TD Test 1

Name : Surname : Group :

Question from the class

Let $(\alpha, \beta) \in \mathbb{R}^2$. Remind of the necessary and sufficient conditions for the convergence of the series $\sum \frac{1}{n^\alpha (\ln(n))^\beta}$.

This is a Bertrand series.
It is CV iff $(\alpha > 1)$ or $(\alpha = 1$ and $\beta > 1)$

Exercise 1

Determine $\lim_{x \to 0} \frac{1 - \ln(1+x^2) - \cos(2x)}{1 - \sqrt{1-x^2}}$.

* $1 - \ln(1+x^2) - \cos(2x) = 1 - \left(x^2 + o(x^2)\right) - \left(1 - \frac{4x^2}{2} + o(x^2)\right)$
$= x^2 + o(x^2)$

* $1 - \sqrt{1-x^2} = 1 - \left(1 - \frac{1}{2}x^2 + o(x^2)\right) = \frac{1}{2}x^2 + o(x^2)$

* $\frac{1 - \ln(1+x^2) - \cos(2x)}{1 - \sqrt{1-x^2}} = \frac{x^2 + o(x^2)}{\frac{1}{2}x^2 + o(x^2)}$

$= \frac{\cancel{x^2}(1 + o(1))}{\cancel{x^2}\left(\frac{1}{2} + o(1)\right)}$

$\xrightarrow{x \to 0} \frac{1}{\frac{1}{2}} = 2$

Exercise 2

1. Détermine the nature of the series $\sum \frac{(n!)^2 3^n}{(2n)!}$.

* $u_{n+1} = \frac{((n+1)n!)^2 \, 3^{n+1}}{(2n+2)(2n+1)(2n)!}$ $\qquad u_n = \frac{(n!)^2 \, 3^n}{(2n)!}$

* So $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} \times 3$

$$= \frac{n^2(1+\frac{1}{n})^2}{n^2(2+\frac{2}{n})(2+\frac{1}{n})} \times 3$$

$$\xrightarrow[]{n\to+\infty} \frac{3}{4}$$

* $\lim \frac{u_{n+1}}{u_n} = \frac{3}{4} < 1$ so according to d'Alembert's rule, $\sum u_n$ is CV.

2. First, determine $\lim_{n\to+\infty} \sqrt[n]{n}$, and then, give the nature of the series $\sum \frac{1}{n\sqrt[n]{n}}$ by reasoning with an equivalent.

* $\sqrt[n]{n} = n^{1/n} = e^{\frac{1}{n}\ln(n)}$. Since $\frac{1}{n}\ln(n) \xrightarrow[n\to+\infty]{} 0$, then $\lim_{n\to+\infty} e^{\frac{1}{n}\ln(n)} = e^0 = 1$

* We have $\frac{1}{n\sqrt[n]{n}} \sim \frac{1}{n}$ because $\frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} \to 1$

So $\sum \frac{1}{n\sqrt[n]{n}}$ and $\sum \frac{1}{n}$ have the same nature. But $\sum \frac{1}{n}$ is DV (Riemann) so $\sum \frac{1}{n\sqrt[n]{n}}$ is DV.