Final exam n°2

Duration : three hours
Documents and calculators not allowed

Name : **First Name :** **Class :**

Exercise 1 (3 points)

Let $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}$. Determine the inverse matrix A^{-1} (don't forget to check - on your draft - the final result).

Exercise 2 (4,5 points)

Expand into partial fractions of $\mathbb{R}(X)$ the following rational fractions :

1. $F(X) = \dfrac{X+1}{(X-1)(X+2)(X+3)}$

2. $G(X) = \dfrac{X^3+X^2+1}{(X-1)(X+2)}$

3. $H(X) = \dfrac{X^2 - X + 2}{(X-1)(X^2+1)}$

Exercise 3 (4 points)

Let $E = \mathbb{R}_2[X]$ and $f : E \longrightarrow E$ defined for every $P \in E$ by $f(P) = 2(X+1)P - (X^2+1)P'$.

Let $\mathscr{B} = \big(1, X, X^2\big)$ and $\mathscr{B}' = \big(1, X-1, (X+1)^2\big)$, two bases of E.

1. Determine $\mathrm{Mat}_{\mathscr{B}}(f)$, the matrix of f with respect to the basis $\mathscr{B}$.

2. Determine $\mathrm{Mat}_{\mathscr{B}'}(f)$, the matrix of f with respect to the basis $\mathscr{B}'$.

3. Determine $\mathrm{Mat}_{\mathscr{B},\mathscr{B}'}(f)$, the matrix of f with respect to the bases $\mathscr{B}$, $\mathscr{B}'$.

4. Determine $\mathrm{Mat}_{\mathscr{B}',\mathscr{B}}(f)$, the matrix of f with respect to the bases $\mathscr{B}'$, $\mathscr{B}$.

Exercise 4 (4 points)

Let us denote I the identity matrix of order 3. Let $J = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

1. Check that $J^2 - J - 2I = 0$. Deduce an expression of J^{-1} as a function of I and J.

   N.B. : We remind you that, if there exists $K \in \mathscr{M}_3(\mathbb{R})$ such that $JK = I$ then J is invertible and $J^{-1} = K$.

2. Let $n \in \mathbb{N}$. We proceed to the Euclidean division of X^n by $X^2 - X - 2$. Thus, there exists $Q(X) \in \mathbb{R}[X]$ and $R(X) \in \mathbb{R}[X]$ such that

   $$X^n = (X^2 - X - 2)Q(X) + R(X)$$

   with the degree of R being strictly inferior to 2.

   Then, there exists $(a, b) \in \mathbb{R}^2$ such that

   $$X^n = (X^2 - X - 2)Q(X) + aX + b$$

   Noticing that 2 and -1 are roots of $X^2 - X - 2$, determine a and b.

3. Let $n \in \mathbb{N}$. Deduce an expression of J^n as a function of n, I and J.

   N.B. : You will substitute J to the indeterminate X of question 2 (knowing that the polynomial 1 becomes I).

   As an example, $X^4 + 2X^3 + 4$ becomes after substitution $J^4 + 2J^3 + 4I$.

Exercise 5 (5,5 points)

Let $E = C^\infty(\mathbb{R}, \mathbb{R})$, the set of smooth functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. functions that are infinitely differentiable on $\mathbb{R}$). Let us denote f_0, f_1 and f_2 the vectors of E defined for every $x \in \mathbb{R}$ by

$$f_0(x) = e^{2x}, \ f_1(x) = xe^{2x} \text{ and } f_2(x) = x^2e^{2x}$$

Let us denote $F = \text{Span}\big(\{f_0, f_1, f_2\}\big)$ i.e. F is the vector subspace of E spanned by the vectors f_0, f_1 and f_2.

1. Show that $B = (f_0, f_1, f_2)$ is a basis of F.

2. Let d be the application defined for every $f \in F$ by $d(f) = f'$.
Show that d is an endomorphism of F.

3. Determine A, the matrix of d with respect to B.

4. Let $n \in \mathbb{N}^*$. Calculate $d^n(f_0)$, $d^n(f_1)$ and $d^n(f_2)$ where, for every $p \in \mathbb{N}^*$, $d^p = \underbrace{d \circ ... \circ d}_{p \text{ times}}$.

N.B. : You can use the general Leibniz rule, giving the n^{th} derivative of the product of two functions u and v of E, denoted $(uv)^{(n)}$:

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}$$

assuming that $u^{(0)} = u$ and where $C_n^k = \binom{n}{k} = \dfrac{n(n-1)\dots(n-k+1)}{k!}$.

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5. Deduce an expression of A^n as a function of n for every $n \in \mathbb{N}^*$.