Revisions

(one week)

(from Monday, 18 September 2017 to Friday, 22 September 2017)

Exercise 1

Let f be of class C^{n+1} on an interval I of $\mathbb{R}$ and $(a,b) \in I^2$. Show by induction on $n \in \mathbb{N}$, Taylor's formula with an integral remainder :

$$f(b) = f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\, \mathrm{d}t$$

Exercise 2

We recall that the Taylor's expansions result from the Taylor-Young theorem (demonstrable from exercise 1) : let $n \in \mathbb{N}$ and f be of class C^n on an interval I of $\mathbb{R}$. Then at a neighborhood of $a \in I$, we have

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \boxed{\frac{(x-a)^n}{n!} f^{(n)}(a)} + o\big((x-a)^n\big)$$

Recall Taylor's expansion near 0 at order 6 of the following functions :

1. $f(x) = e^x$.
2. $g(x) = \ln(1+x)$.
3. $h(x) = (1+x)^\alpha$ with $\alpha \in \mathbb{R}^*$.
4. $i(x) = \sin(x)$.
5. $j(x) = \cos(x)$.

Exercise 3

Find Taylor's expansion near 0 of the following functions :

1. $f(x) = \cos(x)e^x$ at order 4.
2. $g(x) = \dfrac{1}{1-x} - e^x$ at order 3.
3. $h(x) = \dfrac{\cos(x)}{\sqrt{1+x}}$ at order 3.
4. $i(x) = \ln\big(1+\cos(x)\big)$ at order 4.
5. $j(x) = e^{\cos(x)}$ at order 4.
6. $k(x) = \dfrac{xe^x}{1-x^2}$ at order 3.
7. $\ell(x) = \big(\cos(x)\big)^{\sin(x)}$ at order 4.

Exercise 4

Find the following limits :

1. $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x$.

2. $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$.

3. $\lim_{x \to +\infty} \left(\cos\left(\frac{1}{x}\right)\right)^{x^2}$.

4. $\lim_{x \to +\infty} x^3 \sin\left(\frac{1}{x}\right) - x^2$.

5. $\lim_{x \to 0} \frac{e^x - \cos(x) - x}{x - \ln(1+x)}$.

6. $\lim_{x \to 0} \frac{e^x - e^{-x}}{\ln(1+x)}$.

7. $\lim_{x \to 0} \frac{\ln(1+\sin(x)) - \sin(\ln(1+x))}{x^2 \sin(x^2)}$.

Exercise 5

Let $a \in \mathbb{R} \cup \{+\infty\}$, f and g two real functions defined over $\mathbb{R}$.

One denotes e^f the map $x \mapsto e^{f(x)}$ and $\ln(f)$ the map $x \mapsto \ln(f(x))$.

1. Show that :
$$f \underset{a}{\sim} g \not\Rightarrow e^f \underset{a}{\sim} e^g$$

2. Give a necessary and sufficient condition on f and g such that $e^f \underset{a}{\sim} e^g$

3. One assumes f and g to be (strictly) positive. Show that :
$$f \underset{a}{\sim} g \not\Rightarrow \ln(f) \underset{a}{\sim} \ln(g)$$