Modelling Inefficiency of Herd Behaviour

MRes IO Writing Assignment I

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Abstract

This paper focuses on social learning that elicits inefficient herd behaviour. We introduced the common background for social learning, and then focus on two models (BHW 1992 and Banerjee 1992). Through comparison, we show the uniqueness of the assumptions and results of Banerjee (1992). In addition, we extend our paper in three aspects. First, we provide a proposition to illustrate the results in Banerjee (1992) more intuitively, see section 3. Second, we specify the outcome space and provide a recursive approach to calculate the probability of inefficient herding, both of which are absent in Banerjee (1992) original paper. Third, we extend our analysis to include the conditional probability of (in)efficient herding and provide an analytical solution to it in section 5. This describes the decision an agent faces if there is already a herd. Knowing the conditional probability could not only help the individual, but also guide policies attempting to increase social welfare by introduction of a black sheep.

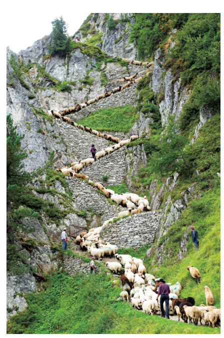

Figure 1: Cute Sheep

1 Introduction to Social Learning

The actions of others convey information about the state of nature. When I see others using an umbrella, I know it is raining. I do so because I know the decision rule of others, and thus, from their actions, I can deduce their private information. This reverse engineering process could have a wide variety of characteristics, and resulting in drastically different social outcomes. In a general way of speaking, this wide varieties of processes in which agents rationally engineer information from others' actions are called social learning.

Social learning is a particular interesting topic attracting efforts from several fields of research. Vives (1993)'s model is probably a good representative of the standard way micro economists study it. In an accounting setup, there have been studies looking at firms' decisions within a financial year during which they learn from each other but not knowing the exact result. In finance, asset price bubbles are studied (Brunnermeier, 2001) using learning models.

In this report, we focus on two nice models which focus on social learning that elicits herd behaviour. The first one by Bikhchandani, Hirshleifer, and Welch (1992, BHW from now on), which develops similarly to the Vives(1993)'s standard model. The second one is by Banerjee (1992). Since both assume we have one agent that participates in each term, we from now on index agents by time t.

There are several key components in the setups of a social learning model.

Firstly, the state of nature. The state of nature is the ultimate target of learning. Whether or not the true state is discovered (and how fast it is discovered) are crucial measurements for the efficiency of social learning. It is intuitively true that a more complicated state of nature is harder (and slower) to be discovered than a binary case. Therefore, the structure of this state of nature is the first and foremost to define. As the state of nature is unknown to agents, we often use a random variable θ to represent it.

Secondly, the signals. To learn θ, agents observe multiple or infinite signals $\{s_t\}_0^\infty$ generated (by nature) based on some distribution $F^\theta(\cdot)$. Imagine one agent who observes all these signals and goes through the learning process him(her)self. This process is the central part we want to model, and it is for this purpose that Bayesian tools are introduced. Thinking as the nature, θ is a parameter and the series of signals $\{s_t\}_0^\infty$ is an event living in the signal space Ω. Thinking as an agent, however, s_t is observed while θ is a random variable. We therefore learn about its distribution using Bayes's Law repeatedly. Notice, importantly, $F^{\theta}(\cdot)$ must be common knowledge.

In reality the signals are unobservable. Instead, actions are; therefore, we need to specify the action space A. An action space could be as rich as the signal space; however, we are more interested in the situation where actions are limited. In the standard model, A is binary. This leads to attrition of information among agents, which plays a role in eliciting inefficient herd behaviour.

Last but not least, the decision rule—is the decision rule homogenous? If not, is each agent's decision rule common knowledge? In the models we focus on here, the decision rule is homogenous and common knowledge to all. This property simplifies our model. Just as Bayes's Law allows the reverse engineering of state of nature from signals, common knowledge of the decision rule allows agents to engineer the unobservable signals from observable actions. Together, an agent combines the history of actions with his(her) private signal to learn θ—the ultimate target to find.

In the next sections, we will set up the two models formally, specifying each of the components above with a brief comparison between them. We continue to discuss the agents' equilibrium behaviour (main results) and provide an intuitive, but nonetheless rigorous explanation for them. In specific, we zoom in to their analysis on herd behaviour and calculate for probability of (in)efficient herding. In section 4, we lay our map further with a discussion on Banerjee (1992)'s contributions and limitation. Being a model somewhat at the knife-edge, we put it in the more standard context of social learning models and explain in detail its uniqueness. For reasons we will show in section 5, somtimes knowing the conditional probability is more useful, and therefore we provide an analytical solution to it. We conclude in section 6.

2 Standard Learning Model (BHW 1992)

2.1 Model Setups

First, denote θ as the state of nature, which is a binary variable taking values from $\Theta = \{\theta_0, \theta_1\}, \theta_0 < \theta_1$. By convention, we call θ_1 *good state of nature.* As θ is unobservable by agents, with slightly confusing notation, we also use θ to denote the random variable. Before period 1, there is a public prior on θ. In this binary case, we can use $\mu := Pr(\theta = \theta_1)$ to parametrize the prior distribution.

Let's suppose $\mu_1 = \frac{1}{2}$ (uniform over Θ).

Second, we define the signals. At each period, a signal s_t is generated according to distribution $F^\theta(s)$ with support S, and $\{s_t\}_0^\infty$ is a series of i.i.d. signals which lives in the signal space $\Omega = S^\infty$. When agent t receives this signal, she forms a private belief using the public belief μ_t (e.g., μ_1 for the first agent) and s_t following Bayes's Law:

$$\mu_t^p := Pr(\theta = \theta_1 | s_t) = \frac{Pr(s_t|\theta = \theta_1)Pr(\theta = \theta_1)}{Pr(s_t|\theta = \theta_1)Pr(\theta = \theta_1) + Pr(s_t|\theta = \theta_0)Pr(\theta = \theta_0)} \tag{1}$$

We introduce the transformation in line with Chamley (2004). Define *log-likelihood ratio (LLR)*:

$$\lambda_t := \ln \frac{Pr(\theta = \theta_1)}{Pr(\theta = \theta_0)} = \ln \frac{\mu_t}{1 - \mu_t} \tag{2}$$

Then λ_1 describes the prior on θ same as μ_1. Let $f^\theta(\cdot)$ be the *pdf*, equation 1 is simplified to

$$\lambda_t^p = \lambda_t + \ln \frac{f^{\theta_1}(s_t)}{f^{\theta_0}(s_t)} \tag{3}$$

Third, we proceed to define the decision rule. After forming private belief, agent t maximizes her utility $u(x) = (\theta - c)x, x \in \{0, 1\}$, in which $\theta_0 < c < \theta_1$ is a cost (non-stochastic) associated with investment. Notice here the action space $A = \{0, 1\}$ is discrete. This is equivalent with the following decision rule (DR):

$$x_t = DR(\lambda_t^p) = \begin{cases} 1 & if \;\; \lambda_t^p > \gamma := \ln \frac{c}{1-c} \\ 0 & if \;\; \lambda_t^p < \gamma \\ \text{invest with probability } 0.5 & if \;\; \lambda_t^p = \gamma \end{cases} \tag{4}$$

Finally, we clarify how her action changes the public belief. Let the history of actions be

$$h_t = [x_1, x_2, ..., x_{t-1}] \tag{5}$$

As agent $t+1$, she observes x_t and uses Bayes's Law to form a new public belief

$$\lambda_{t+1} = \lambda_t + \ln \frac{Pr(x_t|\theta = \theta_1)}{Pr(x_t|\theta = \theta_0)} \tag{6}$$

In order to calculate equation 6, we need to introduce a mild assumption on $F^\theta(\cdot)$. We say $F^{\theta_0}(\cdot), F^{\theta_1}(\cdot)$ satisfy *proportional monotonicity* if $\frac{f^{\theta_1}(s)}{f^{\theta_0}(s)}$ is monotonic increasing in s. Using equation

3, we can calculate the distribution of private belief conditional on λ_t as $F^{\theta}(s) = G^{\theta}(\lambda_t + \ln \frac{f^{\theta_1}(s)}{f^{\theta_0}(s)} | \lambda_t)$. Inferring from agent t's $x_t = DR(\lambda_t^p)$, the public belief is therefore

$$\lambda_{t+1} = \lambda_t + \begin{cases} \ln \frac{1-G^{\theta_1}(\gamma)}{1-G^{\theta_0}(\gamma)} & if \quad x_t = 1 \\ \ln \frac{G^{\theta_1}(\gamma)}{G^{\theta_0}(\gamma)} & if \quad x_t = 0 \end{cases} \tag{7}$$

We finish setup with a restatement of all *common knowledge(CK)*, as always remembering CK is helpful for understanding the model. By common knowledge, we mean not only we all know it, but also we know that others know it, and others know that we know that they know it, so on and so forth. In the standard model above

$$CK_t = \{\Theta, \lambda_1, F^{\theta_1}(\cdot), F^{\theta_0}(\cdot), \gamma, DR(\cdot), A, h_t\} \tag{8}$$

2.2 Main Results

The agents will, after rationally learning from previous actions, herd on one action. The speed that a herd happens is exponential. If the support for G^{θ} is bounded, then there is strictly positive probability that the herd behaviour could be inefficient. To develop the main theorem, we first need to establish the following concepts.

Assumption 1 *Bounded LLR*

The LLR of signals, $\ln \frac{f^{\theta_1}(s)}{f^{\theta_0}(s)}$*, takes values from a bounded interval* $\Lambda = [\lambda_-, \lambda_+]$

The boundedness is crucial for herd behaviour. Without loss of generality, let $\lambda_1 = \gamma = 0$. Let $\Lambda_t = [\lambda_t - \lambda_-, \lambda_t + \lambda_+]$ denote the support for G^{θ} at period t. We also assume $\lambda_- < 0 < \lambda_+$ (otherwise, the first agent will make a decision regardless of s_1). Lastly, denote ν the shift of public belief at the end of each period t:

$$\nu(x_t, \lambda_t) = \begin{cases} \ln \frac{1-G^{\theta_1}(\gamma)}{1-G^{\theta_0}(\gamma)} & if \quad x_t = 1 \\ \ln \frac{G^{\theta_1}(\gamma)}{G^{\theta_0}(\gamma)} & if \quad x_t = 0 \end{cases} \tag{9}$$

Definition 1 *Herding (Chamley 2004, Definition 4.1)*

An agent herds on the public belief when his action is independent of his private signal.

Theorem 1 *Herd Behaviour (Chamley 2004, Proposition 4.3)*

All agents herd on one or the other of the actions after some finite date, almost surely. The probability that a herding of all future agents has not started by date t, converges to 0 like β^t for some β with $0 < \beta < 1$.

We discuss the intuition of the theorem. Since $\Lambda_1 = \Lambda$ is bounded, so is Λ_t for all t, as they are simply shiftings of Λ_1. If, at period t, the whole Λ_t is strictly positive($>> 0$), then agent t will invest regardless of his(her) signal. Rationally knowing that agent t is herding, λ_t will not be changed, and therefore $\Lambda_{t+1} = \Lambda_t$. All agents after t will herd on action $x = 1$ following the same logic.

Such an event as $\Lambda_t >> 0$ has strictly positive probability to happen. It just need $\eta_1 \in N^+$ consecutive upward shifts to happen, where $\sum_1^{\eta_1} v(1, \lambda_t) + \lambda_- > 0$. Symmetrically, It just need $\eta_0 \in N^+$ consecutive downward shifts to happen, where $\sum_1^{\eta_0} v(0, \lambda_t) + \lambda_+ < 0$, to plunge the whole support of private belief below zero. Either case will lead to a herd. The probability that $v(1)$ and $v(0)$ happens alternately (i.e., less then η_1 and η_0 consecutive times) tends to zero exponentially with periods t. The actual speed β will depend on F^θ.

2.3 Probability of Inefficient Herding

Theorem 1 indicates that a herd should happen almost surely. However, it is unsure whether the true state of nature will be discovered. Instead, a herd could form on either action, and thus if we herd on $x = 0$ when true state of nature $\theta = \theta_1$, we call it *inefficient herding*.

We use a graph (Figure 2) to illustrate how signals $\{s_t\}_0^\infty$ leads to an inefficient herding. The true state of nature is 1, while agents herd on 0.

The calculation of probability that an inefficient herding takes place is not trivial. Let us provide an estimate:

Theorem 2 *Probability of Inefficient Herding (Chamley 2004, Proposition 4.5, adjusted slightly)*

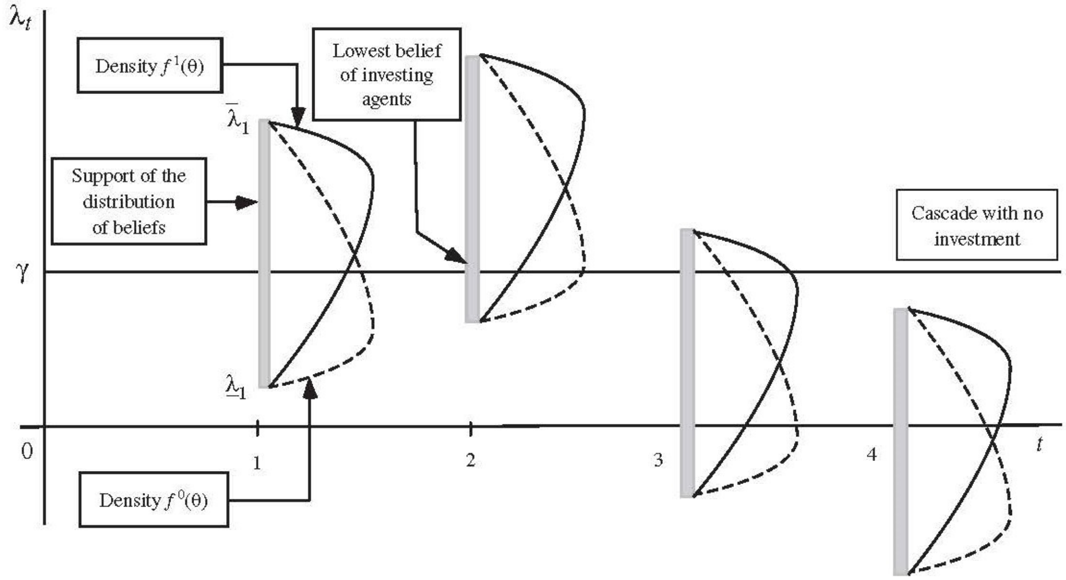

Figure 4.3 The evolution of beliefs. In each period, the support of the distribution of beliefs (LLR) is represented by a segment. The action is $x_t = 1$ if and only if the belief (LLR) of the agent is above γ. If agent t happens to have a belief above (below) γ, the distribution moves up (down) in the next period $t + 1$. If the entire support is above (below) γ, the action is equal to 1 (0) and the distribution stays constant.

Figure 2: Graphical Illustration of Belief Update (Chamley 2004)

If Λ contains $[-\sigma, +\sigma]$, then the probability of an inefficient herding is less than 4β, with

$$ln\frac{\beta}{1-\beta} = -\sigma \tag{10}$$

Intuitively, when the support of private belief is short, the probability of inefficient herding is large. If the support tends to be unbounded, then we always converge to efficient herding, as it is diminishingly likely to have large numbers of consecutive opposite shifts. Also, we see that $\beta < \frac{1}{2}$.

3 the Model of this Paper (Banerjee 1992)

We have seen that the standard model features its discrete action space, on which the herd behaviour is based. Banerjee's model is unique in that it allows continuity of almost every components, except for the signal generating distribution, which was continuous in the standard model. In every way, it is a very interesting exercise to compare the two models, that use different, even opposite setups, but maintain the herd behaviour and, in specific, the inefficient herding.

3.1 Model Setups

Different from the standard model, Banerjee (1992) adopts a continuous space for the state of nature $\Theta = [0, 1]$. The true investment point, i^*, is unobservable to agents. The prior is $U_{[0,1]}$.

Interestingly, the only and crucial twist where Banerjee (1992) does not allow for continuity is the signal space. Instead, agent t would receive a signal with probability α, and it is true with probability β. This signalling system could be **re-modelled as such: each period t, a pre-signal s is drawn** from the discrete space $S = \{0, 1, -1\}$. If $s = 1$, then the true state of nature i^* is revealed; if $s = 0$, then nothing is revealed; if $s = -1$, then one incorrect state of nature $i' \neq i^*$ is drawn randomly from $U_{[0,1]}$ and revealed. Of course one can not tell between $s = \pm 1$; it is the $i'(i^*)$'s that are observed. This equivalent way captures the discreteness.

For the action space, again, Banerjee (1992) allows for continuity. Each agent selects $x_t \in A = [0, 1]$ to maximize payoff, which is positive if $x_t = i^*$ and zero otherwise. Because the signal space is now discrete, the belief is not continuous in general either. We therefore **would not need Bayesian** tools, and indeed we explain all the theorems in the next section without complicated maths. Similarly, $h_t = [x_1, x_2, ..., x_{t-1}]$. Let $\overline{x}[t]$ be the largest element in $h(t)$, $\overline{x}([1]) = 0$.

3.2 Optimal Strategy

Assumption 2 *Tie-breaking Assumptions*

Agents behave in alignment with AA,AB and AC at tie-breaking decisions (see appendix)

Theorem 3 *Optimal Decision Rule (Banerjee 1992, Figure 1)*

For agent t, his(her) optimal strategy is described by the following decision rule DR:

$$x_t = DR(h_t, s_t) = \begin{cases} x^* & if \;\; \exists\, 0 < x^* < \overline{x}[t] \textit{ has been invested twice} \\ x^* & if \;\; s = \pm 1 \textit{ and } \exists\, x^* \in h(t),\; i' = x^* \\ i' & if \;\; s_t = \pm 1 \textit{ and no positive } x \textit{ has been invested twice} \\ \overline{x}[t] & if \;\; s_t = \pm 1, \overline{x}[t] > 0 \textit{ and } \overline{x}[t] \textit{ only has been invested twice} \\ \overline{x}[t] & if \;\; s_t = 0 \textit{ and no positive } x \textit{ except } \overline{x}[t] \textit{ has been invested twice} \end{cases} \tag{11}$$

The logic for such an optimal decision rule is straightforward. Consider the agent 1. If she receives signal i', she will invest at i' hoping $s_1 = 1$. If she does not receive a signal, by AA $x_1 = 0$.

Suppose agent 2 receives a different signal, then by AB she invests accordingly. If she receives the same signal, then she knows that $x_1 = i^*$. If she does not receive a signal, she hopes that x_1 is the true state of nature.

Suppose agent 3 sees that x_1 has been invested twice. Then x_1 is slightly more likely to be the true state of nature than her signal, if she receives one. So she herds on x_1 regardless of her signal. Otherwise, if $x_1 \neq x_2$, then she will behave similarly to agent 2, with the exception that $x_3 = \overline{x}[3]$ if she receives no signal by AC.

Now for generic agent t. If she receives a signal that also matches one of the historical actions, then she knows this is i^*. Similarly, if there is an action x_{t^*} which is not maximal but has been invested (more than) twice, than the only explanation is $x_{t^*} = i^*$. Otherwise, let her signal be i'_t, and by the same logic applied by agent 3, $\overline{x}[t]$ is at least as likely to be the true state of nature as i'_t. Banerjee (1992) introduces AB to diversify the actions such that when indeed indifferent, $x_t = i'_t$. Finally, if she receives no signal and no positive action in $(0, \overline{x}[t])$ has been invested (more than) twice, then she has to hope for $\overline{x}[t]$.

Now since all rational agents behave in alignment with the optimal strategy, the following proposition holds as a direct result

Proposition 1 *History Trajectory*

According to $DR(\cdot)$, h_t could only take one of the following three trajectories. When $t \to \infty$, any other history that does not fall in these three categories happens with probability 0:

1) $\exists\, t^*$, h_{t^*} consists only of initial 0's and non-repeating x_t's, and $x_t = \overline{x}[t^*]$ $\forall t \geqslant t^*$

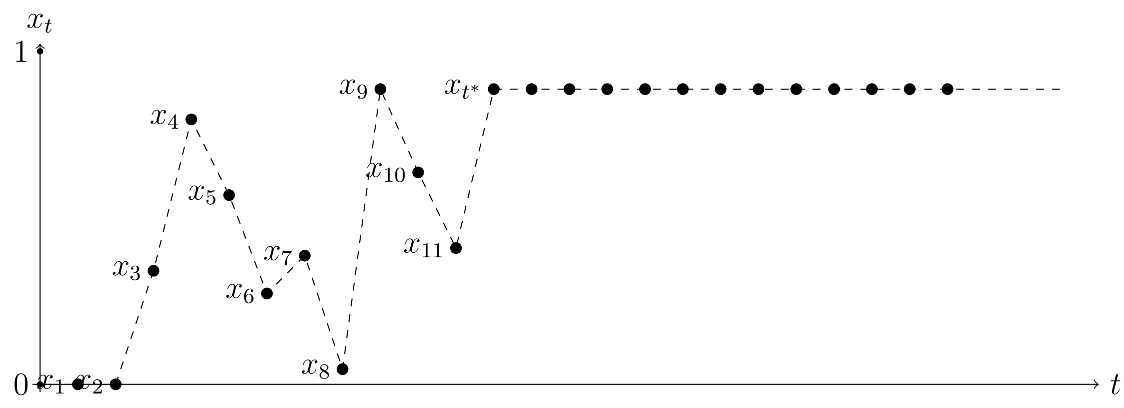

Figure 3: History Trajectory 1)

2) $\exists\, t^*$, h_{t^*} behaves like 1) but includes i^*, and $x_t = i^*$ $\forall t \geqslant t^*$

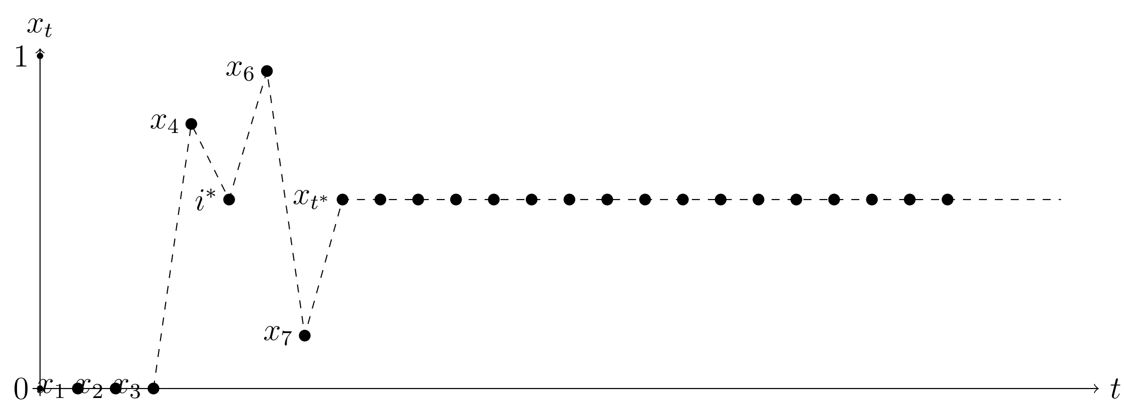

Figure 4: History Trajectory 2)

3) $\exists\, (t_1, t^*), t_1 < t^*$, h_{t_1} behaves like 2), $x_t = \begin{cases} \overline{x}[t_1] & t_1 \leqslant t < t^* \\ i^* & \forall t \geqslant t^* \end{cases}$

The crucial person which allows herd behaviour to start is the first agent who receives no signal. Suppose agent t_0 is the first agent who receives a signal. Let $\kappa > t_0$ be the first agent after t_0 who does not receive a signal. Because of the discreteness of the signal generating process, $\kappa < \infty$.

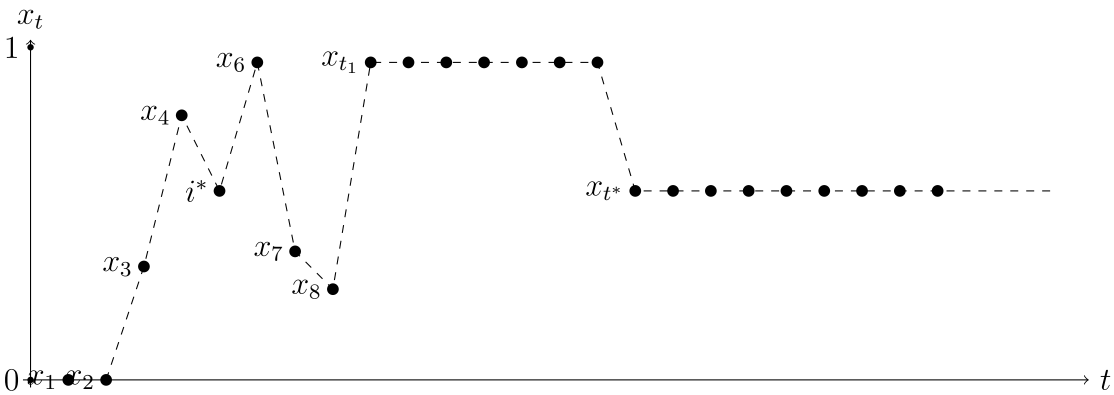

Figure 5: History Trajectory 3)

Proposition 2 *A Sufficient and Necessary Condition for Inefficient Herding*

Inefficient herding happens if and only if $i^* \notin h_\kappa$.

Proof.

Let's first prove the **"if"** part. Formally, we want to show the following: If none of the actions in h_κ is i^*, then for all $t \geqslant \kappa$

$$x_t = \overline{x}[\kappa] \tag{12}$$

Notice, all non-zero actions in h_κ are different (because incorrect signals coincide with probability 0 and assumption AB). Therefore, $x_\kappa = \overline{x}[\kappa]$ (assumption AC).

For $t > \kappa$, she either receives no signal, or her signal will be different from all elements in h_κ. In either case, she will choose $\overline{x}[\kappa]$. **Q.E.D.**

To tackle the **"only if"** part, we prove the following: If there is a critical agent $t^* < \kappa$ such that she is the first to receive the true signal, then almost surely there exists $N > t^*$:

$$s_N = 1, \ \textit{and} \, \forall \, t \geqslant N, \ x_k = i^* \tag{13}$$

We assume $i^* < \overline{x}[\kappa]$, since they coincide with probability zero. Since i^* is the unique correct signal, almost surely there will be someone who receives it again. So let her be agent N. w.l.o.g., let's assume $N > \kappa$ (otherwise h_κ has a trajectory of type 3) in proposition 1 already).

By construction, for all agent $t^* < t < N$, $s_t \neq 1$. Therefore,

$$h_N = [h_\kappa, \overline{x}[\kappa], \overline{x}[\kappa], ..., \overline{x}[\kappa]] \tag{14}$$

Therefore agent N realizes $s_N = 1$, as i^* appeared in h_N once and once only. So $x_N = i^*$. Any agent after N understands that true state of nature has been learned, and will choose i^* regardless of her signal. **Q.E.D.** ■

Corollary 1 *Incorrect Herding and Choosing* i^*

If and only if incorrect herding happens, no agent chooses i^*

Corollary 2 *Partition of Outcome*

In infinity, agents either herd efficiently on i^**, or inefficiently on* $\overline{x}[\kappa]$ *for some* κ*.*

3.3 Probability of Inefficient Herding

Theorem 4 *Probability of Inefficient Herding (Banerjee 1992)*

$$Pr(\textit{Inefficient Herding}) = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)} \tag{15}$$

From the theorem, we see that the probability of inefficient herding goes to 0 when either α or β approaches 1. When $\alpha \to 1$, signals are given almost every period, and $\kappa \to \infty$, ensuring that $i^* \in h(\kappa)$. By proposition 2, inefficient herding does not happen.

When $\beta \to 1$, signals are more likely to be true, then i^* is more likely to appear in $h(\kappa)$. The above graph (figure 6) provides a detailed description of the probability as a function of α and β.

To calculate probability, we formally define the outcome space. Let $\omega = \{s_t\}_1^\infty$ be an infinite series of pre-signals, $s_t \in S = \{1, 0, -1\}$, and h one possible history elicited by ω. Then ω is a basic event in the outcome space $\Omega = S^\infty$, upon which we have a measure Pr well-defined because signals are *i.i.d.*.

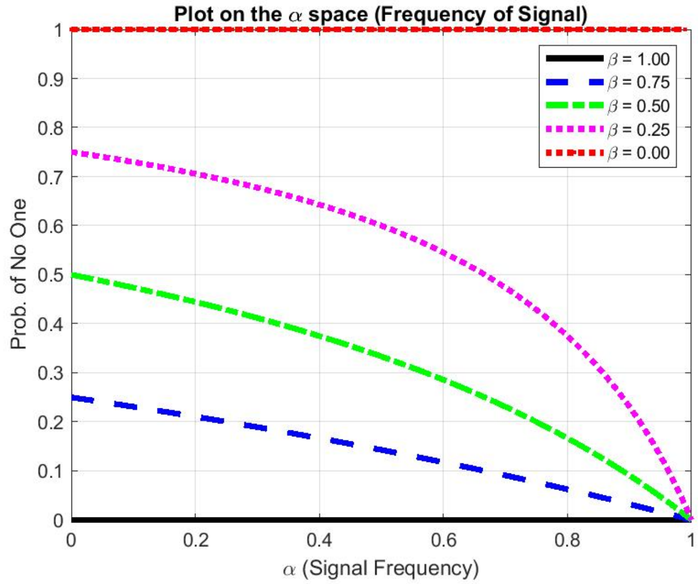

Figure 6: Comparative Statics

If the signal is incorrect, it does not matter what value i' actually takes. In other words, it is sufficient to specify an infinite series of pre-signals ω to determine the herd behaviour of the actions elicited by it. Hence divide Ω into $(\Omega_1, \Omega_0, \Omega_{-1})$ according to the herd behaviour it leads to (Ω_1 for efficient herding, Ω_{-1} for inefficient herding and Ω_0 non-herding). By corollary 2, We know $Pr(\Omega_0) = 0$. According to proposition 2, we have the following properties of Ω_i:

Lemma 1 *Invariant Operations*

1) If $(s, \omega) \in \Omega_i$, $s \neq 0$, *then* $(-1, s, \omega) \in \Omega_i$

2) If $\omega \in \Omega_i$, *then* $(0, \omega) \in \Omega_i$

3) $\forall\ \omega \in \Omega$, $(1, \omega) \in \Omega_1$

4) Combining 1) and 3), in specific, $(-1, 1, \omega) \in \Omega_1$

We are now ready to provide a recursive approach to calculating the probability.

Proof.

From 2) of the lemma

$$\begin{cases} Pr(\Omega_{-1}) \leqslant Pr(\{\omega|(0,\omega) \in \Omega_1\}) \\ Pr(\Omega_1) \leqslant Pr(\{\omega|(0,\omega) \in \Omega_1\}) \end{cases} \tag{16}$$

Since we know $Pr(\Omega_1) + Pr(\Omega_{-1}) = 1$, then

$$Pr(\{\omega|(0,\omega) \in \Omega_1\}) + Pr(\{\omega|(0,\omega) \in \Omega_{-1}\}) \geqslant 1 \tag{17}$$

as $(0,\omega)$ either $\in \Omega_{-1}$ or Ω_1, both inequalities in 16 bind.

From 1) of the lemma we find a similar result. Therefore, denote

$$V := Pr(\Omega_{-1}) = Pr(\{\omega|(0,\omega) \in \Omega_{-1}\}) \tag{18}$$

$$U := Pr(\{\omega|(-1,\omega) \in \Omega_{-1}\}) = Pr(\{\omega|(-1,-1,\omega) \in \Omega_{-1}\}) \tag{19}$$

We want to calculate V. Using law of total probability on V and U, repectively:

$$\begin{aligned} V = Pr(\Omega_{-1}|s_1 = 1)Pr(s_1 = 1) + Pr(\Omega_{-1}|s_1 = 0)Pr(s_1 = 0) \\ + Pr(\Omega_{-1}|s_1 = -1)Pr(s_1 = -1) \end{aligned} \tag{20}$$

in which, by 2), 3) of lemma and equation 18

$$\begin{cases} Pr(\Omega_{-1}|s_1 = 1) = Pr(\{\omega|(1,\omega) \in \Omega_{-1}\}) = 0 \\ Pr(\Omega_{-1}|s_1 = 0) = Pr(\{\omega|(0,\omega) \in \Omega_{-1}\}) = V \end{cases} \tag{21}$$

$$\begin{aligned} U = Pr(\Omega_{-1}|s_1 = -1, s_2 = 1)Pr(s_2 = 1) + Pr(\Omega_{-1}|s_1 = -1, s_2 = 0)Pr(s_2 = 0) \\ + Pr(\Omega_{-1}|s_1 = -1, s_2 = -1)Pr(s_2 = -1) \end{aligned} \tag{22}$$

in which, by proposition 2, 4) of lemma and equation 19

$$\begin{cases} Pr(\Omega_{-1}|s_1 = -1, s_2 = 1) = 0 \\ Pr(\Omega_{-1}|s_1 = -1, s_2 = 0) = 1 \\ Pr(\Omega_{-1}|s_1 = -1, s_2 = -1) = U \end{cases} \tag{23}$$

Therefore, equation 20 and 22 simplifies to the canonical recursive form:

$$V = \alpha(1-\beta)U + (1-\alpha)V \tag{24}$$

$$U = (1-\alpha) + \alpha(1-\beta)U \tag{25}$$

solving this gives us

$$U = \frac{1-\alpha}{1-\alpha(1-\beta)}, \quad V = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)} \tag{26}$$

Q.E.D. ■

4 Main Contributions and Limitations

Banerjee's model differs from the standard learning model mainly from its assumption of continuity. In the standard model, the actions space is discrete and it is for this reason, each agent's information is not communicated fully. In other words, the rich information in the signal s is severely lost when agents only communicate with a discrete action x. Variants of the standard model mostly rely on the discreteness of **action space** to elicit inefficient herd behaviour.

However, in Banerjee's model, people can accurately communicate their signal if they have one, but since there is $1-\alpha$ probability that one has no signal, **there is no action for such a piece of information to be communicated**. The only exceptions are the initial 0's according to the questionable assumption AA, and it is because of this assumption AA that 2) of lemma holds. In other cases, as we have shown in proposition 2, the first agent who has no signal is the reason for inefficient herding, and Banerjee's paper is unique in this sense that it relies on the discreteness of **signal space** to elicit inefficient herd behaviour.

When an inefficient herd takes place, the social welfare could be improved by several strategies. One way in the standard social learning model, is to blindfold the actions of the first several agents, and thus more agents rely on their signals without resorting to herding.

This logic translates smoothly in Banerjee's setup. If more agents in the starting periods are not allowed to herd, then h_κ will contain more *effective actions* (see definition in proposition 3) and there is a higher chance that i^* appears before κ (proposition 2). As a result, it is less likely that we will fall in a trajectory of type 1) in proposition 1, and more likely to be of type 2) and 3).

Another way is to introduce a *black sheep* at period T, and make her arrival common knowledge. A black sheep describes an agent B who is given a signal which is particularly accurate. This is to say, her signal s_B does not follow $F^\theta(\cdot)$ as other signals; instead, it comes from a more reliable source (for example, in a continuous Θ model, then s_B could follow a distribution concentrated highly around the true state of nature). As other knows her arrival, agent B's action will be looked upon differently, and if x_B is different from the current herd action, people easily believe that x_B is more accurate, and therefore the inefficient herd is destroyed.

This methods is particularly useful in the standard social learning environment, as we could imagine that people with more precise signal could naturally persuade others. Also, through introducing a black sheep we easily see the fragility of herd behaviour, which is also a stylized fact.

Whether this method could function in Banerjee's setup, is, however, questionable. Consider an introduction of a black sheep at period B, who enjoys a higher precision level of information (higher $1 > \beta_B > \beta$). And this is common knowledge. First of all, whether agent B will invest according to i'_B is uncertain, even if she receives one with probability α. She may still herd on the (inefficient) action if she believes that's more likely to be true. Secondly, agents after B are not easily believing her action, even if x_B is different from the herd action.

In the next section, therefore, I calculate the analytical solution to the conditional probability. This is the probability that the herd action $\overline{x}[t] = i^*$ given that an agent t already faces a herd of type 1) in proposition 1. We will see that, in order to destroy the (inefficient) herd, β_B has to be larger than the above conditional probability to persuade agents after B to deviate from herding. Of course, if they do deviate, then they form a new herd, led by sheep B.

5 Conditional Probability of Efficient Herding

The unconditional probability of (in)efficient herding is a useful piece of knowledge for social welfare, on the one hand. Knowing the conditional probability of (in)efficient herding given a particular history, on the other hand, is of particular interest for any individual agent.

Consider an agent t_∞ where t_∞ is sufficiently large. If everyone before her has followed the optimal decision rule (theorem 3), we show that there are only three possible kinds of history h_{t_∞} that she could observe (proposition 1). In the 2) and 3) case, efficient social learning has taken place and she knows that her herding is efficient. In case 1), however, she herds only hoping that $\overline{x}[t^*]$ is indeed the true state of nature. What is the probability that her belief is indeed correct? In other words, could she learn from the fact that agents are herding on the largest value, and **infer that the society is probably in an inefficient herd?**

Proposition 3 *Conditional Probability of Efficient Herding*

Consider h_κ described in proposition 2, i.e., it consist only of t_0 initial 0's and T non-repeating actions, with κ being the first agent not receiving a signal after t_0, $t_0 \geqslant 0, T \geqslant 1, \kappa = t_0 + T + 1$. We call these T non-repeating actions ***effective actions***

Then the conditional probability that agent κ is herding efficiently is

$$Pr(\textit{Efficient Herding}\ |s_\kappa = 0, h_\kappa) = \frac{\beta}{1 + (T-1)\beta} \tag{27}$$

The conditional probability that any agent after κ is herding efficiently is $(\forall\, t > \kappa)$

$$Pr(\textit{Efficient Herding}|s_t, h_t) = \begin{cases} \frac{\beta(1-\alpha(1-\beta))}{\beta(\alpha\beta)+(1+(T-1)\beta)(1-a)} & \textit{if her signal } \notin h(\kappa) \textit{ and } h_t \in \textit{case 1)} \\ 1 & \textit{if her signal } \in h(\kappa) \textit{ or } h_t \in \textit{case 3)} \end{cases} \tag{28}$$

The intuition is straightforward for agent κ's conditional probability. She herd efficiently if and only if the $\overline{x}[\kappa]$ is the true state of nature. Since h_κ consists of T effective actions, any of them are equally likely to be true. The more actions (T larger), the less likely a specific guess is correct; the higher the precision (β larger), the more likely the true state of nature has been revealed at all.

The conditional probability of agents after κ are harder to interpret, but the comparative statics follow suit. The more important observation is that all agents after κ enjoy the same conditional probability of efficient herding. This is because agents after κ are herding and **herd actions does not convey any information** (h_t is informationally equivalent with $(h_\kappa, \overline{x}[\kappa])$.

We provide an approach to calculate conditional probability. Bayesian tools are now needed to be applied on the outcome space Ω.

Proof.

Consider agent κ. W.l.o.g, let m be the agent who invests at the largest x in h_κ. Using Bayes's

Law and by independence between h_t and s_t:

$$
\begin{aligned}
&Pr(\textit{Efficient Herding } | s_\kappa = 0, h_\kappa) = Pr(s_m = 1 | s_\kappa = 0, h_\kappa) \\
&= \frac{Pr(s_m = 1, h_\kappa) Pr(s_\kappa = 0)}{Pr(s_m = 1, h_\kappa) Pr(s_\kappa = 0) + Pr(s_m = -1, h_\kappa) Pr(s_\kappa = 0)} \\
&= \frac{\alpha\beta \cdot (\alpha(1-\beta))^{T-1}}{\alpha\beta \cdot (\alpha(1-\beta))^{T-1} + (\alpha(1-\beta))^{T} + (T-1)\alpha\beta \cdot (\alpha(1-\beta))^{T-1}} \\
&= \frac{\beta}{1 + (T-1)\beta}
\end{aligned}
\tag{29}
$$

Consider agent $t = \kappa + 1$. If she receives a signal that coincides with any $x \in h_\kappa$ then she learns i^*. So let's assume $s_t \neq 1$. By optimal decision rule (theorem 3), we know hence she herds regardless of her signal ($x_t = x_m$).

Observing $h_t = (h_\kappa, x_m)$, she is uncertain whether agent κ herds or received a signal coinciding with x_m. If it is the latter case, then she is sure that efficient herding has started. It is still a non-trivial thought process if it is the former. Using Bayes's Law

$$
Pr(\textit{Efficient Herding } | (h_\kappa, x_m)) = \frac{Pr(s_m = 1, (h_\kappa, x_m))}{Pr(s_m = 1, (h_\kappa, x_m)) + Pr(s_m = -1, (h_\kappa, x_m))} \tag{30}
$$

in which

$$
\begin{aligned}
&Pr(s_m = 1, (h_\kappa, x_m)) = Pr(s_m = 1, s_\kappa = 1, h_\kappa) + Pr(s_m = 1, s_\kappa = 0, h_\kappa) \\
&= Pr(s_m = 1, h_\kappa)(Pr(s_\kappa = 1) + Pr(s_\kappa = 0)) \\
&= \alpha\beta \cdot (\alpha(1-\beta))^{T-1} \cdot (1 - \alpha(1-\beta))
\end{aligned}
\tag{31}
$$

so equation 30 becomes

$$
\begin{aligned}
&Pr(s_m = 1 | (h_\kappa, x_m)) = \frac{Pr(s_m = 1, h_\kappa)(1 - \alpha(1-\beta))}{Pr(s_m = 1, h_\kappa)(1 - \alpha(1-\beta)) + Pr(s_m = -1, h_\kappa)(1-\alpha)} \\
&= \frac{\alpha\beta(1 - \alpha(1-\beta))}{\alpha\beta(1 - \alpha(1-\beta)) + \alpha(1-\beta)(1-\alpha) + (T-1)\alpha\beta(1-\alpha)} \\
&= \frac{\beta(1 - \alpha(1-\beta))}{\beta(\alpha\beta) + (1 + (T-1)\beta)(1-a)}
\end{aligned}
\tag{32}
$$

Q.E.D. ∎

6 Conclusion

In this report we looked at two models of social learning, and through comparing their assumptions and results, we showed the uniqueness of Baherjee's model. In specific, we hope to provide a more formal way to describe Banerjee's model (in specific, using pre-signals to formulate outcome space), as it is absent in the original work. As an extra, we provided some insights together with proof of the conditional probability, which could not only be a useful piece of knowledge for individual agent facing a herd, but also useful for any policy trying to improve social welfare through an introduction of black sheep.

7 Appendix I: Tie-breaking Assumptions

Assumption 3 *Point Zero (Assumption A in the paper)*

For agent t, if she receives no signal, and all previous investment actions $\{x_1, x_2, ..., x_{t-1}\}$ equal 0, then she chooses 0.

Assumption 4 *Prioritizing Own Signal (Assumption B in the paper)*

For agent t, if she receives a signal i'_t, and she is indifferent between following her signal and any preceding agent's action, then she will invest at $x_t = i'_t$

Assumption 5 *Towards the Maximum (Assumption C in the paper)*

For agent t, if she receives no signal, and she is indifferent between following one or more of the preceding agent's actions, then she will invest at the one whose value is highest.

8 Appendix III: An Alternative Proof for Theorem 4

Proof. By proposition 2, we decompose the goal as the following double infinite sums

$$P(Incorrect\ Herding) = \sum_{k_0=1}^{\infty} \sum_{\kappa=k_0+1}^{\infty} P(None\ of\ the\ actions\ in\ h_\kappa\ is\ true) \tag{33}$$

First fix $k_0 = 1$, that is, the first person receives a signal. Denote

$$V := \sum_{\kappa=2}^{\infty} P(\textit{None of the actions in } h_\kappa \textit{ is true}) \tag{34}$$

Recall κ is defined to be the first to not receive a signal, so

$$\begin{aligned} V =& \alpha(1-\beta)(1-\alpha) + \alpha(1-\beta)\alpha(1-\beta)(1-\alpha) \\ &+ \alpha(1-\beta)\alpha(1-\beta)\alpha(1-\beta)(1-\alpha) + ... \\ =& \Big[\sum_{\kappa=2}^{\infty} (\alpha(1-\beta))^{\kappa-1} \Big] \cdot (1-\alpha) \\ =& [1-\alpha(1-\beta)]^{-1}\alpha(1-\beta) \cdot (1-\alpha) \end{aligned} \tag{35}$$

in which $(1-\alpha)$ stands for DM κ not receiving a signal.

When $k_0 > 1$, notice she is, in essence, the first DM to start the tree. Returning to equation 33,

$$\begin{aligned} P(\textit{Incorrect Herding}) =& V + (1-\alpha)V + (1-\alpha)(1-\alpha)V + ... \\ =& \Big[\sum_{k_0=1}^{\infty} (1-\alpha)^{k_0-1} \Big] \cdot V \\ =& [1-\alpha(1-\beta)]^{-1}(1-\beta)(1-\alpha) \end{aligned} \tag{36}$$

Q.E.D. ■

References

[1] Abhijit V Banerjee. A simple model of herd behavior. *The quarterly journal of economics*, 107(3):797–817, 1992.

[2] Heski Bar-Isaac and Steven Tadelis. *Seller reputation*. Now Publishers Inc, 2008.

[3] Dirk Bergemann and Juuso Välimäki. Learning and strategic pricing. *Econometrica: Journal of the Econometric Society*, pages 1125–1149, 1996.

[4] Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of political Economy*, 100(5):992–1026, 1992.

[5] Sushil Bikhchandani and Sunil Sharma. Herd behavior in financial markets. *IMF Staff papers*, 47(3):279–310, 2000.

[6] Urs Birchler and Monika Bütler. *Information economics*, chapter 10. Routledge, 1999.

[7] Markus K Brunnermeier and Markus Konrad Brunnermeier. *Asset pricing under asymmetric information: Bubbles, crashes, technical analysis, and herding*. Oxford University Press on Demand, 2001.

[8] Christophe P Chamley. *Rational herds: Economic models of social learning*, chapter 3,4. Cambridge University Press, 2004.

[9] Leonardo Felli and Christopher Harris. Learning, wage dynamics, and firm-specific human capital. *Journal of Political Economy*, 104(4):838–868, 1996.

[10] Marco Ottaviani and Peter Sørensen. Information aggregation in debate: who should speak first? *Journal of Public Economics*, 81(3):393–421, 2001.

[11] David S Scharfstein and Jeremy C Stein. Herd behavior and investment. *The American economic review*, pages 465–479, 1990.

[12] Xavier Vives. How fast do rational agents learn? *The Review of Economic Studies*, 60(2):329–347, 1993.