P1 Vlad Bogdan-Tudor, 917

1) $x_n = \dfrac{3^n \cdot (n!)^2}{(2n)!}$, $n \in \mathbb{N}$

a)(i) $\dfrac{x_{n+1}}{x_n} = \dfrac{3^{n+1} \cdot ((n+1)!)^2}{\cancel{(2n+2)!}} \cdot \dfrac{\cancel{(2n)!}}{\cancel{3^n} \cdot (n!)^2} = \dfrac{3 \cancel{(n!)^2} (n+1)^2}{2 \cancel{(n+1)} \cancel{(n!)^2}}$
$(2n+2)$

$= \dfrac{3}{2}(n+1) > 1$

$\Rightarrow \dfrac{x_{n+1}}{x_n} > 1$, $\forall n \in \mathbb{N} \Rightarrow (x_n)$ increasing

Clearly $x_n > 0$, $\forall n \in \mathbb{N}$

$\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} \dfrac{3}{2}(n+1) = \infty > 1$

$\Rightarrow (x_n)_{n\geq 1}$ has no upper bound

ratio test

$\Rightarrow (x_n)_{n\geq 1}$ divergent

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(ii) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{3}{2}(n+1) = +\infty$

$\exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = +\infty \Longrightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$

Consequence of Stolz - Cesaro

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = +\infty$

(iii) $\sum_{n\geq 1} x_n = \sum_{n\geq 1} \frac{3^n \cdot (n!)^2}{(2n)!}$

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = +\infty > 1 \Rightarrow$ the series $\sum_{n\geq 1} x_n$ is divergent

from (ii)

ratio test

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b) (x_n) seq. in $[0,\infty)$. Is $\sum_{n\geq 1} \frac{x_n}{1+n^2 x_n}$ convergent?

$$a_n = \frac{x_n}{1+n^2x_n} = \frac{\cancel{x_n}}{\cancel{x_n}\left(\frac{1}{x_n}+n^2\right)} = \frac{1}{n^2+\frac{1}{x_n}} = \frac{1}{n^2+\frac{1}{x_n}},$$

$\forall n \in \mathbb{N}$

$\sum_{n\geq 1} a_n$ series with positive terms.

Let $y_n = \frac{1}{n^\alpha}$, $\alpha = 2$

$$\lim_{n\to\infty} \frac{a_n}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+\frac{1}{x_n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n^2}\cdot n^2 = 1 \in (0,\infty)$$

$\left(\lim_{n\to\infty} x_n = \infty \ ((x_n)_{n\geq 1} \text{ divergent}) \Rightarrow \lim_{n\to\infty} \frac{1}{x_n} = 0\right)$

$$\left.\begin{array}{l} \lim_{n\to\infty} \frac{a_n}{y_n} = 1 \in (0,\infty) \\ \sum_{n\geq 1} \frac{1}{n^2} \text{ convergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} a_n \text{ convergent}$$

$\Rightarrow \sum_{n\geq 1} \frac{x_n}{1+n^2x_n}$ convergent.

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2) $f: \mathbb{R}^2 \setminus \{0_2\} \to \mathbb{R}$, $f(x,y) = \dfrac{1-\cos(x-y)}{x^2+y^2}$

$$\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2)'} =$$

$$= \lim_{x\to 0} \frac{\sin x}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

$$\lim_{(0,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{1-\cos(-y)}{y^2} = \lim_{y\to 0} \frac{1-\cos y}{y^2} = \frac{1}{2}$$

as in the previous limit

both limits are $\frac{1}{2}$ $\Rightarrow$ yes, it has a limit at 0_2

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3) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

a) For $(x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = 6x^2 + y^2 + 10x$$

$$\frac{\partial f}{\partial y}(x,y) = 2xy + 2y$$

$$\Rightarrow \nabla f(x,y) = (6x^2 + y^2 + 10x,\ 2xy + 2y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x + 10 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = 2y$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2y \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 2x + 2$$

$$\Rightarrow H_f(x,y) = \begin{pmatrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{pmatrix}$$

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a) $\nabla f(x,y)=0 \iff \begin{cases} 6x^2+y^2+10x=0 \\ 2x\cdot y+2y=0 \end{cases}$

$2xy+2y=0$

$2y(x+1)=0 \Rightarrow y=0$ or $x=-1$

- $y=0$

$6x^2+10x=0$

$2x(3x+5)=0$

$\Rightarrow x=0$ or $3x+5=0$

$3x=-5$

$x=-\frac{5}{3}$

- $x=-1$

$6+y^2-10=0$

$y^2-4=0$

$y^2=4 \Rightarrow y=\pm 2$

$\Rightarrow$ Stationary points: $(0,0), (-\frac{5}{3},0), (-1,2), (-1,-2)$

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$$Hf(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 10 > 0 \\ \Delta_2 = 20 > 0 \end{array}\right\} \Rightarrow Hf(0,0) \text{ pos. def.} \Rightarrow (0,0) \text{ local minimum}$$

$$Hf\left(-\tfrac{5}{3}, 0\right) = \begin{pmatrix} \overset{4}{\cancel{12}} \cdot \left(-\tfrac{5}{\cancel{3}_{1}}\right) + 10 & 0 \\ 0 & 2 \cdot \left(-\tfrac{5}{3}\right) + \overset{3}{2} \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & \frac{-10+6}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = -10 < 0 \\ \Delta_2 = \frac{40}{3} > 0 \end{array}\right\} \Rightarrow Hf\left(-\tfrac{5}{3}, 0\right) \text{ neg. def.} \Rightarrow \left(-\tfrac{5}{3}, 0\right) \text{ local maximum}$$

$$Hf(-1,2) = \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix}$$

$$\Delta_2 = -16 < 0 \Rightarrow Hf(-1,2) \text{ indefinite} \Rightarrow (-1,2) \text{ not a local extremum point}$$

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$$Hf(-1,-2) = \begin{pmatrix} -2 & -4 \\ -4 & 0 \end{pmatrix}$$

$\Delta_1 = -2 < 0$

$\Delta_2 = -16 < 0 \Rightarrow Hf(-1,-2)$ indefinite $\Rightarrow (-1,-2)$ not a local extr. point

$\Rightarrow \begin{cases} (0,0) \text{ local min} \\ (-\frac{5}{3}, 0) \text{ local max} \end{cases}$

c) $f(0,0) = 0$

$\left.\begin{array}{l} f(-3,0) = 2\cdot(-27) + 0 + 5\cdot 9 = -54+45 = -9 < 0 \end{array}\right\} \Rightarrow$

$(0,0)$ <u>not</u> a global min

$$f(-\tfrac{5}{3}, 0) = 2\cdot -\frac{125}{27} + 0 + 5\cdot\frac{25}{9} + 0$$

$$= -\frac{250}{54} + \frac{{}^{6}125}{9} = \frac{-250+750}{54} = \frac{500}{54} = \frac{250}{27}$$

$f(3,0) = 2\cdot 27 + 0 + 5\cdot 9 + 0 = 54+45 = 99$

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$$f(3,0) = 99 > \frac{250}{27} = f\left(-\frac{5}{3}, 0\right)$$

$\Rightarrow \left(-\frac{5}{3}, 0\right)$ <u>not</u> a local max.

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4) a) $f:[0,\infty)\to\mathbb{R}$, $f(x)=\dfrac{1}{(1+x^2)(1+\operatorname{arctg}x)}$

f cont

Let $t\in[0,\infty)$. Then $\displaystyle\int_0^t \frac{1}{(1+x^2)(1+\operatorname{arctg}x)}\,dx \;(=)$

$u=1+\operatorname{arctg}x \Rightarrow \begin{cases} u_1=\operatorname{arctg}(0)+1=1\\ u_2=\operatorname{arctg}t+1\end{cases}$

$du=\dfrac{1}{1+x^2}\,dx.$

$$(=)\int_1^{\operatorname{arctg}(t)+1}\frac{1}{u}\,du=\Big[\ln u\Big]_1^{\operatorname{arctg}(t)+1}$$

$= \ln(\operatorname{arctg}(t)+1)-\ln(1)$

$= \ln(\operatorname{arctg}(t)+1)\xrightarrow{t\to\infty}\ln\left(\frac{\pi}{2}+1\right)$

$\Rightarrow f$ is impr. int. on $[0,\infty)$ and $\displaystyle\int_0^\infty f(x)\,dx=\ln\left(\frac{\pi}{2}+1\right)$

P1 Nicol Bogdan-Tudor
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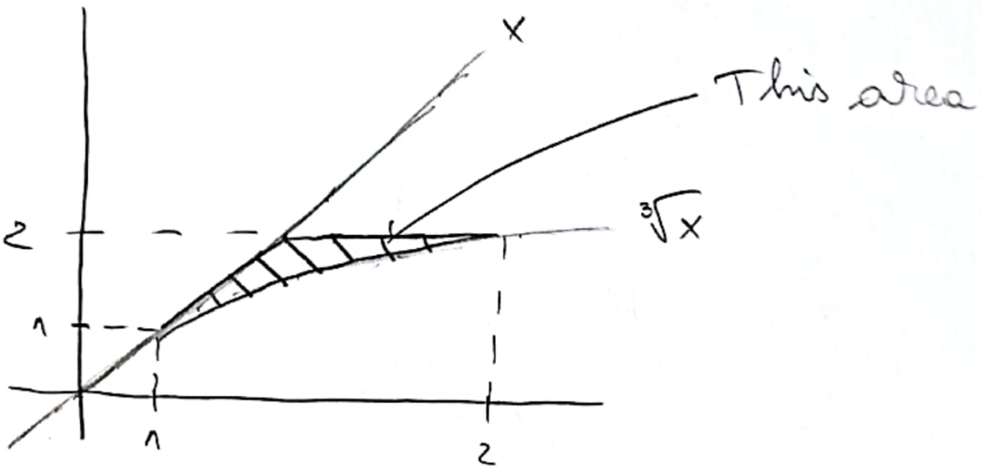
h) $M = \{(x,y)\in\mathbb{R}^2 : 1 \le x \le 8,\ \sqrt[3]{x} \le y \le \min\{x,2\}\}$ simple w.r.t y-axis

$$\iint_M e^{\frac{x}{y}}\,dx\,dy = ?$$



$M = \{(x,y)\in\mathbb{R}^2 : 1\le y\le 2,\ y\le x\le y^3\}$

simple w.r.t. x axis

$$\int_1^2\int_y^{y^3} e^{\frac{x}{y}}\,dx\,dy$$

Inner: $\int_y^{y^3} e^{\frac{x}{y}}\,dx = \left[y\cdot e^{\frac{x}{y}}\right]_{x=y}^{x=y^3} =$

$= y\cdot e^{\frac{y^3}{y}} - y\cdot e^{\frac{y}{y}} = y\left(e^{y^2}-e\right)$

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Outer: $\frac{1}{2}\int_1^2 2y\left(e^{y^2}-e\right)\,dy =$

$$u = y^2 \Rightarrow \begin{cases} u_1 = 1 \\ u_2 = 4 \end{cases}$$
$$du = 2y\,dy$$

$$= \frac{1}{2}\int_1^4 (e^u - e)\,du = \frac{1}{2}\left[e^u - e\cdot u\right]_1^4 =$$

$$= \frac{1}{2}\left[e^4 - 4e - (e^1 - e)\right]$$

$$= \frac{1}{2}\left(e^4 - 4e\right)$$