Class Test n°3

Name : **First Name :** **Class :**

Question from the lesson

Let (u_n) be a numerical sequence and $\ell \in \mathbb{R}$. Give the accurate definition, using the mathematical quantifiers, of : « (u_n) is bounded », « (u_n) tends to $+\infty$ » and « (u_n) does not converge to ℓ ».

* (u_n) bounded : $\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ |u_n| \leq M$
* $\lim (u_n) = +\infty$: $\forall A \in \mathbb{R},\ \exists N \in \mathbb{N},\ \forall n \in \mathbb{N},\ n > N \Rightarrow u_n > A$
* (u_n) does not converge to ℓ:
  $\exists \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > N$ and $|u_n - \ell| \geq \varepsilon$

Question from the lesson

Give an example of a numerical sequence (u_n) that is both increasing and bounded (and prove these two properties).

Let (u_n) be defined on $\mathbb{N}^*$ by $u_n = -\frac{1}{n}$.

Then:

- $\forall n \in \mathbb{N}^*,\ u_{n+1} - u_n = -\frac{1}{n+1} + \frac{1}{n} = \frac{-n + (n+1)}{n(n+1)} = \frac{1}{n(n+1)} > 0$

  So (u_n) is increasing.

- $\forall n \in \mathbb{N}^*$, we have $-\frac{1}{n} < 0$ so $u_n < 0$

  and $n \geq 1 \Rightarrow \frac{1}{n} \leq 1 \Rightarrow -\frac{1}{n} \geq -1$

So $\forall n \in \mathbb{N},\ -1 \leq u_n \leq 0$ and (u_n) is bounded.

Exercise 1

Let (u_n) be defined by $u_0 = 3$ and for every $n \in \mathbb{N}$, $u_{n+1} = 5 - 4u_n$. Determine, for every $n \in \mathbb{N}$, u_n as a function of n.

* We determine the fixed point ℓ: $\ell = 5 - 4\ell \Rightarrow 5\ell = 5 \Rightarrow \ell = 1$
* Let $v_n = u_n - 1$

Then $\forall n \in \mathbb{N},\ v_{n+1} = u_{n+1} - 1 = 5 - 4u_n - 1 = 4 - 4u_n$
$= -4(u_n - 1)$
$= -4 v_n$

So (v_n) is geometric with common ratio -4.

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* Then $\forall n \in \mathbb{N}$, $v_n = v_0 (-4)^n$ and $v_0 = u_0 - 1 = 3 - 1 = 2$.
  So $v_n = 2 \times (-4)^n$
* Since $v_n = u_n - 1$, we have $u_n = v_n + 1$.

Thus, $\forall n \in \mathbb{N}$, $u_n = 2 \times (-4)^n + 1$

Exercise 2

Let $(u_n)_{n \in \mathbb{N}^*}$ be defined for every $n \in \mathbb{N}^*$ by $u_n = \left(\sum_{k=1}^{n} \frac{1}{k!} \right) + \frac{1}{n!}$.

1. Study the monotonicity of (u_n).

Let $n \in \mathbb{N}$. Then

$$u_n = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \right) + \frac{1}{n!}$$

$$u_{n+1} = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!} \right) + \frac{1}{(n+1)!}$$

So
$$u_{n+1} - u_n = \frac{1}{(n+1)!} + \frac{1}{(n+1)!} - \frac{1}{n!}$$
$$= \frac{2}{(n+1)!} - \frac{1}{n!}$$
$$= \frac{2}{(n+1)!} - \frac{n+1}{(n+1)!}$$
$$= \frac{2-n-1}{(n+1)!} = \frac{-n+1}{(n+1)!} \leq 0 \text{ since } n \in \mathbb{N}^* \text{ (so } n \geq 1)$$

So the sequence (u_n) is decreasing.

2. Is (u_n) convergent? Justify your answer.

For any $n \in \mathbb{N}$, u_n is the sum of positive terms, so $u_n \geq 0$.

Thus, we have $\begin{cases} (u_n) \text{ decreasing} \\ (u_n) \text{ bounded below (by 0)} \end{cases}$

So (u_n) is convergent.