Question 1

求证: 当 $n \geqslant 3$ 时有不等式

$$\sum_{k=0}^{n} \frac{1}{k!} - \frac{3}{2n} < \left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!}.$$

证明. 引入

$$e_n = \left(1+\frac{1}{n}\right)^n, \quad s_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad n = 1, 2, \cdots .$$

利用二项式展开, 得到

$$\begin{aligned} e_n &= 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= 1 + \frac{1}{1!} + \sum_{k=2}^{n} \frac{1}{k!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\ &< 1 + \frac{1}{1!} + \sum_{k=2}^{n} \frac{1}{k!} = s_n. \end{aligned}$$

同样地,

$$\begin{aligned} s_n - e_n &= \sum_{k=0}^{n} \frac{1}{k!} - \left[1 + \frac{1}{1!} + \sum_{k=2}^{n} \frac{1}{k!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right] \\ &= \sum_{k=2}^{n} \frac{1}{k!} \left[1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\right] \end{aligned}$$

注意到当实数 $a_1, \cdots, a_j$ $(j \geqslant 2)$ 同号且都大于 -1 时, 有

$$(1+a_1)\cdots(1+a_j) > 1 + a_1 + \cdots + a_j.$$

因此,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) > 1 - \frac{k(k-1)}{2n}, \quad k \geqslant 3.$$

所以

$$\begin{aligned} s_n - e_n &< \frac{1}{2n} + \sum_{k=3}^{n} \frac{1}{k!} \frac{k(k-1)}{2n} \\ &= \frac{1}{2n}\left[1 + \sum_{j=1}^{n-2} \frac{1}{j!}\right] \leqslant \frac{1}{2n}\left[1 + \sum_{j=1}^{n-2} \frac{1}{2^{j-1}}\right] < \frac{3}{2n} \end{aligned}$$

□