Module V

State - space analysis

State variable analysis: state equation, state representation of continuous time systems. Transfer function from state variable representation, solution of the state equation, state transition matrix.
Concept of controllability, Observability, Kalman test, Gilbert test.

→ The state variable approach is a powerful technique for the analysis & Design of control system.

State variable analysis :-

The procedure for determining the state of a system is called state variable analysis. A system can be represented by

- Transfer function Model $H(s) = \frac{Y(s)}{X(s)}$
- Block diagram model
- Differential equation model
- State space model.

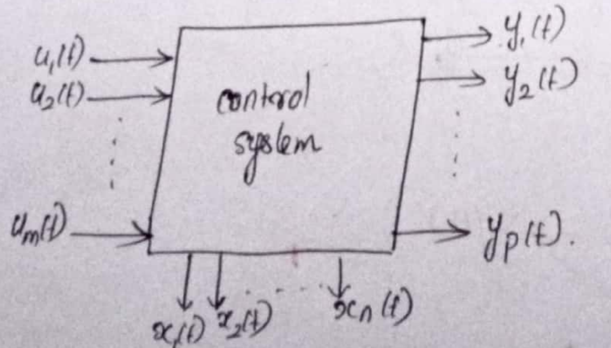
→ **Drawbacks of transfer function Model:-**

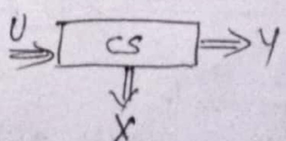
1) Only defined under zero initial conditions
2) It is applicable to linear time invariant systems. (LTI)
3) TF analysis is restricted to SISO (single i/p single o/p) systems.

→ **Advantages of statespace Techniques:-**

1) State variable analysis can be applied to any type of system like linear, non linear systems, Time varying & time invariant systems.
2) Analysis can be carried with initial conditions.
3) can be carried on (MIMO) Multiple i/p & Multiple o/p systems.
4) suitable for digital computer computations.

STATE SPACE REPRESENTATION





⇒ System consists of m-inputs, P-outputs & n-state variables.

ie, I/p variables $= u_1(t), u_2(t) \ldots u_m(t)$

O/p variables $= y_1(t), y_2(t) \ldots y_p(t)$

state variable $= x_1(t), x_2(t) \ldots x_n(t)$

⇒ standard form of state model of a system consists of the state equation & o/p equation

$$\boxed{\begin{aligned} \dot{X} &= AX + BU \rightarrow \text{state equation} \\ Y &= CX + DU \rightarrow \text{o/p equation} \end{aligned}}$$

X - state vector

$\dot{X} \rightarrow$ differential state vector $\left(\frac{dx}{dt}\right)$

$U \rightarrow$ I/p vector

$A \rightarrow$ state matrix

$B \rightarrow$ I/p matrix

$C \rightarrow$ O/p matrix

$D \rightarrow$ Transition Matrix.

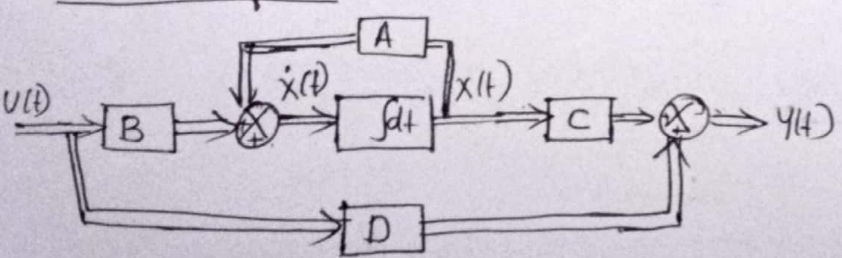
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} \cdots & b_{1m} \\ b_{21} \cdots & b_{2m} \\ \vdots & \\ b_{n1} \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$(n \times 1) \quad (n \times n) \quad (n \times 1) \quad (n \times m) \quad (m \times 1)$

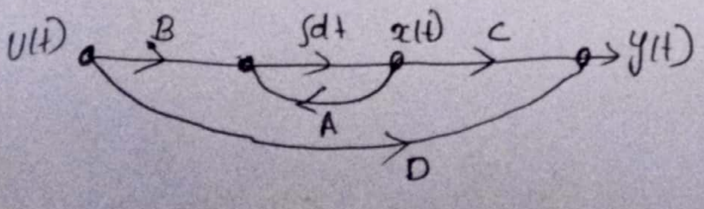
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & & \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \cdots & d_{1m} \\ d_{21} & d_{22} \cdots & d_{2m} \\ \vdots & & \\ d_{p1} & d_{p2} \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$(p \times 1) = (p \times n) \; (n \times 1) + (p \times m) \; (m \times 1)$

Block diagram of state model

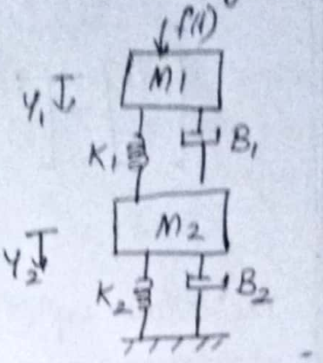


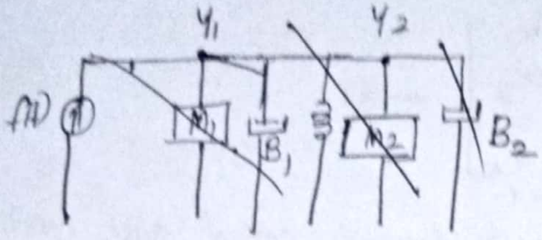
Signal flow graph

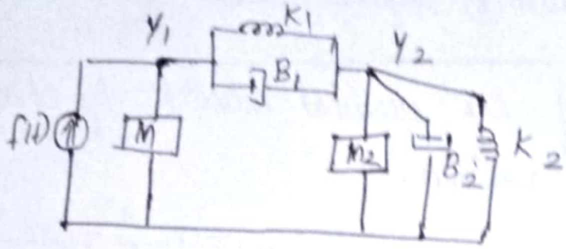


State space representation of continuous time systems

① Construct the state model of mechanical system as shown in fig







$$f(t) = M_1\frac{d^2y_1}{dt^2} + B_1\frac{d}{dt}(y_1-y_2) + K_1(y_1-y_2) \quad -①$$

$$0 = M_2\frac{d^2y_2}{dt^2} + B_2\frac{d}{dt}y_2 + K_2y_2 + B_1\frac{d}{dt}(y_2-y_1) + K_1(y_2-y_1) \quad -②$$

Let us choose 4 state variables x_1, x_2, x_3 & x_4.

Let i/p $f(t) = U$

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = \frac{dy_1}{dt},\ x_4 = \frac{dy_2}{dt},\ \dot{x}_3 = \frac{d^2y_1}{dt^2},\ \dot{x}_4 = \frac{d^2y_2}{dt^2}.$$

$$① \Rightarrow U = M_1\dot{x}_3 + B_1x_3 - B_1x_4 + K_1x_1 - K_1x_2$$

$$① \Rightarrow M_1\dot{x}_3 = -K_1x_1 + K_1x_2 - B_1x_3 + B_1x_4 + U$$

$$\dot{x}_3 = \frac{-K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{U}{M_1} \quad -③$$

$$② \Rightarrow M_2\dot{x}_4 + B_2x_4 + K_2x_2 + B_1x_4 - B_1x_3 + K_1x_2 - K_1x_1 = 0$$

$$\dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4 \quad -④$$

$$\dot{x}_1 = x_3 \quad -⑤$$

$$\dot{x}_2 = x_4 \quad -⑥$$

$\dot{X} = AX + BU$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{-K_1}{M_1} & \frac{K_1}{M_1} & \frac{-B_1}{M_1} & \frac{B_1}{M_1}\\ \frac{K_1}{M_2} & \frac{-(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & \frac{-(B_1+B_2)}{M_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{M_1}\\ 0\end{bmatrix}\begin{bmatrix}u\end{bmatrix} \quad -⑦$$

o/p equation $Y = CX + DU$

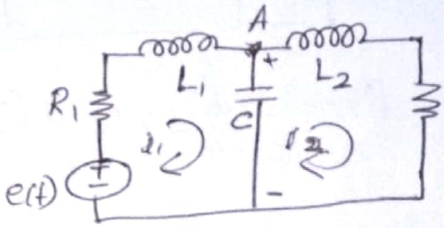
o/ps are displacements Y_1 & Y_2

$Y_1 = X_1$ & $Y_2 = X_2$.

$$\underset{2\times 1}{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}} = \underset{2\times 4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} \quad -⑧$$

The state eq^n ⑦ & o/p equation ⑧ together called state model of the s/m

② Obtain the state model of the electrical network by choosing minimal number of state variables.



⇒ Applying KVL, in loop 1

$$e(t) - R_1 i_1 - L_1 \frac{di_1}{dt} - V_C = 0 \quad -①$$

Let

i_1	$e(t)$	i_2	V_C
↓	↓	↓	↓
x_1	u	x_2	x_3

$$① \Rightarrow u - R_1 x_1 - L_1 \dot{x}_1 - x_3 = 0$$

$$\dot{x}_1 = \frac{u}{L_1} - \frac{R_1}{L_1} x_1 - \frac{1}{L_1} x_3 \quad -②$$

⇒ loop 2

$$V_C - L_2 \frac{di_2}{dt} - R_2 i_2 = 0 \quad -③$$

$$x_3 - L_2 \dot{x}_2 - R_2 x_2 = 0$$

$$\dot{x}_2 = -\frac{R_2}{L_2} x_2 + \frac{x_3}{L_2} \quad -④.$$

KVL loop 1

$$-e(t) + i R_1 + L_1 \frac{di_1}{dt} + V_C = 0$$

Loop 2

$$-V_C + L_2 \frac{di_2}{dt} + R_2 i_2 = 0$$

$$L_2 \dot{x}_2 = x_3 - R_2 x_2$$

$$\dot{x}_2 = -\frac{R_2}{L_2} x_2 + \frac{x_3}{L_2}.$$

⇒ At node A

$$i_1 = i_2 + C \cdot \frac{dV_C}{dt}$$

$$x_1 = x_2 + C \dot{x}_3 \Rightarrow \dot{x}_3 = \frac{x_1}{C} - \frac{x_2}{C} \quad -⑤$$

using ②, ④ & ⑤ ⇒

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{-R_1}{L_1} & 0 & \frac{-1}{L_1} \\ 0 & \frac{-R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & \frac{-1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{-1}{L_1} \\ 0 \\ 0 \end{bmatrix} [u] \quad -⑥$$

o/p equation $V_0 = y = R_2 i_2 = x_2 R_2$

$$\cancel{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \qquad y = \underset{1\times 3}{[0 \quad R_2 \quad 0]} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \quad -⑦$$

State eq^n ⑥ & o/p eq^n ⑦ together constitute the state model of the system.

3. A s/m described by differential equation $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = U(t)$.
where Y is o/p & U is the i/p of s/m.

Type-1 $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = U(t)$

$\dot{X} = AX + BU$ — state equation
$Y = CX + DU$ — o/p equation

Let $y(t) = X_1$

$\frac{dy(t)}{dt} = \dot{X}_1 = X_2$

$\frac{d^2y(t)}{dt^2} = \dot{X}_2$

$\dot{X}_2 + 4X_2 + 5X_1 = U$

$\dot{X}_2 = U - 4X_2 - 5X_1$

$\dot{X}_1 = X_2$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

state equation.

o/p equation $\Rightarrow [1\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

4) Construct state model for a system characterized by the differential equation $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$

$y = X_1$

$\frac{dy}{dx} = \dot{X}_1 = X_2$

$\frac{d^2y}{dx^2} = \dot{X}_2 = X_3$

$\frac{d^3y}{dx^2} = \dot{X}_3$

$\dot{X}_3 + 6X_3 + 11X_2 + 6X_1 + u = 0$

$\dot{X}_3 = -6X_1 - 11X_2 - 6X_3 - U$

$\dot{X}_1 = X_2$

$\dot{X}_2 = X_3$

state equation

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u$$

o/p equation

$Y = X_1$

$Y = [1\,0\,0]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

5) sm is described by the differential equation

$$\frac{d^3x}{dt^2} + \frac{3d^2x}{dt^2} + \frac{4dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)$$

$$y_1 = 4\frac{dx}{dt} + 3u_1$$

$$y_2 = \frac{d^2x}{dt^2} + 4u_2 + u_3.$$

Assume $x = x_1$

$$\frac{dx}{dt} = \dot{x}_1 = x_2$$

$$\frac{d^2x}{dt^2} = \dot{x}_2 = x_3$$

$$\frac{d^3x}{dt^2} = \dot{x}_3$$

$$\dot{x}_3 + 3x_3 + 4x_2 + 4x_1 = u_1(t) + 3u_2(t) + 4u_3(t)$$

State eqⁿ

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -4x_1 - 4x_2 - 3x_3 + u_1(t) + 3u_2(t) + 4u_3(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

O/P equation

$$y_1 = 4\frac{dx}{dt} + 3u_1 = 4x_2 + 3u_1$$

$$y_2 = \frac{d^2x}{dt^2} + 4u_2 + u_3 = x_3 + 4u_2 + u_3.$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underset{2\times 3}{\begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \underset{2\times 3}{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Scanned with CamScanner

Q:- Obtain the state model for the given transfer function,

type 2

$$G(s) = \frac{s^2+3s+3}{s^3+2s^2+3s+1}$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{Y(s)}{R(s)} \cdot \frac{X_1(s)}{X_1(s)}$$

$$= \frac{1}{(s^3+2s^2+3s+1)} (s^2+3s+3)$$

$$\frac{X_1(s)}{R(s)} = \frac{1}{s^3+2s^2+3s+1} \quad \text{— (1)}$$

$$\frac{Y(s)}{X_1(s)} = s^2+3s+3 \quad \text{— (2)}$$

$(1) \Rightarrow \quad R(s) = s^3 X_1(s) + 2s^2 X_1(s) + 3s\, X_1(s) + X_1(s)$

Inverse laplace transform,

$$r(t) = \frac{d^3 x_1(t)}{dt^3} + 2\frac{d^2 x_1(t)}{dt^2} + 3\frac{d x_1(t)}{dt} + x_1(t)$$

Let $x_1 \rightarrow$ 1st variable

$x_2(t) \rightarrow \left(\frac{dx_1}{dt}\right) \dot{x}_1(t) \rightarrow x_2$

$\left(\frac{d^2x_1}{dt^2}\right) \dot{x}_2(t) \rightarrow x_3$

$\left(\frac{d^3x_1}{dt^3}\right) \Rightarrow \dot{x}_3(t)$

State eqution

$\dot{X} = AX + BU$

$$r(t) = \dddot{x}_1(t) + 2\ddot{x}_1(t) + 3\dot{x}_1(t) + x_1(t)$$

$\dot{x}_3 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_1 = x_2 \qquad x_1$

$$\therefore r(t) = \dot{x}_3 + 2x_3 + 3x_2 + x_1$$

$$\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + r(t) \quad , \dot{x}_2 = x_3, \ \dot{x}_1 = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

— (3) state equation

o/p equation

$$Y = CX + DU$$

(2) ⇒ $$\frac{Y(s)}{X_1(s)} = s^2 + 3s + 3$$

$$Y(s) = s^2 X_1(s) + 3s X_1(s) + 3 X_1(s)$$

Inverse Laplace transform,

$$Y(t) = \frac{d^2 x_1(t)}{dt^2} + \frac{3\,d x_1(t)}{dt} + 3 x_1(t).$$

$$\dot{X}_2 = x_3 \qquad \dot{X}_1 \rightarrow X_2 \qquad X_1$$

$$Y(t) = X_3 + 3X_2 + 3x_1$$

$$Y = 3X_1 + 3X_2 + X_3$$

$$Y = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{——— (4)}$$

state equation (3) & o/p equation (4) together constitutes state model of the system.

⑥ A feedback system has a closed loop transfer function $\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$. Construct state model of the system and give blockdiagram representation of state model.

Type-3

$$\frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$A = \left.\frac{10(s+4)}{(s+1)(s+3)}\right|_{s=0} \quad \frac{10\times 4}{1\times 3} = \frac{40}{3}$$

$$B = \left.\frac{10(s+4)}{s(s+3)}\right|_{s=-1} = \frac{10\times 3}{-1\times 2} = -15$$

$$C = \left.\frac{10(s+4)}{s(s+1)}\right|_{s=-3} = \frac{10\times 1}{-3\times -2} = \frac{5}{3}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{40/3}{s} + \frac{-15}{s+1} + \frac{5/3}{s+3}$$

$$Y(s) = \underbrace{\frac{40/3}{s}U(s)}_{X_1(s)} + \underbrace{\frac{-15}{s+1}U(s)}_{X_2(s)} + \underbrace{\frac{5/3}{s+3}U(s)}_{X_3(s)} \Rightarrow Y(s) = X_1(s) + X_2(s) + X_3(s)$$

$$X_1(s) = \frac{40/3}{s}U(s) \Rightarrow sX_1(s) = \frac{40}{3}U(s)$$

$$\frac{d}{dt}x_1(t) = \frac{40}{3}u(t) \qquad x_1(t) \rightarrow x_1$$

$$\dot{x}_1 = \frac{40}{3}u(t) \text{ — ①} \qquad \frac{dx_1(t)}{dt} \rightarrow \dot{x}_1$$

$$X_2(s) = \frac{-15}{s+1}U(s) \Rightarrow sX_2(s) + X_2(s) = -15U(s)$$

$$\dot{x}_2 = -x_2 - 15u(t) \text{ — ②}$$

$$X_3(s) = \frac{5/3}{s+3}U(s) \Rightarrow sX_3(s) + 3X_3(s) = \frac{5}{3}U(s)$$

$$\dot{x}_3 = -3x_3 + \frac{5}{3}u \text{ — ③}$$

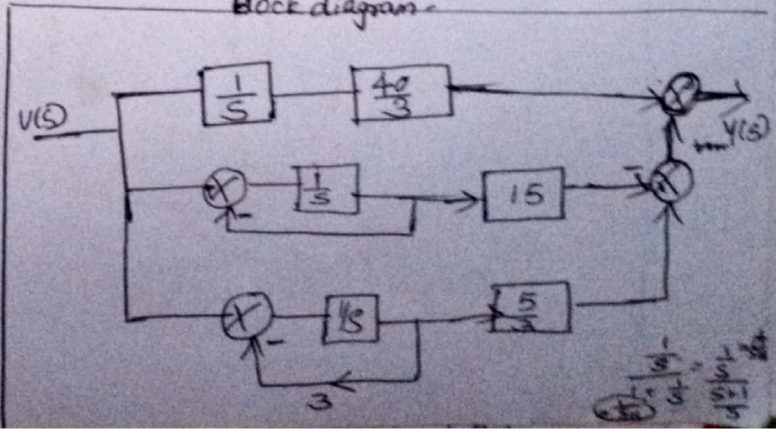
state eqn

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\frac{40}{3}\\ -15\\ \frac{5}{3}\end{bmatrix}[u]$$

o/p eqn.

$$[y] = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Block diagram.



HW

Q:- Determine the canonical state model of the system, whose T.F is $T(s) = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$

$$\frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$A = 3,\ B = -4,\ C = 1$

State model

① State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} u$$

② o/p equation

$$Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Transfer function from state equations

State equations are

$\dot{X} = AX + BU, \quad Y = CX + DU$

i.e., $\dot{X}(t) = AX(t) + BU(t), \quad Y(t) = CX(t) + DU(t)$

Taking LT on Both sides

$$LT\left\{\underbrace{\frac{d}{dt}x(t)}_{\dot{x}(t)}\right\} = SX(S)$$

$$\Rightarrow \quad SX(S) = AX(S) + BU(S) \quad \text{—①}$$

$$Y(S) = CX(S) + DU(S) \quad \text{—②}$$

From ① $\quad SX(S) - AX(S) = BU(S)$

$$X(S)[SI - A] = BU(S)$$

$$X(S) = [SI - A]^{-1} BU(S) \quad \text{—③}$$

Assuming o/p depends on the state not on the i/p then

$$Y(S) = CX(S) \quad \text{—④}$$

Sub. ③ in ④ $\Rightarrow$

$$Y(S) = C[SI - A]^{-1} BU(S)$$ $\Rightarrow$ Y(S) is called transfer matrix.

$$\boxed{\text{Transfer function } \frac{Y(S)}{U(S)} = C[SI - A]^{-1}B}$$

Qn:- Find the transfer function of the sm described by the state eqn

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \quad \& \quad y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{1}{S^2 + 3S + 2}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}$$

$$C[SI - A]^{-1}B \Rightarrow \frac{1}{S^2 + 3S + 2}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1x2 2x2

$$= \frac{1}{S^2 + 3S + 2}\begin{bmatrix} S+3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{S^2 + 3S + 2} \times 1 = \underline{\frac{1}{S^2 + 3S + 2}} \Rightarrow \underline{\frac{1}{(S+1)(S+2)}}$$

Transfer function from state variable representation (State eq)

$$T.F = C[SI-A]^{-1}B.$$

Q:- $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} u$ & $Y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Find func

Here = $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$SI-A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} S+1 & 0 \\ 0 & S+2 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+2 & 0 \\ 0 & S+1 \end{bmatrix}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$

$$TF = \frac{1}{(S+1)(S+2)}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} S+2 & 0 \\ 0 & S+1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$= \frac{1}{(S+1)(S+2)}\begin{bmatrix} S+2 & S+1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$= \frac{1}{(S+1)(S+2)} \quad 4(S+2) + {}^{-}2(S+1)$$

$$= \frac{4S+8-2S-2}{(S+1)(S+2)} = \frac{2S+6}{(S+1)(S+2)}$$

Qn :- Consider the s/m described by $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$ & $Y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Obtain transfer function of the s/m.

$$T.F = C[SI-A]^{-1}B.$$

$$[SI-A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} S+4 & 1 \\ -3 & S+1 \end{bmatrix}$$

$$[SI-A^{-1}] = \frac{1}{(S+1)(S+4)+3}\begin{bmatrix} S+1 & -1 \\ 3 & S+4 \end{bmatrix}$$

$$T.F = C[SI-A^{-1}]B \Rightarrow \frac{1}{(S+1)(S+4)+3}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} S+1 & -1 \\ 3 & S+4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{S^2+5S+7}\begin{bmatrix} S+1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \frac{1}{S^2+5S+7}(S+1+{}^{-}1)$$

$$\boxed{T.f = \frac{S}{S^2+5S+7}}$$

Solution of state Equations

① solution of homogeneous state equations (without input or excitation) $U=0$

WKT $\dot{X}(t) = AX + BU$

$\because U=0 \Rightarrow \dot{X}(t) = AX$

$$LT\left\{\frac{dx}{dt}\right\} = LT\{AX\}$$

$$S\,X(S) - X(0) = A\,X(S)$$

$$X(S)[S-A] = X(0)$$

$$X(S) = \frac{X(0)}{S-A}$$

$$\frac{1}{S-a} \xrightarrow{ILT} e^{at}$$

Take Inverse LT, $\boxed{x(t) = e^{At}\,x(0)}$

Let $\phi(t) = e^{At}$,

$\boxed{x(t) = \phi(t)\,x(0)}$ where $\phi(t) \rightarrow$ state transition matrix.

is the solution for the equation.

Properties of state transition Matrix ($\phi(t)$)

$$\phi(t) = e^{At} = 1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots = \sum_{i=0}^{n} \frac{A^i t^i}{i!}$$

property 1 : $\phi(0) = e^{A\times 0} = 1 = I$ (unity matrix)

property 2 : $\phi(t) = e^{At}$

$$\phi(t_1+t_2) = e^{A(t_1+t_2)} = \left(e^{At_1}\right)\left(e^{At_2}\right) = \phi(t_1)\,\phi(t_2)$$

property 3 : $[\phi(t)]^n = \phi(nt)$

property 4 : $\phi^{-1}(t) = \phi(-t)$

property 5 : $\phi(t_2-t_1)\cdot\phi(t_1-t_0) = \phi(t_2-t_0)$

$$\left[\phi(t_2-t_1)\cdot\phi(t_1-t_0) = \phi[t_2-t_1+t_1-t_0] = \phi(t_2-t_0)\right]$$

---

② solutions for non-homogeneous state equation ($u(t) \neq 0$)

state equation, is,

$$\dot{x}(t) = Ax + BU = A\,x(t) + B\,U(t)$$

Taking LT on both sides,

$$LT[\dot{X}(t)] = LT[AX(t) + BU(t)]$$

$$SX(S) - X(0) = AX(S) + BU(S)$$

$$X(S)[S-A] = X(0) + BU(S)$$

$$X(S) = \frac{X(0)}{S-A} + \frac{BU(S)}{S-A}$$

$$X(S) = (SI-A)^{-1}BU(S) + (SI-A)^{-1}X(0)$$

$$\frac{U(S)}{S-A} = U(S) \cdot \frac{1}{(S-A)} = U(t) * \phi(t)$$

multiplication in freq domain = conv. in time domain

$$U(t) * e^{At} \Rightarrow \int_0^t u(\tau) e^{A(t-\tau)} d\tau$$

$$\because \boxed{x_1(t) * x_2(t) = \int_0^t x_1(\tau) x_2(t-\tau) d\tau}$$

Taking ILT on both sides

$$\boxed{x(t) = e^{At} x(0) + B\int_0^t u(\tau) \cdot e^{A(t-\tau)} d\tau} \quad \text{— (a)}$$

since $\phi(t) = e^{At}$

$$\boxed{x(t) = \phi(t) x(0) + \int_0^t \phi(t-\tau) B U(\tau) d\tau} \quad \text{— (b)}$$

(a)&(b) are the solution of state equations, when initial conditions are known at $\boxed{t=0}$.

$\Rightarrow$ If initial conditions are known at $\boxed{t=t_0}$, then the solution is,

$$\boxed{\begin{aligned} x(t) &= e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} B U(\tau) d\tau. \\ \text{ie, } x(t) &= \phi(t-t_0) \cdot x(t_0) + \int_{t_0}^t \phi(t-\tau) B U(\tau) d\tau \end{aligned}}$$

Equations

$\dot{X} = AX + BU$ — state eqn

$Y = CX + DU$ — o/p eqn

$\phi(t) = e^{At}$ — state transition mx (STM)

$\phi(S) = [SI-A]^{-1}$

$\phi(t) = ILT[SI-A]^{-1}$

$$x(t) = \phi(t) x(0) + \int_0^t B\phi(t-\tau) u(\tau) d\tau.$$

TF (transfer fn.) $= C(SI-A)^{-1}B$

problems to find transition Matrix $\phi(t)$

Qn:- Obtain the state transition matrix $\phi(t)$ of the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$. Also obtain the inverse of the state transition matrix $\phi^{-1}(t)$.

$\Rightarrow$ State Transition matrix $\boxed{\phi(t) = e^{At} = L^{-1}\left[(SI - A)^{-1}\right]}$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad (SI - A) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$(SI - A)^{-1} = \frac{\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}}{S(S+3)+2} = \frac{\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}}{S^2+3S+2} = \frac{\begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix}}{(S+1)(S+2)}$$

$$\therefore (SI - A)^{-1} = \begin{bmatrix} \frac{S+3}{(S+1)(S+2)} & \frac{1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{bmatrix}$$

$L^{-1}[SI - A]^{-1}$:- $\underline{1^{st}\text{ term}}$ $\Rightarrow \frac{S+3}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{(S+2)}$.

$S+3 = A(S+2) + B(S+1)$

if $S = -2$: $1 = B(-2+1) = -B$, $\therefore \underline{B = -1}$ | if $S = -1$: $2 = A(-1+2)$, $\underline{\underline{A = 2}}$

$$\therefore \frac{S+3}{(S+1)(S+2)} = \frac{2}{S+1} - \frac{1}{S+2}$$

$$L^{-1}\left[\frac{S+3}{(S+1)(S+2)}\right] = \underline{\underline{2e^{-t} - e^{-2t}}}$$

$\underline{2^{nd}\text{ term}}$ $\frac{1}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{S+2}$ $\quad A = +1, B = -1$

$$= \frac{1}{S+1} + \frac{-1}{S+2}$$

$$L^{-1}\left[\frac{1}{(S+1)(S+2)}\right] = \underline{\underline{e^{-t} - e^{-2t}}}$$

$\underline{3^{rd}\text{ term}}$ $\frac{-2}{(S+1)(S+2)} = \frac{A}{(S+1)} + \frac{B}{S+2}$ $\Rightarrow -2 = A(S+2) + B(S+1)$

$-2 = B(-2+1)$, $\underline{\underline{B = 2}}$ | $-2 = A(-1+2)$, $\underline{\underline{A = -2}}$

$$L^{-1}\left[\frac{-2}{(S+1)(S+2)}\right] = L^{-1}\left[\frac{-2}{S+1} + \frac{2}{S+2}\right] \Rightarrow \underline{-2e^{-t} + 2e^{-2t}}$$

4th term $\quad \dfrac{s}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$$s = A(s+2) + B(s+1)$$

$s = -2$	$s = -1$
$-2 = B(-2+1)$	$-1 = A(-1+2)$
$\underline{B = 2}$	$-1 = A$
	$A = -1$

$$= \frac{-1}{s+1} + \frac{2}{s+2}$$

$$L^{-1}\left[\frac{s}{(s+1)(s+2)}\right] \Rightarrow -e^{-t} + 2e^{-2t}$$

$$\therefore \ \phi(t) = L^{-1}[sI - A]^{-1} \Rightarrow \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$\phi^{-1}(t) = \phi(-t) \rightarrow$ property.

$$\therefore \ \phi^{-1}(t) = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

Qn:-

Obtain the time response (total response) for the following system (compute state vector $x(t)$ for $t \geq 0$, when the i/p $u(t) = 1$, $t \geq 0$)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

where $u(t)$ is the unit step function. Given $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\boxed{x(t) = \phi(t)\,x(0) + \int_0^t B\,\phi(t-\tau)\,u(\tau)\,d\tau}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cancel{\int_0^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[2e^{-(t-\tau)} - e^{-2(t-\tau)} \quad e^{-(t-\tau)} \right.}$$

$$+ \int_0^t \underset{2\times1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)\,d\tau$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} d\tau \qquad \because u(t) = 1$$

Integration 1st term $\int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$

$$= \int_0^t \left(e^{-t}\cdot e^{\tau} - \left[e^{-2t}\cdot e^{2\tau}\right]\right) d\tau$$

$$= \left[e^{-t}\cdot e^{\tau}\right]_0^t - \left[\left(e^{-2t}\cdot \frac{e^{2\tau}}{2}\right)\right]_0^t$$

$$= e^{0} - e^{-t} - \left(\frac{e^{0}}{2} - \frac{e^{-2t}}{2}\right)$$

$$= 1 - e^{-t} - \frac{1}{2} + \frac{e^{-2t}}{2}$$

$$= .5 - e^{-t} + .5e^{-2t}$$

Integrating 2nd term. $\int_0^t -e^{-(t-\tau)} + 2e^{-2(t-\tau)} d\tau$

$$\int_0^t -(e^{-t}\cdot e^{\tau} + 2e^{-2t}\cdot e^{2\tau}) d\tau$$

$$= -e^{-t}\left[e^{\tau}\right]_0^t + 2e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_0^t$$

$$= -e^{-t}\cdot\left[e^{t} - e^{0}\right] + 2e^{-2t}\left[\frac{e^{2t}}{2} - \frac{e^{0}}{2}\right]$$

$$= -1 + e^{-t} + 1 - \frac{2e^{-2t}}{2}$$

$$= \cancel{5} + e^{-t} - e^{-2t}$$

$$= e^{-t} - e^{-2t}$$

$$x(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} .5 - e^{-t} + .5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$x_1(t) = .5 + e^{-t} - .5e^{-2t}$$

$$x_2(t) = -e^{-t} + e^{-2t}$$

$$x(t) = \begin{bmatrix} .5 + e^{-t} - .5e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}$$

Qn:- Find the time response of the system described by the eqⁿ

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad ; \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} ; \quad u(t) = 1 \text{ for } t$$

$$\boxed{x(t) = \phi(t)\, x(0) + \int_0^t \phi(t-\tau)\, B\, u(\tau)\, d\tau}$$

$$\boxed{\phi(t) = L^{-1}[sI - A]^{-1}} \qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \qquad [sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}}{(s+1)(s+2)} = \begin{bmatrix} 1/s+1 & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$L^{-1}(sI - A)^{-1} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} = \phi(t) \quad \text{— ①}$$

$$\because \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{s+2}$$

$1 = A(s+2) + B(s+1)$

$s = -2, \quad 1 = B(-1) = \underline{B = -1}$

$s = -1, \quad 1 = A \Rightarrow \underline{A = 1}$

$$L^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) \Rightarrow \underline{\underline{e^{-t} - e^{-2t}}}$$

$$x(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{x(0)} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times 1\, d\tau$$

$$= \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ e^{-2(t-\tau)} \end{bmatrix} d\tau$$

Integration term 1

$$\int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau$$

$$= \int_0^t \left(e^{-t} \cdot e^{\tau} - e^{-2t} \cdot e^{2\tau}\right) d\tau$$

$$= e^{-t}\left[e^{\tau}\right]_0^t - e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_0^t$$

$$= e^{-t} \cdot \left[e^{t} - e^{0}\right] - e^{-2t}\left[\frac{e^{2t}}{2} - \frac{e^{0}}{2}\right]$$

$$= e^{0} - e^{-t} - \left[\frac{e^{0}}{2} - \frac{e^{-2t}}{2}\right] = \underline{.5 - e^{-t} + .5e^{-2t}}$$

2nd term

$$\int_0^t e^{-2(t-\tau)} d\tau = \int_0^t e^{-2t} \cdot e^{2\tau} d\tau$$

$$e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_0^t = e^{-2t}\left[\frac{e^{2t}}{2} - \frac{e^{0}}{2}\right]$$

$$= \frac{1}{2} - \frac{e^{-2t}}{2} = .5 - .5e^{-2t}$$

$$x(t) = \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} .5 - e^{-t} + .5e^{-2t} \\ .5 - .5e^{-2t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} .5 - 2e^{-t} + .5e^{-2t} \\ .5 - .5e^{-2t} \end{bmatrix}$$

Controllability & Observability

A system is said to be controllable if any initial state can be transferred to any final state in a finite time interval. Initially $x(t_0)$ and finally $x(t_f)$. $x(t_0)$ can be transfered during the interval $t_f - t_0$.

If a system is said to be observable, if every state can be exactly determined from the measurement of output y over a finite interval time $0 \le t \le t_f$

Concept of controllability and observability were introduced by 'Kalman'. There are 4 possible states for the s/m.

1. Controllable & Observable
2. Controllable & Un Observable
3. Uncontrollable & observable
4. Uncontrollable & Un observable.

KALMANS TEST

An LTI continuous s/m described by the state eq^n $\dot{x} = AX + BU$ and $y = CX$ is completely controllable iff the rank of the controllability matrix is defined as $Q = [B : AB : A^2B : \dots A^{n-1}B]$ is equal to Rank n.

A s/m is observable iff the matrix $Q = [C^T : A^TC^T : (A^T)^2C^T \dots (A^T)^{n-1}C^T]$ is equal to rank n.

Qo:-

① consider the following system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad \& \quad y(t) = [0 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Test for controllability & observability.

(a) controllability :- $Q = [B : AB : A^2B \dots A^{n-1}B]$

Rank $n = 2$ $\therefore Q = [B : AB]$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad AB = \underset{2\times 2}{\begin{bmatrix} -.5 & 0 \\ 0 & -2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$Q = [B : AB]$

$= \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \rightarrow$ Rank = 1 $\therefore$ system is uncontrollable.

(1) Observability

$$Q = \left[C^T : A^T C^T : (A^T)^2 C^T + (A^T)^{n-1} C^T\right]$$

$n = 2 \quad \therefore Q = (C^T : A^T C^T)$

$C = [0\ 1], \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A^T C^T = \begin{bmatrix} -.5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \qquad \text{Rank} = 1$$

$\therefore n = 2$ & Rank $= 1$ $\quad \therefore$ Then the s/m is unobservable.

H.W

(2) Given the s/m $\dot{x}(t) = AX + BU$, $Y = CX$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C[1\ 1]$. Determine the state & o/p controllability

$$AB = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$Q = [B : AB] = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix}.$$

The rank of the controllability matrix is $Q = n = 2$.
Hence the system is completely controllable.

o/p controllability

$$[CB : CAB : CA^2B \dots CA^{n-1}B \quad D] \text{ is of rank } m$$

where m is the no. of outputs.

$$CB = [1\ 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3$$

$$CAB = [1\ 1] \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5$$

$[CB : CAB] = [3 \ -5]$ $\quad$ Rank of $[CB : CAB] = 1$

Here only one o/p.

$\therefore$ system is completely o/p controllable.

Gilbert's Test

Gilbert's method of testing controllability.

case I

When the system matrix has distinct eigen values,

In this case the s/m matrix can be diagonalised and the state model can be converted to canonical model

consider the state Model $\dot{x} = Ax + BU$ & $Y = CX + DU$.

→ using transformation, $X = MZ$ where M is model mx. & z is transformed state variable vector

$$\dot{X} = M\dot{Z}$$

$$M\dot{Z} = AMZ + BU$$

$$\dot{Z} = M^{-1}AMZ + M^{-1}BU$$

$$\boxed{\dot{Z} = AZ + \tilde{B}U} \rightarrow \text{next state eq}^n.$$

where $M^{-1}AM = A$

$M^{-1}B = \tilde{B}$

$$Y = CMZ + DU$$

$$Y = \tilde{C}Z + DU \rightarrow \text{transformed o/p eq}^n$$

where $\tilde{C} = CM$.

In this case the necessary & sufficient condition for completely controllability is that the matrix $\tilde{B}$ must have no rows with all zeros.

case 2 :- When the s/m matrix has repeated eigen values.

In this case the s/m matrix ~~cannot~~ cannot be diagonalise but can be transform to Jordhan canonical model

$$J = M^{-1}AM$$

Gilbert test

$$\dot{z} = M^{-1}AM\, z + M^{-1}BU$$

$$y = CMz$$

Controlable

$\boxed{M^{-1}B \text{ matrix has non zero element.}}$

Observable

$\boxed{CM \text{ Matrix has non zero element.}}$

where $\underline{M \text{ is Model Matrix}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}[u] \quad \& \quad y = [1 \; 0 \; 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

characteristic eq^n $|\lambda I - A| = 0$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \; ; \quad \lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{bmatrix} = \lambda I - A.$$

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} = \lambda \begin{vmatrix} \lambda+3 & 0 \\ -2 & \lambda+3 \end{vmatrix} - 0 + (-1)\begin{vmatrix} 2 & \lambda+3 \\ 0 & -2 \end{vmatrix}$$

$$= \lambda(\lambda^2 + 6\lambda + 9) + 4$$

$$= \lambda^3 + 6\lambda^2 + 9\lambda + 4$$

$$\underline{\underline{\lambda = -4, -1, -1}} \qquad \lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -4$$

$\Rightarrow$ For finding eigen vector

$$|\lambda_1 I - A| = 0$$

for finding eigen vectors $P_1, P_2 \& P_3$ $\Rightarrow$

$$|\lambda I - A| = 0$$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{bmatrix}$$

let $C_{11} = (-1)^{1+1} \begin{vmatrix} \lambda+3 & 0 \\ -2 & \lambda+3 \end{vmatrix} = (-1)^2 (\lambda+3)^2$
$= \underline{\lambda^2 + 6\lambda + 9}$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 0 \\ 0 & \lambda+3 \end{vmatrix} = \underline{-2(\lambda+3)}$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & \lambda+3 \\ 0 & -2 \end{vmatrix} = \underline{-4}$

Eigenvector $P = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$

$$P_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}\Bigg|_{\lambda=\lambda_1=-1} \Rightarrow \begin{bmatrix} \lambda_1^2+6\lambda_1+9 \\ -2(\lambda_1+3) \\ -4 \end{bmatrix}\Bigg|_{\lambda_1=-1} = \begin{bmatrix} 1-6+9 \\ -2\times 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$P_2 |_{\lambda_2=-1}$ $\Rightarrow$ since eigen values are repeated, take differentiate the co-factors,

$$P_2 = \begin{bmatrix} 2\lambda_2+6 \\ -2 \\ 0 \end{bmatrix}\Bigg|_{\lambda_2=-1} = \begin{bmatrix} -2+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

$P_3 |_{\lambda_3=-4}$

$$P_3 = \begin{bmatrix} (-4)^2+6\times-4\times 9 \\ -2(-4+3) \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

Model matrix $M = [P_1 \ P_2 \ P_3] = \begin{bmatrix} 1 & 4 & 1 \\ -1 & -2 & 2 \\ -1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ -1 & 0 & -4 \end{bmatrix}$

$\Rightarrow$ For checking controllability using Gilbert's test
find $M^{-1}B$

$$M^{-1} = \frac{adj(M)}{|M|}$$

$|A| =$

$\cancel{4} + \cancel{4} + 0 - \cancel{4}$
$- 0 - 8$

$$M^{-1} = \frac{\begin{bmatrix} 4 & 8 & 5 \\ -6 & -3 & -3 \\ -1 & -2 & 1 \end{bmatrix}}{-9}$$

$$M^{-1}B = -\frac{1}{9}\begin{bmatrix} 4 & 8 & 5 \\ -6 & -3 & -3 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} 16 \\ -6 \\ -4 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} 8 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8/9 \\ 1/3 \\ -2/9 \end{bmatrix} \neq 0$$ non zero element

$\therefore$ s/m is controllable.

<u>observability checking</u>

$$CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ -1 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Since non zero element

$\therefore$ s/m is observable.

$|A| \rightarrow$ short cut

$$\begin{array}{ccccc} 1 & 2 & 1 & 1 & 2 \\ -1 & -1 & 2 & -1 & -1 \\ -1 & 0 & -4 & -1 & 0 \end{array}$$

$$4 + -4 + 0 - 1 - 0 - 8 = -9$$

$$\boxed{\text{Inverse} = \frac{\text{adj } A}{|A|} = \frac{[\text{Cofactor}]^T}{|A|}}$$

co-factor shortcut

$$\begin{array}{c|cccc} 1 & 2 & 1 & 1 & 2 \\ \hline -1 & -1 & 2 & -1 & -1 \\ -1 & 0 & -4 & -1 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ -1 & -1 & 2 & -1 & -1 \end{array}$$

$$\begin{bmatrix} 4 & -6 & -1 \\ 8 & -3 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

UQ:-

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^3+9s^2+26s+24}$$

test controllability & observability

$$s+2 = \frac{Y(s)}{X_1(s)} \Rightarrow sX_1(s) + 2X_1(s) = Y(s)$$

$\cancel{\dot{x}_1 = 2x}$

$$y(t) = 2x_1 + x_2$$

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^2+9s^2+26s+24}$$

$$s^3X_1(s) + 9s^2X_1(s) + 26sX_1(s) + 24X_1(s) = U(s)$$

$\downarrow \dot{x}_3$ $\quad \downarrow \dot{x}_2 = x_3$ $\quad \downarrow \dot{x}_1 = x_2$ $\quad \downarrow x_1$

$$\dot{x}_3 + 9x_3 + 26x_2 + 24x_1 = u$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

controllability $Q = [B : AB : A^2B]$

rank $n = 3$

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -9 \\ 1 & -9 & 8 \end{bmatrix}$$

Rank of $Q = 3$

∴ The s/m is controllable

observability $Q = [C^T : A^TC^T : (A^T)^2C^T]$

AB

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -9 \end{bmatrix}$$

A^2

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -24 & -26 & -9 \\ 216 & 214 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 8 \end{bmatrix}$$

$Q = [C^T : A^T C^T : (A^T)^2 C^T]$ $\quad C = [2\ 1\ 0]$ $\quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & 9 \end{bmatrix}$

$C^T = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$A^T C^T = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 2 & 0 & -24 \\ 1 & 2 & -26 \\ 0 & 1 & -7 \end{bmatrix}$

$(A^T)^2 = \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix} \begin{bmatrix} 0 & 0 & -24 \\ 1 & 0 & -26 \\ 0 & 1 & -9 \end{bmatrix}$

$= \begin{bmatrix} 0 & -24 & 216 \\ 0 & -26 & 214 \\ 1 & -9 & 55 \end{bmatrix}$

Rank (Q) $\neq 3$

$\therefore$ Rank n $\neq$ Rank Q

$\therefore$ s/m is

~~observable~~

$(A^T)^2 C^T = \begin{bmatrix} 0 & -24 & 216 \\ 0 & -26 & 214 \\ 1 & -9 & 55 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ -26 \\ -7 \end{bmatrix}$

ans:- ~~un~~ observable

Qn:- A system $\frac{Y(s)}{U(s)} = \frac{2}{s^3+6s^2+11s+6}$. Find the state & o/p eqn & test controllability & observability

ans -

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = [1\ 0\ 0]\, x(t)$$

controllability $Q = [b : Ab : A^2 b] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & -12 \\ 2 & -12 & 50 \end{bmatrix} \rightarrow$ s/m controllable

$|Q| \neq 0$ rank = 3.

observability $Q = [C^T : A^T C^T : (A^T)^2 C^T]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Rank = 3

s/m obserable.