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1.2) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x, y, z) = (3x + 3y, 5x + 5z, 3y + 3z)$$

$$Im(f) = \{f(v) \mid v \in \mathbb{R}^3\}$$

$$= \{(3x + 3y, 5x + 5z, 3y + 3z) \mid x, y, z \in \mathbb{R}\}$$

$$= \{(3x, 5x, 0) + (3y, 0, 3y) + (0, 5z, 3z) \mid x, y, z \in \mathbb{R}\}$$

$$= \{x(3, 5, 0) + y(3, 0, 3) + z(0, 5, 3) \mid x, y, z \in \mathbb{R}\}$$

$$= \langle \underbrace{(3, 5, 0)}_{v_1}, \underbrace{(3, 0, 3)}_{v_2}, \underbrace{(0, 5, 3)}_{v_3} \rangle$$

We will check to see if (v_1, v_2, v_3) independent

$$M = \begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix}; \quad \det(M) = \begin{vmatrix} 3 & 5 & 0 \\ 3 & 0 & 3 \\ 0 & 5 & 3 \end{vmatrix} = 3(0 - 15) - 5(9 - 0) + 0(15 - 0)$$

$$\Rightarrow \det(M) = -45 - 45 = -90 \neq 0$$

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$\det(M) = -90 \neq 0 \Rightarrow (v_1, v_2, v_3)$ lin. ind.

$\Rightarrow \boxed{\dim(\operatorname{Im} f) = 3.}$ with basis$(\operatorname{Im} f) = (v_1, v_2, v_3)$

Also notice: $\operatorname{Ker}(f) = \{ v \in \mathbb{R}^3 \mid f(v) = 0 \}$

$$\operatorname{Ker}(f) = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 3x + 3y = 0 \\ 5x + 5z = 0 \\ 3y + 3z = 0 \end{array} \right\}$$

$$(S): \begin{cases} 3x + 3y = 0 \Rightarrow x = -y \\ 5x + 5z = 0 \Rightarrow x = -z \\ 3y + 3z = 0 \Rightarrow y = -z \end{cases}$$

$\left.\begin{array}{l} x = -y \\ x = -z \end{array}\right\} \Rightarrow y = z$

$\left.\begin{array}{l} y = z \\ y = -z \end{array}\right\} \Rightarrow y = z = 0 \Rightarrow x = 0$

$\Rightarrow$ the kernel of f only has the zero vector

$\Rightarrow \dim(\operatorname{Ker} f) = 0$

We know that $\dim(\mathbb{R}^3) = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)$ (from the 1st dimension formula) $\Rightarrow$

$\Rightarrow \boxed{\dim(\operatorname{Im} f) = 3 - 0 = 3}$

This is a 2nd method of finding $\dim(\operatorname{Im} f)$