Assignment #1: Review of Numerical Methods and Basics of Optimization

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Course : Numerical Optimization
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1. Consider the following function and do the following (by hand):
$f(x) = 2x_1^2 - 3x_2^2 + 4x_1x_2 + (x_3 + 2)^2 + 4x_1$
(a) What are the gradient and Hessian of $f(x)$?
(b) What are the stationary point(s) of $f(x)$?
(c) Is the Hessian positive-definite?

2. Let f(x) = sin(x) be a function that you are interested in optimizing. Please answer the following questions completely:
(a) What are the necessary conditions for a solution to be an optimum of $f(x)$?
(b) Using the necessary conditions obtained in (a), and considering the interval $0 \leq x \leq 2\pi$, obtain the stationary point(s)?
(c) Confirm whether the above point(s) are inflection points, maxima, or minima. If they are maxi- mum (or minimum) points, are they global maximum (or minimum) in the given interval?
(d) Plot the function $sin(x)$ over the interval $0 \leq x \leq 2\pi$. Show all the stationary points on it, and label them appropriately (maximum, minimum, or inflection).

3. Consider the single variable function $f(x) = e - ax^2$, where a is a constant. This function is often used as a "radial basis function" for function approximation. Please answer the following questions completely:
(a) Is the point $x = 0$ a stationary point for (i) $a > 0$, and (ii) $a < 0$. What happens if $a = 0$? Is $x = 0$ still a stationary point?
(b) If $x = 0$ is a stationary point, classify it as a minimum, maximum, or an inflection point for (i) $a > 0$,(ii) $a < 0$, and (iii) $a = 0$.
(c) Prepare a plot of f(x) for $a = 1$, $a = 2$, and $a = 3$. Plot all three curves on the same figure. By observing the plot, do you think $f(x) = e - ax^2$, $a > 0$ has a global minimum? If so, what is the value of x and $f(x)$ at the minimum?

4. Let $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

(a) Use the definition to determine whether $\begin{bmatrix} -\pi \\ \pi \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors. of A associated with $\lambda = -1$.

(b) Is either of the given eigenvectors of A associated with $\lambda = 5$?

(c) What is the point of this exercise?

5. Use the Bisection, Fixed-Point, Newton's, and Secant methods to find solutions accurate to within 10^{-5} for the following problems:

(a) $x^2 - 4x + 4 - ln(x) = 0$ for $1 \leq x \leq 2$ and $2 \leq x \leq 4$

(b) $x + 1 - 2sin(\pi x) = 0$ for $0 \leq x \leq 0.5$ and $0.5 \leq x \leq 1$

Write a code to solve the above problems using the specified methods. Provide a plot illustrating the convergence of the error versus the number of iterations. For the fixed-point, Newton's, and Secant methods set x_0 to be the minimum point for the specified range. In addition to the plot, show a table with four entries of the values of $x, f(x)$ and the error $f(x)$. You may treat x as $\bar{x}$ in these cases. Out of the four entries, provide the initial value, the final values and two intermediary values during the convergence of the algorithm.

6. Let $f(x, y) = x^3 - x + y^3 - y$

(a) Graph the surface $z = f(x, y)$.

(b) Verify that the complete list of critical points of f is $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(c) Calculate the Hessian matrix?

(d) Fill in the following table.

Critical points	Hessian at (x, y)	Eigenvalues of Hessian at (x, y)	Concavity at (x, y)
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$			
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$			
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$			
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$			

(e) Examine the table carefully and explain how eigenvalues of Hessian matrices can help you classify the concavity of the surface at each critical point.

7. Find the critical points of $f(x, y) = x^2 + y^3 - x^2y + xy^2$ and classify them all by using the eigenvalues of the appropriate Hessian matrices.