Key to Tutorial 5
Boolean Algebra

Exercise 1

1. Simplify the following expressions:

- $S1 = (a + b).(\overline{a} + \overline{b})$
  $S1 = a.\overline{a} + a.\overline{b} + b.\overline{a} + b.\overline{b}$
  $S1 = 0 + a.\overline{b} + b.\overline{a} + 0$
  $S1 = a.\overline{b} + \overline{a}.b$
  $S1 = a \oplus b$

- $S2 = a.b + \overline{a}.\overline{b} + \overline{a}.b$
  $S2 = a.b + \overline{a}.(\overline{b} + b)$
  $S2 = a.b + \overline{a}.1$
  $S2 = \mathbf{a}.b + \overline{\mathbf{a}}$ → *Theorem 2*
  $S2 = \overline{a} + b$

- $S3 = (a + \overline{b}).(a + b) + c.(\overline{a} + b)$
  $S3 = a.a + a.b + \overline{b}.a + \overline{b}.b + c.\overline{a} + c.b$
  $S3 = \mathbf{a} + \mathbf{a}.b + \overline{b}.\mathbf{a} + c.\overline{a} + c.b$ → *Theorem 1*
  $S3 = \mathbf{a} + c.\overline{\mathbf{a}} + c.b$ → *Theorem 2*
  $S3 = a + \mathbf{c} + \mathbf{c}.b$ → *Theorem 1*
  $S3 = a + c$

- $S4 = (a + c + d).(b + c + d)$
  $S4 = a.b + a.c + a.d + c.b + c.c + c.d + d.b + d.c + d.d$
  $S4 = a.b + a.\mathbf{c} + a.d + \mathbf{c}.b + \mathbf{c} + \mathbf{c}.d + d.b + d.\mathbf{c} + d$ → *Theorem 1*
  $S4 = a.b + a.\mathbf{d} + c + \mathbf{d}.b + \mathbf{d}$ → *Theorem 1*
  $S4 = a.b + c + d$

- $S5 = (a.\overline{b} + a.b + a.c).(\overline{a}.\overline{b} + a.b + a.\overline{c})$
  $S5 = a.(\overline{b} + b + c).(\overline{a}.\overline{b} + a.b + a.\overline{c})$
  $S5 = a.1.(\overline{a}.\overline{b} + a.b + a.\overline{c})$
  $S5 = a.(\overline{a}.\overline{b} + a.b + a.\overline{c})$
  $S5 = a.\overline{a}.\overline{b} + a.a.b + a.a.\overline{c}$
  $S5 = 0.\overline{b} + a.b + a.\overline{c}$
  $S5 = a.b + a.\overline{c}$

- $S6 = (a + \bar{b} + c).(a + \bar{c}).(\bar{a} + \bar{b})$
  $S6 = (a.a + a.\bar{c} + \bar{b}.a + \bar{b}.\bar{c} + c.a + c.\bar{c}).(\bar{a} + \bar{b})$
  $S6 = (a + a.\bar{c} + \bar{b}.a + \bar{b}.\bar{c} + c.a + 0).(\bar{a} + \bar{b})$
  $S6 = (\mathbf{a} + \mathbf{a}.\bar{c} + \bar{b}.\mathbf{a} + \bar{b}.\bar{c} + c.\mathbf{a}).(\bar{a} + \bar{b})$ → *Theorem 1*
  $S6 = (a + \bar{b}.\bar{c}).(\bar{a} + \bar{b})$
  $S6 = a.\bar{a} + a.\bar{b} + \bar{b}.\bar{c}.\bar{a} + \bar{b}.\bar{c}.\bar{b}$
  $S6 = 0 + a.\bar{b} + \bar{b}.\bar{c}.\bar{a} + \bar{b}.\bar{c}$
  $S6 = a.\bar{b} + \mathbf{\bar{b}.\bar{c}}.\bar{a} + \mathbf{\bar{b}.\bar{c}}$ → *Theorem 1*
  $S6 = a.\bar{b} + \bar{b}.\bar{c}$

- $S7 = a.b.c + a.\bar{b}.\bar{c} + \bar{a}.b.\bar{c} + \bar{a}.b.c$
  $S7 = b.c.(a + \bar{a}) + a.\bar{b}.\bar{c} + \bar{a}.b.\bar{c}$
  $S7 = b.c.1 + a.\bar{b}.\bar{c} + \bar{a}.b.\bar{c}$
  $S7 = b.c + a.\bar{b}.\bar{c} + \bar{a}.b.\bar{c}$
  $S7 = b.(\mathbf{c} + \bar{a}.\mathbf{\bar{c}}) + a.\bar{b}.\bar{c}$ → *Theorem 2*
  $S7 = b.\mathbf{(c + \bar{a})} + a.\bar{b}.\bar{c}$ → *De Morgan's Theorem*
  $S7 = b.\overline{\bar{c}.a} + \bar{b}.\bar{c}.a$
  $S7 = b \oplus (\bar{c}.a)$

- $S8 = a.b.c + a.\bar{b}.c + a.b.\bar{c}.d$
  $S8 = a.c.(b + \bar{b}) + a.b.\bar{c}.d$
  $S8 = a.c.1 + a.b.\bar{c}.d$
  $S8 = a.c + a.b.\bar{c}.d$
  $S8 = a.(\mathbf{c} + b.\mathbf{\bar{c}}.d)$ → *Theorem 2*
  $S8 = a.(c + b.d)$
  $S8 = a.c + a.b.d$

- $S9 = \mathbf{a} + b.c + \mathbf{\bar{a}}.(\bar{b} + \bar{c}).(a.d + c)$ → *Theorem 2*
  $S9 = a + b.c + \mathbf{(\bar{b} + \bar{c})}.(a.d + c)$ → *De Morgan's Theorem*
  $S9 = a + \mathbf{b.c} + \mathbf{\overline{b.c}}.(a.d + c)$ → *Theorem 2*
  $S9 = \mathbf{a} + b.c + \mathbf{a}.d + c$ → *Theorem 1*
  $S9 = a + b.\mathbf{c} + \mathbf{c}$ → *Theorem 1*
  $S9 = a + c$

2. Calculate and simplify the complement of S1, S5 and S6.

- $\overline{S1} = \overline{a.\bar{b} + \bar{a}.b}$
  $\overline{S1} = \overline{a.\bar{b}}.\overline{\bar{a}.b}$
  $\overline{S1} = (\bar{a} + b).(a + \bar{b})$
  $\overline{S1} = \bar{a}.a + \bar{a}.\bar{b} + b.a + b.\bar{b}$
  $\overline{S1} = 0 + \bar{a}.\bar{b} + b.a + 0$
  $\overline{S1} = \bar{a}.\bar{b} + b.a$
  $\overline{S1} = \overline{a \oplus b}$

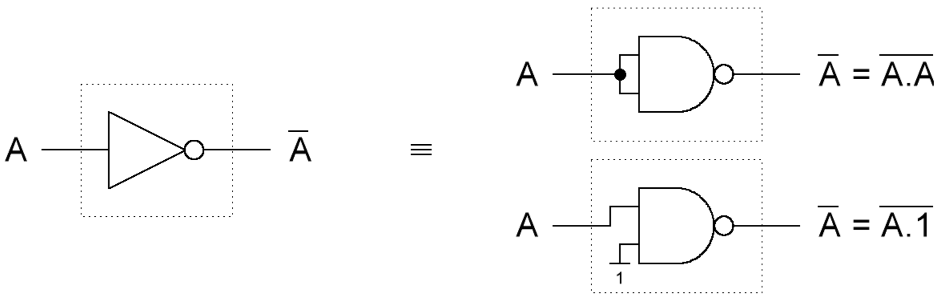
- $\overline{S5} = \overline{a.b + a.\overline{c}}$
  $\overline{S5} = \overline{\mathbf{a.(b + \overline{c})}}$ → *De Morgan's Theorem*
  $\overline{S5} = \overline{a} + \overline{\mathbf{b + \overline{c}}}$ → *De Morgan's Theorem*
  $\overline{S5} = \overline{a} + \overline{b}.c$

- $\overline{S6} = \overline{a.\overline{b} + \overline{b}.\overline{c}}$
  $\overline{S6} = \overline{\mathbf{\overline{b}.(a + \overline{c})}}$ → *De Morgan's Theorem*
  $\overline{S6} = b + \overline{\mathbf{a + \overline{c}}}$ → *De Morgan's Theorem*
  $\overline{S6} = b + \overline{a}.c$

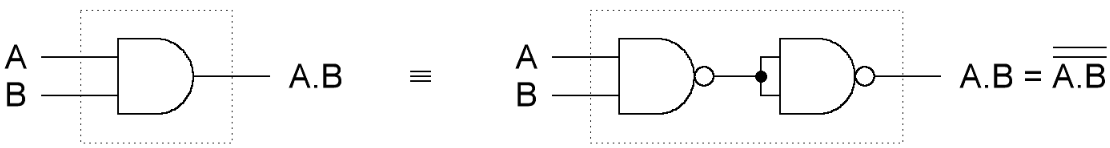
3. Design the NOT, AND, and OR gates by using only NAND gates, then only NOR gates.

NAND gates:

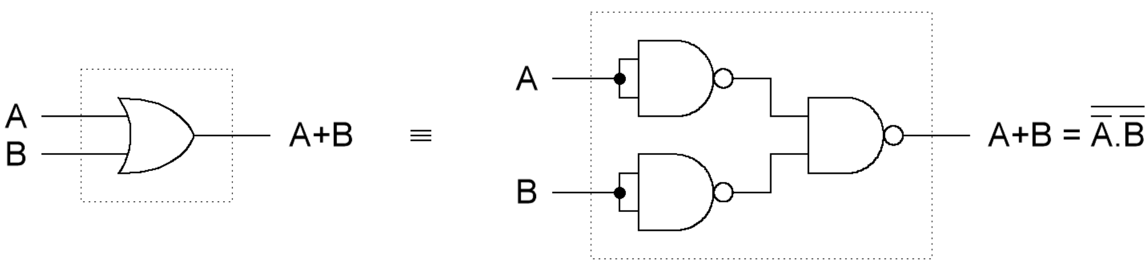
- NOT (two possibilities):
  - $\overline{A} = \overline{A.A}$
  - $\overline{A} = \overline{A.1}$



- AND: $A.B = \overline{\overline{A.B}}$

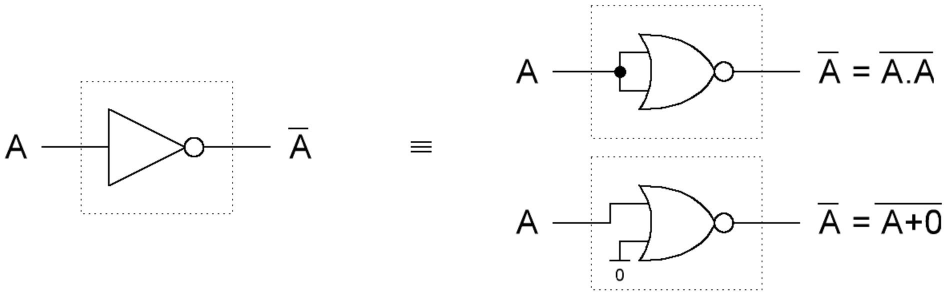


- OR: $A + B = \overline{\overline{A}.\overline{B}}$

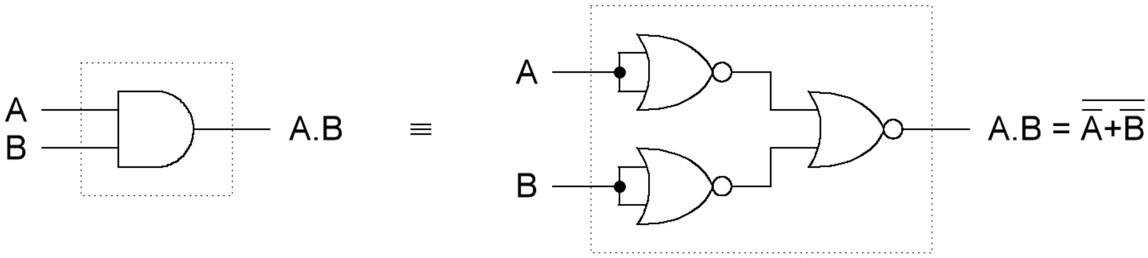


NOR gates:

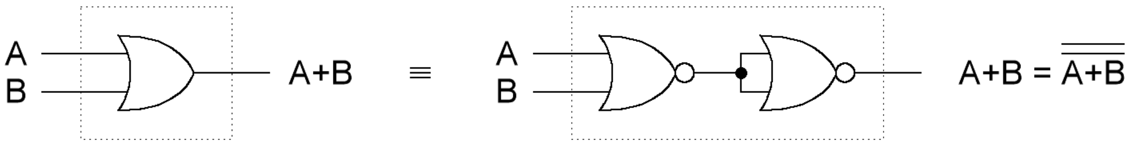
- NOT (two possibilities):
  - $\overline{A} = \overline{A + A}$
  - $\overline{A} = \overline{A + 0}$



- AND: $A.B = \overline{\overline{A} + \overline{B}}$



- OR: $A + B = \overline{\overline{A + B}}$



Exercise 2

1. Write down the following expressions by using minterm canonical forms:

- $S1 = \overline{a}.b.d + a.\overline{b}.\overline{c} + a.b.c$
  $S1 = \overline{a}.b.d.(c + \overline{c}) + a.\overline{b}.\overline{c}.(d + \overline{d}) + a.b.c.(d + \overline{d})$
  $S1 = \overline{a}.b.c.d + \overline{a}.b.\overline{c}.d + a.\overline{b}.\overline{c}.d + a.\overline{b}.\overline{c}.\overline{d} + a.b.c.d + a.b.c.\overline{d}$

- $S2 = a.c.d + b.c.\overline{d} + \overline{b}.\overline{c}.d$
  $S2 = a.c.d.(b + \overline{b}) + b.c.\overline{d}.(a + \overline{a}) + \overline{b}.\overline{c}.d.(a + \overline{a})$
  $S2 = a.b.c.d + a.\overline{b}.c.d + a.b.c.\overline{d} + \overline{a}.b.c.\overline{d} + a.\overline{b}.\overline{c}.d + \overline{a}.\overline{b}.\overline{c}.d$

- $S3 = (\bar{a} + \bar{c}).(a + \bar{d} + c).b.\bar{c}$
  $S3 = (\bar{a}.a + \bar{a}.\bar{d} + \bar{a}.c + \bar{c}.a + \bar{c}.\bar{d} + \bar{c}.c\ ).b.\bar{c}$
  $S3 = (0 + \bar{a}.\bar{d} + \bar{a}.c + \bar{c}.a + \bar{c}.\bar{d} + 0).b.\bar{c}$
  $S3 = (\bar{a}.\bar{d} + \bar{a}.c + \bar{c}.a + \bar{c}.\bar{d}).b.\bar{c}$
  $S3 = \bar{a}.\bar{d}.b.\bar{c} + \bar{a}.c.b.\bar{c} + \bar{c}.a.b.\bar{c} + \bar{c}.\bar{d}.b.\bar{c}$
  $S3 = \bar{a}.b.\bar{c}.\bar{d} + 0 + a.b.\bar{c} + b.\bar{c}.\bar{d}$
  $S3 = \bar{a}.b.\bar{c}.\bar{d} + a.b.\bar{c} + b.\bar{c}.\bar{d}$
  $S3 = \bar{a}.b.\bar{c}.\bar{d} + a.b.\bar{c}.(d + \bar{d}) + b.\bar{c}.\bar{d}.(a + \bar{a})$
  $S3 = \bar{\mathbf{a}}\mathbf{.b.}\bar{\mathbf{c}}\mathbf{.}\bar{\mathbf{d}} + a.b.\bar{c}.d + \mathbf{a.b.}\bar{\mathbf{c}}\mathbf{.}\bar{\mathbf{d}} + \mathbf{a.b.}\bar{\mathbf{c}}\mathbf{.}\bar{\mathbf{d}} + \bar{\mathbf{a}}\mathbf{.b.}\bar{\mathbf{c}}\mathbf{.}\bar{\mathbf{d}}$ → *Remove duplication*
  $S3 = \bar{a}.b.\bar{c}.\bar{d} + a.b.\bar{c}.d + a.b.\bar{c}.\bar{d}$

- $S4 = b.c.(a + \bar{d}) + \bar{b}.d.(a + \bar{c})$
  $S4 = b.c.a + b.c.\bar{d} + \bar{b}.d.a + \bar{b}.d.\bar{c}$
  $S4 = b.c.a.(d + \bar{d}) + b.c.\bar{d}.(a + \bar{a}) + \bar{b}.d.a.(c + \bar{c}) + \bar{b}.d.\bar{c}.(a + \bar{a})$
  $S4 = a.b.c.d + \mathbf{a.b.c.}\bar{\mathbf{d}} + \mathbf{a.b.c.}\bar{\mathbf{d}} + \bar{a}.b.c.\bar{d} + a.\bar{b}.c.d + \mathbf{a.}\bar{\mathbf{b}}\mathbf{.}\bar{\mathbf{c}}\mathbf{.d} + \mathbf{a.}\bar{\mathbf{b}}\mathbf{.}\bar{\mathbf{c}}\mathbf{.d} + \bar{a}.\bar{b}.\bar{c}.d$ → *Remove dup.*
  $S4 = a.b.c.d + a.b.c.\bar{d} + \bar{a}.b.c.\bar{d} + a.\bar{b}.c.d + a.\bar{b}.\bar{c}.d + \bar{a}.\bar{b}.\bar{c}.d$

2. Write down the following expressions by using maxterm canonical forms:

- $S1 = (a + c).(\bar{a} + b + c)$
  $S1 = (a + b.\bar{b} + c).(\bar{a} + b + c)$ → *Distributivity*
  $S1 = (a + b + c).(a + \bar{b} + c).(\bar{a} + b + c)$

- $S2 = a.b + a.\bar{c} + \bar{a}.\bar{b}.c$ → *Distributivity*
  $S2 = (\mathbf{a}.b + \mathbf{a}.\bar{c} + \bar{\mathbf{a}}).(a.\mathbf{b} + a.\bar{c} + \bar{\mathbf{b}}).(a.b + a.\bar{\mathbf{c}} + \mathbf{c})$ → *Theorem 2*
  $S2 = (b + \bar{c} + \bar{a}).(\mathbf{a} + \mathbf{a}.\bar{c} + \bar{b}).(\mathbf{a}.b + \mathbf{a} + c)$ → *Theorem 1*
  $S2 = (\bar{a} + b + \bar{c}).(a + \bar{b}).(a + c)$
  $S2 = (\bar{a} + b + \bar{c}).(a + \bar{b} + c.\bar{c}).(a + b.\bar{b} + c)$ → *Distributivity*
  $S2 = (\bar{a} + b + \bar{c}).\mathbf{(a + \bar{b} + c)}.(a + \bar{b} + \bar{c}).(a + b + c).\mathbf{(a + \bar{b} + c)}$ → *Remove duplication*
  $S2 = (\bar{a} + b + \bar{c}).(a + \bar{b} + c).(a + \bar{b} + \bar{c}).(a + b + c)$

Exercise 3

Prove that the following identities are true:

- $\overline{\mathbf{a.c} + \mathbf{b.}\bar{\mathbf{c}}} = \bar{a}.c + \bar{b}.\bar{c}$ → *De Morgan's Theorem*
  $\overline{\mathbf{a.c}}\ .\ \overline{\mathbf{b.}\bar{\mathbf{c}}} = \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$ → *De Morgan's Theorem*
  $(\bar{a} + \bar{c}).(\bar{b} + c) = \bar{a}.c + \bar{b}.\bar{c}$
  $\bar{a}.\bar{b} + \bar{a}.c + \bar{c}.\bar{b} + \bar{c}.c = \bar{a}.c + \bar{b}.\bar{c}$
  $\bar{a}.\bar{b} + \bar{a}.c + \bar{c}.\bar{b} + 0 = \bar{a}.c + \bar{b}.\bar{c}$
  $\bar{a}.\bar{b} + \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$
  $\bar{a}.\bar{b}.\mathbf{(c + \bar{c})} + \bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$
  $\bar{\mathbf{a}}.\bar{b}.\mathbf{c} + \bar{a}.\bar{b}.\bar{c} + \bar{\mathbf{a}}\mathbf{.c} + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$ → *Theorem 1*
  $\bar{a}.\bar{\mathbf{b}}\mathbf{.}\bar{\mathbf{c}} + \bar{a}.c + \bar{\mathbf{b}}\mathbf{.}\bar{\mathbf{c}} = \bar{a}.c + \bar{b}.\bar{c}$ → *Theorem 1*
  $\bar{a}.c + \bar{b}.\bar{c} = \bar{a}.c + \bar{b}.\bar{c}$

- $(a + b).(\bar{a} + c).(b + c) = (a + b).(\bar{a} + c)$
  $(a.\bar{a} + a.c + b.\bar{a} + b.c).(b + c) = a.\bar{a} + a.c + b.\bar{a} + b.c$
  $(0 + a.c + b.\bar{a} + b.c).(b + c) = 0 + a.c + b.\bar{a} + b.c$
  $(a.c + b.\bar{a} + b.c).(b + c) = a.c + b.\bar{a} + b.c$
  $a.c.b + a.c.c + b.\bar{a}.b + b.\bar{a}.c + b.c.b + b.c.c = a.c + b.\bar{a} + b.c$
  $a.c.b + a.c + \bar{a}.b + b.\bar{a}.c + b.c + b.c = a.c + \bar{a}.b + b.c$
  $\mathbf{a.c}.b + \mathbf{a.c} + \bar{a}.b + b.\bar{a}.c + b.c = a.c + \bar{a}.b + b.c$ → *Theorem 1*
  $a.c + \bar{a}.b + \mathbf{b}.\bar{a}.\mathbf{c} + \mathbf{b.c} = a.c + \bar{a}.b + b.c$ → *Theorem 1*
  $a.c + \bar{a}.b + b.c = a.c + \bar{a}.b + b.c$

- $\overline{\mathbf{(a + c).(b + \bar{c})}} = (\bar{a} + c).(\bar{b} + \bar{c})$ → *De Morgan's Theorem*
  $\overline{\mathbf{(a + c)}} + \overline{\mathbf{(b + \bar{c})}} = \bar{a}.\bar{b} + \bar{a}.\bar{c} + c.\bar{b} + c.\bar{c}$ → *De Morgan's Theorem*
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{b} + \bar{a}.\bar{c} + c.\bar{b} + 0$
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{b} + \bar{a}.\bar{c} + \bar{b}.c$
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{b}.\mathbf{(c + \bar{c})} + \bar{a}.\bar{c} + \bar{b}.c$
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{b}.c + \bar{\mathbf{a}}.\bar{b}.\bar{\mathbf{c}} + \bar{\mathbf{a}}.\bar{\mathbf{c}} + \bar{b}.c$ → *Theorem 1*
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{\mathbf{b}}.\mathbf{c} + \bar{a}.\bar{c} + \bar{\mathbf{b}}.\mathbf{c}$ → *Theorem 1*
  $\bar{a}.\bar{c} + \bar{b}.c = \bar{a}.\bar{c} + \bar{b}.c$

Exercise 4

Let us consider the following binary variables: *A*, *B*, *C*. Write down an expression that is 1 when the number of variables being 1 is odd (simplify with EXCLUSIVE OR).

a	b	c	Output	
0	0	0	0	
0	0	1	1	→ $\bar{A}.\bar{B}.C$
0	1	0	1	→ $\bar{A}.B.\bar{C}$
0	1	1	0	
1	0	0	1	→ $A.\bar{B}.\bar{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	→ $A.B.C$

Output = $\bar{A}.\bar{B}.C + \bar{A}.B.\bar{C} + A.\bar{B}.\bar{C} + A.B.C$
Output = $\bar{A}.(\bar{B}.C + B.\bar{C}) + A.(\bar{B}.\bar{C} + B.C)$
Output = $\bar{A}.(B \oplus C) + A.(\overline{B \oplus C})$
Output = $\mathbf{A \oplus B \oplus C}$