= Homework of Seminar 4 =

Exercise 4.1

Define the seq (x_n) with:

$x_1 \in (0,1)$ and $x_{n+1} = x_n - x_n^2$, $n \in \mathbb{N}$

Prove (x_n) converges; find its limit; study the convergence of $n \cdot x_n$ (Hint: Stolz–Cesàro for (a_n), (b_n), $n \in \mathbb{N}$ with $a_n = n$; $b_n = \frac{1}{x_n}$)

$x_1 \in (0,1)$ $\qquad$ $x_{n+1} = x_n - x_n^2$

$x_{n+1} = x_n(1 - x_n)$

We prove that $x_n \in (0,1)$, $\forall n \in \mathbb{N}$ with mathematical induction:

We need to prove $P(n)$: "$x_n \in (0,1)$, $\forall n \in \mathbb{N}$"

We know $x_1 \in (0,1) \Rightarrow P(1)$ true

Assume $P(k)$ true $\Rightarrow x_k \in (0,1)$

$\Rightarrow$ We have

$0 < x_k < 1 \quad \cdot | (-1)$

$-1 < -x_k < 0 \quad + | 1$

$0 < 1 - x_k < 1$

So $\left.\begin{array}{l} x_k \in (0,1) \\ (1-x_k) \in (0,1) \end{array}\right\} \Rightarrow x_k(1-x_k) \in (0,1)$

(2 subunitary positive numbers multiplied result in a positive subunitary number)

$\Rightarrow x_{k+1} \in (0,1) \Rightarrow P(k+1)$ true

$\Rightarrow P(n)$ true $\Rightarrow x_n \in (0,1), \forall n \in \mathbb{N}$

$\Downarrow$

(x_n) bounded. (1)

$x_{n+1} - x_n = x_n - x_n^2 - x_n = -x_n^2 < 0, \forall n \in \mathbb{N}$

$\Rightarrow x_{n+1} < x_n, \forall n \in \mathbb{N}$

$\Rightarrow (x_n)$ strictly decreasing (2)

(1), (2) $\Rightarrow$ $\underline{(x_n) \text{ convergent}}$

Let $l = \lim\limits_{n\to\infty} x_n \in \overline{\mathbb{R}}$

$l = l - l^2 \Rightarrow -l^2 = 0 \Rightarrow l = 0 \Rightarrow \underline{\lim\limits_{n\to\infty} x_n = 0}$

((x_n) bounded, we can subtract l safely)

Let $y_n = n \cdot x_n = \dfrac{n}{\frac{1}{x_n}}$, $\forall n \in \mathbb{N}$

Let $a_n = n$
$b_n = \dfrac{1}{x_n}$ $\forall n \in \mathbb{N}$

$$\left.\begin{array}{l} x_n \in (0,1),\ \forall n \in \mathbb{N} \\ (x_n) \text{ strictly decreasing} \end{array}\right\} \Rightarrow \frac{1}{x_n} \text{ strictly increasing}$$

$\left(\text{as } x_n \to 0, \text{ we have } \dfrac{1}{x_n} \to \infty\right)$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} x_n = 0 \\ x_n \in (0,1),\ \forall n \in \mathbb{N} \\ (x_n) \text{ strictly decreasing} \end{array}\right\} \Rightarrow \frac{1}{x_n} \to \infty \text{ as } n \to \infty$$

$\Rightarrow$ we have that (b_n) strictly increasing and $\lim\limits_{n\to\infty} b_n = \infty$

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{n+1-n}{\frac{1}{x_{n+1}}-\frac{1}{x_n}} = \frac{1}{\frac{x_n - x_{n+1}}{x_{n+1}\cdot x_n}} =$$

$$= \frac{x_{n+1}\cdot x_n}{x_n - x_{n+1}} = \frac{(x_n - x_n^2)\cdot x_n}{\cancel{x_n} - \cancel{x_n} + x_n^2} = \frac{\cancel{x_n^2}(1-x_n)}{\cancel{x_n^2}} \to 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = 1 \in \overline{\mathbb{R}}$$

$$\xRightarrow{\text{Stolz-Cesàro}} \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n\cdot x_n = 1 \in \overline{\mathbb{R}}$$

$\Rightarrow$ The sequence $(n\cdot x_n)$ is convergent

<u>Exercise 4.3</u>

Find the sum of the series and specify whether they are convergent or divergent.

a) $\sum_{n\geq 1}\left(-\frac{\pi}{4}\right)^n$

$\pi < 4 \Rightarrow -\frac{\pi}{4} \in (-1, 1) \Rightarrow \sum_{n=1}^{\infty}\left(-\frac{\pi}{4}\right)^n = \left(-\frac{\pi}{4}\right)^1 \cdot \frac{1}{1-\left(-\frac{\pi}{4}\right)}$

$$\Rightarrow \sum_{n=1}^{\infty}\left(-\frac{\pi}{4}\right)^n = -\frac{\pi}{4}\cdot\frac{1}{1+\frac{\pi}{4}}$$

$$\sum_{n=1}^{\infty}\left(-\frac{\pi}{4}\right)^n = -\frac{\pi}{\cancel{4}}\cdot\frac{\cancel{4}}{4+\pi}$$

$$\sum_{n=1}^{\infty}\left(-\frac{\pi}{4}\right)^n = -\frac{\pi}{4+\pi} \in \mathbb{R} \Rightarrow \sum_{n\geq 1}\left(-\frac{\pi}{4}\right)^n$$

convergent.

(Note: $\pi < 4 \Rightarrow -\frac{\pi}{4} \in (-1, 1) \Rightarrow$ We notice that we're dealing with a geometric series with $q \in (-1, 1)$, so we could've concluded that the series is convergent directly, without calculating the sum)

b) $\sum_{n \geq 0} \frac{2^{3n}}{5^{n-1}}$

$$\frac{2^{3n}}{5^{n-1}} = \frac{2^{3n}}{5^n \cdot 5^{-1}} = \frac{5 \cdot 2^{3n}}{5^n} = \frac{5 \cdot (2^3)^n}{5^n} =$$

$$= 5 \cdot \left(\frac{8}{5}\right)^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{2^{3n}}{5^{n-1}} = 5 \cdot \sum_{n=0}^{\infty} \left(\frac{8}{5}\right)^n.$$

$$\frac{8}{5} > 1 \Rightarrow \left(\frac{8}{5}\right)^n \longrightarrow \infty \quad \text{as} \quad n \to \infty$$

By the n-th Term Test $\Rightarrow \sum_{n=0}^{\infty} \frac{2^{3n}}{5^{n-1}} = \infty \Rightarrow \sum_{n \geq 0} \frac{2^{3n}}{5^{n-1}}$ divergent

c) $\sum_{n\geq 1} \frac{1}{4n^2-1}$

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\cdot\frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{1}{2}\cdot\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

We denote the partial sum by s_n, for $n\in\mathbb{N}, n\geq 1$

$$s_n = \frac{1}{2}\left[\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} + \ldots + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right]$$

$$s_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \longrightarrow \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \quad \text{convergent}$$

d) $\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$

$\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

We denote the partial by s_n, $\forall\, n \in \mathbb{N}$, $n \geq 1$

$s_n = \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} - \cancel{\ln 2} + \cancel{\ln 4} - \cancel{\ln 3} + \ldots + \ln(n+1) - \cancel{\ln n}$

$s_n = \ln(n+1) - 0 = \ln(n+1) \longrightarrow \infty$

$\Rightarrow \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \infty$ divergent

e) $\sum_{n \geq 1} \frac{3n-2}{2^n}$

We denote the partial sum by s_n; $n \in \mathbb{N}$

$$s_n = \frac{1}{2} + \frac{4}{2^2} + \frac{7}{2^3} + \frac{10}{2^4} + \dots + \frac{3n-2}{2^n} \quad \Big| \cdot \frac{1}{2}$$

$$\frac{1}{2} s_n = \frac{1}{2^2} + \frac{4}{2^3} + \frac{7}{2^4} + \frac{10}{2^5} + \dots \frac{3n-5}{2^n} + \frac{3n-2}{2^{n+1}}$$

$-$

$$\frac{1}{2} s_n = \frac{1}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \frac{3}{2^4} + \frac{3}{2^5} + \dots + \frac{3}{2^n} - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = \frac{1}{2} + 3\left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^n}\right) - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = \frac{1}{2} + 3\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} - \frac{1}{2} - 1\right) - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = \frac{1}{2} + 3\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right) - \frac{3}{2} - 3 - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = -4 + 3 \cdot \frac{1}{1 - \frac{1}{2}} - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = -4 + 3 \cdot 2 - \frac{3n-2}{2^{n+1}}$$

$$\frac{1}{2} s_n = 2 - \frac{3n-2}{2^{n+1}} \quad \Big| \cdot 2$$

$$s_n = 4 - \overbrace{\frac{2\cdot(3n-2)}{2^{n+1}}}^{\to 0} \longrightarrow 4$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3n-2}{2^n} = 4 \quad \text{convergent.}$$

= Homework of Seminar 5 =

Exercise 5.1

Study if the following series are convergent or divergent:

a) $\sum_{n\geq 1} \left(1-\frac{1}{n}\right)^n$

$$\left(1-\frac{1}{n}\right)^n = \left[\left(1+\frac{-1}{n}\right)^{-n}\right]^{\frac{-1}{n}\cdot n} \longrightarrow e^{-1} = \frac{1}{e} \neq 0$$

$\Rightarrow$ By the n^{th} Term Test the series

$$\sum_{n\geq 1} \left(1-\frac{1}{n}\right)^n$$

is divergent.

b) $\sum_{n\geq 1} \sin \frac{1}{n^{5/4}}$

$\frac{1}{n^{5/4}} \in \left(0, \frac{\pi}{2}\right), \forall n \in \mathbb{N} \Rightarrow \sin \frac{1}{n^{5/4}} > 0, \forall n \in \mathbb{N}$

$\Rightarrow$ All terms of the series are positive

We know $\frac{\sin x}{x} \to 1$, as $x \to 0$

$$\Rightarrow \frac{\sin \frac{1}{n^{5/4}}}{\frac{1}{n^{5/4}}} \to 1, \text{ as } n \to \infty$$

$$\left.\begin{array}{l} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\sin \frac{1}{n^{5/4}}}{\frac{1}{n^{5/4}}} = 1 \in (0, \infty) \\ \\ \sum\limits_{n\geq 1} \dfrac{1}{n^{5/4}} \text{ convergent (By the generalised harmonic series)} \end{array}\right\} \Longrightarrow$$

$\Rightarrow$ By the Second Comparison Test the series

$$\sum_{n\geq 1} \sin \frac{1}{n^{5/4}}$$

is convergent.

c) $\sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$ ← All terms are non-negative

Consider the limit

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n\sqrt[3]{n}+2}}{\frac{1}{\sqrt{n}\cdot\sqrt[3]{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n\sqrt[3]{n}+2} \cdot \frac{\sqrt{n}\cdot\sqrt[3]{n}}{1} =$$

$$= \lim_{n\to\infty} \frac{n\sqrt[3]{n}}{n\sqrt[3]{n}+2} \longrightarrow 1 \in (0,\infty)$$

$$\sum_{n\geq 1} \frac{1}{\sqrt{n}\cdot\sqrt[3]{n}} = \sum_{n\geq 1} \frac{1}{n^{5/6}}$$ divergent (By the generalised harmonic series)

$\Longrightarrow$

$\Rightarrow$ By the Second Comparison Test the series

$$\sum_{n\geq 1} \frac{\sqrt{n}}{n\sqrt[3]{n}+2}$$

is divergent.

d) $\sum_{n\geq 1} \frac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$ ← All terms are non-negative

Let $x_n = \frac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}, \forall n \in \mathbb{N}$

$$\frac{x_{n+1}}{x_n} = \frac{\overset{(n+1)}{\cancel{(n+1)!}}}{\cancel{3\cdot 5\cdot \ldots \cdot (2n+1)}(2n+3)} \cdot \frac{\cancel{3\cdot 5\cdot \ldots \cdot (2n+1)}}{\cancel{n!}}$$

$$\Rightarrow \frac{x_{n+1}}{x_n} = \frac{n+1}{2n+3} \longrightarrow \frac{1}{2} < 1$$

$\Rightarrow$ By the Ratio Test we have that the series

$$\sum_{n\geq 1} \frac{n!}{3\cdot 5\cdot \ldots \cdot (2n+1)}$$

is convergent.

e) $\sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$ ← All terms are non-negative

Let $x_n = \frac{n^3 5^n}{2^{3n+1}}, \forall n \in \mathbb{N}$

$$\sqrt[n]{x_n} = \sqrt[n]{\frac{n^3 5^n}{2^{3n+1}}} = \frac{n^{\frac{3}{n}} \cdot 5^{\frac{n}{n}}}{2^{\frac{3n+1}{n}}} \to \frac{1^3 \cdot 5^1}{2^3} = \frac{5}{8} < 1$$

⇒ By the Root Test we conclude that the series

$$\sum_{n\geq 1} \frac{n^3 5^n}{2^{3n+1}}$$

is convergent.

$\left(\text{Note: we know } n^{\frac{1}{n}} \to 1 \Rightarrow n^{\frac{3}{n}} = \left(n^{\frac{1}{n}}\right)^3 \to 1^3 = 1\right)$

f) $\sum_{n \geq 1} \dfrac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3 \cdot 6 \cdot \ldots \cdot (3n)}$ ← All terms are non-negative

Let $x_n = \dfrac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3 \cdot 6 \cdot \ldots \cdot (3n)}$

$$\frac{x_{n+1}}{x_n} = \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)(3n+2)}{3 \cdot 6 \cdot \ldots \cdot (3n) \cdot (3n+3)} \cdot \frac{3 \cdot 6 \cdot \ldots \cdot (3n)}{2 \cdot 5 \cdot \ldots \cdot (3n-1)}$$

$$\frac{x_{n+1}}{x_n} = \frac{3n+2}{3n+3} \longrightarrow 1$$

⇒ The Ratio Test is inconclusive

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) = n \cdot \left(\frac{3n+3}{3n+2} - 1\right) = n \cdot \frac{1}{3n+4}$$

$$\Rightarrow n\left(\frac{x_n}{x_{n+1}} - 1\right) = \frac{n}{3n+4} \longrightarrow \frac{1}{3} < 0$$

⇒ By Raabe's Test we conclude that the series

$$\sum_{n \geq 1} \frac{2 \cdot 5 \cdot \ldots \cdot (3n-1)}{3 \cdot 6 \cdot \ldots \cdot (3n)}$$

is divergent.

Exercise 5.2

Let (x_n), (y_n) be 2 seq. of positive numbers. Suppose that the series $\sum\limits_{n\geq 1} \frac{x_n}{y_n}$ and $\sum\limits_{n\geq 1} y_n$ are both convergent. Is the series $\sum\limits_{n\geq 1} \sqrt{x_n}$ convergent as well?

We know $G\left(\frac{x_n}{y_n}, y_n\right) \leq A\left(\frac{x_n}{y_n}, y_n\right)$

$$\Rightarrow \sqrt{\frac{x_n}{y_n} \cdot y_n} \leq \frac{\frac{x_n}{y_n} + y_n}{2}$$

$$\Rightarrow \sqrt{x_n} \leq \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right), \forall n \in \mathbb{N}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \sum\limits_{n\geq 1} \frac{x_n}{y_n} \text{ convergent} \\ \sum\limits_{n\geq 1} y_n \text{ convergent} \end{array}\right\} \Rightarrow \sum\limits_{n\geq 1} \frac{1}{2}\left(\frac{x_n}{y_n} + y_n\right) \text{ convergent} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ By the First Comparison Test we conclude that the series $\sum\limits_{n\geq 1} \sqrt{x_n}$ is also convergent

Note: the seq. (x_n) and (y_n) are positive $\Rightarrow$ We can apply F.C.T.

= Homework of Seminar 6 =

Exercise 6.1

Study if the following series are abs. conv, semi-conv, or div:

a) $\sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$

$$\left|\frac{(-1)^{n+1}}{n\sqrt{n+1}}\right| = \frac{1}{n\sqrt{n+1}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

$\sum\limits_{n\geq 1} \frac{1}{n^{3/2}}$ convergent (By the generalized harmonic series) $\Rightarrow$

$\Rightarrow$ By the First Comparison Test the series $\sum\limits_{n\geq 1} \frac{1}{n\sqrt{n+1}}$ is convergent $\Rightarrow \sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{n\sqrt{n+1}}$ is absolutely convergent.

b) $\sum_{n\geq 1} \frac{n}{n^2+1} \cos(n\pi) = \sum_{n\geq 1} \frac{n}{n^2+1} (-1)^n$

$$\left| (-1)^n \cdot \frac{n}{n^2+1} \right| = \frac{n}{n^2+1}$$

$$\left.\begin{aligned} &\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \in (0,\infty) \\ &\sum_{n\geq 1} \frac{1}{n} \text{ divergent (harmonic series)} \end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ By the Second Comparison Test the series $\sum_{n\geq 1} \frac{n}{n^2+1}$ is divergent $\Rightarrow$ the given series is <u>not</u> absolutely convergent.

We check if $\left(\frac{n}{n^2+1}\right)$ is decreasing

$$\frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} = \frac{n^3+n+n^2+1}{(n^2+2n+1+1)\,n} = \frac{n^3+n^2+n+1}{n^3+2n^2+2n}$$

$\Rightarrow \left(\frac{n}{n^2+1}\right)$ decreasing $\Leftrightarrow n^3+2n^2+2n \geq n^3+n^2+n+1$

$\Leftrightarrow n^2+n \geq 1$

$\Leftrightarrow n(n+1) \geq 1$, which is True $\forall n \in \mathbb{N}$

$\Rightarrow \left(\frac{n}{n^2+1}\right)$ decreasing.

$$\lim_{n \to \infty} \frac{n}{n^2+1} = 0$$

$\Rightarrow$ By the Alternating Series Test, the given series is convergent

$\Rightarrow$ The given series is semi-convergent.

Exercise 6.3

Find all continuous functions $f : \mathbb{R} \to \mathbb{N}$ and all continuous functions $f : \mathbb{N} \to \mathbb{R}$.

(I) For $f : \mathbb{R} \to \mathbb{N}$,

Let $f : \mathbb{R} \to \mathbb{N}$ be a continuous function

We know

$$\left.\begin{array}{l} \mathbb{R} \text{ interval} \\ f \text{ continuous} \end{array}\right\} \xrightarrow[\text{Value Theorem}]{\text{By } \cancel{\text{the}} \text{ Intermediate}} f(\mathbb{R}) \text{ is an interval}$$

$$\left.\begin{array}{l} f(\mathbb{R}) \text{ an interval} \\ f : \mathbb{R} \to \mathbb{N} \Rightarrow f(\mathbb{R}) \text{ a set of natural numbers} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ the interval $f(\mathbb{R})$ degenerates into a singleton

$\Rightarrow$ The function is

$f : \mathbb{R} \to \mathbb{N}$ $f(x) = n$, for some $n \in \mathbb{N}$

$\Rightarrow f : \mathbb{R} \to \mathbb{N}$ is continuous $\Leftrightarrow f$ constant

Note: We can also notice that if f was <u>not</u> constant, then at the point $x \in \mathbb{R}$ where the function would change its value from $n \in \mathbb{N}$ to $m \in \mathbb{N}$ (where $n \neq m$) we would have either a jump discontinuity or a removable discontinuity, hence the function f would <u>not</u> be continuous.

(II) For $f: \mathbb{N} \to \mathbb{R}$,

Let $f: \mathbb{N} \to \mathbb{R}$

We know:

All points in $\mathbb{N}$ are isolated $\Rightarrow f$ continuous at c, $\forall c \in \mathbb{N} \Rightarrow f$ continuous

$\Rightarrow$ All functions $f: \mathbb{N} \to \mathbb{R}$ are continuous

Note: This result stems from the fact that if we have a function $f: A \to \mathbb{R}$ and c is an isolated point of A, then f is continuous at c.