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1) $e^{-3t} + \sin(3t)$ solution $\Rightarrow e^{-3t}$ and $\sin(3t)$ solutions

e^{-3t} solution $\Rightarrow -3$ is a root of the characteristic equation

$\sin(3t)$ solution $\Rightarrow \cos(3t)$ is also a solution $\Rightarrow \pm 3i$ are roots of the characteristic equation

$\Rightarrow$ we have the characteristic equation

$$(r+3)(r+3i)(r-3i)=0$$

$$(r+3)(r^2+9)=0$$

$$r^3+9r+3r^2+27=0$$

$$r^3+3r^2+9r+27=0$$

The LHDE with constant real coefficients associated with the above characteristic equation is:

$$x''' + 3x'' + 9x' + 27x = 0$$

2) $x_{k+2} - 13x_{k+1} + 30x_k = 0$, $x_0 = 1$, $x_1 = 0$

↳ 2nd order linear hom. diff. equation with cc

The char. equation: $r^2 - 13r + 30 = 0$

$\Delta = 169 - 4 \cdot 30 = 169 - 120 = 49 > 0$

$\Rightarrow r_{1,2} = \dfrac{13 \pm 7}{2} \Rightarrow r_1 = 10$

$r_2 = 3$

$r_1 = 10 \in \mathbb{R} \Rightarrow x_k^1 = 10^k$

$r_2 = 3 \in \mathbb{R} \Rightarrow x_k^2 = 3^k$

The general solution is $x_k = c_1 \cdot x_k^1 + c_2 \cdot x_k^2$, $c_1, c_2 \in \mathbb{R}$
$k \geq 0$

$\Rightarrow x_k = c_1 \cdot 10^k + c_2 \cdot 3^k$, $c_1, c_2 \in \mathbb{R}$, $k \geq 0$

$x_0 = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1$

$x_1 = 0 \Rightarrow 10c_1 + 3c_2 = 0$

$10c_1 + 3(1 - c_1) = 0$

$10c_1 + 3 - 3c_1 = 0$

$7c_1 = -3 \Rightarrow \boxed{c_1 = -\dfrac{3}{7}}$

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$$c_2 = \frac{7}{1} + \frac{3}{7} \Rightarrow \boxed{c_2 = \frac{10}{7}}$$

$\Rightarrow$ the solution of the given IVP is

$$\boxed{x_k = -\frac{3}{7} \cdot 10^k + \frac{10}{7} \cdot 3^k}, \quad k \geq 0$$

3) $g: \mathbb{R}^2 \to \mathbb{R}$ is C^1

$$\begin{cases} \dot{x} = -y + x \cdot g(x,y) = f_1(x,y) \\ \dot{y} = x + y \cdot g(x,y) = f_2(x,y) \end{cases}$$

a) For $x=0, y=0$ we get

$$\begin{cases} \dot{x} = 0 + 0 \cdot g(x,y) = 0 \\ \dot{y} = 0 + 0 \cdot g(x,y) = 0 \end{cases}$$

$\Rightarrow$ $(0,0)$ is indeed an eq. point

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

$$\begin{cases} -y + x \cdot g(x,y) = 0 \\ x + y \cdot g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} x \cdot g(x,y) = y \\ y \cdot g(x,y) = -x \end{cases}$$

$$\Leftrightarrow \begin{cases} g(x,y) = \frac{y}{x} \\ g(x,y) = -\frac{x}{y} \end{cases}$$ For $(x,y) \in \mathbb{R}^2$, $x \neq 0$ and $y \neq 0$

$\Rightarrow$ we have an eq. point whenever $\frac{y}{x} = -\frac{x}{y}$, $x \neq 0, y \neq 0$

$$\frac{y}{x} = -\frac{x}{y}$$

$$\underbrace{y^2}_{>0} = \underbrace{-x^2}_{<0} \Rightarrow \text{no solutions}$$

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Let us consider the cases when $x=0$ OR $y=0$

(I) $\begin{cases} x=0 \\ y\in\mathbb{R}^* \end{cases}$

$$\begin{cases} \dot{x} = -y \\ \dot{y} = y\cdot g(x,y) \end{cases}$$

we get an eq. point when

$$-y = y\cdot g(x,y) \quad \Big|\cdot\frac{1}{y},\ y\in\mathbb{R}^*$$

$$g(x,y) = -1$$

$\Rightarrow A = \{(0,y) \mid y\in\mathbb{R}^*,\ g(0,y)=-1\}$ are eq. points

(II) $\begin{cases} x\in\mathbb{R}^* \\ y=0 \end{cases}$

$$\begin{cases} \dot{x} = x\cdot g(x,y) \\ \dot{y} = x \end{cases}$$

$\Rightarrow$ eq. point when

$$x = x\cdot g(x,y)$$

$$g(x,y)=1$$

$\Rightarrow B = \{(x,0) \mid x\in\mathbb{R}^*,\ g(x,0)=1\}$ are eq. points

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b) $\begin{cases} \dot{x} = -y + x \cdot g(x,y) = f_1(x,y) \\ \dot{y} = x + y \cdot g(x,y) = f_2(x,y) \end{cases}$

$$Jf(x,y) = \begin{pmatrix} g(x,y) + x \cdot \frac{\partial g}{\partial x}(x,y) & -1 + x \cdot \frac{\partial g}{\partial y}(x,y) \\ 1 + y \cdot \frac{\partial g}{\partial x}(x,y) & g(x,y) + y \cdot \frac{\partial g}{\partial y}(x,y) \end{pmatrix}$$

$$Jf(0,0) = \begin{pmatrix} g(0,0) & -1 \\ 1 & g(0,0) \end{pmatrix} = A$$

To apply the linearisation method we need $\eta^* = (0,0)$ to be hyperbolic

$$\det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} g(0,0) - \lambda & -1 \\ 1 & g(0,0) - \lambda \end{vmatrix} = 0$$

$$(g(0,0) - \lambda)^2 + 1 = 0$$

$$(g(0,0) - \lambda)^2 = -1$$

$$g(0,0) - \lambda = \pm i$$

$$\Rightarrow \boxed{\lambda = g(0,0) \pm i}$$

We distinguish 3 cases:

(I) $\underline{\rho(0,0) > 0}$ $\Rightarrow$ hyperbolic $\left(\begin{matrix} Re(\lambda_1) \neq 0 \text{ and} \\ Re(\lambda_2) \neq 0 \end{matrix}\right)$

$\lambda = \underbrace{\rho(0,0)}_{>0} \pm i \Rightarrow$ FOCUS

$\left.\begin{matrix} Re(\lambda_1) > 0 \\ Re(\lambda_2) > 0 \end{matrix}\right\} \Rightarrow$ global repeller

$\Rightarrow$ For $\rho(0,0) > 0$ the eq. point $\eta^* = (0,0)$ of the planar differential system is a repeller, unstable

(II) $\underline{\rho(0,0) < 0}$ $\Rightarrow$ hyperbolic $\left(\begin{matrix} Re(\lambda_1) \neq 0 \text{ and} \\ Re(\lambda_2) \neq 0 \end{matrix}\right)$

$\lambda = \underbrace{\rho(0,0)}_{<0} \pm i \Rightarrow$ FOCUS

$\left.\begin{matrix} Re(\lambda_1) < 0 \\ Re(\lambda_2) < 0 \end{matrix}\right\} \Rightarrow$ global attractor

$\Rightarrow$ For $\rho(0,0) < 0$ the eq. point $\eta^* = (0,0)$ of the planar differential system is an attractor, stable

III $g(0,0)=0 \Rightarrow$ non-hyperbolic
$(Re(\lambda_1) = Re(\lambda_2) = 0)$

$\Rightarrow$ we cannot apply the linearisation method

c) orbit that does not correspond to an eq point

$\Rightarrow x \neq 0$ and $y \neq 0$

$$\begin{cases} \dot{x} = -y + x \cdot g(x,y) \\ \dot{y} = x + y \cdot g(x,y) \end{cases}$$

we transform to polar coordinates

$$\begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$$

$$\Rightarrow \rho\dot{\rho} = -xy + x^2 g(x,y) + xy + y^2 g(x,y)$$
$$= g(x,y) \cdot (x^2+y^2) = g(x,y) \cdot \rho^2$$

$$\Rightarrow \boxed{\dot{\rho} = g(x,y) \cdot \rho}$$

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$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{x^2 + \cancel{xy\cdot g(x,y)} + y^2 - \cancel{xy\cdot g(x,y)}}{x^2}$$

$$\frac{\dot{\theta}}{\cancel{\cos^2\theta}} = \frac{\cancel{\rho^2}}{\cancel{\rho^2}\,\cancel{\cos^2\theta}}$$

$\Rightarrow \dot{\theta} = 1 \Rightarrow$ rotation around $(0,0)$ in the trigonometric sense

Also from $l_1) \Rightarrow (0,0)$ is a focus $\Rightarrow$ rotation around $(0,0)$

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1) $a, b \in \mathbb{R}$

$f(x) = ax^2 + bx + 1$

$f(1) = 2$

$a + b + 1 = 2 \Rightarrow a + b = 1$

$f(2) = 1$

$4a + 2b + 1 = 1 \Rightarrow 4a + 2b = 0 \Rightarrow 2a + b = 0$

$\Downarrow$

$b = -2a$

$a + b = 1$

$a - 2a = 1$

$-a = 1 \Rightarrow \boxed{a = -1} \Rightarrow \boxed{b = 2}$

$\Rightarrow f(x) = -x^2 + 2x + 1$

First we find the fixed points

$f(x) = x$

$-x^2 + 2x + 1 = x$

$-x^2 + x + 1 = 0$

$\Delta = 1 - 4 \cdot (-1) = 5 > 0 \Rightarrow x_{1,2} = \dfrac{-1 \pm \sqrt{5}}{-2} = \dfrac{1 \pm \sqrt{5}}{2}$

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Now we need the fixed points of f^2

$$f^2(x) = -(-x^2+2x+1)^2 + 2(-x^2+2x+1) + 1$$

$$= (-x^2+2x+1)(+x^2-2x-1+2) + 1$$

$$= (-x^2+2x+1)(x^2-2x+1) + 1$$

$$= -x^4 + 2x^3 \cancel{-x^2} + 2x^3 - 4x^2 \cancel{+2x} \cancel{+x^2} \cancel{-2x} + 1 + 1$$

$$= -x^4 + 4x^3 - 4x^2 + 2$$

$$f(x) = x$$

$$-x^4 + 4x^3 - 4x^2 + 2 = 0$$

$$-x^4 + 4x^3 - 4x^2 - x + 2 = 0$$

we know that $\frac{1 \pm \sqrt{5}}{2}$ are solutions of this

$$\left(x - \frac{1-\sqrt{5}}{2}\right)\left(x - \frac{1+\sqrt{5}}{2}\right) = x^2 - x \cdot \frac{1+\sqrt{5}}{2} - x \cdot \frac{1-\sqrt{5}}{2} + \frac{1-5}{4}$$

$$= x^2 - x \cdot \frac{2}{2} + \frac{-4}{4}$$

$$= x^2 - x - 1$$

$$-x^4+4x^3-4x^2-x+2 \;\Big|\; x^2-x-1$$
$$= -x^2+3x-2$$

we now solve $-x^2+3x-2=0$

$$x^2-3x+2=0$$
$$(x-2)(x-1)=0$$

$\Rightarrow \begin{cases} x_1=1 \\ x_2=2 \end{cases}$ Check: $f(1)=-1+2+1=2$ ✓
$f(2)=-4+4+1=1$ ✓

$\Rightarrow \{1,2\}$ is a 2-periodic orbit

$$f'(x)=-2x+2=-2(x-1)$$
$$f'(1)=-2\cdot 0=0$$
$$f'(2)=-2\cdot 1=-2$$
$$|f'(1)\cdot f'(2)|=|0\cdot(-2)|=0<1 \Rightarrow$$

$\{1,2\}$ is an attracting 2-periodic orbit

$\Rightarrow$ YES, the discrete dynamical system $x_{k+1}=f(x_k)$, $k\in\mathbb{N}$ has an attracting 2-periodic orbit

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d) $x^2+y^2=1 \Rightarrow x=\pm\sqrt{1-y^2}$

$$g(\sqrt{1-y^2}, y) = g(-\sqrt{1-y^2}, y) = 0, \forall y \in \mathbb{R}$$

$$\varphi(t, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = \left(\cos\left(t+\tfrac{\pi}{4}\right), \sin\left(t+\tfrac{\pi}{4}\right)\right), \forall t \in \mathbb{R}$$

$$\begin{cases} \dot{x} = -y + x \cdot g(x,y) = f_1(x,y) \\ \dot{y} = x + y \cdot g(x,y) = f_2(x,y) \end{cases}$$

$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ is the sol which passes through $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow \begin{cases} \dot{\varphi}_1(t) = f_1(\varphi_1(t), \varphi_2(t)) \\ \dot{\varphi}_2(t) = f_2(\varphi_1(t), \varphi_2(t)) \\ \varphi_1(0) = \frac{1}{\sqrt{2}} \\ \varphi_2(0) = \frac{1}{\sqrt{2}} \end{cases}$$

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$\varphi_1(0) = \cos\left(0+\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ ✓

$\varphi_2(0) = \sin\left(0+\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ ✓

$\dot{\varphi}_1(t) = -\sin\left(t+\frac{\pi}{4}\right)$

$f_1(\varphi_1(t), \varphi_2(t)) = -\varphi_2(t) + \varphi_1(t) \cdot g(\varphi_1(t), \varphi_2(t))$

$= -\sin\left(t+\frac{\pi}{4}\right) + \cos\left(t+\frac{\pi}{4}\right) \cdot g\left(\cos\left(t+\frac{\pi}{4}\right), \sin\left(t+\frac{\pi}{4}\right)\right)$

$= -\sin\left(t+\frac{\pi}{4}\right) + \cos\left(t+\frac{\pi}{4}\right) \cdot 0$

$= -\sin\left(t+\frac{\pi}{4}\right) = \dot{\varphi}_1(t)$ ✓

$\dot{\varphi}_2(t) = \cos\left(t+\frac{\pi}{4}\right)$

$f_2(\varphi_1(t), \varphi_2(t)) = \varphi_1(t) + \varphi_2(t) \cdot g(\varphi_1(t), \varphi_2(t))$

$= \cos\left(t+\frac{\pi}{4}\right) + \sin\left(t+\frac{\pi}{4}\right) \cdot g\left(\cos\left(t+\frac{\pi}{4}\right), \sin\left(t+\frac{\pi}{4}\right)\right)$

$= \cos\left(t+\frac{\pi}{4}\right) + \sin\left(t+\frac{\pi}{4}\right) \cdot 0$

$= \cos\left(t+\frac{\pi}{4}\right) = \dot{\varphi}_2(t)$ ✓

$\Rightarrow$ the solution $\varphi\left(t, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\cos\left(t+\frac{\pi}{4}\right), \sin\left(t+\frac{\pi}{4}\right)\right)$ is correct.