# Final exam n°2

Duration: three hours

Documents and calculators not allowed

Name: First Name: Class:

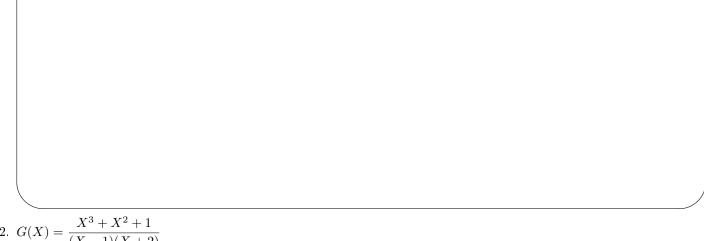
Exercise 1 (3 points)

Let  $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ . Determine the inverse matrix  $A^{-1}$  (don't forget to check - on your draft - the final result).

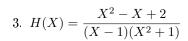
## Exercise 2 (4,5 points)

Expand into partial fractions of  $\mathbb{R}(X)$  the following rational fractions :

1. 
$$F(X) = \frac{X+1}{(X-1)(X+2)(X+3)}$$



2. 
$$G(X) = \frac{X^3 + X^2 + 1}{(X - 1)(X + 2)}$$



### Exercise 3 (4 points)

Let  $E = \mathbb{R}_2[X]$  and  $f: E \longrightarrow E$  defined for every  $P \in E$  by  $f(P) = 2(X+1)P - (X^2+1)P'$ .

Let  $\mathscr{B} = (1, X, X^2)$  and  $\mathscr{B}' = (1, X - 1, (X + 1)^2)$ , two bases of E.

1. Determine  $\operatorname{Mat}_{\mathscr{B}}(f)$ , the matrix of f with respect to the basis  $\mathscr{B}$ .

Determine $\operatorname{Mat}_{\mathscr{B},\mathscr{B}'}(f)$ , the matrix of $f$ with respect to the bases $\mathscr{B},\mathscr{B}$	8'.
Determine $\operatorname{Mat}_{\mathscr{B}',\mathscr{B}}(f)$ , the matrix of $f$ with respect to the bases $\mathscr{B}'$ , $\mathscr{B}$	B.

#### Exercise 4 (4 points)

Let us denote I the identity matrix of order 3. Let  $J=\left(\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right)$ 

1. Check that  $J^2 - J - 2I = 0$ . Deduce an expression of  $J^{-1}$  as a function of I and J.

N.B.: We remind you that, if there exists  $K \in \mathcal{M}_3(\mathbb{R})$  such that JK = I then J is invertible and  $J^{-1} = K$ .

2. Let  $n \in \mathbb{N}$ . We proceed to the Euclidean division of  $X^n$  by  $X^2 - X - 2$ . Thus, there exists  $Q(X) \in \mathbb{R}[X]$  and  $R(X) \in \mathbb{R}[X]$  such that

$$X^{n} = (X^{2} - X - 2)Q(X) + R(X)$$

with the degree of R being strictly inferior to 2.

Then, there exists  $(a, b) \in \mathbb{R}^2$  such that

$$X^{n} = (X^{2} - X - 2)Q(X) + aX + b$$

Noticing that 2 and -1 are roots of  $X^2 - X - 2$ , determine a and b.

3.	Let $n \in \mathbb{N}$ .	Deduce an	expression	of $J^r$	$^{i}$ as a	function	of $n, I$	and $J$ .
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N.B.: You will substitute J to the indeterminate X of question 2 (knowing that the polynomial 1 becomes I).

As an example,  $X^4 + 2X^3 + 4$  becomes after substitution  $J^4 + 2J^3 + 4I$ .

### Exercise 5 (5,5 points)

Let  $E = C^{\infty}(\mathbb{R}, \mathbb{R})$ , the set of smooth functions from  $\mathbb{R}$  to  $\mathbb{R}$  (i.e. functions that are infinitely differentiable on  $\mathbb{R}$ ). Let us denote  $f_0$ ,  $f_1$  and  $f_2$  the vectors of E defined for every  $x \in \mathbb{R}$  by

$$f_0(x) = e^{2x}$$
,  $f_1(x) = xe^{2x}$  and  $f_2(x) = x^2e^{2x}$ 

Let us denote  $F = \text{Span}(\{f_0, f_1, f_2\})$  i.e. F is the vector subspace of E spanned by the vectors  $f_0$ ,  $f_1$  and  $f_2$ .

1. Show that  $B = (f_0, f_1, f_2)$  is a basis of F.

2. Let d be the application defined for every  $f \in F$  by d(f) = f'. Show that d is an endomorphism of F

Show that d is an endomorphism of F.

3. Determine A, the matrix of d with respect to B.

4. Let  $n \in \mathbb{N}^*$ . Calculate  $d^n(f_0)$ ,  $d^n(f_1)$  and  $d^n(f_2)$  where, for every  $p \in \mathbb{N}^*$ ,  $d^p = \underbrace{d \circ ... \circ d}_{p \text{ times}}$ .

N.B.: You can use the general Leibniz rule, giving the  $n^{\text{th}}$  derivative of the product of two functions u and v of E, denoted  $(uv)^{(n)}$ :

$$(uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}$$

assuming that  $u^{(0)} = u$  and where  $C_n^k = \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$ .

5.	Deduce an	expression	of	$A^n$	as a	function	of $n$	for	every	n	$\in \mathbb{N}^*$ .