Exercise 4

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x_1	x_2	x_3	x_4	$f(x_1, x_2, x_3, x_4)$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1

Theorem 4.

A Boolean function $f:(B_2)^n \to B_2$ is unique determined by its values

$f(\alpha_1,\dots,\alpha_n)$, ***where*** $(\alpha_1,\dots,\alpha_n) \in B_2^n$:

1. ***disjunctive canonical form*** **(*DCF*):**

$$\textbf{(1)} \Leftrightarrow \textbf{(1')}\ f(x_1,\dots,x_n) = \bigvee_{(\alpha_1,\dots,\alpha_n)\in B_2^n \text{ and } f(\alpha_1,\dots,\alpha_n)=1} (x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n})$$

- **The disjunctive canonical form, DCF, is the disjunction of the minterms corresponding to the values 1 of the function.**

There are 6 minterms: $m_0, m_1, m_3, m_7, m_{11}, m_{15}$ corresponding to the values 1 of the function (on rows 1, 2, 4, 8, 12, 16).

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$DCF(f) = m_0 \vee m_1 \vee m_3 \vee m_7 \vee m_{11} \vee m_{15}$

$= \overline{x_1}\,\overline{x_2}\,\overline{x_3}\,\overline{x_4} \vee \overline{x_1}\,\overline{x_2}\,\overline{x_3}\,x_4 \vee \overline{x_1}\,\overline{x_2}\,x_3\,x_4 \vee \overline{x_1}\,x_2\,x_3\,x_4 \vee$

$x_1\,\overline{x_2}\,x_3\,x_4 \vee x_1\,x_2\,x_3\,x_4$

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$$x_1 \overline{x_2}\,x_3 x_4 \vee x_1 x_2 x_3 x_4$$

We use the Veitch diagram to simplify the function:

Simplification of Boolean Functions

Informally, to simplify a Boolean function given in a disjunctive canonical form, means to cover all its minterms with a minimum number of maximal monoms and with a minimum number of overlaps.

The simplification process is formalized by the steps below.

The steps 2, 3, 4 are specific to the applied simplification method.

1. **The initial function f is transformed into DCF(f).**
2. **Factorization process => the set of maximal monoms $M(f)$.**
3. **From the set of maximal monoms the central monoms are selected => $C(f)$**
4. **The case of the simplification algorithm (presented below) is identified and all simplified forms are obtained.**

Simplification Algorithm

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<u>Input data</u>: ***f* – a Boolean function in disjunctive canonical form**

<u>Output data</u>: ***f'*$_1$, *f'*$_2$, …, *f'*$_k$ all simplified forms of *f***

<u>Step 1</u>: Compute *M(f)* and *C(f)*

<u>Step 2</u>: If *M(f)* = *C(f)* then $f' = \bigvee_{m \in M(f)} m$ **, <u>STOP1</u> // case1--- one solution**

else

If *C(f)*≠∅ then // there exist central monoms

$g = \bigvee_{m \in C(f)} m$

$f_i' = g \vee h_i$, $i = \overline{1,k}$ **,** h_i **is a disjunction of a minimum number of maximal monoms such that** $S_{h_i} = S_f - S_g$

<u>STOP2</u> // case2 --- k solutions

else // there are no central monoms

$f_i' = h_i$, $i = \overline{1,k}$ **,** h_i **is a disjunction of a minumum number of maximal monoms such that** $S_{h_i} = S_f$

<u>STOP3</u> // case3 --- k solutions

End_if

End_if

End_algorithm

Veitch-Karnaugh Diagrams Method

M.Lupea

- **The cells in a Veitch/Karnaugh diagram are arranged so that there is only a single variable change between neighbor (adjacent cells).**
- **Adjacency (graphical neighborhood) relation is defined by a single variable change.**
- **The diagrams are circular, meaning that the first row/column and the last row/column are neighbors.**
- **Two minterms belonging to two neighbor cells factorize => *simple factorization*.**
- **The result of a *k-factorization* is a monom which contains the common variables of all 2^k neighbor minterms (cells).**
- **The maximal monoms are obtained from the diagram (Veitch/Karnaugh) by applying factorizations as follows: for a Boolean function of *n* variables we try first *n-factorization*, we continue with *n-1 factorizations*, ... , *simple factorizations* and *0-factorizations* (isolated minterms).**

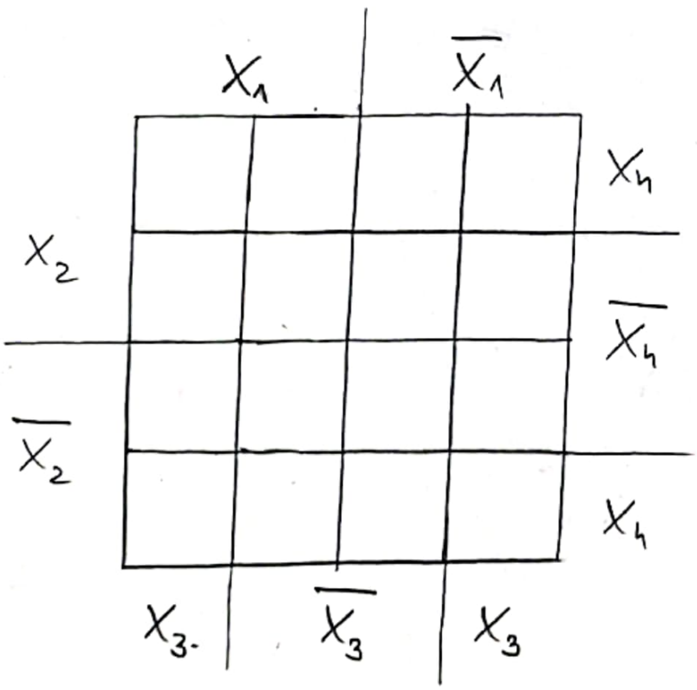
There are 6 minterms: $m_0, m_1, m_3, m_7, m_{11}, m_{15}$ corresponding to the values 1 of the function (on rows 1, 2, 4, 8, 12, 16).

$$DCF(f) = m_0 \vee m_1 \vee m_3 \vee m_7 \vee m_{11} \vee m_{15}$$

$$= \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \vee \bar{x}_1 \bar{x}_2 \bar{x}_3 x_4 \vee \bar{x}_1 \bar{x}_2 x_3 x_4 \vee \bar{x}_1 x_2 x_3 x_4 \vee$$

$$x_1 \bar{x}_2 x_3 x_4 \vee x_1 x_2 x_3 x_4$$

We use the Veitch diagram to simplify the function:



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	x_1		$\overline{x_1}$		
x_2	m_{15}			m_7	x_4
					$\overline{x_4}$
$\overline{x_2}$			m_0		
	m_{11}		m_1	m_3	x_4
	x_3	$\overline{x_3}$		x_3	

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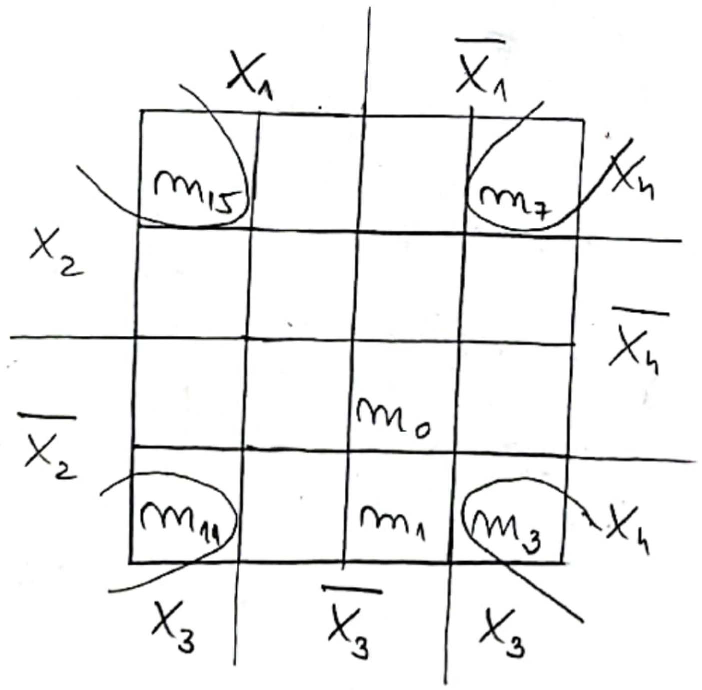

$m_{15} \vee m_{11} \vee m_7 \vee m_3 = x_3 x_4 = max_1$ (double factorization)

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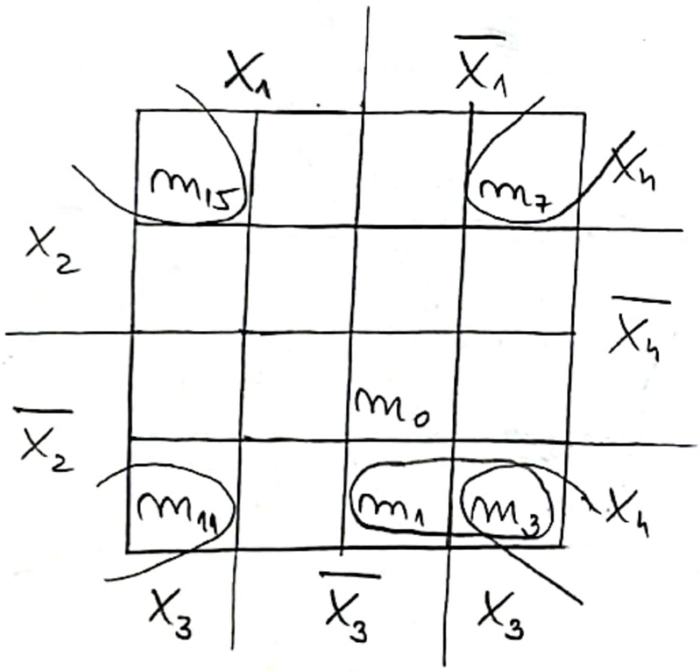

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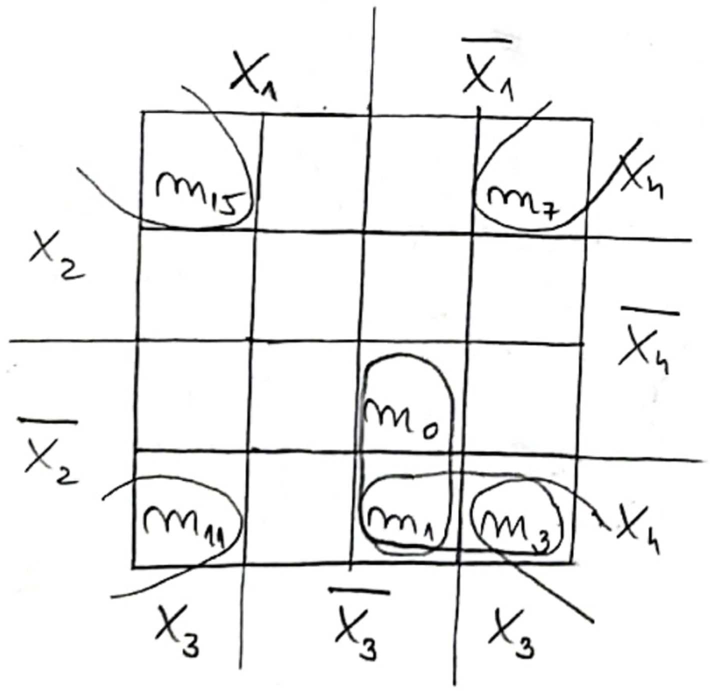

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$m_1 \vee m_0 = \overline{x_1}\,\overline{x_2}\,\overline{x_3} = max_3$ (simple factorisation)

$M(f) = \{max_1, max_2, max_3\}$ - the set of maximal monoms

Veitch-Karnaugh Diagrams Method

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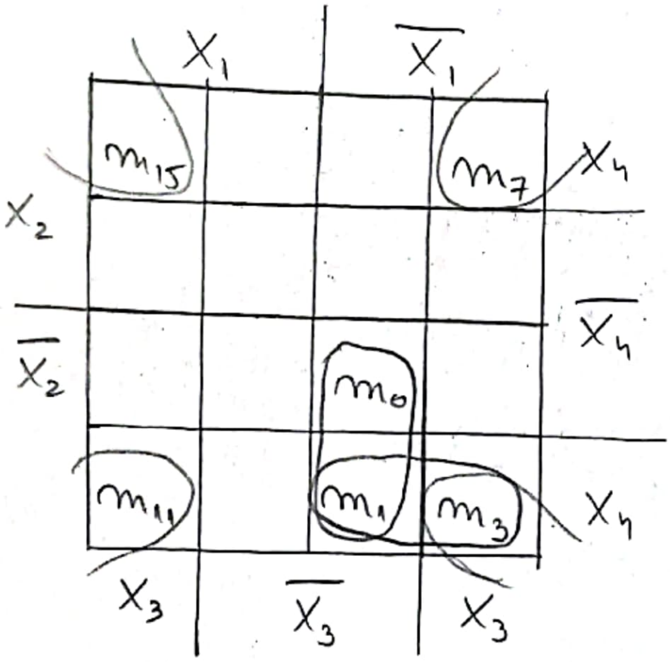
- Factorization process => the set of *maximal monoms M* (f).
- From the set of maximal monoms the *central monoms* are selected => C (f)
- The *central monoms* are the maximal monoms which cannot be covered by the disjunction of all the other maximal monoms.
- Identification of the *central monoms:*

  if the group of the minterms covered by a maximal monom contains at least one cell (minterm) circled once, then the maximal monom is a central one and it belongs to all the simplified forms of the function.
- The groups of the minterms corresponding to the central monoms are shaded in the diagram.
- The minterms from the function's expression which are unshaded, are not covered by the central monoms and they will be covered in all the possible ways using a minimum number of unused maximal monoms and with a minimum number of overlaps, resulting *all the simplified forms* of the initial function.

$M(f) = \{max_1, max_2, max_3\}$ - the set of maximal monoms



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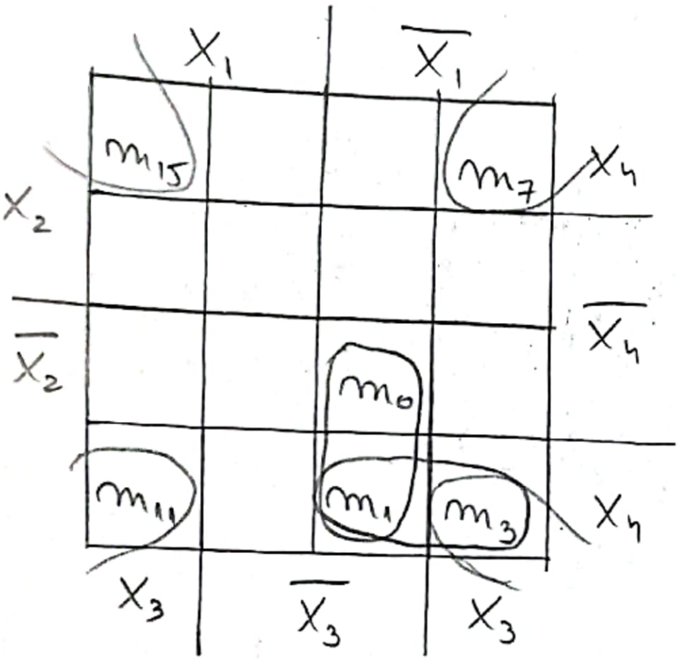

$$\begin{cases} max_1 = x_3 x_4 \\ max_2 = \overline{x_1}\,\overline{x_2}\,x_4 \\ max_3 = \overline{x_1}\,\overline{x_2}\,\overline{x_3} \end{cases}$$

The maximal monom max_1 covers m_7, m_{11}, m_{15}, which are all circled once $\Rightarrow$ max_1 is a central monom

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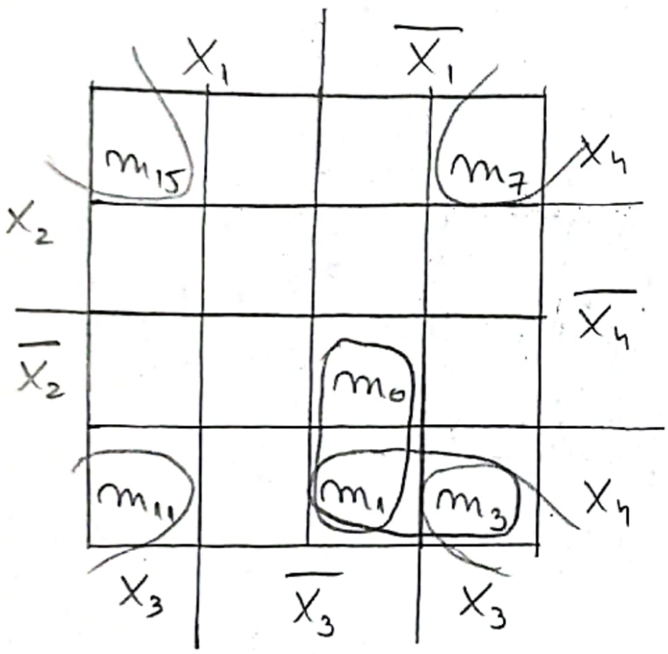

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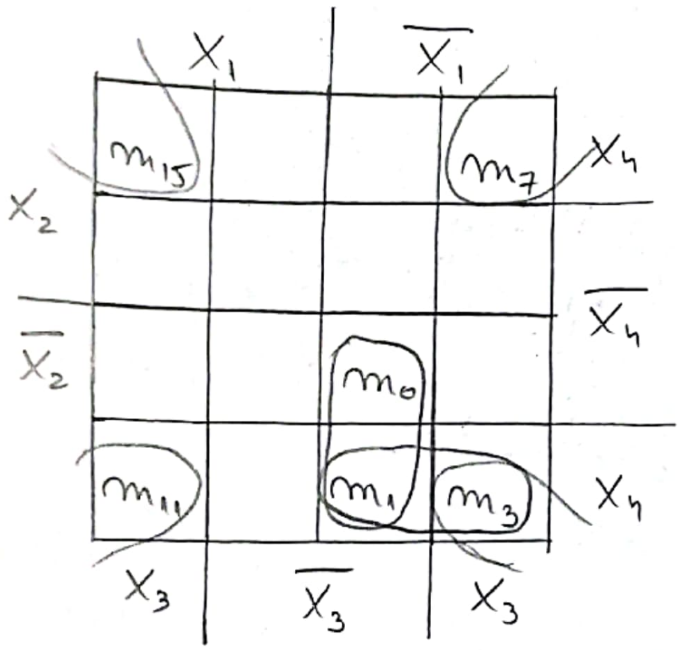
$M(f) = \{max_1, max_2, max_3\}$ - the set of maximal monoms



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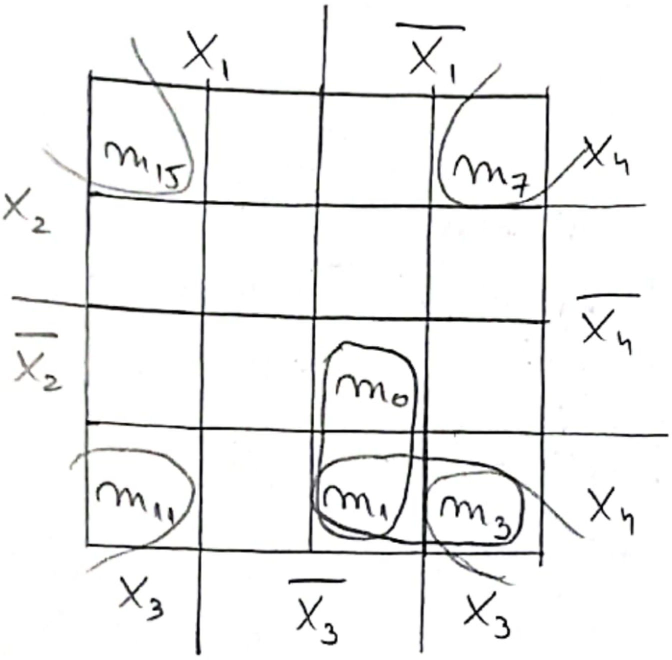
- The maximal monom max_3 covers m_0, which is circled once $\Rightarrow$ max_3 is a central monom

- The maximal monom max_2 covers only minterms which are circled twice $\Rightarrow$ max_2 is <u>not</u> a central monom

$M(f) = \{max_1, max_2, max_3\}$ - the set of maximal monoms



$$\begin{cases} max_1 = X_3 X_4 \\ max_2 = \overline{X_1}\,\overline{X_2}\, X_4 \\ max_3 = \overline{X_1}\,\overline{X_2}\,\overline{X_3} \end{cases}$$

- The maximal monom max_1 covers m_7, m_{11}, m_{15}, which are all circled once $\Rightarrow$ max_1 is a central monom
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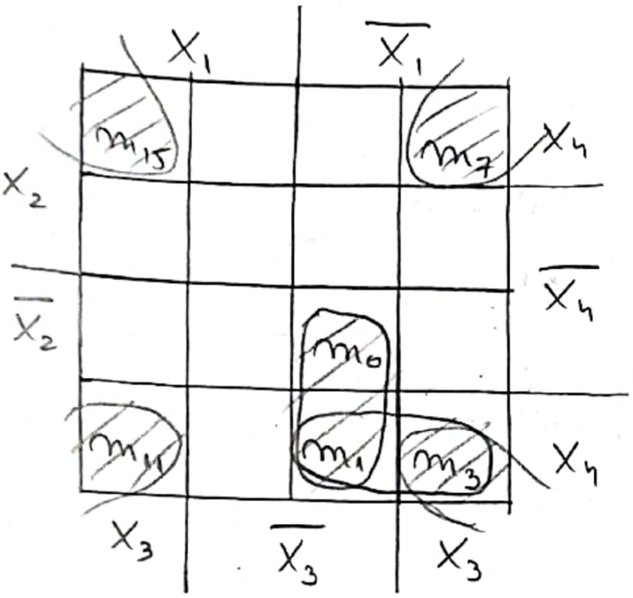
- The maximal monom max_2 covers only minterms which are circled twice $\Rightarrow$ max_2 is <u>not</u> a central monom

$\Rightarrow$ $C(f) = \{max_1, max_3\}$ - the set of central monoms

$M(f) = \{max_1, max_2, max_3\}$ - the set of maximal monoms



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$\Rightarrow$ $C(f) = \{max_1, max_3\}$ - the set of central monoms

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All minterms are already covered by the central monoms $\Rightarrow$ there is a unique simplified form of the initial function

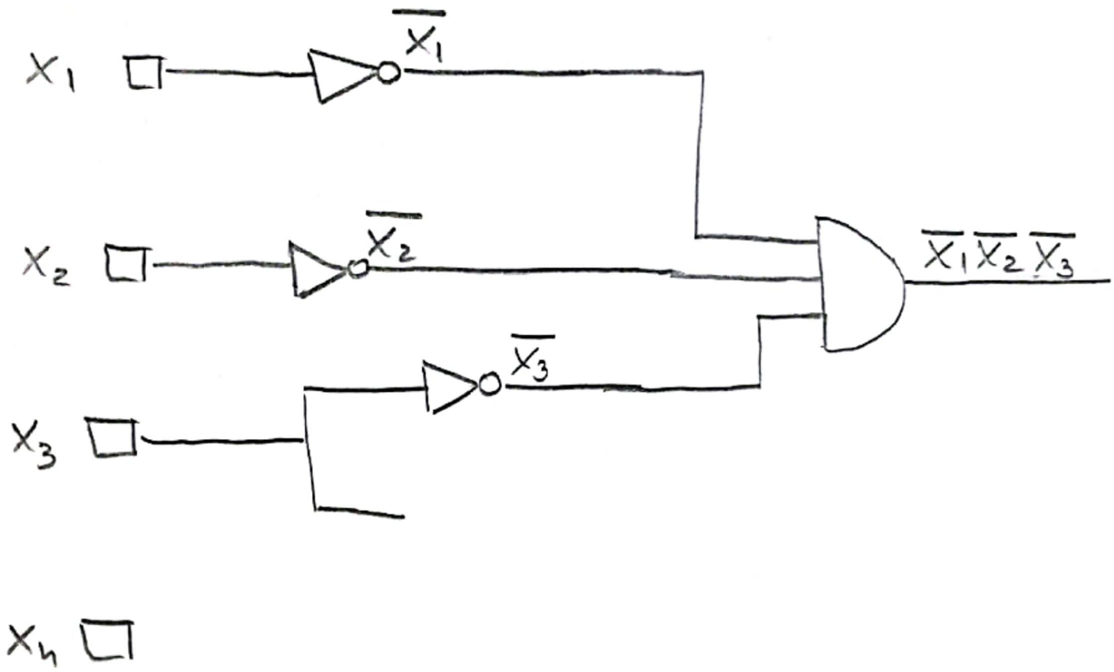
$$f^s(x_1, x_2, x_3) = max_1 \vee max_3$$

$$= x_3 x_4 \vee \overline{x_1} \overline{x_2} \overline{x_3}$$

$$f^{s}(x_1, x_2, x_3) = \max x_1 \vee \max x_3$$

$$= x_3 x_4 \vee \overline{x}_1 \overline{x}_2 \overline{x}_3$$

The simplified circuit is:



$$f^s(x_1, x_2, x_3) = m_a x_1 \vee m_a x_3$$

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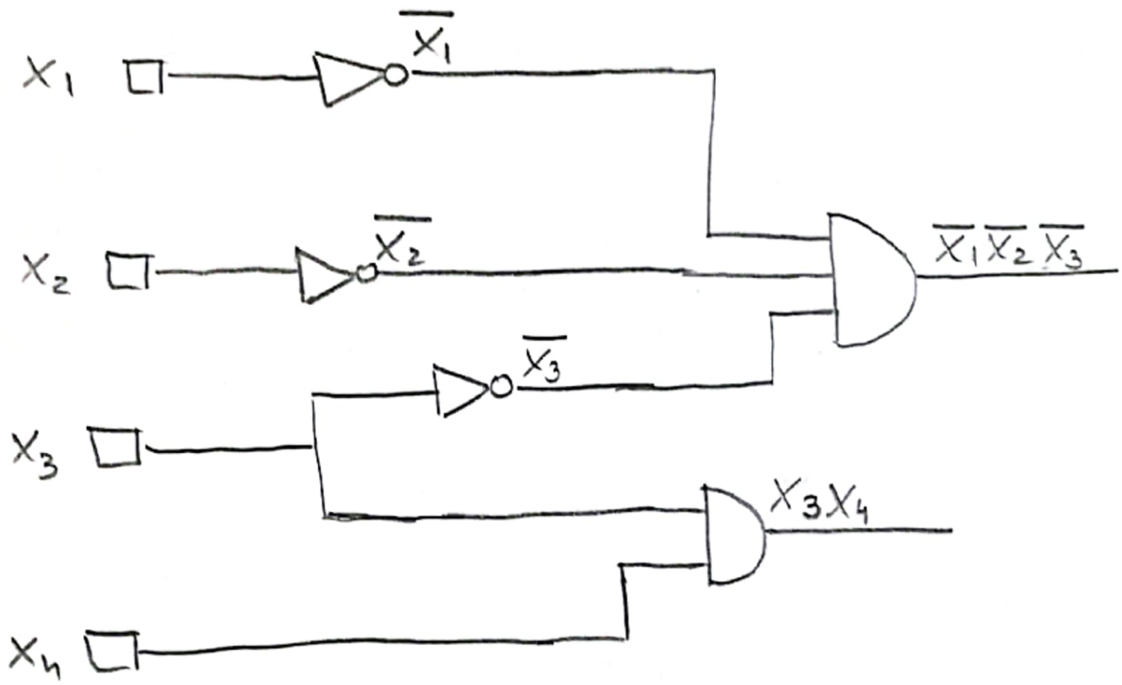
The simplified circuit is:



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