Exercise 4

$$B_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}; \quad B_3 = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}; \quad B_4 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 1 \\ 0 & 8 \end{pmatrix}$$

We'll first show that $\mathcal{B} = (B_1, B_2, B_3, B_4)$ is a basis of $\mathcal{M}_2(\mathbb{R})$.

We'll do this by showing that every matrix $V \in \mathcal{M}_2(\mathbb{R})$ can be uniquely written as a linear comb of B_1, B_2, B_3, B_4.

Let $V \in \mathcal{M}_2(\mathbb{R})$, $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$, $v_1, v_2, v_3, v_4 \in \mathbb{R}$

We have to solve the system

$$\boxed{\sum_{i=1}^{4} k_i B_i = V}$$

We have the system

$$\begin{cases} k_1 + 2k_2 + 3k_3 + 4k_4 = v_1 \\ 2k_1 + 3k_2 + 4k_3 + 1k_4 = v_2 \\ 3k_1 + 4k_2 + k_3 + 2k_4 = v_3 \\ 4k_1 + k_2 + 2k_3 + 3k_4 = v_4 \end{cases}$$

It is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

We'll work on the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & v_1 \\ 2 & 3 & 4 & 1 & v_2 \\ 3 & 4 & 1 & 2 & v_3 \\ 4 & 1 & 2 & 3 & v_4 \end{array}\right) \xrightarrow[\substack{L_3 = L_3 - 3L_1 \\ L_4 = L_4 - 4L_1}]{L_2 = L_2 - 2L_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & v_1 \\ 0 & -1 & -2 & -7 & v_2 - 2v_1 \\ 0 & -2 & -8 & -10 & v_3 - 3v_1 \\ 0 & -7 & -10 & -13 & v_4 - 4v_1 \end{array}\right)$$

$$\xrightarrow[\substack{L_3 = (-1)L_3 \\ L_4 = (-1)L_4}]{L_2 = (-1)L_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & v_1 \\ 0 & 1 & 2 & 7 & 2v_1 - v_2 \\ 0 & 2 & 8 & 10 & 3v_1 - v_3 \\ 0 & 7 & 10 & 13 & 4v_1 - v_4 \end{array}\right)$$

$$\xrightarrow[L_4=L_4-7L_2]{L_3=L_3-2L_2}\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & v_1 \\ 0 & 1 & 2 & 7 & 2v_1-v_2 \\ 0 & 0 & 4 & -4 & 3v_1-v_3-4v_1+2v_2 \\ 0 & 0 & -4 & -36 & 4v_1-v_4-14v_1+7v_2 \end{array}\right)$$

$$\xrightarrow{L_4=L_3+L_4}\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & v_1 \\ 0 & 1 & 2 & 7 & 2v_1-v_2 \\ 0 & 0 & 4 & -4 & 2v_2-v_3-v_1 \\ 0 & 0 & 0 & -40 & -10v_1+7v_2-v_4+2v_2-v_3-v_1 \end{array}\right)$$

$$\Rightarrow \begin{cases} k_1+2k_2+3k_3+4k_4 = v_1 \\ k_2+2k_3+7k_4 = 2v_1-v_2 \\ 4k_3-4k_4 = 2v_2-v_3-v_1 \\ -40k_4 = -11v_1+9v_2-v_3-v_4 \end{cases}$$

$$\Rightarrow \boxed{k_4 = \frac{11v_1-9v_2+v_3+v_4}{40}}$$

$$4k_3 = 2v_2 - v_3 - v_1 + 4\cdot \underbrace{\frac{11v_1 - 9v_2 + v_3 + v_4}{40 \;\; 10}}_{k_4}$$

$$4k_3 = \frac{20v_2 - 10v_3 - 10v_1 + 11v_1 - 9v_2 + v_3 + v_4}{10} \quad \Big| \cdot \frac{1}{4}$$

$$\boxed{k_3 = \frac{v_1 + 11v_2 - 9v_3 + v_4}{40}}$$

$$k_2 = 2v_1 - v_2 - 7k_4 - 2k_3$$

$$k_2 = \frac{80v_1 - 40v_2 - 7(11v_1 - 9v_2 + v_3 + v_4) - 2(v_1 + 11v_2 - 9v_3 + v_4)}{40}$$

$$k_2 = \frac{80v_1 - 40v_2 - 77v_1 + 63v_2 - 7v_3 - 7v_4 - 2v_1 - 22v_2 + 18v_3 - 2v_4}{40}$$

$$\boxed{k_2 = \frac{v_1 + v_2 + 11v_3 - 9v_4}{40}}$$

$$k_1 = v_1 - 4k_4 - 3k_3 - 2k_2$$

$$k_1 = \frac{40v_1 - 4(11v_1 - 9v_2 + v_3 + v_4) - 3(v_1 + 11v_2 - 9v_3 + v_4) - 2(v_1 + v_2 + 11v_3 - 9v_4)}{40}$$

$$k_1 = \frac{40v_1 - 44v_1 + 36v_2 - 4v_3 - 4v_4 - 3v_1 - 33v_2 + 27v_3 - 3v_4 - 2v_1 - 2v_2 - 22v_3 + 18v_4}{40}$$

$$\boxed{k_1 = \frac{-9v_1 + v_2 + v_3 + 11v_4}{40}}$$

$\Rightarrow$ Clearly, the matrix $V \in M_2(\mathbb{R})$ can be written as a unique linear comb. of the vectors (actually matrices) from B.

(That is, any vector from $M_2(\mathbb{R})$ can be written as a lin. comb of B)

$$B = \begin{pmatrix} 5 & 1 \\ 0 & 8 \end{pmatrix}$$

To find its coordinates in $\mathcal{B}$, we simply substitute the values in the previously found formula.

$$[B]_{\mathcal{B}} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$b_1 = \frac{-9\cdot 5 + 1 + 0 + 11\cdot 8}{40} = \frac{-45+88}{40} = \frac{43}{40}$$

$$b_2 = \frac{5 + 1 + 11\cdot 0 - 9\cdot 8}{40} = \frac{6-72}{40} = \frac{-66}{40} = -\frac{33}{20}$$

$$b_3 = \frac{5 + 11\cdot 1 - 9\cdot 0 + 8}{40} = \frac{24}{40} = \frac{12}{20} = \frac{6}{10} = \frac{3}{5}$$

$$b_4 = \frac{11\cdot 5 - 9\cdot 1 + 0 + 8}{40} = \frac{55-1}{40} = \frac{54}{40} = \frac{27}{20}$$

$$\Rightarrow \boxed{[B]_{\mathcal{B}} = \begin{pmatrix} \frac{43}{40} & -\frac{33}{20} \\ \frac{3}{5} & \frac{27}{20} \end{pmatrix}}$$

Is it a basis for $M_2(\mathbb{C})$?

YES! (I think...)

We might think that even tough $\mathcal{B}$ is a basis for $M_2(\mathbb{R})$, it is not a basis for $M_2(\mathbb{C})$ because we can't represent complex numbers. But, if we have a complex matrix $A \in M_2(\mathbb{C})$ we can just use the previously found formulas for k_1, k_2, k_3, k_4 and we'll find the right linear combination of B_1, B_2, B_3, B_4. I.e., if we take complex linear combinations of $\mathcal{B}$ (which we do) we can represent any vector $A \in M_2(\mathbb{C})$ in terms of the vectors of $\mathcal{B}$. So yes, it is a basis for $M_2(\mathbb{C})$.