Integrals

Correction

1 Integration by parts

— $\boxed{\textbf{Find a primitive of } x \longmapsto x^2\mathbf{e}^{2x} \textbf{ using integrations by parts}}$

This function is continuous over $\mathbb{R}$; let us calculate a primitive formally, that is to say

$$x \longmapsto \int^x t^2\mathrm{e}^{2t}\mathrm{d}t.$$

The lower boundary of the integral would just change the integration constant by fixing a point where the chosen primitive would reach 0. Thus it is of no use for this problem.

$$\begin{aligned}
\int^x t^2\mathrm{e}^{2t}\mathrm{d}t &= \left[t^2\frac{1}{2}\mathrm{e}^{2t}\right]^x - \int^x 2t\frac{1}{2}\mathrm{e}^{2t}\mathrm{d}t \\
&= \left[t^2\frac{1}{2}\mathrm{e}^{2t}\right]^x - \left[t\frac{1}{2}\mathrm{e}^{2t}\right]^x + \int^x \frac{1}{2}\mathrm{e}^{2t}\mathrm{d}t \\
&= \left[t^2\frac{1}{2}\mathrm{e}^{2t}\right]^x - \left[t\frac{1}{2}\mathrm{e}^{2t}\right]^x + \left[\frac{1}{4}\mathrm{e}^{2t}\right]^x \\
&= \frac{1}{4}\left[(2t^2 - 2t + 1)\mathrm{e}^{2t}\right]^x
\end{aligned}$$

Thus a primitive of the function $x \longmapsto x^2\mathrm{e}^{2x}$ is the function $x \longmapsto \dfrac{1}{4}(2x^2 - 2x + 1)\mathrm{e}^{2x}$.

— $\boxed{\textbf{Determine } \int_0^x \cos(t)\mathbf{e}^t\mathbf{d}t}$

$$\begin{aligned}
\int_0^x \cos(t)\mathrm{e}^t\mathrm{d}t &= \left[\cos(t)\mathrm{e}^t\right]_0^x + \int_0^x \sin(t)\mathrm{e}^t\mathrm{d}t \\
&= \left[\cos(t)\mathrm{e}^t\right]_0^x + \left[\sin(t)\mathrm{e}^t\right]_0^x - \int_0^x \cos(t)\mathrm{e}^t\mathrm{d}t
\end{aligned}$$

Hence :

$$2\int_0^x \cos(t)\mathrm{e}^t\mathrm{d}t = \left[\cos(t)\mathrm{e}^t\right]_0^x + \left[\sin(t)\mathrm{e}^t\right]_0^x$$

Thus :

$$\int_0^x \cos(t)\mathrm{e}^t\mathrm{d}t = \frac{1}{2}\Big[\big(\cos(t) + \sin(t)\big)\mathrm{e}^t\Big]_0^x = \frac{\mathrm{e}^x\big(\cos(x) + \sin(x)\big) - 1}{2}.$$

2 Substitution

— $\boxed{\displaystyle\int_1^3 \frac{\mathrm{d}t}{\sqrt{t}(1+t)} \textbf{ by using the substitution } u=\sqrt{t}}$

When setting $u=\sqrt{t}$ we get $\mathrm{d}u=\dfrac{\mathrm{d}t}{2\sqrt{t}}$ (or $\mathrm{d}t=2u\mathrm{d}u$).
After changing the boundaries $(1 \longrightarrow 1, 3 \longrightarrow \sqrt{3})$:

$$\int_1^3 \frac{\mathrm{d}t}{\sqrt{t}(1+t)} = \int_1^{\sqrt{3}} \frac{2\mathrm{d}u}{1+u^2} = 2\Big[\mathrm{Arctan}(u)\Big]_1^{\sqrt{3}} = 2\big(\mathrm{Arctan}(\sqrt{3})-\mathrm{Arctan}(1)\big) = 2\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{6}.$$

— $\boxed{\displaystyle\int_2^4 \frac{\ln\big(\ln(t)\big)}{t\ln(t)}\mathrm{d}t \textbf{ by using the substitution } u=\ln(t)}$

When setting $u=\ln t$ we get $\mathrm{d}u=\dfrac{\mathrm{d}t}{t}$ (or $\mathrm{d}t=\mathrm{e}^u\mathrm{d}u$).
After changing the boundaries $\big(2 \longrightarrow \ln(2), 4 \longrightarrow \ln(4)=2\ln(2)\big)$:

$$\int_2^4 \frac{\ln\big(\ln(t)\big)}{t\ln(t)}\mathrm{d}t = \int_{\ln(2)}^{2\ln(2)} \frac{\ln(u)}{u}\mathrm{d}u = \left[\frac{1}{2}\ln^2(u)\right]_{\ln(2)}^{2\ln(2)}$$

Note that, if one does not recognize a known form $\displaystyle\int f'f$ in the integral $\displaystyle\int_{\ln(2)}^{2\ln(2)} \frac{\ln(u)}{u}\mathrm{d}u$, it is possible to set again the same substitution $v=\ln(u)$ and proceed.
By using the relations on the ln function and a remarkable identity, we can even further simplify this expression :

$$\begin{aligned}
\left[\frac{1}{2}\ln^2(u)\right]_{\ln(2)}^{2\ln(2)} &= \frac{1}{2}\Big(\ln^2\big(2\ln(2)\big)-\ln^2\big(\ln(2)\big)\Big) \\
&= \frac{1}{2}\Big(\ln(2)+\ln\big(\ln(2)\big)+\ln\big(\ln(2)\big)\Big)\Big(\ln(2)+\ln\big(\ln(2)\big)-\ln\big(\ln(2)\big)\Big) \\
&= \frac{1}{2}\ln(2)\Big(\ln(2)+2\ln\big(\ln(2)\big)\Big)
\end{aligned}$$

3 Substitution + integration by parts

— $\boxed{\displaystyle\int_0^{\frac{\pi}{2}} \sin^3(t)\mathrm{e}^{\cos(t)}\mathrm{d}t \textbf{ by using the substitution } u=\cos(t)}$

When setting $u=\cos t$ we get $\mathrm{d}u=-\sin(t)\mathrm{d}t$.
We have to change the boundaries : $0 \longrightarrow \cos(0)=1, \frac{\pi}{2} \longrightarrow \cos\left(\frac{\pi}{2}\right)=0$.
Moreover, we need to transform the expression in order to be able to do the substitution ; then, we rewrite $\sin^3(t)$ as $\sin(t)\big(1-\cos^2(t)\big)$ using the known $\cos^2+\sin^2=1$ formula.

Thus :

$$\int_0^{\frac{\pi}{2}} \sin^3(t)\mathrm{e}^{\cos(t)}\mathrm{d}t = \int_0^{\frac{\pi}{2}} \left(1-\cos^2(t)\right)\mathrm{e}^{\cos(t)}\sin(t)\mathrm{d}t = -\int_1^0 (1-u^2)\mathrm{e}^u\mathrm{d}u = \int_0^1 (1-u^2)\mathrm{e}^u\mathrm{d}u$$

We now have to compute the last integral : as the function to be integrated is over the form of a product polynomial $\times$ exponential, it can be calculated by successive integrations by parts.

$$\begin{aligned}
\int_0^1 (1-u^2)\mathrm{e}^u\mathrm{d}u &= \Big[(1-u^2)\mathrm{e}^u\Big]_0^1 + \int_0^1 2u\mathrm{e}^u\mathrm{d}u \\
&= \Big[(1-u^2)\mathrm{e}^u\Big]_0^1 + \Big[2u\mathrm{e}^u\Big]_0^1 - \int_0^1 2\mathrm{e}^u\mathrm{d}u \\
&= \Big[(1-u^2)\mathrm{e}^u\Big]_0^1 + \Big[2u\mathrm{e}^u\Big]_0^1 - \Big[2\mathrm{e}^u\Big]_0^1 \\
&= \Big[(-u^2+2u-1)\mathrm{e}^u\Big]_0^1 \\
&= \Big[-(u-1)^2\mathrm{e}^u\Big]_0^1 = 1
\end{aligned}$$

Thus, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3(t)\mathrm{e}^{\cos(t)}\mathrm{d}t = 1$.

— $\boxed{\displaystyle\int_0^{\sqrt{\pi}} x^5\sin(x^2)\mathrm{d}x \textbf{ by using the substitution } u = x^2}$

When setting $u = x^2$ we get $\mathrm{d}u = 2x\mathrm{d}x$.
We have to change the boundaries : $0 \longrightarrow 0$, $\sqrt{\pi} \longrightarrow \pi$.
Remarking that $x^5 = \frac{1}{2}(2x)\left(x^2\right)^2$:

$$\int_0^{\sqrt{\pi}} x^5\sin(x^2)\mathrm{d}x = \frac{1}{2}\int_0^{\sqrt{\pi}} \left(x^2\right)^2\sin(x^2)2x\mathrm{d}x = \frac{1}{2}\int_0^{\pi} u^2\sin(u)\mathrm{d}u$$

This last integral will be calculated using two successive integrations by parts :

$$\begin{aligned}
\frac{1}{2}\int_0^{\pi} u^2\sin(u)\mathrm{d}u &= \frac{1}{2}\Big[-u^2\cos(u)\Big]_0^{\pi} + \int_0^{\pi} u\cos(u)\mathrm{d}u \\
&= \frac{1}{2}(\pi^2-0) + \Big[u\sin(u)\Big]_0^{\pi} - \int_0^{\pi}\sin(u)\mathrm{d}u \\
&= \frac{\pi^2}{2} + 0 + \Big[\cos(u)\Big]_0^{\pi} \\
&= \frac{\pi^2}{2} - 2
\end{aligned}$$

Thus, $\displaystyle\int_0^{\sqrt{\pi}} x^5\sin(x^2)\mathrm{d}x = \frac{\pi^2}{2} - 2$

— $\boxed{\int_0^1 x^3 \mathrm{e}^{x^2} \mathrm{d}x \textbf{ by using the substitution } u = x^2}$

When setting $u = x^2$ we get $\mathrm{d}u = 2x\mathrm{d}x$.
We have to « change » the boundaries : $0 \longrightarrow 0, 1 \longrightarrow 1$.
As $x^3 = \frac{1}{2}(2x)x^2$:

$$\int_0^1 x^3 \mathrm{e}^{x^2} \mathrm{d}x = \frac{1}{2}\int_0^1 x^2 \mathrm{e}^{x^2} 2x\mathrm{d}x = \frac{1}{2}\int_0^1 u\mathrm{e}^u \mathrm{d}u$$

An integration by parts on the obtained integral gives :

$$\begin{aligned} \frac{1}{2}\int_0^1 u\mathrm{e}^u \mathrm{d}u &= \frac{1}{2}\Big[u\mathrm{e}^u\Big]_0^1 - \frac{1}{2}\int_0^1 \mathrm{e}^u \mathrm{d}u \\ &= \frac{1}{2}\Big[u\mathrm{e}^u\Big]_0^1 - \frac{1}{2}\Big[\mathrm{e}^u\Big]_0^1 \\ &= \frac{1}{2}\Big[(u-1)\mathrm{e}^u\Big]_0^1 = \frac{1}{2} \end{aligned}$$

Thus, $\int_0^1 x^3 \mathrm{e}^{x^2} \mathrm{d}x = \frac{1}{2}$.