Assignment #1: Review of Numerical Methods and Basics of Optimization

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Course : Numerical Optimization
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1. Consider the following function and do the following (by hand):

$$f(\mathbf{x}) = 2x_1^2 - 3x_2^2 + 4x_1x_2 + (x_3+2)^2 + 4x_1$$

(a) What are the gradient and Hessian of $f(\mathbf{x})$?

$$\nabla^T f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 4x_1 + 4x_2 + 4 \\ -6x_2 + 4x_1 \\ 2x_3 + 4 \end{bmatrix}$$

$$\mathbf{H}(\mathbf{x}) = \nabla(\nabla^T f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_3 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_3 \partial x_2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 \\ 4 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) What are the stationary point(s) of $f(x)$?

The gradient of function at the stationary point is zero. Let stationary point $\mathbf{x}^*$

$$\rightarrow \nabla f(\mathbf{x}^*) = \mathbf{0}$$

$$\rightarrow \begin{bmatrix} 4x_1^* + 4x_2^* + 4 \\ -6x_2^* + 4x_1^* \\ 2x_3^* + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4 & 4 & 4 \\ 4 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}$$

By Gaussian elimination process,

$$\rightarrow \begin{bmatrix} 4 & 4 & 4 \\ 4 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0.8 \\ -2 \end{bmatrix}$$

$$\therefore \mathbf{x}^* = \begin{bmatrix} 1.2 \\ 0.8 \\ -2 \end{bmatrix}$$

(c) Is the Hessian positive-definite?

To determine if Hessian is positive-definite, the eigenvalues of Hessian should be evaluated. Let $\mathbf{x}^*$ eigenvector and λ eigenvalues.

$$\rightarrow (\mathbf{H} - \lambda\mathbf{I})\mathbf{x}^* = \mathbf{0}$$

$$\rightarrow \begin{bmatrix} 4-\lambda & 4 & 0 \\ 4 & -6-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow det(\mathbf{H} - \lambda\mathbf{I}) = 0$$

$$\rightarrow (\lambda - 2)(\lambda^2 + 2\lambda - 40) = 0$$

The product of the roots of the second-order equation is negative, so the eigenvalues of the characteristic equation must have different signs.

Therefore the Hessian is indeterminate, rather than positive-definite

2. Let $f(x) = \sin(x)$ be a function that you are interested in optimizing. Please answer the following questions completely:

(a) What are the necessary conditions for a solution to be an optimum of $f(x)$?

If $\mathbf{x}^*$ is a local minimizer of a function f which is continuously differentiable near $\mathbf{x}^*$, then:

$$\nabla f(\mathbf{x}^*) = 0$$

That is, the gradient of f at $\mathbf{x}^*$ must be zero if $\mathbf{x}^*$ is a local minimizer.

Proof by Contradiction)

Assume $\mathbf{x}^*$ is a local minimizer of $f(\mathbf{x})$, but:

$$\nabla f(\mathbf{x}^*) \neq 0$$

Using a second-order Taylor expansion:

$$f(\mathbf{x}^* + \mathbf{p}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^\top \mathbf{p} + \frac{1}{2}\mathbf{p}^\top H(f(\mathbf{x}))\mathbf{p}$$

Let $\mathbf{p} = -\gamma \nabla f(\mathbf{x}^*)$, for some small $\gamma > 0$. Then:

$$f(\mathbf{x}^* + \mathbf{p}) = f(\mathbf{x}^*) - \gamma \|\nabla f(\mathbf{x}^*)\|_2^2 + \mathcal{O}(\gamma^2)$$

For sufficiently small γ, the second-order term is negligible, so:

$$f(\mathbf{x}^* + \mathbf{p}) < f(\mathbf{x}^*)$$

This contradicts the assumption that $\mathbf{x}^*$ is a local minimizer.

$$\therefore \quad \nabla f(\mathbf{x}^*) = 0$$

In this case, $f(x)$ is univariate function so $\frac{df(x^*)}{dx}$ should be zero to be local minimum at x^*. The necessary condition for a local maximum follows in the same way.

(b) Using the necessary conditions obtained in (a), and considering the interval $0 \leq x \leq 2\pi$, obtain the stationary point(s).

$$\begin{aligned} f(x) &= \sin x \\ \frac{df(x)}{dx} &= \cos x \\ \text{Set } \frac{df(x^*)}{dx} &= 0 \Rightarrow \cos x = 0 \\ &\Rightarrow x_1^* = \frac{\pi}{2}, \quad x_2^* = \frac{3\pi}{2} \end{aligned}$$

(c) Confirm whether the above point(s) are inflection points, maxima, or minima. If they are maxi- mum (or minimum) points, are they global maximum (or minimum) in the given interval?

$$\begin{aligned} \frac{d^2 f(x)}{dx^2} &= -\sin x \\ \text{At } x_1^* = \frac{\pi}{2} : \frac{d^2 f(x)}{dx^2} &= -\sin\left(\frac{\pi}{2}\right) = -1 < 0 \\ &\Rightarrow x_1^* \text{ is a local maximum} \\ f(x_1^*) &= \sin\left(\frac{\pi}{2}\right) = 1 \\ \text{At } x_2^* = \frac{3\pi}{2} : \frac{d^2 f(x)}{dx^2} &= -\sin\left(\frac{3\pi}{2}\right) = 1 > 0 \\ &\Rightarrow x_2^* \text{ is a local minimum} \\ f(x_2^*) &= \sin\left(\frac{3\pi}{2}\right) = -1 \end{aligned}$$

$$\begin{aligned} f(x_2^*) = -1 < f(0) = 0 < f(x_1^*) = 1 \\ f(x_2^*) = -1 < f(\pi) = 0 < f(x_1^*) = 1 \end{aligned}$$

$$\therefore \text{At the given interval, } x^* = \frac{\pi}{2} \text{ is the global maximum, and } x^* = \frac{3\pi}{2} \text{ is the global minimum.}$$

(d) Plot the function $\sin(x)$ over the interval $0 \leq x \leq 2\pi$. Show all the stationary points on it, and label them appropriately (maximum, minimum, or inflection).

Please refer to the Figure 1 to see the plot of function $\sin(x)$.

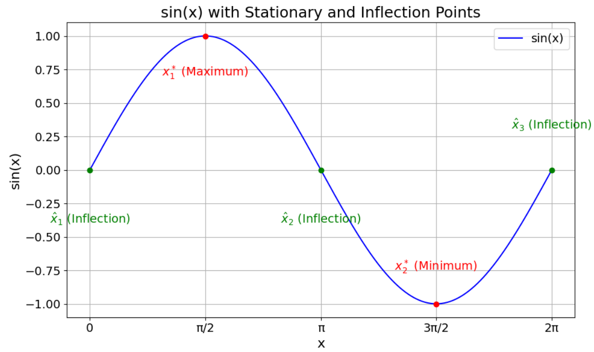

Figure 1: Problem 2.(d) - Graph of $\sin(x)$ with labeled stationary and inflection points.

3. Consider the single variable function $f(x) = e - ax^2$, where a is a constant. This function is often used as a "radial basis function" for function approximation. Please answer the following questions completely:

(a) Is the point $x = 0$ a stationary point for (i) $a > 0$, and (ii) $a < 0$. What happens if $a = 0$? Is $x = 0$ still a stationary point?

$$f'(x) = -2ax \cdot e^{-ax^2}$$

$$f'(0) = 0 \Rightarrow x = 0 \text{ is a stationary point regardless of the sign of } a.$$

$$\text{If } a = 0, \quad f(x) = 0 \quad \text{(constant function).}$$

Since every point on a constant function is a stationary point, $x = 0$ is still a stationary point.

(b) If $x = 0$ is a stationary point, classify it as a minimum, maximum, or an inflection point for (i) $a > 0$,(ii) $a < 0$, and (iii) $a = 0$.

$$\begin{aligned} f''(x) &= -2ae^{-ax^2} + 4a^2x^2e^{-ax^2} = e^{-ax^2}(-2a + 4a^2x^2) \\ &= 4a^2e^{-ax^2}\left(x^2 - \frac{1}{2a}\right) \\ f''(0) &= -2a \end{aligned}$$

$$\begin{cases} < 0 & \text{if } a > 0 \Rightarrow x = 0 \text{ is a local maximum} \\ > 0 & \text{if } a < 0 \Rightarrow x = 0 \text{ is a local minimum} \\ = 0 & \text{if } a = 0 \Rightarrow x = 0 \text{ is an inflection point} \end{cases}$$

(c) Prepare a plot of f(x) for $a = 1$, $a = 2$, and $a = 3$. Plot all three curves on the same figure. By observing the plot, do you think $f(x) = e - ax^2$, $a > 0$ has a global minimum? If so, what is the value of x and $f(x)$ at the minimum?

Please refer to the Figure 2.

I think $f(x)$ would have a global minimum, if certain finite boundary is given. If given boundary is $[-b; b]$, global minimum value will be e^{-ab^2}.

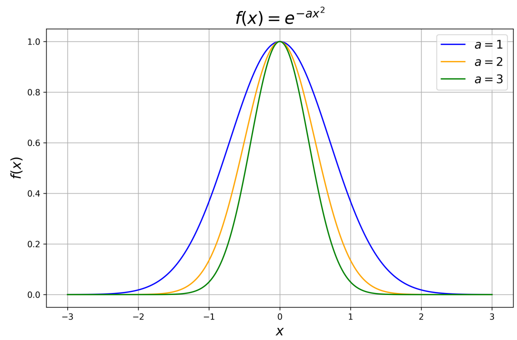

Figure 2: Problem 3.(c) - Graph of e^{-ax^2} with three different positive a values.

4. Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

(a) Use the definition to determine whether $\begin{bmatrix} -\pi \\ \pi \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors. of A associated with $\lambda = -1$.

$$A \begin{bmatrix} -\pi \\ \pi \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\pi \\ \pi \end{bmatrix} = \begin{bmatrix} -3\pi + 4\pi \\ -2\pi + \pi \end{bmatrix} = \begin{bmatrix} \pi \\ -\pi \end{bmatrix} = -1 \cdot \begin{bmatrix} -\pi \\ \pi \end{bmatrix}$$

Therefore, $\begin{bmatrix} -\pi \\ \pi \end{bmatrix}$ is an eigenvector of A with eigenvalue -1.

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector.

(b) Is either of the given eigenvectors of A associated with $\lambda = 5$?

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{cases} 3x_1 + 4x_2 = 5x_1 \Rightarrow x_1 = 2x_2 \\ 2x_1 + x_2 = 5x_2 \Rightarrow x_1 = 2x_2 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore \lambda = 5$ is eigenvalue and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is corresponding eigenvector.

(c) What is the point of this exercise?

If either eigenvector or eigenvalue is known, the other one could be evaluated as well.

5. Use the Bisection, Fixed-Point, Newton's, and Secant methods to find solutions accurate to within 10^{-5} for the following problems:

(a) $x^2 - 4x + 4 - ln(x) = 0$ for $1 \leq x \leq 2$ and $2 \leq x \leq 4$

(b) $x + 1 - 2sin(\pi x) = 0$ for $0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$

Write a code to solve the above problems using the specified methods. Provide a plot illustrating the convergence of the error versus the number of iterations. For the fixed-point, Newton's, and Secant methods set x_0 to be the minimum point for the specified range. In addition to the plot, show a table with four entries of the values of $x, f(x)$ and the error $f(x)$. You may treat x as $\bar{x}$ in these cases. Out of the four entries, provide the initial value, the final values and two intermediary values during the convergence of the algorithm.

Please refer to the Figure 3 to see the plot of convergence error versus iterations of each method, and refert to the Table 1, 2, 3, 4 to see the comparison between Fixed-point iteration method and Newton's method. Please refer to the Appendix section to see the code(Python-based) corresponding to each method.

Bisection Method : This method always found roots regardless of the function that it solved for. The rate of convergence seemed affordable.

Fixed-Point Iteration Method : This method didn't find a root for second function and it ended up with divergence. In the case of first function, the rate of convergence was very low even if it found a root. And it even didn't find a root existing between [2; 4] with regard to the first function.

Newton's Method : This method showed the fastest rate of convergence. But when it attempted to find a root for second function between [2; 4], it ended up with finding a root existing between [1; 2]. It failed to find the root between [2; 4] regardless of the initial guess.

Secant Method : This method found roots regardless of the function being solved, along with the bisection method. But the rate of convergence changed rapidly depending on the case.

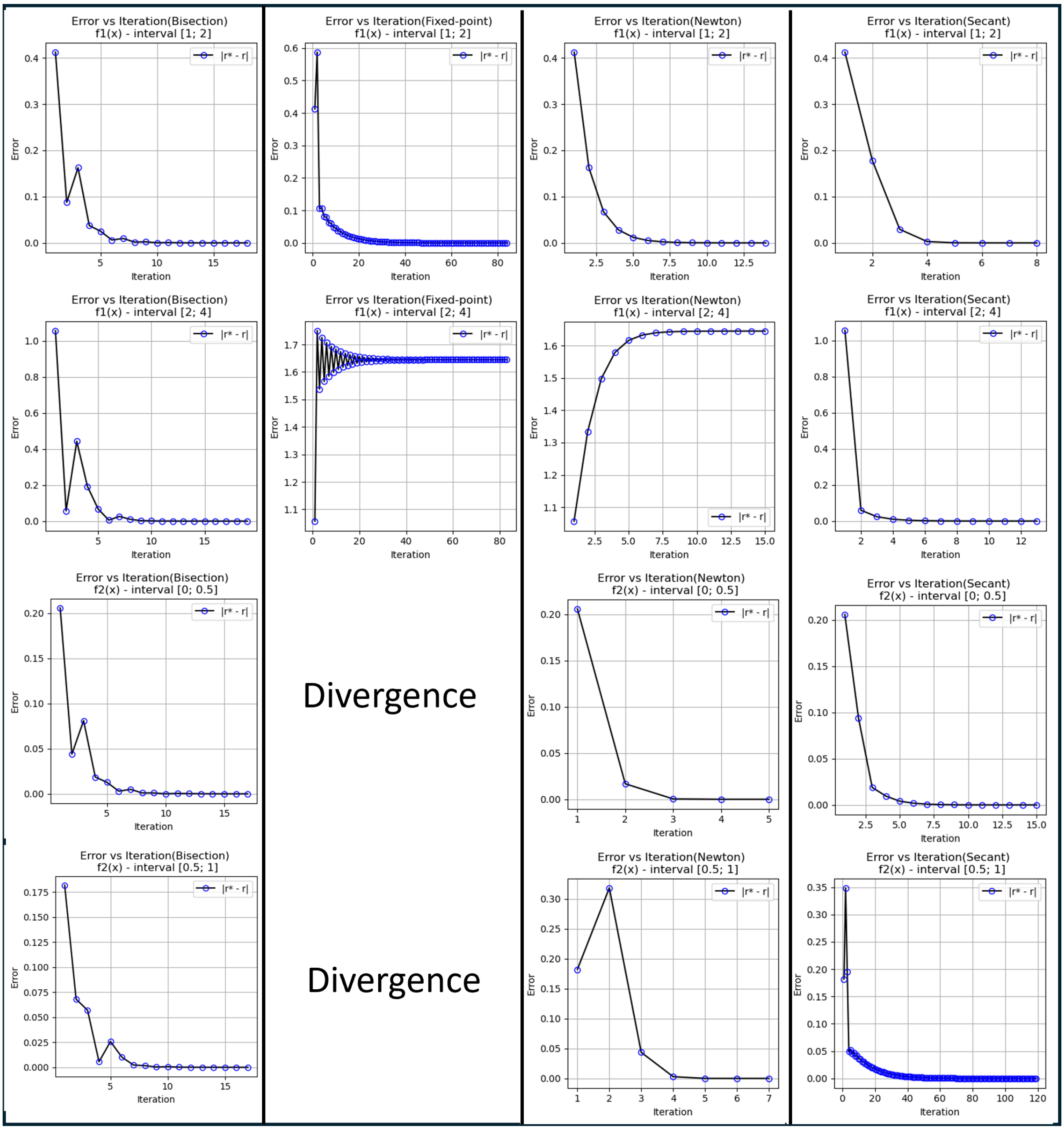

Figure 3: Problem 5 - Error vs Iteration plot of Bisection, Fixed-point iteration, Newton's, Secant method.

Iteration (k)	r	$\lvert f(r)\rvert$	$\lvert r^* - r\rvert$
0	2.000000	0.693147	1.057104
1	1.306853	0.212831	1.750251
2	1.519684	0.187799	1.537420
82(converged)	1.412396	8.36×10^{-6}	1.644708

Table 1: Problem 5 - Fixed-Point Iteration for $f_1(x)$ on $[2, 4]$

Iteration (k)	r	$\lvert f(r)\rvert$	$\lvert r^* - r\rvert$
0	2.000000	0.693147	1.057104
1	1.722741	0.467044	1.334362
2	1.559309	0.250035	1.497794
14(converged)	1.412397	1.15×10^{-5}	1.644706

Table 2: Problem 5 - Newton's Method for $f_1(x)$ on $[2, 4]$

Iteration (k)	r	$\lvert f(r)\rvert$	$\lvert r^* - r\rvert$
0	0.000000	1.000000	0.206035
1	1.000000	2.000000	0.793965
2	3.000000	4.000000	2.793965
∞(diverged)	∞(diverged)	∞(diverged)	∞(diverged)

Table 3: Problem 5 - Fixed-Point Iteration for $f_2(x)$ on $[0, 0.5]$

Iteration (k)	r	$\lvert f(r)\rvert$	$\lvert r^* - r\rvert$
0	0.000000	1.000000	0.206035
1	0.189280	0.068859	0.016755
2	0.205656	0.001520	0.000379
4(converged)	0.206035	2.68×10^{-13}	3.84×10^{-13}

Table 4: Problem 5 - Newton's Method for $f_2(x)$ on $[0, 0.5]$

6. Let $f(x, y) = x^3 - x + y^3 - y$
(a) Graph the surface $z = f(x, y)$.

Please refer to the Figure 4.

(b) Verify that the complete list of critical points of f is $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\text{First derivatives: } \frac{\partial f}{\partial x} = 3x^2 - 1 = 0, \frac{\partial f}{\partial y} = 3y^2 - 1 = 0$$

$$\text{Critical points: } \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$$

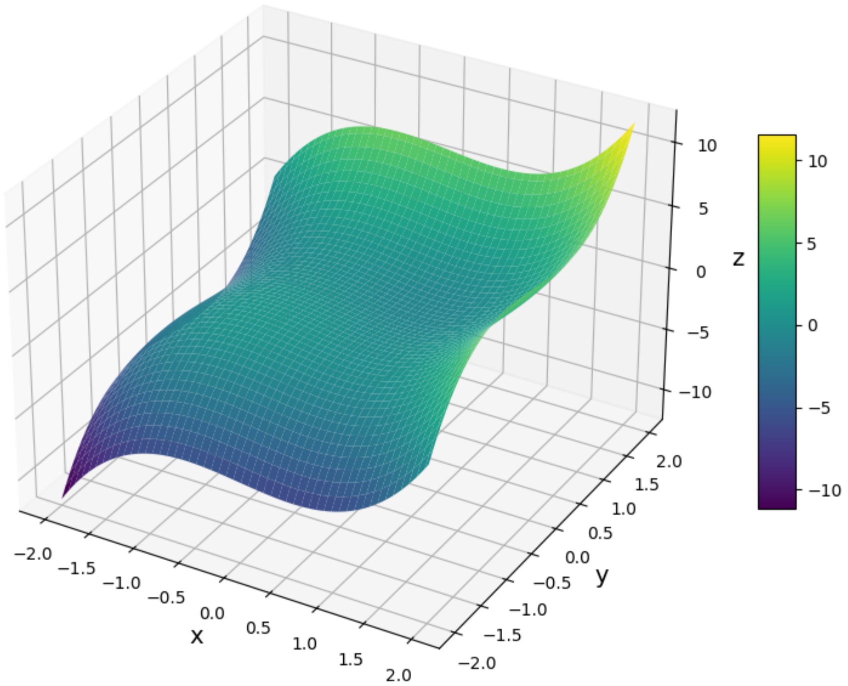

Figure 4: Problem 6.(a) - Graph of $f(x, y) = x^3 - x + y^3 - y$

(c) Calculate the Hessian matrix

$$\text{Hessian: } H = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

$$\text{At } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) : H = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix}, \lambda_1, \lambda_2 > 0 \Rightarrow \text{Local Min}$$

$$\text{At } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) : H = \begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{bmatrix}, \lambda_1, \lambda_2 < 0 \Rightarrow \text{Local Max}$$

$$\text{At } \left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right) : H = \begin{bmatrix} \pm 2\sqrt{3} & 0 \\ 0 & \mp 2\sqrt{3} \end{bmatrix}, \text{Eigenvalues of opposite sign} \Rightarrow \text{Saddle Point}$$

(d) Fill in the following table.

Critical points	Hessian at (x, y)	Eigenvalues of Hessian at (x, y)	Concavity at (x, y)
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{bmatrix}$	$-2\sqrt{3}, -2\sqrt{3}$	Concavity
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix}$	$-2\sqrt{3}, 2\sqrt{3}$	Saddle
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} \end{bmatrix}$	$2\sqrt{3}, -2\sqrt{3}$	Saddle
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} \end{bmatrix}$	$2\sqrt{3}, 2\sqrt{3}$	Convexity

(e) Examine the table carefully and explain how eigenvalues of Hessian matrices can help you classify the concavity of the surface at each critical point.

If eigenvalues are all positive, the function has local minimum and convexity at that point. If eigenvalues are all negative, the function has local maximum and concavity at that point. If eigenvalues are mixed with positive and negative, the function has saddle point.

7. Find the critical points of $f(x,y) = x^2 + y^3 - x^2y + xy^2$ and classify them all by using the eigenvalues of the appropriate Hessian matrices.

The partial derivatives are:

$$\frac{\partial f}{\partial x} = -2xy + 2x + y^2$$
$$\frac{\partial f}{\partial y} = -x^2 + 2xy + 3y^2$$

We solve the coupled equation:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 2x - 2xy + y^2 = 0 \rightarrow 2x(1-y) + y^2 = 0 \rightarrow x = \frac{y^2}{2(y-1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - x^2 + 22xy = 0 \rightarrow 3y^2 - \frac{y^4}{4(y-1)^2} + \frac{2y^3}{2(y-1)} = 0 \rightarrow y^2\left(3 - \frac{y^2}{4(y-1)^2} + \frac{y}{y-1}\right) = 0$$

$$\rightarrow y^2\left(12(y-1)^2 - y^2 + 4y(y-1)\right) = 0 \rightarrow y^2(15y^2 - 28y + 12) = 0 \rightarrow y^2(3y-2)(5y-6) = 0$$

$$\rightarrow y_1 = 0, \quad y_2 = \frac{2}{3}, \quad y_3 = \frac{6}{5}$$

- Critical point: $(x,y) = (-\frac{2}{3}, \frac{2}{3})$
- Critical point: $(x,y) = (0,0)$
- Critical point: $(x,y) = (\frac{18}{5}, \frac{6}{5})$

Hessian matrix at (x,y) :

$$H(x,y) = \begin{bmatrix} 2-2y & -2x+2y \\ -2x+2y & 2x+6y \end{bmatrix}$$

At the critical point $(x,y) = (-\frac{2}{3}, \frac{2}{3})$:
Hessian :

$$H = \begin{bmatrix} \frac{2}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} \end{bmatrix}$$

Eigenvalues:

$$\frac{5}{3} - \frac{\sqrt{73}}{3}, \frac{5}{3} + \frac{\sqrt{73}}{3}$$

Classification: Saddle point

At the critical point $(x,y) = (0,0)$:
Hessian matrix:

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues:

$$2, 0$$

Classification: Indeterminate

At the critical point $(x, y) = (\frac{18}{5}, \frac{6}{5})$:
Hessian matrix:

$$H = \begin{bmatrix} -\frac{2}{5} & -\frac{24}{5} \\ -\frac{24}{5} & \frac{72}{5} \end{bmatrix}$$

Eigenvalues:

$$7 - \frac{\sqrt{1945}}{5}, 7 + \frac{\sqrt{1945}}{5}$$

Classification: Saddle point

Appendix

Here is the Python code regarding the problem 5.

```python
def bisection(func, a0, b0, tol):
    if func(a0)*func(b0) < 0:
        list_k, list_r, list_f = [0], [a0], [func(a0)]
        a_new = a0
        b_new = b0
        k = 0
        err = np.abs(b_new - a_new)
        while err > tol:
            k = k + 1; list_k.append(k)
            a_old = a_new
            b_old = b_new
            mid = .5*(a_old + b_old); list_r.append(mid)
            f_a_old = func(a_old)
            f_b_old = func(b_old)
            f_mid = func(mid); list_f.append(np.abs(f_mid))
            if f_mid == 0:
                err = 0
                print(f"At {k:d}-th iteration : The root of which function value is exactly zero is acquired so iteration ends ! ")
                break
            sign_w_a = f_mid*f_a_old
            sign_w_b = f_mid*f_b_old
            if (sign_w_a < 0) & (sign_w_b > 0):
                b_new = mid
                a_new = a_old
            elif (sign_w_a > 0) & (sign_w_b < 0):
                a_new = mid
                b_new = b_old
            err = abs(a_new - b_new)
            print(f"At {k:d}-th iteration : a_new, b_new of interval are {a_new:.6f}, {b_new:.6f} and the interval length is {err:.6f}")
        if f_mid == 0:
            r = mid
            print(f"Converged to Exact Root. {k:d}-th iteration / r = {r:.6f} / func(r) = {func(r):.6f}")
        else:
            r = .5*(a_new + b_new)
            print(f"Converged by Tolerance. {k:d}-th iteration / r = {r:.6f} / func(r) = {func(r):.6f} / length_interval = {err:.6f}")
        return r, np.array(list_k), np.array(list_r), np.array(list_f)
    else:
        print(f"f(a0)*f(b0) is positive. There may not be root in [{a0}; {b0}]. Try another bracket !")
--------------------------------------------------------------------------------

def fixed_point(func, r0, tol):
```

```
g = lambda x : func(x) + x
r_new = r0
k = 0
err = 1
list_k, list_r, list_f = [k], [r0], [func(r0)]
while err > tol:
    k = k + 1; list_k.append(k)
    r_old = r_new
    r_new = g(r_old); list_r.append(r_new); list_f.append(func(r_new))
    err = np.abs(r_new - r_old)
    if err < tol:
        print(f"Converged By Tolerance. {k:d}-th iteration / r = {r_new:.6f} /
            f(r) = {f(r_new):.6f} / |r_new - r_old| = {err:.6f}")
        return r_new, np.array(list_k), np.array(list_r), np.array(list_f)
    else:
        print(f"At {k:d}-th iteration : r_new is {r_new:.6f} and the |r_new -
            r_old| is {err:.6f}")
--------------------------------------------------------------------------------

def newton(func, Dfunc, r0, tol):
r_new = r0
k = 0
err = 1
list_k, list_r, list_f = [k], [r0], [func(r0)]
while err > tol:
    k = k + 1; list_k.append(k)
    r_old = r_new
    r_new = r_old - func(r_old)/Dfunc(r_old); list_r.append(r_new); list_f.
        append(func(r_new))
    err = np.abs(r_new - r_old)
    if err < tol:
        print(f"Converged By Tolerance. {k:d}-th iteration / r = {r_new:.6f} /
            f(r) = {func(r_new):.6f} / |r_new - r_old| = {err:.6f}")
        return r_new, np.array(list_k), np.array(list_r), np.array(list_f)
    else:
        print(f"At {k:d}-th iteration : r_new is {r_new:.6f} and the |r_new -
            r_old| is {err:.6f}")
--------------------------------------------------------------------------------

def secant(func, r0, r1, tol):
r_old = r0
r_new = r1
k = 0
err = np.abs(r_new - r_old)
list_k, list_r, list_f = [k], [r0], [func(r0)]
while err > tol:
    k = k + 1; list_k.append(k)
    r_older = r_old
    r_old = r_new
```

```python
r_new = r_old - func(r_old)/(func(r_old) - func(r_older)/(r_old - r_older)); list_r.append(r_new); list_f.append(func(r_new))
err = np.abs(r_new - r_old)
if err < tol:
    print(f"Converged By Tolerance. {k:d}-th iteration / r = {r_new:.6f} / f(r) = {func(r_new):.6f} / |r_new - r_old| = {err:.6f}")
    return r_new, np.array(list_k), np.array(list_r), np.array(list_f)
else:
    print(f"At {k:d}-th iteration : r_new is {r_new:.6f} and the |r_new - r_old| is {err:.6f}")
```

Root-Finding Methods(Bisection, Fixed-point iteration, Newton's, Secant) in Python