= Homework of Seminar 7 =

(7.1) Find the n^{th} derivative ($n \in \mathbb{N}$) of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x \sin x$.

$$f'(x) = e^x \sin x + \cos x \cdot e^x = e^x(\sin x + \cos x)$$

$$f''(x) = e^x(\sin x + \cos x) + (\cos x - \sin x)\, e^x = 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x + (-\sin x) \cdot 2e^x = 2e^x(\cos x - \sin x)$$

$$f^{(4)}(x) = 2e^x(\cos x - \sin x) + (-\sin x - \cos x) \cdot 2e^x$$

$$= 2e^x(-2\sin x) = -4e^x \sin x$$

$$f^{(5)}(x) = -4e^x \sin x + \cos x \cdot (-4e^x) = -4e^x(\sin x + \cos x)$$

(Notice the similarity with the 1st derivative)

$$f^{(6)}(x) = -4e^x(\sin x + \cos x) + (\cos x - \sin x) \cdot (-4e^x)$$

$$= -8e^x \cos x$$ (Notice similarity with $f''(x)$)

$$f^{(7)}(x) = -8e^x \cos x + (-\sin x) \cdot (-8e^x) = 8e^x(\sin x - \cos x)$$

(Notice similarity with $f'''(x)$)

$$f^{(8)}(x) = 8e^x(\sin x - \cos x) + (\cos x + \sin x) \cdot 8e^x$$

$$= 16e^x \sin x$$

(Notice similarity with $f^{(4)}(x)$)

$$f^{(9)}(x) = 16e^x \sin x + \cos x \cdot 16e^x = 16e^x(\sin x + \cos x)$$

(Notice similarity with $f'(x)$ and $f^{(5)}(x)$)

. . . . .

The pattern is obvious.

Let $n \in \mathbb{N}$

By mathematical induction $\Rightarrow$

$$f^{(n)}(x) = \begin{cases} (-1)^{k-1} \cdot 4^k \cdot e^x \cdot (\sin x + \cos x), & n = 4k-3 \\ (-1)^{k-1} \cdot 2^{2k-1} \cdot e^x \cos x, & n = 4k-2 \\ (-1)^k \cdot 2^{2k-1} \cdot e^x (\sin x - \cos x), & n = 4k-1 \\ (-1)^k \cdot 4^k \cdot e^x \sin x, & n = 4k \end{cases}$$

where $k \in \mathbb{N}$.

7.2) Compute the following limits:

a) $$\lim_{x\to\infty} \frac{x+\ln x}{x\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1\cdot \ln x + \frac{1}{x}\cdot x} =$$

$$= \lim_{x\to\infty} \frac{\frac{x+1}{x}}{\ln x + 1} = \lim_{x\to\infty} \frac{x+1}{x(1+\ln x)} =$$

$$= \lim_{x\to\infty} \frac{x+1}{x+x\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1+1\cdot\ln x + \frac{1}{x}\cdot x} =$$

$$= \lim_{x\to\infty} \frac{1}{2+\ln x} = 0$$

$$b) \lim_{\substack{x\to 0\\ x>0}} x \ln(\sin x) = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(\sin x)}{\frac{1}{x}} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{\sin x} \cdot \cos x}{(-1)\, x^{-2}}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{\cos x}{\sin x} \cdot (-1) \cdot x^2$$

$$= \lim_{\substack{x\to 0\\ x>0}} - \frac{x^2 \cos x}{\sin x}$$

$$\overset{L'H}{=} \lim_{\substack{x\to 0\\ x>0}} - \frac{2x \cdot \cos x - \sin x \cdot x^2}{\cos x}$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{x^2 \sin x - 2x\cos x}{\cos x} = \frac{0^2 \cdot 0 - 2 \cdot 0 \cdot 1}{1} = 0$$

c) $\lim\limits_{\substack{x \to 0 \\ x > 0}} (\sin x)^x = \lim\limits_{\substack{x \to 0 \\ x > 0}} e^{\ln(\sin x)^x} =$

$= \lim\limits_{\substack{x \to 0 \\ x > 0}} e^{x \ln(\sin x)}$

$= e^{\lim\limits_{\substack{x \to 0 \\ x > 0}} x \ln(\sin x)} = e^0 = 1$

From point b).

7.3) Let $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^3 - 3x^2 + 5x + 1$

Find the third Taylor polynomial $T_3(x)$ of f at 1.

$x \in \mathbb{R}$ $\quad f'(x) = 3x^2 - 6x + 5$

$f''(x) = 6x - 6$

$f'''(x) = 6$

$f^{(k)}(x) = 0, \ \forall k \in \mathbb{N}, \ k \geq 4$

$f(1) = 1 - 3 + 5 + 1 = 4$

$f'(1) = 3 - 6 + 5 = 2$

$f''(1) = 0$

$f'''(1) = 6$

$T_3: \mathbb{R} \to \mathbb{R}$ $\quad T_3(x) = 4 + \frac{2}{1!}(x-1) + \frac{0}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3$

$\Rightarrow T_3(x) = 4 + 2(x-1) + (x-1)^3$

$T_3(x) = 4 + 2x - 2 + x^3 - 3x^2 + 3x - 1$

$T_3(x) = x^3 - 3x^2 + 5x + 1 = f(x)$

= Homework of Seminar 8 =

Ex 8.1. Prove that the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=\frac{1}{x^2}$, can be expanded as a Taylor series around 1 on $[1,2)$ and find the corresponding Taylor series expansion.

$f:(0,\infty)\to\mathbb{R}$

$$f(x)=\frac{1}{x^2}\ ;\ f'(x)=\frac{-2}{x^3}\ ;\ f''(x)=\frac{6}{x^4}\ \dots$$

$$f^{(k)}(x)=\frac{(-1)^k(k+1)!}{x^{k+2}},\ \forall k\in\mathbb{N}$$

$$f(1)=1,\quad f^{(k)}(1)=(-1)^k\cdot(k+1)!,\ \forall k\in\mathbb{N}$$

Let $n\in\mathbb{N}$

$$T_n(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\dots+\frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$T_n(x)=1+\frac{(-1)^1\cdot 2!}{1!}(x-1)+\frac{(-1)^2\cdot 3!}{2!}(x-1)^2+\dots+\frac{(-1)^n\cdot(n+1)!}{n!}(x-1)^n$$

$$T_n(x)=1-2(x-1)+3(x-1)^2+\dots+(-1)^n(n+1)(x-1)^n,\ x\in\mathbb{R}$$

Let $x \in [1,2)$. Then $\exists c$ between 1 and x s.t.

$$\frac{1}{x^2} = T_n(x) + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-1)^{n+1} = (-1)^{n+1}\frac{\cancel{(n+2)!}^{\,n+2}}{c^{n+3}} \cdot \frac{(x-1)^{n+1}}{\cancel{(n+1)!}}$$

$$R_n(x) = (-1)^{n+1}\frac{(x-1)^{n+1}\cdot(n+2)}{c^{n+3}}$$

$$|R_n(x)| = \frac{(x-1)^{n+1}\cdot(n+2)}{c^{n+2}} \leq (x-1)^{n+1}\cdot(n+2)$$

$$\lim_{n\to\infty}(x-1)^{n+1}\cdot(n+2) = \lim_{n\to\infty}\frac{t^{n+1}}{\frac{1}{n+2}}, \text{ where } t \in (-1,0]$$

$$\lim_{n\to\infty}\frac{t^{n+1}}{\frac{1}{n+2}} = 0 \text{ (exponential fn. grows faster)}$$

$\Rightarrow$ By the Squeeze Theorem, $\lim_{n\to\infty} R_n(x) = 0$

$\Rightarrow$ f can be expanded as a Taylor series around 1 on $[1,2)$

$$\frac{1}{x^2} = \sum_{n=0} (-1)^n(n+1)(x-1)^n, \forall x \in [1,2)$$

(8.2.) Let $z \in \mathbb{R}^n$, $r > 0$, and $\varepsilon \in (0, 2]$. Prove that if $x, y \in \overline{B}(z, r)$ s.t. $\|x - y\| \geq \varepsilon r$, then

$$\left\| z - \frac{x+y}{2} \right\| \leq r \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Let $a = z - x$, $b = z - y$

$\Downarrow$ $x = z - a$ $\qquad$ $\Downarrow$ $y = z - b$

$x, y \in \overline{B}(z, r) \Rightarrow \|x - z\| \leq r \Rightarrow \|a\| \leq r$

$\Rightarrow \|y - z\| \leq r \Rightarrow \|b\| \leq r$

$x - y = z - a - (z - b) = b - a \Rightarrow \|b - a\| \geq \varepsilon r$

From Seminar 8, exercise 4, b we have:

$$\|a + b\|^2 + \|a - b\|^2 = 2(\|a\|^2 + \|b\|^2) \quad \text{(the parallelogram law)}$$

$$\Rightarrow \|a + b\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a - b\|^2$$

$$\|z - x + z - y\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a - b\|^2$$

$$\|2z - x - y\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a - b\|^2$$

We have :

$$\left.\begin{array}{l} \|2z - x - y\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a - b\|^2 \\ \|a\| \leq r \Rightarrow \|a\|^2 \leq r^2 \\ \|b\| \leq r \Rightarrow \|b\|^2 \leq r^2 \\ \|a - b\| \geq \varepsilon r \Rightarrow -\|a - b\|^2 \leq -\varepsilon^2 r^2 \end{array}\right\} \Longrightarrow$$

$$\Rightarrow \|2z - x - y\|^2 \leq 2r^2 + 2r^2 - \varepsilon^2 r^2 = r^2(4 - \varepsilon^2)$$

$$\Rightarrow \|2z - x - y\|^2 \leq r^2(4 - \varepsilon^2)$$

$$\|2z - x - y\| \leq r\sqrt{4 - \varepsilon^2}$$

$$2 \cdot \left\|z - \frac{x+y}{2}\right\| \leq r\sqrt{4 - \varepsilon^2}$$

$$\left\|z - \frac{x+y}{2}\right\| \leq r\sqrt{1 - \frac{\varepsilon^2}{4}}.$$

= Homework of seminar 9 =

$f:\mathbb{R}^2 \to \mathbb{R}$

9.1. a) $f(x,y) = \begin{cases} \dfrac{xy + x^2y\ln(x^2+y^2)}{x^2+y^2}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$

$a^k = \left(\frac{1}{k}, \frac{1}{k}\right)$, $k \in \mathbb{N}$, $\lim\limits_{k\to\infty} a^k = 0_2$

$$\lim_{k\to\infty} f(a^k) = \lim_{k\to\infty} \frac{\frac{1}{k^2} + \frac{1}{k^3}\ln\left(\frac{2}{k^2}\right)}{\frac{2}{k^2}} =$$

$$= \lim_{k\to\infty}\left(\frac{1}{2} + \frac{1}{2k}\ln\left(\frac{2}{k^2}\right)\right) = \frac{1}{2}\lim_{k\to\infty}\left(1 + \frac{1}{k}\ln\left(\frac{2}{k^2}\right)\right)$$

$$= \frac{1}{2}(1+0) = \frac{1}{2} \neq f(0_2)$$

$\hookrightarrow \lim\limits_{k\to\infty} \frac{1}{k}\ln\frac{2}{k^2} = \lim\limits_{k\to\infty} \frac{\ln\frac{2}{k^2}}{k} \overset{L'H}{=}$

$$= \lim_{k\to\infty} \frac{\frac{k^2}{\cancel{2}}\cdot \cancel{2}\cdot(-2)\cdot k^{-3}}{1} = \lim_{k\to\infty} -2\cdot\frac{\cancel{k^2}}{k^{\cancel{3}}} = \lim_{k\to\infty}\frac{-2}{k} = 0$$

By the Seq. Charact. of continuity $\Rightarrow f$ not cont. at 0_2.

b) $f(x,y) = \begin{cases} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}, & \text{if } (x,y) \neq 0_2 \\ 0, & \text{if } (x,y) = 0_2 \end{cases}$

$$x^4+y^4 \geq \frac{(x^2+y^2)^2}{2} \iff 2(x^4+y^4) \geq (x^2+y^2)^2$$

$$\iff 2x^4+2y^4 \geq x^4+2x^2y^2+y^4$$

$$\iff x^4+y^4 \geq 2x^2y^2$$

$$\iff x^4+y^4-2x^2y^2 \geq 0$$

$$\iff (x^2-y^2)^2 \geq 0, \text{ True } \forall\, x,y \in \mathbb{R}$$

$$\Rightarrow x^4+y^4 \geq \frac{(x^2+y^2)^2}{2}, \ \forall\, x,y \in \mathbb{R}.$$

$$x^4 + y^4 \geq \frac{(x^2+y^2)^2}{2} \Rightarrow \frac{1}{x^4+y^4} \leq \frac{2}{(x^2+y^2)^2}$$

for $(x,y) \neq 0_2$, $0 \leq f(x,y) = \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} \leq \dfrac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2}$

$$\lim_{(x,y)\to 0_2} \frac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} = \lim_{(x,y)\to 0_2} \frac{2}{(x^2+y^2)^2 \, e^{\frac{1}{x^2+y^2}}} =$$

$$= \lim_{(x,y)\to 0} \frac{2 \cdot \left(\frac{1}{x^2+y^2}\right)^2}{e^{\frac{1}{x^2+y^2}}} = \lim_{t\to\infty} \frac{2t^2}{e^t} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{4t}{e^t}$$

$$\underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{4}{e^t} = 0$$

By the Squeeze Theorem $\Rightarrow \lim\limits_{(x,y)\to 0_2} f(x,y) = 0 \Rightarrow f$ is continuous at 0_2.

9.2. Find the 2nd order partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sin(x \sin y)$

$$\frac{\partial f}{\partial x} = \cos(x \sin y) \cdot \sin y = \sin y \cos(x \sin y)$$

$$\frac{\partial f}{\partial y} = \cos(x \sin y) \cdot x \cdot \cos y = x \cos y \cos(x \sin y)$$

$$\frac{\partial^2 f}{\partial x^2} = \sin y \cdot (-\sin(x \sin y)) \cdot \sin y$$

$$= -\sin^2 y \cdot \sin(x \sin y)$$

$$\frac{\partial^2 f}{\partial y^2} = x \cdot (-\sin y) \cdot \cos(x \sin y) + (-\sin(x \sin y) \cdot x \cos y \cdot x \cos y)$$

$$= -x \sin y \cos(x \sin y) - x^2 \cos^2 y \sin(x \sin y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\sin(x \sin y) \cdot x \cos y \cdot \sin y + \cos y \cdot \cos(x \sin y)$$

$$= -x \sin y \cos y \sin(x \sin y) + \cos y \cos(x \sin y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x \sin y) \cdot \sin y \cdot x \cdot \cos y + \cos y \cdot \cos(x \sin y)$$

$$= -x \sin y \cos y \sin(x \sin y) + \cos y \cos(x \sin y) = \frac{\partial^2 f}{\partial y \partial x}$$

b) $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=(1+x^2)\,y\,e^z$

$$\frac{\partial f}{\partial x}=2x\,y\,e^z$$

$$\frac{\partial f}{\partial y}=(1+x^2)\,e^z$$

$$\frac{\partial f}{\partial z}=(1+x^2)\,y\,e^z$$

$$\frac{\partial^2 f}{\partial x^2}=2y\,e^z \; ; \; \frac{\partial^2 f}{\partial y\,\partial x}=2x\,e^z \; ; \; \frac{\partial^2 f}{\partial z\,\partial x}=2xy\,e^z$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=2x\,e^z=\frac{\partial^2 f}{\partial y\,\partial x} \; ; \; \frac{\partial^2 f}{\partial y^2}=0 \; ; \; \frac{\partial^2 f}{\partial z\,\partial y}=e^z(1+x^2)$$

$$\frac{\partial^2 f}{\partial x\,\partial z}=2xy\,e^z=\frac{\partial^2 f}{\partial z\,\partial x} \; ; \; \frac{\partial^2 f}{\partial y\,\partial z}=(1+x^2)\,e^z=\frac{\partial^2 f}{\partial z\,\partial y} \; ;$$

$$\frac{\partial^2 f}{\partial z^2}=(1+x^2)\,y\,e^z$$