Final exam 1

Duration : three hours
Documents and calculators not allowed

Name : **First name :** **Class :**

Exercise 1 (5 points)

1. Using d'Alembert's test, determine the nature of the series $\sum \frac{(n!)^2}{(3n)!}$.

2. Let $k \in \mathbb{N}^*$. Using d'Alembert's test, determine, depending on k, the nature of the series $\sum \frac{(n!)^2}{(kn)!}$.

[this frame continues on next page]

3. Let $a \in \mathbb{R}$. Using Cauchy's test, determine, depending on a, the nature of the series $\sum \left(\dfrac{n}{n+a}\right)^{n^2}$.

Exercise 2 (4 points)

Let $A = \begin{pmatrix} 0 & 3 & 0 \\ 1 & -2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & -4 \\ 1 & 1 & -3 \end{pmatrix}$.

Are A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, determine D and P.

N.B. : the bases of the eigenspaces must be deduced from a clear reasoning, and not by randomly picking particular values.

[this frame continues on next page]

Exercise 3 (4 points)

Let $a \in \mathbb{R}$ and $A = \begin{pmatrix} -3 & 1 & 0 \\ a-3 & 0 & 1-a \\ -1 & 1 & -2 \end{pmatrix}$. Study the diagonalizability of A in $\mathscr{M}_3(\mathbb{R})$ depending on the value of a.

N.B. : **when A is diagonalizable, the eigenbasis is not required**.

Exercise 4 (4 points)

1. Let $f : \begin{cases} \mathbb{R}_3[X] & \longrightarrow \mathbb{R}_3[X] \\ P(X) & \longmapsto 3XP(X) - (X^2-1)P'(X) \end{cases}$.

   a. Determine (no need to justify) the matrix of f in the standard basis $\mathscr{B} = (1, X, X^2, X^3)$ of $\mathbb{R}_3[X]$.

   b. Is f bijective ? Justify your answer.

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{R})$ and $f : \begin{cases} \mathscr{M}_2(\mathbb{R}) \to \mathscr{M}_2(\mathbb{R}) \\ X \mapsto AX - XA \end{cases}$. Determine (no need to justify) the matrix of f in the standard basis $\mathscr{B} = \left(E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$ of $\mathscr{M}_2(\mathbb{R})$.

Exercise 5 (4 points)

Let $(a, b, c, d, e, f) \in \mathbb{R}^6$ and $A = \begin{pmatrix} 1 & a & b & c \\ 0 & 2 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Study the diagonalizability of A depending on the values of a, b, c, d, e and f.
When A is diagonalizable, the eigenbasis is not required.

[this frame continues on next page]