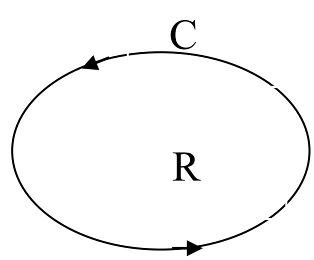
Integral theorems

Green's theorem in the plane

Statement: If R is a closed region in the xy plane bounded by simple closed curve C and M, N are continuous functions of x and y having continuous derivative in R then

$\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$ (Relationship between line integral and area integral)

where C is traversed in the positive (counter-clockwise) direction.



Proof:

Suppose

Equation of AEB is $y=Y_1(x)$ and

Equation of AFB is $y=Y_2(x)$

and R is the region bounded by the curve C

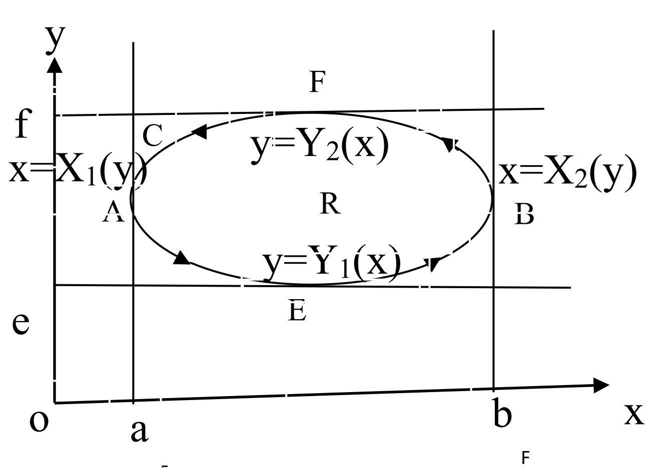


Fig1

$$\iint_R \frac{\partial M}{\partial y} dxdy$$
$$= \int_a^b \left[\int_{y=Y_1(x)}^{y=Y_2(x)} \frac{\partial M}{\partial y} dy \right] dx$$
$$= \int_a^b \left[M(x,y) \right]_{Y_1(x)}^{Y_2(x)} dx$$
$$= \int_a^b \left[M(x, Y_2(x)) - M(x, Y_1(x)) \right] dx$$
$$= - \int_a^b M(x, Y_1(x)) dx - \int_b^a M(x, Y_2(x)) dx$$
$$= - \left[\int_a^b M(x, Y_1(x)) dx + \int_b^a M(x, Y_2(x)) dx \right]$$
$$= - \oint_C M dx$$
$$\oint_C M dx = - \iint_R \frac{\partial M}{\partial y} dxdy$$

Again let the equation of the curves EAF and EBF be $x=X_1(y)$ and $x=X_2(y)$.

Now

$$\iint_R \frac{\partial N}{\partial x} dxdy$$
$$= \int_e^f \left[\int_{x=X_1(y)}^{x=X_2(y)} \frac{\partial N}{\partial x} dx \right] dy$$
$$= \int_e^f \left[N(x,y) \right]_{X_1(y)}^{X_2(y)} dy$$

$$=\int_e^f [N(X_2(y),y)-N(X_1(y),y)]dy$$

$$=\int_e^f N(X_2(y),y)dy-\int_e^f N(X_1(y),y)dy$$

$$=\int_e^f N(X_2(y),y)dy+\int_f^e N(X_1(y),y)dy$$

$$=\int_f^e N(X_1(y),y)dy+\int_e^f N(X_2(y),y)dy$$

$$=\oint_C Ndy$$

$$\oint_C Ndy=\iint_R \frac{\partial N}{\partial x}dxdy$$

Adding $\oint_C$ Mdx+Ndy=$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$ proved

Deduction:

If M=-y/2 and N=x/2 then

$\frac{1}{2}\oint_C$ -ydx+xdy=$\iint_R dxdy$=Area of the region R

Problem-1. Use line integral to find the area enclosed by the curves y=x and y^2=4x.

Discussion on area

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Area=$\int_0^4 (y_1-y_2)dx$

$=\int_0^4 (2\sqrt{x}-x)dx=\frac{8}{3}$

$$\text{Area}=\iint_R dxdy$$

$$=\int_0^4 \int_x^{2\sqrt{x}} dydx$$

$$=\int_0^4 [y]_x^{2\sqrt{x}} dx$$

$$=\int_0^4 (2\sqrt{x}-x)dx=\frac{8}{3}$$

Solution

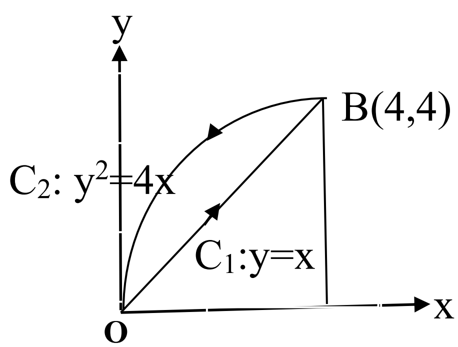

Fig2

$$\text{Area}=\iint_R dxdy$$

$$=\frac{1}{2}\oint_C -ydx+xdy$$

$$=\frac{1}{2}\int_{C_1} -ydx+xdy+\frac{1}{2}\int_{C_2} -ydx+xdy, \text{ along } C_2 \;\; x=y^2/4,\ dx=(y/2)\,dy$$

$$=\frac{1}{2}\int_0^4 -xdx+xdx+\frac{1}{2}\int_4^0 -\frac{y^2}{2}dy+\frac{y^2}{4}dy=\frac{8}{3}$$

Verification of this theorem

Problem-2. Verify Green's theorem for the plane for $\oint_C xydx+x^2dy$ where C is the closed region bounded by y=x and $y=x^2$

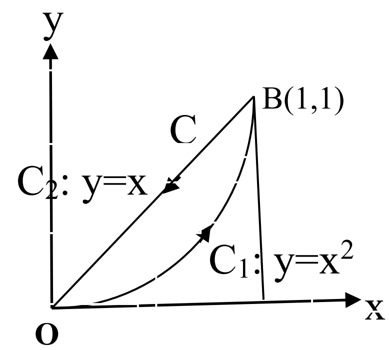

Fig 3

$$\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$LS=\oint_C Mdx+Ndy$$

$$=\oint_C xydx+x^2dy$$

$$=\oint_{C_1} xydx+x^2dy+\oint_{C_2} xydx+x^2dy$$

$$=\int_0^1 xx^2dx+x^2 2xdx+\int_1^0 xxdx+x^2dx$$

$$=3\int_0^1 x^3dx+2\int_1^0 x^2dx$$

$$=\frac{1}{12}$$

$$RS=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$=\iint_R(2x-x)dxdy$$

$$=\int_0^1\int_{x^2}^{x} x\,dy\,dx$$

$$=\int_0^1 x[y]_{x^2}^{x}\,dx$$

$$=\int_0^1 x(x-x^2)\,dx$$

$$=\frac{1}{12}$$

LS=RS verified

Problem-3. Find the work done in moving a particle along the curve C in the force field **F**$=(3x^2-8y^2)i+(4y-6xy)j$ where C is the curve $y=x^2$ from (0,0) to (1,1) and the $y^2=x$ from (1,1) to (0,0) using Green,s theorem.

F.dr = $(3x^2-8y^2)dx+(4y-6xy)dy$

Work= $\int_C$ **F**.d**r**$=\int_C (3x^2-8y^2)dx+(4y-6xy)dy$

$$=\int_{C_1}(3x^2-8y^2)dx+(4y-6xy)dy+\int_{C_2}(3x^2-8y^2)dx+(4y-6xy)dy$$

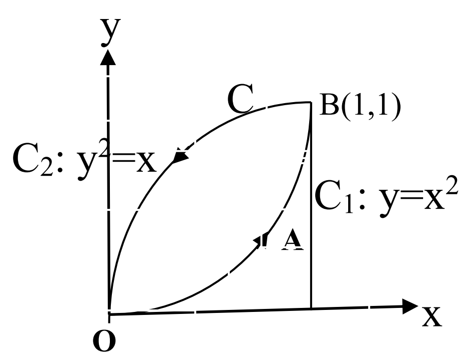

Fig 4

On C_1, OAB $y=x^2$ so that $dy=2xdx$

$$\int_{C_1} \mathbf{F}.d\mathbf{r}=\int_{C_1} (3x^2-8y^2)dx+(4y-6xy)dy$$

$$=\int_0^1 (3x^2-8x^4)dx+(4x^2-6xx^2)2xdx$$

$$=-1$$

On C_2 BCO $x=y^2$ so that $dx=2ydy$

$$\int_{C_1} \mathbf{F}.d\mathbf{r}=\int_{C_1} (3x^2-8y^2)dx+(4y-6xy)dy$$

$$=\int_1^0 (3y^4-8y^2)2ydy+(4y-6y^2y)dy$$

$$=\frac{5}{2}$$

Work done$=\int_C \mathbf{F}.d\mathbf{r}=\int_{C_1} \mathbf{F}.d\mathbf{r}+\int_{C_2} \mathbf{F}.d\mathbf{r}=-1+\frac{5}{2}=\frac{3}{2}$

Alternative method

This problem can also be done by using Green's theorem

Green's theorem is $\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

Work$=\int_C \mathbf{F}.d\mathbf{r}=\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ $M=3x^2-8y^2$ $N=4y-6xy$

$$=\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

$$=\iint_R(-6y+16y))dxdy$$

$$=\iint_R(10y)dxdy$$

$$=10\iint_R ydydx$$

$$=10\int_0^1\int_{x^2}^{\sqrt{x}} ydydx$$

$$=10\int_0^1\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$=10\int_0^1 \frac{1}{2}(x-x^2)dx$$

$$=\frac{3}{2}$$

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Express Green's theorem in the plane in vector form

Green's theorem is $\oint_C$ Mdx+Ndy=$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

Mdx+Ndy=(Mi+Nj).(dxi+dyj)=**A.dr**

where **A=**Mi+Nj and **r**=xi+yj

$$\nabla \times \mathbf{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z} i + \frac{\partial M}{\partial z} j + \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)k$$

(∇ x**A**).k=$\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$

Green's theorem can be written as $\oint_C$ **A.dr** =$\iint_R$ (∇ x**A**).k dR

dR=dxdy

This is for two dimensional case…. curve to region

For three dimensional case…….curve to surface which is known as Stoke's theorem (it will be considered next)

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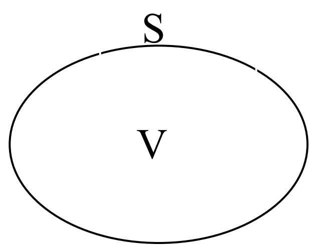
Gauss divergence theorem

Statement: The surface integral of the normal component of the vector **A** taken over a closed surface is equal to the integral of the divergence of **A** taken over the volume enclosed by the surface.

If n be unit outward normal at a point of the boundary of the closed region S and dV be the volume element of the region, then the theorem expresses that

$$\iint_S \mathbf{A}.\mathbf{n}dS = \iiint_V \nabla.\mathbf{A}dV$$ (Relationship between surface and volume integral)



Proof:

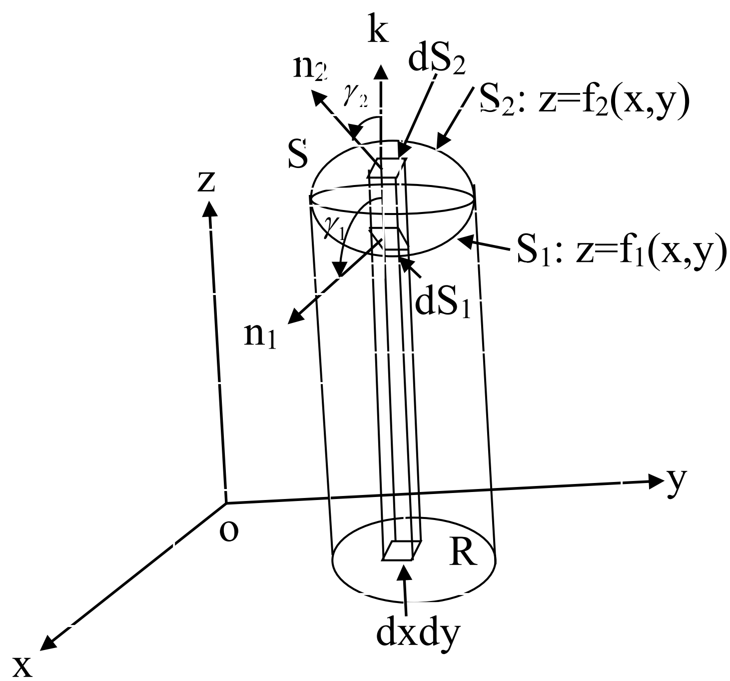


Fig4

If $\mathbf{A}=A_1i+A_2j+A_3k$ then

$$\iint_S \mathbf{A}.\mathbf{n}dS = \iiint_V \nabla.\mathbf{A}dV$$

$$\iint_S (A_1i+A_2j+A_3k).\mathbf{n}dS = \iiint_V \left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)dV$$

$$\iint_S A_1i.\mathbf{n}dS+\iint_S A_2j.\mathbf{n}dS+\iint_S A_3k.\mathbf{n}dS$$

$$=\iiint_V \frac{\partial A_1}{\partial x}dV+\iiint_V \frac{\partial A_2}{\partial y}dV+\iiint_V \frac{\partial A_3}{\partial z}dV$$

Let us suppose that S be a closed surface such that any line parallel to the coordinate axes cuts S in at most two points.
Suppose that
Equation of the lower surface S_1 is $z=f_1(x,y)$ and
equation of the upper surface S_2 is $z=f_2(x,y)$

and R is the projection of the surface S on the xy plane

Consider the third integral of the RHS

$$\iiint_V \frac{\partial A_3}{\partial z}\,\mathrm{dV}$$

$$=\iiint_V \frac{\partial A_3}{\partial z}\,\mathrm{dzdydx}$$

$$=\iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial A_3}{\partial z}\,dz\right]dydx$$

$$=\iint_R \left[A_3(x,y,z)\right]_{z=f_1(x,y)}^{z=f_2(x,y)}dydx$$

$$=\iint_R \left[A_3(x,y,f_2)-A_3(x,y,f_1)\right]dydx$$

$$=\iint_R A_3(x,y,f_2)dydx-\iint_R A_3(x,y,f_1)dydx$$

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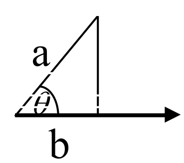
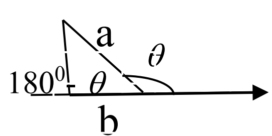
<u>Discussion</u>

When θ is acute



Projection of a is b=acosθ

When θ is obtuse



Projection of a is b=acos(180-θ)=- acosθ

We consider the projection on the xy plane

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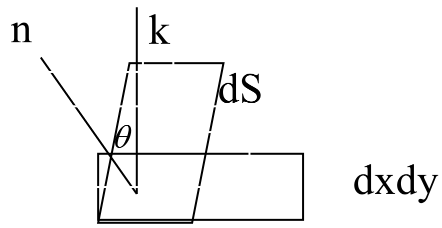
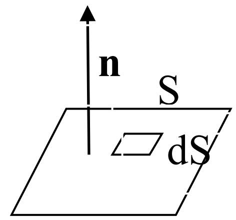
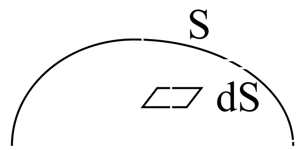
Elementary area in the xy plane is =dxdy

The angle between two planes is equal to the angle between their normals







Projection of dS is $dxdy = dS\cos\theta = dS\,k.\mathbf{n}$ $[k.\mathbf{n} = 1.1\cos\theta]$

For the upper portion of the surface S_2, $dxdy = dS_2\cos\gamma_2 = dS_2\,k.\mathbf{n_2}$ since the outward normal $\mathbf{n_2}$ to S_2 makes an acute angle γ_2 with k.

For the lower portion of the surface S_1, $dxdy = -dS_1\cos\gamma_1 = -dS_1\,k.\mathbf{n_1}$ since the outward normal $\mathbf{n_1}$ to S_1 makes an obtuse angle γ_1 with k.
hence

$$\iint_R A_3(x,y,f_2)\,dydx = \iint_{S_2} A_3\,\mathbf{k.n_2}\,dS_2$$

$$\iint_R A_3(x,y,f_1)\,dydx = -\iint_{S_1} A_3\,\mathbf{k.n_1}\,dS_1$$

Then $$\iint_R A_3(x,y,f_2)\,dydx - \iint_R A_3(x,y,f_1)\,dydx = \iint_{S_2} A_3\,\mathbf{k.n_2}\,dS_2 + \iint_{S_1} A_3\,\mathbf{k.n_1}\,dS_1$$

$$= \iint_S A_3\,k.\mathbf{n}\,dS$$

$$\iiint_V \frac{\partial A_3}{\partial z} dV = \iint_S A_3 k.\mathbf{n}dS$$

Similarly $\iiint_V \frac{\partial A_1}{\partial x} dV = \iint_S A_1 i.\mathbf{n}dS$

And $\iiint_V \frac{\partial A_2}{\partial y} dV = \iint_S A_2 j.\mathbf{n}dS$

Adding all these we get

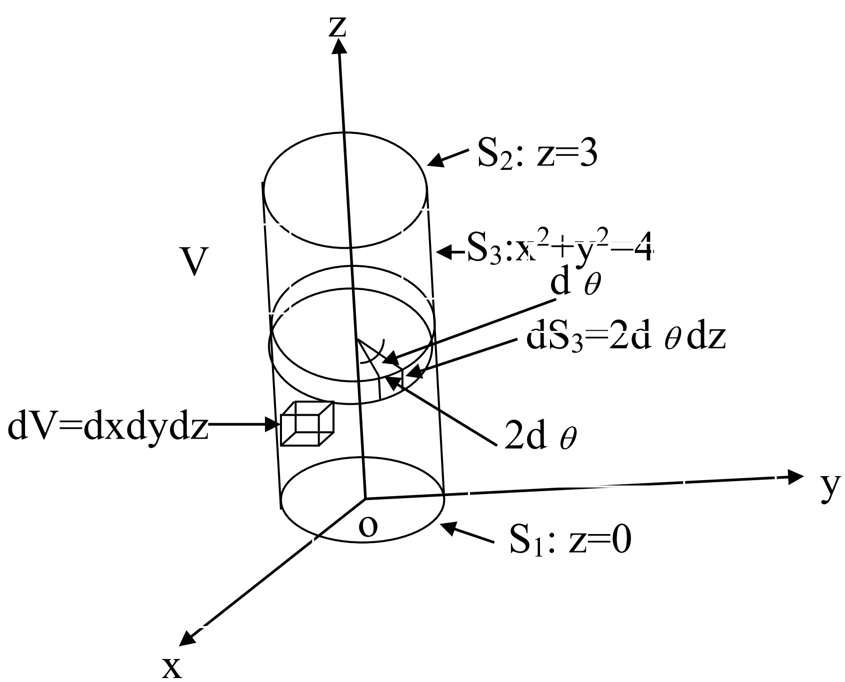
$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1 i + A_2 j + A_3 k).\mathbf{n}dS$$

$\iiint_V \nabla.\mathbf{A}dV = \iint_S \mathbf{A}.\mathbf{n}dS$ proved

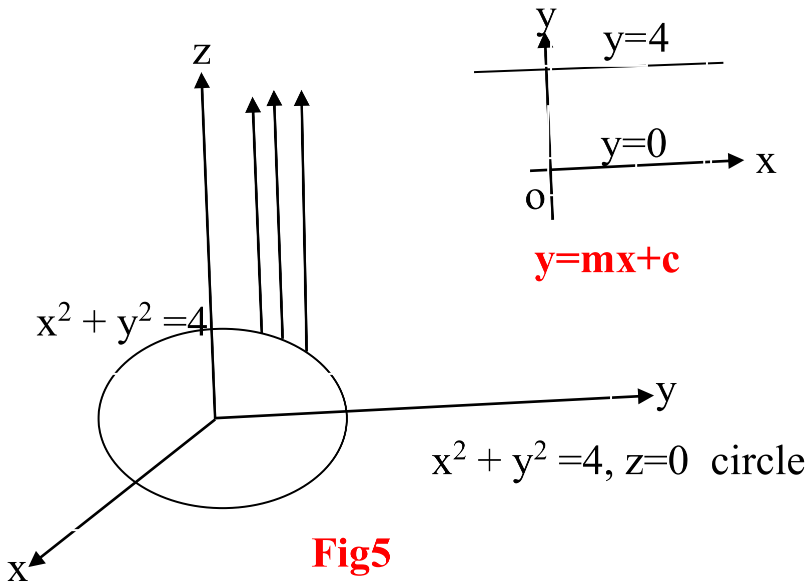
Verification

Problem-4. Verify Gauss divergence theorem for $\mathbf{A}=4xi-2y^2j+z^2k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

$\iiint_V \nabla.\mathbf{A}dV = \iint_S \mathbf{A}.\mathbf{n}dS$ (to be verified)



Discussion



y=mx+c

Fig5

$$\nabla.\mathbf{A}=\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}$$

$$=\frac{\partial(4x)}{\partial x}+\frac{\partial(-2y^2)}{\partial y}+\frac{\partial(z^2)}{\partial z}$$

$$=4-4y+2z$$

$$LS=\iiint_V \nabla.\mathbf{A}dV=\iiint_V (4-4y+2z)\,dV$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dzdydx$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z-4yz+z^2\right]_0^3 dydx$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)dydx$$

$$=\int_{x=-2}^{2}\left[21y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

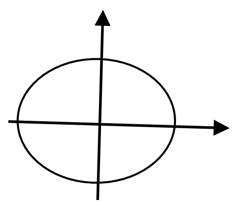
$$= \int_{x=-2}^{2} \left[42\sqrt{4-x^2}\right]dx$$

$$=2 \int_{x=0}^{2} \left[42\sqrt{4-x^2}\right]dx \quad x=2\sin\theta$$

$$=84x4\int_{0}^{\frac{\pi}{2}} \sin^2\theta d\theta$$

$$=84x4\frac{\pi}{4}$$

$$=84\pi$$



RS=$\iint_S$ **A**.**n**dS

The surface S of the cylinder consists of

S_1, the plane z=0

S_2, the plane z=3

S_3, the convex surface $x^2+y^2=4$

$\iint_S$ **A**.**n**dS=$\iint_{S_1}$ **A**.**n**dS_1+$\iint_{S_2}$ **A**.**n**dS_2+$\iint_{S_3}$ **A**.**n**dS_3

On S_1, z=0, n=-k, **A**=4xi-2y^2j **A.n**=0

Therefore $\iint_{S_1}$ **A**.**n**dS_1=0

On S_2, z=3, **n=k**, **A**=4xi-2y^2j+9k **A.n**=9

therefore $\iint_{S_2}$ **A**.**n**dS_2=9$\iint_{S_2}$ dS_2=9x4π=36π

On S_3, $x^2+y^2=4$, $\mathbf{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{2xi+2yj}{\sqrt{(2x)^2+(2y)^2}}=\frac{xi+yj}{2}$

$\mathbf{A.n}=(4xi-2y^2j+z^2k).(\frac{xi+yj}{2})=2x^2-y^3$

From fig.

$x=2\cos\theta$ $y=2\sin\theta$ $dS_3=2d\theta dz$

$$\iint_{S_3} \mathbf{A}.\mathbf{n}dS_3=\iint_{S_3} (2x^2-y^3)dS_3$$

$$=\int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2-(2\sin\theta)^3]\, 2dzd\theta$$

$$=48\int_{\theta=0}^{2\pi} (\cos^2\theta-\sin^3\theta)\, d\theta$$

$$=48\int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta$$

$$=48\pi$$

$$\iint_{S} \mathbf{A}.\mathbf{n}dS=0+36\pi+48\pi=84\pi$$

LS=RS verified

Problem-5. Evaluate $\iint_S$ F.ndS where $F=4xi-2y^2j+z^2k$ and S is surface the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Or,

Find the flux of $F=4xi-2y^2j+z^2k$ through the surface S where S is surface the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Or,

Find the flow of fluid through the surface S where S is surface the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$, if the fluid velocity is $\mathbf{v}=4xi-2y^2j+z^2k$

Or,

Find the flow of heat through the surface S where S is surface the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$, if the heat flux $\mathbf{q}=4xi-2y^2j+z^2k$

Find the flow of current through the surface S where S is surface the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$, if the current density $\mathbf{J}=4xi-2y^2j+z^2k$

These problems can be done either directly or by using Gauss divergence theorem

$$\iiint_V \nabla.\mathbf{A}dV=\iint_S \mathbf{A}.\mathbf{n}dS$$

Directly (using surface integral)

$$\iint_S \mathbf{A}.\mathbf{n}dS=84\pi \quad \text{(done previously)}$$

Or,

by using Gauss divergence theorem

$$\iint_S \mathbf{A}.\mathbf{n}dS=\iiint_V \nabla.\mathbf{A}dV$$

$$\iiint_V \nabla.\mathbf{A}dV=84\pi \quad \text{(done previously)}$$

Stoke's theorem

Statement: The line integral of the tangential component of a vector **A** taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of **A** taken over any Surface S having C as its boundary.

Mathematically

$$\oint_C \mathbf{A}.\mathbf{dr}=\iint_S \nabla x\mathbf{A}.\mathbf{n}dS \quad \text{(Relation between line and surface integral)}$$

A.dr=A.Tds

Proof:

Let S be a surface such that its projection on the xy, yz and zx planes are the regions bounded by simple closed curves and suppose that the equation of the surface can be written as z=f(x,y), x=g(y,z) and y=h(z,x) where f, g, and h are single valued continuous and differentiable function.

Fig6

Let If $\mathbf{A}=A_1i+A_2j+A_3k$ and $\mathbf{r}=xi+yj+zk$

Then the theorem expresses that

$$\oint_C A_1dx+A_2dy+A_3dz=\iint_S \nabla \times (A_1i+A_2j+A_3k).\mathbf{n}dS$$

$$\oint_C A_1dx+\oint_C A_2dy+\oint_C A_3dz=\iint_S \nabla \times (A_1i).\mathbf{n}dS+$$

$$\iint_S \nabla \times (A_2j).\mathbf{n}dS+\iint_S \nabla \times (A_3k).\mathbf{n}dS$$

Consider the first term on the right $\iint_S \nabla \times (A_1i).\mathbf{n}dS$

$$\nabla \times (A_1i)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix}=\frac{\partial A_1}{\partial z}j-\frac{\partial A_1}{\partial y}k$$

$[\nabla \times (A_1 i)].\mathbf{n}dS = \left(\frac{\partial A_1}{\partial z} j.n - \frac{\partial A_1}{\partial y} k.n \right) dS$

Choosing z=f(x,y) as the equation of the surface S, the position vector **r** of any point of S is given by

$$\mathbf{r} = xi + yj + zk = xi + yj + f(x,y)k$$

$$\frac{\partial r}{\partial y} = j + \frac{\partial z}{\partial y} k$$

$\frac{\partial r}{\partial y}$ is a vector tangent to S, therefore it must be perpendicular to **n**

$\mathbf{n}.\frac{\partial r}{\partial y} = \mathbf{n}\, j + \mathbf{n}.\frac{\partial z}{\partial y} k = 0 \qquad \mathbf{n}\, j = -\frac{\partial z}{\partial y} \mathbf{n}.k$

$$[\nabla \times (A_1 i)].\mathbf{n}dS = \left(-\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} n.k - \frac{\partial A_1}{\partial y} k.n \right) dS$$

$$= -\left(\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} \right) \mathbf{n}.kdS$$

On the Surface S $A_1(x,y,z) = A_1(x,y,f(x,y)) = F(x,y)$ (say)
$A_1(x,y,z) = F(x,y)$

Differentiating partially with respect to y

$$\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$$

$$[\nabla \times (A_1 i)].\mathbf{n}dS = -\frac{\partial F}{\partial y} \mathbf{n}.kdS = -\frac{\partial F}{\partial y} dxdy$$

$$\iint_S \nabla \times (A_1 i).\mathbf{n}dS = -\iint_R \frac{\partial F}{\partial y} dxdy$$

where R is the projection of S on the xy plane
Now by Green’s theorem in the plane region R

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

If C_1 is the boundary of R then

$$\oint_{C_1} Fdx+0dy=\iint_R\left(\frac{\partial 0}{\partial x}-\frac{\partial F}{\partial y}\right)dxdy=-\iint_R\left(\frac{\partial F}{\partial y}\right)dxdy$$

$$\iint_S \nabla \times (A_1 i).\mathbf{n}dS=\oint_{C_1} Fdx$$

Since at each point (x,y) of C_1, the value of F is the same as the value of A_1 at each point (x,y,z) of C and since dx is same for both the curves, we have

$$\oint_{C_1} Fdx=\oint_C A_1dx$$

$$\iint_S \nabla \times (A_1 i).\mathbf{n}dS=\oint_C A_1dx$$

Similarly

$$\iint_S \nabla \times (A_2 i).\mathbf{n}dS=\oint_C A_2dx$$

$$\iint_S \nabla \times (A_3 i).\mathbf{n}dS=\oint_C A_3dx$$

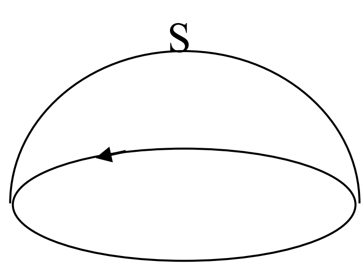
Adding

$$\iint_S \nabla \times (A_1 i+A_2 j+A_3 k).\mathbf{n}dS=\oint_C A_1dx+A_2dy+A_3dz$$

$$\iint_S \nabla \times \mathbf{A}.\mathbf{n}dS=\oint_C \mathbf{A.dr} \qquad \text{proved}$$

Verification

Problem-6. Verify Stoke's theorem for $\mathbf{A}=(2x-y)i-yz^2j-y^2zk$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.



C

The boundary C of S is a circle in the xy plane of radius one and centre at the origin The parametric equation of the circle is x=cost, y=sint, z=0, $0 \le t \le 2\pi$.

We are to show that $\oint_C \mathbf{A}.d\mathbf{r} = \iint_S \nabla \text{x} \mathbf{A}.\mathbf{n} dS$

LS=$\oint_C \mathbf{A}.d\mathbf{r} = \oint_C (2x-y)dx - yz^2 dy - y^2 z dz = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt = \pi$

RS=$\iint_S \nabla \text{x} \mathbf{A}.\mathbf{n} dS$

$$\nabla \text{x} A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = k$$

$\iint_S \nabla \text{x} \mathbf{A}.\mathbf{n} dS$

$= \iint_S k.\mathbf{n} dS$

$= \iint_R dxdy$

$= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dydx$

$=4 \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} dydx$

$=4 \int_{x=0}^{x=1} \sqrt{1-x^2} dx$

$= \pi$

LS=RS verified

END