Question 1

求证:

$$\text{数列}\left\{\left(1+\frac{1}{n}\right)^n\right\}\text{是严格递增的数列;}$$
$$\text{数列}\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}\text{是严格递减的数列.}$$

法一:

证明. 引入

$$x_n=\left(1+\frac{1}{n}\right)^n,\quad y_n=\left(1+\frac{1}{n}\right)^{n+1},\quad n=1,2,\cdots.$$

对任意正整数 n, 应用 Bernoulli 不等式, 有:

$$\begin{aligned}\frac{x_{n+1}}{x_n}&=\frac{(n+2)^{n+1}}{(n+1)^{n+1}}\frac{n^n}{(n+1)^n}\frac{n}{n+1}\frac{n+1}{n}\\&=\left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n}\\&>\left(1-\frac{1}{n+1}\right)\frac{n+1}{n}\\&=1;\end{aligned}$$

$$\begin{aligned}\frac{y_n}{y_{n+1}}&=\left(\frac{n+1}{n}\right)^{n+1}\left(\frac{n+1}{n+2}\right)^{n+2}\\&=\left(1+\frac{1}{n^2+2n}\right)^{n+1}\frac{n+1}{n+2}\\&>\left(1+\frac{n+1}{n^2+2n}\right)\frac{n+1}{n+2}\\&=\frac{n^3+4n^2+4n+1}{n^3+4n^2+4n}\\&>1.\end{aligned}$$

□

Question 1 continued on next page. . .

法二:

证明. 利用算数平均-几何平均不等式得到:

$$\left(1+\frac{1}{n}\right)^n = 1\cdot\underbrace{\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{1}{n}\right)}_{n\text{个}} < \left(\frac{1+n\left(1+\frac{1}{n}\right)}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}.$$

类似地, 有:

$$\left(\frac{n}{n+1}\right)^{n+1} = 1\cdot\underbrace{\left(\frac{n}{n+1}\right)\cdots\left(\frac{n}{n+1}\right)}_{n+1\text{个}} < \left(\frac{1+(n+1)\left(\frac{n}{n+1}\right)}{n+2}\right)^{n+2} = \left(\frac{n+1}{n+2}\right)^{n+2},$$

这等价于

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$$

□