= Homework of Seminar 11 =

Exercise 11.1

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$$

Study the continuity and the partial differentiability of f at 0_2.

$\forall (x,y) \in \mathbb{R}^2 \setminus \{0_2\}$,

$$0 \le |f(x,y) - f(0_2)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|xy|}{\sqrt{x^2+y^2}} =$$

$$= \frac{|x| \cdot |y|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2} \cdot \sqrt{y^2}}{\sqrt{x^2+y^2}} \le \frac{\cancel{\sqrt{x^2+y^2}} \cdot \sqrt{x^2+y^2}}{\cancel{\sqrt{x^2+y^2}}}$$

$$= \underbrace{\sqrt{x^2+y^2}}_{\substack{\downarrow \text{ as } (x,y) \to 0_2 \\ 0}}$$

By the Squeeze Theorem $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0_2)$

$$\Downarrow$$

f is cont. at 0_2

- $\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0}{x} = 0 \Rightarrow f$ is partially diff w.r.t. x at 0_2, $\frac{\partial f}{\partial x}(0,0) = 0$

- $\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0}{y} = 0 \Rightarrow f$ is partially diff w.r.t. y at 0_2, $\frac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow f$ is partially diff. at 0_2.

Exercise 11.2

Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = e^{2x+y} \cos(3z)$. Find the gradient and the Hessian matrix of f at $(0,0,\frac{\pi}{6})$.

For $(x,y) \in \mathbb{R}^2$,

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 2e^{2x+y} \cos(3z) \\ \frac{\partial f}{\partial y}(x,y) = e^{2x+y} \cos(3z) \\ \frac{\partial f}{\partial z}(x,y) = e^{2x+y} \cdot (-\sin(3z)) \cdot 3 = -3\sin(3z) \cdot e^{2x+y} \end{cases}$$

$$\frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = 2 \cdot \cos\frac{\pi}{2} = 0$$

$$\frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = 1 \cdot \cos\frac{\pi}{2} = 0$$

$$\frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -3\sin\left(\frac{\pi}{2}\right) \cdot 1 = -3$$

$$\Rightarrow \nabla f\left(0,0,\frac{\pi}{6}\right) = (0,0,-3)$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x,y) = 4e^{2x+y}\cos(3z) \\ \dfrac{\partial^2 f}{\partial y\,\partial x}(x,y) = 2e^{2x+y}\cos(3z) \\ \dfrac{\partial^2 f}{\partial z\,\partial x}(x,y) = 2e^{2x+y}\cdot(-\sin(3z))\cdot 3 = -6e^{2x+y}\sin(3z) \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x\,\partial y}(x,y) = 2e^{2x+y}\cos(3z) \\ \dfrac{\partial^2 f}{\partial y^2}(x,y) = e^{2x+y}\cos(3z) \\ \dfrac{\partial^2 f}{\partial z\,\partial y}(x,y) = e^{2x+y}\cdot(-\sin(3z))\cdot 3 = -3e^{2x+y}\sin(3z) \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x\,\partial z}(x,y) = -3\sin(3z)\cdot 2e^{2x+y} = -6e^{2x+y}\sin(3z) \\ \dfrac{\partial^2 f}{\partial y\,\partial z}(x,y) = -3\sin(3z)\cdot e^{2x+y} \\ \dfrac{\partial^2 f}{\partial z^2}(x,y) = -3e^{2x+y}\cdot\cos(3z)\cdot 3 = -9\cos(3z)\cdot e^{2x+y} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(0,0,\frac{\pi}{6}\right) = 4\cdot 1\cdot \cos\frac{\pi}{2} = 0 \\[2ex] \dfrac{\partial^2 f}{\partial y \partial x}\left(0,0,\frac{\pi}{6}\right) = 2\cdot 1\cdot \cos\frac{\pi}{2} = 0 \\[2ex] \dfrac{\partial^2 f}{\partial z \partial x}\left(0,0,\frac{\pi}{6}\right) = -6\cdot 1\cdot \sin\frac{\pi}{2} = -6 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x \partial y}\left(0,0,\frac{\pi}{6}\right) = 2\cdot 1\cdot \cos\frac{\pi}{2} = 0 \\[2ex] \dfrac{\partial^2 f}{\partial y^2}\left(0,0,\frac{\pi}{6}\right) = 1\cdot \cos\frac{\pi}{2} = 0 \\[2ex] \dfrac{\partial^2 f}{\partial z \partial y}\left(0,0,\frac{\pi}{6}\right) = -3\cdot 1\cdot \sin\frac{\pi}{2} = -3 \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial z}\left(0,0,\frac{\pi}{6}\right) = -6\cdot 1\cdot \sin\frac{\pi}{2} = -6$$

$$\frac{\partial^2 f}{\partial y \partial z}\left(0,0,\frac{\pi}{2}\right) = -3\cdot \sin\frac{\pi}{2}\cdot 1 = -3$$

$$\frac{\partial^2 f}{\partial z^2}\left(0,0,\frac{\pi}{6}\right) = -9\cdot \cos\left(\frac{\pi}{2}\right)\cdot 1 = 0$$

$$H_f\left(0,0,\frac{\pi}{6}\right) = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 0 & -3 \\ -6 & -3 & 0 \end{pmatrix}$$

= Homework of Seminar 12 =

(12.1) Let $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^3+y^3-3xy$. Find the local extremum points for f and specify their type.

$f:\mathbb{R}^2\to\mathbb{R}$ $\quad f(x,y)=x^3+y^3-3xy$

For $(x,y)\in\mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x,y)=3x^2-3y\ ; \qquad \frac{\partial f}{\partial y}(x,y)=3y^2-3x$$

$$\begin{cases}3x^2-3y=0\\3y^2-3x=0\end{cases} \Leftrightarrow \begin{cases}x^2-y=0\\y^2-x=0\end{cases} \Leftrightarrow \begin{cases}y=x^2\\y^2-x=0\end{cases}$$

Substituting y in the 2nd equation, we get:

$$x^4-x=0$$

$$x(x^3-1)=0 \Rightarrow \begin{cases}1.\ x=0,\ y=0\\2.\ x=1,\ y=1\end{cases}$$

Stationary points: $(0,0)$, $(1,1)$

For $(x,y) \in \mathbb{R}^2$, $\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(x,y) = 6x \\[2ex] \dfrac{\partial^2 f}{\partial y^2}(x,y) = 6y \\[2ex] \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = -3 = \dfrac{\partial^2 f}{\partial x \partial y} \end{cases}$

$$H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

$\Delta_2 = 0 - 9 = -9 < 0$

$\Downarrow$

$H_f(0,0)$ is indefinite

$\Downarrow$

$(0,0)$ is not a local extremum point of f

$$H_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$\Delta_2 = 36 - 9 = 27 > 0$

$\Delta_1 = 6 > 0$

$\Rightarrow H_f(1,1)$ is positive definite

$\Rightarrow (1,1)$ is a local minimum point of f

= Homework of Seminar 13 =

(13.1) Let $\alpha, \beta \in \mathbb{R}$ and $f:[1,\infty) \to \mathbb{R}$, $f(x) = \dfrac{x^{\alpha} \arctan x}{1+x^{\beta}}$.
Study the improper integrability of f on its domain.

f cont

$\forall x \geq 1, \ f(x) \geq 0$

We try to find $p \in \mathbb{R}$ s.t. $\exists\, L = \lim\limits_{x \to \infty} x^p f(x)$ $\left(\begin{array}{c} \in (0,\infty) \\ \text{preferably} \end{array}\right)$

Let $p \in \mathbb{R}$.

$$L = \lim_{x\to\infty} x^p f(x) = \lim_{x\to\infty} \frac{x^{p+\alpha} \arctan x}{1+x^{\beta}} = \lim_{x\to\infty} \frac{x^{p+\alpha}\arctan x}{x^{\beta}(1+x^{-\beta})}$$

$$= \lim_{x\to\infty} \frac{x^{p+\alpha-\beta} \arctan x}{1+x^{-\beta}}$$

For $p = \beta - \alpha$, $L = \lim\limits_{x\to\infty} \dfrac{\arctan x}{1+x^{-\beta}}$

- For $\beta > 0$, $L = \lim\limits_{x\to\infty} \dfrac{\arctan x}{1+x^{-\beta}} = \dfrac{\frac{\pi}{2}}{1} = \dfrac{\pi}{2} \in (0,\infty)$

- For $\beta = 0$, $L = \lim\limits_{x\to\infty} \dfrac{\arctan x}{1+x^{-\beta}} = \dfrac{\frac{\pi}{2}}{2} = \dfrac{\pi}{4} \in (0,\infty)$

- For $\beta < 0$, $L = \lim\limits_{x\to\infty} \dfrac{\arctan x}{1+x^{-\beta}} = 0 \notin (0,\infty)$

$\Rightarrow$ For $p = \beta - \alpha$ and $\beta \geq 0$, we have $L \in (0, \infty)$,

so f impr. int $\Leftrightarrow p > 1 \Leftrightarrow \beta - \alpha > 1$

- For $\beta < 0$, we try another p.

  We have $L = \lim\limits_{x \to \infty} \dfrac{x^{p+\alpha-\beta} \arctan x}{1 + x^{-\beta}}$

  $$L = \lim_{x \to \infty} \frac{x^{p+\alpha} \cdot x^{-\beta} \cdot \arctan x}{1 + x^{-\beta}}$$

  For $p = -\alpha$, $L = \lim\limits_{x \to \infty} \dfrac{x^{-\beta} \arctan x}{1 + x^{-\beta}} =$

  $$= \lim_{x \to \infty} \frac{x^{-\beta} \arctan x}{x^{-\beta}(1 + x^{\beta})} = \lim_{x \to \infty} \frac{\arctan x}{1 + x^{\beta}} = \frac{\pi}{2} \in (0, \infty)$$

  $\Rightarrow f$ impr. integr. $\Leftrightarrow p > 1 \Leftrightarrow -\alpha > 1 \Leftrightarrow \alpha < -1$

$\Rightarrow$ We have that f is impr. integr $\Leftrightarrow \begin{cases} \beta \geq 0 \text{ and } \beta - \alpha > 1 \\ \beta < 0 \text{ and } \alpha < -1 \end{cases}$