …(4 B)

Collecting relations (4 A) and (4 B) together, initial values are given by

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = \lambda/2, \quad p_0 = 1/2 \quad \text{and} \quad q_0 = \lambda, \text{ when } t = t_0 = 0 \qquad \ldots(7)$$

Re-writing (1), let $f(x, y, z, p, q) = xp + yq - pq = 0$ …(8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = x - p \qquad \ldots(9)$$

$$dy/dt = \partial f/\partial q = y - p \qquad \ldots(10)$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(x-q) + q(y-p) = -pq, \text{ using (1)} \qquad \ldots(11)$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = -p - (p \times 0) = -p \qquad \text{...(12)}$$

$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = -q - (q \times 0) = -q \qquad \text{...(13)}$$

From (12), $(1/p)dp = -dt$ so that $\log p - \log c_1 = -t$

Thus, $p = c_1 e^{-t}$, c_1 being an arbitrary constant ...(14)

Similarly, (13) yields $q = c_2 e^{-t}$, c_2 being an a arbitrary constant ...(15)

Using initial values (7), (14) yields $p_0 = c_1 e^{-t_0}$ giving $c_1 = 1/2$

Hence, (14) reduces to $p = (1/2) \times e^{-t}$ ...(16)

Using initial values (7), (15) yields $q_0 = c_2 e^{-t_0}$ giving $c_2 = \lambda$

Hence, (15) reduces to $q = \lambda\, e^{-t}$ ...(17)

From (9) and (17), $dx/dt = x - \lambda e^{-t}$ or $dx/dt - x = -\lambda e^{-t}$,

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$xe^{-t} = c_3 + \int\{(-\lambda e^{-t}) \times e^{-t}\}\,dt, \; c_3 \text{ being an arbitrary constant}$$

or $$xe^{-t} = c_3 + (\lambda/2) \times e^{-2t} \qquad \text{...(18)}$$

Using initial values (7), (18) yields $x_0 e^{-t_0} = c_3 + (\lambda/2) \times e^{-2t_0}$

or $\lambda = c_3 + \lambda/2$ so that $c_3 = \lambda/2$.

Hence, (18) yields $xe^{-t} = (\lambda/2) \times (1 + e^{-2t})$

or $$x = (\lambda/2) \times e^{t}(1 + e^{-2t}) \qquad \text{...(19)}$$

Now, from (10) and (16), $dy/dt = y - (e^{-t})/2$ or $dy/dt - y = -(e^{-t})/2$

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$ye^{-t} = c_4 + \int\{(-e^{-t}/2) \times e^{-t}\}dt, \quad c_4 \text{ being an arbitarary constant}$$

or $$ye^{-t} = c_4 + (1/4) \times e^{-2t} \qquad \text{... (20)}$$

Using initial values (7), (20) yields $y_0 e^{-t_0} = c_4 + (1/4) \times e^{-2t_0}$

or $0 = c_4 + 1/4$ so that $c_4 = -(1/4)$

Hence, (20) reduces to $ye^{-t} = (1/4) \times (e^{-2t} - 1)$

Thus, $y = (1/4) \times e^{t}\,(e^{-2t} - 1)$ ...(21)

From (11), (16) and (17), $dz/dt = -(e^{-t}/2) \times (\lambda e^{-t}) = -(\lambda/2) \times e^{-2t}$

Thus, $(1/z)dz = -(\lambda/2) \times e^{-2t}\,dt$

Integrating, $z = (\lambda/4) \times e^{-2t} + c_5$, c_5 being an arbitrary constant ...(22)

Using initial values (7), (22) yields $z_0 = (\lambda/4) \times e^{-2t_0} + c_5$

or $\lambda/2 = (\lambda/4) \times e^{0} + c_5$ so that $c_5 = \lambda/4$.

Hence, (22) reduces to $z = (\lambda/4) \times (e^{-2t} + 1)$ ...(23)

The required characteristics of (1) are given by (19), (21) are (23)

In order to obtain the desired integral surface of (1), we now proceed to eliminate two parameters t and λ from (19), (21) and (23).

From (19) and (23), we have $x/z = 2e^t$ giving $e^t = x/2z$. ...(24)

From (21), $y = (1/4) \times (1/e^t - e^t) = (1/4) \times (2z/x - x/2z)$, using (24)

or $8xyz = 4z^2 - x^2$, which is the required integral surface of (1).

Ex.12. *Prove that for the equation $z + px + qy - 1 - pq\, x^2y^2 = 0$ the characteristic strips are given by $x = (B + Ce^{-t})^{-1}$, $y = (A + De^{-t})^{-1}$, $z = E - (AC + BD)e^{-t}$, $p = A(B + Ce^{-t})^2$, $q = B(A + De^{-t})^2$, where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z = 0$, $x = y$.* **[I.A.S 2001]**

Sol. The given equation is $z + px + qy - 1 - pq\, x^2y^2 = 0$ ...(1)

Let $f(x, y, z, p, q) = z + px + qy - 1 - pq\, x^2y^2$ ...(2)

Then, the characteristic equations of (1) are given by

$$dx/dt = \partial f/\partial p = x - qx^2y^2 \qquad ...(3)$$

$$dy/dt = \partial f/\partial q = y - px^2y^2 \qquad ...(4)$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(x - qx^2y^2) + q(y - px^2y^2) = px + qy - 2pqx^2y^2 \qquad ...(5)$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = -(p - 2\,pqxy^2) - p = -2p(1 - qxy^2) \qquad ...(6)$$

$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = -(q - 2pqx^2y) - q = -2q\,(1 - px^2y) \qquad ...(7)$$

From (3) and (6), $(1/x)\,(dx/dt) = -(1/2p)\,(dp/dt)$ or $(2/x)\,dx + (1/p)\,dp = 0$

Integrating, $2 \log x + \log p = \log A$ or $x^2p = A$, A being an arbitrary constant ...(8)

From (4) and (7), $(1/y)\,(dy/dt) = (-1/2q)\,(dq/dt)$ or $(2/y)dy + (1/q)dq = 0$

Integrating as before, $y^2q = B$, B being an arbitrary constant ...(9)

From (3) and (9), $dx/dt = x - Bx^2$ or $x^{-2}(dx/dt) - x^{-1} = -B$ ...(10)

Putting $x^{-1} = v$ and $-x^{-2}(dx/dt) = dv/dt$, (10) reduces to

$$-(dv/dt) - v = -B \qquad \text{or} \qquad dv/dt + v = B,$$

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$$ve^t = C + \int Be^t\, dt, \quad C \text{ being an arbitrary constant}$$

$e^t/x = c + Be^t$ or $1/x = Ce^{-t} + B$ or $x = (B + Ce^{-t})^{-1}$ ...(11)

Similarly, (4) and (8) yield $dy/dt = y - Ay^2$ or $y^{-2}(dy/dt) - y^{-1} = -A$ ...(12)

Putting $y^{-1} = u$ and $-y^{-2}(dy/dt) = du/dt$, (12) yields

$$-(du/dt) - u = -A \qquad \text{or} \qquad du/dt + u = A$$

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$$ue^t = D + \int Ae^t\, dt, \quad D \text{ being an arbitrary constant}$$

or $e^t/y = D + Ae^t$ or $1/y = De^{-t} + A$ or $y = (A + De^{-t})^{-1}$ ...(13)

Using (8) and (9), (5) yields $dz/dt = A/x + B/y - 2AB$

or $dz/dt = A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$, using (11) and (13)

or $dz/dt = (AC + BD)e^{-t}$ or $dz = (AC + BD)\, e^{-t}\, dt$

Integrating, $z = E - (AC + BD)e^{-t}$, E being an arbitrary constant ...(14)

From (8) and (11), $p = Ax^{-2} = A(B + C e^{-t})^2$ ...(15)

From (9) and (13), $q = By^{-2} = B(A + De^{-t})^2$ ...(16)

The required characteristics are given by (11), (13), (14), (15) and (16). We now proceed to find the required integral surface passing through the line given by

$$z = 0 \qquad \text{and} \qquad x = y \qquad \text{...(17)}$$

Re-writing (17), $x = \lambda$, $y = \lambda$, $z = 0$, λ being a parameter ...(18)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = \lambda, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = 0 \qquad \text{...(19)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since the initial values statisfy (1), we have

$$z_0 + p_0x_0 + q_0y_0 - 1 - p_0q_0\, x_0^2 y_0^2 = 0 \qquad \text{or} \qquad p_0\lambda + q_0\lambda - 1 - p_0q_0\lambda^4 = 0, \text{ using (19)}$$

Thus, $$\lambda(p_0 + q_0) = p_0q_0\lambda^4 + 1 \qquad \text{...(20)}$$

Also, we have $$z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$$

so that, $0 = p_0 + q_0$ giving $q_0 = -p_0$, using (19) ...(21)

Using (21), (20) yields $p_0q_0\lambda^4 + 1 = 0$ giving $-p_0^2\lambda^4 + 1 = 0$, using (21)

Thus, $p_0 = 1/\lambda^2$ so that $q_0 = -(1/\lambda^2)$, using (21) ...(22)

Using initial values $x = x_0 = \lambda$, $t = t_0 = 0$, (11) reduces to

$$\lambda = (B + C)^{-1} \qquad \text{so that} \qquad B + C = 1/\lambda \qquad \text{...(23)}$$

Using initial values $y = y_0 = \lambda$, $t = t_0 = 0$, (13) reduce to

$$\lambda = (A + D)^{-1} \qquad \text{so that} \qquad A + D = 1/\lambda \qquad \text{...(24)}$$

Using initial values $p = p_0 = 1/\lambda^2$, $t = t_0 = 0$, (15) reduces to

$$p_0 = A(B + C)^2 \qquad \text{or} \qquad 1/\lambda^2 = A \times (1/\lambda)^2, \text{ by (23)} \qquad \text{so that} \qquad A = 1 \qquad \text{...(25)}$$

Using initial values $q = q_0 = -(1/\lambda^2)$, $t = t_0 = 0$, (16) reduces to

$$q_0 = B\,(A + D)^2 \qquad \text{or} \qquad -(1/\lambda^2) = B \times (1/\lambda^2), \text{ by (24)} \qquad \text{so that} \qquad B = -1 \qquad \text{...(26)}$$

From (23) and (26), $-1 + C = 1/\lambda$ so that $C = 1 + 1/\lambda$ ...(27)

From (24) and (25), $1 + D = 1/\lambda$ so that $D = (1/\lambda) - 1$ ...(28)

Using the initial values $z = z_0 = 0$, $t = t_0 = 0$, (14) reduces to

$$0 = E - (AC + BD) \qquad \text{or} \qquad E = 1 + 1/\lambda - (1/\lambda - 1) \qquad \text{or} \qquad E = 2 \qquad \text{...(29)}$$

Substituting the values of A, B, C, D and E given by (25), (26), (27), (28) and (29) in (11), (13) and (14), we obtain

$$x = \{-1 + (1/\lambda + 1)e^{-t}\}^{-1} \qquad \text{...(30)}$$

$$y = \{1 + (1/\lambda - 1)e^{-t}\}^{-1} \qquad \text{...(31)}$$

$$z = 2 - \{1 + 1/\lambda - (1/\lambda - 1)\}e^{-t} = 2(1 - e^{-t}) \qquad \text{...(32)}$$

In order to obtain the required surface, we now eliminate λ from (30), (31) and (32).

From (30), $x^{-1} = -1 + (1/\lambda + 1)e^{-t}$ so that $1/x + 1 = (1/\lambda + 1)e^{-t}$ ...(33)

From (31), $y^{-1} = 1 + (1/\lambda - 1)e^{-t}$ so that $1/y - 1 = (1/\lambda - 1)e^{-t}$ ...(34)

Subtracting (34) from (33), $1/x - 1/y + 2 = 2e^{-t}$ or $1/x - 1/y = -2(1 - e^{-t})$ ...(35)

From (32) and (35), we get $1/x - 1/y = -z$ which is the required integral surface.

Ex. 13. The general solution of the partial differential equation $\partial^2 z/\partial x \partial y = x + y$ is of the form (*a*) $(1/2) \times xy(x+y) + F(x) + G(y)$ (*b*) $(1/2) \times xy(x-y) + F(x) + G(y)$

(*c*) $(1/2) \times xy(x-y) + F(x)\, G(y)$ (*d*) $(1/2) \times xy(x+y) + F(x)\, G(y)$ **(GATE 2010)**

Sol. Ans. (*a*). Integrating the give equation w.r.t '*x*', we get

$$\partial z/\partial y = x^2/2 + xy + g(y),\ g(y) \text{ being an arbitrary function of } y.$$

Integrating the above equation w.r.t. '*y*', we get

$$z = (x^2 y)/2 + (xy^2)/2 + G(y) + F(x), \quad \text{where} \quad G(y) = \int g(y) dy$$

Ex. 14. Find whether the following is hyperbolic, parabolic or elliptic :

(*i*) $x^2 r - y^2 t - px - qy = x^2$ **[Delhi B.A. (Prog) II 2010, 11]**

(*ii*) $x^2 r + (5/2) \times xys + y^2 t + xp + yq = 0$ **[Delhi B.A. (Prog) II 2010]**

(*iii*) $\partial^2 u/\partial t^2 + \partial^2 u/\partial x \partial t + \partial^2 u/\partial x^2 = 0$ **[Meerut 2010]**

(*iv*) $u_{xx} + u_{yy} + u_{zz} = (1/c^2) \times (\partial u/\partial t)$ **[Meerut 2007, 10]**

(*v*) $(1 - x^2) r - 2xys + (1 - y^2) t + xp + 3x^2 yq = 0$ **[Ravishankar 2010]**

(*vi*) $\partial^2 z/\partial x^2 = x^2 (\partial^2 z/\partial y^2)$ **[Bhopal 2010]**

(*vii*) $\partial^2 u/\partial t^2 + (\partial u/\partial x)(\partial u/\partial t) + \partial^2 u/\partial t^2 = 0$ **[Meerut 2011]**

Hint. Use Articles 8.1, 8.2 and 8.2A.

Ans. (*i*) Hyperbolic (*ii*) Hyperbolic (*iii*) Elliptic (*iv*) Parabolic (*v*) Hyperbolic if $x^2 + y^2 > 1$, parabolic if $x^2 + y^2 = 1$, elliptic if $x^2 + y^2 < 1$ (*vi*) Hyperbolic (*vii*) Elliptic

Ex. 15. *The partial differential equation* $x^2(\partial^2 z/\partial x^2) - (y^2 - 1)x(\partial^2 z/\partial x\, \partial y)$ $+ y(y-1)^2\ (\partial^2 z/\partial y^2) + x(\partial z + \partial x) + y(\partial z/\partial y) = 0$ *is hyperbolic in a region in xy–plane if (a)* $x \neq 0$ *and y = 1 (b) x = 0 and* $y \neq 1$ *(c)* $x \neq 0,$ *and* $y \neq 1$ *(d) x = 0 and y = 1.* **[GATE 2011]**

Sol. Ans (*c*) The given can be re-written as

$$x^2 r - x(y^2 - 1)s + y(y-1)^2 t + xp + yq = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = x^2$, $S = -x(y^2 - 1)$ and $T = y(y-1)^2$. Now, in order that (1) may be hyperbolic we must have $S^2 - rRT > 0$, i.e.,

$$x^2(y^2 - 1)^2 - 4x^2 y(y-1)^2 > 0 \qquad \text{or} \qquad x^2(y-1)^2(y+1)^2 - 4x^2 y\,(y-1)^2 > 0$$

or $\quad x^2(y-1)^2\{(y+1)^2-4y\}>0 \quad$ or $\quad x^2(y-1)^2(y-1)^2>0, \quad$ ... (2)

Which is true when $x \neq 0$ and $y \neq 1$.

Ex. 16. *The integral surface for the Cauchy problem* $\partial z/\partial x+\partial z/\partial y=1$ *which passes through the circle* $z = 0$, $x^2 + y^2 = 1$ *is*

(*a*) $x^2+y^2+2z^2+2zx-2yx-2yz-1=0$ (*b*) $x^2+y^2+2z^2+2zx-2yz-1=0$

(*c*) $x^2+y^2+2z^2-2zx-2yz-1=0$ (*d*) $x^2+y^2+2z^2+2zx+2yz-1=0$ **[GATE 2011]**

Sol. Ans. (*c*) In usual symbols, the given equation is $\quad p + q = 1 \quad$ ... (1)

Lagrange's auxiliary equation of (1) are $\quad (dx)/1=(dy)/1=(dz)/1 \quad$ ... (2)

Taking the first two fractions of (2), we get $\quad dx - dy = 0 \quad$ ... (3)

Integrating (3), $\quad x - y = c_{-1}$, c_1 being an arbitrary constant ... (4)

Next, taking the first and third fractions of (2), we get $\quad dx - dz = 0 \quad$ ... (5)

Integrating (5), $\quad x - z = c_2$, c_2 being an arbitrary constant ... (6)

The given curve is defined by $\quad x^2 + y^2 = 1,\ z = 0 \quad$ ... (7)

Putting z = 0 in (6), we have $\quad x = c_2, \quad$ ... (8)

Now, from (4) and (8), we have $\quad c_2 - y = c_1 \quad$ so that $\quad y = c_1 - c_2 \quad$ ... (9)

Substituting the values of x and y given by (8) and (9) in (7), we obtain

$c_2^2+(c_2-c_1)^2=1 \quad$ or $\quad (x-z)^2+(y-z)^2=1, \quad$ using (4) and (6)

or
$$x^2+y^2+2z^2-2zx-2yz-1=0.$$

Ex. 17. *The integral surfaces satisfying the partial differential equation* $(\partial z/\partial y)+z^2(\partial z/\partial y)=0$ *and spassing through the straight line* $x = 1$, $y = z$ *is*

(*a*) $(x-1)z+z^2=y^2$ (*b*) $x^2+y^2-z^2=1$

(c) $(y-z)x+x^2=1$ (*d*) $(x-1)z^2+z=y$ **[GATE 2012]**

Sol. Ans. (d). Given $p + z^2q = 0$, where $\quad p=\partial z/\partial x,\ q=\partial z/\partial y \quad$ ... (1)

Lagrange's auxiliary equation (1) are $\quad (dx)/1=(dy)/z^2=(\partial z)/0 \quad$ ... (2)

From third fraction of (1), $\quad dz = 0 \quad$ so that $\quad z = c^1 \quad$ ... (3)

Using (3), from first and second fractions of (2), $\quad (dx)/1=(dy)/c_1^{\,2} \quad$ or $\quad c_1^2dx-dy=0$

Integrating, $\quad c_1{}^2x-y=c_2 \quad$ or $\quad z^2x-y=c_2$, *as* $c_1=z \quad$ ... (4)

In order to get the required integral surfaces, we shall use method II given on page 2.28.

The given straight line is represented by $\quad x = 1, \quad y = z \quad$ ... (5)

Using (5) in (3) and (4), we get $\quad y=c_1 \quad$ and $\quad y^2-y=c_2 \quad$ ... (6)

Eliminating *y* from the two equation of (6), we get $\quad c_1^2-c_1=c_2 \quad$ ... (7)

Substituting the values of c_1 and c_2 given by (3) and (4) in (7), we get

$z^2-z=z^2x-y \quad$ or $\quad (x-1)z^2+z^2=y$, which is the required integral surface

Ex.18. *The expression* $\dfrac{1}{D_x^2 - D_y^2}\sin(x-y)$ *is equal to*

(*a*) $-(x/2)\times\cos(x-y)$ (*b*) $-(x/2)\times\sin(x-y)+\cos(x-y)$

(*c*) $-(x/2)\times\cos(x-y)+\sin(x-y)$ (*d*) $(3x/2)\times\sin(x-y)$ **[GATE 2012]**

Sol. Ans. (a). Here note that D_x and D_y stand for D and D' respectively. For solution, processd as in Ex. 7(d). Here, we wish to find only P.I. Thus,

$$\frac{1}{D_x^2 - D_y^2}\sin(x-y) = \frac{1}{D^2 - D'^2}\sin(x-y) = \frac{1}{D+D'}\frac{1}{D-D'}\sin(x-y)$$

$$= \frac{1}{D-D'}\frac{1}{1-(-1))}\int \sin v\, dv, \text{ where } v = x-y$$ [Using formula (i) page 4.9]

$$= \frac{1}{2}\ \frac{1}{D+D'}\ [-\cos(x-y)] = \frac{1}{2}\ \frac{1}{(-1)D-(1)\times D}\ \cos(x-y) = \frac{1}{2}\times\frac{x}{(-1)^1\times 1!}\cos\ (x-y)$$

[Using formula (ii), page 4.9]

$$= -(x/2)\times\cos(x-y)$$