Exercise 1

in $\mathbb{R}^3$:

$v_1 = (3, -4, 5)$ $\quad v_2 = (14, 1, 1)$

$v_3 = (-8, -5, 2)$ $\quad v_4 = (1, 1, 7)$

4 vectors in $\mathbb{R}^3$ are always lin. dependent

$\Rightarrow v_1, v_2, v_3, v_4$ lin. dependent

Notice,

$$\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{vmatrix} = \begin{vmatrix} 3 & -4 & 5 \\ 14 & 1 & 1 \\ -8 & -5 & 2 \end{vmatrix} = 3(2+5)+4(28+8) + 5(-70+8)$$

$$= 21 + 4 \cdot 36 + 5 \cdot (-62) = 21 + 144 - 310 = -145 \neq 0$$

$\Rightarrow v_1, v_2, v_3$ lin. ind. vectors in $\mathbb{R}^3 \Rightarrow (v_1, v_2, v_3)$ basis of $\mathbb{R}^3$

That means we can write v_4 in terms of the basis (v_1, v_2, v_3). Let's see

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = v_4$$

$$\begin{pmatrix} 3 & 14 & -8 \\ -4 & 1 & -5 \\ 5 & 1 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = v_4$$

$$\begin{cases} 3k_1 + 14k_2 - 8k_3 = 1 & (1) \\ -4k_1 + k_2 - 5k_3 = 1 & (2) \\ 5k_1 + k_2 + 2k_3 = 7 & (3) \end{cases}$$

$(3) - (2) \Rightarrow 9k_1 + 7k_3 = 6$

$(1) - 14 \cdot (2) \Rightarrow 59k_1 + 62k_3 = -13$

$3 - 14 \cdot (-4) =$
$= 3 + 56$
$= 59$

$-8 - 14 \cdot (-5) =$
$= -8 + 70$
$= 62$

$$\begin{cases} 9k_1 + 7k_3 = 6 & \cdot \frac{62}{7} \\ 59k_1 + 62k_3 = -13 \end{cases}$$

$$\begin{cases} \frac{558}{7}k_1 + 62k_3 = \frac{372}{7} \\ 59k_1 + 62k_3 = -13 \end{cases} \quad \ominus$$

$59 \cdot 7 = 413$ $\qquad$ $13 \cdot 7 = 91$

$$\frac{558 - 413}{7} k_1 = \frac{372 + 91}{7}$$

$$145 k_1 = 463 \Rightarrow \boxed{k_1 = \frac{463}{145}}$$

$$7k_3 = 6 - 9k_1$$

$$7k_3 = 6 - 9 \cdot \frac{463}{145}$$

$$7k_3 = 6 - \frac{4167}{145}$$

$$7k_3 = \frac{870 - 4167}{145}$$

$$k_3 = -\frac{\overset{471}{\cancel{3297}}}{145} \cdot \frac{1}{\cancel{7}} \Rightarrow \boxed{k_3 = -\frac{471}{145}}$$

$$② \Rightarrow k_2 = 1 + 4 \cdot \frac{463}{145} + \cancel{5} \cdot \left(-\frac{471}{\underset{29}{\cancel{145}}}\right)$$

$$k_2 = \frac{29 - 471}{29} + \frac{1852}{145}$$

$$k_2 = -\frac{442}{29} + \frac{1852}{145}$$

$$k_2 = \frac{-2210 + 1852}{145}$$

$$\boxed{k_2 = -\frac{358}{145}}$$

$$\Rightarrow \boxed{v_4 = \frac{463}{145} \cdot v_1 - \frac{358}{145} \cdot v_2 - \frac{471}{145} \cdot v_3}$$

Given such beautiful numbers, we notice 2 possibilities here: either I made a bunch of arithmetic errors or the author wanted to see if I know how to use a calculator!