a) $S_1: z = x^2 + y^2$

$f(x,y,z) = x^2 + y^2 - z$

$$\begin{cases} f'_x(x,y,z) = 2x \\ f'_y(x,y,z) = 2y \\ f'_z(x,y,z) = -1 \end{cases}$$

$T_\varphi(x_0,y_0,z_0): \ 2x_0(x-x_0) + 2y_0(y-y_0) - (z-z_0) = 0$

$T_\varphi(x_0,y_0,z_0): \ 2x_0x + 2y_0y - z - 2x_0^2 - 2y_0^2 + z_0 = 0$

$\vec{n}_{T_\varphi(x_0,y_0,z_0)}(2x_0, 2y_0, -1)$

$T_\varphi(x_0,y_0,z_0) \parallel \vec{v}(1,-1,0) \Rightarrow \vec{n}_{T_\varphi(x_0,y_0,z_0)} \cdot \vec{v} = 0$

$\Rightarrow \ 2x_0 - 2y_0 = 0$

$2x_0 = 2y_0$

$x_0 = y_0$

$(x_0,y_0,z_0) \in S_1 \Rightarrow z_0 = x_0^2 + y_0^2 = x_0^2 + x_0^2 = 2x_0^2$

$\Rightarrow \underline{A = \{(x_0, x_0, 2x_0^2) \mid x_0 \in \mathbb{R}\}}$

b) S_2 conoidal surface

intersect ℓ: $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z-3}{3}$ $\vec{v}(2,1,3)$

$\parallel$ to Π: $x+z=0$

$\mathcal{C}: \begin{cases} x=0 \\ z=x^2+y^2 \end{cases}$

$\ell: \begin{cases} \frac{x-2}{2}=\frac{y-1}{1} \\ \frac{x-2}{2}=\frac{z-3}{3} \end{cases} \Leftrightarrow \ell: \begin{cases} x-2=2y-2 \\ 3x-6=2z-6 \end{cases}$

$\Leftrightarrow \ell: \begin{cases} x-2y=0 \\ 3x-2z=0 \end{cases}$

$\Rightarrow d_{\lambda,\mu}: \begin{cases} \boxed{x+z=\lambda} \\ x-2y+\mu(3x-2z)=0 \Rightarrow \boxed{\mu=\frac{2y-x}{3x-2z}} \end{cases}$

$\begin{cases} x=0 \\ x+z=\lambda \\ x-2y+\mu(3x-2z)=0 \\ x^2+y^2=z \end{cases}$

$$\begin{cases} x=0 \\ z=\lambda \\ x-2y+\mu(3x-2z)=0 \Rightarrow -2y+\mu(-2\lambda)=0 \Rightarrow \\ x^2+y^2=z \end{cases}$$

$$\Rightarrow \quad -2y-2\mu\lambda=0$$

$$2y+2\mu\lambda=0$$

$$y+\mu\lambda=0$$

$$y=\mu\lambda$$

Replace x, y, z

$$0^2+\mu^2\lambda^2=\lambda^2$$

$$\mu^2\lambda^2=\lambda^2$$

Replace μ, λ

$$S_2: \left(\frac{2y-x}{3x-2z}\right)^2 \cdot (x+z)^2 = (x+z)^2$$

(we don't simplify. $x+z$ might be 0. We could factorise, tough)

Final equation

c)

Take $P(1,1,1)$

$$\left(\frac{2-1}{3-2}\right)^2 \cdot 2^2 = 2^2$$

$$1 \cdot 2^2 = 2^2 \Rightarrow P \in S_2$$

$$1+1=2 \neq 0 \Rightarrow P \notin \Pi$$

$$f(x,y,z) = (x+z)^2\left(\left(\frac{2y-x}{3x-2z}\right)^2 - 1\right)$$

$$= (x^2+2xz+z^2)\cdot\left(\frac{4y^2-4xy+x^2}{9x^2-12xz+4z^2} - 1\right)$$

$$= (x^2+2xz+z^2).\ \frac{4y^2-4xy+x^2-9x^2+12xz-4z^2}{9x^2-12xz+4z^2}$$

$$= \frac{(x^2+2xz+z^2)(4y^2-4xy-8x^2+12xz-4z^2)}{9x^2-12xz+4z^2}$$

$$f'_x(1,1,1) = \frac{-144+252-36-32+96-228-120+84+168-40-64-32-80-16}{1} = -192$$

$$f'_y(1,1,1) = \frac{(1+2+1)(8-4)}{9-12+4} = \frac{4\cdot 4}{1} = \underline{\underline{16}}$$

$$f'_z(1,1,1) = \frac{-132+280-120-20+120-40-240+40+80+160-80-32}{1} = \underline{\underline{16}}$$

$$\underbrace{N_\varphi(1,1,1):\ \frac{x-1}{-192} = \frac{y-1}{16} = \frac{z-1}{16}}$$

Used formula $N_\varphi(x_0,y_0,z_0)$: $\dfrac{x-x_0}{f'_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{f'_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{f'_z(x_0,y_0,z_0)}$

Found derivatives with calculator. Didn't write them down, too long. Too little time.