1) $\begin{cases} \dot{x} = ax - 2y - 2y(x^2+y^2) = f_1(x,y) \\ \dot{y} = 2x + ay = f_2(x,y) \end{cases}$

$a \in \mathbb{R}$

a) $\begin{cases} \dfrac{\partial f_1}{\partial x}(x,y) = a - 4xy \\ \dfrac{\partial f_1}{\partial y}(x,y) = -2 - 2x^2 - 2 \cdot 3y^2 \end{cases}$

$\begin{cases} \dfrac{\partial f_2}{\partial x}(x,y) = 2 \\ \dfrac{\partial f_2}{\partial y}(x,y) = a \end{cases}$

$\Rightarrow Jf(x,y) = \begin{pmatrix} a-4xy & -2-2x^2-6y^2 \\ 2 & a \end{pmatrix}$

$Jf(0,0) = \begin{pmatrix} a & -2 \\ 2 & a \end{pmatrix}$

$\begin{vmatrix} a-\lambda & -2 \\ 2 & a-\lambda \end{vmatrix} = 0$

$(a-\lambda)^2+4=0$

$(a-\lambda)^2=(2i)^2$

$a-\lambda=\pm 2i$

$\boxed{\lambda = a \pm 2i}$

I $\underline{a \in (0,+\infty)}$

$\Rightarrow \lambda = \underbrace{a}_{>0} \pm 2i$

$\left.\begin{array}{l} Re(\lambda_1) \neq 0 \\ Re(\lambda_2) \neq 0 \end{array}\right\} \Rightarrow \eta_1^* = (0,0)$ hyperbolic $\Rightarrow$ can apply the lin. method

$\lambda = a \pm 2i \Rightarrow$ <u>FOCUS</u>

$\left.\begin{array}{l} Re(\lambda_1) > 0 \\ Re(\lambda_2) > 0 \end{array}\right\} \Rightarrow$ <u>repeller</u> $\Big\} \overset{\text{LIN}}{\underset{\text{METHOD}}{\Rightarrow}}$ $\boxed{\eta^* = (0,0)}$ <u>is a repeller</u>

II $\underline{a \in (-\infty, 0)}$

$\Rightarrow \lambda = \underbrace{a}_{<0} \pm 2i$

$\left.\begin{array}{l} Re(\lambda_1) \neq 0 \\ Re(\lambda_2) \neq 0 \end{array}\right\} \Rightarrow \eta_1^* = (0,0)$ hyperbolic

$\lambda = a \pm 2i \Rightarrow$ <u>FOCUS</u>

$\left.\begin{array}{l} Re(\lambda_1) < 0 \\ Re(\lambda_2) < 0 \end{array}\right\} \Rightarrow$ <u>attractor</u> $\Big\} \Rightarrow$ $\boxed{\eta^* = (0,0)}$ <u>is an attractor</u>

III $a=0$

$\rightarrow \lambda = \pm 2i$

$Re(\lambda_1) = Re(\lambda_2) = 0 \Rightarrow$ non-hyperbolic

We try to find a first integral in a neighborhood of $\eta_1^* = (0,0)$

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad \begin{cases} \dot{x} = -2y \\ \dot{y} = 2x \end{cases}$$

$$\frac{\partial y}{\partial x} = \frac{2x}{-2y} = -\frac{x}{y}$$

$$\int -y\,dy = \int x\,dx$$

$$-\frac{y^2}{2} = \frac{x^2}{2} + c \qquad \cdot|2$$

$$-y^2 = x^2 + c$$

$$-x^2 - y^2 = c \qquad \cdot|(-1)$$

$$x^2 + y^2 = c$$

Let $H: \mathbb{R}^2 \to \mathbb{R}$ $H(x,y) = x^2 + y^2$ not locally constant, $C^1$ function

Check:

$$\frac{\partial H}{\partial x}(x,y) \cdot (-2y) + \frac{\partial H}{\partial y}(x,y) \cdot 2x =$$

$$= 2x \cdot (-2y) + 2y \cdot 2x = 0 \Rightarrow \text{global first integral}$$

$\lambda_{1,2} = \pm 2i$

The system has a first integral in a neighborhood of $\eta^* = (0,0)$

$\Rightarrow$ $\underline{\eta^* = (0,0) \text{ stable}}$

(ii) $\begin{cases} \dot{x} = ax - 2y(1+x^2+y^2) \\ \dot{y} = 2x + ay \end{cases}$

$\begin{cases} ax - 2y(1+x^2+y^2) = 0 \\ 2x + ay = 0 \end{cases}$

The only real sol of the system is $(0,0)$

The others sols. are complex

$\Rightarrow$ one eq. point, $\eta^* = (0,0)$

We can see that on the next page

$$\begin{cases} \alpha x - 2y(1 + x^2 + y^2) = 0 \\ 2x + \alpha y = 0 \end{cases}$$

$$2x = -\alpha y$$

$$x = -\frac{\alpha}{2} y$$

$$-\frac{\alpha^2}{2} y - 2y\left(1 + \frac{\alpha^2}{2} y^2 + y^2\right) = 0$$

$$-\frac{\alpha^2}{2} y - 2y - \alpha^2 y^3 - 2y^3 = 0$$

$$-(\alpha^2 + 2) y^3 - \left(\frac{\alpha^2}{2} + 2\right) y = 0$$

$$(\alpha^2 + 2) y^3 + \left(\frac{\alpha^2}{2} + 2\right) y = 0$$

$$y\left((\alpha^2 + 2) y^2 + \frac{\alpha^2}{2} + 2\right) = 0$$

$y = 0$ OR $(\alpha^2 + 2) y^2 + \frac{\alpha^2}{2} + 2 = 0$

the only real sol

$$(\alpha^2 + 2) y^2 = -\frac{\alpha^2}{2} - 2$$

$$y^2 = \underbrace{-\left(\frac{\alpha^2}{2} + 2\right)(\alpha^2 + 2)}_{< 0}$$

$\Rightarrow$ complex solutions

$y = 0 \Rightarrow x = 0$

(iii) $a = 0$ ②

$$\begin{cases} \dot{x} = -2y - 2y(x^2+y^2) = -2y - 2x^2y - 2y^3 = f_1(x,y) \\ \dot{y} = 2x = f_2(x,y) \end{cases}$$

$H : \mathbb{R}^2 \to \mathbb{R}$  $H(x,y) = (x^2+y^2)e^{y^2} = x^2e^{y^2} + y^2e^{y^2}$

$$\frac{\partial H}{\partial x}(x,y) \cdot f_1(x,y) + \frac{\partial H}{\partial y} \cdot f_2(x,y) =$$

$$= 2xe^{y^2} \cdot (-2y - 2x^2y - 2y^3) + (2ye^{y^2} + e^{y^2} \cdot 2y \cdot y^2 + x^2 \cdot e^{y^2} \cdot 2y) \cdot 2x$$

$$= -2xe^{y^2}(2y + 2x^2y + 2y^3) + 2xe^{y^2}(2y + 2y^3 + 2x^2y)$$

$$= 0$$

$H$ is a $C^1$ function, not locally constant

$\Rightarrow$ $H$ is a <u>global first integral</u>

$H : \mathbb{R}^2 \to \mathbb{R}$  $H(x,y) = (x^2+y^2)e^{y^2}$

$(x^2+y^2)e^{y^2} = c$

For $a = 0$, $\eta^* = (0,0)$ stable $\Rightarrow$ level curves of $H$ are <u>bounded</u>

(iv) $a=0$ 

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$$\begin{cases} \dot{x} = -2y - 2y(x^2+y^2) = -2y - 2x^2y - 2y^3 \\ \dot{y} = 2x \end{cases}$$

$$\begin{cases} \rho^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

$$\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\cdot\dot{x}}{x^2} \end{cases} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\rho\dot{\rho} = \rho\cos\theta\,(-2y - 2x^2y - 2y^3) + \rho\sin\theta\cdot 2x$$

$$\rho\dot{\rho} = -\rho\cos\theta\,(2\rho\sin\theta + 2\rho\sin\theta(\rho^2\cos^2\theta + \rho^2\sin^2\theta)) + \rho\sin\theta\cdot 2\rho\cos\theta$$

$$= -\rho\cos\theta\,(2\rho\sin\theta + 2\rho\sin\theta\cdot\rho^2) + 2\rho^2\sin\theta\cos\theta$$

$$= -2\rho^2\cancel{\sin\theta\cos\theta} + \cancel{2\rho^2\sin\theta\cos\theta} - 2\rho^4\sin\theta\cos\theta$$

$$\rho\dot{\rho} = -2\rho^4\sin\theta\cos\theta$$

$$\Rightarrow \boxed{\dot{\rho} = -2\rho^3\sin\theta\cos\theta}$$

$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{2x\cdot\rho\cos\theta - \rho\sin\theta\cdot(-2y - 2y(x^2+y^2))}{\rho^2\cos^2\theta}$$

$$= \frac{2\rho\cos\theta\,\rho\cos\theta + \rho\sin\theta\,(2\rho\sin\theta + 2\rho\sin\theta\cdot\rho^2)}{\rho^2\cos^2\theta}$$

$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{2\rho^2\cos^2\theta + 2\rho^2\sin^2\theta + 2\rho^4\sin^2\theta}{\rho^2\cos^2\theta}$$

$$\frac{\dot{\theta}}{\cancel{\cos^2\theta}} = \frac{2\rho^2 + 2\rho^4\sin^2\theta}{\rho^2\cancel{\cos^2\theta}}$$

$$\boxed{\dot{\theta} = 2 + 2\rho^2\sin^2\theta}$$

(v) We know $H : \mathbb{R}^2 \to \mathbb{R}$ $H(x,y) = (x^2+y^2)e^{y^2}$ is a global first integral when $a=0$

$\Rightarrow$ the shape of the orbits is like ellipses.

2) $\dot{x} = 1 - 2x^2$

$f(x) = 0$

$1 - 2x^2 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow \boxed{x = \pm \frac{\sqrt{2}}{2}}$

$f'(x) = 1 - 4x$

$f'\left(\frac{\sqrt{2}}{2}\right) = 1 - \cancel{4}^{2} \frac{\sqrt{2}}{\cancel{2}} = 1 - 2\sqrt{2} < 0 \Rightarrow \underline{x_1^* = \frac{\sqrt{2}}{2}}$

<u>attractor</u>

$f'\left(-\frac{\sqrt{2}}{2}\right) = 1 - \cancel{4}^{2} \cdot \left(-\frac{\sqrt{2}}{\cancel{2}}\right) = 1 + 2\sqrt{2} > 0 \Rightarrow \underline{x_2^* = -\frac{\sqrt{2}}{2}}$

<u>repeller</u>




<u>Orbits</u>: $\left(-\infty, -\frac{\sqrt{2}}{2}\right), \left\{-\frac{\sqrt{2}}{2}\right\}, \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left\{\frac{\sqrt{2}}{2}\right\}, \left(\frac{\sqrt{2}}{2}, \infty\right)$

- $\underline{\varphi(t, -100)}$

$-100 \in \left(-\infty, -\frac{\sqrt{2}}{2}\right) \Rightarrow \gamma_{-100} = \left(-\infty, -\frac{\sqrt{2}}{2}\right) \Rightarrow$

the image of $\varphi(\cdot, -100)$ is $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$

$I_{-100} = (-\infty, \beta_{-100})$

$\varphi(\cdot, -100)$ strictly decreasing

$$\lim_{t \to -\infty} \varphi(t, -100) = -\frac{\sqrt{2}}{2}$$

$$\lim_{t \to \beta_{-100}} \varphi(t, -100) = -\infty$$

- $\underline{\varphi(t, 0)}$

$0 \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \Rightarrow \gamma_0 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \Rightarrow$

the image of $\varphi(\cdot, 0)$ is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$I_0 = \mathbb{R}$

$\varphi(\cdot, 0)$ strictly increasing

$$\lim_{t \to \infty} \varphi(t, 0) = \frac{\sqrt{2}}{2}$$

$$\lim_{t \to -\infty} \varphi(t, 0) = -\frac{\sqrt{2}}{2}$$

- $\underline{\varphi(t, 100)}$

$100 \in \left(\frac{\sqrt{2}}{2}, \infty\right) \Rightarrow \gamma_{100} = \left(\frac{\sqrt{2}}{2}, \infty\right) \Rightarrow$

the image of $\varphi(\cdot, 100)$ is

$\left(\frac{\sqrt{2}}{2}, \infty\right)$

$I_{100} = (\alpha_{100}, \infty)$

$\varphi(\cdot, 100)$ strictly decreasing

$$\lim_{t \to \infty} \varphi(t, 100) = \frac{\sqrt{2}}{2}$$

$$\lim_{t \to \alpha_{100}} \varphi(t, 100) = \infty$$