Exercise 3

$$v_1 = (1+i,\ i,\ 2)$$

$$v_2 = (-1+i,\ 2+i,\ 1+i)$$

$$v_3 = (3+i,\ 1,\ 1+i)$$

(i) $\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{vmatrix} = \begin{vmatrix} 1+i & i & 2 \\ -1+i & 2+i & 1+i \\ 3+i & 1 & 1+i \end{vmatrix} \underline{\underline{C_1 = C_1 - C_2}}$

$$= \begin{vmatrix} 1 & i & 2 \\ -3 & 2+i & 1+i \\ 2+i & 1 & 1+i \end{vmatrix} \underline{\underline{C_3 = C_3 - 2C_1}} \begin{vmatrix} 1 & i & 0 \\ -3 & 2+i & 7+i \\ 2+i & 1 & -3-i \end{vmatrix}$$

$\underline{\underline{C_2 = C_2 - i \cdot C_1}} \begin{vmatrix} 1 & 0 & 0 \\ -3 & 2+i+3i & 7+i \\ 2+i & \underbrace{1-2i-i^2}_{=2-2i} & -3-i \end{vmatrix} = (2+4i)(-3-i) - (7+i)(2-2i)$

$= -6 \cancel{-2i} \cancel{-12i} - 4i^2$

$-14 + \cancel{14i} - 2i + 2i^2$

$= -20 - 2i^2 - 2i$

$= -20 + 2 - 2i$

$= -18 - 2i \neq 0\,!$

$\Rightarrow$ the vectors v_1, v_2, v_3 are lin. ind.
Any 3 lin. ind. vectors in $\mathbb{C}^3$ are a basis of $\mathbb{C}^3$ $\Bigg\} \Rightarrow$

$\Rightarrow$ (v_1, v_2, v_3) a basis of $\mathbb{C}^3$.

(ii) Let $V = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$

We need to solve $\begin{cases} Vx = e_1 \\ Vx = e_2 \\ Vx = e_3 \end{cases}$ for $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

vector.

Let's first solve it as a general case first.

$Vx = a$, where $a = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and

$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ unknowns.

$$Vx = a$$

$$\begin{pmatrix} 1+i & -1+i & 3+i \\ i & 2+i & 1 \\ 2 & 1+i & 1+i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We can manipulate the system directly in matrix notation! It is much easier.

$$\left(\begin{array}{ccc|c} 1+i & -1+i & 3+i & a \\ i & 2+i & 1 & b \\ 2 & 1+i & 1+i & c \end{array}\right) \xrightarrow{L_2=-L_2+L_1} \left(\begin{array}{ccc|c} 1+i & -1+i & 3+i & a \\ 1 & -3 & 2+i & a-b \\ 2 & 1+i & 1+i & c \end{array}\right)$$

$$\xrightarrow{R_1=R_1-i} \left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 1 & -3 & 2+i & a-b \\ 2 & 1+i & 1+i & c \end{array}\right) \xrightarrow[L_3=L_3-2L_1]{L_2=L_2-L_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 0 & -2 & i-1 & -b+i \\ 0 & 3+i & -5+i & c-2a+2i \end{array}\right)$$

$$\xrightarrow{R_3=R_3-i} \left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 0 & -2 & i-1 & -b+i \\ 0 & 3 & -5 & c-2a+i \end{array}\right)$$

$$\xrightarrow{R_2=(-1)\cdot R_2}\left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 0 & 2 & 1-i & b-i \\ 0 & 3 & -5 & c-2a+i \end{array}\right)$$

$$\xrightarrow{R_2=R_2-1}\left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 0 & 1 & -i & b-1-i \\ 0 & 3 & -5 & c-2a+i \end{array}\right)$$

$$\xrightarrow{R_3=R_3-3R_2}\left(\begin{array}{ccc|c} 1 & -1 & 3 & a-i \\ 0 & 1 & -i & b-1-i \\ 0 & 0 & -5+3i & c-2a+i-3b+3+3i \end{array}\right)$$

$$\Rightarrow \begin{cases} x - y + 3 \cdot z = a - i \\ y - i \cdot z = b-1-i \\ (-5+3i)\cdot z = c-2a-3b+3+4i \end{cases}$$

$$\Rightarrow z = \frac{c-2a-3b+3+4i}{-5+3i} = \frac{2a+3b-c-3-4i}{5-3i}$$

$$5+3i$$

$$z = \frac{2a + 3b - c - 3 - 4i}{5 - 3i}$$

$$z = \frac{(2a + 3b - c - 3 - 4i)(5 + 3i)}{25 - 9 \cdot i^2}$$

$$z = \frac{10a + 6ai + 15b + 9bi - 5c - 3ci - 15 - 9i}{25 + 9}$$

$$-20i - 12i^2$$

$$z = \frac{10a + 15b - 5c - 15 + 12 - 29i + 6ai + 9bi - 3ci}{34}$$

$$\boxed{z = \frac{10a + 15b - 5c - 3 + (6a + 9b - 3c - 29)i}{34}}$$

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$$y = b - 1 - i + z \cdot i$$

$$y = b - 1 - i + \frac{10ai + 15bi - 5ci - 3i - (6a + 9b - 3c - 29)}{34}$$

$$y = \frac{34b - 34 - 34i - 6a - 9b + 3c + 29 + 10ai + 15bi - 5ci - 3i}{34}$$

$$y = \frac{25b - 6a + 3c - 5 - 37i + 10ai + 15bi - 5ci}{34}$$

$$\boxed{y = \frac{25b - 6a + 3c - 5 + (10a + 15b - 5c - 37)i}{34}}$$

$$x = a - i + y - 3z$$

$$x = \frac{34a - 34i + 25b - 6a + 3c - 5 + (10a + 15b - 5c - 37)i - 3\left(10a + 15b - 5c - 3 + (6a + 9b - 3c - 29)i\right)}{34}$$

$$x = \frac{34a - 34i + 25b - 6a + 3c - 5 - 30a - 45b + 15c + 9 + (10a + 15b - 5c + 37 - 18a - 27b + 9c + 87)i}{34}$$

$$\boxed{x = \frac{-2a - 20b + 18c + 4 - 34i + (10a + 15b - 5c + 37 - 18a - 27b + 9c + 87)i}{34}}$$

Well wasn't that a lot of fun...

Now we just substitute a, b, c with $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

for $e_1 = (1,0,0)$,

$$x = \frac{-2+4-34i+(10+37-18+87)i}{34}$$

$$x = \frac{2+(47+87-18-34)i}{34}$$

$$x = \frac{2+(134-52)i}{34}$$

$$\boxed{x = \frac{2+82i}{34}}$$

$$y = \frac{-6-5+(10-37)i}{34}$$

$$\boxed{y = \frac{-11-27i}{34}}$$

$$\boxed{z = \frac{10-3+(6-29)i}{34} = \frac{7-23i}{34}}$$

$$\Rightarrow \boxed{e_1 = \frac{2+82i}{34} v_1 + \frac{-11-27i}{34} v_2 + \frac{-11-27i}{34} v_3}$$

for $e_2 = (0, 1, 0)$

$$x = \frac{-20+4-34i+(15+37-27+87)i}{34}$$

$$x = \frac{-16+34i+(102+10)i}{34}$$

$$x = \frac{-16+34i+112i}{34}$$

$$\boxed{x = \frac{-16+146i}{34}} = \frac{-8+73i}{16}$$

$$y = \frac{25-5+(15-37)i}{34}$$

$$\boxed{y = \frac{20-22i}{34}} = \frac{10-11i}{17}$$

$$z = \frac{15 - 3 + (9 - 29)i}{34}$$

$$\boxed{z = \frac{12 - 20i}{34}} = \frac{6 - 10i}{17}$$

$$\Rightarrow \boxed{l_2 = \frac{-8 + 73i}{16} v_1 + \frac{-8 + 146i}{34} v_2 + \frac{10 - 11i}{17} v_3}$$

for $e_3 = (0,0,1)$,

$$x = \frac{18+4-34i+(-5+37+9+87)i}{34}$$

$$x = \frac{22-34i+(96+32)i}{34}$$

$$x = \frac{22-34i+128i}{34}$$

$$\boxed{x = \frac{22+94i}{34}} = \frac{11+47i}{17}$$

$$y = \frac{3-5+(-5-37)i}{34}$$

$$\boxed{y = \frac{-2-42i}{34}} = \frac{-1-21i}{17}$$

$$z = \frac{-5-3+(-3-29)i}{34}$$

$$\boxed{z = \frac{-8-32i}{34}} = \frac{-4-16i}{17}$$

$$\Rightarrow \boxed{e_3 = \frac{11+47i}{17} v_1 - \frac{1+21i}{17} v_2 - \frac{4+16i}{17} v_3}$$

(iii) $u = (i,\ 5-i,\ 1+i)$

E is the canonical bases, where the coordinates of a vector are equal to its components

$$\Rightarrow [u]_E = \begin{pmatrix} i \\ 5-i \\ 1+i \end{pmatrix}$$

To find $[u]_B$ we have to replace (a, b, c) from (ii) with $(i,\ 5-i,\ 1+i)$

$$X = \frac{-2i - 20(5-i) + 18(1+i) + 4 - 34i + \big(10i + 15(5-i) - 5(1+i) + 37 - 18i - 27(5-i) + 9(1+i) + 87\big)\, i}{34}$$

$$X = \frac{-2i - 100 + 20i + 18 + 18i + 4 - 34i + (10i + 75 - 15i - 5 - 5i + 37 - 18i - 135 + 27i + 9 + 9i + 87)\, i}{34}$$

$$X = \frac{2i - 78 + (+8i + 70 + 46 + 87 - 135)\, i}{34}$$

$$X = \frac{2i - 78 + (8i + 78)\, i}{34} = \frac{2i - 78 - 8 + 78i}{34}$$

$$\boxed{X = \frac{-86 + 80i}{34}}$$

$$y = \frac{25(5-i) - 6i + 3(1+i) - 5 + (10i + 15(5-i) - 5(1+i) - 37)i}{34}$$

$$y = \frac{125 - 25i - 6i + 3 + 3i - 5 + (10i + 75 - 15i - 5 - 5i - 37)i}{34}$$

$$y = \frac{123 - 28i + (-10i + 33)i}{34}$$

$$y = \frac{123 - 28i - 10i^2 + 33i}{34}$$

$$\boxed{y = \frac{133 + 5i}{34}}$$

$$z = \frac{10i + 15(5-i) - 5(1+i) - 3 + (6i + 9(5-i) - 3(1+i) - 29)i}{34}$$

$$z = \frac{10i + 75 - 15i - 5i - 5 - 3 + (6i + 45 - 9i - 3 - 3i - 29)i}{34}$$

$$z = \frac{67 - 10i + (-6i + 13)i}{34} = \frac{67 - 10i - 6i^2 + 13i}{34}$$

$$\Rightarrow \boxed{z = \frac{73+3i}{34}}$$

$$\Rightarrow [u]_B = \begin{bmatrix} \frac{-86+80i}{34} \\ \frac{133+5i}{34} \\ \frac{73+3i}{34} \end{bmatrix}$$