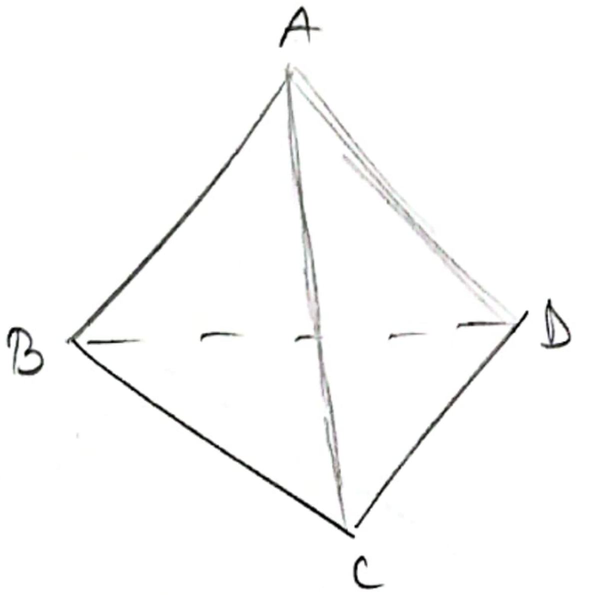
(3)



$A(1,0,1)$; $B(0,-1,1)$;
$C(-1,1,1)$; $D(2,1,0)$

l_1 - common perp. of AB, CD

l_2 - common perp of AD, BC

$\text{dist}(l_1, l_2) = ?$

$$AB: \frac{x-1}{0-1} = \frac{y-0}{-1-0} = \frac{z-1}{1-0}$$

$$AB: \frac{x-1}{-1} = \frac{y}{-1} = \frac{z-1}{1} \qquad \overrightarrow{AB}(-1,-1,1)$$

$$CD: \frac{x+1}{2+1} = \frac{y-1}{1-1} = \frac{z-1}{0-1}$$

$$CD: \frac{x+1}{3} = \frac{y-1}{1-1} = \frac{z-1}{-1} \quad (=t)$$

$\to$ Not good !!

$$\to \boxed{y_{CD} = 1}$$

$$\to \overrightarrow{CD}(3,0,-1)$$

$$\left(CD: \begin{cases} x = 3t+1 \\ y = 1 \\ z = -t+1 \end{cases} \right)$$

$$\begin{cases} A(1,0,1) \in AB \\ C(-1,1,1) \in CD \end{cases}$$

- We find l_1, common perp of AB, CD

$\Pi_1 :=$ plane that contains AB and is $\parallel$ to $\vec{AB} \times \vec{CD}$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 1 \\ 3 & 0 & -1 \end{vmatrix} = (1-0)\vec{i} - (1-3)\vec{j} + (0+3)\vec{k}$$
$$= \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\rightarrow \vec{AB} \times \vec{CD}(1,2,3)$$

$$\Pi_1 : \begin{vmatrix} x-1 & y & z-1 \\ -1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0 \qquad \begin{cases} C_1 := C_1 + C_3 \\ C_2 := C_2 + C_3 \end{cases}$$

$$\begin{vmatrix} x+z-2 & y+z-1 & z-1 \\ 0 & 0 & 1 \\ 4 & 5 & 3 \end{vmatrix} = 0$$

$$-(5(x+z-2) - 4(y+z-1)) = 0$$
$$4y + 4z - 4 - 5x - 5z + 10 = 0$$

$$\boxed{\Pi_1 : \ -5x + 4y - z + 6 = 0}$$

$\Pi_2 :=$ plane that contains CD and is $\parallel$ to $\vec{AB} \times \vec{CD}$

$$\Pi_2 : \begin{vmatrix} x+1 & y-1 & z-1 \\ 3 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 0 \qquad C_1 := C_1 + 3C_3$$

$$\begin{vmatrix} x+1+3z-3 & y-1 & z-1 \\ 0 & 0 & -1 \\ 10 & 2 & 3 \end{vmatrix} = 0$$

$$2(x+3z-2) - 10(y-1) = 0$$

$$2x + 6z - 4 - 10y + 10 = 0$$

$$2x - 10y + 6z + 6 = 0$$

$$\boxed{\Pi_2 : x - 5y + 3z + 3 = 0}$$

$$\Rightarrow l_1 = \Pi_1 \cap \Pi_2 \rightarrow \boxed{(l_1) : \begin{cases} -5x + 4y - z + 6 = 0 \\ x - 5y + 3z + 3 = 0 \end{cases}}$$

• We find l_2, common perp of AD, BC

$$AD: \frac{x-1}{2-1} = \frac{y-0}{1-0} = \frac{z-1}{0-1}$$

$$AD: \frac{x-1}{1} = \frac{y}{1} = \frac{z-1}{-1} \qquad \vec{AD}(1,1,-1)$$

$$BC: \frac{x-0}{-1-0} = \frac{y+1}{1+1} = \frac{z-1}{1-1}$$

$$BC: \frac{x}{-1} = \frac{y+1}{2} = \frac{z-1}{1-1} \quad (=s)$$

$\longrightarrow$ Not good!!

$$\Rightarrow \boxed{z_{BC} = 1}$$

$$\left(BC: \begin{cases} x = -s \\ y = 2s-1 \\ z = 1 \end{cases} \right) \Rightarrow \vec{BC}(-1,2,0)$$

$\pi_3 :=$ plane that contains AD and is $\parallel$ to $\vec{AD} \times \vec{BC}$

$$\vec{AD} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 0 \end{vmatrix} = (0+2)\vec{i} - (0-1)\vec{j} + (2+1)\vec{k}$$

$$= 2\vec{i} + \vec{j} + 3\vec{k}$$

$$\Rightarrow \vec{AD} \times \vec{BC}(2,1,3)$$

$$\Pi_3 : \begin{vmatrix} x-1 & y & z-1 \\ 1 & 1 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 0 \qquad \begin{array}{l} C_1 := C_1 + C_3 \\ C_2 := C_2 + C_3 \end{array}$$

$$\Pi_3 : \begin{vmatrix} x+z-2 & y+z-1 & z-1 \\ 0 & 0 & -1 \\ 5 & 4 & 3 \end{vmatrix} = 0$$

$$4(x+z-2) - 5(y+z-1) = 0$$

$$4x + 4z - 8 - 5y - 5z + 5 = 0$$

$$\boxed{\Pi_3 : 4x - 5y - z - 3 = 0}$$

Π_4 := plane that contains BC and is $\parallel$ to $\vec{AD} \times \vec{BC}$

$$\Pi_4 : \begin{vmatrix} x+1 & y-1 & z-1 \\ -1 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix} = 0 \qquad C_2 := C_2 + 2C_1$$

$$\begin{vmatrix} x+1 & y+2x+2-1 & z-1 \\ -1 & 0 & 0 \\ 2 & 5 & 3 \end{vmatrix} = 0$$

$$3(2x+y+1)-5(z-1)=0$$

$$6x+3y+3-5z+5=0$$

$$\boxed{\Pi_4: 6x+3y-5z+8=0}$$

$$l_2 = \Pi_1 \cap \Pi_2 \Rightarrow \boxed{(l_2): \begin{cases} 4x-5y-z-3=0 \\ 6x+3y-5z+8=0 \end{cases}}$$

We have $(l_1): \begin{cases} -5x+4y-z+6=0 \\ x-5y+3z+3=0 \end{cases}$

$(l_2): \begin{cases} 4x-5y-z-3=0 \\ 6x+3y-5z+8=0 \end{cases}$

$$l_1: \begin{cases} -15x+12y-3z+18=0 \\ x-5y+3z+3=0 \end{cases} \quad (+)$$

$$-14x+7y=-21$$

$$-2x+y=-3$$

$$-2x=-y-3$$

$$2x=y+3$$

$$\boxed{x=\frac{1}{2}y+\frac{3}{2}}$$

$$z = 4x - 5y - 3$$

$$z = 2y + 6 - 5y - 3$$

$$z = -3y + 3$$

$$\Rightarrow (\ell_1): \begin{cases} x = \frac{1}{2}\alpha + \frac{3}{2} \\ y = \alpha \\ z = -3\alpha + 3 \end{cases} \qquad \alpha \in \mathbb{R}$$

$$\Rightarrow \boxed{\vec{\ell_1}(1, 2, -6)}$$

$$\ell_2: \begin{cases} -20x + 25y + 5z + 15 = 0 \\ 6x + 3y - 5z + 8 = 0 \end{cases} \quad (+)$$

$$-14x + 28y + 23 = 0$$

$$14x = 28y + 23$$

$$x = 2y + \frac{23}{14}$$

$$z = 4x - 5y - 3 = 8y + \frac{\cancel{4}^{2} \cdot 23}{\cancel{14}_{7}} - 5y - \frac{7}{3}$$

$$z = 3y + \frac{46 - 21}{7}$$

$$z = 3y + \frac{25}{7}$$

$$\Rightarrow (\ell_2): \begin{cases} x = 2\beta + \frac{23}{14} \\ y = \beta & \beta \in \mathbb{R}. \\ z = 3\beta + \frac{25}{7} \quad \left(= 3\beta + \frac{50}{14}\right) \end{cases}$$

$$\Rightarrow \boxed{\vec{\ell_2}\,(2,1,3)}$$

We pick $A_1 \in \ell_1$ by choosing $\alpha = 1$

$$\Rightarrow A_1(2,1,0)$$

We pick $A_2 \in \ell_2$ by choosing $\beta = 0$

$$\Rightarrow A_2\left(\frac{23}{14}, 0, \frac{25}{7}\right)$$

$$\overrightarrow{A_1A_2}\left(\frac{23}{14} - \overset{\frac{14}{}}{2}, 0-1, \frac{25}{7} - 0\right)$$

$$\overrightarrow{A_1A_2}\left(-\frac{5}{14}, -1, \frac{25}{7}\right)$$

$$\text{dist}(l_1, l_2) = \frac{\left|\left(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}\right)\right|}{\|\vec{l_1} \times \vec{l_2}\|}$$

$$\vec{l_1} \times \vec{l_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -6 \\ 2 & 1 & 3 \end{vmatrix} = (6+6)\vec{i} - (3+12)\vec{j} + (1-4)\vec{k}$$

$$= 12\vec{i} - 15\vec{j} - 3\vec{k}$$

$$\rightarrow \vec{l_1} \times \vec{l_2}\,(4, -5, -1)$$

$$\|\vec{l_1} \times \vec{l_2}\| = \sqrt{16+25+1} = \sqrt{42}$$

$$\left(\overrightarrow{A_1A_2}, \vec{l_1}, \vec{l_2}\right) = \begin{vmatrix} -\frac{5}{14} & -1 & \frac{25}{7} \\ 1 & 2 & -6 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= -\frac{5}{14}(6+6) + 1(3+12) + \frac{25}{7}(1-4)$$

$$= -\frac{5}{\cancel{14}_{7}} \cdot \cancel{12}^{6} + 15 + \frac{25}{7} \cdot (-3)$$

$$= -\frac{30}{7} + 15 - \frac{75}{7}$$

$$= -\frac{105}{7} + {}^{7/}15 = \frac{-105+105}{7} = 0 \rightarrow \boxed{\text{dist}(l_1, l_2) = \frac{0}{\sqrt{42}} = 0}$$

⑥ $l_1: \frac{x+1}{3} = \frac{y}{5} = \frac{z-1}{2}$ $\quad \vec{l_1}(3,5,2)$

$l_2: \frac{x}{-7} = \frac{y+2}{2} = \frac{z}{4}$ $\quad \vec{l_2}(-7,2,4)$

$l_3: \frac{x+2}{6} = \frac{y-1}{2} = \frac{z-1}{5}$ $\quad \vec{l_3}(6,2,5)$

$l_4: \frac{x-1}{4} = \frac{y+6}{2} = \frac{z-4}{4}$ $\quad \vec{l_4}(4,2,4)$

$l_{1,2}$ common perp of l_1 and l_2 $\Rightarrow \begin{cases} l_{1,2} \perp l_1 \\ l_{1,2} \perp l_2 \end{cases} \Rightarrow \begin{cases} \vec{l_{1,2}} \perp \vec{l_1} \\ \vec{l_{1,2}} \perp \vec{l_2} \end{cases}$

Let $\vec{l_{1,2}}(a,b,c)$

$\vec{l_{1,2}} \perp \vec{l_1} \Rightarrow 3a+5b+2c=0 \quad \cdot 2$

$\vec{l_{1,2}} \perp \vec{l_2} \Rightarrow -7a+2b+4c=0$

$\Leftrightarrow \begin{cases} 6a+10b+4c=0 \\ -7a+2b+4c=0 \end{cases} \quad \ominus$

$13a+8b=0$

$13a=-8b$

$\boxed{a=-\frac{8}{13}b}$

$2c=-3a-5b$

$2c=-3\cdot\left(-\frac{8}{13}\right)b-5b = \frac{24}{13}b - \overset{13}{5b} = \frac{24b-65b}{13}$

$2c = \frac{41b}{13} \Rightarrow \boxed{c=\frac{41}{26}b}$

$\Rightarrow \boxed{\overrightarrow{l_{1,2}}(-16, 41, 26)}$

$l_{3,4}$ common perp of l_3 and l_4 $\Rightarrow \begin{cases} l_{3,4} \perp l_3 \\ l_{3,4} \perp l_4 \end{cases} \Rightarrow \begin{cases} \overrightarrow{l_{3,4}} \perp \overrightarrow{l_3} \\ \overrightarrow{l_{3,4}} \perp \overrightarrow{l_4} \end{cases}$

Let $\overrightarrow{l_{3,4}}(x, y, z)$

$\overrightarrow{l_{3,4}} \perp \overrightarrow{l_3} \Rightarrow \quad 6x + 2y + 5z = 0$

$\overrightarrow{l_{3,4}} \perp \overrightarrow{l_4} \Rightarrow \quad 4x + 2y + 4z = 0$ $\ominus$

$2x + z = 0$

$\boxed{z = -2x}$

$2y = -4x - 4z = -4x - 4\cdot(-2x) = -4x + 8x = 4x$

$\Rightarrow \boxed{y = 2x}$

$\Rightarrow \boxed{\overrightarrow{l_{3,4}}(1, 2, -2)}$

l common perp of $l_{1,2}$ and $l_{3,4}$ $\Rightarrow \begin{cases} l \perp l_{1,2} \\ l \perp l_{3,4} \end{cases} \Rightarrow \begin{cases} \vec{l} \perp \overrightarrow{l_{1,2}} \\ \vec{l} \perp \overrightarrow{l_{3,4}} \end{cases}$

Let $\vec{l}(m, n, p)$

$\vec{l} \perp \overrightarrow{l_{1,2}} \Rightarrow -16m + 41n + 26p = 0$

$\vec{l} \perp \overrightarrow{l_{3,4}} \Rightarrow m + 2n - 2p = 0 \quad \cdot | 13$

$$\begin{array}{r} -16m + 41n + 26p = 0 \\ 13m + 26n - 26p = 0 \\ \hline -3m + 15n = 0 \end{array} \oplus$$

$-m + 5n = 0$

$\boxed{m = 5n}$

$2p = m + 2n$

$2p = 5n + 2n = 7n \Rightarrow \boxed{p = \frac{7}{2} n}$

$\Rightarrow \boxed{\vec{l}(10, 2, 7)}$

$\vec{l}(10,2,7)$

Let $\pi: Ax+By+Cz+D=0$

$l \perp \pi \Rightarrow \vec{l} \perp \pi \Rightarrow \pi: 10x+2y+7z+D=0$

$P(1,3,9) \in \pi \Rightarrow 10\cdot 1+2\cdot 3+7\cdot 9+D=0$

$D=-10-6-63$

$D=-79$

$\Rightarrow \boxed{\pi: 10x+2y+7z-79=0}$