1.2.1

a) $x'' + t^2 x = 0 \qquad x(0) = 0.$

2nd order LHDE $\xrightarrow[\text{for LHDE}]{\text{Fundam. Th.}}$ we need 2 lin. ind. solutions

$$X = c_1 X_1 + c_2 X_2 \quad , \quad c_1, c_2 \in \mathbb{R}$$

We have only one condition, namely $x(0)=0$
$\Rightarrow$ infinitely many solutions

b) $x'' + t^2 x = 0 \quad , \quad x(0) = 0, \; x'(0) = 0$

This is an IVP (Initial Value Problem)

By the existance and uniqueness th. $\Rightarrow$ one solution

c) $x'' + t^2 x = 0, \quad x(0) = 0, \; x'(0) = 0, \; x''(0) = 1$
Can have at most one solution. Let's check.
$x(0) = 0 \Rightarrow$ for $t = 0$ we have

$$x''(0) + 0^2 \cdot x(0) = 0$$

$x''(0) = 0$ contradiction with the problem statement

$\Rightarrow$ no solutions

1.2.5

a) $x' - 2x = e^t$

$x_p = a e^t \quad , \quad a \in \mathbb{R}$

$(a e^t)' - 2 a e^t = e^t$

$a e^t - 2 a e^t = e^t$

$-a e^t = e^t$

$\Rightarrow -a = 1 \Rightarrow \boxed{a = -1} \Rightarrow \boxed{x_p = -e^t}$

b) $x' - 2x = e^{-t}$

$x_p = b e^{-t} \quad , \quad b \in \mathbb{R}.$

$(b e^{-t})' - 2 b e^{-t} = e^{-t}$

$-b e^{-t} - 2 b e^{-t} = e^{-t}$

$-3 b e^{-t} = e^{-t}$

$\Rightarrow -3b = 1 \Rightarrow \boxed{b = -\frac{1}{3}} \Rightarrow \boxed{x_p = -\frac{1}{3} e^{-t}}$

c) $x' - 2x = 5e^{t} - 3e^{-t}$

We know $x' - 2x = e^{t}$ has $x_p = -e^{t}$ particular sol (from a))

$x' - 2x = e^{-t}$ has $x_p = -\frac{1}{3}e^{-t}$ particular sol (from b))

We need $f = \alpha_1 f_1 + \alpha_2 f_2$

$$f = \underbrace{5}_{\alpha_1}\overbrace{e^{t}}^{f_1} \underbrace{-3}_{\alpha_2}\overbrace{e^{-t}}^{f_2}$$

$x_{p_1} = -e^{t}$ particular sol. of $x' - 2x = f_1 = e^{t}$

$x_{p_2} = -\frac{1}{3}e^{-t}$ particular sol. of $x' - 2x = f_2 = e^{-t}$

By the Superposition Principle $\Rightarrow$ $x_p = \underbrace{5}_{\alpha_1}\cdot\overbrace{(-e^{t})}^{x_{p1}} + \underbrace{(-3)}_{\alpha_2}\cdot\overbrace{\left(-\frac{1}{3}e^{-t}\right)}^{x_{p2}}$

is a particular solution of $x' - 2x = 5e^{t} - 3e^{-t}$

$$\boxed{x_p = -5e^{t} + e^{-t}}$$

d) general solution of $x' - 2x = 5e^{t} - 3e^{-t}$

From c) $\Rightarrow$ $x_p = -5e^{t} + e^{-t}$

We solve the 1st order LHDE $x' - 2x = 0$

$x' - 2x = 0$ 1st order LHDE with c.c.

$r - 2 = 0 \Rightarrow r = 2 \mapsto e^{2t}$

By the Fundam. Th. for LHDE $\Rightarrow$ $x_h = c_1 e^{2t}$, $c_1 \in \mathbb{R}$.

By the Fundam. Th. for Ln-HDE $\Rightarrow$ the general solution is

$$\boxed{x = c_1 e^{2t} - 5e^{t} + e^{-t}}, \quad c_1 \in \mathbb{R}$$

1.3.4 the general solution of $x' - x = e^{t-1}$ in 2 ways.

I For $x = t \cdot e^{t-1}$,

$$x' = 1 \cdot e^{t-1} + e^{t-1} \cdot t = e^{t-1}(t+1)$$

$$x' - x = e^{t-1}(t+1) - te^{t-1} = e^{t-1}$$

$\Rightarrow \underbrace{x_p = t \cdot e^{t-1}}$ particular solution

We now solve the 1st order LHDE $x' - x = 0$

$$\frac{dx}{dt} - x = 0 \quad \leftarrow \text{separable diff. eq.}$$

$$\frac{dx}{dt} = x$$

$$\frac{1}{x} dx = dt \quad \Big| \text{Integrate}$$

$$\int \frac{1}{x} dx = \int dt$$

$$\ln|x| = t + C$$

$|x| = e^{t+C} = e^{C} \cdot e^{t} \Rightarrow x_h = \pm e^{c_1} \cdot e^{t}$

$x_h = 0$ is a solution $\Big\} \Rightarrow x_h = c_1 \cdot e^{t}, \; c_1 \in \mathbb{R}$.

$\Rightarrow$ the general solution is $\boxed{x = t \cdot e^{t-1} + c_1 \cdot e^{t}, \; c_1 \in \mathbb{R}}$

(II) $x' - x = e^{t-1}$

$$\frac{dx}{dt} - x = e^{t-1}$$

We use the integrating factor $\mu(t) = e^{\int -1\, dt} = e^{-t}$

$$\frac{dx}{dt} - x = e^{t-1} \qquad \cdot \big| e^{-t}$$

$$\underbrace{e^{-t}\frac{dx}{dt} - xe^{-t}}_{\text{product rule}} = e^{-1}$$

$$(x \cdot e^{-t})' = e^{-1} \qquad \big| \text{Integrate w.r.t. } t$$

$$\int (x \cdot e^{-t})'\, dt = \int e^{-1}\, dt$$

$$x \cdot e^{-t} = te^{-1} + C \qquad \cdot \big| e^{t}$$

$$x = t \cdot e^{t-1} + Ce^{t}$$

$\Rightarrow$ the general sol. is $\underline{x = t \cdot e^{t-1} + c_1 e^{t}}$, $c_1 \in \mathbb{R}$

III 3rd method (just to be extra sure)

$x' - x = e^{t-1}$

$x_h$: $x' - x = 0$

$r - 1 = 0$

$r = 1 \mapsto e^t$

$\Rightarrow x_h = c_1 \cdot e^t \xmapsto{\text{vary}} x_p = c_1(t) \cdot e^t$

$x_p' = c_1'(t) \cdot e^t + e^t \cdot c_1(t)$

$x_p' - x_p = c_1'(t) e^t + \cancel{e^t \cdot c_1(t)} - \cancel{c_1(t) \cdot e^t} = e^{t-1}$

$\Rightarrow c_1'(t) \cdot e^t = e^{t-1}$

$c_1'(t) = e^{-1} \Rightarrow c_1(t) = \int e^{-1} dt = \frac{t}{e} + c_2, \ c_2 \in \mathbb{R}.$

$\Rightarrow x_p = \left(\frac{t}{e} + c_2\right) \cdot e^t = \frac{t e^t}{e} + c_2 e^t = t e^{t-1} + c_2 e^t$

Fundam. Th $\Rightarrow$ for Ln-HDE

$$x = \underbrace{t e^{t-1} + c_2 e^t}_{x_p} + \underbrace{c_1 e^t}_{x_h}$$

$$\Rightarrow \boxed{x = t e^{t-1} + c e^t, \ c \in \mathbb{R}}$$

1.7.5 $\boxed{e^{ix} = \cos x + i \sin x}$ $\leftarrow$ Euler's Formula

$x \in \mathbb{R}$.

- $e^{it} = \cos t + i \sin t$

- $e^{i\pi} = \cos \pi + i \sin \pi = -1 + i \cdot 0 = -1$

- $e^{i\frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i \cdot 1 = i$

- $e^{(-1+i)t} = e^{-t+it} = e^{-t} \cdot e^{it} = e^{-t}(\cos t + i \sin t)$

---

1.7.8 LHDE of min. order that has $1+t(1+e^{-t})$ as sol.

$1 + t(1+e^{-t}) = 1 + t + te^{-t}$

$\Rightarrow$ 1 solution, $t$ solution $\Rightarrow r = 0$ double root of the ch. eq.

$te^{-t}$ solution $\Rightarrow e^{-t}$ sol $\Rightarrow r = -1$ double root of the ch. eq.

$\Rightarrow$ We get the ch. eq.

$$r^2(r+1)^2 = 0$$

$$r^2(r^2+2r+1) = 0$$

$$r^4 + 2r^3 + r^2 = 0$$

4th order LHDE with c.c,

$\Rightarrow x^{(4)} + 2x''' + x'' = 0$

$x = c_1 + c_2 t + c_3 e^{-t} + c_4 t e^{-t}$

(1.7.9) $k, \eta \in \mathbb{R}$ fixed params. Find solutions of the IVP $x' = k(21 - x)$, $x(0) = \eta$

$x' = k(21 - x)$ 1st order LnHDE

$$\frac{dx}{dt} = 21k - xk$$

$$\frac{dx}{dt} + kx = 21k$$

We will use the integrating factor $\mu(t) = e^{\int k dt} = e^{kt}$

$$\frac{dx}{dt} + kx = 21k \quad \cdot \Big| \, e^{kt}$$

$$\underbrace{e^{kt}\frac{dx}{dt} + e^{kt} \cdot k \cdot x}_{} = 21k \cdot e^{kt}$$

$$(x \cdot e^{kt})' = 21k\, e^{kt} \quad \Big| \text{Integrate w.r.t. } t$$

$$x \cdot e^{kt} = \int 21k\, e^{kt}\, dt$$

$$x \cdot e^{kt} = 21 e^{kt} + C$$

$$x = 21 + \frac{C}{e^{kt}} \Rightarrow \underline{x = 21 + c_1 e^{-kt}}, \; c_1 \in \mathbb{R}$$

$$X = 21 + c_1 e^{-kt}$$

$$x(0) = \eta$$

$$\left.\begin{aligned} x(0) &= \eta \\ x(0) &= 21 + c_1 \cdot e^{0} \end{aligned}\right\} \Rightarrow c_1 = \eta - 21$$

$$\Rightarrow \boxed{X = 21 + (\eta - 21) e^{-kt}}$$

(1.7.10) $x'' - x = t e^{-2t}$ 2nd order Ln-HDE

a) $x_p(t) = (at + b)e^{-2t}$, $a, b \in \mathbb{R}$

$x_p(t)' = a \cdot e^{-2t} + (-2)e^{-2t} \cdot (at + b)$

$x_p(t)' = e^{-2t}(a - 2at - 2b)$

$x_p(t)'' = -2e^{-2t}(a - 2at - 2b) + (-2a) \cdot e^{-2t}$

$= -2e^{-2t}(a - 2at - 2b + a)$

$= -2e^{-2t}(2a - 2at - 2b)$

$= -4e^{-2t}(a - at - b)$

$\Rightarrow x_p(t)'' = 4e^{-2t}(at + b - a)$

$x_p(t)'' - x_p(t) = te^{-2t}$

$4e^{-2t}(at + b - a) - (at + b)e^{-2t} = te^{-2t}$

$e^{-2t}(4at + 4b - 4a - at - b) = te^{-2t}$ $\quad \cdot | e^{2t}$

$3at + 3b - 4a = t$

$3at - t + 3b - 4a = 0$

$(3a - 1)t + 3b - 4a = 0$

Take $a = \frac{1}{3}$, $b = \frac{4}{9}$ $\Rightarrow$ $\boxed{x_p(t) = \left(\frac{1}{3}t + \frac{4}{9}\right)e^{-2t}}$

b) We solve the 2nd order LHDE $x''-x=0$ (with c.c.)

$$x''-x=0$$

$r^2-1=0$ characteristic eq.

$$r^2=1 \Rightarrow r_1=1 \mapsto e^{-t}$$

$r_2=-1 \mapsto e^{t}$ lin. ind. solutions

$\xrightarrow[\text{for LHDE}]{\text{Fundam Th}}$ $x_h=c_1e^{-t}+c_2e^{t}$ ; $c_1, c_2 \in \mathbb{R}$

a), b) $\Rightarrow$ the general solution is:

$$\boxed{x=\left(\frac{1}{3}t+\frac{4}{9}\right)\cdot e^{-2t}+c_1e^{-t}+c_2e^{t}}\ ;\ c_1, c_2 \in \mathbb{R}$$

c) $x(0)=0$, $x'(0)=0$

$$x'=\frac{1}{3}\cdot e^{-2t}+(-2)e^{-2t}\left(\frac{1}{3}t+\frac{4}{9}\right)-c_1e^{-t}+c_2e^{t}$$

$$\begin{cases} x(0)=\frac{4}{9}+c_1+c_2=0 \Rightarrow c_1+c_2=-\frac{4}{9} \\ x'(0)=\frac{1}{3}-2\cdot\frac{4}{9}-c_1+c_2=\frac{3}{1}\!\!\!/\,\frac{1}{3}-\frac{8}{9}+c_2-c_1=0 \end{cases}$$

$-\frac{5}{9}+c_2-c_1=0 \Rightarrow$ We have $\begin{cases} c_2-c_1=\frac{5}{9} \\ c_2+c_1=-\frac{4}{9} \end{cases}$ $\oplus$

$2c_2=\frac{1}{9} \Rightarrow \boxed{c_2=\frac{1}{18}}$

$\Rightarrow \boxed{x=\left(\frac{1}{3}t+\frac{4}{9}\right)e^{-2t}+\frac{1}{18}e^{t}-\frac{1}{2}e^{-t}}$ $\qquad \Rightarrow \boxed{c_1=-\frac{1}{2}}$

1.7.12 $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$ $\quad \mathcal{L}(x) = x'' - 2x' + x, \forall x \in C^2(\mathbb{R})$

(a) $\mathcal{L}(x) = x'' - 2x' + x$

$\mathcal{L}(x)$ linear map iff. $\mathcal{L}(\alpha x + \beta y) = \alpha \mathcal{L}(x) + \beta \mathcal{L}(y)$,

$\forall x, y \in C^2(\mathbb{R})$
$\forall \alpha, \beta \in \mathbb{R}$.

Let $x, y \in C^2(\mathbb{R})$
$\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned}
\mathcal{L}(\alpha x + \beta y) &= (\alpha x + \beta y)'' - 2(\alpha x + \beta y)' + \alpha x + \beta y \\
&= \alpha x'' + \beta y'' - 2\alpha x' - 2\beta y' + \alpha x + \beta y \\
&= \alpha(x'' - 2x' + x) + \beta(y'' - 2y' + y) \\
&= \alpha \mathcal{L}(x) + \beta \mathcal{L}(y) \Rightarrow \mathcal{L}(x) \text{ linear map}
\end{aligned}$$

The fundamental theorem for LHDEs $\Rightarrow$ $\boxed{\dim(\ker(\mathcal{L}(x))) = 2}$

b) $x'' - 2x' + x = \cos 2t$

First, we find a particular solution of the form

$$x_p = a\cos 2t + b\sin 2t, \quad a, b \in \mathbb{R}$$

$$x_p' = -2a\sin 2t + 2b\cos 2t$$

$$x_p'' = -2a \cdot 2\cos 2t + 2b \cdot (-2)\sin 2t$$

$$= -4a\cos 2t - 4b\sin 2t$$

$$x_p'' - 2x_p' + x_p = \cos 2t$$

$$-4a\cos 2t - 4b\sin 2t - 2(-2a\sin 2t + 2b\cos 2t) +$$
$$a\cos 2t + b\sin 2t = \cos 2t$$

$$-4a\cos 2t - 4b\sin 2t + 4a\sin 2t - 4b\cos 2t + a\cos 2t + b\sin 2t$$
$$= \cos 2t$$

$$-3a\cos 2t - 4b\cos 2t - 4b\sin 2t + 4a\sin 2t = \cos 2t$$

$$-3a\cos 2t - 4b\cos 2t - \cos 2t + 4a\sin 2t - 4b\sin 2t = 0$$

$$-(3a + 4b + 1)\cos 2t + 4(a - b)\sin 2t = 0$$

Take $\begin{cases} a - b = 0 \Rightarrow a = b \\ 3a + 4b + 1 = 0 \end{cases}$

$$3a + 4a + 1 = 0$$

$$7a = -1 \Rightarrow \boxed{a = -\frac{1}{7}} \Rightarrow \boxed{b = -\frac{1}{7}}$$

$\Rightarrow x_p = -\frac{1}{7}\cos 2t - \frac{1}{7}\sin 2t$

Now we solve the 2nd order LHDE with c.c.

$$x'' - 2x' + x = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \Rightarrow r = 1 \text{ double root} \mapsto e^t, te^t$$

lin. ind.

Fundam. Th. for LHDE $\Rightarrow x_h = c_1 e^t + c_2 t e^t;\ c_1, c_2 \in \mathbb{R}$

$\Rightarrow$ the general solution is

$$\boxed{x = -\frac{1}{7}\cos 2t - \frac{1}{7}\sin 2t + c_1 e^t + c_2 t e^t};\quad c_1, c_2 \in \mathbb{R}$$

(c) $f_1(t) = e^{2t}$ ; $f_2(t) = e^{-2t}$ , $\forall t \in \mathbb{R}$

Find a particular solution of $\mathcal{L}(x) = 3f_1 + 5f_2$

$$\mathcal{L}(x) = x'' - 2x' + x$$

To simplify our task, we will use the superposition principle.

$$f = \overset{\alpha_1}{\widehat{3}} f_1 + \overset{\alpha_2}{\widehat{5}} f_2$$

We will find $x_{p_1}$ and $x_{p_2}$ particular solutions of $\mathcal{L}(x) = f_1$ , $\mathcal{L}(x) = f_2$ respectively.

- $\mathcal{L}(x) = f_1$

$$x'' - 2x' + x = e^{2t}$$

Take $\boxed{x_{p_1} = e^{2t}}$

$$(e^{2t})'' - 2(e^{2t})' + e^{2t} = (2e^{2t})' - 4e^{2t} + e^{2t}$$

$$= \cancel{4e^{2t}} - \cancel{4e^{2t}} + e^{2t} = e^{2t} \Rightarrow x_{p_1} \text{ particular solution of } \mathcal{L}(x) = f_1$$

• $\mathcal{L}(x) = f_2$

$x'' - 2x' + x = e^{-2t}$

Take $\boxed{x_{p_2} = \frac{1}{9} e^{-2t}}$

$$\left(\frac{1}{9} e^{-2t}\right)'' - 2\left(\frac{1}{9} e^{-2t}\right)' + \frac{1}{9} e^{-2t} =$$

$$= \left(-\frac{2}{9} e^{-2t}\right)' + \frac{4}{9} e^{-2t} + \frac{1}{9} e^{-2t}$$

$$= \frac{4}{9} e^{-2t} + \frac{4}{9} e^{-2t} + \frac{1}{9} e^{-2t}$$

$= e^{-2t} \Rightarrow x_{p_2}$ particular solution of $\mathcal{L}(x) = f_2$

By the superposition principle $\Rightarrow$

$$\boxed{x_p = 3e^{2t} + \frac{5}{9} e^{-2t}}$$

is a particular solution of

$$\mathcal{L}(x) = f = 3e^{2t} + 5e^{-2t}$$

1.17.19

$$x'' + 4x = \cos 2t$$

a) $x_p = t(a\cos 2t + b\sin 2t)$, $a, b \in \mathbb{R}$.

$$x_p' = a\cos 2t + b\sin 2t + (-2a\sin 2t + 2b\cos 2t)t$$

$$x_p'' = -2a\sin 2t + 2b\cos 2t + (-4a\cos 2t - 4b\sin 2t)t$$
$$+ 1\cdot(-2a\sin 2t + 2b\cos 2t)$$

$$x_p'' = -4a\sin 2t + 4b\cos 2t - 4at\cos 2t - 4bt\sin 2t$$

$$x_p'' = -4(a+bt)\sin 2t - 4(at-b)\cos 2t$$

$$x_p'' + 4x_p = -4(a+bt)\sin 2t - 4(at-b)\cos 2t + 4ta\cos 2t$$
$$+ 4tb\sin 2t$$

$$= (-4a - \cancel{4bt} + \cancel{4tb})\sin 2t + (-\cancel{4at} + 4b + \cancel{4at})\cos 2t$$

$$= -4a\sin 2t + 4b\cos 2t \quad (= \cos 2t)$$

Take $a=0$, $b=\frac{1}{4}$

$$\Rightarrow x_p = \frac{t}{4}\sin 2t$$

ii) We now solve the 2nd order LHDE with c.c

$$x'' + 4x = 0$$

$$r^2 + 4 = 0$$

$$r^2 = -4 = 4i^2 = (2i)^2$$

$$\Rightarrow \begin{cases} r_1 = 2i \\ r_2 = -2i \end{cases} \longmapsto \cos(2t),\ \sin(2t) \text{ lin. ind.}$$

Fundamental Th for LHDE $\Rightarrow$ $x_h = c_1 \cos(2t) + c_2 \sin(2t)$

$c_1, c_2 \in \mathbb{R}$

Fundamental th for Ln-HDE $\Rightarrow$ the general solution is

$$\boxed{x = \frac{t}{4}\sin(2t) + c_1\cos(2t) + c_2(\sin 2t)},\quad c_1, c_2 \in \mathbb{R}$$

c) motion of a spring mass system governed by the eq.

$$x'' + 4x = \cos 2t$$

We have the case with motion without damping, with an external force:

$$\boxed{x'' + \frac{k}{m}x = A\cos\omega t}, \quad A>0, \omega>0$$

We know $x_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

Our equation is $x'' + 4x = \cos 2t$,

From b) $\Rightarrow x_h = c_1\cos 2t + c_2 \sin 2t \Rightarrow \underline{\underline{\omega = \omega_0 = 2}}$

General solution for a diff. eq. describing motion without damping, with an external force (when $\omega = \omega_0$):

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2\omega_0} \cdot t \cdot \sin(\omega_0 t)$$

$\Rightarrow$ General sol for our equation:

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} t \sin 2t$$

Unbounded function, it oscillates with an amplitude which increases to $\infty$

$\rightarrow$ Resonance (ext. frequency = int. frequency)

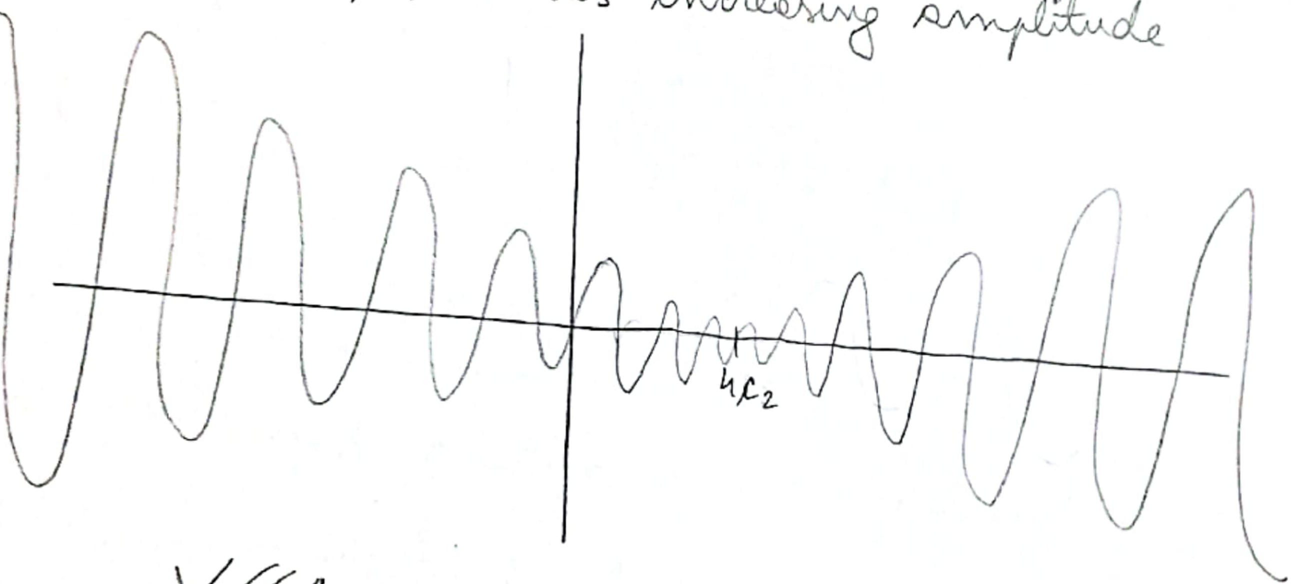
This can also be observed by studying the graph of the solution (next page)

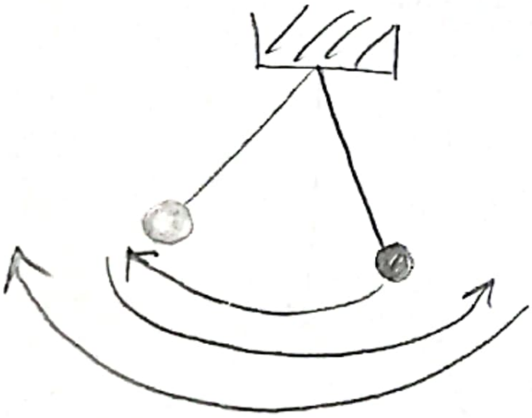
$$x = \underbrace{\left(\frac{t}{4} + c_2\right) \sin(2t)}_{\text{unbounded}} + c_1 \cos(2t)$$

x oscillatory around 0.

For $t \in (-\infty, 4c_2)$ x has decreasing amplitude

For $t \in [4c_2, \infty)$ x has increasing amplitude



The pendulum will oscillate around the vertical with linearly decreasing amplitude up to $t = 4c_2$, after which the amplitude will start to increase linearly.

Unbounded function $\Rightarrow$ resonance

1.7.24

$$m x'' + 25x = 12\cos(36\pi t)$$

$m \to$ mass $\Rightarrow m > 0$

$$m x'' + 25x = 12\cos(36\pi t) \qquad :/m$$

$$\underbrace{x'' + \frac{25}{m}x = \frac{12}{m}\cos(36\pi t)}$$

The given equation is of the type

$$\boxed{x'' + \frac{k}{m}x = A\cos \omega t}$$

$\Rightarrow$ undamped motion with an external force

We have $k = 25$, $A = \frac{12}{m}$, $\omega = 36\pi$

We know

$$x_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t),$$

where $\omega_0 = \sqrt{\frac{25}{m}} = \frac{5}{\sqrt{m}}$ (natural frequency)

Case 1: Suppose $\omega_0 \neq \omega$

A particular solution is

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos \omega t = \frac{\frac{12}{m}}{\frac{25}{m} - \frac{m}{36^2\pi^2}}\cos(36\pi t) =$$

$$= \frac{12}{25 - m(36\pi)^2}\cos(36\pi t)$$

The general solution is

$$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t, \quad c_1, c_2 \in \mathbb{R}$$

$$x = \underbrace{c_1 \cos\left(\frac{5t}{\sqrt{m}}\right)}_{\in [-1,1]} + c_2 \underbrace{\sin\left(\frac{5t}{\sqrt{m}}\right)}_{\in [-1,1]} + \frac{12}{25 - m(36\pi)^2} \underbrace{\cos(36\pi t)}_{\in [-1,1]}$$

$\Rightarrow$ the general solution is bounded, oscillations will occur with <u>bounded amplitude</u>

$\Rightarrow$ <u>no resonance</u>

<u>Case 2</u>: Suppose $\omega = \omega_0$

A particular solution is

$$x_p = \frac{1}{2\omega_0} t \sin(\omega_0 t)$$

$$x_p = \frac{1}{2 \cdot \frac{5}{\sqrt{m}}} \cdot t \sin\left(\frac{5t}{\sqrt{m}}\right)$$

$$x_p = \frac{\sqrt{m}}{10} t \sin\left(\frac{5t}{\sqrt{m}}\right)$$

The general solution is

$$x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{2\omega_0} t \sin(\omega_0 t),$$

$c_1, c_2 \in \mathbb{R}$.

$$x = c_1 \cos\left(\frac{5t}{\sqrt{m}}\right) + c_2 \sin\left(\frac{5t}{\sqrt{m}}\right) + \underbrace{\frac{\sqrt{m}}{10} t \sin\left(\frac{5t}{\sqrt{m}}\right)}_{\text{ubounded}}$$

$\Rightarrow$ oscillations will occur with an amplitude that increases to $\infty \Rightarrow$ resonance

Remember, we need

$$\omega = \omega_0$$

$$36\pi = \frac{5}{\sqrt{m}} \quad (\ )^2$$

$$\frac{25}{m} = 36^2 \pi^2$$

$$\Rightarrow \boxed{m = \frac{25}{36^2 \pi^2}}$$

1.7.25

$\ddot{\theta} + \dot{\theta} + \theta = 0$ 2nd order LHDE with cc.

$r^2 + r + 1 = 0$

$\Delta = 1 - 4 \cdot 1 = -3 = 3i^2$

$$r_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} \Rightarrow \begin{cases} r_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \mapsto e^{-\frac{t}{2}} \cos\left(\frac{t\sqrt{3}}{2}\right) \\ r_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \mapsto \underbrace{e^{-\frac{t}{2}} \sin\left(\frac{t\sqrt{3}}{2}\right)}_{\text{lin. ind.}} \end{cases}$$

By the Fundam. Th. for LHDE $\Rightarrow$

$$\theta = c_1 e^{-\frac{t}{2}} \cos \frac{t\sqrt{3}}{2} + c_2 e^{-\frac{t}{2}} \sin \frac{t\sqrt{3}}{2}, \quad c_1, c_2 \in \mathbb{R}$$

$$\lim_{t\to\infty} \theta(t) = \lim_{t\to\infty} \underbrace{e^{-\frac{t}{2}}}_{\to 0} \underbrace{\left(c_1 \cos \frac{t\sqrt{3}}{2} + c_2 \sin \frac{t\sqrt{3}}{2}\right)}_{\text{bounded}} = 0$$

1.7.29

$$t^2 x'' + 2t x' - 2x = 0 \qquad t \in (0, \infty)$$

a) $x(t) = t^r$

$$\begin{cases} x = t^r \\ x' = r\, t^{r-1} \\ x'' = r(r-1)\, t^{r-2} \end{cases}$$

$$t^2 r(r-1) t^{r-2} + 2t\, r\, t^{r-1} - 2 t^r = 0$$

$$r(r-1) t^r + 2r\, t^r - 2t^r = 0$$

$$t^r (r^2 - r + 2r - 2) = 0$$

$$t^r (r^2 + r - 2) = 0$$

$$t^r (r+2)(r-1) = 0 \Rightarrow \begin{cases} r_1 = 1 \mapsto t \text{ solution} \\ r_2 = -2 \mapsto t^{-2} \text{ solution.} \end{cases}$$

b) $t^2 x'' + 2t x' - 2x = 0$ 2nd order LHDE

We prove that $\{t, t^{-2}\}$ are lin. ind.

$\{t, t^{-2}\}$ lin. ind. if we have:

$$a_1 t + a_2 t^{-2} = 0 \iff a_1 = a_2 = 0, \ \forall t \in \mathbb{R}^*$$

take $t = 1$, $\quad a_1 + a_2 = 0$

take $t = 2$, $\quad -a_1 + a_2 = 0$

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0$$

$\Downarrow$

unique solution

$\oplus$

$2a_2 = 0$

$\boxed{a_2 = 0} \Rightarrow \boxed{a_1 = 0}$

$\Rightarrow a_1 = a_2 = 0$ is the only solution of the system

$\Rightarrow \{t, t^{-2}\}$ lin. ind.

$t^2 x'' + 2t x' - 2x = 0$ 2nd order LHDE $\Bigg\} \xRightarrow[\text{for LHDE}]{\text{Fundam. Th}}$

$\Rightarrow$ the general solution is $x = c_1 t + c_2 t^{-2}$; $c_1, c_2 \in \mathbb{R}$

c) IVP:

$t^2 x'' + 2t x' - 2x = 0$ , $x(1) = 0$ , $x'(1) = 1$

b) $\Rightarrow$ $x = c_1 t + c_2 t^{-2}$ , $x' = c_1 - 2c_2 t^{-3}$

$x(1) = 0 \Rightarrow c_1 + c_2 = 0$

$x'(1) = 1 \Rightarrow c_1 - 2c_2 = 1$

$\ominus$

$3c_2 = -1 \Rightarrow \boxed{c_2 = -\frac{1}{3}} \Rightarrow \boxed{c_1 = \frac{1}{3}}$

$\Rightarrow \boxed{x = \frac{1}{3} t - \frac{1}{3} t^{-2}}$

(1.7.34)

$$\mathcal{L}(x) = x'' + 25x$$

(i) $\mathcal{L}(x) = 0$ , $x(0) = 0$, $x'(0) = 1$

$x'' + 25x = 0$ 2nd order LHDE with cc.

$r^2 + 25 = 0$

$r^2 = -25 = 25i^2$

$\Rightarrow r = \pm 5i \longmapsto \cos 5t, \sin 5t$ lin. ind.

$\xrightarrow[\text{for LHDE}]{\text{Fundam. Th}}$ $x = c_1 \cos(5t) + c_2 \sin(5t)$ ; $c_1, c_2 \in \mathbb{R}$

$x' = -5c_1 \sin(5t) + 5c_2 \cos(5t)$

$x(0) = 0$

$c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow \boxed{c_1 = 0}$

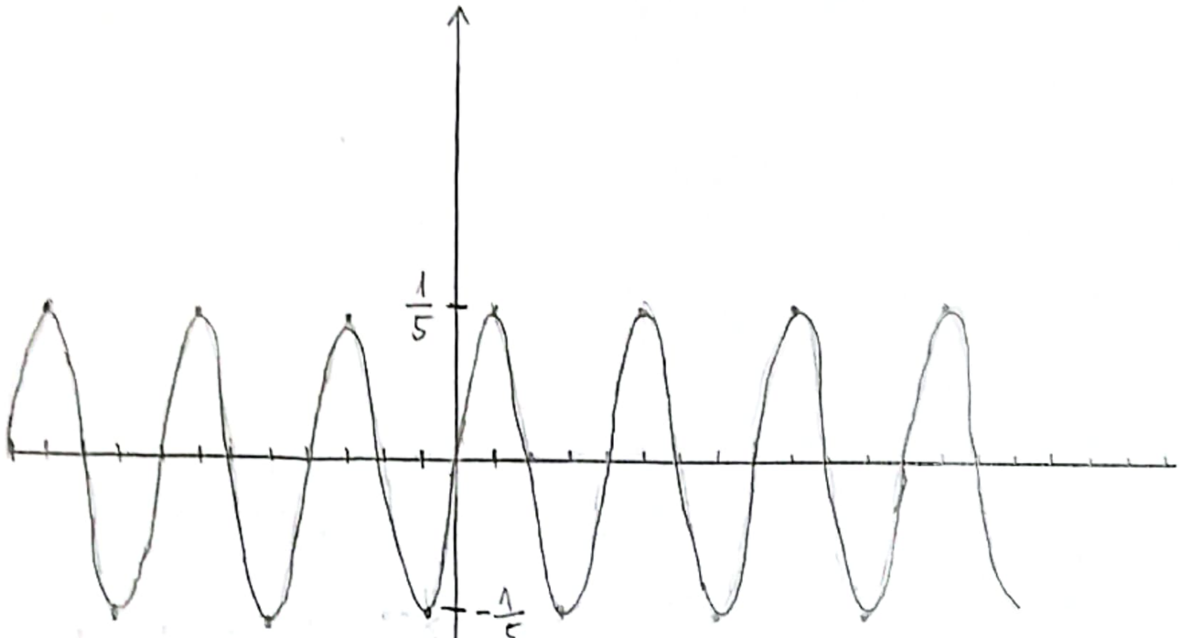
$x'(0) = 1$

$-5c_1 \cdot 0 + 5c_2 \cdot 1 = 1$

$5c_2 = 1 \Rightarrow \boxed{c_2 = \frac{1}{5}}$

$$\Rightarrow x = \frac{\sin 5t}{5}$$

$x = \frac{1}{5}\sin(5t)$




**Long term behaviour:**

Periodic, main period $T = \frac{2\pi}{5}$
oscillatory around 0 with
constant amplitude $\frac{1}{5}$

$\frac{1}{5}\sin(5t) = \frac{1}{5} \Leftrightarrow \sin(5t) = 1 \Leftrightarrow 5t = \frac{\pi}{2} + 2k\pi = \frac{(4k+1)\pi}{2}$

$\Leftrightarrow t = \frac{(4k+1)\pi}{10}$ ; $t = \dots, \frac{-7\pi}{10}, \frac{-3\pi}{10}, \frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10}, \dots$

$\frac{1}{5}\sin(5t) = -\frac{1}{5} \Leftrightarrow \sin(5t) = -1 \Leftrightarrow 5t = \frac{3\pi}{2} + 2k\pi = \frac{(4k+3)\pi}{2}$

$\Leftrightarrow t = \frac{(4k+3)\pi}{10}$ ; $t = \dots, \frac{-5\pi}{10}, \frac{-\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{11\pi}{10}, \frac{15\pi}{10}, \dots$

$\frac{1}{5}\sin(5t) = 0 \Leftrightarrow \sin(5t) = 0 \Leftrightarrow 5t = k\pi \Leftrightarrow t = \frac{k\pi}{5}$

$t = \dots, \frac{-2\pi}{5}, \frac{-\pi}{5}, 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \dots$

(ii) $\varphi_1(t) = t\cos(5t)$

$\varphi_2(t) = t\sin(5t)$

$\mathcal{L}(x) = x'' + 25x$

- $\mathcal{L}(5) = (5)'' + 25 \cdot 5 = 125$

- $\mathcal{L}(\varphi_1) = (t\cos(5t))'' + 25t\cos(5t)$

$= (1 \cdot \cos(5t) + (-5)\sin(5t) \cdot t)' + 25t\cos 5t$

$= (\cos 5t - 5t\sin 5t)' + 25t\cos 5t$

$= -5\sin 5t - (5\sin 5t + 5\cos(5t) \cdot 5t) + 25t\cos 5t$

$= -5\sin 5t - 5\sin 5t \cancel{-25t\cos 5t} \cancel{+25t\cos 5t}$

$= -10\sin 5t$

- $\mathcal{L}(\varphi_2) = (t\sin 5t)'' + 25t\sin 5t$

$= (\sin 5t + 5t\cos 5t)' + 25t\sin 5t$

$= (5\cos 5t + 5\cos 5t + (-5)\sin(5t) \cdot 5t) + 25t\sin 5t$

$= 5\cos 5t + 5\cos 5t \cancel{-25t\sin 5t} \cancel{+25t\sin 5t}$

$= 10\cos 5t$

(iii) constant solution for $\mathcal{L}(x) = 5$

$$x'' + 25x = 5, \quad x \text{ constant}$$

$$\Rightarrow \boxed{x = \frac{1}{5}} \quad \left( \text{Since } \left(\frac{1}{5}\right)'' = 0 \right)$$

(iv) the general solution for $\mathcal{L}(x) = 25 - 25\sin(5t)$

From a) $\Rightarrow x_h = c_1 \cos(5t) + c_2 \sin(5t)$

Now we look for a particular solution

$$x'' + 25x = 25 - 25\sin(5t)$$

We will apply the superposition principle

$$x'' + 25x = f, \text{ where}$$

$$f = \underbrace{5}_{\alpha_1} \cdot \underbrace{5}_{f_1} + \underbrace{\tfrac{5}{2}}_{\alpha_2} \underbrace{(-10)\sin 5t}_{f_2}$$

From (iii), we know $x_{p_1} = \frac{1}{5}$ is a part. solution of $x'' + 25x = 5$

From (ii), we know $x_{p_2} = y_1 = t\cos(5t)$ is a particular solution of $x'' + 25x = -10\sin 5t$ $\Rightarrow$

$\Rightarrow$ By the Superposition Principle we know that

$$x_p = \alpha_1 x_{p_1} + \alpha_2 x_{p_2} = 5 \cdot \frac{1}{5} + \frac{5}{2} \cdot t\cos 5t = 1 + \frac{5t\cos 5t}{2}$$

is a particular solution of $\mathcal{L}(x) = 25 - 25\sin(5t)$

By the Fundamental Th. for Ln-HDE $\Rightarrow$ the general solution is

$$\boxed{x = 1 + \frac{5t\cos 5t}{2} + c_1 \cos 5t + c_2 \sin 5t}, \quad c_1, c_2 \in \mathbb{R}$$

1.7.35

$$x' + \frac{1}{t^2} x = 0 \quad , \quad t \in (-\infty, 0)$$

a) For $x = e^{\frac{1}{t}}$ we have:

$$\left(e^{\frac{1}{t}}\right)' + \frac{1}{t^2} e^{\frac{1}{t}} = (t^{-1})' e^{\frac{1}{t}} + \frac{1}{t^2} e^{\frac{1}{t}} =$$

$$= -1 \cdot \frac{1}{t^2} e^{\frac{1}{t}} + \frac{1}{t^2} e^{\frac{1}{t}} = 0$$

$\Rightarrow x = e^{\frac{1}{t}}$ is a solution

b) IVP: $x' + \frac{1}{t^2} x = 0 \quad , \quad x(-1) = 1$

$$\frac{dx}{dt} + \frac{1}{t^2} x = 0$$

We will use the integrating factor $\mu(t) = e^{\int \frac{1}{t^2} dt}$

$$\mu(t) = e^{\int t^{-2} dt} = e^{-t^{-1}}$$

$$\frac{dx}{dt} + \frac{1}{t^2} x = 0 \quad \cdot \Big| e^{-\frac{1}{t}}$$

$$e^{-\frac{1}{t}} \frac{dx}{dt} + \frac{1}{t^2} e^{-\frac{1}{t}} x = 0$$

$\left(x \cdot e^{-\frac{1}{t}}\right)' = 0$ | Integrate w.r.t. $t$

$x \cdot e^{-\frac{1}{t}} = C$

$x = Ce^{\frac{1}{t}}$

(Note: We also could've just used the answer from a), apply the Fundam. Th. for LHDE, and arrive at the same result)

$\Rightarrow$ the solution is $x = c_1 e^{\frac{1}{t}}$ ; $c_1 \in \mathbb{R}$

$x(-1) = 1$

$c_1 e^{-1} = 1 \quad \cdot | e$

$c_1 = e \Rightarrow \quad x = e \cdot e^{\frac{1}{t}}$

$x = e^{t/t + \frac{1}{t}} \Rightarrow \boxed{x = e^{\frac{t+1}{t}}}$

c) general solution of $x' + \frac{1}{t^2} x = 1 + \frac{1}{t}$, $t \in (-\infty, 0)$

From b) $\Rightarrow x_h = c_1 e^{\frac{1}{t}}$

For $x_p = t$ we have

$t' + \frac{1}{t^2} \cdot t = 1 + \frac{1}{t} \Rightarrow x_p = t$ particular solution

Fundam Th. of Ln-HDE $\Rightarrow$ the general sol is $\boxed{x = t + c_1 e^{\frac{1}{t}}}$, $c_1 \in \mathbb{R}$

1.6.1

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$\boxed{X' = AX}$

• characteristic equation method

$\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = 2$

$Au = \lambda u$

• $Au = 2u$

$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix}$

$\begin{cases} 2a = 2a \\ a + 2b = 2b \Rightarrow a = 0,\ b \in \mathbb{R} \end{cases}$

$\Rightarrow u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We have that:

$\lambda_{1,2} = 2 \in \mathbb{R}$

$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A has only one eigenvector, but it needs 2 in order to be diagonalizable $\Rightarrow$ A is <u>not</u> diagonalizable

- reduction to second order equation method

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\boxed{X' = AX}$$

$$\begin{cases} x' = 2x \\ y' = x + 2y \end{cases}$$ Coupled system

$x' = 2x \Rightarrow \boxed{x = c_1 e^{2t}}$, $c_1 \in \mathbb{R}$

$$y' = c_1 e^{2t} + 2y$$

$$y' - 2y = c_1 e^{2t}$$

$$\frac{dy}{dt} - 2y = c_1 e^{2t}$$

We'll use the integrating factor $\mu(t) = e^{\int -2\,dt} = e^{-2t}$

$$\frac{dy}{dt} - 2y = c_1 e^{2t} \quad \Big| \cdot e^{-2t}$$

$$e^{-2t}\frac{dy}{dt} - 2y\,e^{-2t} = c_1$$

$$(e^{-2t} \cdot y)' = c_1 \quad \Big| \text{Integrate w.r.t. } t$$

$$\int (e^{-2t} \cdot y)' \, dt = \int c_1 \, dt$$

$$e^{-2t} \cdot y = c_1 t + c_2 \quad , c_2 \in \mathbb{R}.$$

$$\Rightarrow \boxed{y = c_1 t \cdot e^{2t} + c_2 e^{2t}} , \quad c_1, c_2 \in \mathbb{R}$$

$\Rightarrow$ the general solution is $X = c_1 e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}\right) + c_2 e^{2t}\begin{pmatrix}0\\1\end{pmatrix}$

Let $X_1 = c_1 e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}\right)$, $c_1 \in \mathbb{R}$.

$X_2 = c_2 e^{2t}\begin{pmatrix}0\\1\end{pmatrix}$, $c_2 \in \mathbb{R}$.

$$\Rightarrow X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}\right) & c_2 e^{2t}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}$$

Fundamental matrix solution

$X(0) = \begin{pmatrix}1\\0\end{pmatrix}$

$$\begin{cases} c_1 \cdot 1 (1+0) + c_2 \cdot 1 \cdot 0 = 1 \\ c_1 \cdot 1 (0+0) + c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

$X(0) = \begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{cases} c_1 \cdot 1 (1+0) + c_2 \cdot 1 \cdot 0 = 0 \\ c_1 \cdot 1 (0+0) + c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = 0 \\ c_2 = 1 \end{cases}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) & e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} e^{2t} & 0 \\ t & e^{2t} \end{pmatrix}$$ The principal matrix solution

k) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$

$\boxed{X' = AX}$

- <u>characteristic eq method</u>

$\det(A - \lambda I_2) = 0$

$\begin{vmatrix} -\lambda & 4 \\ 5 & 1-\lambda \end{vmatrix} = 0$

$-\lambda + \lambda^2 - 20 = 0$

$\lambda^2 - \lambda - 20 = 0$

$(\lambda - 5)(\lambda + 4) = 0 \Rightarrow \lambda_1 = 5; \; \lambda_2 = -4$

$Au = \lambda u$

- $Au_1 = 5u_1$

$\begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5a \\ 5b \end{pmatrix}$

$\begin{cases} 4b = 5a \\ 5a + b = 5b \Rightarrow 4b = 5a \end{cases}$

Take $a = 4$, $b = 5$

$\Rightarrow u_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

- $Au_2 = -4u_2$

$\begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4a \\ -4b \end{pmatrix}$

$\begin{cases} 4b = -4a \Rightarrow a = -b \\ 5a + b = -4b \Rightarrow a = -b \end{cases}$

Take $a = 1$, $b = -1$

$\Rightarrow u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_1 = 5, \ \lambda_2 = -4 \in \mathbb{R}.$

$u_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{vmatrix} 4 & 1 \\ 5 & -1 \end{vmatrix} = -4 - 5 = -9 \neq 0 \Rightarrow u_1, u_2$ lin ind

$\Rightarrow$

$\Rightarrow A$ diagonalizable.

We have that $e^{5t}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are lin. ind. sol. of $X' = AX$.

$\Rightarrow X = c_1 e^{5t}\begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ the general solution of $X' = AX$. $(c_1, c_2 \in \mathbb{R})$

$U(t) = \begin{pmatrix} 4c_1 e^{5t} & c_2 e^{-4t} \\ 5c_1 e^{5t} & -c_2 e^{-4t} \end{pmatrix}$ Fundamental matrix sol.

$U_1(t) = \begin{pmatrix} 4e^{5t} & e^{-4t} \\ 5e^{5t} & -e^{-4t} \end{pmatrix}$

$P = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \qquad \det(P) = -4 - 5 = -9$

$\Rightarrow P^{-1} = \frac{1}{-9}\begin{pmatrix} -1 & -1 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{5}{9} & -\frac{4}{9} \end{pmatrix}$

$$\operatorname{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{5}{9} & -\frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} \frac{e^{5t}}{9} & \frac{e^{5t}}{9} \\ \frac{5e^{-4t}}{9} & -\frac{4e^{-4t}}{9} \end{pmatrix} = \begin{pmatrix} \frac{4e^{5t}}{9} + \frac{5e^{-4t}}{9} & \frac{4e^{5t}}{9} - \frac{4e^{-4t}}{9} \\ \frac{5e^{5t}}{9} - \frac{5e^{-4t}}{9} & \frac{5e^{5t}}{9} + \frac{4e^{-4t}}{9} \end{pmatrix}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} \frac{1}{9}\left(4e^{5t} + 5e^{-4t}\right) & \frac{4}{9}\left(e^{5t} - 4e^{-4t}\right) \\ \frac{5}{9}\left(e^{5t} - e^{-4t}\right) & \frac{1}{9}\left(5e^{5t} + 4e^{-4t}\right) \end{pmatrix}$$

the principal matrix solution

• reduction to second order equation method

$$A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$$

$$X' = AX$$

$$\begin{cases} x' = 4y \\ y' = 5x + y \end{cases}$$ Coupled system

$$\boxed{y = \frac{1}{4}x'}$$

$$x'' = 4y' = 4(5x + y) = 4\left(5x + \frac{1}{4}x'\right)$$

$$\Rightarrow x'' - 20x - x' = 0$$

$$x'' - x' - 20x = 0$$

$$r^2 - r - 20 = 0$$

$$(r-5)(r+4) = 0$$

$$\Rightarrow r_1 = 5 \longmapsto e^{5t}$$
$$r_2 = -4 \longmapsto e^{-4t}$$

$$\Rightarrow \boxed{x = c_1 e^{-4t} + c_2 e^{5t}}, \quad c_1, c_2 \in \mathbb{R}.$$

$$x' = -4c_1 e^{-4t} + 5c_2 e^{5t}$$

$$y = \frac{1}{4}x' = \frac{1}{4}\left(-4c_1 e^{-4t} + 5c_2 e^{5t}\right) \Rightarrow \boxed{y = -c_1 e^{-4t} + \frac{5}{4}c_2 e^{5t}},$$

$c_1, c_2 \in \mathbb{R}$

⇒ the general solution is

$$X = c_1 e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{5t}\begin{pmatrix}1\\\frac{5}{4}\end{pmatrix}$$

$$X = c_1 e^{-4t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{5t}\begin{pmatrix}4\\5\end{pmatrix} \quad , \quad c_1, c_2 \in \mathbb{R}$$

m) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

· <u>characteristic equation method</u>

$X' = AX$

$\det(A - \lambda I_2) = 0.$

$$\begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = 0$$

$\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = (2i)^2 \Rightarrow \lambda_{1,2} = \pm 2i$

$\Rightarrow$ the eigenvalues of A are <u>not</u> real

$\Rightarrow$ A is <u>not</u> diagonalisable.

- reduction to second order equation method

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$\begin{cases} x' = -2y \\ y' = 2x \end{cases}$$ coupled system

$$x'' = -2y' = -2 \cdot 2x = -4x$$

$$\Rightarrow x'' + 4x = 0$$

$$r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm 2i \longmapsto \cos(2t),\ \sin(2t)$$

$$\Rightarrow \boxed{x = c_1 \cdot \cos(2t) + c_2 \cdot \sin 2t}, \quad c_1, c_2 \in \mathbb{R}.$$

$$x' = -2y \Rightarrow y = -\frac{1}{2}x'$$

$$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

$$\Rightarrow \boxed{y = c_1 \sin(2t) - c_2 \cos(2t)}, \quad c_1, c_2 \in \mathbb{R}$$

$\Rightarrow$ the general solution is

$$\boxed{X = c_1\left(\cos(2t)\begin{pmatrix}1\\0\end{pmatrix} + \sin(2t)\begin{pmatrix}0\\1\end{pmatrix}\right) + c_2\left(\sin(2t)\begin{pmatrix}1\\0\end{pmatrix} - \cos(2t)\begin{pmatrix}0\\1\end{pmatrix}\right)},$$

Let $X_1 = c_1\left(\cos(2t)\begin{pmatrix}1\\0\end{pmatrix} + \sin(2t)\begin{pmatrix}0\\1\end{pmatrix}\right)$, $c_1 \in \mathbb{R}$

$X_2 = c_2\left(\sin(2t)\begin{pmatrix}1\\0\end{pmatrix} - \cos(2t)\begin{pmatrix}0\\1\end{pmatrix}\right)$, $c_2 \in \mathbb{R}$.

$$\Rightarrow X = (X_1 \;\; X_2) = \left(c_1\left(\cos(2t)\begin{pmatrix}1\\0\end{pmatrix} + \sin(2t)\begin{pmatrix}0\\1\end{pmatrix}\right) \quad c_2\left(\sin(2t)\begin{pmatrix}0\\1\end{pmatrix} - \cos(2t)\begin{pmatrix}0\\1\end{pmatrix}\right)\right),$$

$c_1, c_2 \in \mathbb{R}$

Fundamental matrix solution

$$X(0) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{cases} c_1 \cdot (1\cdot 1 + 0 \cdot 0) + c_2(0\cdot 0 - 1 \cdot 0) = 1 \\ c_1 \cdot (1 \cdot 0 + 0 \cdot 1) + c_2(0 \cdot 1 - 1 \cdot 1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 1 \\ -c_2 = 0 \Rightarrow c_2 = 0 \end{cases}$$

$$X(0) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{cases} c_1 \cdot (1\cdot 1 + 0 \cdot 0) + c_2(0\cdot 0 - 1 \cdot 0) = 0 \\ c_1 \cdot (1 \cdot 0 + 0 \cdot 1) + c_2(0 \cdot 1 - 1 \cdot 1) = 1 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ -c_2 = 1 \Rightarrow c_2 = -1 \end{cases}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$$ the principal matrix solution