Class Test n°4

Name : **First Name :** **Class :**

Question from the lesson (2 points)

Let E be a vector space over $\mathbb{R}$ and $S = (e_1, \ldots, e_n)$ be a family of vectors of E. Give the precise mathematical definition of «S is a spanning family of E».

$\forall w \in E, \exists (\lambda_1 \ldots \lambda_n) \in \mathbb{R}^n, w = \lambda_1 e_1 + \ldots + \lambda_n e_n$

Exercise 1 (2 points)

Let $E = \{(u_n) \in \mathbb{R}^{\mathbb{N}}$ such that (u_n) is bounded$\}$ and $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}$ such that (u_n) is divergent$\}$.
Are E and F some $\mathbb{R}$-vector spaces? Justify your answer.

We know that $\mathbb{R}^{\mathbb{N}}$ is a $\mathbb{R}$-vs, so we wonder weither E and F are linear subspaces of $\mathbb{R}^{\mathbb{N}}$

1/ E :

- 0_E is the zero sequence : $\forall n \in \mathbb{N}, u_n = 0 \Rightarrow (u_n)$ is bounded.
  So $0_E \in E$
- Let (u_n) and (v_n) be two elements of E. Let $\lambda \in \mathbb{R}$
  (u_n) is bounded : $\exists M_1 \in \mathbb{R}, \forall n \in \mathbb{N}, |u_n| \leq M_1$
  (v_n) is bounded : $\exists M_2 \in \mathbb{R}, \forall n \in \mathbb{N}, |v_n| \leq M_2$

  Then $\forall n \in \mathbb{N}$, $|\lambda u_n + v_n| \leq |\lambda u_n| + |v_n|$
  $\leq |\lambda| \cdot |u_n| + |v_n|$
  $\leq |\lambda| \cdot M_1 + M_2$

  So $\lambda(u_n) + (v_n)$ is bounded, $\lambda(u_n) + (v_n) \in E$
  E is hence a linear subspace of $\mathbb{R}^{\mathbb{N}}$

2/ F :

0_E is the zero sequence : $\forall n \in \mathbb{N}, u_n = 0$.
Then $\lim_{n\to\infty} u_n = 0$, so 0_E is not divergent. $0_E \notin F \Rightarrow F$ is not a linear subspace.

Exercise 2 (3 points)

Let $u = (2, 2, 6)$, $v = (3, 1, -3)$ and $w = (7, 5, 9)$. Is $\{u, v, w\}$ a linearly independent set of $\mathbb{R}^3$? Justify your answer.

Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda_1 u + \lambda_2 v + \lambda_3 w = 0$

Then $\begin{cases} 2\lambda_1 + 3\lambda_2 + 7\lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + 5\lambda_3 = 0 \\ 6\lambda_1 - 3\lambda_2 + 9\lambda_3 = 0 \end{cases} \iff \begin{cases} 2\lambda_1 + 3\lambda_2 + 7\lambda_3 = 0 \\ -2\lambda_2 - 2\lambda_3 = 0 \quad (Eq2 - Eq1) \\ -12\lambda_2 - 12\lambda_3 = 0 \quad (Eq3 - 3Eq1) \end{cases}$

$\iff \begin{cases} 2\lambda_1 + 3\lambda_2 + 7\lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases}$

We can choose $\lambda_3 = 1$; $\lambda_2 = -1$; $\lambda_1 = -2$

Then $\lambda_1 u + \lambda_2 v + \lambda_3 w = 0$ and $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$

So $\{u, v, w\}$ is not linearly independant

Exercise 3 (3 points)

Let $E = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{lcl} x - 2y - z & = & 0 \\ 2x - 3y - 2z & = & 0 \\ -2x + 2y + 2z & = & 0 \end{array} \right. \right\}$. Write E as a spanned subspace, using the Span notation.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E \iff \begin{cases} x - 2y - z = 0 \\ 2x - 3y - 2z = 0 \\ -2x + 2y + 2z = 0 \end{cases} \iff \begin{cases} x - 2y - z = 0 \\ y = 0 \quad (Eq2 - 2Eq1) \\ -2y = 0 \quad (Eq3 + 2Eq1) \end{cases}$

$\iff \begin{cases} x = z \\ y = 0 \end{cases}$

We can take z as free parameter:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$E = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$