Vlad Bogdan-Tudor, 917

1) $\begin{cases} x' = 5x - 7y \\ y' = -2x \end{cases}$

We will use the characteristic equation method to find the general solution

$$A = \begin{pmatrix} 5 & -7 \\ -2 & 0 \end{pmatrix}$$

$$Au = \lambda u$$

$$\det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} 5-\lambda & -7 \\ -2 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(5-\lambda) - 14 = 0$$

$$\lambda^2 - 5\lambda - 14 = 0$$

$$(\lambda - 7)(\lambda + 2) = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 7 \in \mathbb{R} \\ \lambda_2 = -2 \in \mathbb{R} \end{cases} \quad (1)$$

- $\lambda_1 = 7$

$$Au = 7u$$

$$\begin{pmatrix} 5 & -7 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7x \\ 7y \end{pmatrix}$$

$$\begin{cases} 5x - 7y = 7x \\ -2x = 7y \Rightarrow y = -\frac{2}{7}x \Rightarrow \vec{u}_1 \begin{pmatrix} 1 \\ -\frac{2}{7} \end{pmatrix} \end{cases}$$

, OR we can use $\vec{u}_1 \begin{pmatrix} 7 \\ -2 \end{pmatrix}$

- $\lambda_2 = -2$

$Au = -2u$

$$\begin{pmatrix} 5 & -7 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$$\begin{cases} 5x - 7y = -2x \\ -2x = -2y \Rightarrow x = y \Rightarrow \vec{u}_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

$\left.\begin{array}{l} \vec{u}_1, \vec{u}_2 \text{ lin. ind.} \\ (1) \Rightarrow \lambda_1, \lambda_2 \in \mathbb{R} \end{array}\right\} \Rightarrow$ A is diagonalizable, we can continue the procedure

$\Rightarrow$ we have that $\left\{\begin{array}{l} e^{\lambda_1 t} u_1 = e^{7t}\begin{pmatrix} 7 \\ -2 \end{pmatrix} \\ e^{\lambda_2 t} u_2 = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}\right.$ are solutions of the system $X' = AX$

$\Rightarrow$ The general solution is $\underline{X = c_1 e^{7t}\begin{pmatrix} 7 \\ -2 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$, $c_1, c_2 \in \mathbb{R}$

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Let $U(t) = \begin{pmatrix} 7e^{7t} & e^{-2t} \\ -2e^{7t} & e^{-2t} \end{pmatrix}$ (we took $c_1 = c_2 = 1$)

$\left.\begin{array}{l}\text{Both columns are of } U(t) \text{ are solutions of the} \\ \quad \text{system } X' = AX \\ \\ \text{The columns are lin. ind.}\end{array}\right\} \Rightarrow$

$\Rightarrow U(t)$ is a fundamental matrix solution

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2) $x' + 2\lambda x = t^2 - t$

We will use the integrating factor

$$\mu = e^{\int 2\lambda \, dt} = e^{2\lambda t}$$

$$x' + 2\lambda x = t^2 - t \quad \Big| \cdot \mu = e^{2\lambda t}$$

$$\underbrace{\frac{dx}{dt} \cdot e^{2\lambda t} + e^{2\lambda t} \cdot 2\lambda x}_{} = (t^2 - t) \cdot e^{2\lambda t}$$

$$\left(x \cdot e^{2\lambda t}\right)' = (t^2 - t) \cdot e^{2\lambda t}$$

$$x \cdot e^{2\lambda t} = \underbrace{\int (t^2 - t) e^{2\lambda t} dt}_{\text{we need integration by parts (twice)}}$$

$$\int (t^2 - t) e^{2\lambda t} dt = \frac{(t^2 - t) e^{2\lambda t}}{2\lambda} - \int \frac{(2t-1) e^{2\lambda t}}{2\lambda} dt$$

$f = t^2 - t \qquad g = \dfrac{e^{2\lambda t}}{2\lambda}$

$\downarrow \qquad\qquad \uparrow$

$f' = 2t - 1 \qquad g' = e^{2\lambda t}$

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$$= \frac{(t^2-t)e^{2\lambda t}}{2\lambda} - \frac{1}{2\lambda}\int (2t-1)e^{2\lambda t}\,dt$$

$$f = 2t-1 \qquad g = \frac{e^{2\lambda t}}{2\lambda}$$

$$\downarrow \qquad \uparrow$$

$$f' = 2 \qquad g' = e^{2\lambda t}$$

$$= \frac{(t^2-t)e^{2\lambda t}}{2\lambda} - \frac{1}{2\lambda}\left(\frac{(2t-1)e^{2\lambda t}}{2\lambda} - \int \cancel{2}\cdot\frac{e^{2\lambda t}}{\cancel{2}\lambda}\,dt\right)$$

$$= \frac{(t^2-t)e^{2\lambda t}}{2\lambda} - \frac{(2t-1)e^{2\lambda t}}{4\lambda^2} - \frac{1}{2\lambda^2}\cdot\frac{e^{2\lambda t}}{2\lambda} + c$$

$$= \frac{(t^2-t)e^{2\lambda t}}{2\lambda} - \frac{(2t-1)e^{2\lambda t}}{4\lambda^2} - \frac{e^{2\lambda t}}{4\lambda^3} + c,\ c\in\mathbb{R}$$

$$\Rightarrow x\cdot e^{2\lambda t} = \frac{(t^2-t)e^{2\lambda t}}{2\lambda} - \frac{(2t-1)e^{2\lambda t}}{4\lambda^2} - \frac{e^{2\lambda t}}{4\lambda^3} + c \quad \Big|\cdot e^{-2\lambda t}$$

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$$\Rightarrow \boxed{x = \frac{t^2 - t}{2\lambda} - \frac{2t-1}{4\lambda^2} - \frac{1}{4\lambda^3} + \frac{c}{e^{2\lambda t}}}, \quad c \in \mathbb{R}$$

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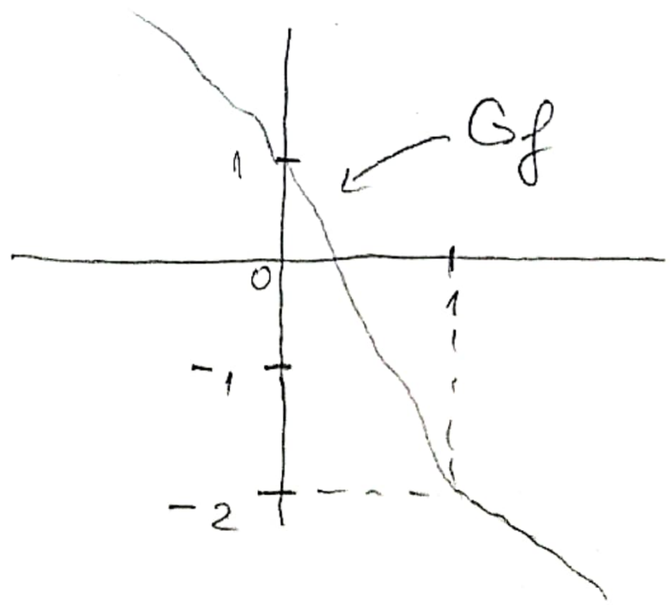
3) $f:\mathbb{R}\to\mathbb{R}$ injective, C^1

$$\begin{cases} f(0)=1 \\ f(1)=-2 \end{cases}$$

a) $\dot{x}=f(x)$

we have an eq. point $\Leftrightarrow f(x)=0$

$\left.\begin{array}{l} f(0)=1 \\ f(1)=-2 \\ f \text{ is } C^1 \text{ injective} \end{array}\right\} \Rightarrow$ the function look something like this:



$\Rightarrow \exists\, x_0 \in (0,1)$ s.t. $f(x)=0$

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We can also justify this by using the <u>Darboux Theorem</u>

$$\left.\begin{array}{l} f \text{ is } C^1 \\ f(0) = 1 > 0 > f(1) = -2 \end{array}\right\} \xrightarrow{\text{Darboux}} f(x_0) = 0 \text{ has at least one sol with } x_0 \in (0,1)$$

But, f is injective

$\Rightarrow f(x_0) = 0$ has <u>one</u> sol with $x_0 \in (0,1)$.

$$\left.\begin{array}{l} f(0) = 1 \\ f(1) = -2 \\ f \text{ injective} \end{array}\right\} \Rightarrow f \text{ is strictly decreasing on } (0,1)$$

$$\Downarrow$$

$$f'(x) < 0, \ \forall x \in (0,1)$$

$x_0 \in (0,1) \Rightarrow f'(x_0) < 0$

So we have:

$$\left.\begin{array}{l} f \text{ is } C^1(\mathbb{R}) \\ f(x_0)=0 \Rightarrow x_0\in(0,1) \text{ eq. point} \\ f'(x_0)<0 \end{array}\right\} \Rightarrow x_0 \text{ attractor}$$

The linearisation method

$$\left.\begin{array}{l} x_0 \text{ attractor} \\ f \text{ is } C^1 \text{ and injective} \end{array}\right\} \Rightarrow x_0 \text{ global attractor eq. point}$$

$\Rightarrow \dot{x}=f(x)$ has a global attractor eq point $x_0\in(0,1)$

<u>Note</u>: We can also see that since f is C^1 and injective and $f(0)=1$ and $f(1)=-2 \Rightarrow$ f is strictly decreasing on $\mathbb{R}$

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b) $\begin{cases} \dot{\rho} = f(\rho) \\ \dot{\theta} = -2 \end{cases}$

$\dot{\rho} = f(\rho)$

from a) $\Rightarrow$ $f(\rho)$ strictly decreasing

$f(x_0) = 0, \quad x_0 \in (0,1)$

$\boxed{f(\rho) > 0}$

$\Rightarrow \begin{cases} \text{on } \rho \in (-\infty, x_0) \text{ the length is increasing} \\ \text{on } \rho = x_0 \text{ the length is constant} \\ \text{on } \rho \in (x_0, \infty) \text{ the length is decreasing} \end{cases}$

$\boxed{f(\rho) < 0}$

$\dot{\theta} = -2 < 0 \Rightarrow$ clockwise behavior

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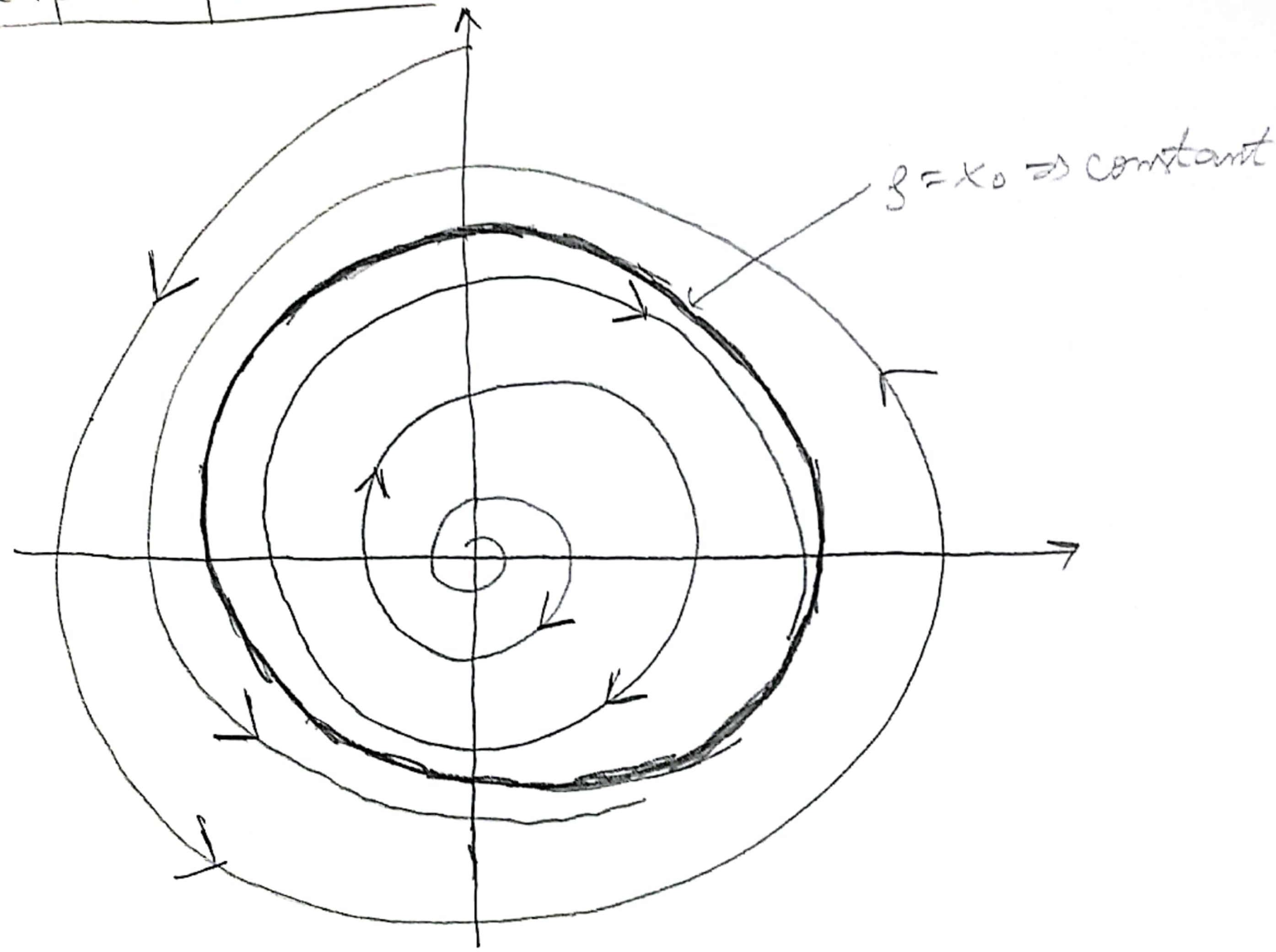
the phase portrait



(0,0) is the only eq point, it is a repeller

$\rho = 1$ is an "attractor"

Vlad Bogdan-Tudor, 917 (12)

c) $f(0)=1$
$f(1)=-2$ $\Rightarrow m = \frac{-2-1}{1-0} = -3$

slope

$\Rightarrow y = -3x + n \leftarrow G_f$

$(0,1) \in G_f \Rightarrow 1 = -3 \cdot 0 + n \Rightarrow n = 1$

$\Rightarrow y = -3x+1$

Let $f: \mathbb{R} \to \mathbb{R}$ $\underline{f(x) = -3x+1}$

f is strictly decreasing, $C^1(\mathbb{R})$, injective

$f(x_0)=0$

$-3x_0+1=0$

$3x_0 = 1 \Rightarrow \boxed{x_0 = \frac{1}{3}}$

$$\begin{cases} \dot{\rho} = -3\rho + 1 \\ \dot{\theta} = -2 \end{cases}$$

$\Rightarrow \boxed{\theta(t) = -2t + c_1, \ c_1 \in \mathbb{R}}$, we can take $c_1 = 0$

$\dot{\rho} = -3\rho + 1$

$\dot{\rho} + 3\rho = 1$

$\dot{\rho} + 3\rho = 0$

$\dot{\rho} = -3\rho$

$\Rightarrow \rho_h = c \cdot e^{-3t}$

$\rho_p = \frac{1}{3}$

$\Rightarrow \boxed{\rho = \frac{1}{3} + c_2 \cdot e^{-3t}, \ c_2 \in \mathbb{R}.}$, we can take $c_2 = 1$

$\Rightarrow$ We consider $\begin{cases} \theta(t) = -2t \\ \rho(t) = \frac{1}{3} + e^{-3t} \end{cases}$

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We know the relations:

$$\begin{cases} x(t) = \rho(t) \cdot \cos\theta(t) \\ y(t) = \rho(t) \cdot \sin\theta(t) \end{cases}$$

$$\Rightarrow \begin{cases} x(t) = \left(\frac{1}{3} + e^{-3t}\right) \cdot \cos(-2t) \\ y(t) = \left(\frac{1}{3} + e^{-3t}\right) \cdot \sin(-2t) \end{cases}$$

We also know

$$\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x + y \cdot \dot{x}}{x^2} \end{cases}$$

$$\rho\dot{\rho} = x\dot{x} + y\dot{y}$$

$$y\dot{y} = \rho\dot{\rho} - x\dot{x}$$

$$\boxed{\dot{y} = \frac{\rho\dot{\rho} - x\dot{x}}{y}}$$

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$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{\frac{\rho\dot{\rho} - x\dot{x}}{y}\cdot x + y\cdot\dot{x}}{x^2}$$

$$\frac{x(\rho\dot{\rho} - x\dot{x})}{y} + y\cdot\dot{x} = \frac{x^2\dot{\theta}}{\cos^2\theta} \qquad \cdot | y$$

$$x\rho\dot{\rho} - x^2\dot{x} + y^2\dot{x} = \frac{x^2 y\dot{\theta}}{\cos^2\theta}$$

$$\dot{x}(y^2 - x^2) = \frac{x^2 y\dot{\theta}}{\cos^2\theta} - x\rho\dot{\rho}$$

$$\boxed{\dot{x} = \frac{1}{y^2 - x^2}\left(\frac{x^2 y\dot{\theta}}{\cos^2\theta} - x\rho\dot{\rho}\right)}$$

$$\dot{x} = \frac{1}{y^2 - x^2}\left(\frac{-2x^2 y}{\cos^2(-2t)} - x\left(\frac{1}{3} + e^{3t}\right)(-3\rho + 1)\right.$$

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$$\boxed{\dot{y} = \frac{z\dot{z} - x\dot{x}}{y}}$$

$$\dot{y} = \frac{\left(\frac{1}{3}+e^{-3t}\right)(-3z+1) - x\cdot\dot{x}}{y}$$

$$\dot{y} = \frac{1}{y}\left(\frac{1}{3}+e^{-3t}\right)(-3z+1) - \frac{x}{y}\dot{x}$$

here we can replace the previously found $\dot{x}$ and find $\dot{y}$ exactly, also.

4) $h > 0$

$$\dot{x} = x^2 + 9x - 10 = f(x)$$

$$f(x) = 0$$

$$x^2 + 9x - 10 = 0$$

$$(x+10)(x-1) = 0$$

$$\Rightarrow \begin{cases} x_1 = -10 \\ x_2 = 1 \end{cases} \Rightarrow \left.\begin{matrix} \eta_1^* = -10 \\ \eta_2^* = 1 \end{matrix}\right\} \text{eq. points.}$$

f is $C^1(\mathbb{R})$

$$f'(x) = 2x + 9$$

$$f'(\eta_1^*) = f'(-10) = -20 + 9 = -11 < 0$$

$$\Downarrow$$

$\underline{\eta_1^* = -10 \text{ attractor}}$

$$f'(\eta_2^*) = f'(1) = 2 + 9 = 11 > 0$$

$$\Downarrow$$

$\underline{\eta_2^* = 1 \text{ repeller}}$

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$$g(x_n) = h x_n^2 + (1+9h) x_n - 10h$$

$$\Rightarrow g'(x_n) = 2h x_n + 1 + 9h$$

-10 is attractor if $|f(-10)| < 1$

$$\Rightarrow |-20h + 9h + 1| < 1$$

$$\Rightarrow |-11h + 1| < 1$$

$$\Rightarrow \begin{cases} -11h + 1 < 1 \\ \text{and} \\ -11h + 1 > -1 \end{cases} \Rightarrow \begin{cases} -11h < 0 \\ -11h > -2 \end{cases}$$

$$\Rightarrow \begin{cases} h > 0 \\ h < \frac{2}{11} \end{cases} \Rightarrow h \in \left(0, \frac{2}{11}\right)$$