RESOLUTION PROOF METHOD IN PROPOSITIONAL LOGIC

Exercise 1

Using general resolution prove that the following formulas are theorems.

7. $U_7 = (A \to B) \to ((\neg A \to C) \to (\neg B \to C))$

RESOLUTION - A REFUTATION PROOF METHOD

Theorem

A propositional formula U is a **theorem** (tautology) *if and only if* the empty clause can be derived from the conjunctive normal form of $\neg U$, using the resolution algorithm.

U is a theorem (tautology) *if and only if* $\mathrm{CNF}(\neg U) \vdash_{Res} \square$.

RESOLUTION PROOF METHOD IN PROPOSITIONAL LOGIC

Exercise 1

Using general resolution prove that the following formulas are theorems.

7. $U_7 = (A \to B) \to ((\neg A \to C) \to (\neg B \to C))$

We will apply the NORMALIZATION ALGORITHM in order to get the equivalent CNF of $\neg U_7$.

M.Lupea

Normalization algorithm

Aim: to transform a formula into another logically equivalent formula, having a certain character of "normal" or "canonical" form.

Transformations which **preserve the logical equivalence** are applied:

Step1: The formulas of " $X \to Y$ " type are replaced by the equivalent form $\neg X \vee Y$.

The formulas of " $X \leftrightarrow Y$ " type are replaced by the equivalent form

$$(\neg X \vee Y) \wedge (\neg Y \vee X).$$

Step2: De Morgan laws are applied ==> push negations in until they apply only to propositional variables

Multiple negations are eliminated by the reduction rule: $\neg\neg X \equiv X$.

Step3: The distribution laws are applied.

$$\neg U_7 = \neg\left((A \xrightarrow{1} B) \xrightarrow{2} \left((\neg A \xrightarrow{3} C) \xrightarrow{4} (\neg B \xrightarrow{5} C) \right) \right)$$

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\left(A \vee C\right) \rightarrow \left(B \vee C\right)\right)\right)$$

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\left(A \vee C\right) \rightarrow \left(B \vee C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formula, denoted by 4)

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\left(A \vee C\right) \rightarrow \left(B \vee C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formula, denoted by 4)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\neg\left(A \vee C\right) \vee \left(B \vee C\right)\right)\right)$$

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\left(A \vee C\right) \rightarrow \left(B \vee C\right)\right)\right)$$

(replace $\rightarrow$ from the inside formula, denoted by 4)

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\neg\left(A \vee C\right) \vee \left(B \vee C\right)\right)\right)$$

$$\equiv \neg\left(\left(\neg A \vee B\right) \rightarrow \left(\left(\neg A \wedge \neg C\right) \vee \left(B \vee C\right)\right)\right)$$ (applied DeMorgan's law)

$$\neg U_7 = \neg((A \xrightarrow{1} B) \xrightarrow{2} ((\neg A \xrightarrow{3} C) \xrightarrow{4} (\neg B \xrightarrow{5} C)))$$

(replace → from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg((\neg A \vee B) \to ((A \vee C) \to (B \vee C)))$$

(replace → from the inside formula, denoted by 4)

$$\equiv \neg((\neg A \vee B) \to (\neg(A \vee C) \vee (B \vee C)))$$

$$\equiv \neg((\neg A \vee B) \to ((\neg A \wedge \neg C) \vee (B \vee C)))$$ (applied DeMorgan's law)

(replace the main connective →, denoted by 2)

$\neg U_7 = \neg((A \overset{1}{\rightarrow} B) \overset{2}{\rightarrow} ((\neg A \overset{3}{\rightarrow} C) \overset{4}{\rightarrow} (\neg B \overset{5}{\rightarrow} C)))$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$\equiv \neg((\neg A \vee B) \rightarrow ((A \vee C) \rightarrow (B \vee C)))$

(replace $\rightarrow$ from the inside formula, denoted by 4)

$\equiv \neg((\neg A \vee B) \rightarrow (\neg(A \vee C) \vee (B \vee C)))$

$\equiv \neg((\neg A \vee B) \rightarrow ((\neg A \wedge \neg C) \vee (B \vee C)))$ (applied DeMorgan's law)

(replace the main connective $\rightarrow$, denoted by 2)

$\equiv \neg(\neg(\neg A \vee B) \vee ((\neg A \wedge \neg C) \vee (B \vee C)))$

$$\neg U_7 = \neg\left(\left(A \xrightarrow{1} B\right) \xrightarrow{2} \left(\left(\neg A \xrightarrow{3} C\right) \xrightarrow{4} \left(\neg B \xrightarrow{5} C\right)\right)\right)$$

(replace $\to$ from the inside formulas, denoted by 1, 3, and 5)

$$\equiv \neg\left(\left(\neg A \vee B\right) \to \left(\left(A \vee C\right) \to \left(B \vee C\right)\right)\right)$$

(replace $\to$ from the inside formula, denoted by 4)

$$\equiv \neg\left(\left(\neg A \vee B\right) \to \left(\neg\left(A \vee C\right) \vee \left(B \vee C\right)\right)\right)$$

$$\equiv \neg\left(\left(\neg A \vee B\right) \to \left(\left(\neg A \wedge \neg C\right) \vee \left(B \vee C\right)\right)\right)$$ (applied DeMorgan's law)

(replace the main connective $\to$, denoted by 2)

$$\equiv \neg\left(\neg\left(\neg A \vee B\right) \vee \left(\left(\neg A \wedge \neg C\right) \vee \left(B \vee C\right)\right)\right)$$

$$\equiv \neg\left(\left(A \wedge \neg B\right) \vee \left(\neg A \wedge \neg C\right) \vee B \vee C\right)$$ (applied DeMorgan's law; removed parantheses)

$\neg U_7 = \neg((A \xrightarrow{1} B) \xrightarrow{2} ((\neg A \xrightarrow{3} C) \xrightarrow{4} (\neg B \xrightarrow{5} C)))$

(replace $\to$ from the inside formulas, denoted by 1, 3, and 5)

$\equiv \neg((\neg A \vee B) \to ((A \vee C) \to (B \vee C)))$

(replace $\to$ from the inside formula, denoted by 4)

$\equiv \neg((\neg A \vee B) \to (\neg(A \vee C) \vee (B \vee C)))$

$\equiv \neg((\neg A \vee B) \to ((\neg A \wedge \neg C) \vee (B \vee C)))$ (applied DeMorgan's law)

(replace the main connective $\to$, denoted by 2)

$\equiv \neg(\neg(\neg A \vee B) \vee ((\neg A \wedge \neg C) \vee (B \vee C)))$

$\equiv \neg((A \wedge \neg B) \vee (\neg A \wedge \neg C) \vee B \vee C)$ (applied DeMorgan's law; removed parantheses)

$\equiv \neg(A \wedge \neg B) \wedge \neg(\neg A \wedge \neg C) \wedge (\neg B) \wedge (\neg C)$

$\equiv (\neg A \vee B) \wedge (A \vee C) \wedge (\neg B) \wedge (\neg C)$ (applied DeMorgan's law)

$\neg U_7 = \neg((A \xrightarrow{1} B) \xrightarrow{2} ((\neg A \xrightarrow{3} C) \xrightarrow{4} (\neg B \xrightarrow{5} C)))$

(replace $\rightarrow$ from the inside formulas, denoted by 1, 3, and 5)

$\equiv \neg((\neg A \vee B) \rightarrow ((A \vee C) \rightarrow (B \vee C)))$

(replace $\rightarrow$ from the inside formula, denoted by 4)

$\equiv \neg((\neg A \vee B) \rightarrow (\neg(A \vee C) \vee (B \vee C)))$

$\equiv \neg((\neg A \vee B) \rightarrow ((\neg A \wedge \neg C) \vee (B \vee C)))$ (applied DeMorgan's law)

(replace the main connective $\rightarrow$, denoted by 2)

$\equiv \neg(\neg(\neg A \vee B) \vee ((\neg A \wedge \neg C) \vee (B \vee C)))$

$\equiv \neg((A \wedge \neg B) \vee (\neg A \wedge \neg C) \vee B \vee C)$ (applied DeMorgan's law; removed parantheses)

$\equiv \neg(A \wedge \neg B) \wedge \neg(\neg A \wedge \neg C) \wedge (\neg B) \wedge (\neg C)$

$\equiv (\neg A \vee B) \wedge (A \vee C) \wedge (\neg B) \wedge (\neg C)$ (applied DeMorgan's law)

$\Rightarrow$ CNF with 4 clauses.

We have

$$CNF(\neg U_7) = (\neg A \vee B) \wedge (A \vee C) \wedge (\neg B) \wedge (\neg C)$$

We have

$$CNF(\neg U_7) = (\neg A \vee B) \wedge (A \vee C) \wedge (\neg B) \wedge (\neg C)$$

We build the set of clauses S :

$$S = \{ \neg A \vee B, A \vee C, \neg B, \neg C \}$$

$C_1 = \neg A \vee B$; $C_2 = A \vee C$; $C_3 = \neg B$; $C_4 = \neg C$

Using the resolution method we will prove that the set S of clauses is inconsistent.

M.Lupea

RESOLUTION METHOD

- FORMAL SYSTEM FOR PROPOSITIONAL LOGIC -

$Res = (\Sigma_{Res}, F_{Res}, A_{Res}, R_{Res})$, where:

- $\Sigma_{Res} = \Sigma_P - \{\to, \leftrightarrow, \wedge\}$ - the alphabet;
- $F_{Res} \cup \{\square\}$
  - F_{Res} - the set of all clauses built using the alphabet Σ_{Res};
  - $\square$ is the empty clause, does not contain any literal,

    it symbolizes inconsistency;
- $A_{Res} = \varnothing$ is the set of axioms ;
- R_{Res} is the set of inference rules containing the ***resolution rule*** *(res)*:

  $f \vee l, g \vee \neg l \vdash_{res} f \vee g$, where l is a literal and $f, g \in F_{Res}$.

We will use this resolution rule

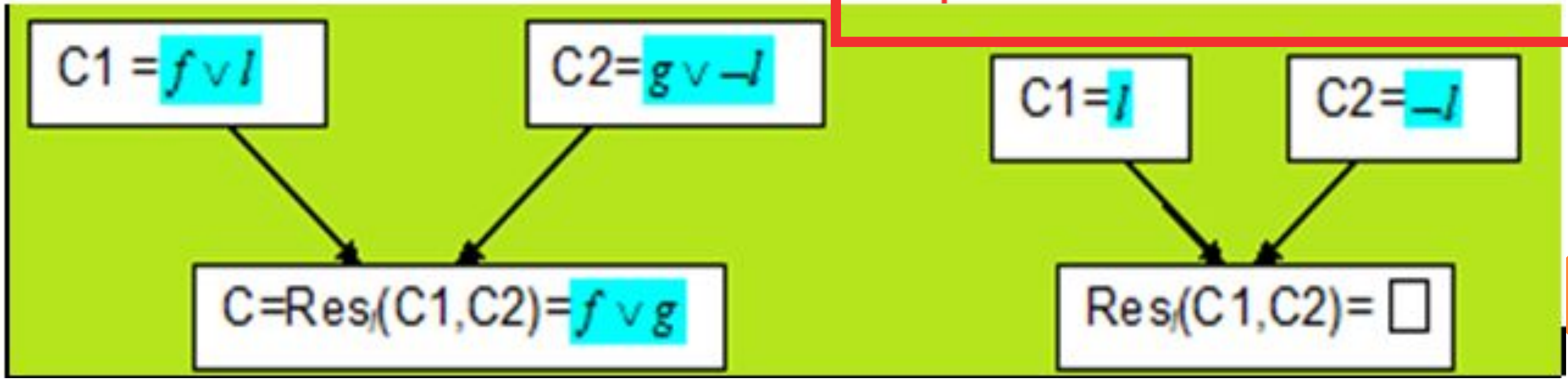
Definitions

Let l be a literal and $f, g \in F_{\text{Res}}$.

- the clauses $C_1 = f \vee l$ and $C_2 = g \vee \neg l$ are called ***clashing clauses*** and they ***resolve upon the literal*** l.
- notation: $C = \text{Res}_l(C_1, C_2) = f \vee g$, where C is called the ***resolvent*** of the ***parent clauses*** C_1 and C_2.
- if $C_1 = l$ and $C_2 = \neg l$, then $\text{Res}_l(C_1, C_2) = \square$ (empty clause) which is inconsistent.

Graphical representation:

We will use this graphical repersentation



$C_1 = \neg A \vee B; \quad C_2 = A \vee C; \quad C_3 = \neg B; \quad C_4 = \neg C$

$\boxed{C_1 = \neg A \vee B} \quad \boxed{C_3 = \neg B}$

$C_1 = \neg A \vee B$; $C_2 = A \vee C$; $C_3 = \neg B$; $C_4 = \neg C$

$$
\begin{tikzcd}
\boxed{C_1 = \neg A \vee B} \arrow[dr] & & \boxed{C_3 = \neg B} \arrow[dl] \\
& \boxed{C_5 = Res_B(C_1, C_3) = \neg A} &
\end{tikzcd}
$$

$C_1 = \neg A \vee B; \quad C_2 = A \vee C; \quad C_3 = \neg B; \quad C_4 = \neg C$

$$
\begin{tikzcd}
\boxed{C_1 = \neg A \vee B} \arrow[dr] & & \boxed{C_3 = \neg B} \arrow[dl] & \boxed{C_2 = A \vee C} \arrow[dr] & & \boxed{C_4 = \neg C} \arrow[dl] \\
& \boxed{C_5 = Res_B(C_1, C_3) = \neg A} & & & \boxed{} &
\end{tikzcd}
$$

$C_1 = \neg A \vee B;$ $\quad C_2 = A \vee C;$ $\quad C_3 = \neg B;$ $\quad C_4 = \neg C$

$$
\begin{tikzcd}
\boxed{C_1 = \neg A \vee B} \arrow[dr] & & \boxed{C_3 = \neg B} \arrow[dl] \\
& \boxed{C_5 = Res_B(C_1, C_3) = \neg A} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\boxed{C_2 = A \vee C} \arrow[dr] & & \boxed{C_4 = \neg C} \arrow[dl] \\
& \boxed{C_6 = Res_C(C_2, C_4) = A} &
\end{tikzcd}
$$

$C_1 = \neg A \vee B; \quad C_2 = A \vee C; \quad C_3 = \neg B; \quad C_4 = \neg C$

$$
\begin{tikzcd}
\boxed{C_1 = \neg A \vee B} \arrow[dr] & & \boxed{C_3 = \neg B} \arrow[dl] & & \boxed{C_2 = A \vee C} \arrow[dr] & & \boxed{C_4 = \neg C} \arrow[dl] \\
& \boxed{C_5 = Res_B(C_1, C_3) = \neg A} \arrow[drr] & & & & \boxed{C_6 = Res_C(C_2, C_4) = A} \arrow[dll] & \\
& & & \boxed{C_7 = Res_A(C_5, C_6) = \square} & & &
\end{tikzcd}
$$

$C_1 = \neg A \vee B$; $\quad C_2 = A \vee C$; $\quad C_3 = \neg B$; $\quad C_4 = \neg C$

$\boxed{C_1 = \neg A \vee B}$ $\quad \boxed{C_3 = \neg B}$ $\quad \boxed{C_2 = A \vee C}$ $\quad \boxed{C_4 = \neg C}$

$\boxed{C_5 = Res_B(C_1, C_3) = \neg A}$ $\quad \boxed{C_6 = Res_C(C_2, C_4) = A}$

$\boxed{C_7 = Res_A(C_5, C_6) = \square}$

We derived the empty clause from $CNF(\neg U_7)$ using the resolution algorithm $\Rightarrow$ The propositional formula <u>U_7 is a theorem</u>.