Exercise 1.2

Let $x > 0$ and $n \in \mathbb{N}$

Use $G(x_1, \dots x_m) \leq A(x_1 \dots x_m)$ for some $m \in \mathbb{N}$

and some $x_1, x_2, \dots x_m > 0$ to deduce

a) $$\frac{x^n}{1 + x + \dots + x^{2n}} \leq \frac{1}{2n+1}$$

$$G(x_1, \dots, x_m) \leq A(x_1, \dots, x_m)$$

$$\sqrt[m]{x_1 \cdot x_2 \cdot \dots \cdot x_m} \leq \frac{x_1 + x_2 + \dots + x_m}{m}.$$

We'll use $\begin{cases} m = 2n+1 \\ x_i = x^i, \text{ for } i = 0, 1, 2, \dots, 2n \end{cases}$

$$\sqrt[2n+1]{x^0 \cdot x^1 \cdot x^2 \cdot \dots \cdot x^{2n}} \leq \frac{x^0 + x^1 + x^2 + \dots + x^{2n}}{2n+1}$$

$$\sqrt[2n+1]{x^{1+2+\dots+2n}} \leq \frac{1 + x + x^2 + \dots + x^{2n}}{2n+1}$$

$$1 + 2 + \dots + 2n = \frac{2n(2n+1)}{2} = n(2n+1)$$

$$\Rightarrow \sqrt[2n+1]{x^{n(2n+1)}} \leq \frac{1+x+\dots+x^{2n}}{2n+1}$$

$$x^{\frac{n(2n+1)}{2n+1}} \leq \frac{1+x+\dots+x^{2n}}{2n+1}$$

$$x^n \leq \frac{1+x+\dots+x^{2n}}{2n+1}$$

$$\Rightarrow \frac{x^n}{1+x+\dots+x^{2n}} \leq \frac{1}{2n+1}$$

b) $1+(n+1)x \leq (1+x)^{n+1}$

$(1+x)^{n+1} \geq 1+(n+1)x$ (Bernoulli's inequality)

We have $A(x_1,\dots,x_m) \geq G(x_1,\dots x_m)$

$$\frac{x_1+x_2+\dots+x_m}{m} \geq \sqrt[m]{x_1\cdot x_2\cdot\dots\cdot x_m}$$

We have $x>0$ and $n\in\mathbb{N}$

We'll use $\begin{cases} m = n \\ x_i = (1+x)^i \end{cases}$

$$\frac{(1+x)^0+(1+x)^1+\dots+(1+x)^n}{n+1} \geq \sqrt[n+1]{(1+x)^0\cdot(1+x)^1\cdot\dots\cdot(1+x)^n}$$

We know
$$\begin{cases} (1+x)^0+(1+x)^1+\dots+(1+x)^n = 1\cdot\dfrac{(1+x)^{n+1}-1}{1+x-1} = \dfrac{(1+x)^{n+1}-1}{x} \\ 0+1+\dots+n = \dfrac{n(n+1)}{2} \end{cases}$$

$$\frac{\frac{(1+x)^{n+1}-1}{x}}{n+1} \geq \sqrt[n+1]{(1+x)^{\frac{n(n+1)}{2}}}$$

$$\frac{(1+x)^{n+1}-1}{(n+1)x} \geq (1+x)^{\frac{n\cancel{(n+1)}}{2\cancel{(n+1)}}}$$

$$\frac{(1+x)^{n+1}-1}{(n+1)x} \geq \sqrt{(1+x)^n}$$

$$\left(x>0 \Rightarrow 1+x>1 \Rightarrow (1+x)^n>1 \Rightarrow \sqrt{(1+x)^n}>1 \right)$$

$\Rightarrow$ We have

$$\frac{(1+x)^{n+1}-1}{(n+1)x} \geqslant \sqrt{(1+x)^n} > 1$$

$$\Rightarrow \frac{(1+x)^{n+1}-1}{(n+1)x} \geqslant 1$$

$$(1+x)^{n+1}-1 \geqslant (n+1)x$$

$$(1+x)^{n+1} \geqslant 1+(n+1)x$$

Exercise 1.3

Prove that $\forall n \in \mathbb{N}$, $n \geq 2$ and $\forall x_1, x_2, \dots x_n \in [-1, +\infty)$

all of the same sign, we have

$$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_n) \geq 1+x_1+x_2+\ldots+x_n$$

(the generalised Bernoulli inequality)

Consider the following statement for $n \in \mathbb{N}$, $n \geq 2$:

$P(n)$: " $\forall x_1, x_2, \dots, x_n \in [-1, +\infty)$ all of the same sign, we have $(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_n) \geq 1+x_1+x_2+\ldots+x_n$"

for $n = 2 \in \mathbb{N}$,

Let $x_1, x_2 \in [-1, +\infty)$ of the same sign

$$(1+x_1)(1+x_2) = 1+x_1+x_2+\underbrace{x_1 x_2}_{\geq 0}$$

$$\geq 1+x_1 x_2 \quad \Rightarrow P(2) \text{ valid}$$

Let $k \in \mathbb{N}$, $k \geq 2$

Assume $P(k)$ valid. That is,

$P(k)$: " $\forall x_1, x_2, \dots x_k \in [-1, +\infty)$ all of the same sign, we have $(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k) \geq 1+x_1+x_2+\ldots+x_k$ "

valid (1

We want to prove $P(k+1)$ valid, where

$P(k+1)$: " $\forall x_1, x_2, \dots, x_{k+1} \in [-1, +\infty)$ all of the same sign, we have

$(1+x_1)(1+x_2)(1+x_3)\cdot\ldots\cdot(1+x_{k+1}) \geq 1+x_1+x_2+x_3+\ldots+x_{k+1}$ "

from $(1) \Rightarrow (1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k) \geq 1+x_1+x_2+\ldots+x_k \quad \cdot \Big| \underbrace{1+x_{k+1}}_{\geq 0}$

$\Rightarrow$ sign doesn't change

$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_k)(1+x_{k+1}) \geq (1+x_1+\ldots+x_k)(1+x_{k+1})$

$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{k+1}) \geq (1+x_1+x_2+\ldots+x_k) + (1+x_1+\ldots+x_k)x_{k+1}$

$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{k+1}) \geq 1+x_1+x_2+\ldots+x_{k+1} + (x_1+x_2+\ldots+x_k)x_{k+1}$

$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{k+1}) \geq 1+x_1+x_2+\ldots+x_{k+1} + \underbrace{x_1x_{k+1}}_{\geq 0} + \ldots + \underbrace{x_k x_{k+1}}_{\geq 0}$

We have

$$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{k+1}) \geq 1+x_1+x_2+\ldots+x_{k+1}+x_1x_{k+1}+\ldots+x_kx_{k+1}$$

All x_i have the same sign

$$\Rightarrow \quad x_i x_{k+1} \geq 0 \ , \ \forall i \in \{1, 2, \ldots, k\}$$

$$\Rightarrow \sum_{i=1}^{k} x_i x_{k+1} \geq 0$$

$$\Rightarrow (1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_{k+1}) \geq 1+x_1+x_2+\ldots+x_{k+1}$$

$\Rightarrow P(k+1)$ valid

$\Rightarrow$ We can conclude that the statement $P(n)$ is valid

Exercise 2.2

Find 2 sets $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ s.t. the following conditions are simultaneously met:

- (i) one of the sets is unbounded (but not an interval) and the other is finite;

(ii) $\sup A = \inf B = 2 \in A$;

(iii) for every $a \in A$ and every $b \in B$, there exists $c \in \mathbb{R}$ with $a < c < b$.

Is it possible to choose B the finite set?

Suppose it is possible to choose B the finite set such that conditions (i), (ii), and (iii) are met.

B finite, $B \neq \emptyset \Rightarrow \exists \min(B), \max(B)$

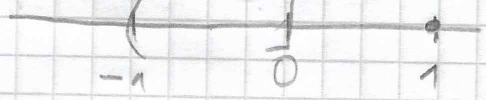
$$\left.\begin{array}{l} \exists \min(B) \\ \inf(B) = 2 \end{array}\right\} \Rightarrow \min(B) = 2 \Rightarrow 2 \in B$$

$$\left.\begin{array}{l} 2 \in A \\ 2 \in B \end{array}\right\} \Rightarrow \exists\, a \in A,\ b \in B \text{ s.t. } \nexists\, c \in \mathbb{R} \text{ with } a < c < b \ (\text{for } a = b = 2)$$

$\Rightarrow$ Contradiction $\Rightarrow$ B cannot be the finite set

Exercise 2.3 Which of the following are neighborhoods of 0. Justify.

- $A_1 = (-1, 0] \cup \{1\}$



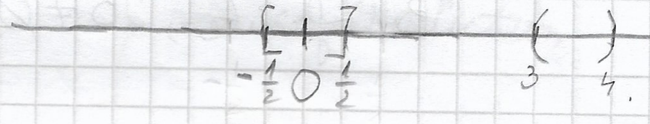
False

$$\forall\ \varepsilon \in \mathbb{R},\ \varepsilon > 0 \text{ we have } 0 + \varepsilon > 0$$

$$\Rightarrow \nexists\ \varepsilon \in \mathbb{R},\ \varepsilon > 0 \text{ s.t. } (0 - \varepsilon, 0 + \varepsilon) \subseteq A$$

- $A_2 = \left[1 - \frac{3}{2}, 1 + \frac{3}{2}\right] \cup (3, 4)$

$$\Leftrightarrow A_2 = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup (3, 4)$$



True

$$\left(-\frac{1}{3}, \frac{1}{3}\right) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right] \cup (3, 4)\ ;\ \left(-\frac{1}{3}, \frac{1}{3}\right) \in V(0)$$

$$\Rightarrow \exists\ \varepsilon \in \mathbb{R}, \varepsilon > 0\ \left(\text{e.g., } \varepsilon = \frac{1}{3}\right) \text{ such that}$$

$$(-\varepsilon, \varepsilon) \subseteq A_2$$

- $A_3 = \mathbb{R}$

True

$(-1,1) \subset \mathbb{R}$; $(-1,1) \in \mathcal{V}(0)$

$\Rightarrow \exists\ \varepsilon \in \mathbb{R},\ \varepsilon > 0$ (e.g., $\varepsilon = 1$) such that

$(-\varepsilon, \varepsilon) \in A_3$

- $A_4 = \mathbb{R} \setminus \mathbb{Q}$

False

By the Density Property of $\mathbb{Q}$ in $\mathbb{R}$, we know that $\forall\ x, y \in \mathbb{R},\ x > y$, $\exists\ q \in \mathbb{Q}$ s.t.

$x < q < y$

$\Rightarrow \mathbb{R} \setminus \mathbb{Q}$ is not an interval

$\Rightarrow A_4$ not an interval

$\Rightarrow$ It cannot contain a neighborhood

Homework of Seminar 3

Exercise 3.1

Find the limit (as $n\to\infty$) of the seq. with general term x_n, $n\in\mathbb{N}$ is:

a) $x_n = \dfrac{n+\sin(n^2)}{\cos(n)-3n}$

$$x_n = \underbrace{\frac{n}{\cos(n)-3n}}_{L_1} + \underbrace{\frac{\sin(n^2)}{\cos(n)-3n}}_{L_2}$$

$$L_1 = \frac{n}{-3n+\cos(n)} = \frac{n}{-3n\left(1-\frac{\cos(n)}{n}\right)} = \frac{-1}{3\left(1-\frac{\cos(n)}{n}\right)}$$

We know $\left\{\begin{array}{l} -1 \le \cos(n) \le 1, \ \forall n\in\mathbb{N} \Rightarrow \text{bounded} \\ \frac{1}{n}\to 0 \text{ as } n\to\infty \end{array}\right\} \Rightarrow$

$\Rightarrow \dfrac{\cos(n)}{n} \to 0 \Rightarrow L_1 \to -\dfrac{1}{3}$ as $n\to\infty$

$$L_2 = \frac{\sin(n^2)}{\cos(n) - 3n}$$

We know $\begin{cases} -1 \leq \sin(n^2) \leq 1, \ \forall n \in \mathbb{N} \\ -1 \leq \cos(n) \leq 1, \ \forall n \in \mathbb{N} \end{cases}$

$\left.\begin{array}{l} -3n \to -\infty \quad \text{as } n \to \infty \\ \cos(n) \in [-1, 1], \ \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \cos(n) - 3n \to -\infty$ as $n \to \infty$

$$\Rightarrow L_2 = \frac{\overbrace{\sin(n^2)}^{\text{bounded}}}{\underbrace{\cos(n) - 3n}_{-\infty}} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow x_n \to -\frac{1}{3} + 0, \quad \text{as } n \to \infty$$

$$\Rightarrow x_n \to -\frac{1}{3}, \quad \text{as } n \to \infty$$

$$\Rightarrow \lim_{n \to \infty} x_n = -\frac{1}{3}$$

b) $x_n = (n^2+n)^{-\frac{n}{n+1}}$

$$x_n = (n^2+n)^{-\frac{n+1-1}{n+1}}$$

$$x_n = (n^2+n)^{-1+\frac{1}{n+1}}$$

$$x_n = \frac{\sqrt[n+1]{n^2+n}}{n^2+n} = \sqrt[n+1]{\frac{\cancel{n^2+n}}{(n^2+n)^{n+\cancel{1}}}}$$

$$\Rightarrow x_n = \sqrt[n+1]{\frac{1}{(n^2+n)^n}}$$

We know $n^2+n > n, \forall n \in \mathbb{N}$

$$\Downarrow$$

$$(n^2+n)^n > n^n, \forall n \in \mathbb{N}$$

$$\Downarrow$$

$$\frac{1}{(n^2+n)^n} < \frac{1}{n^n}, \forall n \in \mathbb{N}$$

We have $\frac{1}{(n^2+n)^n} < \frac{1}{n^n}$, $\forall n \in \mathbb{N}$

$$\Downarrow$$

$$0 \leq \sqrt[n+1]{\frac{1}{(n^2+n)^n}} < \sqrt[n+1]{\frac{1}{n^n}}, \quad \forall n \in \mathbb{N}$$

Obvious

$$0 \leq \sqrt[n+1]{\frac{1}{(n^2+n)^n}} < \left(\frac{1}{n}\right)^{\frac{n}{n+1}}, \quad \forall n \in \mathbb{N}$$

$$\left.\begin{array}{l} \frac{1}{n} \to 0 \quad \text{as } n \to \infty \\ \frac{n}{n+1} \to 1 \quad \text{as } n \to \infty \end{array}\right\} \Rightarrow \left(\frac{1}{n}\right)^{\frac{n}{n+1}} \to 0^1 \quad \text{as } n \to \infty$$

By the Squeeze Theorem as $n \to \infty$ $\Rightarrow \sqrt[n+1]{\frac{1}{(n^2+n)^n}} \to 0$

$$\Rightarrow \lim_{n \to \infty} x_n = 0$$

c) $x_n = \left(1 + \frac{1}{n^3+2n^2}\right)^{n-n^3}$

$$x_n = \left[\left(1+\frac{1}{n^3+2n^2}\right)^{n^3+2n^2}\right]^{\frac{1}{n^3+2n^2}\cdot(n-n^3)}$$

$$x_n = \underbrace{\left[\left(1+\frac{1}{n^3+2n^2}\right)^{n^3+2n^2}\right]}_{\to e}^{\underbrace{\frac{-n^3+n}{n^3+2n^2}}_{\to -1}}$$

$\Rightarrow x_n \to e^{-1}$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} x_n = \frac{1}{e}$

d) $$x_n = \frac{1\cdot 1! + 2\cdot 2! + \ldots + n\cdot n!}{(n+1)!}$$

Take $\begin{cases} a_n = 1\cdot 1! + 2\cdot 2! + \ldots + n\cdot n! \\ b_n = (n+1)! \end{cases}$

$\begin{cases} (b_n) \text{ strictly increasing} \\ \lim\limits_{n\to\infty} b_n = \infty . \end{cases}$

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{(n+1)(n+1)!}{(n+2)! - (n+1)!} =$$

$$= \frac{\cancel{(n+1)!}\,(n+1)}{\cancel{(n+1)!}\,(n+2-1)} = \frac{n+1}{n+1} = 1 \in [0,\infty) \cup \{\infty\}$$

Stolz – Cesàro $\Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$

$\Rightarrow \lim x_n = 1$

e) $x_n = \sqrt[n]{1+2+\ldots+n} = \sqrt[n]{\frac{n(n+1)}{2}}$

We will apply Corollary 3 of Stolz - Cesàro

Take $a_n = \frac{n(n+1)}{2}$

$\forall\, n \in \mathbb{N},\ a_n > 0$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)}(n+2)}{1\cancel{2}} \cdot \frac{\cancel{2}}{n\cancel{(n+1)}} = \frac{n+2}{n}$$

$$\left.\begin{array}{l} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{n+2}{n} \longrightarrow 1 \text{ as } n \to \infty \\ 1 \in [0,\infty) \cup \{\infty\} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} = 1$$

$$\Rightarrow \lim_{n\to\infty} x_n = 1$$

f) $x_n = n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right)$

Bernoulli's inequality $\Rightarrow \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \geq 1+\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)$

valid since $1+\frac{1}{n} \geq -1, \forall n \in \mathbb{N}$

$\Rightarrow \left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} \geq 1+\left(1+\frac{1}{n}\right)^2 \quad -|1$

$\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1 \geq \left(1+\frac{1}{n}\right)^2 \quad \cdot|n, \; n \in \mathbb{N}$

$\Downarrow$

sign doesn't flip

$n\left(\left(1+\frac{1}{n}\right)^{1+\frac{1}{n}} - 1\right) \geq n\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$

$x_n \geq \underbrace{n+2+\frac{1}{n}}_{\downarrow \; \infty}, \; \forall n \in \mathbb{N}$

$\Rightarrow \lim x_n = \infty$

$\Rightarrow \forall\, y \in \mathbb{R},\ y \neq x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$

$\exists\, n_1 \in \mathbb{N}$ s.t. $y \notin [a_{n_1}, b_{n_1}]$

$\Rightarrow \bigcap_{n=1}^{\infty} [a_n, b_n]$ cannot contain more than one point

Exercise 3.3

Let (x_n) be a sequence in $\mathbb{Z}$. If (x_n) is convergent, is it eventually constant (i.e., $\exists\, n_0 \in \mathbb{N}$ s.t. $\forall\, m, n \in \mathbb{N}$ with $m, n \geq n_0$ we have $x_m = x_n$)?

$$\left.\begin{array}{l} (x_n) \text{ seq in } \mathbb{Z} \Rightarrow x_n \in \mathbb{Z},\ \forall\, n \in \mathbb{N} \\ (x_n) \text{ convergent} \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists \lim_{n\to\infty} (x_n) = x \in \mathbb{Z}$

We have

$$\exists \lim_{n\to\infty}(x_n) = x \in \mathbb{Z} \Rightarrow \forall\ \varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}$$

such that $\forall\ n \in \mathbb{N},\ n \geq n_\varepsilon$ we have $|x_n - x| < \varepsilon$

Take $\varepsilon = \frac{1}{2}$,

$$\Rightarrow \exists\ n_\varepsilon \in \mathbb{N} \text{ s.t. } \forall\ n \in \mathbb{N},\ n \geq n_\varepsilon$$

we have $|x_n - x| < \frac{1}{2}$

$$\Rightarrow x_n \in \left(x - \tfrac{1}{2},\ x + \tfrac{1}{2}\right),\ \forall\ n \in \mathbb{N},\ n \geq n_\varepsilon$$

(1)

We know

$$x \in \mathbb{Z} \Rightarrow \left(x - \tfrac{1}{2},\ x + \tfrac{1}{2}\right) \cap \mathbb{Z} = \{x\} \qquad (2)$$

(i.e., the only integer from the interval is the limit x itself)

We also know

$$x_n \in \mathbb{Z}, \ \forall n \in \mathbb{N} \qquad (3)$$

From (1), (2), (3),

$$\Rightarrow \forall n \geq n_\varepsilon, \ x_n = x = \lim_{n \to \infty} x_n \in \mathbb{Z}$$

$\Rightarrow$ For $\forall m, n \in \mathbb{N}$ with $m, n \geq n_\varepsilon$

we have $x_m = x_n$

$\Rightarrow$ the sequence (x_n) is eventually constant