Mathematics
Sup Course

Contents

Chapter 1

Revisions and complements on complex numbers

1.1 Definitions

Definition 1
We call complex number any number of type $a+ib$ where $(a,b)\in\mathbb{R}^2$ and $i^2=-1$. The set of complex numbers is denoted $\mathbb{C}$.

If $z=a+ib\in\mathbb{C}$, a is called real part of z (denoted $\mathrm{Re}(z)$) and b imaginary part of z (denoted $\mathrm{Im}(z)$).

Remarks

1. The rules on operations are the same as for $\mathbb{R}$ with the supplementary condition $i^2=-1$.

   For example if $z_1=1+2i$ and $z_2=4-3i$ then $z_1+z_2=5-i$ and $z_1z_2=10+5i$.

2. $z_1=z_2 \Longleftrightarrow \mathrm{Re}(z_1)=\mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1)=\mathrm{Im}(z_2)$.

   In particular $a+ib=0 \Longleftrightarrow a=0$ and $b=0$.

Definition 2
Let $z=a+ib\in\mathbb{C}$. We call conjugate of z the complex number denoted $\overline{z}$ defined by $\overline{z}=a-ib$.

Proposition 1
Let $(z,z')\in\mathbb{C}^2$. Then

1. $\mathrm{Re}(z)=\dfrac{z+\overline{z}}{2}$ and $\mathrm{Im}(z)=\dfrac{z+\overline{z}}{2i}$
2. $z\in\mathbb{R} \Longleftrightarrow z=\overline{z}$ and $z\in i\mathbb{R} \Longleftrightarrow \overline{z}=-z$
3. $\overline{z+z'}=\overline{z}+\overline{z'}$
4. $\overline{zz'}=\overline{z}\,\overline{z'}$
5. If $z\neq 0$, the conjugate of $\dfrac{z'}{z}$ is $\dfrac{\overline{z'}}{\overline{z}}$

1.2 Trigonometric and exponential form

Let $(O, \overrightarrow{u}, \overrightarrow{v})$ be an orthonormal space.

For each complex $z = a + ib$, we associate the point M of coordinates (a, b) in $(O, \overrightarrow{u}, \overrightarrow{v})$.

OM is called the modulus of z and is denoted $|z|$.

A measure of the angle $\theta = (\overrightarrow{u}, \overrightarrow{OM})$ is called an argument of z denoted $\mathrm{Arg}(z)$. It is defined up to 2π.
We write $\mathrm{Arg}(z) \equiv \theta\,[2\pi]$.

Proposition 2
Let $(z, z') \in \mathbb{C}^2$. Then

1. $|z|^2 = z\overline{z}$
2. $|z| = 0 \iff z = 0$
3. $\big|\mathrm{Re}(z)\big| \leqslant |z| \quad \text{and} \quad \big|\mathrm{Im}(z)\big| \leqslant |z|$
4. $|zz'| = |z||z'|$
5. if $z' \neq 0$, $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|}$

Notation

Let $\theta \in \mathbb{R}$. We denote $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
We have in particular $\big(e^{i\theta}\big)^n = e^{in\theta}$ so that $\big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta)$.
Similarly $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Proposition 3
Any complex number z can be written as

$$z = |z|\big(\cos(\theta) + i\sin(\theta)\big)$$

Remark
If $z' \neq 0$, $\mathrm{Arg}\left(\dfrac{z}{z'}\right) = \mathrm{Arg}(z) - \mathrm{Arg}(\text{z'})$.

1.3 Quadratic equations with complex coefficients

1.4 Square roots of a complex number

We look for a square root of $u + iv \in \mathbb{C}$. Therefore, we look for $z = a + ib$ such that $z^2 = u + iv$, that is $\begin{cases} a^2 - b^2 = u \\ a^2 + b^2 = \sqrt{u^2 + v^2} \\ 2ab = v \end{cases}$

The third equation allows to know whether a and b are of same sign or of opposite sign and the first two equations allow to determine a and b.

1.5 Solving quadratic equations with complex coefficients

Let $az^2 + bz + c = 0$ where $(a, b, c) \in \mathbb{C}^3$ and $a \neq 0$.

Let $\Delta = b^2 - 4ac$ and δ be a complex root of Δ. Then the roots of the equation are $\dfrac{-b \pm \delta}{2a}$

Example

We solve in $\mathbb{C}$ the equation $z^2 + z + 1 - i = 0$.

$\Delta = 1 - 4(1 - i) = -3 + 4i$. We determine a root of Δ. We look for $z = a + ib$ such that $z^2 = -3 + 4i$.

Hence, $\begin{cases} a^2 - b^2 = -3 \\ a^2 + b^2 = \sqrt{(-3)^2 + 4^2} \\ 2ab = 4 \end{cases}$ Let $\begin{cases} a^2 - b^2 = -3 \\ a^2 + b^2 = 5 \\ ab > 0 \end{cases}$

Hence, $z = 1 + 2i$ is a square root of $-3 + 4i$.

Then $z = \dfrac{1}{2}(-1 + 1 + 2i)$ or $z = \dfrac{1}{2}(-1 - 1 - 2i)$ that is $z = i$ or $z = -1 - i$.

1.6 n^{th} roots

We look for the n n^{th} roots of $re^{i\phi}$.

Therefore, we look for $z = \rho e^{i\theta}$ such that $z^n = re^{i\phi}$, that is $\rho^n = r$ and $n\theta \equiv \phi[2\pi]$.

Hence, the n n^{th} roots of $re^{i\phi}$ are the $\sqrt[n]{r}e^{i(\phi/n + 2k\pi/n)}$ for $k \in \{0, 1, 2, \cdots, n-1\}$.

Chapter 2

Revisions and complements on integration

2.1 Preliminaries

2.1.1 Composed function

Let I and J be two intervals of $\mathbb{R}$.

Definition 3
Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be such that, for all $x \in I$, $f(x) \in J$ (i.e $f(I) \subset J$). We define $g \circ f : I \to \mathbb{R}$ by

$$g \circ f(x) = g\left(f(x)\right)$$

Example

Let $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x - 1 \end{cases}$ and g: $\begin{cases} [1; +\infty[\to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$

We look for $x \in \mathbb{R}$ such that $f(x) \in [0; +\infty[$. We find $x \in [1; +\infty[$. Thus, $g \circ f$ is defined on $[1; +\infty[$ by

$$g \circ f(x) = \sqrt{x - 1}$$

2.1.2 Reciprocal function

Let I and J be two intervals of $\mathbb{R}$.

Definition 4
Let $f : I \to J$.

1. We say that f is surjective from I to J if

$$\forall\, y \in J\ \exists x \in I\ y = f(x)$$

2. We say that f is injective from I to J if

$$\forall (x, x') \in I^2, \quad f(x) = f(x') \ \Rightarrow\ x = x'$$

Remarks

1. f surjective from I to J means that the equation $y = f(x)$, with unknown variable $x \in I$, admits at least one solution.

2. f injective from I to J means that the equation $y = f(x)$, with unknown variable $x \in I$ admits at most one solution.

Examples

1. f: $\left\{ \begin{array}{l} \mathbb{N} \to \mathbb{R} \\ x \mapsto x^2 \end{array} \right.$ is not surjective but injective.

2. g: $\left\{ \begin{array}{l} [-\frac{\pi}{2}; \frac{\pi}{2}] \to [-1; 1] \\ x \mapsto \cos(x) \end{array} \right.$ is surjective but not injective.

Definition 5

Let $f : I \to J$. We say that f is bijective from I to J if f is surjective and injective from I to J. This is equivalent to say that

$$\forall\, y \in J \ \exists\,!\, x \in I \ y = f(x)$$

Examples

1. $x \mapsto \ln x$ is bijective from $]0; +\infty[$ to $\mathbb{R}$.

2. $x \mapsto e^x$ is bijective from $\mathbb{R}$ to $]0; +\infty[$.

3. $x \mapsto \cos x$ is bijective from $[0; \pi]$ to $[-1; 1]$.

4. $x \mapsto \sin x$ is bijective from $[-\frac{\pi}{2}; \frac{\pi}{2}]$ to $[-1; 1]$.

5. $x \mapsto \tan x$ is bijective from $]-\frac{\pi}{2}; \frac{\pi}{2}[$ to $\mathbb{R}$.

Proposition 4

Let $f : I \to J$. Then,
f is bijective from I to J if and only if there exists a unique function $g : J \to I$ such that

$$f \circ g = Id_J \quad \text{and} \quad g \circ f = Id_I$$

If g exists then g is unique. We denote $g = f^{-1}$.
Hence, if f is bijective from I to J, $f^{-1} : J \to I$ verifies

$$\left\{ \begin{array}{l} x = f^{-1}(y) \\ y \in J \end{array} \right. \iff \left\{ \begin{array}{l} y = f(x) \\ x \in I \end{array} \right.$$

Example 1

For all $x \in \mathbb{R}$, $\ln(e^x) = x$ and for all $x \in]0; +\infty[$, $e^{\ln x} = x$. Hence, the logarithm and exponential functions are reciprocal.

Example 2

The function tan: $\left\{\begin{array}{l}]-\frac{\pi}{2};\frac{\pi}{2}[\to\mathbb{R} \\ x\mapsto\tan(x)\end{array}\right.$ is bijective. We denote $\tan^{-1}=\arctan$ its reciprocal bijection.

Thus, arctan: $\left\{\begin{array}{l} \mathbb{R}\to]-\frac{\pi}{2};\frac{\pi}{2}[\\ x\mapsto\arctan(x)\end{array}\right.$ verifies

$$\left\{\begin{array}{l} x=\arctan(y) \\ y\in\mathbb{R}\end{array}\right. \iff \left\{\begin{array}{l} y=\tan x \\ x\in]-\frac{\pi}{2};\frac{\pi}{2}[\end{array}\right.$$

From this, we deduce, for example that $\arctan(0)=0$, $\arctan(1)=\frac{\pi}{4}$ and $\arctan\sqrt{3}=\frac{\pi}{3}$.

2.1.3 Operations on derivatives

We recall the following results:

Proposition 5

1. Let f, g be two differentiable functions on I and $\lambda\in\mathbb{R}$. Then
   a. $(f+g)'=f'+g'$
   b. $(\lambda f)'=\lambda f'$
   c. $(fg)'=f'g+fg'$
   d. If g does not nullify on I, $\left(\dfrac{f}{g}\right)'=\dfrac{f'g-fg'}{g^2}$

2. Let $f:I\to J\subset\mathbb{R}$ and $g:J\to\mathbb{R}$ be respectively differentiable on I and J. Then
$$(g\circ f)'=(g'\circ f).f'$$

Remark

Part 2. of the above proposition means that for all $x\in I$,

$$(g\circ f)'(x)=(g'\circ f)(x)\times f'(x)$$

that is

$$(g\circ f)'(x)=g'\big(f(x)\big)\times f'(x)$$

Example

Let $f:x\mapsto\sin\big(\ln(x^2+1)\big)$. Then for all $x\in\mathbb{R}$

$$f'(x)=\cos\big(\ln(x^2+1)\big)\times\frac{1}{x^2+1}\times 2x$$

Proposition 6

Let f be a differentiable function at $x_0\in I$ such that $f'(x_0)\neq 0$. Then, f^{-1} is differentiable at $y_0=f(x_0)$ and

$$\left(f^{-1}\right)'(y_0)=\frac{1}{f'\big(f^{-1}(y_0)\big)}$$

Application

$x \mapsto \arctan x$ is differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$(\arctan x)' = \frac{1}{1+x^2}$$

2.2 Primitive of a continuous function

In the rest of this course, I is an interval of $\mathbb{R}$ and all the functions are real-valued.

2.2.1 definition

Definition 6

Let f be a continuous function on I. We call primitive of f on I any function F of I to $\mathbb{R}$, differentiable on I such that $F' = f$. We then write for all $t \in I$,

$$F(t) = \int f(t)\,\mathrm{d}t$$

Observation

Do not mix up the concept of primitive and the concept of integral (studied below). We note that there is no boundary in the notation of the above definition.

Example

Let $f : \begin{cases} \mathbb{R}_*^+ \to \mathbb{R} \\ t \longmapsto \dfrac{1}{t} \end{cases}$

then $F : t \mapsto \ln(t)$ is a primitive of f on $\mathbb{R}_*^+$ as $F' = f$ i.e. for all $t \in \mathbb{R}_*^+$, $F'(t) = f(t)$. One can also write that for all $t \in \mathbb{R}_*^+$,

$$\ln(t) = \int \frac{1}{t}\,\mathrm{d}t$$

2.2.2 Properties

Proposition 7

Let f be a continuous function on I and F a primitive of f on I. Then any primitive of f on I is under the form $F + \lambda$ where $\lambda \in \mathbb{R}$.

Example

Using the previous example, a primitive of f on $\mathbb{R}_*^+$ is $t \mapsto \ln(t)$ and the primitives of f on $\mathbb{R}_*^+$ are the functions $t \mapsto \ln(t) + \lambda$ where $\lambda \in \mathbb{R}$.

Classical primitives

We recall the primitives (up to a constant) of the following elementary functions :

1. For all $\alpha \in \mathbb{R} - \{-1\}$, $\displaystyle\int t^\alpha \, \mathrm{d}t = \frac{1}{\alpha+1}\, t^{\alpha+1}$

   and $\displaystyle\int t^{-1} \, \mathrm{d}t = \ln(t)$

2. $\displaystyle\int e^t \, \mathrm{d}t = e^t$

3. $\displaystyle\int \sin(t) \, \mathrm{d}t = -\cos(t)$

4. $\displaystyle\int \cos(t) \, \mathrm{d}t = \sin(t)$

5. $\displaystyle\int \frac{1}{1+t^2} \, \mathrm{d}t = \arctan(t).$

2.2.3 Integral of a continuous function

Definition 7

Let f be a continuous function on I and F a primitive of f on I. We call integral of f between a and b, denoted $\displaystyle\int_a^b f(t)\, \mathrm{d}t$, the real number defined by

$$\int_a^b f(t)\, \mathrm{d}t = F(b) - F(a)$$

Observations

1. We sometimes denote $F(b) - F(a)$ as $[F(t)]_a^b$.

2. We also recall that the integration variable is « mute », which means that

$$\int_a^b f(t)\, \mathrm{d}t = \int_a^b f(x)\, \mathrm{d}x = \int_a^b f(u)\, \mathrm{d}u$$

Example

We compute $\int_0^1 t^2\,\mathrm{d}t$. A primitive of $t \mapsto t^2$ is $t \mapsto \frac{t^3}{3}$. So,

$$\begin{aligned}\int_0^1 t^2\,\mathrm{d}t &= \left[\frac{t^3}{3}\right]_0^1 \\ &= \frac{1}{3}\end{aligned}$$

Properties 1

Let f and g be continuous on $[a,b]$ with $a < b$ and $\lambda \in \mathbb{R}$. Then

1. $\int_a^b (f+\lambda g)(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \lambda \int_a^b g(t)\,\mathrm{d}t$
2. For all $c \in [a,b]$, $\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$
3. $f \geqslant 0 \Rightarrow \int_a^b f(t)\,\mathrm{d}t \geqslant 0$
4. $f \leqslant g \Rightarrow \int_a^b f(t)\,\mathrm{d}t \leqslant \int_a^b g(t)\,\mathrm{d}t$
5. $\left|\int_a^b f(t)\,\mathrm{d}t\right| \leqslant \int_a^b |f(t)|\,\mathrm{d}t$
6. Let $a \in \mathbb{R}$.

   If f is even, $\int_{-a}^a f(t)\,\mathrm{d}t = 2\int_0^a f(t)\,\mathrm{d}t$

   If f is odd, $\int_{-a}^a f(t)\,\mathrm{d}t = 0$

2.2.4 Geometric interpretation

Definition 8

In the plan $(0, \vec{i}, \vec{j})$, we call area unit, the area of the rectangle defined by $\vec{i}$ and $\vec{j}$.

Proposition 8

Let f be continuous and positive on $[a,b]$ with $a \neq b$. Then $\int_a^b f(t)\,\mathrm{d}t$ is the area, in area unit, of the part of the plan bounded by the axis $0x$, the graph of f and the lines of equations $x = a$ and $x = b$.

2.3 Computational methods for primitives or integrals

2.3.1 Integration by parts

Proposition 9 (Integration by parts)
Let f and g be two functions of class C^1 on $[a,b]$ (i.e. f and g differentiable on I and their derivative is continuous on $[a,b]$). Then

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = \big[f(t)g(t)\big]_a^b - \int_a^b f'(t)g(t)\,\mathrm{d}t$$

Observation

The assumption « of class C^1 » is here only to say that f' are g' are continuous on $[a,b]$ so that it is possible to consider the integral from a to b of $f'g$ and fg'.

Example 1

We determine $I = \displaystyle\int_0^1 te^t\,\mathrm{d}t$.

We set $f(t) = t \Rightarrow f'(t) = 1$ and $g'(t) = e^t \Rightarrow g(t) = e^t$. We then have

$$\begin{aligned}
I &= \int_0^1 f(t)g'(t)\,\mathrm{d}t \\
&= \big[f(t)g(t)\big]_0^1 - \int_0^1 f'(t)g(t)\,\mathrm{d}t \\
&= \big[te^t\big]_0^1 - \int_0^1 e^t\,\mathrm{d}t \\
&= e - \big[e^t\big]_0^1 \\
&= e - (e-1) \\
&= 1
\end{aligned}$$

Example 2

We determine $I = \displaystyle\int_0^{\frac{\pi}{4}} t\cos(2t)\,\mathrm{d}t$.

We set $f(t) = t \Rightarrow f'(t) = 1$ and $g'(t) = \cos(2t) \Rightarrow g(t) = \frac{1}{2}\sin(2t)$. We then have

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} f(t)g'(t)\,\mathrm{d}t \\
&= \big[f(t)g(t)\big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} f'(t)g(t)\,\mathrm{d}t \\
&= \left[\frac{t}{2}\sin(2t)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(2t)\,\mathrm{d}t \\
&= \frac{\pi}{8} - \left[-\frac{1}{4}\cos(2t)\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8} - \frac{1}{4}
\end{aligned}$$

2.3.2 Integration by substitution

The following proposition is not to be memorized « by heart » but you have to know how to use it.

Proposition 10
let I and J be two intervals of $\mathbb{R}$, $(\alpha, \beta) \in J^2$, f continuous on I and φ of class C^1 from J to I. Then

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t = \int_\alpha^\beta f\big(\varphi(u)\big)\varphi'(u)\,\mathrm{d}u$$

Remark

When you will have to use an integration by substitution, the change of variable will always be told to you. Three steps are necessary to do a change of variable :

- Determine the new « $\mathrm{d}t$ » if t is the new variable.
- Change the integration bounds.
- Make explicit the function of the old variable by a function of the new variable.

Example

We compute $I = \displaystyle\int_e^3 \frac{1}{x(\ln(x))^3}\,\mathrm{d}x$ using the change of variable $t = \ln(x)$.

We then have $x = e^t$.

The derivative of x with respect to t is e^t, which we write under the « physician » form $\dfrac{\mathrm{d}x}{\mathrm{d}t} = e^t$.

Hence $\mathrm{d}x = e^t\,\mathrm{d}t$.

We now change the boundaries : When x is equal to e, then $t\,(=\ln(x))$ is equal to $\ln(e)$ i.e. 1. When x is equal to 3, $t\,(=\ln(x))$ is equal to $\ln(3)$.

Finally $\dfrac{1}{x(\ln(x))^3} = \dfrac{1}{e^t\,t^3}$

$$\begin{aligned} \text{Then}\quad I &= \int_1^{\ln 3} \frac{1}{e^t\,t^3}\,e^t\,\mathrm{d}t \\ &= \int_1^{\ln 3} \frac{1}{t^3}\,\mathrm{d}t \\ &= \left[-\frac{1}{2t^2}\right]_1^{\ln 3} \\ &= -\frac{1}{2(\ln 3)^2} + \frac{1}{2} \end{aligned}$$

Chapter 3

Functions of a real variable

3.1 Definitions

Until today, you have worked a lot with functions from $\mathbb{R}$ to $\mathbb{R}$. But do you know the definition of a function?

3.1.1 Cartesian product

Definition 9
Let E and F be two sets. We call cartesian product of E by F denoted $E \times F$ the set of couples (x, y) with $x \in E$ and $y \in F$ i.e.

$$E \times F = \{(x, y);\, x \in E,\, y \in F\}$$

Example

$u \in \mathbb{N}^2 \times \mathbb{R}$ means that $u = ((n, p)\,,\, x)$ where $(n, p) \in \mathbb{N}^2$ and $x \in \mathbb{R}$ i.e. $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $x \in \mathbb{R}$.

3.1.2 Graph

Definition 10
Let E and F be two sets. We call graph of E to F any part of $E \times F$.

Example

If $E = F = \mathbb{R}$, a graph of E to F is any given part of the plan, for example a circle, a triangle or a line.

3.1.3 Function

In all the course on functions, we will make no distinction between the words «functions» and «applications».

Definition 11
We call function (defined) of E to F, any triplet $f = (E, F, \Gamma)$ where Γ is a graph from E to F such that for all $x \in E$, there exists a unique $y \in F$ where $(x, y) \in \Gamma$.

Remarks

1. If f is a function from E to F, E is called the starting domain (or domain of definition or domain of source) of f, F is called co-domain of f.
   A function f from E to F will be denoted in the usual way as $f \in F^E$ or $f : E \to F$ or $f : \begin{cases} E \to F \\ x \mapsto f(x) \end{cases}$ and the graph Γ of f will then be the set of $(x, f(x))$ for x covering E i.e. the graph of f models what you used to call « representative curve » of f

2. If f is a function (defined) from E to F, the domain of definition of f, $\mathscr{D}_f$, is equal to E. This is the reason why E is also called domain of definition of f.

3. in particular f function from $\mathbb{R}$ to $\mathbb{R}$ means that any vertical line (i.e. parallel to the y-axis) crosses the graph of f at exactly one point.

For the rest of the course, any function $f = (E, F, \Gamma)$ will be denoted $f \in F^E$ or f: $E \to F$. The two notations will be used to get you used to them.

3.2 Concepts of limits

In the rest of this chapter, all the functions will be defined on a part I of $\mathbb{R}$ i.e. $f : I \subset \mathbb{R} \to \mathbb{R}$. i.e. f is defined at $a \in \mathbb{R}$ means that $a \in I$.

3.2.1 Neighborhood of a real number

Definition 12
Let $a \in \mathbb{R}$. We call neighborhood of a any interval of type $]a - h, a + h[$ where $h > 0$.

Remark

A neighborhood of $a \in \mathbb{R}$ simply is an open interval centered in a.

3.2.2 Function defined in a neighborhood of a real number or the infinity

Definition 13
We say that f is defined in the neighborhood of $a \in \mathbb{R}$ if for all $h > 0$, $]a - h, a + h[$ meets I i.e. if

$$\forall h > 0,\ ]a - h, a + h[\cap I \neq \emptyset$$

We say that f is defined in the neighborhood of $+\infty$ (resp. $-\infty$) if for all $A \in \mathbb{R}$, $]A, +\infty[$ meets I (resp. $]-\infty, A[$ meets I) i.e. if

$$\forall A \in \mathbb{R},\ ]A, +\infty[\cap I \neq \emptyset$$

$$(\text{resp. } \forall A \in \mathbb{R},\ ]-\infty, A[\cap I \neq \emptyset)$$

Examples

1. $f : \begin{cases} \mathbb{R}^+ \to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$ is defined in the neighborhood of 0. Indeed any open interval (even very small) centered at 0 meets $\mathbb{R}^+$. More precisely for all $h > 0$, we have

$$]-h, h[\cap \mathbb{R}^+ = [0, h[$$

so

$$]-h, h[\cap \mathbb{R}^+ \neq \emptyset$$

2. $g : \begin{cases} [1, +\infty[\to \mathbb{R} \\ x \mapsto \sqrt{x-1} \end{cases}$ is not defined in the neighborhood of 0 as for example

$$\left]-\frac{1}{2}, \frac{1}{2}\right[\cap [1, +\infty[= \emptyset$$

3. $h : \begin{cases} \mathbb{R}^+ \to \mathbb{R} \\ x \mapsto \sqrt{x} \end{cases}$ is defined in the neighborhood of $+\infty$ because any interval $]A, +\infty[$ meets $\mathbb{R}^+$.

Indeed for all $A \in \mathbb{R}$,

$$]A, +\infty[\cap \mathbb{R}^+ = \begin{cases}]A, +\infty[\text{ if } A \geqslant 0 \\ \mathbb{R}^+ \text{ if } A < 0 \end{cases}$$

so we have for all $A \in \mathbb{R}$

$$]A, +\infty[\cap \mathbb{R}^+ \neq \emptyset$$

3.2.3 Finite limit of a function at a point

Definition 14

f has a limit $l \in \mathbb{R}$ at $a \in \mathbb{R}$ if f is defined in the neighborhood of a and

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in I, |x - a| < \eta \Rightarrow |f(x) - l| < \varepsilon$$

Remark

Saying that f has a limit l at $a \in \mathbb{R}$ simply means that the gap between $f(x)$ and l is as small as we want, provided that x is sufficiently close to a.

Proposition 11

If f has a limit $l \in \mathbb{R}$ at $a \in \mathbb{R}$, then l is unique and we note

$$l = \lim_{x \to a} f(x) \quad \text{or} \quad l = \lim_a f$$

Example

Let $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$. We prove that $\lim\limits_{x \to 0} x^2 = 0$. This result is natural but we have to prove it here using quantifiers.

Let $\varepsilon > 0$. We look for $\eta > 0$ such that for all $x \in \mathbb{R}$, $|x - 0| < \eta \Rightarrow |x^2 - 0| < \varepsilon$ i.e.

$$|x| < \eta \Rightarrow x^2 < \varepsilon$$

It is sufficient to choose $\eta = \sqrt{\varepsilon}$. Indeed

$$|x| < \sqrt{\varepsilon} \Rightarrow x^2 < \varepsilon$$

Remarks

1. If f is defined at $a \in \mathbb{R}$ (and not defined only *in a neighborhood* of a) and f admits a limit $l \in \mathbb{R}$ at a then $l = f(a)$.

2. However, the definition of limit still keeps sense even if f is not defined at a but only defined in the neighborhood of a as the following example shows it.

   Let $f : \begin{cases} \mathbb{R} - \{1\} \to \mathbb{R} \\ x \mapsto \dfrac{x^3 - 1}{x - 1} \end{cases}$.

   then f is defined in the neighborhood of 1 (but not defined at 1). The limit of f at 1 is nevertheless computable. We have

   $$\lim_{x \to 1} f(x) = 3$$

   Indeed,

   $$f(x) = \frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1.$$

   Hence

   $$\lim_{x \to 1} f(x) = 1 + 1 + 1 = 3.$$

Proposition 12
If f has a finite limit at a then f is bounded in the neighborhood of a.

Proposition 13
If $\lim\limits_a f = l$ and $\lim\limits_a g = m$ then $\lim\limits_a (\lambda f + \mu g) = \lambda l + \mu m$ and $\lim\limits_a fg = lm$ where λ and μ are two real numbers.

Proposition 14
If $\lim\limits_a f = l \neq 0$, then f does not nullify in the neighborhood of a and $\lim\limits_a \dfrac{1}{f} = \dfrac{1}{l}$.

Proposition 15
let I and J be two intervals of $\mathbb{R}$, a, b and l be three real numbers and $f : I \longrightarrow J$ and $g : J \longrightarrow \mathbb{R}$ be such that $\lim\limits_a f = b$ and $\lim\limits_b g = l$. Then $\lim\limits_a (g \circ f) = l$.

3.2.4 Other types of limit

Definition 15

1. We say that f admits a limit $l \in \mathbb{R}$ at $+\infty$ (and we denote $\lim\limits_{x \to +\infty} f(x) = l$) if f is defined in the neighborhood of $+\infty$ and

$$\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in I,\ x > A \Rightarrow |f(x) - l| < \varepsilon$$

2. We say that f tends to $+\infty$ at $a \in \mathbb{R}$ (and we denote $f(x) \xrightarrow[x \to a]{} +\infty$) if f is defined in the neighborhood of a and

$$\forall A \in \mathbb{R},\ \exists \eta > 0,\ \forall x \in I,\ |x - a| < \eta \Rightarrow f(x) > A$$

3. We say that f tends to $+\infty$ at $+\infty$ (and we denote $f(x) \xrightarrow[x \to +\infty]{} +\infty$) if f is defined in the neighborhood of $+\infty$ and

$$\forall A \in \mathbb{R},\ \exists B \in \mathbb{R},\ \forall x \in I,\ x > B \Rightarrow f(x) > A$$

Example

We prove that $x^3 - 1 \xrightarrow[x \to +\infty]{} +\infty$.

Let $A \in \mathbb{R}$. We look for $B \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x > B \Rightarrow x^3 - 1 > A$. Since $x^3 - 1 > A \Leftrightarrow x > \sqrt[3]{A+1}$, it is sufficient to choose $B = \sqrt[3]{A+1}$. We then have, for all $x \in \mathbb{R}$,

$$x > B = \sqrt[3]{A+1} \Rightarrow x^3 - 1 > A$$

3.3 Continuity

Until today, your definition of continuity of a function f was maybe like : «f is continuous if its graph can be drawn without lifting your pencil from the paper ». One of the goals of this paragraph is to define the continuity of a function f on an interval using quantifiers.

Definition 16
We say that f is continuous at $a \in I$ if

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in I,\ |x - a| < \eta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

We say that f is continuous on I if f is continuous at every point of I.

Remark

We note that f is continuous on I simply means that for all $a \in I$, $\lim\limits_{x \to a} f(x) = f(a)$.

3.3.1 Intermediate value theorem

One of the key theorems on continuity is the intermediate value theorem.

Theorem 1 (Intermediate value theorem)
Let f be continuous on an interval I of $\mathbb{R}$ and $(a,b) \in I^2$. If $f(a)f(b) < 0$ then there exists (at least one) $c \in]a,b[$ such that $f(c) = 0$.

Remark

The assumption $f(a)f(b) < 0$ simply means that $f(a)$ and $f(b)$ are of opposite sign.

Example

We prove that the equation $x^2\cos(x) + x\sin(x) + 1 = 0$ admits at least one solution $x \in \mathbb{R}$.
Let $f : x \mapsto x^2\cos(x) + x\sin(x) + 1$. Then f is continuous on $\mathbb{R}$, $f(0) = 1 > 0$ and $f(\pi) = 1 - \pi^2 < 0$. Using the intermediate value theorem, there exists at least one $x \in]0,\pi[$ such that $f(x) = 0$ i.e. such that $x^2\cos(x) + x\sin(x) + 1 = 0$.

3.3.2 Image of an interval by a continuous function

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $A \subset I$. We recall that the image of f by A, denoted $f(A)$ is defined by

$$f(A) = \{f(x); x \in A\}$$

Hence $y \in f(A) \Leftrightarrow$ there exists $x \in A$ such that $y = f(x)$.

Example : We take $f : x \mapsto x^2$. Then $f([-1,2]) = [0,4]$

Proposition 16
The image of an interval by a continuous function is an interval.

3.3.3 Image of a segment by a continuous function

Proposition 17
The image of a segment $[a,b]$ by a continuous function is a segment.

Remark

The assumption «segment» is fundamental, as the following counter-example shows it:

$f : \begin{cases}]0,1] \to \mathbb{R} \\ x \mapsto \dfrac{1}{x} \end{cases}$. Then $f(]0,1]) = [1,+\infty[$. But $]0,1]$ is not a segment !

Corollary 1
Let f be a continuous function on a segment $[a,b]$. Then

$$f([a,b]) = [m,M]$$

where m (resp. M) is the minimum (resp. maximum) of f on $[a,b]$.

Remark

In particular, we have for all $x \in [a,b]$, $m \leqslant f(x) \leqslant M$. We say that f is bounded and reaches its boundaries.

3.4 Differentiability

All the functions of this chapter are of type $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ containing at least two points.

3.4.1 Definitions

Definition 17

We say that f is differentiable at a if the increasing rate $\tau_a : x \mapsto \dfrac{f(x)-f(a)}{x-a}$ has a finite limit at a. If this is the case, we denote this limit $f'(a)$ (called derivative number of f at a) i.e.

$$f'(a) = \lim_{x \to a} \tau_a(x)$$

If f is differentiable at any point of I, we say that f is differentiable on I and the function $x \mapsto f'(x)$ is called derivative of f.

Remarks

1. Setting $h = x - a$, f differentiable at a is equivalent to $h \mapsto \dfrac{f(a+h)-f(a)}{h}$ has a finite limit at 0. If this is the case
$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$
2. f is differentiable at a if and only if the graph of f admits a non vertical tangent at $A(a, f(a))$. In this case, $f'(a)$ represents the slope of the tangent of the graph of f at a.
3. If $\tau_a(x) \xrightarrow[x \to a]{} +\infty$ or $\tau_a(x) \xrightarrow[x \to a]{} -\infty$, then the graph of f admits a vertical tangent at $A(a, f(a))$.

3.4.2 Differentiability and continuity

Is there a link between differentiability and continuity? This section answers the question.

Proposition 18

Let f be differentiable at a. Then f is continuous at a.

Remark

The reciprocal is false, as the following counter-example shows it. We consider the function $f : x \mapsto \sqrt{x}$. Then f is continuous on $\mathbb{R}^+$, and in particular at 0 but is not differentiable at 0. Indeed

$$\frac{f(x)-f(0)}{x-0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \xrightarrow[x \to 0]{} +\infty$$

3.4.3 Local extremum

Definition 18
We say that f admits a local maximum (resp. minimum) at a if $f(x) \leqslant f(a)$ (resp. $f(x) \geqslant f(a)$) provided that x be sufficiently close a i.e. if

$$\exists \eta > 0, \forall x \in I, |x - a| < \eta \Rightarrow f(x) \leqslant f(a) \quad (\text{resp.}\, f(x) \geqslant f(a))$$

We say that f admits a local extremum at a if f admits a local mimimum or a local maximum at a.

Proposition 19
We assume that a is not a boundary of the interval I, that f is differentiable at a and that f presents a local extremum at a. then $f'(a) = 0$.

Remarks

This proposition has to be carefully used, as the following remarks show it:

1. If a is a boundary of the interval I then the proposition is false, as the following counter-example shows it :

   Let $f : \begin{cases} [0,1] \to \mathbb{R} \\ x \mapsto x \end{cases}$. Then f is differentiable on $[0,1]$, in particular at 0 and 1, f admits a local minimum at 0 and a local maximum at 1 and yet $(f)'(0) \neq 0$ and $(f)'(1) \neq 0$ as for all $x \in [0,1]$, $f'(x) = 1$.

2. A function can have one extremum at a without being differentiable at a. For example, the function $x \mapsto \sqrt{x}$ admits a minimum at 0 but is not differentiable at 0 (cf remark of prvsious section).

3. The reciprocal of the proposition is false, as the following counter-example shows it:

   Let $f : \begin{cases} [-2,2] \to \mathbb{R} \\ x \mapsto x^3 \end{cases}$ Then f admits no extremum and yet $f'(0) = 0$ because for all $x \in [-2,2]$, $f'(x) = 3x^2$.

3.5 Classical theorems

3.5.1 Rolle's theorem

Theorem 2 (Rolle)
Let a, b be two real distinct numbers, f continuous on $[a,b]$, differentiable on $]a,b[$ such that $f(a) = f(b)$. Then there exists (at least one) $c \in]a,b[$ such that $f'(c) = 0$.

Example

Let $f : I \to \mathbb{R}$ be two times differentiable (i.e. f' and f'' exist) admitting three zeros x_0, x_1 and x_2 (i.e. $f(x_0) = f(x_1) = f(x_2) = 0$). Then f'' admits at least one zero. Indeed, it is sufficient to apply three times the Rolle's as follows :

f is continuous, differentiable on I and $f(x_0) = f(x_1)\,(=0)$ so using Rolle's theorem, there exists $y_1 \in]x_0, x_1[$ such that $f'(y_1) = 0$. Similarly $f(x_1) = f(x_2)$ so there exists again $y_2 \in]x_1, x_2[$ such that $f'(y_2) = 0$. Now we have a function f' continuous and differentiable on I such that $f'(y_1) = f'(y_2)\,(=0)$. Using for a last time Rolle's theorem, we conclude that there exists $z \in]y_1, y_2[$ such that $(f')'(z) = 0$ i.e such that $f''(z) = 0$.

3.5.2 Mean value theorem

What happens if we remove $f(a) = f(b)$ from the assumptions of Rolle's theorem ? The following theorem gives the answer.

Theorem 3 (Mean value theorem)
Let a, b be two distinct real numbers, f be continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists (at least one) $c \in]a, b[$ such that $f(b) - f(a) = (b - a)f'(c)$.

Remark

The previous theorem is often used with $a = 0$ and $b = x$, as the following example shows it.

Example

We want to prove that for all $x \in \mathbb{R}_*^+$, $\dfrac{x}{x+1} < \ln(1 + x) < x$.

We set $f : x \mapsto \ln(1 + x)$. Let $x > 0$. Then f is continuous and differentiable on $[0, x]$.
Using the Mean value theorem on $[0, x]$, there exists $c \in]0, x[$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

Yet $f(0) = 0$ and for all $x \in \mathbb{R}_*^+$, $f'(x) = \dfrac{1}{1+x}$. So there exists $c \in]0, x[$ such that

$$\ln(1 + x) = x.\frac{1}{1+c} = \frac{x}{1+c}$$

Yet

$$\begin{aligned} 0 < c < x &\Rightarrow 1 < 1 + c < 1 + x \\ &\Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < 1 \\ &\Rightarrow \frac{x}{1+x} < \frac{x}{1+c} < x \end{aligned}$$

Hence, for all $x > 0$,

$$\frac{x}{x+1} < \ln(1 + x) < x$$

3.6 Local comparison of functions

3.6.1 Definitions of Landau notations

Let a be a real number or $+\infty$ or $-\infty$ (which is sometimes denoted $-\infty \leqslant a \leqslant +\infty$).

Definition 19 (Landau Notations)

1. We say that f is bounded above by g (up to constant factor) at a neighborhood of a (and we write : At a neighborhood of a, $f = O(g)$) if at a neighborhood of a, $f = g.h$ where h is bounded at a neighborhood of a.

2. We say that f is dominated by g at a neighborhood of a (and we write : At a neighborhood of a, $f = o(g)$) if at a neighborhood of a, $f = g.\varepsilon$ where $\varepsilon(t)$ tends to 0 when $t \to a$.

3. We say that f is asymptotically equal to g at a neighborhood of a (and we write $f \underset{a}{\sim} g$) if at a neighborhood of a, $f = g.k$ where $k(t)$ tends to 1 when $t \to a$.

Remark

$f = O(g)$ is read f is a big «O» of g.

$f = o(g)$ is read f is a small «o» of g.

$f \underset{a}{\sim} g$ is read f is equivalent to g at a.

Examples

1. At a neighborhood of $+\infty$, $\sin(t) = O(1)$ as the function $t \mapsto \dfrac{\sin(t)}{1} = \sin(t)$ is bounded (by 1). At a neighborhood of $+\infty$.

2. At a neighborhood of 0, $t^2 = o(t)$ as $\dfrac{t^2}{t} = t \xrightarrow[t\to 0]{} 0$.

3. $t+1 \underset{+\infty}{\sim} t$ as $\dfrac{t+1}{t} \xrightarrow[t\to+\infty]{} 1$. Indeed $\dfrac{t+1}{t} = 1 + \dfrac{1}{t} \xrightarrow[t\to+\infty]{} 1$.

3.6.2 properties

Properties 2
We focus on a neighborhood of a where $-\infty \leqslant a \leqslant +\infty$.

1. $$\left.\begin{array}{l} f = o(h) \\ g = o(h) \end{array}\right\} \Longrightarrow f + g = o(h)$$

2. $$\left.\begin{array}{l} f = o(g) \\ h = o(l) \end{array}\right\} \Longrightarrow fh = o(gl)$$

3. $$\left.\begin{array}{l} f \underset{a}{\sim} g \\ h \underset{a}{\sim} l \end{array}\right\} \Longrightarrow fh \underset{a}{\sim} gl$$

3.7 Taylor's expansion

The concept of Taylor's expansion is essential and very useful to find difficult limits of functions. It derives from the following theorem.

3.7.1 Taylor-Young's theorem

Theorem 4 (Taylor-Young at order n)
Let $n \in \mathbb{N}$ and f be of class C^n on I (i.e. f is n-times derivable on I and each of its derivatives is continuous). Then at a neighborhood of $a \in I$, we have

$$f(x) = f(a) + (x-a)f'(a) + ... + \frac{(x-a)^n}{n!}f^{(n)}(a) + o((x-a)^n)$$

Remarks

1. We recall that for all integer n, $n! = 1 \times 2 \times ... \times n$ with the convention $0! = 1$.
   For example $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

2. The symbol $f^{(n)}$ means dérivative n^{th} of f with the convention $f^{(0)} = f$.
   For example $f^{(2)} = f''$.

3. Under the hypothesis of this theorem, f can be written locally (i.e. at a neighborhood of a) as a polynomial.

4. The «$o((x-a)^n)$» means that the rest of the expansion is negligible compared with $(x-a)^n$.

5. Theorem is mostly used for $a = 0$.

3.7.2 Definition of Taylor's expansion

Definition 20
Let $n \in \mathbb{N}$. We say that f admits a Taylor's expansion of order n at a neighborhood of 0 (or at 0) if there exists real numbers a_0, ..., a_n such that in a neighborhood of 0

$$f(x) = a_0 + a_1x + ... + a_nx^n + o(x^n)$$

Remark

The coefficients a_0, ..., a_n are obtained using Taylor-Young's theorem for f. Let us take for example the function $f : x \mapsto e^x$ and let us determine the Taylor's expansion of order 2 of f at 0.

Using Taylor-Young's theorem, we have at a neighborhood of 0,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + o(x^2)$$

Yet $f(0) = 1$, $2! = 2$ and for all $x \in \mathbb{R}$, $f'(x) = f''(x) = e^x$. Hence $f'(0) = f''(0) = 1$. Then at a neighborhood of 0, we have

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

Classical examples of Taylor's expansions

The following examples are to be known or found back using Taylor-Young's theorem.

1. $\sin(x) = x - \frac{x^3}{3!} + ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

2. $\cos(x) = 1 - \frac{x^2}{2!} + ... + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n})$

3. $e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + o(x^n)$

4. $\ln(1+x) = x - \frac{x^2}{2} + ... + \frac{(-1)^{n-1} x^n}{n} + o(x^n)$

5. With $\alpha \in \mathbb{R}$, $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + ... + \frac{\alpha(\alpha-1)...(\alpha-(n-1))x^n}{n!} + o(x^n)$

3.7.3 Operations on Taylor's expansions

How to sum, multiply or compose two Taylor's expansions at a neighborhood of 0 ? This section answers it.

Proposition 20

Let us assume that at a neighborhood of 0, we know the Taylor's expansions of order n of f and g i.e. f and g are in the neighborhood of 0 of type $f(x) = P(x) + o(x^n)$ and $g(x) = Q(x) + o(x^n)$ where P and Q are two polynomials of degree less than or equal to n. Then at a neighborhood of 0 :

1. $(f+g)(x) = P(x) + Q(x) + o(x^n)$

2. $(fg)(x) = R(x) + o(x^n)$ where $R(x)$ is the polynomial obtained by keeping only in $P(x)Q(x)$ the terms of degree less than or equal to n.

3. If $f(0) = 0$, $(g \circ f)(x) = T(x) + o(x^n)$ where $T(x)$ is the polynomial obtained by only keeping in $(Q \circ P)(x)$ the terms of degree less than or equal to n.

Examples

1. We determine the Taylor's expansion of order 3 of $x \mapsto \sin(x) + \cos(x)$ at a neighborhood of 0. we have
$$\sin(x) = x - \frac{x^3}{3!} + o(x^3)$$
and
$$\cos(x) = 1 - \frac{x^2}{2!} + o(x^3)$$

so

$$\begin{aligned}\sin(x)+\cos(x) &= x-\frac{x^3}{3!}+1-\frac{x^2}{2!}+o(x^3)\\ &= 1+x-\frac{x^2}{2}-\frac{x^3}{6}+o(x^3)\end{aligned}$$

2. We determine the Taylor's expansion of order 3 of $x \mapsto e^x \sin(x)$ at a neighborhood of 0. We have

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3) = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3)$$

and

$$\sin(x) = x-\frac{x^3}{3!}+o(x^3) = x-\frac{x^3}{6}+o(x^3)$$

We have now to compute the product

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}\right)$$

by keeping only the terms of degree less than or equal to 3 (as the other terms will be negligible compared to x^3 i.e. they will «go» in the $o(x^3)$). We then have

$$\begin{aligned}e^x\sin(x) &= x-\frac{x^3}{6}+x^2+\frac{x^3}{2}+o(x^3)\\ &= x+x^2+\frac{x^3}{3}+o(x^3)\end{aligned}$$

3. We determine the Taylor's expansion of $x \mapsto e^{\sin(x)}$ of order 3 at a neighborhood of 0. We have

$$e^{\sin(x)} = e^{x-\frac{x^3}{3!}+o(x^3)} = e^{x-\frac{x^3}{6}+o(x^3)}$$

(note that $x-\frac{x^3}{6}$ is null at 0).
We use the Taylor polynomial of e^u at 0 of order 3 which is

$$e^u = 1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+o(u^3) = 1+u+\frac{u^2}{2}+\frac{u^3}{6}+o(u^3)$$

Yet, here $u = x-\dfrac{x^3}{6}+o(x^3)$ hence

$$u^2 = \left(x-\frac{x^3}{6}+o(x^3)\right)^2 = x^2+o(x^3)$$

since all the others terms are negligible compared to x^3,

$$u^3 = \left(x-\frac{x^3}{6}+o(x^3)\right)^3 = x^3+o(x^3)$$

for the same reason. Then

$$\begin{aligned}e^{\sin(x)} &= 1+x-\frac{x^3}{6}+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\\ &= 1+x+\frac{x^2}{2}+o(x^3)\end{aligned}$$

3.7.4 Applications of Taylor's expansions

Taylor's expansions allow to determine tricky limits and find equivalents.

Examples

1. We determine the following limit:

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x$$

Be careful, the limit is not 1 as we could imagine because any term of type «1^∞» is indeterminate (as «$1^\infty = e^{\infty \ln(1)} = e^{\infty \times 0}$» and the limit «$\infty \times 0$» is indeterminate). We have

$$\left(1 + \frac{1}{x}\right)^x = e^{x \ln\left(1 + \frac{1}{x}\right)}$$

When $x \longrightarrow +\infty$, $\dfrac{1}{x} \longrightarrow 0$, so

$$\left(1 + \frac{1}{x}\right)^x = e^{x\left(\frac{1}{x} + o\left(\frac{1}{x}\right)\right)} = e^{1+o(1)}$$

so

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

2. Let us prove that at a neighborhood of 0, $\ln(\cos(x)) \sim -\dfrac{x^2}{2}$.

We have

$$\ln(\cos(x)) = \ln\left(1 - \frac{x^2}{2} + o(x^2)\right) = -\frac{x^2}{2} + o(x^2)$$

As at a neighborhood of 0, $\ln(1-x) = -x + o(x)$ then at a neighborhood of 0,

$$\ln(\cos(x)) \sim -\frac{x^2}{2}$$

as

$$\frac{\ln(\cos(x))}{-\frac{x^2}{2}} = \frac{-\frac{x^2}{2} + o(x^2)}{-\frac{x^2}{2}} = 1 + o(1) \underset{x \to 0}{\longrightarrow} 1$$

Chapter 4

Differential equations

A differential equation is an equation where the unknown is a function and where its derivatives appear. Solving a differential equation is to determine all functions which are solutions of this equation.
In the whole chapter, I is an interval of $\mathbb{R}$.

4.1 Linear differential equation of first order with constant coefficients

4.1.1 Resolution of $ay' + by = 0$

Definition 21

1. We call linear differential equation of first order without second member and with constant coefficients any equation of type

$$(E_0) \quad ay'(t) + by(t) = 0$$

   where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$.

2. Let (E_0) $ay' + by = 0$.

   We call solution on I of (E_0) any function f, differentiable I, such that, for all $t \in I$

$$af'(t) + bf(t) = 0$$

Example

The function $f(t) = 5e^{-2t}$ is a solution of equation $3y' + 6y = 0$. The function $g(t) = e^{-2t}$ is another solution of this equation.

Theorem 5
Let $(a, b) \in \mathbb{R}^* \times \mathbb{R}$ and (E_0) $ay' + by = 0$. Let us denote $\mathcal{S}_0$ the set of solutions of (E_0). Then

$$\mathcal{S}_0 = \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-\frac{b}{a}t} \end{array} ; \ k \in \mathbb{R} \right\}$$

Example

The set of solutions of the equation $3y' - 2y = 0$ is

$$\mathcal{S}_0 = \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{\frac{2}{3}t} \end{array} ; \ k \in \mathbb{R} \right\}$$

Proposition 21

Let $(a, b) \in \mathbb{R}^* \times \mathbb{R}$, $(E_0)\ \ ay' + by = 0$ and $(\alpha, \beta) \in \mathbb{R}^2$. Then, (E_0) admits a unique solution y on $\mathbb{R}$ satisfying the condition $\beta = y(\alpha)$.

Example

The differential equation $3y' - 2y = 0$ admits a unique solution y satisfying $y(0) = 1$. It is the function defined on $\mathbb{R}$by$y(t) = e^{\frac{2}{3}t}$.

4.1.2 Resolution of $ay' + by = c$

Definition 22

1. We call linear differential equation of first order with second member and constant coefficients any equation of type
$$(E) \quad ay'(t) + by(t) = c$$
where $a \in \mathbb{R}^*$ and $(b, c) \in \mathbb{R}^2$.

2. Let $(E)\ \ ay' + by = 0$.
We call solution on I of (E) any function f, differentiable on I, such that, for all $t \in I$
$$af'(t) + bf(t) = c$$

Example

1. Let us assume that $b \neq 0$. Then, The constant function, defined on $\mathbb{R}$, by$f(t) = \frac{c}{b}$ is a solution of $(E)\ \ ay' + by = c$.
2. Let us assume that $b = 0$. Then, the function, defined on $\mathbb{R}$, by $f(t) = \frac{c}{a}t$ is a solution of (E).

Theorem 6

Let $(a, b) \in \mathbb{R}^* \times \mathbb{R}$ and $(E)\ \ ay' + by = 0$. We denote $\mathcal{S}$ the set of solutions of (E). Then,

1. if $b = 0$,
$$\mathcal{S} = \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \frac{c}{a}t + k \end{array} ; \ k \in \mathbb{R} \right\}$$
2. if $b \neq 0$,
$$\mathcal{S} = \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-\frac{b}{a}t} + \frac{c}{b} \end{array} ; \ k \in \mathbb{R} \right\}$$

Remark

We note that the general solution of (E) is the sum of a particular solution of (E) and of a general solution of (E_0).

Example

The set of solutions of equation $3y' - 2y = 5$ is

$$\mathcal{S}_0 = \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{\frac{2}{3}t} - \frac{5}{2} \end{array} ;\ k \in \mathbb{R} \right\}$$

Proposition 22

Let $(a, b, c) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}$, $(E)\ \ ay' + by = c$ and $(\alpha, \beta) \in \mathbb{R}^2$. Then, (E) admits a unique solution y on $\mathbb{R}$ verifying the condition $\beta = y(\alpha)$.

Example

The differential equation $3y' - 2y = 5$ admits a unique solution y satisfying $y(0) = 1$. It is the function defined on $\mathbb{R}$ by $y(t) = \frac{7}{2}e^{\frac{2}{3}t} - \frac{5}{2}$.

4.2 Linear differential equations of first order with constant coefficients

4.2.1 Generalities

Definition 23

1. We call linear differential equation of first order any equation of type

$$a(t)y'(t) + b(t)y(t) = c(t)$$

where a, b and c are three continuous functions on I.

2. Let $(E)\ :\ a(t)y'(t) + b(t)y(t) = c(t)$.

We call solution of (E) on I any function f differentiable and continuous and I such that

$$\forall\, t \in I, \quad a(t)f'(t) + b(t)f(t) = c(t)$$

3. Let $(E)\ :\ a(t)y' + b(t)y = c(t)$.

We call homogeneous equation associated to (E) the equation

$$(E_0)\ :\ a(t)y' + b(t)y = 0$$

Notations

We denote $\mathcal{S}$ the set of solutions of (E) and $\mathcal{S}_0$ the set of solutions of (E_0).
We assume that $\mathcal{S} \neq \emptyset$.

Theorem 7

Let $y_p \in \mathcal{S}$ be a particular solution of (E).
Then,

$$\mathcal{S} = \{\ y_p + y_0;\ y_0 \in \mathcal{S}_0\ \}$$

The general solution of (E) is the sum of a particular solution of (E) and the general solution of (E_0).

To conclude, to solve (E), there are three steps :

- Step 1 : We solve (E_0) and we find $\mathcal{S}_0$.
- Step 2 : We look for a particular solution of (E).
- Step 3 : We conclude by giving $\mathcal{S}$.

4.2.2 Resolution of (E_0)

Let (E_0) : $a(t)y' + b(t)y = 0$
where a and b continuous on I.
We assume that $\forall\ t \in I$, $a(t) \neq 0$.

Theorem 8

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} I & \longrightarrow & \mathbb{R} \\ x & \longmapsto & ke^{-\int \frac{b(t)}{a(t)}\,\mathrm{d}t} \end{array} \quad ; \ k \in \mathbb{R} \right\}$$

Example

Solve (E_0) $(1+x^2)y' + 4xy = 0$ in $I = \mathbb{R}$.

we have
$$\int \frac{b(x)}{a(x)}\,dx = 2\int \frac{2x}{1+x^2}\,dx = 2\ln(1+x^2) = \ln\left((1+x^2)^2\right)$$
Using the previous theorem, we obtain

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{k}{(1+x^2)^2} \end{array} \quad ; \ k \in \mathbb{R} \right\}$$

4.2.3 Resolution of (E)

Let (E) $ay' + by = c$ where a, b and c are three functions continuous sur I.
We have seen that the general solution of (E) is the sum of the general solution of (E_0) and a particular solution of (E).
We then have the two following possibilities :

1. a particular solution of (E) is obvious.

   #### Example

   Solve (E) $xy' + y = 3x^2$ in $I =]0, +\infty[$.

   • Step 1 : We solve (E_0) $xy' + y = 0$ on I.
   We find
   $$\mathcal{S}_0 = \left\{ \begin{array}{lll}]0,+\infty[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{k}{x} \end{array} \quad ; \ k \in \mathbb{R} \right\}$$

   • Step 2 : We easily see that $y_p(x) = x^2$ is a particular solution of (E).

   • Step 3 : Conclusion

   $$\mathcal{S} = \left\{ \begin{array}{lll}]0,+\infty[& \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{k}{x} + x^2 \end{array} \quad ; \ k \in \mathbb{R} \right\}$$

2. There is no trivial particular solution of (E).

   We use then **the method of variation of parameters**.

   We set $y_0 = e^{-\int \frac{b(t)}{a(t)}\,\mathrm{d}t}$ a non-null solution of (E_0) and we look for a solution y_p of (E) under the form

$$y_p(t) = k(t)y_0(t)$$

   where $k : I \to \mathbb{R}$ is an unknown function differentiable on I.

   We then have

$$y_p \in \mathcal{S} \iff ay_p' + by_p = c \iff ak'y_o + aky_0' + bky_0 = c \iff ak'y_0 = c$$

   as $ay_0' + by_0 = 0$.

   We deduce that $k' = \dfrac{c}{ay_0}$.

   We then choose a primitive k of k'. We then deduce y_p.

Example

Solve $(E) \quad y' + 2ty = e^{t-t^2}$ in $I = \mathbb{R}$.

• Step 1 : We solve $(E_0) \quad y' + 2ty = 0$.

We find

$$\mathcal{S}_0 = \left\{ \begin{array}{lcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-t^2} \end{array} \quad ;\ k \in \mathbb{R} \right\}$$

• Step 2 : We look for a particular solution y_p of (E) of type

$$y_p(t) = k(t)e^{-t^2}$$

where $k : \mathbb{R} \to \mathbb{R}$ is differentiable.

We have

$$y_p \in \mathcal{S} \iff y_p' + 2ty_p = e^{t-t^2} \iff k'(t)e^{-t^2} - 2tk(t)e^{-t^2} + 2tk(t)e^{-t^2} = e^{t-t^2}$$

We obtain that $k'(t) = e^t$.

We then take

$$k(t) = e^t$$

Finally,

$$y_p(t) = e^t e^{-t^2} = e^{t-t^2}$$

• Step 3 : Conclusion

$$\mathcal{S} = \left\{ \begin{array}{lcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-t^2} + e^{t-t^2} \end{array} \quad ;\ k \in \mathbb{R} \right\}$$

Remark

(E) has an infinity of solutions.
If we impose initial conditions then we have a unique solution.

4.3 Second order linear differential equations with constant coefficients

4.3.1 Generalities

Definition 24

1. We call linear differential equation of second order with constant coefficients any equation of type
$$ay''(t) + by'(t) + cy(t) = d(t)$$
where $(a, b, c) \in \mathbb{R}^* \times \mathbb{R}^2$ and d is a function continuous on I.

2. Let $(E)\ :\ ay''(t) + by'(t) + cy(t) = d(t)$.
We call solution of (E) on I any function f two times differentiable on I such that
$$\forall\, t \in I, \quad af''(t) + bf'(t) + cf(t) = d(t)$$

3. Let $(E)\ :\ ay'' + by' + cy = d$.
We call homogeneous equation associated with (E) the equation
$$(E_0)\ :\ ay'' + by' + cy = 0$$

Notations

We denote $\mathcal{S}$ the set of solutions of (E) and $\mathcal{S}_0$ the set of solutions of (E_0).
We assume that $\mathcal{S} \neq \emptyset$.

Theorem 9
Let $y_p \in \mathcal{S}$ be a particular solution of (E). Then,
$$\mathcal{S} = \{\ y_p + y_0;\ y_0 \in \mathcal{S}_0\ \}$$

The general solution of (E) is the sum of a particular solution of (E) and of the general solution of (E_0).

The technique of resolution of (E) is the same as the one used to solve first order differential equations!

4.3.2 Resolution of (E_0)

Let $(E_0)\quad ay'' + by' + c = 0$ where $(a, b, c) \in \mathbb{R}^* \times \mathbb{R}^2$.
The goal is to look for real-valued solutions of (E_0).
By analogy with what we found for the first order equations, we look for solutions of (E_0) of type
$$y_0 = e^{rt}$$

We have

$$\begin{aligned} y_0 \in \mathcal{S}_0 &\iff ay_0'' + by_0' + cy_0 = 0 \\ &\iff (ar^2 + br + c)e^{rt} = 0 \\ &\iff ar^2 + br + c = 0 \end{aligned}$$

Definition 25

We call characteristic equation of (E_0) the equation

$$(C) \quad ar^2 + br + c = 0$$

of unknown $r \in \mathbb{R}$ or $\mathbb{C}$.

Theorem 10

Let $\Delta = b^2 - 4ac$ be the discriminant of (C).
• 1st case : $\Delta > 0$.
We denote r_1 and r_2 the two real and distinct solutions of (C).
Then,

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & k_1 e^{r_1 t} + k_2 e^{r_2 t} \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

• 2nd case : $\Delta = 0$.
We denote r_1 the real double root of (C).
Then,

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & (k_1 t + k_2) e^{r_1 t} \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

• 3rd case : $\Delta < 0$.
We denote $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ ($(\alpha, \beta) \in \mathbb{R}^2$) the two conjugate complex roots of (C).
Then,

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & e^{\alpha t}\left(k_1 \cos(\beta t) + k_2 \sin(\beta t)\right) \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

Examples

1. Solve $(E_0) \quad y'' + y' - 6y = 0$ in $\mathbb{R}$.

   The characteristic equation $(C) \quad r^2 + r - 6 = 0$ admits two real distinct solutions : 2 and -3.

   Hence,

   $$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & k_1 e^{2t} + k_2 e^{-3t} \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

2. Solve $(E_0) \quad y'' - 2y + y = 0$ in $\mathbb{R}$.

   The characteristic equation $(C) \quad r^2 - 2r + 1 = 0$ admits a double root: 1.

   Hence,

   $$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & (k_1 t + k_2) e^{t} \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

3. Solve $(E_0)\quad y'' + y' + y = 0$ in $\mathbb{R}$.

   The characteristic equation $(C)\quad r^2 + r + 1 = 0$ admits two complex solutions: $\dfrac{-1}{2} + i\dfrac{\sqrt{3}}{2}$ and $\dfrac{-1}{2} - i\dfrac{\sqrt{3}}{2}$.
   Hence,

$$\mathcal{S}_0 = \left\{ \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & e^{-\frac{1}{2}t}\left(k_1 \cos(\frac{\sqrt{3}}{2}t) + k_2 \sin(\frac{\sqrt{3}}{2}t)\right) \end{array} \ ;\ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

4.3.3 Case where the second member is polynomial or exponential-polynomial

Let

$$(E)\quad ay'' + by' + cy = d$$

where $(a, b, c) \in \mathbb{R}^* \times \mathbb{R}^2$ and $d : I \to \mathbb{R}$ is continuous.

Proposition 23
Let $(E)\quad ay'' + by' + cy = P$ where P is a polynomial function of degree n.
We look for a particular solution of (E) as a polynomial function of degree
-n if $c \neq 0$.
-$n + 1$ if $c = 0$ and $b \neq 0$.
-$n + 2$ if $c = b = 0$.

Example

Solve $(E)\quad y'' - 4y' + 4y = x^2 + 1$ in $I = \mathbb{R}$.

• Step 1: Resolution of $(E_0)\quad y'' - 4y' + 4y = 0$.
The characteristic equation $(C)\quad r^2 - 4r + 4 = 0$ admits a double real root: 2.
Hence,

$$\mathcal{S}_0 = \left\{ \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (k_1 x + k_2)e^{2x} \end{array} \ ;\ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

• Step 2: We look for a particular solution y_p of (E) of the form

$$yp(x) = \alpha x^2 + \beta x + \gamma$$

We have

$$y_p \in \mathcal{S} \iff 4\alpha x^2 + (4\beta - 8\alpha)x + 2\alpha - 4\beta + 4\gamma = x^2 + 1$$

We find $\alpha = \dfrac{1}{4}$, $\beta = \dfrac{1}{2}$ and $\gamma = \dfrac{5}{8}$.
Thus,

$$y_p(x) = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{5}{8}$$

• Step 3: Conclusion

$$\mathcal{S} = \left\{ \begin{array}{ccl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (k_1 x + k_2)e^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{5}{8} \end{array} \ ;\ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

Proposition 24
We look for a particular solution y_p of (E) of type $y_p(t) = e^{mt}Q(t)$ where Q is a polynomial function of degree

-n if m is not a root of (C).
-$n+1$ if m is a simple root of (C).
-$n+2$ if m is a double root of (C).

Example

Solve $(E) \quad y'' - 2y' + y = e^t$ in $I = \mathbb{R}$.

• Step 1: The characteristic equation $(C) \quad r^2 - 2r + 1 = 0$ admits 1 as double root.
Hence,

$$\mathcal{S}_0 = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & (k_1 t + k_2)e^t \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

• Step 2: We look for a particular solution y_p of (E) of type

$$y_p(t) = \left(\alpha t^2 + \beta t + \gamma\right) e^t$$

After computation, we find that $\alpha = \dfrac{1}{2}$, β and γ arbitrary.
Let us take $\beta = \gamma = 0$.
We conclude that

$$y_p(t) = \frac{1}{2} t^2 e^t$$

• Step 3: Conclusion

$$\mathcal{S} = \left\{ \begin{array}{lll} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & (k_1 t + k_2)e^t + \frac{1}{2}t^2 e^t \end{array} ; \ (k_1, k_2) \in \mathbb{R}^2 \right\}$$

4.4 Examples of non linear differential equations

4.4.1 Bernoulli differential equation

Definition 26
It is the differential equation

$$(E) \quad ay' + by + cy^\alpha = 0$$

where $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and a, b and c three functions continuous on I.

Method of formal resolution

We look for solutions of (E) which do not nullify I.
We set $z = y^{1-\alpha}$.
We have $z' = (1-\alpha)\frac{y'}{y^\alpha}$ and

$$y \text{ solution of } (E) \Longleftrightarrow a\frac{y'}{y^\alpha} + by^{1-\alpha} + c = 0 \Longleftrightarrow \frac{a}{1-\alpha} z' + bz + c = 0$$

Then, z is solution of the linear equation of order 1

$$(E') \quad \frac{a}{1-\alpha} z' + bz + c = 0$$

Example

Solve $(E)\quad y' + \frac{y}{x} - y^3 = 0$.
We set $z = \frac{1}{y^2}$. Then, $z' = -2\frac{y'}{y^3}$.

$$y \text{ solution of } (E) \Longleftrightarrow \frac{y'}{y^3} + \frac{1}{x}\frac{1}{y^2} - 1 = 0 \Longleftrightarrow -\frac{1}{2}z' + \frac{1}{x}z = 1$$

Let $(E')\quad -\frac{1}{2}z' + \frac{1}{x}z = 1$ $(I = \mathbb{R}^*)$.
The set of solutions of (E'_0) is

$$\mathcal{S'}_0 = \left\{ \begin{array}{lcl} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & kx^2 \end{array} \quad ; \; k \in \mathbb{R} \right\}$$

Moreover, $z_p(x) = 2x$ is a particular solution of (E'). Hence, the set of solutions of (E') is

$$\mathcal{S'} = \left\{ \begin{array}{lcl} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & kx^2 + 2x \end{array} \quad ; \; k \in \mathbb{R} \right\}$$

We conclude that the solutions of (E) are functions of type

$$y(x) = \pm\sqrt{kx^2 + 2x}$$

4.4.2 Riccati differential equation

Definition 27
It is the differential equation of type

$$(E) \quad ay' + by + cy^2 + d = 0$$

where a, b, c and d are continuous functions on I.

Remark

If $d = 0$, then we have a Bernoulli equation for $\alpha = 2$.

Method of formal resolution

We assume the solution y_0 of (E) to be known.
We set

$$Y = y - y_0 \Longleftrightarrow y = Y + y_0$$

We have

$$y \text{ solution of } (E) \Longleftrightarrow a(y_0'+Y')+b(y_0+Y)+c(y_0+Y)^2+d = 0 \Longleftrightarrow aY'+(b+2cy_0)Y+cY^2 = 0$$

Hence, we are lead to solve a Bernoulli equation, using $Z = \frac{1}{Y}$.

Example

Solve $(E)\ \ (x^2+1)y' - y^2 + 1 = 0$.
$y_0 = 1$ is a trivial solution of (E).
We set $Y = y - 1$.

$$y \ \text{ solution } \ \text{of } \ (E) \Longleftrightarrow (x^2+1)Y' - Y^2 - 2Y - 1 + 1 = 0 \Longleftrightarrow (x^2+1)Y' - 2Y - Y^2 = 0$$

We set $Z = \frac{1}{Y}$. We are lead to solve

$$(E') \quad -(x^2+1)Z' - 2Z = 1$$

The set of solutions of (E'_0) is

$$\mathcal{S}'_0 = \left\{ \begin{array}{lll} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-2\arctan x} \end{array} ;\ k \in \mathbb{R} \right\}$$

Moreover, $z_p(x) = -\frac{1}{2}$ is a particular solution of (E'). Hence, the set of solutions of (E') is

$$\mathcal{S}' = \left\{ \begin{array}{lll} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & ke^{-2\arctan x} - \frac{1}{2} \end{array} ;\ k \in \mathbb{R} \right\}$$

Then,

$$Y(x) = \frac{1}{ke^{-2\arctan x} - \frac{1}{2}}$$

We conclude that the solutions of (E) are functions of type

$$y(x) = \frac{1}{ke^{-2\arctan x} - \frac{1}{2}} + 1$$

Chapter 5

Logic

5.1 On propositions

5.1.1 Basic notions

Definition 28
A proposition (or assertion) is a combination of words which construction follows a certain syntax, and for which we can say if under given conditions, it is true or false.

Examples

1. «3 is a prime number» is a true assertion.
2. «$(100+2)^2 = 100^2 + 2^2$» is false.
3. «$x < 3$» is true if $x = 1$ but is false if $x = 10$.
4. «$1 = 1 + ($» is not an assertion.

5.1.2 The logic connectors

Let P and Q be two propositions.

Definition 29
The **negation** of P, denoted $Non(P)$ or $\neg P$. It is the proposition which is true when P is false, and false when P is true.

Example

Let the proposition P be: «The square root of a natural number is a natural number».
P is false.
Hence, its negation $Non(P)$ is true and is
$Non(P)$: «there exists a natural number which square root is not a natural number».

Definition 30
The **conjonction** P and Q, is denoted $P \wedge Q$. It is the property which is true when the two propositions P and Q are simultaneously true.

Example

Let P :«$x < 4$» and Q : «$x \geqslant -1$».
Then, $P \wedge Q$:« $x \in [-1, 4[$»

Definition 31

The **disjonction** P or Q, is denoted $P \vee Q$. It is the property which is true when at least one of two propositions P or Q is true.

Example

Let P :«$x < 0$» and Q :«$x \geqslant 1$».
Then, $P \vee Q$:«$x \in]-\infty, 0[\cup [1, +\infty[$»

We can sum up all these notions under the form of a table of truth:

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$\neg(P \vee Q)$	$\neg(P) \wedge \neg(Q)$	$\neg P \vee \neg Q$
T	T	T	T	F	F	F	F	F	F
T	F	F	T	F	T	T	F	F	T
F	T	F	T	T	F	T	F	F	T
F	F	F	F	T	T	T	T	T	T

Proposition 25

1. $Non(P \wedge Q) \iff Non(P) \vee Non(Q)$. Saying that P and Q are false, means that at least one of the two propositions is false.
2. $Non(P \vee Q) \iff Non(P) \wedge Non(Q)$. Negating the fact that at least one of the two propositions is true, means they are both false.

5.1.3 Implication, reciprocal, equivalence

Let P and Q be two propositions.

Definition 32

The implication $P \Longrightarrow Q$ means $Non(P) \vee Q$.

We can formulate $P \Longrightarrow Q$ in the following ways:
-To have P, Q is needed.
-To have Q, P is sufficient.
-If P is true then Q is true. We say that P is a sufficient condition for Q or that Q is a necessary condition for P.
The table of truth is as follows :

P	Q	$P \Longrightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We conclude that $P \Longrightarrow Q$ is true when P is false. Actually, $P \Longrightarrow Q$ is false if P is true and Q is false. It is true in all the other cases.

Examples

Let $x \in \mathbb{R}$.
The following implications are true:

1. $\sqrt{x^2+1} = 0 \implies x^2+1 = 0$.
2. $x = \dfrac{\pi}{2}\,[2\pi] \implies x = \dfrac{\pi}{2}\,[\pi]$.

Definition 33
The reciprocal $P \Longleftarrow Q$ is the implication read inside out.

Definition 34
The equivalence $P \Longleftrightarrow Q$ means $(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$.

$P \Longleftrightarrow Q$ is read:
-For P, it is necessary and sufficient that Q.
-P if Q.
We say that P is a necessary and sufficient condition for Q.

Example

Let $x \in \mathbb{R}$.
We have

$$\sqrt{x^2+1} = 0 \iff x^2+1 = 0$$

We obtain the new table of truth:

P	Q	$P \Longrightarrow Q$	$Non(P \Longrightarrow Q)$	$Non(Q)$	$P \wedge non(Q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Proposition 26

1. $Non(P \Longrightarrow Q) \Longleftrightarrow P \wedge Non(Q)$.
2. $Non(P \Longleftrightarrow Q) \Longleftrightarrow (P \wedge Non(Q)) \vee (Q \wedge Non(P))$.

Examples

1. The negation of «If the weather is nice, I go to to the beach» is «The weather is nice and I do not go to to the beach».
2. The negation of «$x < 0 \Longrightarrow x \leqslant 0$» is «$x < 0$ and $x > 0$».

Definition 35
The contrapositive of $P \Longrightarrow Q$ is $Non(Q) \Longrightarrow Non(P)$.

Example

The contrapositive of «If the weather is nice, I go to to the beach» is «If I do not go to the beach, then the weather is not nice».

Proposition 27

$P \Longrightarrow Q$ is true if and only if its contrapositive is true.
To prove that $P \Longrightarrow Q$ is true, we thus can prove that its contrapositive is true.

5.1.4 Quantifiers

Let $P(x)$ be a proposition depending on an object x which belongs to a certain set E.
there exists two quantifiers.

Definition 36

The universal quantifier: $\forall$ is read «for all».

Definition 37

The existential quantifier: $\exists$ is read «there exists (at least)».
$\exists\,!$ is read «there exists a unique».

Examples

1. $\forall\ x \in \mathbb{R},\ x^2 + 1 > 0$ is true.
2. $\exists\ x \in \mathbb{R},\ x^2 + 1 = 0$ is false.
3. $\exists\ x \in \mathbb{C},\ x^2 + 1 = 0$ is true.

Exercise

Be careful of the order of quantifiers in a same proposition.
A $\forall$ followed by a $\exists$ does not mean the same thing than $\exists$ followed by a $\forall$.
Let us take the following example: Let f_1 and f_2 be two functions of $\mathbb{R} \to \mathbb{R}$.
Illustrate by a figure the following propositions:

1. $\forall\ i \in \{1,2\},\ \exists\ a \in \mathbb{R}$ such that $f_i(a) = 1$.
2. $\exists\ a \in \mathbb{R},\ \forall\ i \in \{1,2\},\ f_i(a) = 1$.
3. $\forall\ i \in \{1,2\},\ \forall\ a \in \mathbb{R},\ f_i(a) = 1$.
4. $\forall\ a \in \mathbb{R},\ \forall i \in \{1,2\},\ f_i(a) = 1$.

Proposition 28

Let E be a set.

1. $Non(\forall\ x \in E,\ P(x)) \iff \exists\ x \in E,\ Non(P(x))$.
2. $Non(\exists\ x \in E,\ P(x)) \iff \forall\ x \in E,\ Non(P(x))$.

Example

$$Non\,(\exists\, x \in \mathbb{R},\ \forall\, y \in \mathbb{R},\ x + y > 0) \Longleftrightarrow \forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y \leqslant 0$$

5.2 Mathematical proofs

5.2.1 Direct reasoning

We want for example to prove that $P \Longrightarrow Q$.
Our hypothesis is then P. We want to prove then that Q is true.

Example

Let $x \in \mathbb{R}$.
Let us prove that

$$x > 0 \Longrightarrow \frac{x}{3} \leqslant \frac{x}{\cos x + 2} \leqslant x$$

We assume $x > 0$.
Then,

$$1 \leqslant \cos x + 2 \leqslant 3$$

So,

$$\frac{1}{3} \leqslant \frac{1}{\cos x + 2} \leqslant 1$$

So, since $x > 0$, we obtain

$$\frac{x}{3} \leqslant \frac{x}{\cos x + 2} \leqslant x$$

Remark

Such inequality can be used for example to apply the Squeeze theorem when x tends to $+\infty$.
We conclude that

$$\lim_{x \to +\infty} \frac{x}{\cos x + 2} = +\infty$$

5.2.2 Proof by contrapositive

To prove that $P \Longrightarrow Q$, we can prove its contrapositive: $Non(Q) \Longrightarrow Non(P)$.

Example

Let $n \in \mathbb{N}$.
Let us prove that

$$n^2 \quad \text{even} \quad \Longrightarrow \quad n \quad \text{even}$$

To do so, we show that n odd $\Longrightarrow n^2$ odd.
Let us assume that n is odd.
Then,

$$\exists\, k \in \mathbb{N} \ \text{ such that } \ n = 2k + 1$$

So,

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

We set $k' = 2k^2 + 2k$.

We have $k' \in \mathbb{N}$ and $n^2 = 2k' + 1$.
So, n^2 is odd.
Then the contrapositive is true. Hence, the proposition is true.

5.2.3 Proof by contradiction

This consists in assuming the conclusion to be false. Then, we want to reach a contradiction.

Example

Let us prove that $\sqrt{2}$ is irrationnal.
We assume that $\sqrt{2}$ is not irrationnal.
Then, $\exists\ (p, q) \in \mathbb{N} \times \mathbb{N}^*$ coprime such that

$$\sqrt{2} = \frac{p}{q}$$

that is

$$p^2 = 2q^2$$

Thus, p^2 is even.
From the previous example, we then have that p is even.
Thus,

$$\exists\ k \in \mathbb{N} \ \text{ such that } \ p = 2k$$

Then, $2q^2 = 4k^2$ and $q^2 = 2k^2$.
So q^2 is even and so, q is even.
Finally, we have obtained: p and q even. This contradicts the fact that p and q are coprime.
To conclude, $\sqrt{2}$ is not rational.

5.2.4 Proof by induction

The principe is as follows:
Let $P(n)$ be a property depending on the natural integer n.
Let $n_0 \in \mathbb{N}$ be fixed.
We want to prove that

$$\forall\ n \geqslant n_0, \quad P(n) \ \text{ is true}$$

The proof takes three steps:
• Step 1: Basis.
We prove that $P(n_0)$ is true.
• Step 2: Inductive step.
We assume $P(n)$ true for $n \geqslant n_0$. We then prove that $P(n+1)$ is true.
• Step 3: Conclusion

Example

Let $q \in \mathbb{R} - \{1\}$.
Let us prove that

$$\forall\ n \in \mathbb{N}, \quad \sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}$$

Let $P(n)$ be the property: $\displaystyle\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$.

• Step 1 :

We have $\displaystyle\sum_{k=0}^{0} q^k = q^0 = 1$.

On the other part, if $n = 0$, $\dfrac{1-q^{n+1}}{1-q} = 1$.

Hence, $P(0)$ is true.

• Step 2 :

Let us assume $P(n)$ true and let us prove that $P(n+1)$ is true.

We have

$$\begin{aligned}
\sum_{k=0}^{n+1} q^k &= \sum_{k=0}^{n} q^k + q^{n+1} \\
&= \frac{1-q^{n+1}}{1-q} + q^{n+1} \text{ as } P(n) \text{ is true} \\
&= \frac{1-q^{n+1} + q^{n+1}(1-q)}{1-q} \\
&= \frac{1-q^{n+2}}{1-q}
\end{aligned}$$

• Step 3 :

We conclude that

$$\forall\, n \in \mathbb{N}, \quad \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

Chapter 6

Arithmetic in $\mathbb{Z}$

6.1 Divisibility in $\mathbb{Z}$

6.1.1 Divisors, multiples

Definition 38
Let $(a, b) \in \mathbb{Z}^2$.
We say that a divides b, and we note $a \mid b$, if

$$\exists\ k \in \mathbb{Z} \text{ such that } b = ak$$

We say that a is a divisor of b, or that b is a multiple of a (i.e. $b \in a\mathbb{Z}$).

Remarks

1. $\forall\ a \in \mathbb{Z}, \quad a \mid 0$.
2. Let $b \in \mathbb{Z}$. $0 \mid b \iff b = 0$.
3. Let $(a, b) \in \mathbb{Z}^2$. $a \mid b \implies |b| \geqslant |a|$.

Examples

1. $\forall\ b \in \mathbb{Z}, \quad 1 \mid b \quad \text{and} \quad -1 \mid b$.
2. Let $a \in \mathbb{Z}$. $a \mid 8 \iff a \in \{-8, -4, -2, -1, 1, 2, 4, 8\}$.

Proposition 29
Let $(a, b, c) \in \mathbb{Z}^3$.
Then,

1. $a \mid a$ (reflexivity).
2. $a \mid b$ and $b \mid a \iff |a| = |b|$.
3. $a \mid b$ and $b \mid c \implies a \mid c$ (transitivity).

Remark

In $\mathbb{Z}$, $a \mid b$ and $b \mid a$ does not imply $a = b$.
Let us take for example $a = 2$ and $b = -2$.
We have $2 \mid -2$ as $-2 = (-1) \times 2$ and $-2 \mid 2$ as $2 = (-1) \times (-2)$ and yet $2 \neq -2$!

Proposition 30

Let $(a, b, c, d) \in \mathbb{Z}^4$.
Then,

1. $a \mid b \implies a \mid bc$.
2. $a \mid b$ and $a \mid c \iff \forall\ (u, v) \in \mathbb{Z}^2\ a \mid bu + cv$.
3. $a \mid b$ and $c \mid d \implies ac \mid bd$.
4. If $a \mid b$ then, $\forall\ n \in \mathbb{N}$, $a^n \mid b^n$.

Remark

Let $(a, b, c) \in \mathbb{Z}^3$.
If $a \mid c$ and $b \mid c$ then we do not necessarily have $ab \mid c$.
Indeed, for $a = 2$, $b = 4$ and $c = 28$ for example, we have $2 \mid 28$, $4 \mid 28$ but $4 \times 2 = 8$ does not divide 28!

Example

Let $d \in \mathbb{N}$ be a common divisor of two consecutive integers n and $n + 1$.
Let us prove that $d = 1$.
We have
$$d \mid n \quad \text{and} \quad d \mid n + 1$$
Using point 2 of the previous proposition, we conclude that $d \mid (-1).n + 1.(n + 1)$ i.e. $d \mid 1$.
Hence, $d = 1$ (This example will be considered again in the next sections of this chapter).

6.1.2 Euclidean division in $\mathbb{Z}$

Theorem 11

1. Let $(a, b) \in \mathbb{Z} \times \mathbb{N}^*$.

   Then,
   $$\exists\ !\ (q, r) \in \mathbb{Z}^2 \ \text{ such that } \ a = bq + r \quad \text{and} \quad 0 \leqslant r < b$$

2. Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$.

   Then,
   $$\exists\ !\ (q, r) \in \mathbb{Z}^2 \ \text{ such that } \ a = bq + r \quad \text{and} \quad 0 \leqslant r < |b|$$

This is the Euclidean division of a by b.
q is called quotient of the Euclidean division of a by b and r is called its remainder.

Examples

1. Let us take $a = 24$ and $b = 5$.

   Since $24 = 4 \times 5 + 4$ and $4 < 5$, we deduce that $q = 4$ and $r = 4$.

   However, we can note that we also have $24 = 5 \times 5 + (-1)$. Actually, this is not what we call euclidean division of a by b, since -1 is negative.

2. For $a = 8$ and $b = -3$, we have $q = -2$ and $r = 2$.

3. For $a = 5$ and $b = 24$, we have $q = 0$ and $r = 5$.

Proposition 31
Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$.
$a \mid b$ if the remainder of the Euclidean division of b by a is null.

6.2 GCD (and LCM) in $\mathbb{N}$

6.2.1 Definitions

• GCD
Let $(a, b) \in \mathbb{N}^2$ be distinct.
Let us consider the set $\mathcal{D}$ of common divisors of a and b.
We clearly have $\mathcal{D} \subset \mathbb{N}$.
Moreover, $\mathcal{D} \neq \emptyset$ as $1 \in \mathcal{D}$.
Finally, $\mathcal{D}$ is bounded above by$Min(a, b)$.
We conclude that $\mathcal{D}$ admits a unique largest element (larger than or equal to 1).
Thus, we give the following definition:

Definition 39
Let $(a, b) \in \mathbb{N}^2$.
We call GCD of a and of b the largest common divisor of a and of b (larger than or equal to 1).
We denote it $a \wedge b$.

Examples

1. It is easy to see that $4 \wedge 6 = 2$, $16 \wedge 28 = 4$, $3 \wedge 5 = 1$.

2. We have, $\forall\ n \in \mathbb{N}^*$, $n \wedge (n + 1) = 1$.

   Indeed, we have seen that if $d \mid n$ and $d \mid n + 1$ then $d \mid 1$.

   In particular for $d = n \wedge (n + 1)$, we obtain that $n \wedge (n + 1) = 1$.

3. Let us prove that, $\forall\ n \in \mathbb{N}^*$, $(n + n^2) \wedge (2n + 1) = 1$.

   Let d a common divisor of $n + n^2$ and of $2n + 1$.

   Then, using point 2 of proposition 1.2 (for $u = 2n + 1$ and $v = -4$), we have

   $$d \mid (2n + 1)(2n + 1) - 4(n + n^2)$$

   Hence, $d \mid 1$.

   In particular, for $d = (n + n^2) \wedge (2n + 1)$, we obtain $(n + n^2) \wedge (2n + 1) = 1$.

4. $\forall a \in \mathbb{N}^*$, $a \wedge 1 = 1$ and $a \wedge 0 = a$.

Remark

For $(a, b) \in \mathbb{Z}^2$, we define $a \wedge b = |a| \wedge |b|$.

• LCM
Let $(a, b) \in \mathbb{N}^2$.
Let us consider the set $\mathcal{M}$ of positive common multiples of a and of b.
We clearly have $\mathcal{M} \subset \mathbb{N}$.
Moreover, $\mathcal{M} \neq \emptyset$ as $ab \in \mathcal{M}$.
We conclude that $\mathcal{M}$ has a unique smallest element, which leads to define :

Definition 40
Let $(a, b) \in \mathbb{N}^2$.
We call LCM of a and of b the smallest of common positive multiples of a and of b. We denote it $a \vee b$.

Examples

$4 \vee 6 = 12$, $16 \vee 28 = 112$, $3 \vee 5 = 15$.

6.2.2 Euclidean algorithm

Proposition 32
Let $(a, b) \in (\mathbb{N}^*)^2$ such that $a > b$. We denote q and r the quotient and the remainder of the euclidean division of a by b. Then,

$$a \wedge b = b \wedge r$$

Remark

If $r = 0$, we have in particular, $a \wedge b = b$.

Euclidean algorithm

It is a method to determine the GCD of two integers.
Let $(a, b) \in \mathbb{N}^2$ be such that $a > b$.
Using Euclidean division of a by b, $\exists\ (q_1, r_1) \in \mathbb{Z}^2$ such that

$$\begin{aligned} a &= bq_1 + r_1 \\ 0 &\leqslant r_1 < |b| \end{aligned}$$

Using the previous proposition,
• If $r_1 = 0$ then $a \wedge b = |b|$.
• If $r_1 > 0$ then $a \wedge b = b \wedge r_1$.
In this case, using Euclidean division of b by r_1, $\exists\ (q_2, r_2) \in \mathbb{Z}^2$ such that

$$\begin{aligned} b &= r_1 q_2 + r_2 \\ 0 &\leqslant r_2 < r_1 \end{aligned}$$

Using again the previous proposition, we have
• If $r_2 = 0$ then $a \wedge b = b \wedge r_1 = r_1$.
• If $r_1 > 0$ then $a \wedge b = b \wedge r_1 = r_1 \wedge r_2$. We then repeat the operation.

As $|b| > r_1 > r_2 \ldots$, we construct a sequence $(r_k)_{k\in\mathbb{N}^*}$ of natural numbers strictly decreasing and these integers are all comprised between 0 and $|b|$. This sequence converges thus to 0 and the process stops after a finite number of steps.
So there exists $N \in \mathbb{N}^*$, $(q_1, r_1), \ldots, (q_N, r_N)$ in $\mathbb{Z} \times \mathbb{N}$ and $q_{N+1} \in \mathbb{Z}$ such that

$$\begin{cases} a = bq_1 + r_1 \\ 0 < r_1 < |b| \end{cases}, \begin{cases} b = bq_2 + r_2 \\ 0 < r_2 < r_1 \end{cases}, \ldots, \begin{cases} r_{N-2} = r_{N_1}q_N + r_N \\ 0 < r_N < r_{N-1} \end{cases}, r_{N-1} = r_N q_{N+1} + 0.$$

We then have

$$a \wedge b = b \wedge r_1 = r_1 \wedge r_2 = \ldots = r_{N-1} \wedge r_N = r_N$$

To conclude, $a \wedge b$ is the last non null remainder obtained.

Examples

1. Let us compute $3420 \wedge 222$.

   Successive Euclidean divisions give

$$\begin{aligned} 3420 &= 222 \times 15 + 90 \\ 222 &= 90 \times 2 + 42 \\ 90 &= 42 \times 2 + 6 \\ 42 &= 6 \times 7 + 0 \end{aligned}$$

   Thus we have $3420 \wedge 222 = 6$.

2. Let us compute $3140 \wedge 241$.

   Successive Euclidean divisions give

$$\begin{aligned} 3140 &= 241 \times 13 + 7 \\ 241 &= 7 \times 34 + 3 \\ 7 &= 3 \times 2 + 1 \\ 3 &= 1 \times 3 + 0 \end{aligned}$$

   Thus we have $3140 \wedge 241 = 1$.

6.2.3 Coprime integers

Definition 41

Let $(a, b) \in (\mathbb{N}^*)^2$.
We say that a and b are coprime if

$$a \wedge b = 1$$

i.e. the only common divisors of a and of b are 1 and -1.

Example

1. 3140 and 241 are coprime.

2. $\forall\, n \in \mathbb{N}^*$, n and $n + 1$ are coprime.

Definition 42
Let $n \in \mathbb{N}^*$.
Let $(x_1, \ldots, x_n) \in (\mathbb{Z}^*)^n$.
We say that $x_1, \ldots, x_n$ are pairwise relatively prime if

$$\forall\ (i,j) \in [\![1,n]\!]^2,\ (i \neq j \Longrightarrow x_i \wedge x_j = 1)$$

Proposition 33
Let $(a,b) \in (\mathbb{N}^*)^2$ and $\delta = a \wedge b$ so that $a = \delta a'$ and $b = \delta b'$ where $(a',b') \in (\mathbb{N}^*)^2$. Then,

$$a' \wedge b' = 1$$

Theorem 12 (Bézout)
Let $(a,b) \in (\mathbb{Z}^*)^2$.
Then,

$$a \wedge b = 1 \Longleftrightarrow \exists\ (u,v) \in \mathbb{Z}^2\ au + bv = 1$$

Corollary 2
Let $(a,b) \in (\mathbb{Z}^*)^2$. if $\delta = a \wedge b$ then $\exists\ (u,v) \in \mathbb{Z}^2$ such that $au + bv = \delta$.

Remark

The Euclidean algorithm allows to find such couple (u,v). Indeed,

1. By tracking back the algorithm to find $3420 \wedge 222$, we get

$$\begin{aligned} 6 &= 90 - 42 \times 2 \\ &= 90 - (222 - 90 \times 2) \times 2 = 90 \times 5 - 222 \times 2 \\ &= (3420 - 222 \times 15) \times 5 - 222 \times 2 \\ &= 3420 \times 5 + 222 \times (-77) \end{aligned}$$

   Hence the couple $(u,v) = (5, -77)$ is valid.

2. Similarly, by tracking back the algorithm to find $3140 \wedge 241$, we have

$$3140 \times 69 + 241 \times (-899) = 1$$

   The couple $(u,v) = (69, -899)$ is valid.

Proposition 34 (Characterization of the GCD)
Let $(a,b) \in (\mathbb{Z}^*)^2$ and $d \in \mathbb{Z}^*$. We have

$$d \mid a \quad \text{and} \quad d \mid b \quad \Longleftrightarrow \quad d \mid a \wedge b$$

Proposition 35
Let $(a,b,c) \in (\mathbb{Z}^*)^3$. We have

1. $ac \wedge bc = |c|(a \wedge b)$.
2. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

6.2.4 Gauss's theorem and consequences

Theorem 13 (**Gauss**)
Let $(a, b, c) \in (\mathbb{Z}^*)^3$.
Then,

$$a \mid bc \quad \text{and} \quad a \wedge b = 1 \implies a \mid c$$

Application

Resolution of the equation (E) $9x + 15y = 18$ with unknown variables $(x, y) \in \mathbb{Z}^2$.

1. First of all, we have $9 \wedge 15 = 3$.

   By tracking back the Euclidean algorithm, $3 = -15 + 2 \times 9$.

   Hence, $18 = -6 \times 15 + 12 \times 9$.

   Thus,
   $$(x_0, y_0) = (12, -6)$$
   is a particular solution of (E).

2. Let $(x, y) \in \mathbb{Z}^2$ be a solution of (E).

   Then,
   $$\begin{aligned} 9x + 15y = 18 = 9x_0 + 15y_0 &\iff 3x + 5y = 3x_0 + 5y_0 \\ &\iff 3(x - x_0) = 5(y_0 - y) \end{aligned}$$

   We deduce for example that $3 \mid 5(y_0 - y)$.

   Yet, $5 \wedge 3 = 1$. Using Gauss's theorem, we obtain $3 \mid y_0 - y$.

   Thus, $\exists\ k \in \mathbb{Z},\ y_0 - y = 3k$ i.e.
   $$y = y_0 - 3k = -6 - 3k$$

   From $3(x - x_0) = 5(y_0 - y)$, we obtain then that $x - x_0 = 5k$ i.e.
   $$x = x_0 + 5k = 12 + 5k$$

   Finally, if (x, y) is a solution of (E) then $\exists\ k \in \mathbb{Z}$ such that $(x, y) = (12 + 5k, -6 - 3k)$, i.e., denoting $\mathcal{S}$ the set of solutions of (E),
   $$\mathcal{S} \subset \{\ (12 + 5k, -6 - 3k),\ k \in \mathbb{Z}\ \}$$

3. Reciprocally, if $(x, y) \in \{\ (12 + 5k, -6 - 3k),\ k \in \mathbb{Z}\ \}$ then, $\exists\ k \in \mathbb{Z}$ such that $x = 12 + 5k$ and $y = -6 - 3k$.

   Then,
   $$\begin{aligned} 9x + 15y &= 9 \times 12 + 45k + 15 \times (-6) - 45k \\ &= 9x_0 + 15y_0 \\ &= 18 \end{aligned}$$

   Hence, $(x, y) \in \mathcal{S}$ and $\{\ (12 + 5k, -6 - 3k),\ k \in \mathbb{Z}\ \} \subset \mathcal{S}$.

4. Conclusion :
   $$\mathcal{S} = \{\ (12 + 5k, -6 - 3k),\ k \in \mathbb{Z}\ \}$$

Consequences

Proposition 36

1. Let $(a, b, c) \in (\mathbb{Z}^*)^3$.

   Then,
$$a \wedge b = 1 \quad \text{and} \quad a \wedge c = 1 \iff a \wedge bc = 1$$

2. Let $n \in \mathbb{N}^*$.

   Let $(a, b_1, \ldots, b_n) \in (\mathbb{Z}^*)^{n+1}$.

   If $\forall\ i \in [\![1, n]\!]$, $a \wedge b_i = 1$ then
$$a \wedge \prod_{i=1}^{n} b_i = 1$$

3. Let $(a, b) \in (\mathbb{Z}^*)^2$ and $(p, q) \in (\mathbb{N}^*)^2$.

   If $a \wedge b = 1$ then $a^p \wedge b^q = 1$.

Proposition 37
Let $n \in \mathbb{N}^*$.
Let $(a, b_1, \ldots, b_n) \in (\mathbb{Z}^*)^{n+1}$.
If $\forall\ i \in [\![1, n]\!]$, $b_i \mid a$ and if $\forall\ (i, j) \in [\![1, n]\!]^2$ such that $i \neq j$, $b_i \wedge b_j = 1$ then
$$\prod_{i=1}^{n} b_i \mid a$$

Remark

$4 \mid 8$, $8 \mid 8$ and yet 32 does not divide 8! This is because 4 and 8 are not coprime.

6.3 Prime numbers in $\mathbb{N}$

6.3.1 Definition and properties

Definition 43
Let $p \in \mathbb{N} - \{0, 1\}$.
We say that p is a prime number if its only divisors are 1 and p.

Example

$2, 3, 5, 7, 11, 13, 17, 19, 23...$ are prime numbers.

Notation
We note $\mathcal{P}$ the set of prime numbers.

Proposition 38
Let $p \in \mathcal{P}$ and $n \in \mathbb{Z}^*$.
Then,
$$p \mid n \quad \text{or} \quad p \wedge n = 1$$

Proposition 39
Let $p \in \mathcal{P}$ and $(x_1, \ldots, x_n) \in (\mathbb{Z}^*)^n$ (where $n \in \mathbb{N}^*$).
Then,

$$p \mid \prod_{i=1}^{n} x_i \iff \exists\ i_0 \in [\![1, n]\!] \quad p \mid x_{i_0}$$

6.3.2 The set $\mathcal{P}$

Proposition 40
Any natural number larger than or equal to 2 is divisible by a nombre premier.

Theorem 14
The set $\mathcal{P}$ is infinite.

6.3.3 Decomposition into product of prime factors

Theorem 15
Any natural number larger than or equal to 2 can be decomposed into a product of prime factors and this decomposition is unique, up to the the factors order i.e.
$\forall\ n \in \mathbb{N} - \{0, 1\}$, $\exists\ r \in \mathbb{N}^*$, $\exists\ (p_1, \ldots, p_r) \in \mathcal{P}^r$ and $\exists\ (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ such that

$$n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} = \prod_{i=1}^{r} p_i^{\alpha_i}$$

Examples

1. $7007 = 7^2 \times 11 \times 13$.
2. $9100 = 2^2 \times 5^2 \times 7 \times 13$.

Theorem 16
Let $(a, b) \in (\mathbb{N} - \{0, 1\})^2$ such that $a = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $b = \prod_{i=1}^{r} p_i^{\beta_i}$.
Then,

$$a \wedge b = \prod_{i=1}^{r} p_i^{Min(\alpha_i, \beta_i)} \quad \text{and} \quad a \vee b = \prod_{i=1}^{r} p_i^{Max(\alpha_i, \beta_i)}$$

Example

We have in fact $7007 = 2^0 \times 5^0 \times 7^2 \times 11^1 \times 13^1$ and $9100 = 2^2 \times 5^2 \times 7^1 \times 11^0 \times 13^1$.
Hence,

$$\begin{aligned} 7007 \wedge 9100 &= 2^0 \times 5^0 \times 7^1 \times 11^0 \times 13^1 = 91 \\ 7007 \vee 9100 &= 2^2 \times 5^2 \times 7^2 \times 11^1 \times 13^1 = 700700 \end{aligned}$$

6.4 Congruence in $\mathbb{Z}$

In the next paragraph, $n \in \mathbb{N}^*$

6.4.1 Definitions and properties

Definition 44
Let $(a, b) \in \mathbb{Z}^2$.
We say that a and b are congruent modulo n, and we denote $a \equiv b\,[n]$ if $n \mid a - b$.

Examples

$4 \equiv 12\,[2]$ as $12 - 4 = 8 = 4 \times 2$ and $7 \equiv 4\,[3]$ as $4 - 7 = -3 = (-1) \times 3$.

Remark

$a \equiv b\,[n]$ if and only if the remainder of the euclidean division of a by n is equal to the remainder of the euclidean division of b by n.

Proposition 41
Let $a \in \mathbb{Z}$. then, $\exists!\, r \in [|0, n-1|]$ such that $a \equiv r\,[n]$.

Proposition 42
Let $(a, b, c) \in \mathbb{Z}^3$.
Then,

1. $a \equiv a\,[n]$ (Reflexivity).
2. $a \equiv b\,[n] \iff b \equiv a\,[n]$ (Symmetry).
3. $a \equiv b\,[n]$ and $b \equiv c\,[n] \Longrightarrow a \equiv c\,[n]$ (Transitivity).

We say that $\equiv\ [n]$ is a relation of equivalence.

6.4.2 Compatibility of congruence with operations in $\mathbb{Z}$

Proposition 43
Let $(a, b, c, d) \in \mathbb{Z}^4$.
If $a \equiv b\,[n]$ and $c \equiv d\,[n]$ then,

$$\begin{array}{rcl} a + c & \equiv & (b + d)\,[n] \\ ac & \equiv & bd\,[n] \end{array}$$

Corollary 3
Let $(a, b) \in \mathbb{Z}^2$.
If $a \equiv b\,[n]$ then, $\forall\, m \in \mathbb{N}$, $a^m \equiv b^m\,[n]$.

Example

Let us prove that $\forall\, n \in \mathbb{N}$, $5 \mid 2^{2n+1} + 3^{2n+1}$.
We have

$$2^{2n+1} + 3^{2n+1} = 4^n \times 2 + 9^n \times 3$$

Or $4^n \times 2 \equiv 4^n \times 2\,[5]$.
Moreover, $9 \equiv 4\,[5]$ Thus $9^n \equiv 4^n\,[5]$ and hence $9^n \times 3 \equiv 4^n \times 3\,[5]$ (as $3 \equiv 3\,[5]$).
So,

$$4^n \times 2 + 9^n \times 3 \equiv 4^n(2 + 3)\,[5] \equiv 0\,[5]$$

so

$$5 \mid 4^n \times 2 + 9^n \times 3$$

6.4.3 Fermat's little theorem

Theorem 17 (Fermat's little theorem)
Let $p \in \mathcal{P}$.
Then, for all $n \in \mathbb{Z}$, $n^p \equiv n\ [p]$.

Example

Let us prove that $\forall\ n \in \mathbb{Z}$, $42 \mid n^7 - n$.

$$42 = 2 \times 3 \times 7$$

Yet, 2, 3 and 7 are pairwise relatively prime. Hence using proposition 2.5, it is sufficient to show that $2 \mid n^7 - n$, $3 \mid n^7 - n$ and $7 \mid n^7 - n$.
Using Fermat's little theorem, we have $n^2 \equiv n\,[2]$. Thus, $(n^2)^3 \equiv n^3\,[2]$.
Yet $n^7 = (n^2)^3 n$.
Thus,

$$n^7 \equiv n^3.n\,[2] \equiv n^4\,[2] \equiv n^2\,[2] \equiv n\,[2]$$

We conclude that that

$$n^7 - n \equiv 0\,[2]$$

Similarly, using Fermat, $n^3 \equiv n\,[3]$.
So,

$$n^7 = (n^3)^2 n \equiv n^3\,[3] \equiv n\,[3]$$

and

$$n^7 - n \equiv 0\,[3]$$

Finally, using Fermat, we directly get

$$n^7 \equiv n\,[7]$$

So,

$$n^7 - n \equiv 0\,[7]$$

To conclude, we have obtained that $2 \mid n^7 - n$, $3 \mid n^7 - n$ and $7 \mid n^7 - n$. Thus, $2.3.7 \mid n^7 - n$. i.e.

$$42 \mid n^7 - n$$

Chapter 7

Polynomials

In this chapter, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

7.1 Set of univariate polynomials with coefficients in $\mathbb{K}$

7.1.1 Generalities

Definition 45
We call univariate polynomials with coefficients in $\mathbb{K}$ every sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ null above a certain rank.
i.e. : $P = (a_n)_{n\in\mathbb{N}}$ is a polynomial with coefficients in $\mathbb{K}$ if

$$\forall\, n \in \mathbb{N},\ a_n \in \mathbb{K} \ \text{ and } \ \exists\, N \in \mathbb{N}, \ \ \forall\, n \in \mathbb{N}, \ \ (n > N \Rightarrow a_n = 0)$$

We denote $P = (a_0, a_1, \ldots, a_N, 0, \ldots, 0, \ldots)$.
The numbers $a_0, \ldots, a_N$ are called coefficients of P.

Examples

1. $P_1 = (1, 2, 3, 0, \ldots, 0, \ldots)$ is a polynomial.
2. $X = (0, 1, 0, \ldots, 0, \ldots)$ is a polynomial.

Notations

1. The set of univariate polynomials and with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}[X]$.
2. The polynomial defined by the sequence null is called the zero polynomial. It is denoted $0_{\mathbb{K}[X]}$.

Definition 46

1. We call constant polynomial in $\mathbb{K}[X]$ any polynomial $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ such that

$$\forall\, n \in \mathbb{N}^*, \ \ a_n = 0$$

   i.e. $P = (a_0, 0, \ldots, 0, \ldots)$.

2. We call monomial in $\mathbb{K}[X]$ any polynomial $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ such that

$$\exists\ n_0 \in \mathbb{N},\quad \forall\ n \in \mathbb{N},\quad (n \neq n_0 \Rightarrow a_n = 0)$$

i.e. $P = (0, \ldots, 0, a_{n_0}, 0, \ldots, 0, \ldots)$.

3. We say that two polynomials $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $Q = (b_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ are equal if $\forall\ n \in \mathbb{N},\quad a_n = b_n$.

Definition 47 (degree d'un polynomials)
Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$.

1. If $P \neq 0$, We call degree of P the largest natural number N such that $a_N \neq 0$. We denote $N = d(P)$.

   We have

$$N = d(P) \Longleftrightarrow \begin{cases} a_N \neq 0 \\ \forall\ n \in \mathbb{N}, (n > N \Rightarrow a_n = 0) \end{cases}$$

2. If $P = 0$, we denote $d(0) = -\infty$.

Example

Taking up the previous example, $d(P_1) = 2$ and $d(X) = 1$.

7.1.2 Sum of two polynomials

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $Q = (b_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ with $N_1 = d(P)$ and $N_2 = d(Q)$.
Let us consider the sequence $(a_n + b_n)_{n\in\mathbb{N}}$.
We have

- $\forall\ n \in \mathbb{N}$, $a_n + b_n \in \mathbb{K}$.
- for all $n > Max(N_1, N_2)$, $a_n + b_n = 0$.

We conclude that $(a_n + b_n) \in \mathbb{K}[X]$,
from which we get the following definition :

Definition 48
Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $Q = (b_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$.
We define $P + Q \in \mathbb{K}[X]$ by

$$P + Q = (a_n + b_n)_{n\in\mathbb{N}}$$

Example

If $P = (1, 2, 3, -4, 0, \ldots, 0, \ldots)$, $Q_1 = (-1, 2, 20, 0, \ldots, 0, \ldots)$ and $Q_2 = (-1, 2, 20, 4, 0, \ldots, 0, \ldots)$ then

$$P + Q_1 = (0, 4, 23, -4, 0, \ldots, 0, \ldots) \quad \text{and} \quad P + Q_2 = (0, 4, 23, 0, \ldots, 0, \ldots)$$

We note that $d(P) = 3$, $d(Q_1) = 2$, $d(Q_2) = 3$, $d(P + Q_1) = 3$ and $d(P + Q_2) = 2$.

Proposition 44
Let $(P, Q) \in \mathbb{K}[X]^2$.
Then,

1. $d(P + Q) \leqslant Max(d(P), d(Q))$.
2. If $d(P) \neq d(Q)$ then $d(P + Q) = Max(d(P), d(Q))$.

Remark

For all $(P,Q) \in \mathbb{K}[X]^2$, $P+Q = Q+P$ and $P + 0_{\mathbb{K}[X]} = P$.

7.1.3 External product

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ with $N = d(P)$ and $\lambda \in \mathbb{K}$.
Let us consider the sequence $(\lambda\, a_n)_{n\in\mathbb{N}}$.
We have

- $\forall\, n \in \mathbb{N}$, $\lambda\, a_n \in \mathbb{K}$.
- $\forall\, n \in \mathbb{N}$, $(n > N \Longrightarrow \lambda\, a_n = 0)$.

We deduce that $(\lambda\, a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$,
from which we get the following definition:

Definition 49

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $\lambda \in \mathbb{K}$.
We define $\lambda\, P \in \mathbb{K}[X]$ by

$$\lambda\, P = (\lambda\, a_n)_{n\in\mathbb{N}}$$

Example

If $P = (-1, 2, 10, 3, 0, \ldots, 0, \ldots)$ then $3P = (-3, 6, 30, 9, 0, \ldots, 0, \ldots)$.

Proposition 45

Let $P \in \mathbb{K}[X]$ and $\lambda \in \mathbb{K}^*$.
Then,

$$d(\lambda\, P) = d(P)$$

7.1.4 Internal product

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $Q = (b_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ with $N_1 = d(P)$ and $N_2 = d(Q)$.
Let us consider the sequence $(c_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j$$

We have

- $\forall\, n \in \mathbb{N}$, $c_n \in \mathbb{K}$.
- for all $n > N_1 + N_2$,

$$\begin{aligned} c_n &= \sum_{k=0}^{N_1} a_k b_{n-k} + \sum_{k=N_1+1}^{n} a_k b_{n-k} \\ &= \sum_{k=0}^{N_1} a_k b_{n-k} \quad \text{as} \quad \forall\, k > N_1, \quad a_k = 0 \\ &= 0 \quad \text{as} \quad n-k > N_2 \quad \Longrightarrow \quad b_{n-k} = 0 \end{aligned}$$

We conclude that $(c_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$.
Thus, we get the following definition:

Definition 50

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ and $Q = (b_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$.
We define $PQ \in \mathbb{K}[X]$ by $PQ = (c_n)$ where

$$\forall\ n \in \mathbb{N},\quad c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j$$

Example

If $P = (1, 2, -3, 0, \dots, 0, \dots)$ and $Q = (2, 5, 4, 3, 0, \dots, 0, \dots)$ then

$$PQ = (2, 9, 8, -4, -6, -9, 0, \dots, 0, \dots)$$

Proposition 46

Let $(P, Q) \in \mathbb{K}[X]^2$.
Then,

$$d(PQ) = d(P) + d(Q)$$

Proposition 47

Let $(P, Q, R) \in \mathbb{K}[X]^3$.
Then,

1. $PQ = QP$ (Commutativity).
2. $(PQ)R = P(QR)$ (Associativity).
3. $P(Q + R) = PQ + PR$ (Distributivity).

Proposition 48

$$\forall\ (P, Q) \in \mathbb{K}[X]^2,\quad (PQ = 0 \Longleftrightarrow P = 0 \quad \text{or} \quad Q = 0)$$

7.1.5 Final notation for a polynomial

Definition 51

We call X the polynomial of $\mathbb{K}[X]$ defined by the sequence $(a_n)_{n\in\mathbb{N}}$ where

$$a_1 = 1 \quad \text{and} \quad \forall n \in \mathbb{N} \setminus \{1\},\quad a_n = 0$$

X is called the unknown variable.

Proposition 49

$$\forall n \in \mathbb{N},\quad X^n = (b_p)_{p\in\mathbb{N}} \quad \text{where} \quad b_{n+1} = 1 \quad \text{and} \quad \forall p \in \mathbb{N} \setminus \{n+1\},\quad b_p = 0$$

Conclusion

Let $P = (a_n)_{n\in\mathbb{N}} \in \mathbb{K}[X]$ with $N = d(P)$.
We have

$$\begin{aligned}
P &= (a_0, a_1, \dots, a_N, 0, \dots, 0, \dots) \\
&= a_0(1, 0, \dots, 0, \dots) + a_1(0, 1, 0, \dots, 0, \dots) + \dots + a_N(0, \dots, 0, 1, 0, \dots, 0) \\
&= a_0X^0 + a_1X^1 + \dots + a_NX^N
\end{aligned}$$

So

$$P = \sum_{k=0}^{N} a_k X^k$$

7.1.6 Other operations on polynomials

Definition 52

Let $P = \sum_{k=0}^{N} a_k X^k \in \mathbb{K}[X]$ and $Q \in \mathbb{K}[X]$.
We define $P \circ Q \in \mathbb{K}[X]$ by

$$P \circ Q = P(Q) = \sum_{k=0}^{N} a_k Q^k$$

Example

If $P = X^3 + 3X - 4$ and $Q = X + 1$ then,

$$P(X+1) = (X+1)^3 + 3(X+1) - 4$$

Definition 53

Let $P = \sum_{k=0}^{N} a_k X^k \in \mathbb{K}[X]$.

We call derivative polynomial of P the polynomial

$$P' = \sum_{k=1}^{N} k a_k X^{k-1}$$

Similarly, we define the derivative polynomial of P' by

$$P'' = \sum_{k=2}^{N} k(k-1) a_k X^{k-2}$$

We denote $P^{(0)} = P$, $P^{(1)} = P'$, $P^{(2)} = P'' = (P')'$ and for all $\alpha \in \mathbb{N}^*$, $P^{(\alpha)} = (P^{(\alpha-1)})'$.

Proposition 50

1. $\forall\ P \in \mathbb{K}[X] \setminus \mathbb{K}$, $d(P') = d(P) - 1$.
2. $\forall\ (P, Q) \in \mathbb{K}[X]^2$ and $\forall\ \lambda \in K$, $(P + \lambda Q)' = P' + \lambda Q'$ and $(PQ)' = P'Q + PQ'$.

7.1.7 Polynomial functions

Definition 54

Let $P = \sum_{k=0}^{N} a_k X^k \in \mathbb{K}[X]$.

We define the function

$$\begin{array}{rcl} \widetilde{P} : \mathbb{K} & \longrightarrow & \mathbb{K} \\ x & \longmapsto & \displaystyle\sum_{k=0}^{N} a_k x^k \end{array}$$

$\widetilde{P}$ is called polynomial function associated with P.

Proposition 51
$\forall\ (P,Q) \in \mathbb{K}[X]^2$ and $\forall\ \lambda \in K$,

$$\widetilde{P+\lambda Q} = \widetilde{P} + \lambda\widetilde{Q} \quad \text{and} \quad \widetilde{PQ} = \widetilde{P}\widetilde{Q}$$

Example

If $P = (-1, 2, 10, 3, 0, \dots, 0, \dots) = -1 + 2X + 10X^2 + 3X^3$ then $\widetilde{P}(-1) = -1 - 2 + 10 - 3 = 4$.

7.1.8 Arithmetic in $\mathbb{K}[X]$

Definition 55
Let $(A, B) \in \mathbb{K}[X]^2$.
We say that A divides B, and we write $A \mid B$, if

$$\exists\ Q \in \mathbb{K}[X],\ B = AQ$$

Examples

$X + 1 \mid X^2 - 1$ in $\mathbb{R}[X]$ and $X + i \mid X^2 + 1$ in $\mathbb{C}[X]$.

Remarks

1. $\forall\ A \in \mathbb{K}[X]$, $A \mid 0$.
2. Let $B \in \mathbb{K}[X]$, $0 \mid B \iff B = 0$.
3. Let $(A, B) \in \mathbb{K}[X]^2$.
   If $A \mid B$ then $d(A) \leqslant d(B)$.

Proposition 52
Let $(A, B, C) \in (\mathbb{K}[X]^*)^3$.
then,

1. $A \mid A$ (reflexivity).
2. $A \mid B$ and $B \mid A \iff \exists\ \lambda \in \mathbb{K}^*$, $B = \lambda\, A$.
3. $A \mid B$ and $B \mid C \implies A \mid C$ (transitivity)

Remark

In $\mathbb{K}[X]$, $P \mid Q$ and $Q \mid P$ does not imply $P = Q$.
For example, in $\mathbb{R}[X]$, $2X^2 \mid 5X^2$ and $5X^2 \mid 2X^2$ and yet $2X^2 \neq 5X^2$!

Proposition 53
Let $(A, B, C, D) \in (\mathbb{K}[X]^*)^4$.
then,

1. $A \mid B \implies A \mid BC$.
2. $A \mid B$ and $A \mid C \iff \forall\ (U, V) \in \mathbb{K}[X]^2$, $A \mid BU + CV$.

3. $A \mid B$ and $C \mid D \Longrightarrow AC \mid BD$.
4. If $A \mid B$ then $\forall\, n \in \mathbb{N}^*$, $A^n \mid B^n$.

Euclidean division in $\mathbb{K}[X]$

Theorem 18

$\forall\,(A,B) \in \mathbb{K}[X] \times \mathbb{K}[X]^*$, $\exists\,!\,(Q,R) \in \mathbb{K}[X]^2$ such that

$$A = BQ + R \quad \text{and} \quad d(R) < d(B)$$

It is the euclidean division of $A \text{by} B$.
Q is called quotient of the euclidean division of A by B. R is the remainder of this division.

Practical method to find Q and R

Let $(A,B) \in \mathbb{K}[X] \times \mathbb{K}[X]^*$.
• 1st case : $A = 0$ or $d(A) < d(B)$. Then $A = 0B + A$. Hence $Q = 0$ and $R = A$.
• 2nd case : $d(A) \geqslant d(B)$.
We order the two polynomials A and B by order of decreasing power.

Examples

1. For $A = X^3 + 2X + 1$ and $B = X + 1$, we find that

$$Q = X^2 - X + 3 \quad \text{and} \quad R = -2$$

2. For $A = X^4 + 2X^3 - X + 6$ and $B = X^3 - 6X^2 + X + 4$, we have

$$Q = X + 8 \quad \text{and} \quad R = 47X^2 - 13X - 26$$

Remark

Consequently, $A \mid B$ if the remainder of the euclidean division of B by A is null.

7.2 Roots of a polynomial

7.2.1 Definition and properties

Definition 56

Let $P \in \mathbb{K}[X]$ and $a \in \mathbb{K}$.
We say that a is a root (or a zero) of P if $\widetilde{P}(a) = 0$.

Example

2 is a root of $X^2 - X - 2$ in $\mathbb{R}[X]$ as $2^2 - 2 - 2 = 0$.

Proposition 54

Let $P \in \mathbb{K}[X]$ and $a \in \mathbb{K}$.
Then,

$$a \;\text{ root of }\; P \Longleftrightarrow X - a \mid P$$

Proposition 55
Let $P \in \mathbb{K}[X]$, $n \in \mathbb{N}^*$ and $(a_1, \ldots, a_n) \in \mathbb{K}^n$ be pairwise distinct.
If $a_1, \ldots, a_n$ are roots of P then

$$\prod_{i=1}^{n}(X - a_i) \mid P$$

Corollary 4

1. Let $P \in \mathbb{K}[X]$ and $n \in \mathbb{N}^*$.

   If $d(P) < n$ and if P admits at least n distinct roots, then $P = 0$ (so a polynomial of degree n admits at most n distinct roots).

2. If $P \in \mathbb{K}[X]$ nullifies an infinite number of times, then $P = 0$.

7.2.2 Taylor's formula

Theorem 19
Let us consider

$$\mathbb{K}_N[X] = \{\ P \in \mathbb{K}[X],\ d(P) \leqslant N\ \}$$

Let $P \in \mathbb{K}_N[X]$ and $a \in \mathbb{K}$.
Then,

$$P = \sum_{k=0}^{N} \frac{\widetilde{P^{(k)}(a)}}{k!}(X-a)^k$$

7.2.3 Order of multiplicity of a root

Definition 57
Let $P \in \mathbb{K}[X]$, $a \in \mathbb{K}$ and $\alpha \in \mathbb{N}^*$.

1. We say that a is at least a root of order α of P if

   $$(X-a)^\alpha \mid P$$

   i.e. $\exists\ Q \in \mathbb{K}[X],\ P = (X-a)^\alpha Q$.

2. We say that a is exactly a root of order α of P if

   $$(X-a)^\alpha \mid P \quad \text{and} \quad (X-a)^{\alpha+1} \nmid P$$

   i.e. $\exists\ Q \in \mathbb{K}[X],\ P = (X-a)^\alpha Q$ and $\widetilde{Q}(a) \neq 0$.

Theorem 20
Let $P \in \mathbb{K}[X]$, $a \in \mathbb{K}$ and $\alpha \in \mathbb{N}^*$.
Then,

a is exactly a root of order α of $P \Longleftrightarrow \widetilde{P}(a) = \widetilde{P}'(a) = \ldots = \widetilde{P^{(\alpha-1)}}(a) = 0 \quad \text{and} \quad \widetilde{P^{(\alpha)}}(a) \neq 0$

Example

Let $P = X^5 - 4X^4 + 14X^3 - 22X + 17X - 5$. We note that $\widetilde{P}(1) = 0$. We look for the order of multiplicity of 1 as a root of P. We have $P' = 5X^4 - 20X^3 + 42X^2 - 44X + 17$. Thus, $\widetilde{P}'(1) = 0$, so 1 is at least a root of order 2 of P.
$P'' = 20X^3 - 60X^2 + 84X - 44$. Thus, $\widetilde{P}''(1) = 0$, so 1 is at least a root of order 3 of P.
$P''' = 60X^2 - 120X + 84$. Thus, $\widetilde{P}'''(1) = 24$, so 1 is exactly a root of order 2 of P.

7.2.4 Irreducible polynomials in $\mathbb{R}[X]$ and $\mathbb{C}[X]$ (admitted)

Theorem 21
Let $P \in \mathbb{C}[X]$ non constant.
Then, P admits at least one root in $\mathbb{C}$.

Definition 58
Let $P \in \mathbb{K}[X]$.
We say that P is irreducible in $\mathbb{K}[X]$ if $d(P) \geqslant 1$ and the only divisors of P are the constant polynomials of $\mathbb{K}[X]^*$ and polynomials of type λP where $\lambda \in \mathbb{K}^*$.

Theorem 22
Any polynomial of $\mathbb{K}[X]$ of degree larger than or equal to 1 admits a unique decomposition into a product of irreducible polynomials in $\mathbb{K}[X]$.

Definition 59
Let $P \in \mathbb{K}[X]$.
We say that P is split over $\mathbb{K}$ if $\exists\ \lambda \in \mathbb{K}^*$, $n \in \mathbb{N}^*$ and $x_1, \ldots, x_n \in \mathbb{K}$ such that

$$P = \lambda \prod_{i=1}^{n} (X - x_i)$$

Theorem 23

1. Every polynomial of $\mathbb{C}[X]$ non constant is split over $\mathbb{C}$.
2. Irreducible polynomials of $\mathbb{C}[X]$ are polynomials of degree 1.
3. Irreducible polynomials of $\mathbb{R}[X]$ are polynomials of degree 1 and polynomials of degree 2 with strictly negative discriminant.

Chapter 8

Numerical sequences

8.1 Definitions and examples

8.1.1 Generalities

Definition 60
A numerical sequence is an application from $\mathbb{N}$ to $\mathbb{R}$ (or from $\mathbb{N} \cap [n_0, +\infty[$ to $\mathbb{R}$ where $n_0 \in \mathbb{N}$ is fixed).

We note $u : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u(n) = u_n \end{array}$.
u_n is called the general term of the sequence $(u_n)_{n\in\mathbb{N}}$.

Notation

The set of real sequences is denoted $\mathbb{R}^{\mathbb{N}}$.

Definition 61
A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be

- constant if
$$\forall\, n \in \mathbb{N}, \quad u_n = u_{n+1}$$
- stationary if it is constant above a certain rank, i.e.
$$\exists\, N \in \mathbb{N}, \quad \forall\, n \in \mathbb{N}, \quad (n \geqslant N \quad \Longrightarrow \quad u_n = u_{n+1})$$

Example

The sequence (u_n), defined for all integer $n \in \mathbb{N}^*$ by $u_n = E(\frac{5}{n})$ is stationary.

8.1.2 Definitions related with order

Definition 62
A sequence $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is

- bounded above if
$$\exists\, M \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}, \quad u_n \leqslant M$$

- bounded below if
$$\exists\, m \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}, \quad u_n \geqslant m$$
- bounded if it is bounded below and bounded above

Remark

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.
We have
$$(u_n) \quad \text{bounded} \quad \Longleftrightarrow \quad \exists\, M \in \mathbb{R}^+, \quad \forall\, n \in \mathbb{N}, \quad |u_n| \leqslant M$$

Examples

The sequences (u_n) and (v_n), defined for all natural number n by $u_n = \cos(e^n)$ and $v_n = (-1)^n$ are bounded. The sequence (w_n) defined, for all natural number n by $w_n = n^2$ is not bounded. It is bounded below, not bounded above.

Definition 63

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

- We say that (u_n) is increasing if
$$\forall\, n \in \mathbb{N}, \quad u_{n+1} \geqslant u_n$$
- We say that (u_n) is strictly increasing if
$$\forall\, n \in \mathbb{N}, \quad u_{n+1} > u_n$$
- We say that (u_n) is decreasing if
$$\forall\, n \in \mathbb{N}, \quad u_{n+1} \leqslant u_n$$
- We say that (u_n) is strictly decreasing if
$$\forall\, n \in \mathbb{N}, \quad u_{n+1} < u_n$$
- We say that (u_n) is monotone if it is increasing or decreasing

Remark

To study the monotony of a sequence (u_n), it is sufficient to find the sign of
$$u_{n+1} - u_n$$
If $u_n \neq 0$ for all n, we can also compare
$$\frac{u_{n+1}}{u_n}$$
to 1.

Examples

1. Let $(u_n)_{n\in\mathbb{N}}$ be defined by
$$\forall\, n \in \mathbb{N}, \quad u_n = \frac{1}{n+1}$$
Then, for all integer n,
$$u_{n+1} - u_n = -\frac{1}{(n+1)(n+2)} < 0$$
Hence, (u_n) is strictly decreasing.

2. Let $(u_n)_{n\in\mathbb{N}^*}$ be defined by
$$\forall\, n \in \mathbb{N}^*, \quad u_n = \frac{n!}{2^n}$$
We have, for all integer n strictly positif,
$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2} \geqslant 1$$
Hence, (u_n) is increasing.

8.1.3 Examples of sequences

1- Arithmetic sequences

Definition 64

We say that the sequence (u_n) is arithmetic if $\exists\, r \in \mathbb{R}$ such that
$$\forall\, n \in \mathbb{N} \;\; u_{n+1} = u_n + r$$
where $u_0 \in \mathbb{R}$ is given.
r is called common difference of the sequence (u_n).

Example

The sequence (u_n) such that, for all n, $u_{n+1} = u_n + 2$ and $u_0 = 0$ is arithmetic of common difference 2. It is actually the sequence of even integers.

Remark

If the sequence (u_n) is arithmetic 0, then it is constant.

Expression of u_n as a function of n

Let (u_n) be an arithmetic sequence of common difference r. Then, for all natural number n, we have
$$\begin{array}{rcl} u_n &=& u_{n-1} + r \\ u_{n-1} &=& u_{n-2} + r \\ \vdots & & \vdots \\ u_1 &=& u_0 + r \end{array}$$
Adding these equalities, we obtain :
$$\forall\, n \in \mathbb{N}, \quad u_n = u_0 + nr$$
More generally, for $n_0 \in \mathbb{N}$ fixed, we have, for all natural number $n \geq n_0$,
$$u_n = u_{n_0} + (n - n_0)r$$

Example

Let (u_n) be arithmetic of common difference 3 such that $u_10 = 15$. Then, $u_{100} = 15+90\times3 = 285$.

Sum of $n+1$ first terms

Let (u_n) be arithmetic of common difference r. We denote $S_n = \sum_{k=0}^{n} u_k$. We have

$$\begin{array}{rcl} S_n & = & u_0 + u_1 + \ldots + u_{n-1} + u_n \\ & = & u_n + u_{n-1} + \ldots + u_1 + u_0 \end{array}$$

Then, $2S_n = (u_n+u_0)+(u_{n-1}+u_1)+\ldots+(u_0+u_n)$. Yet, for all integer $p \in [|0;n|]$, $u_{n-p}+u_p = 2u_0 + nr = u_0 + u_n$. Then, we obtain

$$S_n = \frac{n+1}{2}(u_0+u_n) = \frac{n+1}{2}(2u_0+nr)$$

Remark

Let $n_0 \in \mathbb{N}$ be fixed. For all integer $n \geq n_0$, we have

$$u_{n_0} + \ldots + u_n = \frac{n-n_0+1}{2}(u_{n_0}+u_n)$$

Example

For all non null natural number n, we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2- Geometric sequences

Definition 65

We say that the sequence (u_n) is geometric if $\exists\, q \in \mathbb{R}$ such that

$$\forall\, n \in \mathbb{N} \quad u_{n+1} = qu_n$$

where $u_0 \in \mathbb{R}$ is given.
q is called common ratio of the sequence (u_n).

Example

The sequence (u_n) such that, for all n, $u_{n+1} = 3u_n$ and $u_0 = 1$ is geometric of common ratio 3.

Remark

If $q = 0$ or if $q = 1$ or if $u_0 = 0$ then the sequence (u_n) is constant....

Expression of u_n as a function of n

Let (u_n) a geometric sequence of common ratio q non null. Then, for all natural number n, we have

$$\begin{array}{rcl} u_n & = & qu_{n-1} \\ u_{n-1} & = & qu_{n-2} \\ \vdots & & \vdots \\ u_1 & = & qu_0 \end{array}$$

After multiplying these equalities and simplifying them we obtain :

$$\forall\, n \in \mathbb{N}, \quad u_n = q^n u_0$$

More generally, for $n_0 \in \mathbb{N}$ fixed, we have, for all natural number $n \geq n_0$,

$$u_n = q^{n-n_0} u_{n_0}$$

Example

Let (u_n) be geometric of common difference -2 such that $u_{10} = 4$. Then, $u_{100} = (-2)^{90} \times 4 = 2^{92}$.

Sum of $n+1$ first terms

Let (u_n) be an arithmetic sequence of common difference q. We denote $S_n = \displaystyle\sum_{k=0}^{n} u_k$.

If $q = 1$, we have

$$S_n = (n+1)u_0$$

We assume $q \neq 1$. We have

$$\begin{array}{rcl} S_n & = & u_0 + \ldots + u_n = u_0 + qu_0 + \ldots q^n u_0 \\ qS_n & = & qu_0 + q^2 u_0 + \ldots + q^{n+1} u_0 \end{array}$$

Then, $S_n - qS_n = u_0(1 - q^{n+1})$. Thus,

$$S_n = u_0 \frac{1 - q^{n+1}}{1 - q}$$

Remark

Let $n_0 \in \mathbb{N}$ be fixed. For all integer $n \geq n_0$, we have

$$u_{n_0} + \ldots + u_n = u_{n_0} \frac{1 - q^{n-n_0+1}}{1 - q}$$

Example

For all natural number n, we have

$$\sum_{k=1}^{n} \frac{1}{2^k} = 2 - \frac{1}{2^n}$$

3- Arithmetico-geometric sequences

Definition 66
We say that the sequence (u_n) is arithmetico-geometric if $\exists\,(a,b) \in (\mathbb{R}^*)^2$ where $a \neq 1$ such that

$$\forall\, n \in \mathbb{N} \;\; u_{n+1} = a u_n + b$$

where $u_0 \in \mathbb{R}$ is given.

Example

The sequence (u_n) is such that, for all n, $u_{n+1} = 3u_n + 2$ and $u_0 = 0$ is arithmetico-geometric.

Remark

For $a = 1$, the sequence (u_n) is arithmetic. For $b = 0$, it is geometric.

Expression of u_n as a function of n

Proposition 56
Let (u_n) be an arithmetico-geometric sequence. Then, $\exists\, l \in \mathbb{R}$ such that the sequence $(u_n - l)$ is geometric of common ratio a.

Remark

l is actually solution of the equation $x = ax + b$.

Example

Let (u_n) be defined, for all natural number n, by $u_{n+1} = 6u_n + 10$ and $u_0 = 2$.
Let us find u_n as a function of n.
Let $l \in \mathbb{R}$ be the solution of the equation $x = 6x + 10$. We find $l = -2$. We verify that the sequence $(v_n) = (u_n + 2)$ is a geometric sequence. We have

$$v_{n+1} = u_{n+1} + 2 = 6u_n + 10 + 2 = 6u_n + 12 = 6(u_n + 2) = 6v_n$$

Hence, the sequence (v_n) is geometric with common difference 6 and first term $v_0 = u_0 + 2 = 4$.
Then, for all $n \in \mathbb{N}$, $v_n = 6^n \times 4$.
We conclude that, for all $n \in \mathbb{N}$,

$$u_n = 6^n \times 4 - 2$$

8.2 Convergence and divergence

8.2.1 Definitions

Definition 67
Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

1. Let $l \in \mathbb{R}$.

   We say that (u_n) converges to l if

$$\forall\, \varepsilon > 0,\; \exists\, N \in \mathbb{N},\; \forall\, n \in \mathbb{N},\;\; (n \geqslant N \implies |u_n - l| < \varepsilon)$$

2. We say that (u_n) converges if

$$\exists\, l \in \mathbb{R},\ \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ \ (n \geqslant N \implies |u_n - l| < \varepsilon)$$

3. We say that (u_n) diverges if it does not converge i.e.

$$\forall\, l \in \mathbb{R},\ \exists\, \varepsilon > 0,\ \forall\, N \in \mathbb{N},\ \exists\, n \in \mathbb{N},\ \ (n \geqslant N \text{ and } |u_n - l| \geqslant \varepsilon)$$

Proposition 57

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $l \in \mathbb{R}$.
If (u_n) converges to l then l is unique.
We then denote

$$l = \lim_{n\to+\infty} u_n$$

Definition 68

Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

- We say that (u_n) tends to $+\infty$ if

  $$\forall\, A > 0, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N},\ \ (n \geqslant N \implies u_n > A)$$

  We then denote

  $$\lim_{n\to+\infty} u_n = +\infty$$

- We say that (u_n) tends to $-\infty$ if

  $$\forall\, B < 0, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N},\ \ (n \geqslant N \implies u_n < B)$$

  We then denote

  $$\lim_{n\to+\infty} u_n = -\infty$$

Remarks

1. Divergent sequences are thus that which tend to $+\infty$, that which tend to $-\infty$ and that which have no limit.

2. Let $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $l \in \mathbb{R}$.
   We have

   $$\lim_{n\to+\infty} u_n = 0 \iff \lim_{n\to+\infty} |u_n| = 0$$

   and

   $$\lim_{n\to+\infty} u_n = l \iff \lim_{n\to+\infty} |u_n| = |l|$$

8.2.2 Examples

Example 1

Let $(u_n)_{n\in\mathbb{N}^*}$ be defined for all positive integer n by

$$u_n = \frac{1}{n}$$

Let $\varepsilon > 0$ be fixed.
We note that

$$\frac{1}{n} > \varepsilon \quad \Longleftrightarrow \quad n > \frac{1}{\varepsilon}$$

Let $N_\varepsilon = E[\frac{1}{\varepsilon}] + 1$.
Let $n \geqslant N_\varepsilon$.
Then,

$$\frac{1}{\varepsilon} < N_\varepsilon \leqslant n$$

So,

$$\frac{1}{n} < \varepsilon$$

So, (u_n) converges to 0.

Example 2

Let $(u_n)_{n\in\mathbb{N}}$ defined for all integer n par

$$u_n = n^2$$

Let $A > 0$ be fixed.
We note that

$$n^2 > A \quad \Longleftrightarrow \quad n > \sqrt{A}$$

Let $N_A = E[\sqrt{A}] + 1$.
Let $n \geqslant N_A$.
Then,

$$n > \sqrt{A}$$

So,

$$n^2 > A$$

So, (u_n) diverges to $+\infty$.

8.2.3 Properties of convergent or divergent sequences

Proposition 58

Every convergent sequence is bounded.

Remark

The reciprocal is false! For example, the sequence (u_n) defined by $u_n = (-1)^n$ for all integer n is a divergent bounded sequence.

Proposition 59

1. Every sequence convergent to $+\infty$ is bounded below and not bounded above.
2. Every sequence convergent to $-\infty$ is bounded above and not bounded below.

Remark

The reciprocal is false. For example, the sequence (u_n) defined for all integer n by $u_n = (-1)^n n$ is not bounded above but diverges to $+\infty$.

8.2.4 Cesàro's theorem

Definition 69

Let $(u_n)_{n\in\mathbb{N}^*}$.

We call Cesàro mean of (u_n) the sequence (v_n) defined by

$$\forall\, n \in \mathbb{N}^*, \quad v_n = \frac{u_1 + \ldots + u_n}{n}$$

Theorem 24 (Cesàro's theorem)

Let $(u_n)_{n\in\mathbb{N}^*}$ and $l \in \mathbb{R}$.

If $(u_n)_{n\in\mathbb{N}^*}$ converges to l then its Cesàro mean (v_n) also converges to l i.e.

$$\lim_{n\to+\infty} u_n = l \quad \Longrightarrow \quad \lim_{n\to+\infty} \frac{u_1 + \ldots + u_n}{n} = l$$

Remarks

1. The reciprocal is false. Indeed, let us consider the example of the sequence (u_n) defined, for all integer n, by
$$u_n = (-1)^n$$
Then, (u_n) diverges and yet its Cesàro mean (v_n) converges to 0.

2. The theorem is also true for $(u_n)_{n\in\mathbb{N}}$ and $v_n = \dfrac{u_0 + \ldots + u_{n-1}}{n}$.

Example

Let $(u_n)_{n\in\mathbb{N}}$ and $a \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} u_{n+1} - u_n = a$$

Then,

$$\lim_{n\to+\infty} \frac{u_n}{n} = a$$

Indeed, let us consider the sequence $(w_n)_{n\in\mathbb{N}^*}$ defined by

$$w_n = u_n - u_{n-1}$$

We note that

$$\frac{w_1 + \ldots + w_n}{n} = \frac{u_n}{n} - \frac{u_0}{n}$$

Thus,

$$\frac{u_n}{n} = \frac{w_1 + \ldots + w_n}{n} + \frac{u_0}{n}$$

Thus, as $\lim\limits_{n\to+\infty} \dfrac{u_0}{n} = 0$ and $\lim\limits_{n\to+\infty} \dfrac{w_1 + \ldots + w_n}{n} = a$ using Cesàro, we obtain the result.

8.3 Limit and order relation

8.3.1 Passage to the limit in inequalities

Proposition 60
Let $(u_n)_{n\in\mathbb{N}}$ and $l \in \mathbb{R}$ be such that (u_n) converges to l.

1. Let $a \in \mathbb{R}$ be such that $a < l$.

   Then,
$$\exists\, N_1 \in \mathbb{N},\ \forall\, n \in \mathbb{N}, \quad (n \geqslant N_1 \implies a < u_n)$$

2. Let $b \in \mathbb{R}$ be such that $l < b$.

   Then,
$$\exists\, N_2 \in \mathbb{N},\ \forall\, n \in \mathbb{N}, \quad (n \geqslant N_2 \implies u_n < b)$$

Proposition 61
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(l, l') \in \mathbb{R}^2$ such that (u_n) converges to l and (v_n) converges to l'.
Let $a \in \mathbb{R}$.

1. If $\exists\, N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $(n \geqslant N \implies u_n > a)$ then $l \geqslant a$.
2. If $\exists\, N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $(n \geqslant N \implies a > u_n)$ then $a \geqslant l$.
3. If $\exists\, N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $(n \geqslant N \implies u_n > v_n)$ then $l \geqslant l'$.

Example

Let us consider $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ defined par
$$u_n = \frac{1}{n} \quad \text{and} \quad v_n = -\frac{1}{n}$$
We have, for all integer $n > 0$, $v_n < u_n$.
Nevertheless,
$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = 0$$

8.3.2 Squeeze theorem

Theorem 25
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be such that
$$\exists\, N \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ (\, n \geqslant N \implies u_n \leqslant v_n \leqslant w_n)$$
Let $l \in \mathbb{R}$.

1. If (u_n) and (w_n) converge to l then (v_n) converges to l.
2. If (v_n) diverges to $-\infty$ then (u_n) diverges to $-\infty$.
3. If (u_n) diverges to $+\infty$ then (v_n) diverges to $+\infty$.

Corollary 5
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be such that
$$\exists\, N \in \mathbb{N},\ \forall\, n \in \mathbb{N},\ (\, n \geqslant N \implies |u_n| \leqslant v_n)$$
If (v_n) converges to 0 then (u_n) converges to 0.

Example 1

Let (u_n) be defined, for all integer $n \in \mathbb{N}^*$ by $u_n = \frac{\sin(e^n}{\sqrt{n}}$. Then, (u_n) converges to 0 as

$$|u_n| \leq \frac{1}{\sqrt{n}}$$

Example 2

Let us consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined for all integer n strictly positive by

$$u_n = \sum_{k=1}^{n} \frac{1}{n^2 + 2k^2}$$

Then, for all $k \in [\![1, n]\!]$, we have

$$\begin{aligned}
2 \leqslant 2k^2 \leqslant 2n^2 &\implies 2 + n^2 \leqslant n^2 + 2k^2 \leqslant 3n^2 \\
&\implies \frac{1}{3n^2} \leqslant \frac{1}{n^2 + 2k^2} \leqslant \frac{1}{2 + n^2} \\
&\implies \sum_{k=1}^{n} \frac{1}{3n^2} \leqslant \sum_{k=1}^{n} \frac{1}{n^2 + 2k^2} \leqslant \sum_{k=1}^{n} \frac{1}{2 + n^2} \\
&\implies \frac{n}{3n^2} \leqslant u_n \leqslant \frac{n}{2 + n^2}
\end{aligned}$$

Since $\lim\limits_{n\to+\infty} \frac{n}{3n^2} = \lim\limits_{n\to+\infty} \frac{1}{3n} = 0$ and $\lim\limits_{n\to+\infty} \frac{n}{2+n^2} = 0$, we deduce that

$$\lim_{n\to+\infty} u_n = 0$$

so the sequence (u_n) converges to 0.

8.4 Operations on the limits of sequences

8.4.1 For convergent sequences

Proposition 62

Let $((u_n), (v_n)) \in (\mathbb{R}^{\mathbb{N}})^2$, $(l, l') \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

1. If $\lim\limits_{n\to+\infty} u_n = l$ and $\lim\limits_{n\to+\infty} v_n = l'$ then,

$$\lim_{n\to+\infty} \lambda\, u_n + v_n = \lambda\, l + l'$$

2. If $\lim\limits_{n\to+\infty} u_n = 0$ and (v_n) bounded then,

$$\lim_{n\to+\infty} u_n v_n = 0$$

3. If $\lim\limits_{n\to+\infty} u_n = l$ and $\lim\limits_{n\to+\infty} v_n = l'$ then,

$$\lim_{n\to+\infty} u_n v_n = ll'$$

4. If $\lim_{n\to+\infty} u_n = l \neq 0$ then the sequence $\left(\dfrac{1}{u_n}\right)$ is defined above a certain rank and
$$\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l}$$
5. If $\lim_{n\to+\infty} u_n = l$ and $\lim_{n\to+\infty} v_n = l' \neq 0$ then the sequence $\left(\dfrac{u_n}{v_n}\right)$ is well defined above a certain rank and
$$\lim_{n\to+\infty} \frac{u_n}{v_n} = \frac{l}{l'}$$

Example

Let us consider the sequence $\left(\dfrac{\sin(n^8)}{\sqrt{n}}\right)$.

Since $(sin(n^8))$ is bounded by 1 and that $(\dfrac{1}{\sqrt{n}})$ converges to 0, we conclude that the sequence $\left(\dfrac{\sin(n^8)}{\sqrt{n}}\right)$ converges to 0.

8.4.2 For divergent sequences

Proposition 63

Let $((u_n),(v_n)) \in (\mathbb{R}^{\mathbb{N}})^2$ and $l' \in \mathbb{R}$.

1. if $\lim_{n\to+\infty} u_n = +\infty$ and (v_n) is bounded below (above a certain rank) then,
$$\lim_{n\to+\infty} u_n + v_n = +\infty \quad \text{and} \quad \lim_{n\to+\infty} u_n v_n = +\infty$$
In particular,
   (a) If $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = +\infty$ then,
$$\lim_{n\to+\infty} u_n + v_n = +\infty$$
   (b) If $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = l'$ then,
$$\lim_{n\to+\infty} u_n + v_n = +\infty$$
   (c) If $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = +\infty$ then,
$$\lim_{n\to+\infty} u_n v_n = +\infty$$
   (d) If $\lim_{n\to+\infty} u_n = +\infty$ and $\lim_{n\to+\infty} v_n = l'$ then,
$$\lim_{n\to+\infty} u_n v_n = +\infty$$
2. If $\lim_{n\to+\infty} u_n = +\infty$ then,
$$\lim_{n\to+\infty} \frac{1}{u_n} = 0$$
3. If $\lim_{n\to+\infty} u_n = 0^+$ then,
$$\lim_{n\to+\infty} \frac{1}{u_n} = +\infty$$

Remark

There are 4 indeterminate forms : $+\infty - \infty$, $0 \times \infty$, $\dfrac{\infty}{\infty}$, $\dfrac{0}{0}$ and 1^{∞}.

Examples

1. $\displaystyle\lim_{n\to+\infty} \frac{2n^3 - 4n + 7}{1 - n^3}$ is indéterminée.

   To solve the indeterminate form, we put the terms of highest degree in factor at the numerator and the denominator.

   Then
   $$\frac{2n^3 - 4n + 7}{1 - n^3} = \frac{n^3(2 - \frac{4}{n^2} + \frac{7}{n^3})}{n^3(\frac{1}{n^3} - 1)} = \frac{2 - \frac{4}{n^2} + \frac{7}{n^3}}{\frac{1}{n^3} - 1}$$
   and so,
   $$\lim_{n\to+\infty} \frac{2n^3 - 4n + 7}{1 - n^3} = \frac{2}{-1} = -2$$

2. $\displaystyle\lim_{n\to+\infty} \frac{7^n + 6^n}{7^{n+1} + 6^{n+1}}$ is indeterminate.

   To solve the indeterminate form, we apply the same idea as previously.

   We have
   $$\frac{7^n + 6^n}{7^{n+1} + 6^{n+1}} = \frac{7^n(1 + (\frac{6}{7})^n)}{7^{n+1}(1 + (\frac{6}{7})^{n+1})} = \frac{1 + (\frac{6}{7})^n}{7(1 + (\frac{6}{7})^{n+1})}$$
   Yet, $\dfrac{6}{7} < 1$. So
   $$\lim_{n\to+\infty} \left(\frac{6}{7}\right)^n = \lim_{n\to+\infty} \left(\frac{6}{7}\right)^{n+1} = 0$$
   Then,
   $$\lim_{n\to+\infty} \frac{7^n + 6^n}{7^{n+1} + 6^{n+1}} = \frac{1}{7}$$

8.5 Monotony

8.5.1 Properties of monotonic sequences

Proposition 64

1. Every sequence real increasing and bounded above converges.
2. Every sequence real decreasing and bounded below converges.
3. Every sequence increasing and not bounded above diverges to $+\infty$.
4. Every sequence decreasing and not bounded below diverges to $-\infty$.

Remark

If (u_n) is an increasing sequence which converges to $l \in \mathbb{R}$ then

$$l = Sup\{\ u_n;\ n \in \mathbb{N}\ \}$$

Hence,

$$\forall\, n \in \mathbb{N}, \quad u_n \leqslant l$$

Example

Let us consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined for all integer n par

$$u_n = \sum_{k=0}^{n} \frac{1}{k!}$$

Let us prove that this sequence converges.
Since $u_{n+1} - u_n = \dfrac{1}{(n+1)!} \geqslant 0$, we deduce that (u_n) is increasing.
Moreover,

$$\forall\, n \geqslant 1, \quad \frac{1}{n!} \leqslant \frac{1}{2 \times \ldots \times 2} = \frac{1}{2^{n-1}}$$

So,

$$u_n \leqslant 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \leqslant 3$$

To conclude, (u_n) is increasing and bounded above by 3 hence it converges.

8.5.2 Adjacent sequences

Definition 70
Let $((u_n), (v_n)) \in \left(\mathbb{R}^{\mathbb{N}}\right)^2$.
We say that (u_n) and (v_n) are two adjacent sequences if
-(u_n) is increasing,
-(v_n) is decreasing,
-and $\lim\limits_{n\to+\infty} u_n - v_n = 0$.

Example

Let us prove that the sequences (u_n) and (v_n) defined for all integer $n \geqslant 3$ by

$$\begin{aligned} u_n &= \sum_{k=3}^{n} \frac{1}{k^2+1} \\ v_n &= u_n + \frac{1}{n} - \frac{1}{2n^2} \end{aligned}$$

are adjacent.
We have

- $u_{n+1} - u_n = \dfrac{1}{(n+1)^2+1} \geqslant 0$.

So, (u_n) is increasing.

• $v_{n+1} - v_n = \dfrac{-(n-1)^2 + 3}{2n^2(n^2+2n+2)(n+1)^2} \leqslant 0.$

Hence, (v_n) is decreasing.

• $u_n - v_n = \dfrac{1}{2n^2} - \dfrac{1}{n}.$

So, $\lim\limits_{n \to +\infty} u_n - v_n = 0.$

We conclude that these sequences are adjacent.

Theorem 26

If two real sequences (u_n) and (v_n) are adjacent then they converge to the same limit l and

$$\forall\, n \in \mathbb{N}, \quad u_n \leqslant u_{n+1} \leqslant l \leqslant v_{n+1} \leqslant v_n$$

Example

The two previous sequences (u_n) and (v_n) converge to the same limit l.

8.6 Subsequences

8.6.1 Definition and examples

Definition 71

Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$.
Let $\varphi\ :\ \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing application .
The sequence defined by

$$\begin{array}{ccccc} \mathbb{N} & \longrightarrow & \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & \varphi(n) & \longmapsto & u_{\varphi(n)} \end{array}$$

is called a sub-sequence of (u_n), denoted $(u_{(\varphi(n)})_{n\in\mathbb{N}}$.

Examples

Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$.

1. Let $\varphi : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & n+1 \end{array}$.

   This application is strictly increasing from $\mathbb{N}$ to $\mathbb{N}$.

   So $(u_{\varphi(n)}) = (u_{n+1})$ is a subsequence of (u_n)

   For example, let us consider the sequence (u_n) defined for all natural number n by

   $$u_n = n^2 - 1$$

   Then, the subsequence (u_{n+1}) of (u_n) is defined for all natural number n by

   $$u_{n+1} = n^2 + 2n$$

2. Let $\varphi_1 : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & 2n \end{array}$ and $\varphi_2 : \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N} \\ n & \longmapsto & 2n+1 \end{array}$.

   φ_1 and φ_2 are two applications strictly increasing from $\mathbb{N}$ to $\mathbb{N}$.

   So $(u_{\varphi_1(n)}) = (u_{2n})$ and $(u_{\varphi_2(n)}) = (u_{2n+1})$ are two sub-sequences of (u_n).

Proposition 65
Let $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing application.
Then,

$$\forall\, n \in \mathbb{N}, \quad \varphi(n) \geqslant n$$

8.6.2 Properties

Proposition 66
Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$ and $l \in \mathbb{R}$.
If (u_n) converges to l then every subsequence of (u_n) converges also to l.

Remark

The contrapositive of this proposition is important.
If $\exists\, \varphi : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing such that $(u_{\varphi(n)})$ diverges, then (u_n) diverges.

Examples

1. A method to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ defined for all integer n by

   $$u_n = (-1)^n$$

   diverges is the following :

   We use a proof by contradiction and assume that $(u_n)_{n\in\mathbb{N}}$ converges to $l \in \mathbb{R}$.

   Then, every subsequence of $(u_n)_{n\in\mathbb{N}}$ also converges to l.

   In particular, the two sub-sequences (u_{2n}) and (u_{2n+1}) converge to l.

   Yet, $u_{2n} = 1$ and $u_{2n+1} = -1$. Hence, (u_{2n}) converges to 1 and (u_{2n+1}) converges to -1.

   We have reached a contradiction.

2. Let us prove that the sequence $(u_n)_{n\in\mathbb{N}}$ defined for all integer n by

   $$u_n = \cos\left(\frac{n\pi}{4}\right)$$

   diverges.

   (u_{4n}) is a subsequence of (u_n). This subsequence is defined for all integer n par

   $$u_{4n} = \cos(n\pi) = (-1)^n$$

   Consequently, the subsequence (u_{4n}) diverges.

   Hence, (u_n) diverges.

Proposition 67
Let $(u_n) \in \mathbb{R}^{\mathbb{N}}$ and $l \in \mathbb{R}$.
We have

$$(u_n) \text{ converges to } l \iff (u_{2n}) \text{ and } (u_{2n+1}) \text{ converge to } l$$

Example

Let us consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by

$$u_n = \sum_{k=1}^{n} \frac{(-1)^k}{k^2}$$

Let us prove that (u_n) converges.
We introduce the two subsequences $(v_n) = (u_{2n})$ and $(w_n) = (u_{2n+1})$ of (u_n).
We have

- $v_{n+1} - v_n = u_{2n+2} - u_{2n} = \dfrac{-4n-3}{(2n+2)^2(2n+1)^2}$.

We deduce that

$$v_{n+1} - v_n \leqslant 0$$

and so $(v_n) = (u_{2n})$ is decreasing.

- $w_{n+1} - w_n = u_{2n+3} - u_{2n+1} = \dfrac{4n+5}{(2n+3)^2(2n+2)^2}$.

We deduce that

$$w_{n+1} - w_n \geqslant 0$$

and so $(w_n) = (u_{2n+1})$ is increasing.

- We note also that

$$w_n - v_n = u_{2n+1} - u_{2n} = -\frac{1}{(2n+1)^2}$$

Thus,

$$\lim_{n\to+\infty} w_n - v_n = 0$$

- Conclusion : The sequences (u_{2n}) and (u_{2n+1}) are two adjacent sequences. So, they converge to the same limit l.

Using the previous proposition, we conclude that (u_n) converges (to l).

8.6.3 Bolzano-Weierstrass's theorem

Theorem 27 (Bolzano-Weierstrass)
From any bounded real sequence, we can extract a convergent subsequence.

Example

Let $(u_n)_{n\in\mathbb{N}}$ be defined by

$$u_n = \cos(n)$$

(u_n) is divergent but bounded. So, there is at least one sub-sequence of (u_n) which converges...

8.7 Comparison of sequences

8.7.1 Relations of predominance

Definition 72
Let $((u_n),(v_n)) \in \left(\mathbb{R}^{\mathbb{N}}\right)^2$.

1. We say that (u_n) is negligible compared to (v_n) if

$$\forall\, \varepsilon > 0,\, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N}, \quad (n \geqslant N \implies |u_n| \leqslant \varepsilon |v_n|)$$

   We note $u_n = o(v_n)$.

2. We say that (u_n) is dominated by (v_n) if

$$\exists\, M > 0,\, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N}, \quad (n \geqslant N \implies |u_n| \leqslant M |v_n|)$$

   We note $u_n = O(v_n)$.

Remarks

1. $u_n = o(1) \quad \iff \quad \lim_{n\to+\infty} u_n = 0.$
2. $u_n = O(1) \quad \iff \quad (u_n) \quad \text{bounded.}$

Proposition 68
$u_n = o(v_n) \quad \implies \quad u_n = O(v_n).$

Theorem 28

1.

$$u_n = o(v_n) \iff \exists\,(\varepsilon_n) \in \mathbb{R}^{\mathbb{N}} \text{ which converges to } 0, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N} \;\; (n \geqslant N \implies u_n = \varepsilon_n v_n)$$

2.

$$u_n = O(v_n) \iff \exists\,(\varepsilon_n) \in \mathbb{R}^{\mathbb{N}} \;\text{ bounded}, \exists\, N \in \mathbb{N}, \forall\, n \in \mathbb{N} \quad (n \geqslant N \implies u_n = \varepsilon_n v_n)$$

Interpretation

If $v_n \neq 0$ above a certain rank, then

$$u_n = o(v_n) \quad \iff \quad \lim_{n\to+\infty} \frac{u_n}{v_n} = 0$$

and

$$u_n = O(v_n) \quad \iff \quad \left(\frac{u_n}{v_n}\right) \quad \text{bounded}$$

Examples

1. We have

$$\frac{1}{n^2} = o(\frac{1}{n}), \quad \ln n = o(n^\alpha) \quad \text{where} \quad \alpha > 0, \quad n^a = o(a^n) \quad \text{where} \quad a > 0$$

2. Let us consider

$$u_n = \frac{2n^2+1}{n\sqrt{n+1}}$$

Then

$$u_n = \frac{n^2\left(2+\frac{1}{n^2}\right)}{n\sqrt{n+1}} = \frac{n}{\sqrt{n+1}}\left(2+\frac{1}{n^2}\right) \leqslant \frac{3n}{\sqrt{n+1}} \leqslant 3\sqrt{n}$$

So,

$$u_n = O(\sqrt{n})$$

Yet $u_n \neq o(\sqrt{n})$ as

$$\frac{u_n}{\sqrt{n}} = \frac{2n^2+1}{n\sqrt{n^2+n}} = \frac{n^2(2+\frac{1}{n^2})}{n^2\sqrt{1+\frac{1}{n}}} = \frac{2+\frac{1}{n^2}}{\sqrt{1+\frac{1}{n}}}$$

Then,

$$\lim_{n\to+\infty} \frac{u_n}{\sqrt{n}} = 2 \neq 0$$

Proposition 69

Let (u_n), (v_n), (w_n) and (t_n) in $\mathbb{R}^{\mathbb{N}}$.
Then,

1. $$u_n = o(v_n) \text{ and } v_n = o(w_n) \implies u_n = o(w_n)$$

2. $$u_n = o(w_n) \text{ and } v_n = o(w_n) \implies u_n + v_n = o(w_n)$$

3. $$\forall\, \alpha \in \mathbb{R}^*, \quad u_n = o(v_n) \implies \alpha u_n = o(v_n)$$

4. $$u_n = o(w_n) \text{ and } v_n = o(t_n) \implies u_n v_n = o(w_n t_n)$$

8.7.2 Relation of equivalence

Definition 73

Let $((u_n), (v_n)) \in \left(\mathbb{R}^{\mathbb{N}}\right)^2$.
We say that (u_n) is equivalent to (v_n) if

$$u_n - v_n = o(v_n)$$

We write $u_n \sim v_n$.

Remark

Let $a \in \mathbb{R}^*$.

1. $$u_n \sim a \quad \Longleftrightarrow \quad \lim_{n \to +\infty} u_n = a$$

2. $$u_n \sim v_n \quad \Longleftrightarrow \quad v_n \sim u_n$$

Theorem 29

$$u_n \sim v_n \iff \exists\,(\varepsilon_n) \in \mathbb{R}^{\mathbb{N}} \text{ which converges to } 0,\ \exists\, N \in \mathbb{N},\ \forall\, n \in \mathbb{N}\ \ (n \geqslant N \implies u_n = (1+\varepsilon_n)v_n)$$

Interpretation

If $v_n \neq 0$ above a certain rank, then

$$u_n \sim v_n \quad \Longleftrightarrow \quad \lim_{n \to +\infty} \frac{u_n}{v_n} = 1$$

Examples

1. $3n^2 + 2n - 8 \sim 3n^2$ as
$$u_n = 3n^2 \left(1 + \frac{2}{3n} - \frac{8}{3n^2}\right)$$
Then
$$\lim_{n \to +\infty} \frac{u_n}{3n^2} = 1$$

2. Classical equivalents :
$$\begin{aligned} \ln\left(1+\frac{1}{n}\right) &\sim \frac{1}{n} \\ e^{\frac{1}{n}} - 1 &\sim \frac{1}{n} \\ \sin\left(\frac{1}{n}\right) &\sim \frac{1}{n} \\ \cos\left(\frac{1}{n}\right) - 1 &\sim -\frac{1}{2n^2} \end{aligned}$$

Proposition 70

Let (u_n), (v_n), (w_n) and (t_n) be in $\mathbb{R}^{\mathbb{N}}$.
then,

1. $$u_n \sim v_n \text{ and } v_n \sim w_n \implies u_n \sim w_n$$

2. $$u_n \sim v_n \implies \forall\, \alpha \in \mathbb{R}^+,\ \ u_n^\alpha \sim v_n^\alpha$$

3. $$u_n \sim v_n \implies \frac{1}{u_n} \sim \frac{1}{v_n}$$

4.
$$u_n \sim w_n \text{ and } v_n \sim t_n \implies u_n v_n \sim w_n t_n$$

5.
$$u_n \sim v_n \text{ and } \lim_{n\to+\infty} u_n = l \cup \{\pm\infty\} \implies \lim_{n\to+\infty} v_n = l \cup \{\pm\infty\}$$

8.7.3 Taylor's expansions and asymptotic expansions

Taylor's expansions

In the following section, the variable n always tends to $+\infty$.
To use Taylor's expansion, we have to make appear, if needed, quantities which tend to 0 as n tends to $+\infty$, such as $\dfrac{1}{n}$ for example.

Examples

1. Let us find the Taylor's expansion at order 4 at $+\infty$ of
$$u_n = \ln\left(1+\cos\left(\frac{1}{n}\right)\right)$$
The variable $\dfrac{1}{n}$ tends to 0 when n tends to $+\infty$.
Thus,
$$\begin{aligned}
u_n &= \ln\left(1+1-\frac{1}{2n^2}+\frac{1}{4!n^4}+o\left(\frac{1}{n^4}\right)\right)\\
&= \ln\left(2\left(1-\frac{1}{4n^2}+\frac{1}{48n^4}+o\left(\frac{1}{n^4}\right)\right)\right)\\
&= \ln 2+\ln\left(1-\frac{1}{4n^2}+\frac{1}{48n^4}+o\left(\frac{1}{n^4}\right)\right)\\
&= \ln 2+\left(-\frac{1}{4n^2}+\frac{1}{48n^4}\right)-\frac{1}{2}\left(-\frac{1}{4n^2}\right)+o\left(\frac{1}{n^4}\right)\\
&= \ln 2-\frac{1}{4n^2}-\frac{1}{96n^4}+o\left(\frac{1}{n^4}\right)
\end{aligned}$$

2. Let us compute
$$\lim_{n\to+\infty}\left(\frac{n}{n+1}\right)^n$$
We have
$$\begin{aligned}
\left(\frac{n}{n+1}\right)^n &= e^{n\ln\left(\frac{n}{n+1}\right)}\\
&= e^{-n\ln\left(\frac{n+1}{n}\right)}\\
&= e^{-n\ln\left(1+\frac{1}{n}\right)}\\
&= e^{-n\left(-\frac{1}{n}+o\left(\frac{1}{n}\right)\right)}\\
&= e^{-1+o(1)}
\end{aligned}$$
Thus,
$$\lim_{n\to+\infty}\left(\frac{n}{n+1}\right)^n = e^{-1}$$

Asymptotic series

Example 1

Let us consider
$$u_n = \sqrt{n+\sqrt{n}} - \sqrt{n}$$
We have
$$\begin{aligned} u_n &= \sqrt{n\left(1+\frac{1}{\sqrt{n}}\right)} - \sqrt{n} \\ &= \sqrt{n}\left(\sqrt{1+\frac{1}{\sqrt{n}}} - 1\right) \\ &= \sqrt{n}\left(\frac{1}{2\sqrt{n}} - \frac{1}{8n} + \frac{1}{16n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)\right) \\ &= \frac{1}{2} - \frac{1}{8\sqrt{n}} + \frac{1}{16n} + o\left(\frac{1}{n}\right) \end{aligned}$$
We say that it is an **asymptotic expansion** of u_n at $+\infty$ of order n (with precision $o\left(\frac{1}{n}\right)$).

Example 2

Let us consider
$$u_n = \ln\left(n \ln n + 1\right)$$
We have
$$\begin{aligned} u_n &= \ln(n\ln n) + \ln\left(1+\frac{1}{n\ln n}\right) \\ &= \ln n + \ln\ln n + \frac{1}{n\ln n} - \frac{1}{2n^2(\ln n)^2} + o\left(\frac{1}{n^2(\ln n)^2}\right) \end{aligned}$$
which constitutes an asymptotic expansion at a neighborhood of $+\infty$ with precision $o\left(\frac{1}{n^2(\ln n)^2}\right)$.

Chapter 9

Vector spaces

In the whole chapter, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

9.1 Generalities

9.1.1 Structure of vector space

Let E be a set with the two operations

$$\begin{aligned} + : E \times E &\longrightarrow E \\ (u, v) &\longmapsto u + v \end{aligned}$$

and

$$\begin{aligned} \cdot : \mathbb{K} \times E &\longrightarrow E \\ (\alpha, u) &\longmapsto \alpha \cdot u \end{aligned}$$

Definition 74
We say that $(E, +, \cdot)$ is a vector space on $\mathbb{K}$, or a $\mathbb{K}$-vector space, if
$\forall\,(u, v, w) \in E^3$ we have

1. $(u + v) + w = u + (v + w)$ (associativity of $+$)
2. $u + v = v + u$ (commutativity of $+$)
3. There exists a vector of E, denoted 0_E such that $\forall\, u_i n E$, $u + 0_E = u$
4. For all $u \in E$, there exists a vector of E, denoted $-u$, such that $u + (-u) = 0_E$

and $\forall\ (u, v) \in E^2$ and $\forall\ (\alpha, \beta) \in \mathbb{K}^2$

- $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot v$
- $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$
- $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$
- $1_{\mathbb{K}} \cdot u = u$

Remark

1. Instead of saying that E is a $\mathbb{K}$-vector space, we often abbreviate using : E is a $\mathbb{K}$-vs.

2. The operation $+$ is called internal law of E. The operation $\cdot$ is called external law.

Definition 75

Let $(E, +, \cdot)$ be a $\mathbb{K}$-vs.
We call vectors the elements of E and scalar the elements of $\mathbb{K}$.
Moreover, the vector 0_E is called null vector.

Property 1

Let E be a $\mathbb{K}$-vs, $u \in E$ and $\alpha \in \mathbb{K}$.
then,

1. $\alpha \cdot 0_E = 0_E$.

2. $0_{\mathbb{K}} \cdot u = 0_E$.

3. $\alpha \cdot u = 0_E \iff \alpha = 0_{\mathbb{K}}$ or $u = 0_E$.

Property 2

Let $(\alpha, \beta) \in \mathbb{K}^2$ and $(u, v) \in E^2$.
Then,

1. $(\alpha - \beta) \cdot u = \alpha \cdot u - \beta \cdot u$

2. $\alpha \cdot (u - v) = \alpha \cdot u - \alpha \cdot v$

3. $-(\alpha \cdot u) = \alpha \cdot (-u) = (-\alpha) \cdot u$

9.1.2 Examples of reference

Example 1

$\mathbb{C}$ and $\mathbb{R}$ are $\mathbb{R}$-vs.

Example 2

In $E = \mathbb{R}^2$, we define the laws

- $+$ by: $\forall\ u = (x_1, y_1) \in E$ and $v = (x_2, y_2) \in E$, $u + v = (x_1 + x_2, y_1 + y_2) \in E$

- $\cdot$ by: $\forall\ u = (x_1, y_1) \in E$ and $\alpha \in \mathbb{R}$, $\alpha \cdot u = (\alpha x_1, \alpha y_1) \in E$.

Then, $(E, +, \cdot)$ is a $\mathbb{R}$-vs.
More generally, for all natural number n larger than 1, $\mathbb{R}^n$ is a $\mathbb{R}$-vs.

Example 3

Let I be an interval of $\mathbb{R}$.
Let

$$\mathbb{R}^I = \{\ f : I \to \mathbb{R}\ \}$$

We define on $\mathbb{R}^I$ the laws

- $+$ by: $\forall\ (f,g) \in (\mathbb{R}^I)^2$ and $\forall\ x \in I$, $(f+g)(x) = f(x) + g(x)$
- $\cdot$ by: $\forall\ f \in \mathbb{R}^I$, $\forall\ \alpha \in \mathbb{R}$ and $\forall\ x \in I$, $(\alpha \cdot f)(x) = \alpha f(x)$.

Then, $(\mathbb{R}^I, +, \cdot)$ is a $\mathbb{R}$-vs.

Example 4

Let $\mathbb{R}^{\mathbb{N}}$ be the set of numerical sequences.
We define on $\mathbb{R}^{\mathbb{N}}$ the laws

- $+$ by: $\forall\ ((u_n),(v_n)) \in (\mathbb{R}^{\mathbb{N}})^2$, $(u_n) + (v_n) = (u_n + v_n)$
- $\cdot$ by: $\forall\ (u_n) \in \mathbb{R}^{\mathbb{N}}$, $\forall\ \alpha \in \mathbb{R}$, $\alpha \cdot (u_n) = (\alpha u_n)$.

Then, $(\mathbb{R}^{\mathbb{N}}, +, \cdot)$ is a $\mathbb{R}$-vs.

Example 5

From the course on polynomials, we can deduce that $(\mathbb{R}[X], +, \cdot)$ is a $\mathbb{R}$-vs.

9.1.3 Vector subspaces

Definition 76

Let $(E, +, \cdot)$ be a $\mathbb{K}$-vs.
Let $F \subset E$.
We say that F is a vector subspace of E (we also say linear subspace) if $(F, +, \cdot)$ is a $\mathbb{K}$-vs.

Examples

1. $\mathbb{R}$ is a linear subspace of $\mathbb{C}$.
2. $\mathbb{R}_n[X]$ is a linear subspace of $\mathbb{R}[X]$.

Theorem 30

Let $(E, +, \cdot)$ be a $\mathbb{K}$-vs.
Then,

$$F \text{ is a linear subspace of } E \Longleftrightarrow \begin{cases} F \subset E \\ F \neq \emptyset\ (0_E \in F) \\ \forall\ (\alpha,\beta) \in \mathbb{K}^2, \forall\ (u,v) \in F^2 \quad \alpha \cdot u + \beta \cdot v \in F \end{cases}$$

Remark

For clarity's sake, the symbol $\cdot$ of the external law of composition is now omitted.

Examples

1. Let $F = \{\, (x, y, z) \in \mathbb{R}^3,\ x + 3y - z = 0 \,\}$.

   Let us prove that F is a linear subspace of $\mathbb{R}^3$.

   By definition, $F \subset \mathbb{R}^3$.

   Moreover, $F \neq \emptyset$ as $(0, 0, 0) \in F$ since $0 + 3 \times 0 - 0 = 0$!

   Moreover, Let $u = (x_1, y_1, z_1) \in F$ and $v = (x_2, y_2, z_2) \in F$.

   We have

   $$\begin{aligned} x_1 + 3y_1 - z_1 &= 0 \\ x_2 + 3y_2 - z_2 &= 0 \end{aligned}$$

   Let $(\alpha, \beta) \in \mathbb{R}^2$.

   Let us prove that $\alpha u + \beta v \in F$.

   We clearly have $\alpha u + \beta v \in \mathbb{R}^3$.

   Moreover, $\alpha u + \beta v = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$ satisfies

   $$\begin{aligned} (\alpha x_1 + \beta x_2) + 3(\alpha y_1 + \beta y_2) - (\alpha z_1 + \beta z_2) &= (\alpha x_1 + 3\alpha y_1 - \alpha z_1) + (\beta x_2 + 3\beta y_2 - \beta z_2) \\ &= \alpha\,(x_1 + 3y_1 - z_1) + \beta\,(x_2 + 3y_2 - z_2) \\ &= \alpha \times 0 + \beta \times 0 \quad \text{as} \quad (u, v) \in F^2 \\ &= 0 \end{aligned}$$

   Hence we have $\alpha u + \beta v \in F$ and we conclude that F is a linear subspace of $\mathbb{R}^3$.

2. Similarly, one can prove that $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ is a linear subspace of $\mathbb{R}^{\mathbb{R}}$.

Counter-Example

1. $G = \{\, (x, y, z) \in \mathbb{R}^3,\ x + 3y - z = 1 \,\}$ is not a linear subspace of $\mathbb{R}^3$ as $(0, 0, 0) \notin G$.

2. $H = \{\, (x, y, z) \in \mathbb{R}^3,\ xyz = 0 \,\}$ is not a linear subspace of $\mathbb{R}^3$.

   Indeed, let us assume that H is a linear subspace of $\mathbb{R}^3$.

   Then,

   $$\forall\,(u, v) \in H, \quad u + v \in H$$

   Let us take for example $u = (1, 1, 0) \in H$ and $v = (0, 1, 3) \in H$.

   We have $u + v = (1, 2, 3)$ and $1 \times 2 \times 3 \neq 0$! So, $u + v \notin H$ which is absurd.

Proposition 71

Let E be a $\mathbb{K}$-vector space and F and G be two linear subspaces of E.
Then, $F \cap G$ is a linear subspace of E.
More generally, the finite intersection linear subspaces of E is a linear subspace of E.

Counter-Example

The union of linear subspaces of E is not a linear subspace of E!!
Indeed, let us consider for example $E = \mathbb{R}^2$ and the two following linear subspaces of E :

$$F = \{\, (x, y) \in \mathbb{R}^2,\, x + 2y = 0 \,\}$$

and

$$G = \{\, (x, y) \in \mathbb{R}^2,\, x = 0 \,\}$$

We use a proof by contradiction and assume that $F \cup G$ is a linear subspace of E.
Then,

$$\forall\, (x, y) \in F \cup G^2, \quad x + y \in F \cup G$$

Let us take for example $x = (-2, 1)$ and $y = (0, 3)$. We have $x \in F \subset F \cup G$ and $y \in G \subset F \cup G$.
However, $x + y = (-2, 4)$. Hence, $x + y \notin F$ and $x + y \notin G$.
So, $x + y \notin F \cup G$, which is a contradiction.

9.1.4 Sum of sub-vector spaces

Let E be $\mathbb{K}$-vs.
Let F and G be two linear subspaces of E.

Definition 77
We define the set $F + G$ by

$$F + G = \{\, u \in E;\ \exists\, (u_1, u_2) \in F \times G,\ u = u_1 + u_2 \,\}$$

Proposition 72
$F + G$ is a linear subspace of E.

Example

Let $E = \mathbb{R}^2$, $F = \{\, (x, y) \in \mathbb{R}^2,\, y = 0 \,\}$ and $G = \{\, (x, y) \in \mathbb{R}^2,\, y = x \,\}$.
F and G are two linear subspaces of E.
Moreover, let $u = (x, y) \in E$.
Then,

$$u = (x - y, 0) + (y, y)$$

Since $(x - y, 0) \in F$ and $(y, y) \in G$, we have $u \in F + G$.
So $E \subset F + G$.
The inverse inclusion being trivial, we have $E = F + G$.

Definition 78
We say that F and G are in direct sum if $F \cap G = \{0_E\}$.
We note then $F \oplus G$ instead of $F + G$.

Examples

1. F and G defined previously are in direct sum.
2. Let $E = \mathbb{R}^3$, $F = \{\, (x, y, z) \in \mathbb{R}^3,\, z = 0 \,\}$ and $G = \{\, (x, y, z) \in \mathbb{R}^3,\, y = 0 \,\}$.

   F and G are two linear subspace of E but they are not in direct sum $(1, 0, 0) \in F \cap G$.

Theorem 31
F and G are in direct sum if

$$\forall\ u \in F+G,\ \exists!\,(u_1,u_2) \in F \times G,\ u = u_1 + u_2$$

Definition 79
We say that F and G are supplementary in E if

$$E = F + G \quad \text{and} \quad F \cap G = \{0_E\}$$

We denote then $E = F \oplus G$.

Theorem 32

$$E = F \oplus G \iff \forall\ u \in E,\ \exists!\,(u_1,u_2) \in F \times G,\ u = u_1 + u_2$$

Examples

1. Let $E = \mathcal{C}^0(\mathbb{R},\mathbb{R})$.

   Let
   $$F = \{\ f \in E,\ \int_0^1 f(t)\,\mathrm{d}t = 0\ \}$$
   and
   $$G = \{\ f \in E,\ \exists\ a \in \mathbb{R},\ \forall\ x \in \mathbb{R},\ f(x) = ax\ \}$$

   It is easy to prove that F and G are two linear subspace of E.

   Let us prove that they are supplementary in E.

   Let us first prove that $F \cap G = \{0_E\}$.

   Let $f \in F \cap G$.

   Then, as $f \in G$, there exists $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,
   $$f(x) = ax$$

   Yet, we also have $f \in F$. Since, in this case,
   $$\int_0^1 f(t)\,\mathrm{d}t = \int_0^1 a\,t\,\mathrm{d}t = \frac{a}{2}$$

   we conclude that $a = 0$.

   Thus, for all $x \in \mathbb{R}$, $f(x) = 0$.

   So, $f = 0_E$ (0_E being the null function).

   We have shown that $F \cap G \subset \{0_E\}$.

   Since the inverse inclusion is obvious, we have
   $$F \cap G = \{0_E\}$$

   It remains to prove that
   $$E = F + G$$

   The inclusion $F + G \subset E$ is obvious.

Let us now prove that $E \subset F + G$.

To do so, we use a proof by analysis and synthesis.

• Analysis

Let us assume that $E \subset F + G$.

Let $f \in E$.

Then, $\exists\, (f_1, f_2) \in F \times G$ such that

$$f = f_1 + f_2$$

i.e.

$$\forall\, x \in \mathbb{R} \quad f(x) = f_1(x) + f_2(x)$$

As $f_2 \in G$, there exists $a \in \mathbb{R}$ such that, for all real number x, $f_2(x) = ax$.

Hence,

$$f(x) = f_1(x) + ax$$

Let us compute $\displaystyle\int_0^1 f(t)\mathrm{d}t$.

We have

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t &= \int_0^1 f_1(t)\mathrm{d}t + \int_0^1 at\mathrm{d}t \\ &= 0 + \frac{a}{2} \quad \text{as} \quad f_1 \in F \end{aligned}$$

So,

$$a = 2\int_0^1 f(t)\mathrm{d}t$$

So,

$$f_2(x) = 2\left(\int_0^1 f(t)\mathrm{d}t\right)x$$

and

$$f_1(x) = f(x) - f_2(x) = f(x) - 2\left(\int_0^1 f(t)\mathrm{d}t\right)x$$

• Synthesis

Let $f \in E$ and $x \in \mathbb{R}$.

We have

$$f(x) = f(x) - 2\left(\int_0^1 f(t)\mathrm{d}t\right)x + 2\left(\int_0^1 f(t)\mathrm{d}t\right)x$$

Let us set

$$f_1(x) = f(x) - 2\left(\int_0^1 f(t)\mathrm{d}t\right)x$$

and

$$f_2(x) = 2\left(\int_0^1 f(t)\mathrm{d}t\right)x$$

Then, we have, for all $x \in \mathbb{R}$, $f(x) = f_1(x) + f_2(x)$.

It remains to prove that $f_1 \in F$ and $f_2 \in G$.

We obviously have $f_2 \in G$.

Moreover,

$$\begin{aligned}
\int_0^1 f_1(t)\mathrm{d}t &= \int_0^1 \left(f(t) - 2\left(\int_0^1 f(t)\mathrm{d}t\right)t\right)\mathrm{d}t \\
&= \int_0^1 f(t)\mathrm{d}t - 2\left(\int_0^1 f(t)\mathrm{d}t\right)\int_0^1 t\mathrm{d}t \\
&= \int_0^1 f(t)\mathrm{d}t - 2\left(\int_0^1 f(t)\mathrm{d}t\right)\frac{1}{2} \\
&= \int_0^1 f(t)\mathrm{d}t - \int_0^1 f(t)\mathrm{d}t \\
&= 0
\end{aligned}$$

So, $f_1 \in F$.

To conclude, $f = f_1 + f_2$ where $(f_1, f_2) \in F \times G$. Thus, $f \in F + G$.

We have proven that $E \subset F + G$.

2. We can also prove that in $E = \mathbb{R}^3$, F and G are supplementary with

$$F = \{\ u = (x, y, z) \in E,\ x = y = z\ \}$$

and

$$G = \{\ (x, y, z) \in E,\ x + y + z = 0\ \}$$

9.1.5 Linear subspace spanned by a part

Definition 80

Let E be a $\mathbb{K}$-vector space and $A \subset E$.
We call linear subspace spanned by A the intersection of all the linear subspaces of E which contain A.
We denote it $Vect(A)$.

Proposition 73

$Vect(A)$ is the smallest linear subspace of E which contains A.

Examples

1. Let E be a $\mathbb{K}$-vector space.

   Then,
$$Vect(\emptyset) = \{0_E\}$$

2. For the $\mathbb{R}$-vector space $E = \mathbb{C}$,
$$Vect(\{1\}) = \mathbb{R}$$

Proposition 74

Let E be a $\mathbb{K}$-vector space, $n \in \mathbb{N}^*$ and $(u_1, \ldots, u_n) \in E^n$ a finite family of vectors of E.
Then,

$$Vect(\{u_1, \ldots, u_n\}) = \left\{ u \in E,\ \exists\ (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n,\ u = \sum_{i=1}^{n} \lambda_i\, u_i \right\}$$

Examples

1. Let $E = \mathbb{R}^2$ and $u = (1, 2)$.

   Then, $Vect(u) = \{ \alpha u, \alpha \in \mathbb{R} \}$ so$Vect(u)$ is the set of vectors collinear to u.

2. Let us consider $E = \mathbb{R}^n$.

   For all $u = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have
   $$u = x_1(1, 0, \ldots, 0) + \ldots + x_n(0, \ldots, 0, 1) := x_1e_1 + \ldots + x_ne_n$$
   So, $u \in Vect(\{ e_1, \ldots, e_n \})$ and so $E \subset Vect(\{ e_1, \ldots, e_n \})$.

   Yet, $Vect(\{ e_1, \ldots, e_n \})$ is a linear subspace of E.

   We conclude that
   $$E = Vect(\{ e_1, \ldots, e_n \})$$

3. Similarly, we show that
   $$\mathbb{R}_n[X] = Vect(\{1, X, \ldots, X^n\})$$

properties

Proposition 75

Let E be a $\mathbb{K}$-vector space, A and B two subsets of E.
Then,

1. $A \subset B \implies Vect(A) \subset Vect(B)$.
2. A linear subspace of $E \iff Vect(A) = A$.
3. $Vect(A \cup B) = Vect(A) + Vect(B)$.

9.2 Linearly independent families, Spanning families, basis of a vector space

Let E be a $\mathbb{K}$-vector space.

Definition 81

Let $n \in \mathbb{N}^*$.
Let $(u_1, \ldots, u_n) \in E^n$.
We call linear combination of $(u_1, \ldots, u_n)$ any vector $u \in E$ such that
$$\exists\ (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n, \quad u = \sum_{i=1}^{n} \lambda_i u_i$$

Example

Let us consider $E = \mathbb{R}^2$, $u_1 = (1, -1)$ and $u_2 = (3, 4)$.
Then, $u = (8, 6)$ is a linear combination of (u_1, u_2), as $u = 2u_1 + 2u_2$.

9.2.1 Linearly independent families

Definition 82

Let $n \in \mathbb{N}^*$.
Let $(u_1, \dots, u_n) \in E^n$.

1. We say that the family $(u_1, \dots, u_n)$ is a linearly independent family of E if

$$\forall\ (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n, \quad \left(\sum_{i=1}^{n} \lambda_i\, u_i = 0_E \implies \lambda_1 = \ldots = \lambda_n = 0 \right)$$

2. We say that the family $(u_1, \dots, u_n)$ is a linearly dependent family of E if it is not linearly independent i.e.

$$\exists\ (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \setminus \{(0, \dots, 0)\}, \ \sum_{i=1}^{n} \lambda_i\, u_i = 0_E$$

Examples

1. Let E be a $\mathbb{K}$-vector space.

$$\{u\} \quad \text{linearly independent} \quad \iff u \neq 0_E$$

and

$$\{u, u\} \quad \text{is linearly dependent}$$

2. Let us take $E = \mathbb{C}$.

   $(1, i)$ is linearly independent in E as for all $(a, b) \in \mathbb{R}^2$, $a + ib = 0 \Longrightarrow a = b = 0$.

3. Let $E = \mathbb{R}^3$.

   Let $u_1 = (1, 0, -1)$, $u_2 = (1, 1, 1)$ and $u_3 = (0, 1, -1)$.

   Let us prove that $\{u_1, u_2, u_3\}$ is a linearly independent family.

   Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0_{\mathbb{R}^3}$$

   Then,

$$\begin{cases} \lambda_1 + \lambda_2 & = & 0 \\ \lambda_2 + \lambda_3 & = & 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 & = & 0 \end{cases}$$

   We easily obtain that the solution of this system is $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

4. $E = \mathbb{R}^n$.

   Let, for all $i \in [\![1, n]\!]$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 being at the i-th place).

   We easily show that the family $(e_1, \dots, e_n)$ is linearly independent in E.

5. $E = \mathbb{R}_n[X]$.

   We show that the family $(1, X, \dots, X^n)$ is linearly independent in E.

6. $E = \mathbb{R}^2$.

   Let $u_1 = (1, 1)$, $u_2 = (2, 1)$ and $u_3 = (-1, 0)$.

   Then, the family (u_1, u_2, u_3) is linearly dependent as $u_1 - u_2 - u_3 = (0, 0)$.

Proposition 76

1. Every sub-family of a linearly independent family is linearly independent.
2. Every sur-family of a linearly dependent family is linearly dependent.

Proposition 77

Let $(u_1, \ldots, u_n) \in E^n$ be a linearly independent family and $u \in E$.
Then,

$$(u_1, \ldots, u_n, u) \text{ linearly dependent } \iff u \text{ is a combination linear of } u_i$$

Definition 83

Let $(u_i)_{i \in I}$ be a family possibly infinite of E.
Then,

1. $(u_i)_{i\in I}$ is linearly independent if every finite sub-family of $(u_i)_{i\in I}$ is linearly independent.
2. $(u_i)_{i\in I}$ is linearly dependent if there exists a finite sub-family of $(u_i)_{i\in I}$ which is linearly dependent.

Example

Let

$$\begin{array}{rcl} f_\alpha : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & e^{\alpha x} \end{array}$$

Then, the family $(f_\alpha)_{\alpha\in\mathbb{R}}$ is linear independent in $\mathbb{R}^{\mathbb{R}}$.
Indeed, we use a proof by contradction and assume that there exists a finite sub-family of (f_α) which is linearly independent.
Then, $\exists\,(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\exists\,(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \setminus \{(0, \ldots, 0\}$ such that

$$\sum_{i=1}^{n} \lambda_i f_{\alpha_i} = 0$$

Removing some terms if necessary, we can assume that $\forall\, i \in [\![1, n]\!]$, $\lambda_i \neq 0$.
Reordering some terms if necessary, we can assume $\alpha_1 > \alpha_2 > \ldots > \alpha_n$.
We have

$$\begin{array}{rcl} \lim\limits_{x\to+\infty} e^{-\alpha_1 x} \sum\limits_{i=1}^{n} \lambda_i e^{\alpha_i x} & = & \lim\limits_{x\to+\infty} \sum\limits_{i=1}^{n} \lambda_i e^{(\alpha_i - \alpha_1)x} \\ & = & \lambda_1 \end{array}$$

Yet, $\sum\limits_{i=1}^{n} \lambda_i f_{\alpha_i} = 0$. hence, $\lambda_1 = 0$ which is absurd.

9.2.2 Spanning family

Definition 84
Let $(u_1, \ldots, u_n) \in E^n$.
We say that the family $(u_1, \ldots, u_n)$ spans E if

$$E = Vect(u_1, \ldots, u_n)$$

i.e.

$$\forall\ u \in E,\ \exists\ (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n,\ u = \sum_{i=1}^{n} \lambda_i u_i$$

Examples

1. $E = \mathbb{C}$.

   $(1, i)$ spans E.

2. $E = \mathbb{R}^n$.

   $(e_1, \ldots, e_n)$ spans E where for all $i \in [\![1, n]\!]$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 being at the i-th place).

3. $E = \mathbb{R}_n[X]$. The family $(1, X, \ldots, X^n)$ spans E.

4. $E = \mathbb{R}^3$.

   Let $F = \{\, u = (x, y, z) \in \mathbb{R}^3,\ 2x + y - z = 0\,\}$ be a linear subspace of E.

   Then,

   $$\begin{aligned} F &= \{\,(x, y, 2x + y), (x, y) \in \mathbb{R}^2\,\} \\ &= \{\,x(1, 0, 2) + y(0, 1, 1), (x, y) \in \mathbb{R}^2\,\} \end{aligned}$$

   So,

   $$F = Vect((1, 0, 2), (0, 1, 1))$$

Proposition 78
Each finite over-family of a spanning family of E spans E.

9.2.3 Basis

Definition 85
Let $n \in \mathbb{N}^*$.
Let $(e_1, \ldots, e_n) \in E^n$ a family of vectors of E.
We say that $(e_1, \ldots, e_n)$ is a basis of E if $(e_1, \ldots, e_n)$ is a linearly independent family and spans E.

Examples

1. $E = \mathbb{C}$ (as a $\mathbb{R}$-vector space).

   $(1, i)$ is a basis of $\mathbb{C}$.

2. $E = \mathbb{R}^n$.

   $(e_1, \ldots, e_n)$ is a basis of E where for all $i \in [\![1, n]\!]$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 being at the i-th place).

   We call $(e_1, \ldots, e_n)$ the **standard basis of** $\mathbb{R}^n$.

3. $E = \mathbb{R}_n[X]$.

   $(1, X, \ldots, X^n)$ is a basis of E also called **standard basis of** $\mathbb{R}_n[X]$

Remark

In a vector space, we have several possible basis.
For example, if $E = \mathbb{R}^2$, (e_1, e_2) where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ is the standard basis.
However, let us consider $u_1 = (1, 1)$ and $u_2 = (2, 3)$.
It is easy to see that (u_1, u_2) is a linearly independent family of $\mathbb{R}^2$.
In addition, it spans $\mathbb{R}^2$ as $\forall\, u = (x, y) \in \mathbb{R}^2$,

$$u = (-x + 2y)u_1 + (x - y)u_2$$

So, (u_1, u_2) is also a basis of $\mathbb{R}^2$.

Theorem 33

Let $(e_1, \ldots, e_n) \in E^n$ be a family of vectors of E.
We have the following equivalency:

$$(e_1, \ldots, e_n) \text{ is a basis of } E \iff \left(\forall\ u \in E,\ \exists!(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n,\ u = \sum_{i=1}^{n} \lambda_i u_i \right)$$

$(\lambda_1, \ldots, \lambda_n)$ are the coordinates of u in the basis $(e_1, \ldots, e_n)$.

9.3 Linear maps

9.3.1 Definitions and examples

Let E and F two $\mathbb{K}$-vector space.

Definition 86

Let $f : E \longrightarrow F$ be an application.
We say that f is linear if

$$\forall\,(u, v) \in E^2,\ \forall\,\lambda \in \mathbb{K},\ f(\lambda u + v) = \lambda f(u) + f(v)$$

Notation

The set of linear maps from E to F is denoted $\mathcal{L}(E, F)$.

Definition 87

Let $f \in \mathcal{L}(E, F)$.

1. If f is bijective, we say that f is an isomorphism.

2. Case $E = F$.

   f is called endomorphism of E.

   $\mathcal{L}(E, E)$ is simply denoted $\mathcal{L}(E)$.

3. If $f \in \mathcal{L}(E)$ and if f is bijective, we say that f is an automorphism.

Property 3

Si $f \in \mathcal{L}(E, F)$ then $f(0_E) = 0_F$.

Examples

1. Let $E = \mathbb{R}^2$. Let us consider the application

$$\begin{array}{rlcl} f: & E & \longrightarrow & E \\ & (x,y) & \longmapsto & (ax+by, cx+dy) \quad \text{where} \quad (a,b,c,d) \in \mathbb{R}^4 \quad \text{are fixed} \end{array}$$

   Let us prove that $f \in \mathcal{L}(E)$.

   Let $u = (x, y) \in E$, $v = (x', y') \in E$ and $\lambda \in \mathbb{R}$.

   Then, $\lambda u + v = (\lambda x + x', \lambda y + y')$.

   Thus

$$\begin{aligned} f(\lambda u + v) &= \big(a(\lambda x + x') + b(\lambda y + y'), c(\lambda x + x') + d(\lambda y + y')\big) \\ &= \big(\lambda(ax + by) + (ax' + by'), \lambda(cx + dx') + (cy + dy')\big) \\ &= \lambda(ax + by, cx + dy) + (ax' + by', cx' + dy') \\ &= \lambda f(u) + f(v) \end{aligned}$$

   and so, f is actually linear.

2. The following map is linear :

$$\begin{array}{rcl} \phi : \mathcal{C}^1(\mathbb{R}) & \longrightarrow & \mathcal{C}^0(\mathbb{R}) \\ f & \longmapsto & f' \end{array}$$

3. Similarly, we show that

$$\begin{array}{rcl} \psi : \mathcal{C}^0([a,b], \mathbb{R}) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \displaystyle\int_a^b f(t)\,\mathrm{d}t \end{array}$$

   is linear.

4. Finally,

$$\begin{array}{rcl} Id_E : E & \longrightarrow & E \\ u & \longmapsto & u \end{array}$$

   is linear.

   We call it **identity function** of E.

9.3.2 Properties

Let E, F and G be three $\mathbb{K}$-vector spaces.

Proposition 79
Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(E, F)$.
Then, $\forall\,(\alpha, \beta) \in \mathbb{K}^2$, $\alpha f + \beta g \in \mathcal{L}(E, F)$.

Proposition 80
Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$.
Then, $g \circ f \in \mathcal{L}(E, G)$.
Moreover, if f is bijective, then, $f^{-1} \in \mathcal{L}(F, E)$.

Proposition 81
$\mathcal{L}(E, F)$ is a $\mathbb{K}$-vector space.

9.3.3 Kernel and image of a linear map

Definition 88
Let E and F be two sets and $f : E \longrightarrow F$ be an application.

1. Let $A \subset E$.
   We call $f(A)$ the subset of F defined by
   $$f(A) = \{\, v \in F, \exists\, u \in A,\, v = f(u)\,\}$$
2. Let $B \subset F$.
   We call $f^{-1}(B)$ the subset of E defined by
   $$f^{-1}(B) = \{\, u \in E,\ f(u) \in B\,\}$$

Proposition 82
Let E and F be two $\mathbb{K}$-vector space and $f \in \mathcal{L}(E, F)$.

1. Let A be a linear subspace of E.
   Then, $f(A)$ is a linear subspace of F.
2. Let B a linear subspace of F.
   Then, $f^{-1}(B)$ is a linear subspace of E.

Definition 89
Let E and F be two $\mathbb{K}$-vector space and $f \in \mathcal{L}(E, F)$.

1. We call kernel of f the subset of E, denoted $Ker(f)$, defined by
   $$Ker(f) = \{\, u \in E,\ f(u) = 0_F\,\} = f^{-1}\left(\{0_F\}\right)$$
2. We call image of f the subset of F, denoted $Im(f)$, defined by
   $$Im(f) = \{\, v \in F, \exists\, u \in E,\ v = f(u)\,\} = f(E)$$

Proposition 83

1. $Ker(f)$ is a linear subspace of E.
2. $Im(f)$ is a linear subspace of F.

Example

Let

$$\begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (x + y, x + y) \end{array}$$

We have

$$\begin{array}{rcl} Ker(f) & = & \{\ u = (x, y) \in \mathbb{R}^2, f(u) = (0, 0)\ \} \\ & = & \{\ u = (x, y) \in \mathbb{R}^2, (x + y, x + y) = (0, 0)\ \} \\ & = & \{\ u = (x, y) \in \mathbb{R}^2, x + y = 0\ \} \\ & = & \{\ (x, -x), x \in \mathbb{R}\ \} \\ & = & \{\ x(1, -1), x \in \mathbb{R}\ \} \end{array}$$

So

$$Ker(f) = Vect((1, -1))$$

Moreover, let $v = (X, Y) \in Im(f)$. Then, $\exists\, u = (x, y) \in \mathbb{R}^2$ such that

$$\left\{ \begin{array}{ccccc} x & + & y & = & X \\ x & + & y & = & Y \end{array} \right.$$

So, $X = Y$ and $v = (X, X) = X(1, 1)$.
Thus,

$$Im(f) = Vect(1, 1)$$

Proposition 84

Let E and F be two $\mathbb{K}$-vector spaces and $f \in \mathcal{L}(E, F)$.
then,

1. f injective $\Longleftrightarrow Ker(f) = \{0_E\}$.
2. f surjective $\Longleftrightarrow Im(f) = F$.

9.3.4 Projectors and symmetries

Let E be a $\mathbb{K}$-vector space.
Let F and G be two supplementary linear subspaces of E i.e. $E = F \oplus G$.
Then, $\forall\, u \in E$, $\exists!\, (u_1, u_2) \in F \times G$ such that $u = u_1 + u_2$.
Let us consider the application

$$\begin{array}{rcl} p : E & \longrightarrow & E \\ u & \longmapsto & u_1 \end{array}$$

Proposition 85

1. $p \in \mathcal{L}(E)$.
2. $p \circ p = p$.
3. $Ker(p) = G$ and $Im(p) = F$.

Definition 90
We call projector any endomorphism p of E verifying $p \circ p = p$.
p is actually the projection onto F along G.
We have so
$$E = Ker(p) \oplus Im(p)$$

Let the application
$$\begin{array}{rcl} s : E & \longrightarrow & E \\ u & \longmapsto & (2p - Id_E)(u) \end{array}$$

Proposition 86

1. $s \in \mathcal{L}(E)$.
2. $\forall\, u \in E$, $s(u) = u_1 - u_2$.
3. $s \circ s = Id_E$.

Definition 91
s is the symmetry onto F along G.

9.4 Finite-dimensional vector spaces

Let E be a $\mathbb{K}$-vector space.

9.4.1 Fundamental theorem

Theorem 34
Let $n \in \mathbb{N}^*$.
If E is spanned by n vectors then every family of more than n vectors is linearly dependent.

9.4.2 finite-dimensional $\mathbb{K}$-vector spaces

Definition 92
We say that E is of finite dimension if it admits a finite spanning family.

Examples

1. $\mathbb{C} = Vect(1, i)$ so $\mathbb{C}$ is a finite-dimensional $\mathbb{R}$-vector space.
2. $\mathbb{R}^n = Vect(e_1, \ldots, e_n)$ so $\mathbb{R}^n$ is a finite-dimensional $\mathbb{R}$-vector space.
3. $\mathbb{R}_n[X] = Vect(1, X, \ldots, X^n)$ so $\mathbb{R}_n[X]$ is a finite-dimensional $\mathbb{R}$-vector space.
4. $\mathbb{R}[X]$ is not a finite-dimensional $\mathbb{R}$-vector space since if it admitted a finite spanning family $(P_1, \ldots, P_n)$ then $\forall\, P \in \mathbb{R}[X]$, $\exists\, (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that
$$P = \lambda_1\, P_1 + \ldots + \lambda_n\, P_n$$
and so, we would have $d(P) \leqslant Max(d(P_1), \ldots, d(P_n))$ which is absurd.
5. $\mathbb{R}^{\mathbb{R}}$ is not a finite-dimensional $\mathbb{R}$-vector space.

Dimension of a finite-dimensional vector space

Proposition 87

Let E be a finite-dimensional $\mathbb{K}$-vector space. Then,

1. E admits at least one basis.
2. All the basis of E have the same cardinal.

Definition 93

Let E be a $\mathbb{K}$-vector space finite-dimensional.
• If $E = \{0_E\}$, we say that the dimension of E, denoted $dim(E)$, is null i.e. $dim(E) = 0$.
• If $E \neq \{0_E\}$, let $(e_1, \ldots, e_n)$ be a basis of E. We then say that E is of dimension n and we note $dim(E) = n$.

Examples

1. $dim(\mathbb{R}^n) = n$.
2. $dim(\mathbb{C}) = 2$ if $\mathbb{C}$ is considered as a $\mathbb{R}$-vector space.
3. $dim(\mathbb{R}_n[X]) = n + 1$.

Consequences

Proposition 88

Let E be a finite-dimensional $\mathbb{K}$-vector space with $dim(E) = n$.
Then,

1. every linearly independent family of E has at most n vectors.
2. every family which spans E has at least n vectors.
3. every family of E having at least $n + 1$ vectors is linearly dependent.

9.4.3 NSC for a family of vectors of E to be a basis of E

Let E be finite-dimensional $\mathbb{K}$-vector space with $dim(E) = n$.

Proposition 89

Let $\mathcal{B}$ be a family of vectors of E.
Then,

1. $\mathcal{B}$ is a basis of $E \iff \mathcal{B}$ is a linearly independent family of E and $card(\mathcal{B}) = n$.
2. $\mathcal{B}$ is a basis of $E \iff \mathcal{B}$ is a family which spans E and $card(\mathcal{B}) = n$.

Examples

1. In $E = \mathbb{R}^3$, let us prove that $u_1 = (1, -1, 0)$, $u_2 = (-1, 0, 1)$ and $u_3 = (0, -1, 2)$ form a basis of E.

   For that purpose, we first prove that this family is linearly independent.

   Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = (0, 0, 0)$$

We have to solve the following system :

$$\begin{cases} \lambda_1 & - & \lambda_2 & & & = & 0 \\ -\lambda_1 & + & & - & \lambda_3 & = & 0 \\ & & \lambda_2 & + & 2\lambda_3 & = & 0 \end{cases},$$

which gives $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$.

Hence, (u_1, u_2, u_3) is a linearly independent family of **three** vectors in $\mathbb{R}^3$ which is of dimension 3.

We conclude that it is a basis of $\mathbb{R}^3$.

2. In $E = \mathbb{R}_2[X]$, $P_0 = 1$, $P_1 = X + 1$ and $P_2 = (X - 1)^2$ form a basis of E.

   Indeed, let $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$ such that

$$\lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 = 0$$

   i.e.

$$\lambda_0 + \lambda_1 + \lambda_2 + (\lambda_1 - 2\lambda_2)X + \lambda_2 X^2 = 0$$

   We have to solve the following system:

$$\begin{cases} \lambda_0 & + & \lambda_1 & + & \lambda_2 & = & 0 \\ & & \lambda_1 & - & 2\lambda_2 & = & 0 \\ & & & & \lambda_2 & = & 0 \end{cases},$$

   which gives $(\lambda_0, \lambda_1, \lambda_2) = (0, 0, 0)$.

   We conclude that (P_0, P_1, P_2) is a linearly independent family of 3 vectors in $\mathbb{R}_2[X]$ which is of dimension 3.

   Hence, it is a basis of $\mathbb{R}_2[X]$.

9.4.4 The incomplete basis theorem and its consequences

Theorem 35

Every linearly independent family of a finite-dimensional $\mathbb{K}$-vector space E can be completed into a basis of E.

Consequences : dimension of vector subspaces

Proposition 90

Let E be a finite-dimensional $\mathbb{K}$-vector space and F be a linear subspace of E.
Then, F is a finite-dimensional $\mathbb{K}$-vector space and

$$dim(F) \leqslant dim(E)$$

Proposition 91

Let $(n, p) \in \mathbb{N}^2$.
Let E be a finite-dimensional $\mathbb{K}$-vector space n and F be a linear subspace of E such that $dim(F) = p$.
Then,

1. F admits at least one supplementary in E.

2. Any supplementary of F in E is of dimension $n - p$.

Corollary 6
Let E be a finite-dimensional $\mathbb{K}$-vector space.
Let F and G be two subspaces of E in direct sum.
Then,

$$dim(F \oplus G) = dim(F) + dim(G)$$

Corollary 7
Let E be a finite-dimensional $\mathbb{K}$-vector space .
Let F and G be two linear subspaces of E.

1. If $F \subset G$ and $dim(F) = dim(G)$ then $F = G$.
2. $dim(F + G) = dim(F) + dim(G) - dim(F \cap G)$.

9.4.5 The rank-nullity theorem and its consequences

Proposition 92
Let E be a finite-dimensional $\mathbb{K}$-vector space finite-dimensional, $\mathcal{B} = (e_1, \dots, e_n)$ a basis of E and F a $\mathbb{K}$-vector space (not necessarily of finite dimension).
Let $f \in \mathcal{L}(E, F)$.
Then,

$$Im(f) = Vect\,(f(e_1), \dots, f(e_n))$$

and so $Im(f)$ is a linear subspace of F of finite dimension.

We deduce the following theorem:

Theorem 36 (Rank-nullity theorem)
Let E be a finite-dimensional $\mathbb{K}$-vector space and F be a $\mathbb{K}$-vector space.
Let $f \in \mathcal{L}(E, F)$.
Then,

$$dim(E) = dim(Ker(f)) + dim(Im(f))$$

$dim(Im(f))$ is called rank of f, denoted $Rg(f)$.

Corollary 8
Let E and F be two finite-dimensional $\mathbb{K}$-vector spaces such that $dim(E) = dim(F)$.
Let $f \in \mathcal{L}(E, F)$.
Then,

$$f \text{ injective} \iff f \text{ surjective} \iff f \text{ bijective}$$

Example

We define the endomorphism of $\mathbb{R}^3$

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & (x + y + z, -5y + 2z, 5y + z) \end{array}$$

We have

$$\begin{aligned} ker(f) &= \left\{ (x,y,z) \in \mathbb{R}^3,\ f(x,y,z) = (0,0,0) \right\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3,\ x+y+z=0, -5y+2z=0, 5y-z=0 \right\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3,\ x=y=z=0 \right\} \end{aligned}$$

We deduce that $Ker(f) = 0_{\mathbb{R}^3}$ and f is injective and $dim(Ker(f)) = 0$.
Using the rank-nullity theorem theorem, we then obtain that

$$dim(Im(f)) = 3$$

Yet, $Im(f)$ is a linear subspace of $\mathbb{R}^3$. So, $Im(f) = \mathbb{R}^3$, and f is surjective. To conclude, f is bijective.

Chapter 10

Matrices

In the whole chapter, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
Let $(n, p) \in (\mathbb{N}^*)^2$.

10.1 Generalities

10.1.1 Definitions

Definition 94
We call matrix with n rows, p columns and with coefficients in $\mathbb{K}$ any application of

$$[\![1, n]\!] \times [\![1, p]\!] \quad \text{in} \quad \mathbb{K}$$

Such application

$$\begin{array}{rcl} A : [\![1, n]\!] \times [\![1, p]\!] & \to & \mathbb{K} \\ (i, j) & \mapsto & a_{ij} \end{array}$$

is denoted as the following table

$$A = (a_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant p} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \dots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{pmatrix}.$$

$\forall (i, j) \in [\![1, n]\!] \times [\![1, p]\!]$, a_{ij} (or $a_{i,j}$) is the term (or coefficient) located on the i-th row, j-th column.

Notations

The set of matrices with n rows, p columns and coefficients in $\mathbb{K}$ is denoted $\mathcal{M}_{n,p}(\mathbb{K})$.
$\mathcal{M}_n(\mathbb{K}) = \mathcal{M}_{n,n}(\mathbb{K})$ is called set of square matrices of order n with coefficients in $\mathbb{K}$.

Example

$$A = \begin{pmatrix} 5 & 3 \\ 0 & 7 \\ 4 & -8 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$$

10.1.2 Particular matrices

Let $A = (a_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$.

1. Null matrix :
$$\forall\ (i,j) \in [\![1,n]\!] \times [\![1,p]\!], \quad a_{ij} = 0$$
We denote $A = 0_{np}$.

2. Row matrix :
$$n = 1 \quad \text{and} \quad A \in \mathcal{M}_{1,p}(\mathbb{K})$$

Example

$$A = (1 \ \ 2 \ \ 3) \in \mathcal{M}_{1,3}(\mathbb{R})$$

3. Column matrix :
$$p = 1 \quad \text{and} \quad A \in \mathcal{M}_{n,1}(\mathbb{K})$$

Example

$$A = \begin{pmatrix} 2 \\ -5 \\ -8i \\ 0 \end{pmatrix} \in \mathcal{M}_{4,1}(\mathbb{C})$$

4. Transposed matrix :

Definition 95
We call transposed matrix of A the matrix, denoted ${}^tA = (b_{ij}) \in \mathcal{M}_{p,n}(\mathbb{K})$, defined by
$$\forall\ (i,j) \in [\![1,p]\!] \times [\![1,n]\!] \quad b_{ij} = a_{ji}$$

Example

If $A = \begin{pmatrix} 2 & 1 \\ -5 & 0 \\ -12 & 4 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$ then ${}^tA = \begin{pmatrix} 2 & -5 & -12 \\ 1 & 0 & 4 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$.

5. Case of square matrices :
$$n = p$$
Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{K})$.
   - If $\forall\ i \neq j$, $a_{ij} = 0$, we say that A is a diagonal matrix.
   If, moreover, $\forall\ i \in [\![1,n]\!]$, $a_{ii} = 1$, A is called **identity matrix** of $\mathcal{M}_n(\mathbb{K})$. It is denoted I_n.
   - A is called upper triangular if $\forall\ (i,j) \in ([\![1,n]\!])^2$,
$$(i > j \quad \Rightarrow \quad a_{ij} = 0)$$

- A is called lower triangular if $\forall\ (i,j) \in (\llbracket 1,n \rrbracket)^2$,

$$(i < j \quad \Rightarrow \quad a_{ij} = 0)$$

- A is called symmetric if ${}^tA = A$.

  Example

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -5 \\ 1 & -5 & \pi \end{pmatrix}$$

- A is called antisymmetric if ${}^tA = -A = (-a_{ij})$.

  Example

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -5 \\ 1 & 5 & 0 \end{pmatrix}$$

10.1.3 Operations on matrices

Addition and external product

Definition 96

1. We call addition in $\mathcal{M}_{n,p}(\mathbb{K})$ the internal law $+$ defined by
   $\forall\ A = (a_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$ and $\forall\ B = (b_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$

$$A + B = (a_{ij} + b_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$$

2. We call product by scalar the external law

$$\begin{array}{rcl} \mathbb{K} \times \mathcal{M}_{n,p}(\mathbb{K}) & \to & \mathcal{M}_{n,p}(\mathbb{K}) \\ (\lambda, A = (a_{ij})) & \mapsto & \lambda\, A = (\lambda\, a_{ij}) \end{array}$$

Example

In $\mathcal{M}_{3,2}(\mathbb{R})$, if $A = \begin{pmatrix} 1 & -8 \\ 0 & 1 \\ 2 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 4 \\ 2 & 44 \\ 5 & -4 \end{pmatrix}$ then,

$$A + 3B = \begin{pmatrix} 28 & 4 \\ 6 & 133 \\ 17 & -16 \end{pmatrix}.$$

Proposition 93
With these two laws, $\mathcal{M}_{n,p}(\mathbb{K})$ is a $\mathbb{K}$-vector space.

Definition 97

For $(n,p) \in (\mathbb{N}^*)^2$ and $(i,j) \in [\![1,n]\!] \times [\![1,p]\!]$, we denote E_{ij} the matrix of $\mathcal{M}_{n,p}(\mathbb{K})$ which (i,j)-th term is 1 and all the others are null.
The matrices E_{ij} are called **elementary matrices**.

Proposition 94

1. $(E_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant p}$ form a basis of $\mathcal{M}_{n,p}(\mathbb{K})$ called **standard basis** of $\mathcal{M}_{n,p}(\mathbb{K})$.
2. $dim(\mathcal{M}_{n,p}(\mathbb{K})) = np$.

Internal product

Definition 98

Let $(n,p,q) \in (\mathbb{N}^*)^3$.
Let $A = (a_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$ and $B = (b_{ij}) \in \mathcal{M}_{p,q}(\mathbb{K})$.
We call product of A by B the matrix $C = (c_{ij}) \in \mathcal{M}_{n,q}(\mathbb{K})$ defined by

$$\forall\, i \in [\![1,n]\!], \quad \forall\, j \in [\![1,q]\!], \quad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}\,.$$

Example

Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & -4 & 10 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$ and $B = \begin{pmatrix} 0 & -2 & 2 & 4 \\ 3 & -1 & 0 & 3 \\ 1 & 2 & 4 & -8 \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$.
Then,

$$AB = \begin{pmatrix} -2 & -6 & -6 & 20 \\ -2 & 18 & 46 & -80 \end{pmatrix} \in \mathcal{M}_{2,4}(\mathbb{R})\,.$$

Remarks

We can do AB only if the number of columns of A is equal to the number of rows of B.
If we can do the product AB, this does not imply that we can do the product BA.
For rectangular matrices ($n \neq p$), the products AB and BA are possible only if
$A \in \mathcal{M}_{n,p}(\mathbb{K})$ and $B \in \mathcal{M}_{p,n}(\mathbb{K})$. In this case, we do not in general have $AB = BA$. The matrix product **does not commute**.
For square matrices, the same applies : $AB \neq BA$ in general.

10.1.4 Properties of matrix calculus

Property 4

$AB = 0$ **does not imply** $A = 0$ or $B = 0$.

Example

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We have $AB = 0$ and yet $A \neq 0$ and $B \neq 0$!!!!

Property 5

Let $(n,p,q,r) \in (\mathbb{N}^*)^4$.

1. $\forall A \in \mathcal{M}_{n,p}(\mathbb{K})$, $\forall B \in \mathcal{M}_{p,q}(\mathbb{K})$ and $\forall C \in \mathcal{M}_{q,r}(\mathbb{K})$,
$$A(BC) = (AB)C$$
The matrix product is associative.
2. $\forall A \in \mathcal{M}_{n,p}(\mathbb{K})$ and $\forall (B, C) \in (\mathcal{M}_{p,q}(\mathbb{K}))^2$,
$$A(B + C) = AB + AC\,.$$
The matrix product is left-distributive over the addition.
3. $\forall (A, B) \in (\mathcal{M}_{n,p}(\mathbb{K}))^2$ and $\forall C \in \mathcal{M}_{p,q}(\mathbb{K})$,
$$(A + B)C = AC + BC\,.$$
The matrix product is right-distributive over the addition.
4. $\forall A \in \mathcal{M}_{n,p}(\mathbb{K})$, $\forall B \in \mathcal{M}_{p,q}(\mathbb{K})$ and $\forall \lambda \in \mathbb{K}$,
$$(\lambda\, A)B = \lambda(AB) = A(\lambda\, B)$$

Case of square matrices

Property 6
$\forall A \in \mathcal{M}_n(\mathbb{K}), \quad AI_n = I_nA = A$

Property 7
Let $(A, B) \in (\mathcal{M}_n(\mathbb{K}))^2$ such that $AB = BA$.
Let $m \in \mathbb{N}$.
Then,
$$(A + B)^m = \sum_{k=0}^{m} C_m^k A^k B^{m-k}$$
with the convention $A^0 = I_n$.

Property 8
$\forall\ (A, B) \in (\mathcal{M}_n(\mathbb{K}))^2, \quad {}^t(AB) = {}^tB\,{}^tA.$

10.1.5 Inverse of a square matrix

Definition 99
Let $A \in \mathcal{M}_n(\mathbb{K})$.
We say that A is invertible if
$$\exists\ B \in \mathcal{M}_n(\mathbb{K}) \quad \text{such that} \quad AB = BA = I_n$$
If A is invertible, its inverse is unique and we denote it A^{-1}.
Hence, if A is invertible,
$$AA^{-1} = A^{-1}A = I_n\,.$$
The set of invertible matrices of $\mathcal{M}_n(\mathbb{K})$ is denoted $GL_n(\mathbb{K})$.

Practical calculation of A^{-1}

We start from the following fact : Let $(U, V) \in (\mathcal{M}_{n,1}(\mathbb{K}))^2$.
Then,

$$AU = V \quad \Longleftrightarrow \quad U = A^{-1}V$$

We then have to find an expression for U as a function of V by solving a linear system. To do so, we use Gaussian elimination method.

Proposition 95

1. $\forall\, (A, B) \in GL_n(\mathbb{K})^2$, AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$
2. $\forall\, A \in GL_n(\mathbb{K})$, ${}^tA$ is invertible and
$$({}^tA)^{-1} = {}^t(A^{-1})$$

10.2 Matrix of a linear map

10.2.1 Definitions and examples

Context

Let E and F be two finite-dimensional $\mathbb{K}$-vector space such that $dim(E) = p$ and $dim(F) = n$.
Let $\mathcal{B} = (e_1, \ldots, e_p)$ be a basis of E and $\mathcal{B}' = (\varepsilon_1, \ldots, \varepsilon_n)$ be a basis of F.
Let $u \in E$.
Then,

$$\exists\,!\,(\lambda_1, \ldots, \lambda_p) \in \mathbb{K}^p \quad \text{such that} \quad u = \sum_{j=1}^{p} \lambda_j e_j$$

Definition 100

The column matrix $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} \in \mathcal{M}_{p,1}(\mathbb{K})$ is called column matrix of coordinates of u in the basis $\mathcal{B}$. We note

$$Mat_{\mathcal{B}}(u) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix}$$

Example

In $\mathbb{R}^2$, let $u = (2, 1)$.
Let $\mathcal{B}_1$ be the standard basis of $\mathbb{R}^2$ and $\mathcal{B}_2 = ((1, 1), (1, 0))$ be another basis of $\mathbb{R}^2$.
Then, $Mat_{\mathcal{B}_1}(u) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $Mat_{\mathcal{B}_2}(u) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $f \in \mathcal{L}(E, F)$.
We then have

$$f(u) = \sum_{j=1}^{p} \lambda_j f(e_j)$$

Hence, f is entirely determined by giving the vectors $f(e_j) \in F$ for all $j \in [\![1,p]\!]$.
Thus,

$$\exists\,!\,(a_{1j},\dots,a_{nj}) \in \mathbb{K}^n \quad \text{such that} \quad f(e_j) = \sum_{i=1}^{n} a_{ij}\varepsilon_i$$

Definition 101

We call matrix of f relatively to the basis $\mathcal{B}$ and $\mathcal{B}'$, denoted $Mat_{\mathcal{B},\mathcal{B}'}(f)$, the matrix which j-th column is composed of the coordinates $f(e_j)$ in the basis $\mathcal{B}'$ for all $j \in [\![1,p]\!]$.
It is a matrix with n rows and p columns

$$A = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1p} \\ a_{21} & \dots & a_{2j} & \dots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots & \\ a_{n1} & \dots & a_{nj} & \dots & a_{np} \end{pmatrix}$$

such that

$$\forall\, j \in [\![1,p]\!], \quad f(e_j) = \sum_{i=1}^{n} a_{ij}\varepsilon_i$$

Remark

Instead of denoting $Mat_{\mathcal{B},\mathcal{B}}(f)$, we only denote $Mat_{\mathcal{B}}(f)$.

Examples

1. Let the linear map f be defined by $f: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x,y) & \longmapsto & (x+y, 2x+4y, -3y) \end{array}$

   Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$ and $\mathcal{B}'$ be the standard basis of $\mathbb{R}^3$.

   Then,

   $$Mat_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}$$

2. Let the linear map g be defined by $g: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (-10x+8y; -12x+8y) \end{array}$.

   Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$.

   Then,

   $$Mat_{\mathcal{B}}(g) = \begin{pmatrix} -10 & 8 \\ -12 & 8 \end{pmatrix}$$

   Let $\mathcal{B}_1 = (u_1, u_2)$ where $u_1 = (1,2)$ and $u_2 = (1,1)$.

   We easily check that $\mathcal{B}_1$ is another basis of $\mathbb{R}^2$.

   Then,

   $$Mat_{\mathcal{B}_1}(g) = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

3. Let the linear map h be defined by $h: \begin{array}{lcl} \mathbb{R}_4[X] & \longrightarrow & \mathbb{R}_5[X] \\ P & \longmapsto & XP - P' \end{array}$.

   Let $\mathcal{B}$ be the standard basis of $\mathbb{R}_4[X]$ and $\mathcal{B}'$ be the standard basis of $\mathbb{R}_5[X]$.
   Then,
$$Mat_{\mathcal{B},\mathcal{B}'}(h) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Let E be a finite-dimensional $\mathbb{K}$-vector space n and $\mathcal{B}$ be a basis of E.
   Then,
$$Mat_{\mathcal{B}}(Id_E) = I_n$$

10.2.2 Matrix interpretation of $v = f(u)$

Proposition 96
Let E and F be two finite-dimensional $\mathbb{K}$-vector spaces with $\mathcal{B}$ a basis of E and $\mathcal{B}'$ a basis of F.
Let $u \in E$ and $f \in \mathcal{L}(E, F)$.
Then,
$$Mat_{\mathcal{B}'}(f(u)) = Mat_{\mathcal{B},\mathcal{B}'}(f) \times Mat_{\mathcal{B}}(u)$$

10.2.3 Matrix of $g \circ f$

Example

Let us consider the linear maps $f: \begin{array}{lcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (x+y, x-y) \end{array}$ and $g: \begin{array}{lcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x,y) & \longmapsto & (x+2y, x, -x+y) \end{array}$.

We denote $\mathcal{B}$ the standard basis of $\mathbb{R}^2$ and $\mathcal{B}'$ the standard basis of $\mathbb{R}^3$.
We have
$$A = Mat_{\mathcal{B}}(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
and
$$B = Mat_{\mathcal{B},\mathcal{B}'}(g) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$g \circ f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is defined by $g \circ f: \begin{array}{lcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (x,y) & \longmapsto & (3x - y, x + y, -2y) \end{array}$.

So,
$$C = Mat_{\mathcal{B},\mathcal{B}'}(g \circ f) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 0 & -2 \end{pmatrix}$$
We note that
$$C = BA$$

Proposition 97

Let E, F and G be three finite-dimensional $\mathbb{K}$-vector space , where $\mathcal{B}$ is a basis of E, $\mathcal{B}'$ is a basis of F and $\mathcal{B}''$ is a basis of G.
Let $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(F,G)$.
Then, $g \circ f \in \mathcal{L}(E,G)$ and

$$Mat_{\mathcal{B},\mathcal{B}''}(g \circ f) = Mat_{\mathcal{B}'',\mathcal{B}}(g) \times Mat_{\mathcal{B},\mathcal{B}'}(f)$$

10.2.4 Matrix of the inverse of a linear map when it is bijective

Example

Let us consider the following linear map $f: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (2x+y, x-4y) \end{array}$.

Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^2$.
We have

$$A = Mat_{\mathcal{B}}(f) = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$$

Moreover, it is easy to see that $Ker(f) = \{0_{\mathbb{R}^2}\}$. We deduce that f is injective.
Hence, f is bijective.

Using calculus, we find that $f^{-1}: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & \left(\frac{2}{5}x + \frac{1}{10}y, \frac{1}{5}x - \frac{1}{5}y\right) \end{array}$.

Thus,

$$B = Mat_{\mathcal{B}}(f^{-1}) = \begin{pmatrix} \frac{2}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$$

We note then that

$$B = A^{-1}$$

Proposition 98

Let E and F be two $\mathbb{K}$-vector space of same dimension.
Let $\mathcal{B}$ be a basis of E and $\mathcal{B}'$ a basis of F.
Let $f \in \mathcal{L}(E,F)$.
Then,

$$f \quad \text{bijective} \quad \Longleftrightarrow \quad Mat_{\mathcal{B},\mathcal{B}'}(f) \quad \text{invertible}$$

In this case, we have

$$\big(Mat_{\mathcal{B},\mathcal{B}'}(f)\big)^{-1} = Mat_{\mathcal{B}',\mathcal{B}}(f^{-1})$$

Exercise

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

1. Compute A^{-1}.

2. Let

$$f: \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x,y,z) & \longmapsto & (x+y; x+z; x+y+z) \end{array}$$

Prove that f is bijective and give the expression of f^{-1}.

Chapter 11

Rational fractions

In the whole chapter, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

11.1 Generalities

11.1.1 Definitions and rules of calculus

Definition 102
We call rational fraction with coefficients in $\mathbb{K}$ each element F written under the form

$$F = \frac{P}{Q} \quad \text{where} \quad (P, Q) \in \mathbb{K}[X]^2 \quad \text{and} \quad Q \neq 0$$

Such couple (P, Q) is called element of the rational fraction F.

Example

$F = \dfrac{\sqrt{3}X - i}{X^5 + 4}$ is a rational fraction with coefficients in $\mathbb{C}$.

Rules of calculus

Let $(P_1, P_2, Q_1, Q_2) \in \mathbb{K}[X]^4$ with $Q_1 \neq 0$ and $Q_2 \neq 0$.

• Sum :

$$\frac{P_1}{Q_1} + \frac{P_2}{Q_2} = \frac{P_1 Q_2 + P_2 Q_1}{Q_1 Q_2}$$

• External product :

$$\forall \lambda \in \mathbb{K}, \quad \lambda \cdot \frac{P_1}{Q_1} = \frac{\lambda P_1}{Q_1}$$

• Internal product :

$$\frac{P_1}{Q_1} \times \frac{P_2}{Q_2} = \frac{P_1 P_2}{Q_1 Q_2}$$

• Equality :

1. $$\frac{P_1}{Q_1} = \frac{P_2}{Q_2} \iff P_1Q_2 = P_2Q_1$$

2. $$\forall\, R \in \mathbb{K}[X] \quad \text{such that} \quad R \neq 0, \quad \text{we have} \quad \frac{P_1}{Q_1} \times \frac{R}{R} = \frac{P_1}{Q_1}$$

Notation

The set of rational fractions with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}\,(X)$.

11.1.2 Irreducible representation of a rational fraction

Example

Let
$$F = \frac{X-1}{X^2-1} \in \mathbb{R}(X)$$
$(X-1, X^2-1)$ and $(1, X+1)$ are two representations of F but $(1, X+1)$ is an irreducible representation of F.

Definition 103

We say that (P, Q) is an irreducible representation of F if no root of Q is a root of P.

Remark

One should be careful that the rational fraction is irreducible.
For example, the fraction
$$F = \frac{X^2 + (i-1)X - i}{X^4 - 1} \in \mathbb{C}(X)$$
is not irreducible as
$$F = \frac{(X+i)(X-1)}{(X-i)(X+i)(X-1)(X+1)} = \frac{1}{(X+1)(X-i)}$$

11.1.3 Degree of a rational fraction

Definition 104

1. Let $(P, Q) \in (\mathbb{K}[X]^*)^2$ such that $F = \dfrac{P}{Q}$.

   We define the degree of F by
$$d(F) = d(P) - d(Q) \in \mathbb{Z}$$

2. If $F = 0$, then $d(F) = -\infty$.

Examples

1. $$d\left(\frac{2X}{X+5}\right) = 1-1=0$$

2. $$d\left(\frac{2X}{X^4+5}\right) = 1-4=-3$$

3. $$d\left(\frac{X^3-2X+8}{1-2X}\right) = 3-1$$

Proposition 99

Let $(F,G) \in \mathbb{K}(X)^2$.
Then,

1. $d(F+G) \leqslant Max(d(F), d(G))$.
2. $d(FG) = d(F) + d(G)$.

11.1.4 Roots and poles of a rational fraction

Definition 105

Let $F \in \mathbb{K}(X)$.
Let (P,Q) be an irreducible element of F.

1. We call root (or zero) of F any root of P.
2. We call pole of F any root of Q.
3. Let $a \in \mathbb{K}$.
   If a is a root (resp. pole) of $F \neq 0$, the order of multiplicity of a is the order of multiplicity of a as a root of P (resp. of Q).

Remark

Once again, one should be careful that (P,Q) is an irreducible representation of F.
For example, 1 is neither a root, nor a pole of

$$F = \frac{X^3-1}{X^2-1}$$

Definition 106

Let $F = \dfrac{P}{Q} \in \mathbb{K}(X)$ be irreducible.

Let $\mathcal{P}$ be the set of poles of F.
For all $\alpha \in \mathbb{K} \setminus \mathcal{P}$, one can then define $\widetilde{F}(\alpha)$ by

$$\widetilde{F}(\alpha) = \frac{\widetilde{P}(\alpha)}{\widetilde{Q}(\alpha)}$$

The function $x \longmapsto \dfrac{\widetilde{P}(x)}{\widetilde{Q}(x)}$, defined on $\mathbb{K} \setminus \mathcal{P}$ is called the rational function associated with the rational fraction F.

11.1.5 A tool : Division by increasing power order

Theorem 37
Let $n \in \mathbb{N}$.
Let $(A, B) \in \mathbb{K}[X]^2$ with $\widetilde{B}(0) \neq 0$.
Then,

$$\exists!\,(Q, R) \in \mathbb{K}[X]^2 \ \text{ such that } \ A = BQ + X^{n+1}R \ \text{ where } \ R = 0 \ \text{ ou } \ d(R) \leqslant n$$

Q is called quotient of the division of A by B following increasing powers until order n. R is called the remainder of division of A by B following increasing powers until order n.

Examples

1. The division of $A = 2 + 3X - X^2 + X^4$ by $B = 1 + X + X^2$ by increasing powers until order 3 gives
$$A = (2 + X - 4X^2 + 3X^3)B + X^4(2 - 3X)$$
2. The division of $A = 1 + 4X^3$ by $B = -2 + X$ by increasing powers until order 2 gives
$$A = (-\frac{1}{2} - \frac{1}{4}X - \frac{1}{8}X^2)B + X^3(\frac{33}{8})$$
A possible application of this division is as follows :
Give a primitive of
$$f(x) = \frac{4x^3 + 1}{x^4 - 2x^3}$$
Using this division, We write $f(x) = -\dfrac{1}{2x^3} - \dfrac{1}{4x^2} - \dfrac{1}{8x} + \dfrac{33}{8(x-2)}$.
Thus, a primitive of f is
$$F(x) = \frac{1}{4x^2} + \frac{1}{4x} - \frac{1}{8}\ln|x| + \frac{33}{8}\ln|x-2| + K$$

11.2 Floor of a rational fraction

11.2.1 definition

Let $F = \dfrac{P}{Q} \in \mathbb{K}(X)$.

We perform the euclidean division of P by Q.
Then, there exists a unique couple (E, R) in $\mathbb{K}[X]^2$ such that $P = EQ + R$ with $d(R) < d(Q)$.
thus,
$$F = E + \frac{R}{Q}$$
E is called the floor of F.
To conclude, every rational fraction F can be written in a unique way as the sum of a polynomial (called floor of F) and a rational fraction of negative degree.

11.2.2 Method of research of the floor

Let $F = \dfrac{P}{Q} \in \mathbb{K}(X)$ with $P \neq 0$ and $Q \neq 0$.

• If $d(F) > 0$, we perform the euclidean division of P by Q and E is the obtained quotient.
• If $d(F) = 0$, We can perform the euclidean division of P by Q. We find that

$$\text{if} \quad F = \frac{a_n X_n + \ldots + a_0}{b_n X^n + \ldots + b_0} \quad \text{then} \quad E = \frac{a_n}{b_n}$$

• If $d(F) < 0$ then $E = 0$.

Examples

1.
$$F = \frac{X+4}{X-5} = \frac{X-5+9}{X-5} = 1 + \frac{9}{X-5}$$

2.
$$F = \frac{X^4+1}{X^3-X^2} = X + 1 + \frac{X^2+1}{X^3-X^2}$$

11.3 Partial fraction decomposition of a rational fraction

11.3.1 General theorem

Theorem 38
Let $F \in \mathbb{K}(X)$ be such that

$$F = \frac{A}{Q_1^{\alpha_1} \ldots Q_n^{\alpha_n}}$$

with

-$n \in \mathbb{N}^*$,
-$Q_1, \ldots, Q_n \in \mathbb{K}[X]^n$ irreducible and pairwise relatively prime,
-$A \in \mathbb{K}[X]$,
-$(\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}^*)^n$.

Then, $\exists\,!\,(E, C_{\alpha_1,1}, \ldots, C_{\alpha_1,\alpha_1}, C_{\alpha_2,1}, \ldots, C_{\alpha_2,\alpha_2}, \ldots, C_{\alpha_n,1}, \ldots, C_{\alpha_n,\alpha_n})$ in $\mathbb{K}[X]$ such that

$$F = E + \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i} \frac{C_{\alpha_i,j}}{Q_i^j}$$

where $\forall\ i \in [\![1, n]\!]$ and $\forall\ j \in [\![1, \alpha_i]\!]$, $d(C_{\alpha_i,j}) < d(Q_i)$.

This is the **partial fraction decomposition** of the rational fraction F in $\mathbb{K}(X)$.

Case of $\mathbb{C}(X)$

Irreducible polynomials of $\mathbb{C}[X]$ are polynomials of order 1.
Using D'Alembert-Gauss's theorem, every rational fraction of $\mathbb{C}(X)$ can be written

$$F = \frac{A}{\prod_{i=1}^{n}(X - a_i)^{\alpha_i}}$$

where for all $i \in [\![1, n]\!]$, $a_i \in \mathbb{C}$ and $\alpha_i \in \mathbb{N}$.

Using the previous theorem, we obtain the partial fraction decomposition of every rational fraction $F \in \mathbb{C}(X)$:

$$F = E + \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i} \frac{b_{i,j}}{(X - a_i)^j}$$

where $\forall\ i \in [|1, n|]$ and $\forall\ j \in [|1, \alpha_i|]$, $b_{i,j} \in \mathbb{C}$ are unique.

Examples

1. The partial fraction decomposition of $F = \dfrac{X}{X^4 - 1}$ in $\mathbb{C}(X)$ is

$$F = \frac{a}{X-1} + \frac{b}{X+1} + \frac{c}{X-i} + \frac{d}{X+i}$$

where $(a, b, c, d) \in \mathbb{C}^4$ are unique (to be determined).

2. The partial fraction decomposition of $F = \dfrac{X+1}{(X-i)^3(X+i)(X-4)^2}$ in $\mathbb{C}(X)$ is

$$F = \frac{a}{X-i} + \frac{b}{(X-i)^2} + \frac{c}{(X-i)^3} + \frac{d}{X+i} + \frac{e}{X-4} + \frac{f}{(X-4)^2}$$

where $(a, b, c, d, e, f) \in \mathbb{C}^6$ are unique (to be determined).

Case of $\mathbb{R}(X)$

Irreducible polynomials of $\mathbb{R}[X]$ are polynomials of degree 1 and polynomials of degree 2 with strictly negative discriminant.
Every rational fraction of $\mathbb{R}(X)$ can be written

$$F = \frac{A}{\prod_{i=1}^{n}(X - a_i)^{\alpha_i} \prod_{k=1}^{m}(X^2 + q_k X + r_k)^{\beta_k}}$$

where for all $i \in [\![1, n]\!]$, $a_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{N}$ and for all $k \in [\![1, m]\!]$, $(q_k, r_k) \in \mathbb{R}^2$ such that $q_k^2 - 4r_k < 0$.
Using the previous theorem, we obtain the partial fraction decomposition of every rational fraction $F \in \mathbb{R}(X)$:

$$F = E + \sum_{i=1}^{n} \sum_{j=1}^{\alpha_i} \frac{c_{i,j}}{(X - a_i)^j} + \sum_{k=1}^{m} \sum_{l=1}^{\beta_k} \frac{d_{k,l} X + e_{k,l}}{(X^2 + q_k X + r_k)^l}$$

where $\forall\ i \in [\![1, n]\!]$ and $\forall\ j \in [\![1, \alpha_i]\!]$, $c_{i,j} \in \mathbb{R}$ are unique and $\forall\ k \in [\![1, m]\!]$ and $\forall\ l \in [\![1, \beta_k]\!]$, $(d_{k,l}, e_{k,l}) \in \mathbb{R}^2$ are unique.

Examples

1. The partial fraction decomposition of $F = \dfrac{X}{X^4 - 1}$ in $\mathbb{R}(X)$ is

$$F = \frac{a}{X-1} + \frac{b}{X+1} + \frac{cX + d}{X^2 + 1}$$

where $(a, b, c, d) \in \mathbb{R}^4$ are unique (to be determined).

2. The partial fraction decomposition of $F = \dfrac{X-6}{(X-1)X^2(X^2+1)(X^2+4)^3}$ in $\mathbb{R}(X)$ is

$$F = \frac{a}{X-1} + \frac{b}{X} + \frac{c}{X^2} + \frac{dX+e}{X^2+1} + \frac{fX+g}{X^2+4} + \frac{hX+j}{(X^2+4)^2} + \frac{kX+l}{(X^2+4)^3}$$

where $(a,b,c,d,e,f,g,h,j,k,l) \in \mathbb{R}^{11}$ are unique (to be determined).

11.3.2 Methods to determine the coefficients

Case of simple poles

Example 1

Let $F = \dfrac{X}{X^2-1} \in \mathbb{R}(X)$

We have

$$F = \frac{X}{(X-1)(X+1)}$$

The floor of F is null as $d(F) < 0$.
The decomposition of F in $\mathbb{R}(X)$ is

$$F = \frac{a}{X-1} + \frac{b}{X+1}$$

where $(a,b) \in \mathbb{R}^2$.
To find a, it is sufficient to compute

$$\widetilde{(X-1)F}(1)$$

Indeed,

$$(X-1)F = \frac{X}{X+1} = a + \frac{b(X-1)}{X+1}$$

Thus,

$$\widetilde{(X-1)F}(1) = \frac{1}{1+1} = a + 0$$

Thus, $a = \frac{1}{2}$.
Similarly, to find b, it is sufficient to compute

$$\widetilde{(X+1)F}(-1)$$

We then have

$$\widetilde{(X+1)F}(-1) = \frac{-1}{-1-1} = 0 + b$$

so $b = \frac{1}{2}$.
To conclude,

$$F = \frac{1}{2(X-1)} + \frac{1}{2(X+1)}$$

Example 2

Let $F = \dfrac{3X^2}{X^2-4} \in \mathbb{R}(X)$.

$d(F) = 0$ so the floor of F is $\dfrac{3}{1} = 3$ and so

$$F = 3 + \frac{3}{X^2-1}$$

The decomposition of $F_1 = \dfrac{3}{X^2-1}$ in $\mathbb{R}(X)$ is

$$F_1 \frac{3}{X^2-1} = \frac{3}{(X-1)(X+1)} = \frac{a}{X-1} + \frac{b}{X+1}$$

We have

$$(\widetilde{X-1})F_1(1) = \frac{3}{1+1} = a + 0$$

so $a = \frac{3}{2}$.
Moreover,

$$(\widetilde{X+1})F_1(-1) = \frac{3}{-1-1} = 0 + b$$

so $b = -\frac{3}{2}$.
Finally,

$$F_1 = \frac{3}{2(X-1)} - \frac{3}{2(X+1)}$$

To conclude,

$$F = 3 + \frac{3}{2(X-1)} - \frac{3}{2(X+1)}$$

Case of multiples poles

Example 1 [Using even function]

Let $F = \dfrac{4}{(X^2-1)^2}$.

$d(F) = -4$ so the floor of F is null.
The decomposition of F is

$$F(X) = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{c}{X+1} + \frac{d}{(X+1)^2}$$

Yet, $F(-X) = F(X)$ and

$$F(-X) = \frac{-a}{X+1} + \frac{b}{(X+1)^2} + \frac{-c}{X-1} + \frac{d}{(X-1)^2}$$

By uniqueness of the partial fraction decomposition, we conclude that

$$\begin{aligned} a &= -c \\ b &= d \end{aligned}$$

Thus,

$$F(X) = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{-a}{X+1} + \frac{b}{(X+1)^2}$$

We then have

$$(X-1)^2 F = \frac{4}{(X+1)^2} = a(X-1) + b - \frac{a(X-1)^2}{X+1} + \frac{b(X-1)^2}{(X+1)^2}$$

so

$$\widetilde{(X-1)^2 F}(1) = \frac{4}{4} = b$$

It remains to find a.
To do so, we can take a particular value for X.
For example, let us take $X = 0$. We have

$$\widetilde{F}(0) = \frac{4}{1} = -a + b - a + b = -2a + 2b$$

so $a = -1$. Finally,

$$F = \frac{-1}{X-1} + \frac{1}{(X-1)^2} + \frac{1}{X+1} + \frac{1}{(X+1)^2}$$

Example 2

Let $F = \dfrac{X}{(X-1)^3(X+1)}$.

$d(F) = -3$ hence the floor of F is null.
The decomposition of F is

$$F = \frac{a}{X-1} + \frac{b}{(X-1)^2} + \frac{c}{(X-1)^3} + \frac{d}{X+1}$$

The simple constants to compute are c and d.
We have

$$\widetilde{(X-1)^3 F}(1) = \frac{1}{2} = c$$

and

$$\widetilde{(X+1) F}(-1) = \frac{-1}{-8} = d$$

Moreover, let us compute $\lim\limits_{X \to +\infty} XF(X)$.
We have

$$XF(X) = \frac{X^2}{(X-1)^3(X+1)} = \frac{aX}{X-1} + \frac{bX}{(X-1)^2} + \frac{cX}{(X-1)^3} + \frac{dX}{X+1}$$

We find

$$\lim_{X \to +\infty} XF(X) = 0 = a + 0 + 0 + d$$

Thus, $a = -d = -\dfrac{1}{8}$. Finally, it remains to find b.
To do so, let us set $X = 0$. We find then

$$0 = -a + b - c + d$$

so $b = a + c - d = \frac{1}{4}$.
Finally,

$$F = \frac{-1}{8(X-1)} + \frac{1}{4(X-1)^2} + \frac{1}{2(X-1)^3} + \frac{1}{8(X+1)}$$

Example 3 [Case of pole 0]

Let $F = \dfrac{X^4+1}{X^2(X-1)}$.

$d(F) = 1$. Using euclidean division, we have

$$F = X + 1 + \frac{X^2+1}{X^2(X-1)}$$

Let us set $F_1 = \dfrac{X^2+1}{X^2(X-1)}$.

The decomposition of F_1 is

$$F_1 = \frac{a}{X} + \frac{b}{X^2} + \frac{c}{X-1}$$

• Method 1 :
The constants b and c are simple to compute.
Indeed,

$$\widetilde{X^2F_1}(0) = \frac{1}{-1} = b$$

and

$$\widetilde{(X-1)F_1}(1) = \frac{2}{1} = c$$

Moreover,

$$\lim_{X\to+\infty} XF_1(X) = 1 = a + c$$

Thus, $a = -1$.
Finally,

$$F_1 = \frac{-1}{X} + \frac{-1}{X^2} + \frac{2}{X-1}$$

Conclusion :

$$F = X + 1 + \frac{-1}{X} + \frac{-1}{X^2} + \frac{2}{X-1}$$

• Method 2 :
When 0 is a pole, we can also use the division by increasing powers.
Indeed, the division by increasing powers at order 1 of X^2+1 by $X-1$ gives

$$X^2 + 1 = (X-1)(-X-1) + 2X^2$$

Thus

$$\begin{aligned} F_1 &= \frac{(X-1)(-X-1)+2X^2}{X^2(X-1)} \\ &= \frac{-X-1}{X^2} + \frac{2}{X-1} \\ &= \frac{-1}{X} + \frac{-1}{X^2} + \frac{2}{X-1} \end{aligned}$$

Remarks

1. Example 2 can also be done using the division by increasing powers, by reducing to the case 0 via the change of variable

$$Y = X - 1 \iff X = Y + 1$$

Indeed, we then have

$$F(Y) = \frac{Y+1}{Y^3(Y+2)}$$

Performing the division by increasing powers at order 2 of $Y+1$ by $Y+2$, we find

$$Y + 1 = (Y+2)\left(\frac{1}{2} + \frac{1}{4} - \frac{1}{8}Y^2\right) + \frac{1}{8}Y^3$$

so

$$\begin{aligned} F &= \frac{1}{2Y^3} + \frac{1}{4Y^2} - \frac{1}{8Y} + \frac{1}{8(Y+2)} \\ &= \frac{1}{2(X-1)^3} + \frac{1}{4(X-1)^2} - \frac{1}{8(X-1)} + \frac{1}{8(X+1)} \end{aligned}$$

2. Limit arguments can only be used with fractions of strictly negative degree.

Example 4

Let $F = \dfrac{X^4+1}{(X+1)^2(X^2+1)} \in \mathbb{C}(X)$.

$d(F) = 1$. Using Euclidean division, we have

$$F = 1 - 2\frac{X^3+X^2+X}{(X+1)^2(X^2+1)} = 1 - 2\frac{X^3+X^2+X}{(X+1)^2(X+i)(X-i)}$$

Let us set $F_1 = \dfrac{X^3+X^2+X}{(X+1)^2(X+i)(X-i)}$.

The decomposition of F_1 is

$$F_1 = \frac{a}{X+1} + \frac{b}{(X+1)^2} + \frac{c}{X+i} + \frac{d}{X-i}$$

we have

$$\widetilde{(X+1)}F_1(-1) = -\frac{1}{2} = b$$

$$\widetilde{(X+i)}F_1(-i) = \frac{1}{4} = c$$

and

$$\widetilde{(X-i)}F_1(i) = \frac{1}{4} = d$$

Moreover,

$$\lim_{X \to +\infty} XF_1(X) = 1 = a + c + d$$

Thus, $a = \frac{1}{2}$.

Finally,

$$F_1 = \frac{1}{2(X+1)} + \frac{-1}{2(X+1)^2} + \frac{1}{4(X+i)} + \frac{1}{4(X-i)}$$

so

$$\begin{aligned} F &= 1 - 2\left(\frac{1}{2(X+1)} + \frac{-1}{2(X+1)^2} + \frac{1}{4(X+i)} + \frac{1}{4(X-i)}\right) \\ &= 1 - \frac{1}{X+1} + \frac{-1}{(X+1)^2} + \frac{1}{2(X+i)} + \frac{1}{2(X-i)} \end{aligned}$$

Case of second order elements

Example

Let $F = \dfrac{X^3}{(X-1)(X^2+1)} \in \mathbb{R}(X)$.

$d(F) = 0$. Using euclidean division, we have

$$F = 1 + \frac{X^2 - X + 1}{(X-1)(X^2+1)}$$

Let us set $F_1 = \dfrac{X^2 - X + 1}{(X-1)(X^2+1)}$.

The decomposition of F_1 is

$$F_1 = \frac{a}{X-1} + \frac{bX + c}{X^2+1}$$

we have

$$(\widetilde{X-1})F_1(1) = \frac{1}{2} = a$$

Moreover,

$$(\widetilde{X^2+1})F(i) = \frac{-i}{i-1} = bi + c$$

i.e.

$$bi + c = -\frac{1}{2} + \frac{1}{2}i$$

We conclude that $b = \dfrac{1}{2}$ and $c = -\dfrac{1}{2}$.
Finally,

$$F_1 = \frac{1}{2(X-1)} + \frac{X-1}{2(X^2+1)}$$

and so,

$$F = 1 + \frac{1}{2(X-1)} + \frac{X-1}{2(X^2+1)}$$