TD Test 3

Name : **Surname :** **Group :**

Question from the class (2 points)

Let (u_n) be a geometric sequence with common ratio $q \in \mathbb{R}\backslash\{1\}$ and (v_n) be an arithmetic sequence with common difference $r \in \mathbb{R}$.
Give the expression of $u_0 + u_1 + \cdots + u_n$ and $v_0 + v_1 + \cdots + v_n$.

$$u_0 + u_1 + \dots + u_n = u_0 \frac{1-q^{n+1}}{1-q}$$

$$v_0 + v_1 + \dots + v_n = \frac{v_0 + v_n}{2} \times (n+1)$$

Question from the class (2 points)

Let (u_n) be a real sequence and $\ell \in \mathbb{R}$. Give the precise definition, with the quantifiers, of « (u_n) converges to ℓ ».

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}, \ \forall n \in \mathbb{N}, \ n > N \Rightarrow |u_n - \ell| < \varepsilon$$

Exercise 1 (2 points)

Let (u_n) be defined by $u_0 = 1$ and for all $n \in \mathbb{N}$, $u_{n+1} = 7u_n + 12$. Determine, for all $n \in \mathbb{N}$, u_n as a function of n.

* We determine the fixed point ℓ:

$$\ell = 7\ell + 12 \Rightarrow -6\ell = 12 \Rightarrow \ell = -2$$

* Let $v_n = u_n - \ell = u_n + 2$.

Then $\forall n \in \mathbb{N}$, $v_{n+1} = u_{n+1} + 2 = 7u_n + 12 + 2 = 7u_n + 14$
$= 7(u_n + 2) = 7v_n$

So (v_n) is geometric with common ratio 7.

* $\forall n \in \mathbb{N}$, $v_n = v_0 \times 7^n$, but $v_0 = u_0 + 2 = 3$.

So $\forall n \in \mathbb{N}$, $v_n = 3 \times 7^n$

* Since $v_n = u_n + 2$, we have $\forall n \in \mathbb{N}$, $u_n = v_n - 2 = 3 \times 7^n - 2$

Exercise 2 (4 points)

Let $(u_n)_{n\in\mathbb{N}^*}$ be defined for all $n \in \mathbb{N}^*$ by $u_n = \frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{2n-1}{2n}$

1. Give the expression of u_{n+1} in terms of n?

$$u_{n+1} = \frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{2n-1}{2n} \times \frac{\overset{= 2(n+1)-1}{2n+1}}{\underset{= 2(n+1)}{2n+2}}$$

2. Study the monotony of (u_n) by using $\frac{u_{n+1}}{u_n}$.

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2}$ because $u_{n+1} = \underbrace{\frac{1}{2} \times \frac{3}{4} \times \cdots \times \frac{2n-1}{2n}}_{u_n} \times \frac{2n+1}{2n+2}$

So ~~st~~ $\forall n \in \mathbb{N}$, $\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} < 1$

Now, $\forall n \in \mathbb{N}$, u_n is the product of ~~pos~~ strictly positive terms, so $u_n > 0$.

Thus, (u_n) is decreasing.

3. Is (u_n) convergent? Justify your answer.

We have $\left|\begin{array}{l} (u_n) \text{ decreasing} \\ \forall n \in \mathbb{N},\ u_n > 0, \text{ so } (u_n) \text{ is bounded below.} \end{array}\right.$

The sequence (u_n) is hence convergent.