control system – Module III

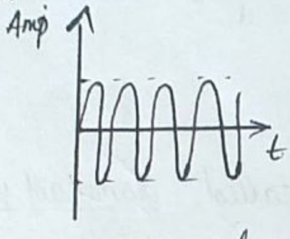
Stability of linear control system: Methods of determining stability, Routh's Hurwitz Criterion*

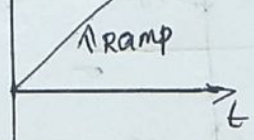
*Root Locus Technique: Introduction, properties & its construction.

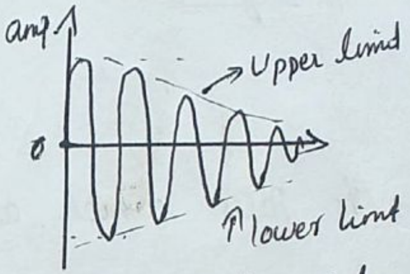
Frequency Domain analysis: Frequency domain specifications, correlation between time & frequency response

STABILITY CRITERIA FOR s/m

A system is stable, if o/p is bounded wrt bounded input.





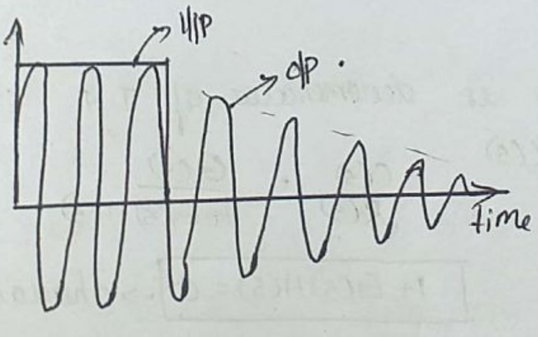


* It has upper & lower limit
s/m is stable

* o/p has no upper limit
s/m is not stable

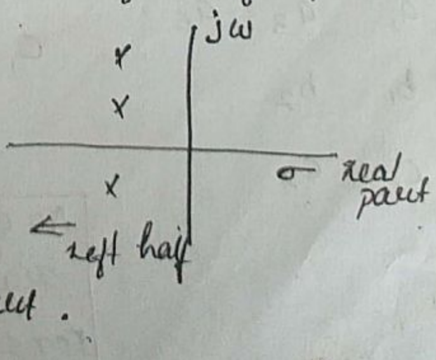
* o/p is bounded
s/m is stable

⇒ A system is asymptotical stable, if in absence of i/p the o/p tends towards zero irrespective of initial condition



STABILITY

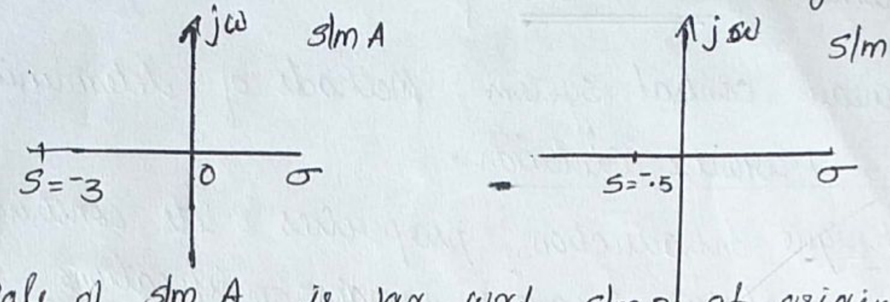
* stability of s/m depends upon poles
* If all the poles are located in left half of s-plane, then the system is stable



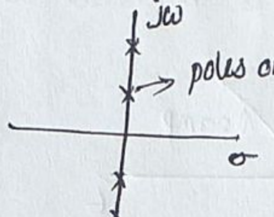
$$T(s) = \frac{N(s)}{D(s)} = \frac{\text{zeros}}{\text{Poles}}$$

* Poles should have (-ve) real part.

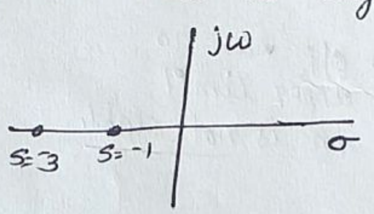
* As poles approaches zero, the stability decreases



* Pole of s/m A is far wrt s/m B at origin. ∴ s/m A is having greater stability.
* When Poles are located on imaginary axis, then the s/m is marginally stable.



* The Poles which are close to origin are called dominant poles



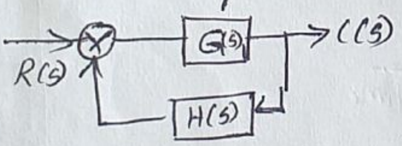
S = -1 is dominant pole.

Techniques used to calculate stability.

1) Routh's Hurwitz criteria 2) Root locus 3) Bode Plot 4) Nyquist Plot

Routh's Hurwitz criteria

1) characteristic equation is denomenator of T.F



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

$\boxed{1+G(S)H(S) = 0}$ → characteristic eq^n.

std. characteristic eq^n

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

s^n	a_0	a_2	a_4
s^{n-1}	a_1	a_3	a_5
s^{n-2}	b_1	b_2	
s^{n-3}	c_1	c_2	
⋮			
s^0			

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1} \qquad c_2 = \frac{a_1 a_5 - a_1 b_3}{b_1}$$

No. of poles on RHS of s plane = no. of sign changes in 1st column of Routh array.
For stable s/m → No sign changes (All +ve / All -ve)

Routh - Hurwitz Criterion Problems.

① Type 1 Investigate the stability of a closed loop s/m whose characteristic equation is given by,

$$s^5+3s^4+7s^3+20s^2+6s+15=0.$$

s^5	1	7	6
s^4	3	20	15
s^3	$\frac{1}{3}$	1	
s^2	11	15	
s	6/11		
s^0	15		

$$b_1 = \frac{3\times 7 - 1\times 20}{3} = \frac{1}{3}$$

$$b_2 = \frac{3\times 6 - 1\times 15}{3} = 1$$

$$c_1 = \frac{1/3\times 20 - 3\times 1}{1/3} = 11$$

$$c_2 = \frac{1/3\times 15 - 3\times 0}{1/3} = 15$$

$$d_1 = \frac{11\times 1 - 1/3\times 15}{11} = 6/11$$

$$e_1 = 15$$

All the elements in column 1 are same sign. Therefore s/m is stable.

⇒ Or No sign changes in the first column of the Routh array. ∴ No roots on RHS. Hence the s/m is stable.

② Investigate the stability of a closed loop s/m whose chara. eqn is given by

$$s^4+2s^3+3s^2+8s+2=0$$

s^4	1	3	2
s^3	2	8	
s^2	-1	2	
s^1	12		
s^0	2		

There are two sign changes (+ to -ve & -ve to +ve) in the first column of the Routh array. ∴ There are two roots with +ve real part (RHS). Hence the s/m in question is unstable.

Qn3 Type2 Investigate the stability of a closed loop s/m whose characteristic equation is given by

$$s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$$

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2			
s^1			
s^0			

$$b_1 = \frac{2\times1 - 2\times1}{1} = 0$$

$$a_2 = \frac{1\times3 - 1\times5}{1} = -2.$$

* When zero occurs at a, substitute $s = 1/p$ in characa. eqn

$$\left(\frac{1}{p}\right)^5 + \left(\frac{1}{p}\right)^4 + 2\times\left(\frac{1}{p}\right)^3 + 2\times\left(\frac{1}{p}\right)^2 + 3\left(\frac{1}{p}\right) + 5 = 0$$

Multiply with highest power ie, p^5

$$1 + p + 2p^2 + 2p^3 + 3p^4 + 5p^5$$

$$\Rightarrow 5p^5 + 3p^4 + 2p^3 + 2p^2 + p + 1 = 0$$

⇒ characteristic equation will become,

$$5p^5 + 3p^4 + 2p^3 + 2p^2 + p + 1 = 0$$

p^5	5	2	1
p^4	3	2	1
p^3	$-4/3$	$-2/3$	
p^2	$1/2$	1	
p^1	2		
p^0	1		

$$b_1 = \frac{3\times2 - 5\times2}{3} = \frac{-4}{3}$$

$$b_2 = \frac{3\times1 - 5\times1}{3} = \frac{-2}{3}$$

$$c_1 = \frac{\frac{-4}{3}\times2 - {}^-2/3\times3}{-4/3} = 1/2$$

$$c_2 = \frac{-4/3\times1 - 3\times0}{-4/3} = 1$$

$$d_1 = \frac{1/2\times -2/3 - {}^-4/3\times1}{1/2} = 2$$

$$e_1 = \frac{2\times1 - 1/2\times0}{2} = 1$$

Here two sign changes in the first column. ie, There are two ~~to~~ roots in Right half of s-plane. Therefore s/m is unstable

Consider the characteristic equation $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$ Using Rouths criterion, investigate the stability of the closed loop s/m

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	0 (8)	0 (24)		
s^2	6	16		
s^1	8/3			
s^0	16			

When any one row with complete zeros. then take the previous row and write it as auxiliary equation. Then take derivative and with that characteristic eqn; continue the process.

Here Auxiliary eqn is

$$A = 2s^4+12s^2+16=0$$

$$\frac{dA}{ds} = 8s^3+24s$$

Since there are no sign changes in the first column, of the entire routh array, the given characteristic eqn has no roots with +ve real parts. Therefore the s/m is marginally stable

ROOT LOCUS

* Purpose

1) To find the closed loop system stability
2) To find the range of K- value for closed loop system stability
3) To find the K marginal and co-marginal value.
4) To find the K values for undamped, Underdamped, critical damped and overdamped systems.
5) To find the relative stability if the root locus branches moving towards left, then the system is more relatively stable. If the root locus branches moving towards right, then the system is less relatively stable.

* Definition

'Root' means roots of characteristic equation. 'Locus' means path. 'Root locus' means closed loop poles path by varying K from 0 to ∞.

* Construction rules of Root Locus.

Rule ① : Symmetrical – The root locus is always symmetric wrt real axis

Rule ② : Total Loci ~ (Root locus Branches)
= max(P, Z)

Rule ③ : Total no. of asymptotes – [Asymptotes are the root locus Branches]

$$N = P - Z$$

Rule ④ : Angle of asymptotes

$$\theta = \frac{(2q+1)180^\circ}{P-Z} \quad \text{where } q = 0, 1, 2 \ldots (\underbrace{P-Z-1}_{N-1})$$

[Asymptotes are symmetrical about real axis. The asymptotes gives direction of zeroes when poles are greater than zero]

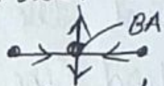
Rule ⑤ : Calculate centroid of asymptotes

Centroid is the intersection point of asymptotes on the real axis

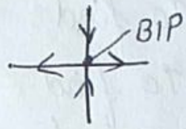
$$\sigma = \frac{\Sigma(\text{real parts of poles}) - \Sigma(\text{real part of zeros})}{P-Z}$$

The centroid may be located any where on the real axis. It may or maynot on root locus branch.

Rule ⑥ : Break away points :- The point at which the root locus branches leave the real axis



Break in Point :- The point at which the root locus branches enter in to the real axis



→ Identify characteristic equation $1+G(s)H(s)=0$

→ Compute k = Polynomial

→ compute $\frac{dk}{ds} = 0$

$s \Rightarrow$ Break away Point

Rule ⑦ : Intersection point to imaginary axis

→ characteristic equation $1+G(s)H(s)=0$

→ Construct Routh array.

→ Find k for marginal stable s/m.

→ Form auxiliary equation (2nd order routh array)

→ Place k in auxiliary equation.

→ S = Intersection to imaginary axis.

Rule ⑧ : Angle of departure & angle of arrival

⇒ Angle of departure is calculated at complex conjugate poles.

⇒ Angle of arrival is calculated at complex conjugate zeroes.

⇒ It gives that with what angle the pole depart or leaves.

$$\phi_d = 180^\circ + \angle GH \Big|_{\text{at a +ve imaginary complex pole}}$$

$$\phi_d = 180^\circ - \phi$$

$$\phi = \Sigma\left[\text{angle controlled by poles}\right] - \Sigma\left[\text{angle controlled by zeroes}\right]$$

$$\phi_A = 180^\circ - \angle GH \Big|_{\text{at a +ve imag. complex zero}}$$

$$\phi_A = 180^\circ + \phi$$

$$\phi = \Sigma(\phi_p) - \Sigma(\phi_z)$$

Sketch the root locus of the system whose open loop transfer function is $G(s) = \dfrac{k}{s(s+1)(s+5)}$.

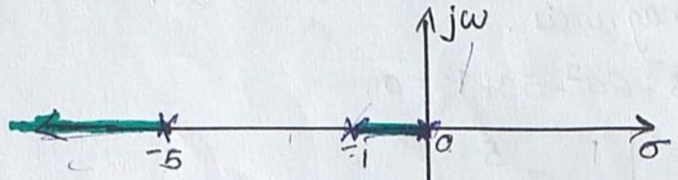
Step 1: Obtain Loci

No. of poles $= 3 = P$ $\quad s=0, s=-1, s=-5$

No. of zeroes $= 0 = Z$

No. of Root Locus Branches $= Max(P, Z) = \underline{\underline{3}}$

~~Step 2~~: Real axis root locus



Step 2: Identify No. of Asymptotes.

$N = P - Z = 3 - 0 = \underline{\underline{3}}$

Step 3: Angle of asymptotes

$\theta = \dfrac{(2q+1)180°}{N}$ $\quad q = 0 \rightarrow N-1 = 0, 1, 2$

$\theta_0 = \dfrac{(2\times 0+1)\times 180°}{3} = 60°$

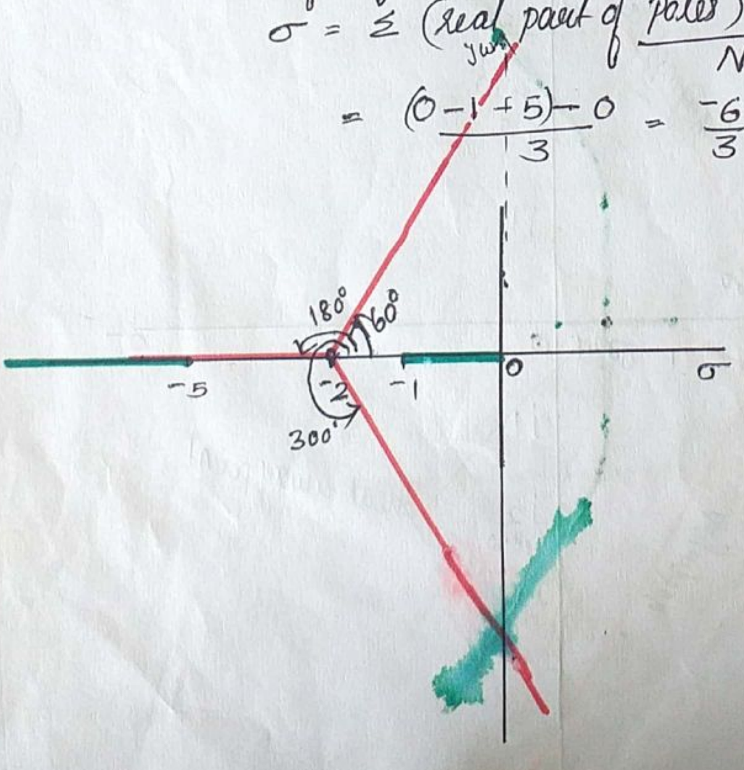
$\theta_1 = \dfrac{(2\times 1+1)180}{3} = 180°$

$\theta_2 = \dfrac{(2\times 2+1)180}{3} = 300°$.

Step 4: Centroid of Asymptotes.

$\sigma = \dfrac{\Sigma \text{(real part of poles)} - \Sigma \text{(real part of zeros)}}{N}$

$= \dfrac{(0-1-5)-0}{3} = \dfrac{-6}{3} = \underline{\underline{-2}}$



Step 5: Angle of Break away points

$\rightarrow$ chara. eq^n - $1+G(s)H(s)=0$ $\quad (\because H(s)=1)$

$$1+\frac{k}{s(s+1)(s+5)} \Rightarrow s(s+1)(s+5)+k=0$$

$$s^3+s^2+5s^2+5s+k=0$$

$$s^3+6s^2+5s+k=0$$

$$\therefore k = -(s^3+6s^2+5s) \;\; \therefore =0$$

$$\frac{dk}{ds}=0 \Rightarrow 3s^2+12s+5=0 \text{ (calc)}$$

$s=-0.47$ (valid Break Point)

$s=-3.57$ (Invalid BP).

Step 6: Intersection with imag. axis.

$\rightarrow$ chara. eq^n $\Rightarrow$ $s^3+6s^2+5s+k=0$.

$\rightarrow$ Routh array

s^3	1	5
s^2	6	k
s^1	$\frac{30-k}{6}$	
s^0	k	

For marginal stable s/m

$$\frac{30-k}{6}=0$$

$k=30$

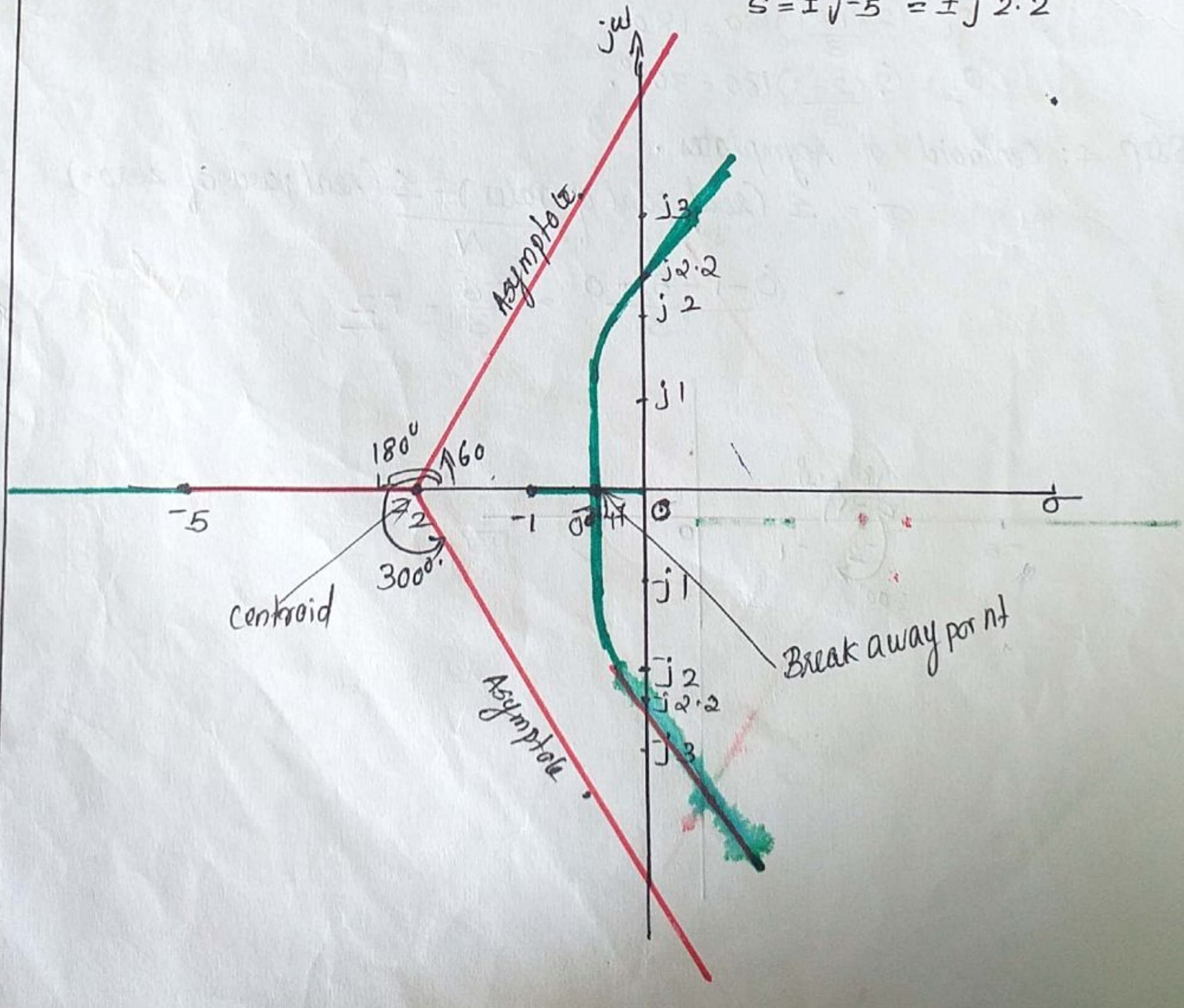
$\rightarrow$ Auxilliary eq^n - 2nd order

$A(s)=0$

$$6s^2+k=0$$

$$6s^2+30=0 \Rightarrow 6s^2=-30$$

$$s=\pm\sqrt{-5}=\pm j\,2.2$$



A unity feedback control system has openloop transfer function

$G(s) = \dfrac{k}{s(s+2)(s+8)}$. sketch the root locus.

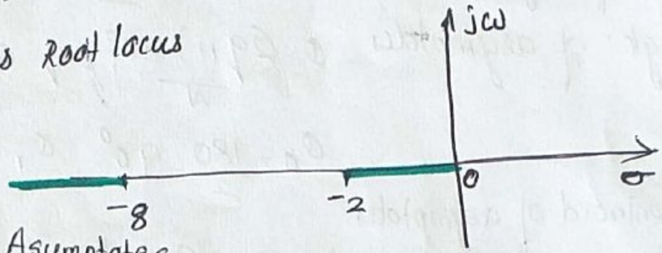
Step 1: Obtain Loci

No. of poles = 3 $\Rightarrow (s=0,\ s=-2,\ s=-8) = P$

No. of zeroes = 0 = Z

No. of Root locus Branches = Max (P, Z) = 3.

Real axis Root locus



Step 2: Identify No. of Asymptotes

$N = P - Z = 3$

Step 3: Angle of asymptotes $\quad \theta = \dfrac{(2q+1)180^\circ}{N} \quad q = 0, 1, 2$

$\theta_0 = 60^\circ$

$\theta_1 = 180^\circ$

$\theta_2 = 300^\circ$.

Step 4: Centroid of asymptotes

$\sigma = \dfrac{\Sigma(\text{Real part of poles}) - \Sigma(\text{Real zeroes})}{N}$

$= \dfrac{(0-2-8)-0}{3} = \dfrac{-10}{3} = -3.33$

Step 5: Break away point:

charac. eq^n $\quad 1 + G(s)H(s) = 0 \qquad 1 + \dfrac{k}{s(s+2)(s+8)}$

$s(s+2)(s+8) + k = 0$

$(s^2+2s)(s+8) + k = 0$

$s^3 + 10s^2 + 16s + k = 0$

$k = -(s^3 + 10s^2 + 16s)$

$\dfrac{dk}{ds} = 0 = 3s^2 + 20s + 16$ (calculator)

$s = -.926 \rightarrow$ valid.

$s = -5.737 \rightarrow$ Invalid

Step 6: Intersection point on Imag. axis

$\rightarrow 1 + G(s)H(s) = 0$.

$s^3 + 10s^2 + 16s + K = 0$.

$\Rightarrow$ Routh array

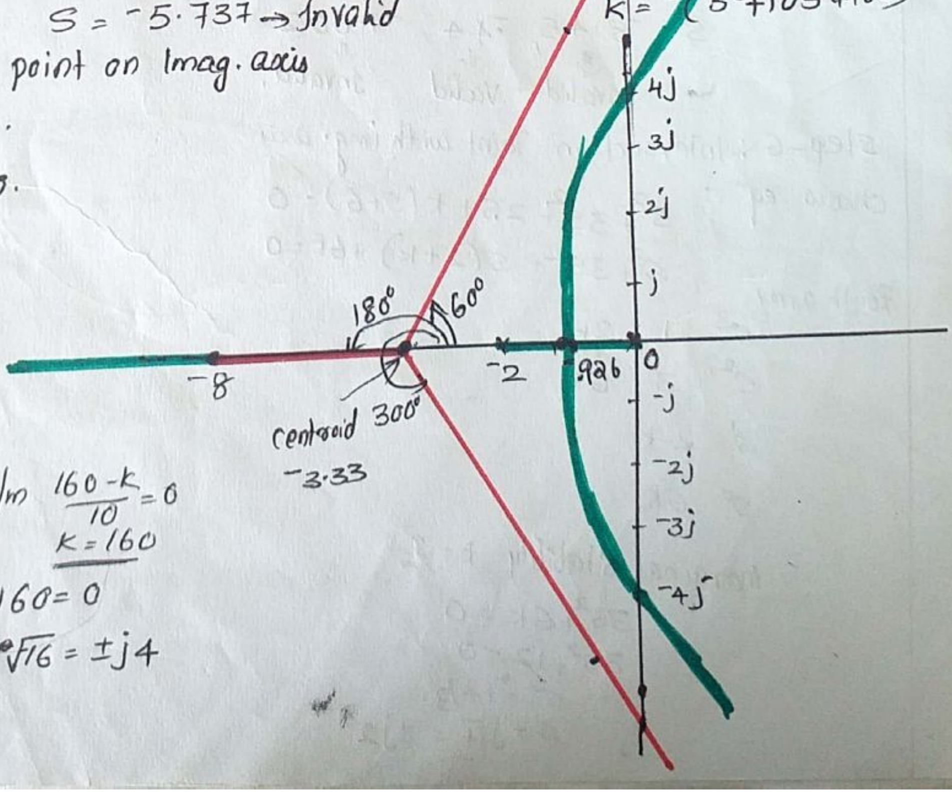
s^3	1	16
s^2	10	K
s^1	$\dfrac{160-K}{10}$	
s^0	K	

marginally stable s/m $\quad \dfrac{160-K}{10} = 0$

$K = 160$

Auxiliary eq^n $\quad 10s^2 + 160 = 0$

$s = \sqrt{-16} = \pm j4$



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3) $G(s) = \frac{k(s+6)}{s(s+1)(s+2)}$

step 1 : obtain Loci

No. of poles $P = 3$, $s = 0, s = -1, s = -2$

No. of zeroes $Z = 1$, $s = -6$.

No. of Root locus Branches $\max(P, Z) = \underline{3}$

step 2 : No. of asymptotes $N = P - Z = 3 - 1 = \underline{2}$

step 3 : Angle of asymptotes $\theta = \frac{(2q+1)180^\circ}{N}$ $q = 0, 1$

$\theta_0 = \frac{180}{2} = 90^\circ$, $\theta_1 = 270^\circ$

step 4 : Centroid of asymptotes

$\sigma = \frac{\Sigma(\text{Real } P) - \Sigma(\text{real } z)}{N} = \frac{(0-1-2)-(-6)}{2} = \frac{3}{2} = \underline{\underline{1.5}}$

step 5 : Break away point

$1 + G(s)H(s) = 0$

$1 + \frac{k(s+6)}{s(s+1)(s+2)} = 0$

$s(s+1)(s+2) + k(s+6) = 0$

$k(s+6) = -(s^3 + 3s^2 + 2s)$

$k = \frac{-(s^3 + 3s^2 + 2s)}{s+6}$

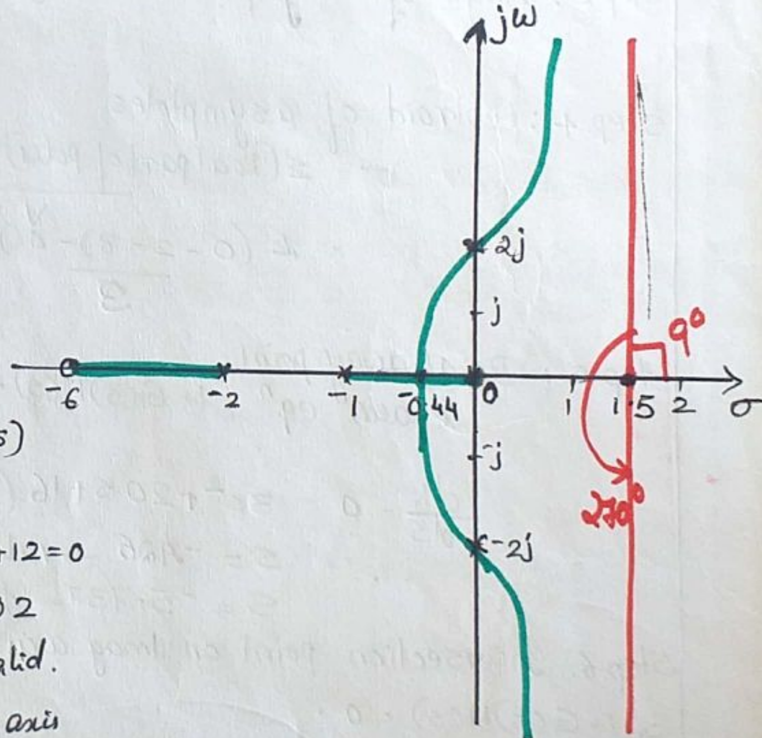
$\frac{dk}{ds} = 0$

$\frac{(s+6)(3s^2 + 6s + 2) - (s^3 + 3s^2 + 2s)}{(s+6)^2}$

$\frac{dk}{ds} = 0 \Rightarrow 2s^3 + 21s^2 + 36s + 12 = 0$

$s = -8.45, -.44, -1.602$

invalid valid invalid.



step-6 : Intersection Point with img. axis

chara. eqⁿ. $s^3 + 3s^2 + 2s + k(s+6) = 0$

$s^3 + 3s^2 + s(2+k) + 6k = 0$

Routh array

s^3	1	$2+k$
s^2	3	$6k$
s^1	$2-k$	
s^0	$6k$	

marginal stability $k = \underline{2}$

$3s^2 + 6k = 0$

$3s^2 + 12 = 0$

$s^2 = -12/3$

$s = j\sqrt{4} = \pm j2$

$$G(s) = \frac{K}{s(s^2+4s+13)}$$

Step-1 No. of poles $= P = 3$ $s=0$, $s=-2+3i$, $s=-2-3i$

No. of zeros $= 0 = Z$

No. of Root locus $(\max(P, Z)) = 3$.

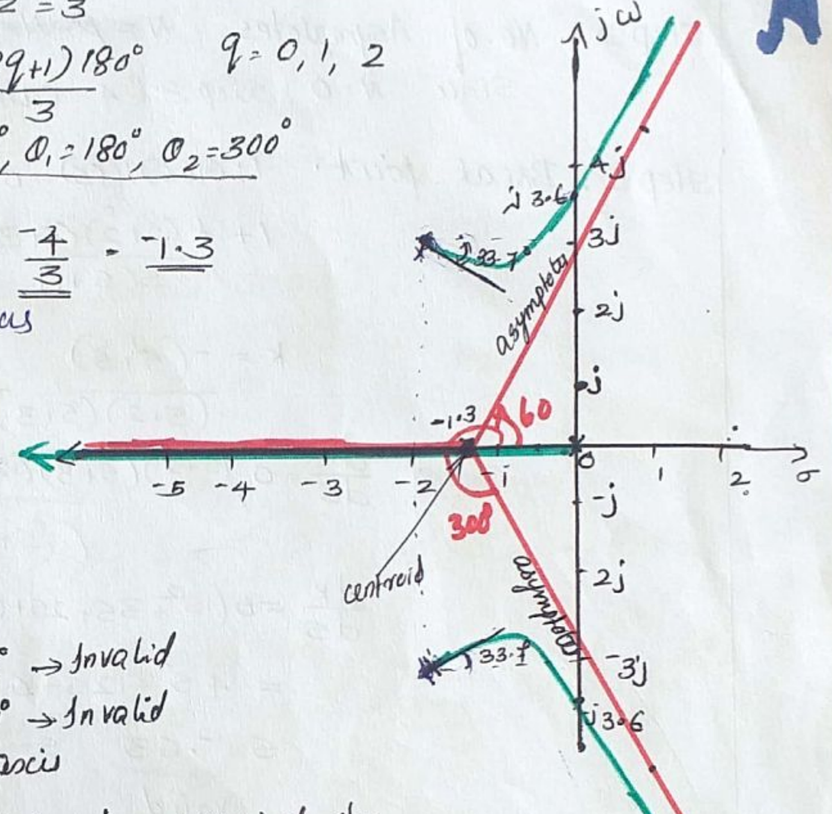
Step 2 No. of asymptotes $N = P - Z = 3$

Step 3 angle of asymptotes. $\theta = \frac{(2q+1)180°}{3}$ $q = 0, 1, 2$

$\theta_0 = 60°$, $\theta_1 = 180°$, $\theta_2 = 300°$

Step 4: Centroid of asymptote

$$\sigma = \frac{(0 + -2 + -2) - 0}{3} = \frac{-4}{3} = -1.3$$



Step 5: Breakaway points of Root locus

$\Rightarrow 1 + G(s)H(s) = 0$

$$1 + \frac{K}{s(s^2+4s+13)} = 0$$

$s(s^2+4s+13) + K = 0$

$K = -(s^3+4s^2+13s)$

$\Rightarrow \frac{dK}{ds} = 0 \Rightarrow 3s^2+8s+13 = 0$

$s = -1.33 + 1.59i \rightarrow$ Invalid

$s = -1.33 - 1.59i \rightarrow$ Invalid

Step 6: Intersection on imaginary axis

$\Rightarrow$ chara. eqn $s^3+4s^2+13s+K = 0$

$\Rightarrow$ Routh array

s^3	1	13
s^2	4	K
s^1	$\frac{52-K}{4}$	
s^0	K	

$\Rightarrow$ marginal s/m,

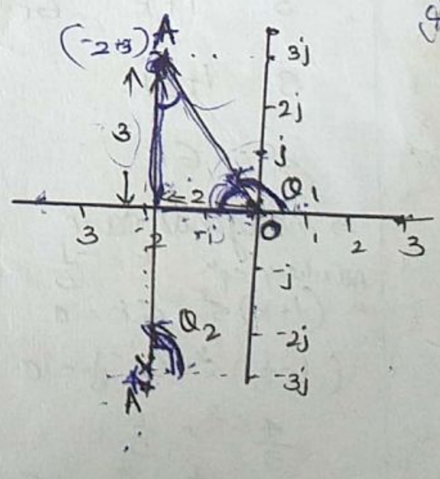
$\frac{52-K}{4} = 0$ $K = 52$

$\Rightarrow$ Auxiliary $4s^2 + K = 0 \Rightarrow 4s^2+52 = 0$

$s = \sqrt{\frac{-52}{4}} = \pm j3.6$

Step 7: Angle of departure & arrival

(complex poles) (complex zeros)



$$G(s) = \frac{K}{s(s^2+4s+13)} = \frac{K}{s(s-(-2+3j))(s-(-2-3j))}$$

$\boxed{\theta_d = 180° - \phi}$

$\phi = \Sigma(\text{angle contributed by poles}) - \Sigma(\text{angle contributed by zeros})$

$\phi = \theta_1 + \theta_2$, $\theta_1 = \cancel{180}(180 - \tan^{-1} 3/2)$

$= 123.7°$

$\theta_2 = 90°$

$\theta_d = 180° - (123.7 + 90) = -33.7°$

[angle of departure at A^* is -ve of angle of departure at complex pole A) $= 33.7°$

$\phi =$

Qn.5. $G(s) = \frac{k(s+2)(s+3)}{s(s+1)}$

Step-1: No. of poles $P = 2$, $s = 0, s = -1$

No. of zeroes $z = 2$ $s = -2, s = -3$.

No. of Root locus Branches $Max(P, z) = \underline{2}$

Step 2: No. of Asymptotes $N = \cancel{P-N} = P - Z = 0$.

Since $N = 0$, Step 3 & 4 can be avoided.

Step 5: Break point. $1 + G(s)H(s) = 0$

$1 + \frac{k(s+2)(s+3)}{s(s+1)} = 0 \Rightarrow s(s+1) + k(s+2)(s+3) = 0$

$\Rightarrow s^2 + s + k(s+2)(s+3)$

$k = \frac{-(s^2+s)}{(s+2)(s+3)}$

$\frac{dk}{ds} = 0, \frac{(s+2)(s+3)(2s+1) - (s^2+s)(2s+5)}{((s+2)(s+3))^2} = 0$

$\frac{dk}{ds} = 0\ (s^2 + 3s + 2s + 6)(2s+1) - (2s^3 + 5s^2 + 2s^2 + 5s) = 0$

$= 4s^2 + 12s + 6 = 0$

$s = -.63$, $s = -2.36$

$\Downarrow$ valid $\quad \Downarrow$ valid

Step 6: Intersection with imag. axis

$\rightarrow$ chara. eq^n $s^2 + s + k(s+2)(s+3) = 0$

$s^2(1+k) + s(1+5k) + 6k = 0$

$\rightarrow$ Routh array

s^2	$1+k$	$6k$
s^1	$1+5k$	
s^0	$6k$.	

$\Rightarrow$ Marginal case

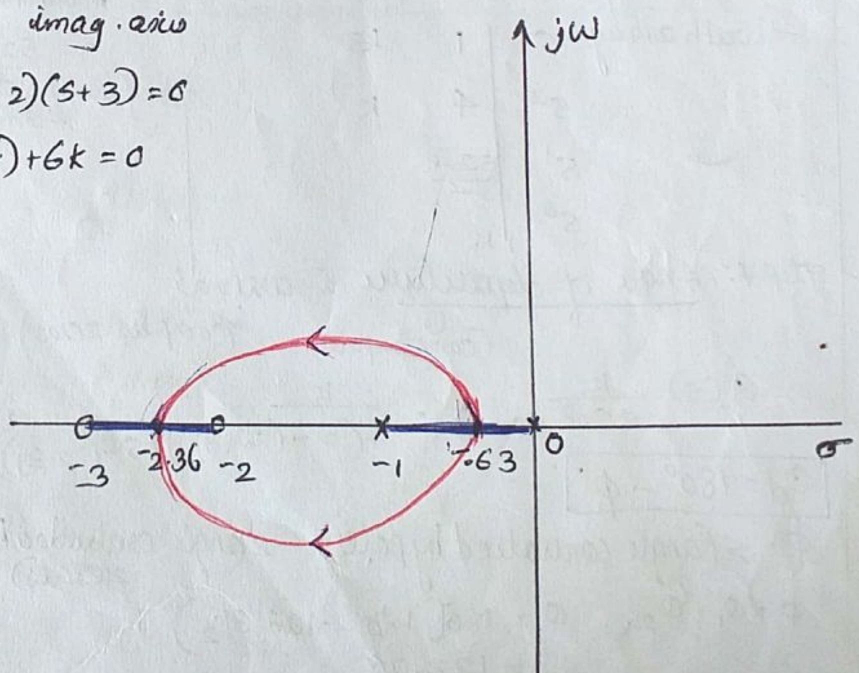
Auxiliary eq^n. $k = \frac{-1}{5}$

$= (1+k)s^2 + 6k = 0$

$(1 - \frac{1}{5})s^2 + 6 \times \frac{-1}{5} = 0$

$\frac{4}{5}s^2 = \frac{6}{5}$

$s^2 = \frac{6}{4} \Rightarrow s = \pm 1.22 \Rightarrow$ Invalid



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Qn: 6 Sketch Root locus for the unity feedback s/m whose open loop transfer function is

$$G(s)H(s) = \frac{k}{s(s+4)(s^2+4s+20)}.$$

step 1: No. of poles $P = 4$, $s = 0$
$s = -4$
$s = -2+4i$
$s = -2-4i$

No. of zeroes $= 0$, ie $z = 0$.

No. of root locus Branches $\max(P, z) = \underline{\underline{4}}$.

Step 2: No. of asymptotes $N = P - Z = \underline{\underline{4}}$

step 3: Angle of asymptote $\theta = \frac{(2q+1)180^\circ}{N}$ $q = 0, 1, 2, 3.$

$\therefore$ $\theta_0 = 45^\circ, \theta_1 = 135^\circ, \theta_2 = 225^\circ, \theta_3 = 315^\circ$

step 4: centroid of asymptote

$$\sigma = \frac{\Sigma(\text{Real } P) - \Sigma(\text{real zeroes})}{N} = \frac{(0 - 4 - 2 - 2) - 0}{4} = \underline{-2}$$

step 5: Break away point.

$1 + G(s)H(s) = 0.$

$$1 + \frac{k}{s(s+4)(s^2+4s+20)}$$

$s(s+4)(s^2+4s+20) + k = 0$

$(s^2+4s)(s^2+4s+20) + k = 0.$

$s^4 + 4s^3 + 20s^2 + 4s^3 + 16s^2 + 80s + k = 0$

$s^4 + 8s^3 + 36s^2 + 80s + k = 0$

$k = -(s^4 + 8s^3 + 36s^2 + 80s)$

$\frac{dk}{ds} = 0$ $4s^3 + 24s^2 + 72s + 80$

$s = -2 \rightarrow$ valid

$s = -2 + 2.4i$

$s = -2 - 2.4i$

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Step 6

Intersection point with imag. axis

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0.$$

s^4	1	36	K
s^3	8	80	
s^2	26	K	
s^1	$\frac{2080-8K}{26}$	0	
s^0	K.		

$$\frac{2080-8K}{26} = 0$$

$$8K = 2080$$

$$\underline{K = 260}$$

Auxillary eq^n

$$26s^2 + K = 0$$

$$26s^2 + 260 = 0$$

$$s^2 = \frac{-260}{26} = -10$$

$$s = \underline{\pm j3.16}.$$

Step 7 Angle of departure.

$$\boxed{\phi_d = 180^\circ - \phi}$$

$\phi = \Sigma$(angle controlled by poles) $- \Sigma$(angle controlled by zeroes).

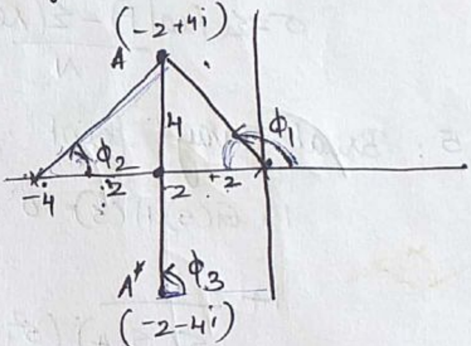
At point A

$$\phi = \phi_1 + \phi_2 + \phi_3$$

$$\phi_1 = \left(180^\circ - \tan^{-1}\frac{4}{2}\right) = 116.6^\circ$$

$$\phi_2 = \tan^{-1} 4/2 = \underline{63.4^\circ}$$

$$\phi_3 = 90^\circ.$$



$$\phi = 116.6 + 63.4 + 90 = 270$$

$\phi_d = 180 - 270^\circ = \underline{-90^\circ}$ | At point A* $\underline{\phi_d = 90^\circ}$