# Numerical series

*(three weeks)*

*(from Monday, 25 September 2017 to Friday, 13 October 2017)*

## Exercise 1

We consider the series $\sum \frac{1}{n}$ and we denote by $(S_n)_{n\in\mathbb{N}^*}$ the sequence $\left(\sum_{k=1}^{n} \frac{1}{k}\right)$.

1. Show that, for all $n \in \mathbb{N}^*$, $S_{2n} - S_n \geqslant \frac{1}{2}$.
2. Deduce that the series $\sum \frac{1}{n}$ is divergent.

## Exercise 2

Let $(u_n)$ be a real, positive and decreasing sequence.

We define $(v_n) = (2^n u_{2^n})$, $(S_n) = \left(\sum_{k=0}^{n} u_k\right)$ and $(T_n) = \left(\sum_{k=0}^{n} v_k\right)$.

1. Show that, for all $k \in \mathbb{N}$,
$$\frac{1}{2} v_{k+1} \leqslant S_{2^{k+1}} - S_{2^k} \leqslant 2^k u_{2^k+1}$$
2. Deduce that
$$\frac{1}{2}(T_{n+1} - v_0) \leqslant S_{2^{n+1}} - S_1 \leqslant T_n$$
3. Deduce that $\sum u_n$ and $\sum v_n$ have the same nature.
4. Let $\alpha \in \mathbb{R}$.
Using the previous question, retrieve the general rule about Riemann series $\sum \frac{1}{n^\alpha}$.

## Exercise 3

Study the nature of the series with the general term $(u_n)$ in the following cases :

1. $u_n = \ln\left(\frac{n^2+2n+1}{n^2+2n}\right)$
2. $u_n = \left(\ln(n)\right)^{-\sqrt{n}}$
3. $u_n = e - \left(1 + \frac{1}{n}\right)^n$
4. $u_n = \sqrt{n^3+n+1} - \sqrt{n^3+n-1}$
5. $u_n = \frac{2 \times 4 \times \cdots \times 2n}{(n!)^2}$

6. $u_n = \dfrac{(n!)^\alpha}{n^n}$ where $\alpha \in \mathbb{R}$

7. $u_n = \left(\dfrac{n}{n+a}\right)^{n^2}$ where $a \in \mathbb{R}$

8. $u_n = \dfrac{n^2}{2^{n^2}}$

9. $u_n = \dfrac{(n!)^2}{(2n)!}a^n$ where $a \in \mathbb{R}_+^*$

10. $u_n = \dfrac{n^{\ln(n)}}{(\ln(n))^n}$

## Exercise 4

Let us consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined for every $n \in \mathbb{N}^*$ by

$$u_n = \ln((n-1)!) - \left(n - \frac{1}{2}\right)\ln(n) + n$$

1. Prove that

$$u_{n+1} - u_n = 1 - \left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right)$$

2. Prove that

$$u_{n+1} - u_n \underset{+\infty}{\sim} -\frac{1}{12n^2}$$

3. Deduce that $(u_n)$ is convergent. We denote by $l$ its limit.

4. Show that

$$e^{u_n} = \frac{n!e^n}{n^n\sqrt{n}}$$

then deduce the following equivalent :

$$n! \underset{+\infty}{\sim} e^l n^n e^{-n}\sqrt{n}$$

## Exercise 5

Let $a \in \mathbb{R}_+^*$ and $\sum u_n$ where $u_n = \ln\left(1 + \dfrac{(-1)^n}{n^a}\right)$

1. Discuss the nature of the series $\sum \dfrac{(-1)^n}{n^a}$ depending on the value of $a$.

2. We know that $u_n \underset{+\infty}{\sim} \dfrac{(-1)^n}{n^a}$. Can we then conclude that the series $\sum u_n$ and $\sum \dfrac{(-1)^n}{n^a}$ have the same nature? Justify your answer.

3. Find $k \in \mathbb{R}$ such that $u_n = \dfrac{(-1)^n}{n^a} + \dfrac{k}{n^{2a}} + o\left(\dfrac{1}{n^{2a}}\right)$.

4. Deduce the nature of $\sum u_n$ depending on the value of $a$.

# Exercise 6

1. Let $N \in \mathbb{N}$, and let $(u_n)$ and $(v_n)$ be two strictly positive sequences such that, for all $n \geqslant N$,

$$\frac{u_{n+1}}{u_n} \leqslant \frac{v_{n+1}}{v_n}$$

Prove that $\sum v_n$ convergent $\Longrightarrow \sum u_n$ convergent.

2. Let $(u_n)$ be a strictly positive sequence such that $\dfrac{u_{n+1}}{u_n} = 1 - \dfrac{\alpha}{n} + o\left(\dfrac{1}{n}\right)$ where $\alpha \in \mathbb{R}$.

   a. Let $(v_n) = \left(\dfrac{1}{n^\beta}\right)$ where $\beta \in \mathbb{R}$.

   Show that $\dfrac{v_{n+1}}{v_n} = 1 - \dfrac{\beta}{n} + o\left(\dfrac{1}{n}\right)$.

   b. We suppose that $\alpha > 1$. Prove that $\sum u_n$ is convergent.

   N.B. : we may consider $\beta \in \mathbb{R}$ such $1 < \beta < \alpha$ and use the sequence $(v_n)$ defined in the previous question.

   c. We suppose now that $\alpha < 1$. Prove that $\sum u_n$ is divergent.

   N.B. : we may consider $\beta \in \mathbb{R}$ such that $\alpha < \beta < 1$ and use the sequence $(v_n)$ defined in the question a.

3. Study the nature of $\sum u_n$ where $u_n = \dfrac{2 \times 4 \times \cdots \times 2n}{3 \times 5 \times \cdots \times (2n+1)}$.

4. Discuss, depending on the value of $a \in \mathbb{R}$, the nature of $\sum u_n$ where $u_n = \dfrac{n \times n!}{(a+1) \times \cdots \times (a+n)}$.

# Exercise 7

The purpose of this exercice is to determine the nature of the series with the general term :

$$u_n = (-1)^n n^\alpha \left(\ln\left(\frac{n+1}{n-1}\right)\right)^\beta$$

where $(\alpha, \beta) \in \mathbb{R}^2$ and $n \in \mathbb{N} \setminus \{0, 1\}$.

1. Show that

$$\ln\left(\frac{n+1}{n-1}\right) = \frac{2}{n}\left(1 + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$$

2. Deduce that

$$u_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}}\left(1 + \frac{\beta}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$$

3. Show that in case $\beta \leqslant \alpha$, then the series $\sum u_n$ diverges.

4. We focus now on the case $\beta > \alpha$.

a. Check that

$$u_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}} + v_n \quad \text{with} \quad v_n = (-1)^n \frac{\beta 2^\beta}{3n^{2+\beta-\alpha}} + o\left(\frac{1}{n^{2+\beta-\alpha}}\right).$$

b. Prove that the series $\sum v_n$ converges absolutely.

c. Show that the series of general term $w_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}}$ converges.

d. Deduce that $\sum u_n$ converges.

## Exercise 8

Let $(\alpha, \beta) \in \mathbb{R}^2$. We consider the series $\sum u_n$ where $u_n = \dfrac{\ln(1+n^\alpha)}{n^\beta}$.

1. Show that the series $\sum \dfrac{1}{n^\alpha (\ln(n))^\beta}$ converges iff $((\alpha > 1) \text{ or } (\alpha = 1 \text{ and } \beta > 1))$.

   N.B. : we will separate the cases $\alpha < 0$ and $\alpha \geqslant 0$. For the later, we will use the results of the exercise 2.

2. Assume that $\alpha < 0$. Find an equivalent of $\ln(1+n^\alpha)$ near $+\infty$. Deduce an equivalent of $u_n$ near $+\infty$. Conclude about the nature of $\sum u_n$ in this case.

3. Assume that $\alpha > 0$. Show that $\ln(1+n^\alpha) \underset{+\infty}{\sim} \alpha \ln(n)$. Deduce an equivalent of $u_n$ near $+\infty$. Conclude about the nature of $\sum u_n$ in this case.

4. Assume that $\alpha = 0$. Find an equivalent of $u_n$ near $+\infty$. Conclude about the nature of $\sum u_n$ in this case.

5. Conclude about the nature of $\sum u_n$ depending $\alpha$ and $\beta$.

## Exercise 9

In this exercise, we propose to compare d'Alembert rule with Cauchy rule.

1. Show Cesàro theorem : let $(u_n)$ be a sequence which converges to $\ell \in \mathbb{R}$. Then

$$\frac{1}{n} \sum_{k=1}^{n} u_k \xrightarrow[n \to +\infty]{} \ell$$

2. Deduce that if $u_{n+1} - u_n \xrightarrow[n \to +\infty]{} \ell \in \mathbb{R}$ then $\dfrac{u_n}{n} \xrightarrow[n \to +\infty]{} \ell$.

3. Deduce (for a strictly positive sequence $(u_n)$) that

$$\frac{u_{n+1}}{u_n} \xrightarrow[n \to +\infty]{} \ell \in \mathbb{R}_+^* \implies \sqrt[n]{u_n} \xrightarrow[n \to +\infty]{} \ell$$

4. What do you conclude about d'Alembert and Cauchy rules ?

5. Let $(a, b) \in (\mathbb{R}_+^*)^2$ with $a \neq b$ and $(u_n)$ defined by $\begin{cases} u_{2p} = a^p b^p \\ u_{2p+1} = a^{p+1} b^p \end{cases}$

   Compare for this sequence d'Alembert rule with Cauchy rule.

# Exercise 10

In this exercise, we propose to prove Abel's rule.

Let $(u_n)$ and $(v_n)$ be two sequences such that

- $(u_n)$ is decreasing and converges to 0.
- The sequence $(V_n) = \left(\sum_{k=0}^{n} v_k\right)$ is bounded.

1. Show that $(u_n)$ converges iff $\sum(u_n - u_{n+1})$ converges.
2. Show that for all $n \in \mathbb{N}$,
$$\sum_{k=0}^{n} u_k v_k = \left(\sum_{k=0}^{n}(u_k - u_{k+1})V_k\right) + u_{n+1}V_n$$
3. Deduce that $\sum u_n v_n$ converges.
4. Let $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. Determine the nature of $\sum \frac{\cos(n\theta)}{n}$.