Correction of final exam n°2

Exercise 1 (3 points)

Let $U = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $V = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ two vectors of $\mathbb{R}^3$. Then

$$AU = V \Longleftrightarrow U = A^{-1}V$$

Yet

$$AU = V \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcll} x + y - 2z & = & X & (1) \\ x - y + z & = & Y & (2) \\ -2x + y - z & = & Z & (3) \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} y + x - 2z & = & X \\ 2x - z & = & X + Y \\ 3x - z & = & X - Z \end{array} \right.$$

(We keep (1) untouched, replace (2) by (1) + (2) and (3) by (1) − (3)). Hence

$$AU = V \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} y + x - 2z & = & X \\ 2x - z & = & X + Y \\ x & = & -Z - Y \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{rcl} y & = & -X - 5Y - 3Z \\ z & = & -X - 3Y - 2Z \\ x & = & -Y - Z \end{array} \right. .$$

Finally

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -5 & -3 \\ -1 & -3 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} .$$

Thus, $A^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -5 & -3 \\ -1 & -3 & -2 \end{pmatrix}$.

Exercise 2 (4,5 points)

1. $F(X) = \dfrac{a}{X-1} + \dfrac{b}{X+2} + \dfrac{c}{X+3}$

   By multiplying by $X - 1$ then evaluating in $X = 1$ we find that $a = 1/6$.

   By multiplying by $X + 2$ then evaluating in $X = -2$ we find that $b = 1/3$.

   By multiplying by $X + 3$ then evaluating in $X = -3$ we find that $c = -1/2$.

   Thus
$$F(X) = \frac{1}{6(X-1)} + \frac{1}{3(X+2)} - \frac{1}{2(X+3)}$$

2. $G(X) = aX + b + \dfrac{c}{X-1} + \dfrac{d}{X+2}$

   By multiplying by $X - 1$ then evaluating in $X = 1$ we find that $c = 1$.

   By multiplying by $X + 2$ then evaluating in $X = -2$ we find that $d = 1$.

   Moreover, $\lim\limits_{X \to +\infty} \dfrac{G(X)}{X} = 1 = a$ then $a = 1$.

   Then, by evaluating in $X = 0$ we find that $b = 0$.

   Thus
$$G(X) = X + \frac{1}{X-1} + \frac{1}{X+2}$$

3. $H(X) = \dfrac{a}{X-1} + \dfrac{bX+c}{X^2+1}$

   By multiplying by $X-1$ then evaluating in $X=1$ we find that $a=1$.

   By multiplying by X^2+1 then evaluating in $X=i$ we find that $-1 = bi + c$, hence $b=0$ and $c=-1$.

   Thus
$$H(X) = \frac{1}{X-1} - \frac{1}{X^2+1}$$

Exercise 3 (4 points)

1. $f(1) = 2X+2$

   $f(X) = X^2 + 2X - 1$

   $f(X^2) = 2X^2 - 2X$

   thus $\mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} 2 & -1 & 0 \\ 2 & 2 & -2 \\ 0 & 1 & 2 \end{pmatrix}$

2. $f(1) = 2X+2 = 2(X-1)+4$

   $f(X-1) = X^2 - 3 = (X+1)^2 - 2(X-1) - 6$

   $f\big((X+1)^2\big) = 4X^2 + 4X = 4(X+1)^2 - 4(X-1) - 8$

   thus $\mathrm{Mat}_{\mathscr{B}'}(f) = \begin{pmatrix} 4 & -6 & -8 \\ 2 & -2 & -4 \\ 0 & 1 & 4 \end{pmatrix}$

3. $f(1) = 2X+2 = 2(X-1)+4$

   $f(X) = X^2 + 2X - 1 = (X+1)^2 - 2$

   $f(X^2) = 2X^2 - 2X = 2(X+1)^2 - 6(X-1) - 8$

   thus $\mathrm{Mat}_{\mathscr{B},\mathscr{B}'}(f) = \begin{pmatrix} 4 & -2 & -8 \\ 2 & 0 & -6 \\ 0 & 1 & 2 \end{pmatrix}$

4. $f(1) = 2X+2$

   $f(X-1) = X^2 - 3$

   $f\big((X+1)^2\big) = 4X^2 + 4X$

   thus $\mathrm{Mat}_{\mathscr{B}',\mathscr{B}}(f) = \begin{pmatrix} 2 & -3 & 0 \\ 2 & 0 & 4 \\ 0 & 1 & 4 \end{pmatrix}$

Exercise 4 (4 points)

1. As
$$\frac{1}{2}(J^2 - J) = I$$
   we find that
$$\frac{1}{2}J(J-I) = I$$
   thus
$$J^{-1} = \frac{1}{2}(J-I)$$

2.
$$X^n = (X^2 - X - 2)Q(X) + aX + b \qquad (*)$$

As 2 and -1 are roots of the polynomial $X^2 - X - 2$, we can deduce that $2^n = 2a + b$ and $(-1)^n = -a + b$, which leads to
$$a = \frac{1}{3}\big(2^n + (-1)^{n+1}\big) \text{ and } b = \frac{1}{3}\big(2^n + 2(-1)^n\big)$$

3. As $J^2 - J - 2I = 0$, we can deduce by substituting J to the indeterminate X in $(*)$ that
$$J^n = aJ + bI = \frac{1}{3}\Big(\big(2^n + (-1)^{n+1}\big)J + \big(2^n + 2(-1)^n\big)I\Big)$$

Exercise 5 (5,5 points)

1. By definition of F, B spans F. Let us show that B is linearly independent.
Let $(\lambda, \mu, \nu) \in \mathbb{R}^3$ such that $\lambda f_0 + \mu f_1 + \nu f_2 = 0$. Then
$$\forall x \in \mathbb{R},\ e^{2x}(\lambda + \mu x + \nu x^2) = 0$$
As for every $x \in \mathbb{R}$, $e^{2x} \neq 0$, the polynomial $\lambda + \mu X + \nu X^2$ has an infinity of roots, thus it must be the zero polynomial. Thus $\lambda = \mu = \nu = 0$.

2. Linearity is obvious. Let us show that for every $f \in F$, $d(f) \in F$.

Let $f \in F$. Then
$$\exists (a_0, a_1, a_2) \in \mathbb{R}^3,\ f = a_0 f_0 + a_1 f_1 + a_2 f_2$$
Thus
$$d(f) = f' = a_0 f_0' + a_1 f_1' + a_2 f_2'$$
Yet
$$f_0' = 2f_0,\ f_1' = f_0 + 2f_1 \text{ and } f_2' = 2f_1 + 2f_2$$
Hence
$$d(f) = (2a_0 + a_1)f_0 + 2(a_1 + a_2)f_1 + 2a_2 f_2$$
Thus
$$d(f) \in F$$
and d is an endomorphism of F.

3. According to the previous question,
$$d(f_0) = 2f_0,\ d(f_1) = f_0 + 2f_1 \text{ and } d(f_2) = 2f_1 + 2f_2$$
Then, the matrix of d in B is
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

4. Via the given formula,
$$d^n(f_0) = 2^n f_0,\ d^n(f_1) = n2^{n-1} f_0 + 2^n f_1 \text{ and } d^n(f_2) = \frac{n(n-1)}{2} 2^{n-1} f_0 + n2^n f_1 + 2^n f_2$$

5. A^n is the matrix of d^n in B then
$$A^n = \begin{pmatrix} 2^n & n2^{n-1} & n(n-1)2^{n-2} \\ 0 & 2^n & n2^n \\ 0 & 0 & 2^n \end{pmatrix}$$