Einführung in die Computergrafik
(Aufgabenblatt 1)

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von

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1 Matrizen

Aufgabe 1. Matrizen. **(4 Punkte)**

Zeigen Sie, dass für alle Matrizen $A \in M^{n \times m}$, alle Vektoren $v, w \in \mathbb{R}^m$ und alle $\lambda \in \mathbb{R}$ gilt:

(a) $A(v + w) = Av + Aw$.

(b) $A(\lambda v) = \lambda Av$.

1.1 Lösung (a)

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_j \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_j \end{pmatrix}$$

Es soll gelten: $A(v + w) = Av + Aw$

$$A(v + w) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix} \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ ... \\ v_j + w_j \end{pmatrix}$$

$$A(v + w) = \begin{pmatrix} a_{11}(v_1 + w_1) + a_{12}(v_2 + w_2) + ... + a_{1j}(v_j + w_j) \\ a_{21}(v_1 + w_1) + a_{22}(v_2 + w_2) + ... + a_{2j}(v_j + w_j) \\ ... \\ a_{i1}(v_1 + w_1) + a_{i2}(v_2 + w_2) + ... + a_{ij}(v_j + w_j) \end{pmatrix}$$

$$A(v + w) = \begin{pmatrix} a_{11}v_1 + a_{11}w_1 + a_{12}v_2 + a_{12}w_2 + ... + a_{1j}v_j + a_{1j}w_j \\ a_{21}v_1 + a_{21}w_1 + a_{22}v_2 + a_{22}w_2 + ... + a_{2j}v_j + a_{2j}w_j \\ ... \\ a_{i1}v_1 + a_{i1}w_1 + a_{i2}v_2 + a_{i2}w_2 + ... + a_{ij}v_j + a_{ij}w_j \end{pmatrix}$$

$$A(v+w) = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + ... + a_{1j}v_j \\ a_{21}v_1 + a_{22}v_2 + ... + a_{2j}v_j \\ ... \\ a_{i1}v_1 + a_{i2}v_2 + ... + a_{ij}v_j \end{pmatrix} + \begin{pmatrix} a_{11}w_1 + a_{12}w_2 + ... + a_{1j}w_j \\ a_{21}w_1 + a_{22}w_2 + ... + a_{2j}w_j \\ ... \\ a_{i1}w_1 + a_{i2}w_2 + ... + a_{ij}w_j \end{pmatrix}$$

$$A(v+w) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_j \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_j \end{pmatrix}$$

$\Rightarrow A(v + w) = Av + Aw$

1.2 Lösung (b)

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix}, \, v = \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_j \end{pmatrix}$$

Es soll gelten: $A(\lambda v) = \lambda A v$

$$A(\lambda v) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix} \left(\lambda \begin{pmatrix} v_1 \\ v_2 \\ ... \\ v_j \end{pmatrix} \right)$$

$$A(\lambda v) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ... & a_{ij} \end{pmatrix} \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ ... \\ \lambda v_j \end{pmatrix}$$

$$A(\lambda v) = \begin{pmatrix} a_{11}\lambda v_1 + a_{12}\lambda v_2 + ... + a_{1j}\lambda v_j \\ a_{21}\lambda v_1 + a_{22}\lambda v_2 + ... + a_{2j}\lambda v_j \\ ... \\ a_{i1}\lambda v_1 + a_{i2}\lambda v_2 + ... + a_{ij}\lambda v_j \end{pmatrix}$$

$$A(\lambda v) = \begin{pmatrix} \lambda(a_{11} v_1 + a_{12} v_2 + ... + a_{1j} v_j) \\ \lambda(a_{21} v_1 + a_{22} v_2 + ... + a_{2j} v_j) \\ ... \\ \lambda(a_{i1} v_1 + a_{i2} v_2 + ... + a_{ij} v_j) \end{pmatrix}$$

$$A(\lambda v) = \lambda \begin{pmatrix} a_{11} v_1 + a_{12} v_2 + ... + a_{1j} v_j \\ a_{21} v_1 + a_{22} v_2 + ... + a_{2j} v_j \\ ... \\ a_{i1} v_1 + a_{i2} v_2 + ... + a_{ij} v_j \end{pmatrix}$$

$\Rightarrow A(\lambda v) = \lambda A v$

2 Basisdarstellung und Basiswechsel

Aufgabe 2. Basisdarstellung und Basiswechsel. **(4 Punkte)**

Gegeben seien die Basen

$$B_1 := \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},\ B_2 := \left\{ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

des $\mathbb{R}^3$ und $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(a) Zeigen Sie, dass B_2 eine Orthonormalbasis ist.

(b) Berechnen Sie $\theta_{B_1}(v)$ und $\theta_{B_2}(v)$.

(c) Bestimmen Sie die Basiswechselmatrix $M_{B_1}^{B_2}$.

2.1 Lösung (a)

Eine Orthonormalbasis muss folgende Bedingungen erfüllen:

- Orthogonal (Skalarprodukt = 0)
- Normiert auf Länge 1 (Skalarprodukt = 1)
- Anzahl der Vektoren entspricht Anzahl der Dimensionen

Nachweis der Orthogonalität:

$$B_2 = \left\{ b_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}; b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; b_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

$\langle b_1, b_2 \rangle = \frac{1}{\sqrt{2}} \cdot 0 + 0 \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 = 0 => \textbf{orthogonal}$

$\langle b_2, b_3 \rangle = 0 \cdot \left(-\frac{1}{\sqrt{2}}\right) + 1 \cdot 0 + 0 \cdot \frac{1}{\sqrt{2}} = 0 => \textbf{orthogonal}$

$\langle b_1, b_3 \rangle = \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) + 0 \cdot 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0 => \textbf{orthogonal}$

$\Rightarrow$ **Alle Basisvektoren vonB_2 sind orthogonal**

Prüfen der Länge 1:

$$||b_1|| = \sqrt{\langle b_1, b_1\rangle} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1 \checkmark$$

$$||b_2|| = \sqrt{\langle b_2, b_2\rangle} = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1 \checkmark$$

$$||b_3|| = \sqrt{\langle b_3, b_3\rangle} = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1 \checkmark$$

$\Rightarrow \checkmark$ B_2 **ist eine Orthonormalbasis**

2.2 Lösung (b)

$$\theta_{B_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$$

Berechnung der Inversen nach Gauß-Jordan**:**

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\theta_{B_1}(v) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\theta_{B_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}$$

Berechnung der Inversen nach Gauß-Jordan**:**

$$\left(\begin{array}{ccc|ccc} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 \end{array}\right) \text{Zeile 1 und 3 } \cdot \frac{2}{\sqrt{2}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & \frac{2}{\sqrt{2}} \end{array}\right) \text{Zeile 1 + 3, Zeile 3 - 1}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -\frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right)$$

$$\theta_{B_2}(v) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

2.3 Lösung (c)

$$M_{B_1}^{B_2} = \theta_{B_1}\theta_{B_2}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

3 Affine Basis und affiner Basiswechsel

Aufgabe 3. Affine Basis und affiner Basiswechsel. **(4 Punkte)**

Gegeben seien die affine Basen (p_1, B_1) und (p_2, B_2) des $\mathbb{A}^3$ mit B_1, B_2 aus Aufgabe 2, $p_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $p_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ und $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(a) Berechnen Sie $\theta_{(p_1,B_1)}(v)$ und $\theta_{(p_2,B_2)}(v)$.

(b) Es sei $\theta_{(p_1,B_1)}(w) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Berechnen Sie $\theta_{(p_2,B_2)}(w)$.

3.1 Lösung (a)

$\theta_{(p_1,B_1)}(v) = \theta_{B_1} \cdot v - \theta_{B_1} \cdot p_1$

$$\theta_{(p_1,B_1)}(v) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\theta_{(p_1,B_1)}(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\theta_{(p_2,B_2)}(v) = \theta_{B_2} \cdot v - \theta_{B_2} \cdot p_2$

$$\theta_{(p_2,B_2)}(v) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta_{(p_1,B_1)}(v) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \\ -\frac{2}{\sqrt{2}} \end{pmatrix}$$

3.2 Lösung (b)

$$\theta_{(p_1,B_1)}(w) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w - \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Big| + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} w \Rightarrow w = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\theta_{(p_2,B_2)}(w) = \theta_{B_2} \cdot w - \theta_{B_2} \cdot p_2$$

$$\theta_{(p_2,B_2)}(v) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta_{(p_2,B_2)}(v) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 2 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

4 Kreuzprodukt

Aufgabe 4. Kreuzprodukt. **(4 Punkte)**

Gegeben seien die Vektoren $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{R}^3$. Zeigen Sie:

(a) $x \times y = -y \times x$.

(b) x und y sind genau dann linear abhängig, wenn $x \times y = 0$ ist.

(a) $\langle x, x \times y \rangle = \langle y, x \times y \rangle = 0$.

4.1 Lösung (a)

$$-y \times x = \begin{pmatrix} -y_1 \\ -y_2 \\ -y_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -y_2x_3 + y_3x_2 \\ -y_3x_1 + y_1x_3 \\ -y_1x_2 + y_2x_1 \end{pmatrix}$$

$$-y \times x = \begin{pmatrix} y_3x_2 - y_2x_3 \\ y_1x_3 - y_3x_1 \\ y_2x_1 - y_1x_2 \end{pmatrix}$$

$$-y \times x = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x \times y \checkmark$$

4.2 Lösung (b)

x und y sind linear abhängig wenn $\exists \lambda : \lambda \neq 0 \wedge x = \lambda y$

$$\Rightarrow x = \lambda y \Leftrightarrow x \times y = 0$$

$$\Rightarrow x \times \lambda y \overset{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_2\lambda y_3 - x_3\lambda y_2 \\ x_3\lambda y_1 - x_1\lambda y_3 \\ x_1\lambda y_2 - x_2\lambda y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda(x_2y_3 - x_3y_2) \\ \lambda(x_3y_1 - x_1y_3) \\ \lambda(x_1y_2 - x_2y_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \lambda \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda\,(x \times y) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$Wenn $\lambda \neq \mathbf{0}$ MUSS $x \times y = 0$ sein

4.3 Lösung (c)

$$\langle x, x \times y \rangle = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

$$\langle x, x \times y \rangle = x_1(x_2y_3 - x_3y_2) + x_2(x_3y_1 - x_1y_3) + x_3(x_1y_2 - x_2y_1)$$

$$\langle x, x \times y \rangle = (x_1x_2y_3 - x_1x_2y_3) + (x_1x_3y_2 - x_1x_3y_2) + (x_2x_3y_1 - x_2x_3y_1)$$

$$\langle x, x \times y \rangle = 0$$

$$\langle y, x \times y \rangle = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

$$\langle y, x \times y \rangle = y_1(x_2y_3 - x_3y_2) + y_2(x_3y_1 - x_1y_3) + y_3(x_1y_2 - x_2y_1)$$

$$\langle y, x \times y \rangle = x_2y_1y_3 - x_3y_1y_2 + x_3y_1y_2 - x_1y_2y_3 + x_1y_2y_3 - x_2y_1y_3$$

$$\langle y, x \times y \rangle = (x_1y_2y_3 - x_1y_2y_3) + (x_2y_1y_3 - x_2y_1y_3) + (x_3y_1y_2 - x_3y_1y_2)$$

$$\langle y, x \times y \rangle = 0$$

$$\Rightarrow \langle x, x \times y \rangle = 0 = \langle y, x \times y \rangle$$

$$\Rightarrow \langle x, x \times y \rangle = \langle y, x \times y \rangle = 0$$