Mathimatical Courses

V A P E N A T I O N

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Contents

Chapter 1

Numerical series

1.1 Definitions

1.1.1 Series

Let $(U_n), n \in \mathbb{N}$, be a numerical sequence. We call series associated with (U_n) / of general term (U_n) the sequence

$$\sum_{k=0}^{n} U_n, n \in \mathbb{N},$$

of the partial sums of the U_k terms. The series is usually denoted $\sum U_n$.

1.1.2 Convergence

Let us denote $\sum\limits_{k=0}^{n} U_k$ by S_n. We say that $\sum U_n$ is convergent iff (S_n) has a finite limit when $n \to +\infty$. Otherwise, we say that it is divergent. (when the limit $\pm\infty$ or when there is no limit).

Proposition: Let $\sum U_n$ and $\sum V_n$ be two series.

$$\left.\begin{array}{r} \sum U_n \ convergent \\ \\ \sum V_n \ convergent \end{array}\right\} => \sum U_n + V_n \ convergent$$

$$\left.\begin{array}{r} \sum U_n \ convergent \\ \\ \sum V_n \ divergent \end{array}\right\} => \sum U_n + V_n \ divergent$$

$$\left.\begin{array}{r} \sum U_n \ divergent \\ \\ \sum V_n \ divergent \end{array}\right\} => \sum U_n + V_n \ not\ determined$$

Proof: We use the known results on the sequences of partial sums.

1.1.3 Example: geometric series

Let $q \in \mathbb{R}$. We call geometric series of ratio q the series q^n, whose sequence of partial sums is $\sum_{k=0}^{n} q^k$.

$$\sum_{k=0}^{n} q^k = \begin{cases} n+1 & if\ q = 1 \\ \frac{1-q^{n+1}}{1-q} & if\ q \neq 1 \end{cases}$$

Thus this series is convergent iff $|q| < 1$ $(or\ \ -1 < q < 1)$.

If so, as $q^{n+1} \underset{+\infty}{\longrightarrow} 0$, then $\lim_{n \to +\infty} \sum_{k=0}^{n} q_k = \frac{1}{(1-q)}$.

1.1.4 Sum of a series / remainder

Defintion: If the series $\sum U_n$ is convergent, we call "sum of the series $\sum U_n$" the limit of its sequence of partial sums.

It is denoted by :

$$\sum_{n=0}^{+\infty} U_n = \lim_{n \to +\infty} \sum_{k=0}^{n} U_k$$

Definition: The remainder of a series is the sequence (R_n) defined by the difference between the sum and the partial sum:

$$Rn = \sum_{n=0}^{+\infty} U_n - S_n = \sum_{k=n+1}^{+\infty} U_k$$

A series will be convergent iff its remainder tends towards 0.

1.1.5 Important remark

Changing a finite number of terms in the sequence (U_n) might change the sum of $\sum U_n$ but not its nature. For most of the remainders of the lesson, as long as the hypotheses are true from a certain rank (ie: in the neighborhood of $+\infty$) then the results will hold.

1.1.6 Nth term test

Proposition: $\sum U_n\ convergent \Rightarrow U_n \underset{+\infty}{\longrightarrow} 0$

This is a necessary but not sufficient condition as the reciprocal is false.

Proof: Let $\sum U_n$ be a convergent series of sum S.

Then:

$$\sum_{k=0}^{n} U_k \underset{n \to \infty}{\longrightarrow} S\ ,\ \sum_{k=0}^{n-1} U_k \underset{n \to \infty}{\longrightarrow} S$$

Thus

$$\sum_{k=0}^{n} U_k - \sum_{k=0}^{n-1} U_k \underset{n \to \infty}{\longrightarrow} S - S$$

Hence

$$sU_n \underset{n\to\infty}{\longrightarrow} 0$$

However, $U_n \to 0 \neq \sum U_n$convergent ei: the harmonic series $\sum \frac{1}{n}$
$\frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$ however, $\sum \frac{1}{n}$ is divergent.

Proof: Let us denote $S_n = \sum\limits_{k=1}^{n} \frac{1}{k}$, $S_{2n} - S_n = \sum\limits_{k=n+1}^{2n} \frac{1}{n}$
$= \frac{1}{n+1} + ... + \frac{1}{2n} \geq n * \frac{1}{2n} = \frac{1}{2}$ (a sum is greater than its number of terms times the smallest one)

If the series were convergent, then (S_{2n}) and (S_n) would have the same limit, thus $S_{2n} - S_n$would tend towards 0, which is impossible here as it is always greater than $\frac{1}{2}$. Thus, $\sum \frac{1}{n}$ is divergent.

1.1.7 Telescopic lemma

$\sum U_{n+1} - U_n$ is of the same nature as (U_n)

Proof:

$$\sum_{k=0}^{n} U_{k+1} - U_k = U_{n+1} - U_0$$

The sequence of partial sums associated with $\sum U_{n+1} - U_n$ is $(U_{n+1} - U_0)$, which is of the same nature as (U_n).

1.2 Series of positive terms

$\sum U_n$ with $U_n \geq 0$

1.2.1 Convergence

Proposition: Let $\sum U_n$ be a series of positive terms of partial sums (S_n). Then $\sum U_n$ convergent $\iff$ (S_n) is bounded above.

Proof: $S_{n+1} - S_n = U_{n+1} \geq 0$ So (S_n) is an increasing sequence, it converges iff it is bounded above.

1.2.2 Comparison of positive series

Proposition: Let (U_n) and (V_n) be two positive sequences such that $\forall n \in \mathbb{N}, 0 \leq U_n \leq V_n$.

Then $\sum V_n$ convergent $\Rightarrow \sum U_n$ convergent

Contraposition:

$\sum U_n$ divergent $\Rightarrow \sum V_n$ divergent

Proof: Let us suppose that $\sum V_n$ is convergent, let V bet its sum.

$$\sum_{k=0}^{n} U_k \geq \sum_{k=0}^{n} V_k \geq V$$

thus the sequence of partial sums of $\sum U_n$ is bounded above by V. Hence $\sum U_n$ is convergent.

Reminder: If (X_n) is an increasing sequence that tends towards l, then $\forall n \in \mathbb{N}, X_n \leq l$ Thus here,

$$\sum_{k=0}^{n} V_k \leq V = \sum_{n=0} +\infty V_n$$

Proposition: Let $\sum U_n$ be any series (not necessarily positive) and $\lambda \in \mathbb{R}\{0\}$. Then $\sum U_n$ and $\sum \lambda U_n$ are of the same nature.

Proof:

$$\sum_{k=0}^{n} \lambda U_k = \lambda \sum_{k=0}^{n} U_k$$

Proposition: Let (U_n) and (V_n) be two positive sequences.
1)Domination: If $U_n = O(V_n)$, then :
$\sum V_n$ convergent $\Rightarrow \sum U_n$ convergent
$\sum U_n$ divergent $\Rightarrow \sum V_n$ divergent
2)Negligeability: If $U_n = o(V_n)$, then
$\sum V_n$ convergent $\Rightarrow \sum U_n$ convergent
$\sum U_n$ divergent $\Rightarrow \sum V_n$ divergent
3) If $U_n \approx V_n$, then $\sum U_n$ and $\sum V_n$ are of same nature.

Remark: These results only hold for positive series (or always negative series, or even series that change sigs a finite number of times)

Proof: 1) $U_n = O(V_n)$

$$\exists k \in \mathbb{R}^+, \forall n \in \mathbb{N}, |U_n| \leq |V_n|$$

$$\exists k \in \mathbb{R}^+, \forall n \in \mathbb{N}, |U_n| \leq k|V_n|$$

$$0 \leq U_n \leq kV_n$$

$\sum V_n$ convergent $\Rightarrow \sum kV_n$ convergent$\Rightarrow \sum U_n$ convergent (using the two previous results)
2) $U_n = o(V_n) \Rightarrow U_n = O(V_n)$
3)

$$U_n \approx V_n \Rightarrow \begin{cases} U_n = O(V_n) \\ V_n = O(U_n) \end{cases}$$

Thus using 1):

$$\begin{array}{ll} \sum V_n \ convergent & \Rightarrow \sum U_n \ convergent \\ \sum U_n \ convergent & \Rightarrow \sum V_n \ convergent \\ \sum U_n \ convergent & \Leftrightarrow \sum V_n \ convergent \end{array}$$

$\sum U_n$ and $\sum V_n$ are of the same nature

Reminder: Landau notations

Domination: $U_n = O(V_n)$, $\exists k \in \mathbb{R}^+, |U_n| \leq k|V_n|$, $U_n = K_n V_n$ with (K_n) bounded , $\frac{U_n}{V_n}$ is bounded.

Negligeability: $U_n = o(U_n)$, $U_n = \epsilon n V_n$ with $\epsilon n \underset{n\to+\infty}{\longrightarrow} 0$, $\frac{U_n}{V_n} \underset{n\to+\infty}{\longrightarrow} 0$

Equivalence: $Un \approx Vn$, $U_n = K_n V_n$ with $\underset{n\to+\infty}{\longrightarrow} 1$, $\frac{U_n}{V_n} \underset{n\to+\infty}{\longrightarrow} 1$

Remark: Any convergent sequence is bounded.

1.2.3 Riemann series (p-series / hyperharmonic series)

Definition: We call Riemann series, any series if the form $\sum \frac{1}{n^\alpha}$, where $\alpha \in \mathbb{R}$.

Theorem: $\sum \frac{1}{n^\alpha}\ CV \Leftrightarrow \alpha > 1$

Proof: - 1st Method, comparison with integrals.

$$\int_0^n \frac{dt}{t+1^\alpha} \leq \sum_{k=1}^n \frac{1}{k^\alpha} \leq 1 + \int_1^n \frac{dt}{t^\alpha}$$

$$A\ primitive\ of\ t \to \frac{1}{t^\alpha}\ is \begin{cases} if \alpha \neq 1 & , \quad t \to \frac{t^{1-\alpha}}{1-\alpha} \\ if \alpha = 1 & , \quad t \to ln(t) \end{cases}$$

if $a = 1$

$$\int_1^n \frac{dt}{t^\alpha} = [ln(t)]_1^{n+1} = ln(n+1) \longrightarrow +\infty$$

Thus $\sum \frac{1}{n}$ is DV

if $\alpha > 0, \alpha \neq 1$

$$[\frac{t^{1-\alpha}}{1-\alpha}]_1^{n+1} \leq \sum_{k=1}^n \frac{1}{k^\alpha} \leq 1 + [\frac{t^{1-\alpha}}{1-\alpha}]_1^n$$

$$\frac{(n+1)^{1-\alpha}}{1-\alpha} \leq \sum_{k=1}^n \frac{1}{k^\alpha} \leq 1 + \frac{n^{1-\alpha}}{1-\alpha}$$

if $\alpha < 1$,

$$\frac{(n+1)^{1-\alpha}}{1-\alpha} \longrightarrow +\infty \ so \ \sum \frac{1}{t^\alpha} \ is \ DV$$

if $\alpha > 1$,

$$1 + \frac{n^{1-\alpha} - 1}{1 - \alpha} \longrightarrow 1 + \frac{-1}{1-\alpha} \ thus \sum \frac{1}{n^\alpha} \ is\ CV$$

if $\alpha \leq 0$, $\frac{1}{n^\alpha} \not\longrightarrow 0$, thus using the nth term test $\sum \frac{1}{t^\alpha}$ is DV
Conclusion : $\sum \frac{1}{n^\alpha}\ CV \Leftrightarrow \alpha > 1$

- 2nd Method, Using Cauchy's condensation lemma *not discribed here*

Examples: Using the comparison with a Riemann series, determine the nature of :
1)

$$\sum \frac{cos(n)}{n}^2$$

2)

$$\sum ne^{-n}$$

3)

$$\sum \frac{1}{n|cos(n)|}$$

Solutions: 1)

$$\frac{cos(n)}{n}^2 = \frac{cos^2(n)}{n^2}$$

$$0 \leq cos^2(n) \leq 1$$

$$0 \leq \frac{cos^2(n)}{n^2} \leq 1$$

$\sum \frac{1}{n^2}$ is a convergent Riemann series
By comparison of positive series, $\sum \frac{cos(n)}{n}^2$ converges.

2) By compared growth

$$n^3 e^{-n} \underset{+\infty}{\longrightarrow} 0$$

thus

$$ne^{-n} = o(\frac{1}{n^2})$$

As $\sum \frac{1}{n^2}$ is CV, by comparison of positive series, $\sum ne^{-n}$ converges as well.
3)

$$0 \leq |cos(n)| \leq 1$$

$$0 \leq n|cos(n)| \leq 1$$

$$\frac{1}{n} \leq \frac{1}{n|cos|}$$

thus it is divergent

Riemann's rule U_n positive, let $\alpha \in \mathbb{R}, l \in \mathbb{R}^+ \cup \{+\infty\}$
we suppose that $n^\alpha U_n \longrightarrow l$ then:
if $\alpha > 1$ and $l \neq +\infty, \sum U_n\ CV$
if $\alpha \leq 1$ and $l \neq 0, \sum U_n\ DV$
The resusts are obtained by comparison between U_n and $\frac{1}{n^\alpha}$.

1.2.4 Betrand series

They are another kind of series of reference.

We call Betrand series, any series of the form

$$\sum \frac{1}{n^\alpha ln^\beta(n)}$$

Theorem:

$$\sum \frac{1}{n^\alpha ln^\beta(n)} CV \Leftrightarrow \begin{cases} \alpha > 1 \\ \quad or \\ \alpha = 1 \quad and \quad \beta > 1 \end{cases}$$

Proof: First Method : By comparison with Riemanns series
If $\alpha = 1$

$$\frac{1}{n^\alpha ln^\beta(n)} = o(\frac{1}{n^{\frac{1+\alpha}{2}}})$$

and $\sum \frac{1}{n^{\frac{1+\alpha}{2}}}$ CV
If $\alpha < 1$

$$\frac{1}{n} = 0(\frac{1}{n^\alpha ln^\beta(n)})$$

and

$$\sum \frac{1}{n}\ DV \ \Rightarrow \sum \frac{1}{n^\alpha ln^\beta(n)}\ DV$$

If $\beta > 0$

$$n^\alpha = o(n^\alpha ln^\beta(n))$$

If $\beta = 0$, we get

$$\frac{1}{n^\alpha}$$

If $\beta < 0$

$$\frac{ln^{-\beta}(n)}{n^\alpha} \neq \frac{ln^{-\beta}(n)}{n^{\frac{\alpha-1}{2}}}$$

We thus can use the condensation test $\sum \frac{1}{n^\alpha ln^\beta(n)}$ is of the same nature as $\sum \frac{2^n}{(2^n)^\alpha Ln^\beta(2n)}$

Let $\alpha = 1$, we get

$$\sum \frac{1}{ln^\beta(2n)} = \sum \frac{1}{(nln(2))^\beta} = \frac{1}{ln^\beta(2)} = \frac{1}{n^\beta}$$

which is a Riemann serie that converges iff $\beta > 1$

Conclusion:
$$\begin{cases} if\ \alpha > 1 & CV \\ if\ \alpha = 1 & CV \quad iff\ \beta > 1 \\ if\ \alpha < 1 & DV \end{cases}$$

Second Method : comparison with integrals *not described here*

1.2.5 Ratio test/ Root test

Ratio test (Alembert's test) Let (U_n) be a strictly positive sequence, so that
$$\frac{U_{n+1}}{U_n} \Longrightarrow l \in \mathbb{R}^+ \cup \{+\infty\}$$
then
$$\begin{cases} if & l < 1 & \sum U_n\ CV \\ if & l > 1 & \sum U_n\ DV \end{cases}$$
We cannot conclude anything if the limit is 1

Root test (Cauchy's test) Let (U_n) be a positive sequence, so that
$$(U_n)^{\frac{1}{n}} \Longrightarrow l \in \mathbb{R}^+ \cup \{+\infty\}$$
then
$$\begin{cases} if & l < 1 & \sum U_n\ CV \\ if & l > 1 & \sum U_n\ DV \end{cases}$$
We cannot conclude anything if the limit is 1

These two tests are called logarithmic comparison tests (or limit comparison tests) as they hide the comparision with a geometric sequence.

Logarithimic comparison test Let (U_n) and (V_n) be two stricly positive sequences.

We supose that
$$\forall n \in \mathbb{N}, \frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n}$$
Then,
$$\sum V_n\ CV \Rightarrow \sum U_n\ CV$$
$$\sum V_n\ DV \Rightarrow \sum U_n\ DV$$

Proof:
$$0 \leq \frac{U_n}{U_{n-1}} \leq \frac{V_n}{V-n-1}$$
...
$$0 \leq \frac{U_1}{U_0} \leq \frac{V_1}{V_0}$$
By multiplying these inequalities we get a telescopic product:

$$\frac{U_n}{U_0} \leq \frac{V_n}{V_0}$$

$$U_n \leq \frac{U_0}{V_0} V_n$$

$$U_n = O(V_n)$$

Hence the conclusion.

Proof : D'Alembert's test Let us have $\frac{U_{n+1}}{U_n} \longrightarrow l$
if $l < 1$: Let λ be such that $l < \lambda < 1$ we use the logarithmic comparison test between (U_n) and the geometric sequence (λ^n)

$$\frac{U_{n+1}}{U_n} \leq \frac{\lambda^{n+1}}{\lambda^n} = \lambda \; (after\ a\ certain\ rank)$$

Let $|\lambda| < 1$ thus $\sum \lambda^n\ CV$ by comparison $\sum U_n\ CV$. If $l > 1$, after a certain rank $\frac{U_{n+1}}{U_n} > 1$ thus (U_n) is increasing stricly; as it is positive and stricly increasing it cannot tend towards 0. If the limit is 1, the comparison with the geometric sequences is not accurate enough to determine the convergence. That is the case of all the Riemann and Bertrand series.