Exercise 2

in $\mathbb{C}^3$:

$$v_1 = (a, 1, 0)$$
$$v_2 = (1, a, a-1)$$
$$v_3 = (1, 1, a)$$
$$v_4 = (3, -a-2, -2)$$

$a \in \mathbb{C}$

at most 2 lin ind. vectors among v_1, v_2, v_3, v_4.

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v_1, v_2, v_3, v_4 – 4 vectors in $\mathbb{C}^3 \Rightarrow$ they are automatically independent

In order to have at most 2 lin. ind. vectors among v_1, v_2, v_3, v_4 we need every 3×3 determinant formed by (v_i, v_j, v_k), $i, j, k \in \{1, 2, 3, 4\}$, $i \neq j$, $i \neq k$, $j \neq k$ to be zero!

$$D_1 = \begin{vmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{vmatrix} = \begin{vmatrix} a & 1 & 0 \\ 1 & a & a-1 \\ 1 & 1 & a \end{vmatrix} = a(a^2 - a + 1) - 1(a - a + 1)$$

$$= a(a^2 - a + 1) - 1$$

$$= a^3 - a^2 + a - 1$$

$$= a^2(a-1) + (a-1)$$

$$= (a-1)(a^2+1)$$

$D_1 = 0 \Rightarrow \boxed{a = 1}$ or $a^2 + 1 = 0$

$$a^2 = -1$$

$$\boxed{a = \pm i}$$

$\Rightarrow$ for $D_1 = 0$ we have $a \in \{-i, i, 1\}$ (1)

$$D_2 = \begin{vmatrix} - v_1 - \\ - v_2 - \\ - v_4 - \end{vmatrix} = \begin{vmatrix} a & 1 & 0 \\ 1 & a & a-1 \\ 3 & -a-2 & -2 \end{vmatrix} =$$

$$= a(-2a - (a-1)(-a-2)) - 1(-2 - 3(a-1))$$

$$= a(-2a + (a-1)(a+2)) - (3 - 2 - 3a)$$

$$= a(\cancel{-2a} + a^2 \cancel{+2a} - a - 2) + 3a - 1$$

$$= a^3 - a^2 - 2a + 3a - 1$$

$$= a^3 - a^2 + a - 1 = a^2(a-1) + (a-1) = (a-1)(a^2+1)$$

$D_2 = 0 \Rightarrow \boxed{a = 1}$ or $a^2 + 1 = 0$

$\boxed{a = \pm i}$

$\Rightarrow$ for $D_2 = 0$ we have $a \in \{1, -i, i\}$ (2)

$$D_3 = \begin{vmatrix} \text{---} & v_2 & \text{---} \\ \text{---} & v_3 & \text{---} \\ \text{---} & v_4 & \text{---} \end{vmatrix} = \begin{vmatrix} 1 & a & a-1 \\ 1 & 1 & a \\ 3 & -a-2 & -2 \end{vmatrix}$$

$$= 1(-2 - (a)(-a-2)) - a(-2 - 3a) + (a-1)(-a-2-3)$$

$$= -2 + a(a+2) + 2a + 3a^2 - (a-1)(a+5)$$

$$= -2 + a^2 + 2a + 2a + 3a^2 - (a^2 + 5a - a - 5)$$

$$= 4a^2 + \cancel{4a} - 2 - a^2 - \cancel{4a} + 5$$

$$= 3a^2 + 3$$

$$= 3(a^2 + 1)$$

$D_3 = 0 \Rightarrow a \in \{\pm i\}$ (3)

Intersecting cases (1), (2), (3) we get

$$\boxed{a \in \{\pm i\}}$$