A Simple Labour Matching Model

Computational Macroeconomics ECON5086

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1 Labour Matching Model: Theory

In this section, we introduce the theory for a simple labour matching model. Broadly classifying, this model belongs to a real business cycle model. We discuss the agents' problems and the labour matching technology. We define the genral equilibrium and the rational expectation equilibrium.

1.1 Households

Infinite homogenous households live for infinite discrete periods. The atemporal utility function is

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma},\ \sigma > 0 \tag{1}$$

Thus a representative household's problem is to maximize

$$\mathbb{E}\big[\sum_{0}^{\infty} \beta^t u(c_t)\big] \tag{2}$$

subject to the budget constraint

$$c_t + q_t b_{t+1} + \tau_t = w_t n_t + b(1-n_t) + b_t \tag{3}$$

in which the choice variables are (c_t, b_{t+1}), state variables are $(b_t,)$, (q_t, τ_t, w_t, n_t) are exogenous, and (β, b) are parameters. b is the unemployment benefits.

We have several things to explain. Firstly, the objective excludes (dis)utility of labour. Each individual is exogenously assigned to be either employed or unemployed. The labour supply is discrete from an individual worker's perspective. When considering a representative household, however, the individual-level labour supply is aggregated into n_t, and normalized to be $\in [0,1]$.

Secondly, b_t is a one-period maturity bond, freely borrowed or purchased. We do not assume borrowing limits. Lastly, w_t is real wage.

The representative household's FOC gives the Euler equation

$$u'(c_t) = \beta \mathbb{E}_t[\frac{u'(c_{t+1})}{q_t}] \tag{4}$$

with a relevant transversality condition.

1.2 Labour Matching Technology

New jobs are created at the beginning of period t and some jobs are exogenously dissolved at the end of period $t-1$, therefore, the law of employment motion is

$$n_t = (1-\delta)n_{t-1} + m_t \tag{5}$$

in which n_t is the employment involved in production this period, and m_t are new employment created by matching. Also, define unemployment to be the level at the start of the period

$$u_t = 1 - (1-\delta)n_{t-1} \tag{6}$$

The matching is determined by demand (job vacancy v_t) and supply (unemployment u_t) with

$$m_t = m u_t^{\alpha} v_t^{1-\alpha} \tag{7}$$

in which $\alpha \in (0,1)$ represents the elasticity of matches with respect to the unemployment rate, and $m \in (0,1)$ denotes the matching efficiency.

Define job market tightness as vacancy over unemployment, therefore tighter laboour market is favourable for workers:

$$\theta_t = \frac{v_t}{u_t} \tag{8}$$

and therefore the matching technology could be re-written as

$$m_t = m v_t \theta_t^{-\alpha} \tag{9}$$

1.3 Firms

Infinitely many homogenous firms produce the same product. A representative firm's production:

$$Y_t = e^{z_t} L_t \tag{10}$$

in which labour L_t is the only input. In general equilibrium, $L_t = n_t$.

Similar to the household's problem, a representative firm's problem is to maximize

$$\mathbb{E}_0[\sum_0^\infty \beta^t \frac{u'(c_t)}{u'(c_0)}(Y_t - w_t n_t - \kappa v_t)] \tag{11}$$

subject to law of employment motion (equation 5) and matching technology (equation 9):

$$n_t = (1-\delta)n_{t-1} + m v_t \theta_t^{-\alpha} \tag{12}$$

in which the choice variables are (v_t, n_t), the state variables are $(n_{t-1},)$, (θ_t, w_t) are assumed to be exogenous, and parameter κ is the cost of posting a vacancy.

In general, we simplify the firms' problems to a representative firm's problem. This is because essentially all firms are homogenous, and operate in a perfect competition market. As a result, their decisions must be identical. In addition, we have several things to explain.

Firstly, the discounting factor $\beta^t \frac{u'(c_t)}{u'(c_0)}$ originates from the firm's owner's utility. In a full-fledged model, the ownership of the firm will be pinned down, but here we could easily attribute the single representative firm's ownership to the single representative household. $u'(c_t)$ transfers value of real consumption in period t to utils, $\beta^t u'(c_t)$ discounts it to period 0 utils, and finally $\beta^t \frac{u'(c_t)}{u'(c_0)}$ transfers utils back to values of real consumption in period 0.

Secondly, we impose a timeline where firms post vacancies, and then matching takes place.

The representative firm's FOC gives

$$w_t = e^{z_t} - \frac{\kappa}{m}\theta_t^{-\alpha} + \beta(1-\delta)\mathbb{E}_t[\frac{u'(c_t)}{u'(c_0)}\frac{\kappa}{m}\theta_{t+1}^{-\alpha}] \tag{13}$$

1.4 Wage Determination Process

To complete the labour matching process, the representative firm and the representative household engage in a Nash-bargaining with a pre-determined parameter ζ:

$$\max_{w} \; (S_t^H)^{\zeta}(S_t^F)^{1-\zeta} \tag{14}$$

in which S_t are surpluses for the representative household and firm in units of real consumption. ζ reflects the representative household's bargaining power.

Of special notice is the comparative statics. **When ζ is larger**, households has a stronger bargaining power, and thus wage w_t is higher. Recall from the **WS-PS model**, the wage-setting curve is an upward-sloping curve. When households have stronger barganing power, **the whole WS curve shifts up, consequently squeezing employment lower** along the PS curve.

The Nash-bargaining process's FOC gives the wage determination equation[1]

$$w_t = \zeta\big(e^{z_t} + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_t)}{u'(c_0)}\kappa\theta_{t+1}\big]\big) + (1-\zeta)b \tag{15}$$

1.5 Government and General Equilibrium

An RBC model allows the existence for a government (fiscal policy). The role of the government is mainly to provide the unemployment benefit, by collecting a lump-sum tax

$$\tau_t = b(1-n_t), \; \forall t \tag{16}$$

It is worth noting that, **as the tax is lump-sum, the existence of fiscal policy does not change** the behaviour of employment. The only effect it has is to reduce the representative household's consumption level by exactly the amount of τ_t. Intuitively, the balanced budget above is equivalent to a transfer among the employed and unemployed workers.

Now, fix an initial employment level n_0 and a sequence of technology shocks $\{z_t\}_0^\infty$. **Define a general equilibrium** as an infinite sequence of the representative household's choice variables $\{c_t, b_{t+1}\}_0^\infty$, the representative firm's choice variables $\{v_t, n_t, Y_t\}_0^\infty$, the market condition variables $\{u_t, \theta_t\}_0^\infty$ derived from the choice variables, and the market prices $\{w_t\}_0^\infty$, such that they satisfy

[1]Please see appendix for derivation details

the Euler equation (equation 4), the **FOC for the representative firm** (equation 13), the **law of employment motion** (equation 12), the exogenously determined **Nash-bargaining result** (equation 15), the bonds-market clearing ($b_t = 0, \ \forall t$) and the following FP-adjusted **goods market clearing** condition:[2]

$$Y_t = c_t + \kappa(1 - (1-\delta)n_{t-1})\theta_t \tag{17}$$

A rational expectation equilibrium is then all the above variables, but **as functions** of $(n_0, \{z_t\}_0^\infty)$, as now the technology process is stochastic. We reduce our focus to the so-called **policy functions** of all these functions, namely consumption $c_t(\cdot,\cdot)$, employment $n_t(\cdot,\cdot)$ and tightness $\theta_t(\cdot,\cdot)$ as Markovian functions of (n_{t-1}, z_t).

2 Computational Derivation

As we have defined above, we finally reduce our focus to a three-variable system: (c_t, n_t, θ_t) is determined by the **law of employment motion**:

$$n_t = (1-\delta)n_{t-1} + m(1 - (1-\delta)n_{t-1})\theta_t^{1-\alpha} \tag{18}$$

the **good market clearing** condition:

$$e^{z_t} n_t = c_t + \kappa(1 - (1-\delta)n_{t-1})\theta_t \tag{19}$$

and the **wage determination equation** combining the FOC for the representative firm (equation 13) and the exogenously determined Nash-bargaining result (equation 15):

$$\frac{\kappa}{m}\theta_t^\alpha = (1-\zeta)(e^{z_t} - b) + \beta(1-\delta)\mathbb{E}_t\big[\frac{c_{t+1}^{-\sigma}(1 - \zeta m \theta_{t+1}^{1-\alpha})}{c_t^{-\sigma}}\frac{\kappa}{m}\theta_{t+1}^\alpha\big] \tag{20}$$

given state variable $(n_{t-1},)$ and parameters ($\zeta = 0.8$ with other parameters of standard calibrations).

2.1 Policy Function Iteration

Equation 18, 19 and 20 form the system we solve for (n_t, θ_t, c_t). We first find the steady state.

[2]taking the budget constraint (equation 3) and fiscal policy (equation 16) into consideration

At steady state, $(n_{t-1}, c_t, \theta_t) = (n_t, c_{t+1}, \theta_{t+1})$. At steady state, we remove stochasticity, fixing $z_t = 0$. Therefore, **a steady state is a tuple** (n^*, θ^*, c^*) that solves the system:

$$\begin{cases} n^* = (1-\delta)n^* + m(1-(1-\delta)n^*)(\theta^*)^{1-\alpha} \\ e^0 n^* = c^* + \kappa(1-(1-\delta)n^*)\theta^* \\ \frac{\kappa}{m}(\theta^*)^\alpha = (1-\zeta)(e^0 - b) + \beta(1-\delta)(1-\zeta m(\theta^*)^{1-\alpha})\frac{\kappa}{m}(\theta^*)^\alpha \end{cases} \tag{21}$$

equivalently, we can include the technology process

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad \epsilon \sim (0, \sigma_\epsilon^2) \tag{22}$$

and solve for **the fix point of this 4-equation system**, as performed during lectures. In either case, we employ a fix point finder, e.g. newton method, to solve the system of equations. With standard calibration, $(n^*, \theta^*, c^*) = (0.86, 1.25, 0.82)$.

Next we proceed to solving the policy functions $(n_t(\cdot,\cdot), \theta_t(\cdot,\cdot), c_t(\cdot,\cdot))$. We start with constructing the state space. In this labour matching model, there are two states: current period technology z_t, and start-of-period employment n_{t-1}.

Technology z_t follows an AR(1) process (equation 22), thus its unconditional variance

$$\sigma_z^2 = \frac{\sigma_\epsilon^2}{1-\rho^2} \tag{23}$$

define the domain of $z_t \in [-4\sigma_z, +4\sigma_z]$. It covers $> 99.99\%$ of z_t's realization.

Lag-employment n_{t-1} has an economic meaning. By definition, we have $n_{t-1} \in [0, 1]$. From an economic motivation, we interpret it as also employment ratio, and thus we are not interested in particularly low values. It would be a miracle to observe any major economy whose employment rate lies below 50%. A reasonable construction will be:

$$n_{t-1} \in [\frac{1}{2}n^*, 1] \tag{24}$$

Besides the economic motivation, shifting away from 0 also has a computational implication. A fix point finder usually performs unsatisfactorily around zero, and after crude attempts, we confirm that the employment policy function $n_t(\cdot,\cdot)$ falls between 0.5 and 1 always. Hence it will be more than enough to focus our attention to $n_{t-1} \in [\frac{1}{2}n^*, 1]$. [3][4]

[3]The computational motivation also arises since policy functions refuse to converge under the domain $n_{t-1} \in [0, 1]$.

[4]Special thanks attributed to ID2487169

We sample from domain using Chebyshev nodes, i.e., the zeros of the Chebyshev polynomials. Given an order of approximation N, the nodes are given by[5]

$$x_i = -\cos\left(\frac{2i-1}{2N}\pi\right),\ i = 1, 2, ..., N \tag{25}$$

One major advantage of using Chebyshev nodes are that the Chebyshoev polynomials remain orthogonal with each other when nodes are selected according to equation (25). Also, Chebyshev nodes sample more points around the centre.

After constructing the state space, we initialize the three policy functions. A flat guess at steady state values will suffice.

Now that we have the initial guess, we proceed to the iteration. At each state (of z_t and n_{t-1}), we solve for $(n_{ij}, \theta_{ij}, c_{ij})$ that solves the three-equation system (equation 18,19,20). Similarly, we employ a fix point finder (newton method) to find the solution.

Of specific notice is how we integrate the expectation. Observe

$$\mathbb{E}_t\left[c_{t+1}^{-\sigma}(1-\zeta m\theta_{t+1}^{1-\alpha})\theta_{t+1}^{\alpha}\right] = \int_{-\infty}^{\infty} c_{t+1}^{-\sigma}(z_{t+1}, n_t)(1-\zeta m\theta_{t+1}^{1-\alpha}(z_{t+1}, n_t))\theta_{t+1}^{\alpha}(z_{t+1}, n_t)dF(z_{t+1}|z_t) \tag{26}$$

Using Gauss-Hermite quadrature and reforming the distribution of $z_{t+1}|z_t$ to standard normal:

$$\mathbb{E}_t\left[c_{t+1}^{-\sigma}(1-\zeta m\theta_{t+1}^{1-\alpha})\theta_{t+1}^{\alpha}\right] \approx \frac{1}{\sqrt{\pi}}\sum_{0}^{N} c_{t+1}^{-\sigma}(z_i, n_t)(1-\zeta m\theta_{t+1}^{1-\alpha}(z_i, n_t))\theta_{t+1}^{\alpha}(z_i, n_t)\cdot w_i \tag{27}$$

in which z_i's are the N Chebyshev nodes we sample on the state space $[-4\sigma_z, +4\sigma_z]$, and w_i's are the Gauss-Hermite weights attached to each node z_i:

$$w_i = \frac{2^{N-1}N!\sqrt{\pi}}{N\Gamma_{N-1}(z_i)} \tag{28}$$

where $\Gamma_{N-1}()$ is the Chevyshev polynomial of order $N-1$.

Iterate through all states (of z_t and n_{t-1}), and we update the three policy functions. If the updated policy function is close enough to the original ones, we stop and conclude that we have found the policy functions. Otherwise, repeat the above until convergence.

[5]Default order for technology space is 5, and for lag-employment space is 6.

2.2 Policy Functions

We present the results with standard calibration below. The horizontal axis is the lag-employment, and we plot the policy function for different level of production technology.

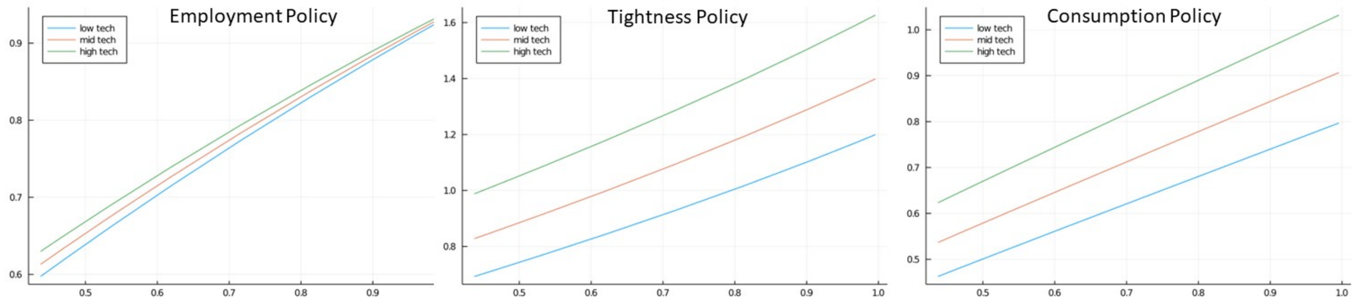

Figure 1: Policy Functions at $\zeta = 0.8$

Some comments: all three policy functions are increasing in both technology and lag-employment. It follows from intuition. **A higher technology is associated with more demand for labour**, thus tightening the job market, and benefits the representative household's consumptions in the end. **A higher lag-employment would keep current employment high**, keep unemployment low (thus tighter job market) and benefit the representative household's consumption.

It is worth noting that **the positive relationship between policy functions and lag-employment** $(n_t(\cdot, n_{t-1}), \theta_t(\cdot, n_{t-1}), c_t(\cdot, n_{t-1}))$ reflects **the internal persistence** of an exogenous positive technology shock. Facing a positive z_t shock, the representative households increase n_t. In the next period, due to higher n_t, c_{t+1} and n_{t+1} will be further higher in addition to the channel of external persistence (highr z_{t+1}). One reasonable explanation for this internal persistence is because the job retain rate $1-\delta$ is a constant, and knowing he/she will be more likely employed next period, the representative households' consumption increases.

3 Simulation Results

3.1 Economic Variables in One Time Series

In this section, we calculate the economic variables' processes, $(n_t, \theta_t, c_t, y_t, u_t, v_t)$, over a time series of 50 periods. Let me explain what I do in **one** time series. Before the simulation, we assume that the economy is in the steady state $(n_0 = n^*)$. All calibrations are standard.

To calculate the economic variables, we first generate a a time-series of random technology shocks $\{\epsilon_t\}_1^{50}$. With equation (22), we calculate the technology process $(\{z_t\}_1^{50})$. The technology shocks simulated in this **one** time series looks like the following figure 2:[6]

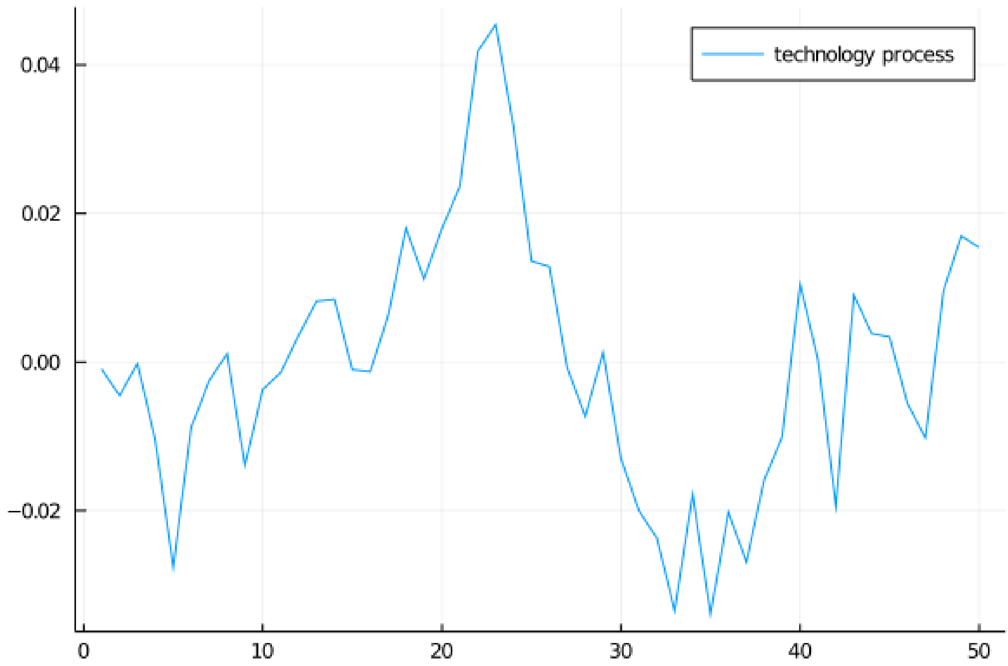

Figure 2: Technology Process of **One** time series

With n_0 and $\{z_t\}_1^{50}$, we now use the policy functions to calculate the processes for other economic variables. For brevity, in this subsection, we only show the process of consumption $(\{c_t\}_1^{50})$ and employment $(\{n_t\}_1^{50})$ for **one** time series over 50 periods.

[6]For replication, we fix the seed for random numbers. Please see the codes for detail

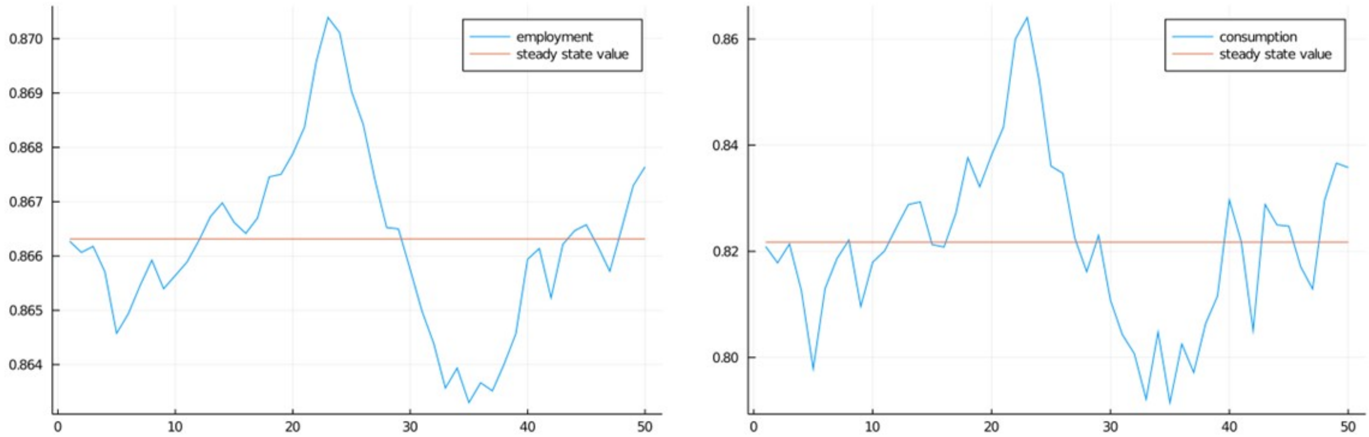

Figure 3: Economic Variables' Processes (n_t and c_t) of **One** time series

3.2 100_000 Time-series' Mean Values of Economics Variables

For each time series, we calculate one **mean value** for employment, tightness, consumption, output, unemployment and vacancy. **We calculate one value of consumer's welfare W**, which is based on all $\{c_t\}_1^{50}$:

$$W = \sum_{t=1}^{50} \beta^{t-1} \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \tag{29}$$

Compactly, **for each time series i, we calculate** $(\overline{n}, \overline{\theta}, \overline{c}, \overline{y}, \overline{u}, \overline{v}, W)_i$.

And we repeat this for $i = 1, 2, ..., 100\_000$, i.e., **we simulate 100_000 time series**. We are interested in the distribution of the **mean value** of each economic variable **and the welfare**. The plots below are the distribution of the **mean value** of each variables calculated by simulation of 100_000 time series. As in the previous section, we only show the distribution of mean employment $\{\overline{n}\}_{i=1}^{100\_000}$, mean consumptions $\{\overline{c}\}_{i=1}^{100\_000}$, and welfare $\{W\}_{i=1}^{100\_000}$, for brevity. [7]

Non-surprisingly, we see from figure 4 the **mean value** of each economic variable follows a normal distribution. This is guaranteed by the **law of large numbers**, because what we are plotting is the distribution of a mean variable. For example, we are estimating the **mean value** of consumption $\overline{c} = \frac{1}{50}\sum_{0}^{50} c_t$, which converges in distribution towards a standard normal centred around the steady state value of consumption ($\overline{c} \rightarrow^d N(c^*, \sigma_c)$). In other words, the fact that $\overline{c}$ follows a normal distribution **is independent from the fact that shocks are normally distributed.**

[7] For replication, we also fix the seed for random numbers.

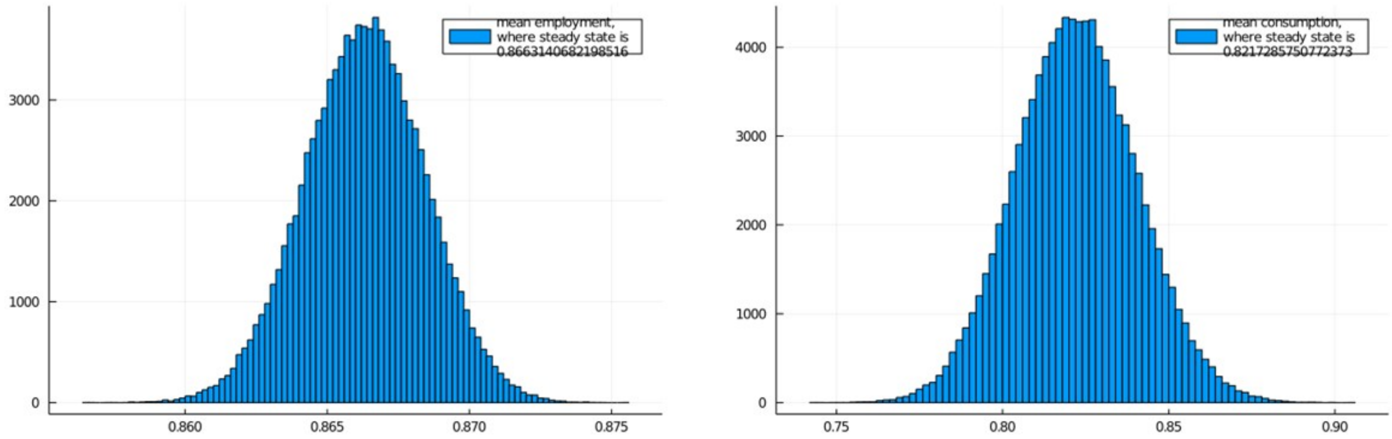

Figure 4: Economic Variables' **Mean Value**'s distribution ($\overline{n}$, $\overline{c}$), sampled from 100_000 time series

Furthermore, we plot the distribution of welfare (figure 5). Again, W follows a normal distribution centred around the representative household's welfare at steady state. Notice that welfare is negative because consumption is always $c_t < 1$ and $\sigma = 2$, therefore $\frac{c_t^{1-\sigma}-1}{1-\sigma}$ is negative.

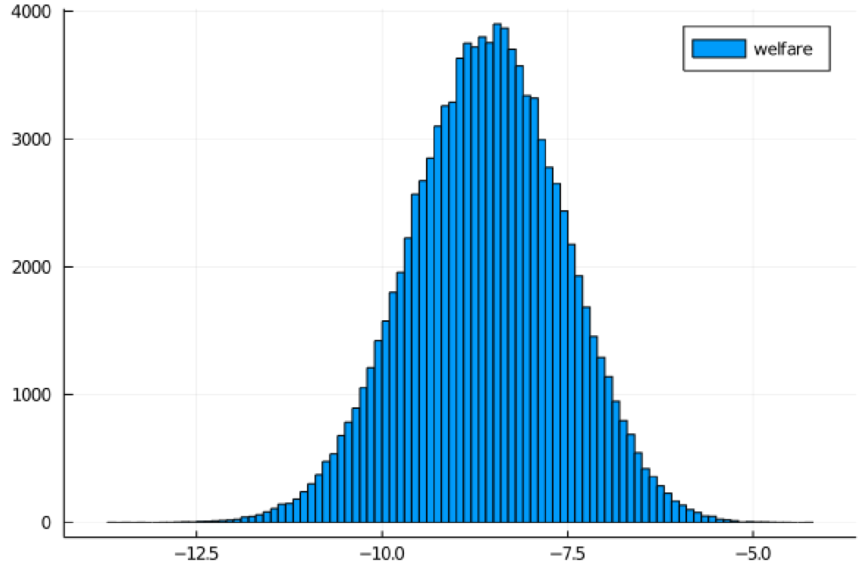

Figure 5: Welfare's distribution (W), sampled from 100_000 time series

The exact value of welfare, non-surprisingly, depends on how many periods the representative household lives. In each time series we simulate, we fix the time periods at 50, so the center of W is around -8.6. If we quadruple the time periods to 200 per each simulation, then the center of W is around -15.

4 Hosios Condition and Comparative Statics

In this section, we change the value of ζ. A higher bargaining power for households increases the wage and **decreases employment, intuitively using the WS-PS model** as we did in the theory section. We will axplain ζ's influence on employment, tightness, consumption and welfare in detail.

First we compare different steady states when households have different bargaining power. As predicted by theory, we observe that with higher bargaining power, the steady state value of employment falls. θ^* falls as well because unemployment increases (see figure 6)

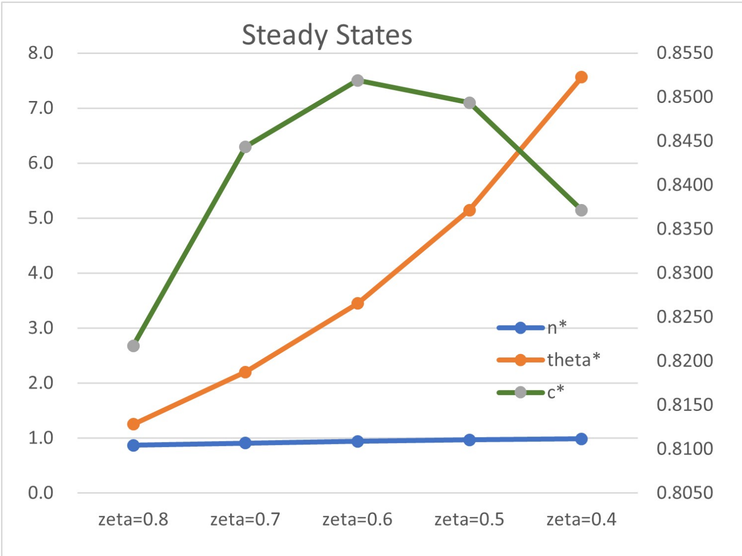

Figure 6: Steady States, n^*, θ^*'s values on the left axis, c^*'s values on the right

We also see c^* peaked when $\zeta = \alpha = 0.6$, and decreases both when bargaining power is too high and too low. When bargaining power is high, a representative household enjoys higher wage but suffers lower employment. When bargaining power is low, a representative household suffers lower wage but more employment.

While a simple WS-PS model delivers the intuition, we here provide a more detailed explanation. When workers' bargaining power is too low, a representative firm makes profit from the matching technology by posting too many vacancies, which leads to low wages and an **inefficiently high** employment rate. On the other hand, when workers have too much bargaining power, a representative firm has little incentive to post vacancies, and the employment rate is **inefficiently low**. Intuitively, we see the matching (and the Nash-bargaining) is efficient if and only if the representative firm has an appropriate incentive to post vacancies, i.e., they are having the same power on bargaining $(1-\zeta)$ as their share of the matching technology $(1-\alpha)$.

Rigorously, assume a benevolent social planner is choosing (n_t, θ_t, c_t) to maximize

$$W = \mathbb{E}_0 \Big[\sum_{0}^{\infty} \beta^t u(c_t) \Big] \tag{30}$$

subject to law of employment motion (equation 18) and goods market clearing (19). The FOC is

$$\frac{\kappa}{m}\theta_t^{\alpha} = (1-\alpha)(e^{z_t} - b) + \beta(1-\delta)\mathbb{E}_t \Big[\frac{c_{t+1}^{-\sigma}(1 - \alpha m \theta_{t+1}^{1-\alpha})}{c_t^{-\sigma}} \frac{\kappa}{m} \theta_{t+1}^{\alpha} \Big] \tag{31}$$

comparing equation (31) with equation (20), we obtain the Hosios condition

$$\zeta = \alpha \tag{32}$$

Hosios condition is reflected also in the simulation results. We now turn to simulation results for other values of bargaining power, for example, $\zeta = 0.6$. We calculate, in each simulation of 50 time periods, **one value of welfare** W, and then repeat this process 100_000 times to get

1) a distribution of $\{W\}_{i=1}^{100\_000}$ and

2) the **mean value of welfare** over all 100_000 simulations ($\overline{W} = \frac{1}{100\_000} \sum \{W\}_{i=1}^{100\_000}$). the distributions are plotted as followed:

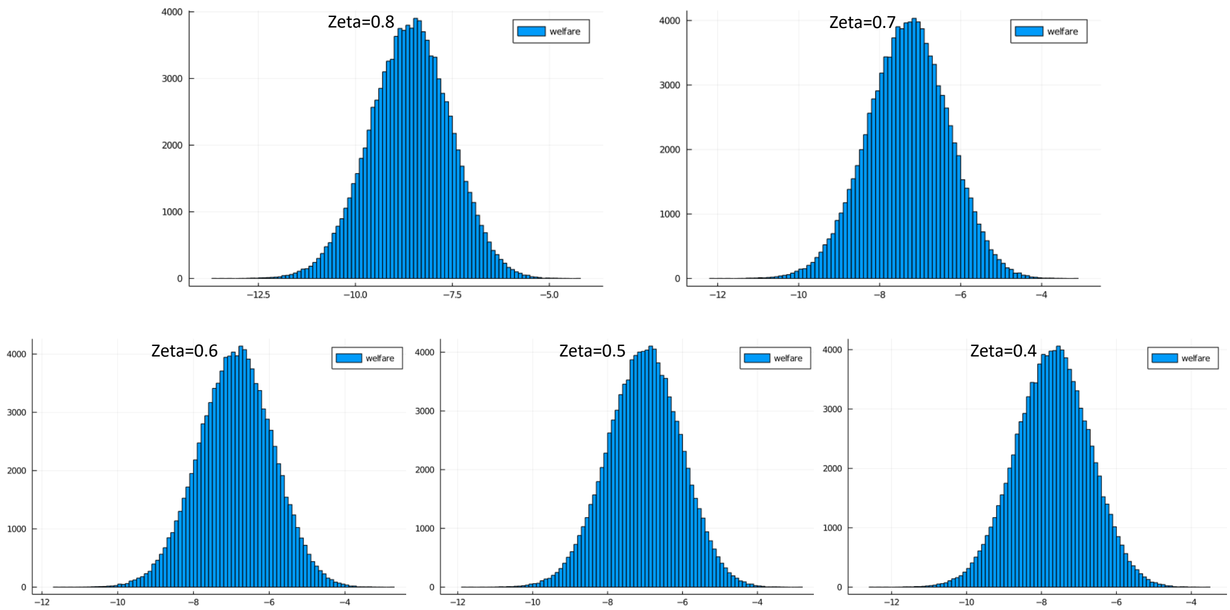

Figure 7: Distribution of W of different ζ, sampled from 100_000 time series

The **mean values** of welfare ($\overline{W}$) with different ζ's are:

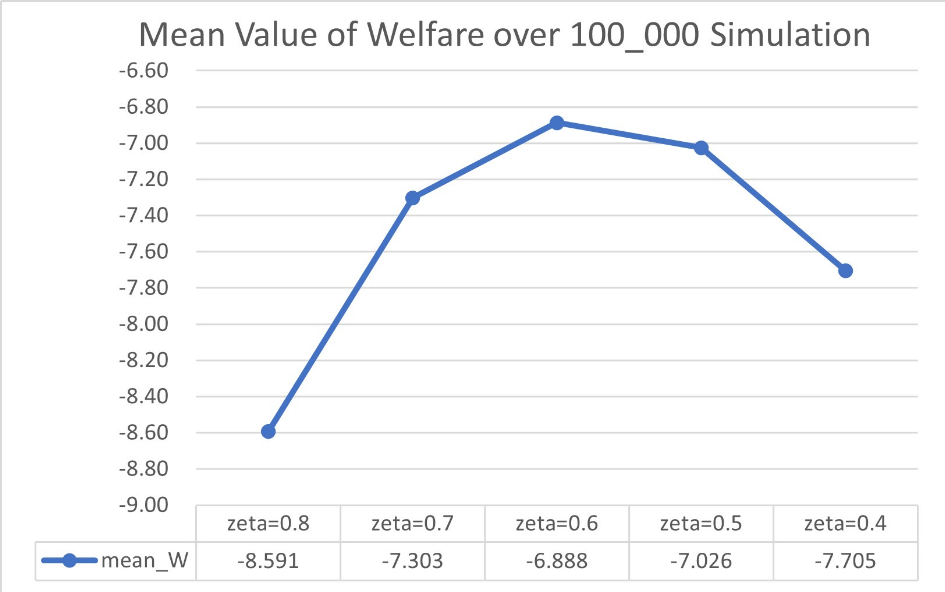

Figure 8: $\overline{W}$ of different ζ, sampled from 100_000 time series

As predicted, we observe the mean value of the representative household's welfare peaked when the Hosias condition is satisfied (figure 7). It reflects that the employment in the economy is at efficient level, averagely over the 50 time periods.

5 Limitations and Discussions

This simple labour matching model explains well the existence of unemployment even in steady state, and captures the efficient level of employment by Hosias condition. In our policy functions $c_t(z_t, n_{t-1})$ and $n_t(z_t, n_{t-1})$, we model a positive response of employment and consumption decision in face of a positive technology shock. Also, we see the internal persistence of a positive technology shock, because both policy functions are increasing functions of lag-employment as well. Finally, the consumer welfare is maximized if and only if Hosias condition is satisfied.

This model subjects to several aspects of limitation. First, as an RBC model, monetary authority does not play a role. Instead, we can develop a production sector with monopolistic competition and sticky prices to allow for a full-fledged New-Keynesian model with labour matching.

Second, as a representative household model, the results do not explain individual level policy. It is generally preferable to have an heterogeneous agent model capturing distribution of consumption, but in this model the limitation is critical subject to another concern. Because of the nature of a representative household, workers in this model perfectly insure each other within the household, and thus n_t is a continuous variable. However, employment is famous for its discreteness from an individual perspective. If we were to model this discreteness, we have to develop a heterogeneous agent model.

Finally, from a computational point of consideration, this model is suitable to explain 'normal' economy where employment rate $n \in [0.5, 1]$. The policy function iteration method (and I suppose neither would VFI) fails to converge on domain of employment close to 0. If we are interested in extreme cases, the validity of policy functions together with the model is under question. Through modifying the labour matching technology, this problem may potentially be relieved.

6 Appendix: Derivation of the Nash-bargaining Condition

Theorem 1 *Nash-bargaining Condition*

When the representative firm and the representative household maximizes the joint surplus

$$(S_t^H)^\zeta (S_t^F)^{1-\zeta} \tag{33}$$

for parameter $\zeta \in (0,1)$, the wage as a result of the Nash-bargaining is

$$w_t = \zeta\big(e^{z_t} + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_t)}{u'(c_0)}\kappa\theta_{t+1}\big]\big) + (1-\zeta)b \tag{34}$$

Proof.

In order to calculate S_t^F, we derive the representative firm's maximization problem:

$$\mathbb{E}_0\big[\sum_0^\infty \beta^t \frac{u'(c_t)}{u'(c_0)}(e^{z_t}n_t - w_t n_t - \kappa v_t)\big] \tag{35}$$

subject to

$$n_t = (1-\delta)n_{t-1} + m v_t \theta_t^{-\alpha} \tag{36}$$

taking θ_t, w_t as given and choosing n_t, v_t. The Lagrangian is

$$\mathcal{L}(n_t, v_t, \eta_t) = \mathbb{E}_0\big[\sum_0^\infty \beta^t \frac{u'(c_t)}{u'(c_0)}\big((e^{z_t}n_t - w_t n_t - \kappa v_t) - \eta_t(n_t - (1-\delta)n_{t-1} - m v_t \theta_t^{-\alpha})\big)\big] \tag{37}$$

and the FOC's are

$$\eta_t m\theta_t^{-\alpha} = \kappa \tag{38}$$

$$e^{z_t} - w_t - \eta_t + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)}\eta_{t+1}\big] = 0 \tag{39}$$

The Lagrangian multiplier η_t has a specific meaning. It reflects the 'marginal value of relaxing the constraint η'. This is a vague maening. However, it could be cleanly interpreted by our setup.

On the one hand, the 'relaxing of constraint' part corresponds to having one extra unit of labour. This is because the Lagrangian is designed specifically such that relaxing the constraint is equivalent to increasing one unit of n_t.

On the other hand, the 'marginal value' has been specially transformed to be measure in real consumption. This follows the same logic applied in the theory section (firm problem). eta_t's value of real consumption is transformed into period t utils through $u'(c_t)$, $\beta^t u'(c_t)$ discounts it to period 0 utils, and finally $\beta^t \frac{u'(c_t)}{u'(c_0)}$ transfers utils back to values of real consumption in period 0.

To sum up, therefore, $S_t^F = \eta_t$, measure in values of real consumption.

In order to calculate S_t^H, we derive the continuation value for the representative household when employed and unemployed:

$$V_t^U = b + \beta\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)}\big((1-f(\theta_{t+1}))V_{t+1}^U + f(\theta_{t+1})V_{t+1}^E\big)\big] \tag{40}$$

$$V_t^E = w_t + \beta\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)}\big(\delta(1-f(\theta_{t+1}))V_{t+1}^U + (1-\delta(1-f(\theta_{t+1})))V_{t+1}^E\big)\big] \tag{41}$$

in which using equation (9)

$$f(\theta_t) := \frac{m_t}{u_t} = m\theta_t^{1-\alpha} \tag{42}$$

is the job-filling probability, capturing the **exogenous** probability that an individual worker, when unemployed, finds a job in period t. It is worth special notice that there are two events under which an employed worker keeps being employed. With probability $(1-\delta)$ he/she keeps the job. With probability $\delta f(\theta)$ he/she gets separated from the 'original' job and find a 'new' job immediately. They add up to $1-\delta(1-f(\theta_{t+1}))$

Therefore the surplus of being matched a new job for the representative household is recursively

$$S_t^H := V_t^E - V_t^H = w_t - b + \beta\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)}(1-\delta)(1-f(\theta_{t+1}))S_{t+1}^H\big] \tag{43}$$

Now the FOC os the Nash-bargining process is

$$S_t^H = \frac{\zeta}{1-\zeta} S_t^F \tag{44}$$

for any period t. Plugging the FOC (equation 44) for t and $t+1$ into equation (43):

$$\eta_t = \frac{1-\zeta}{\zeta}(w_t - b) + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)}(1 - f(\theta_{t+1}))\eta_{t+1}\big] \tag{45}$$

To deal with equation (45), we need some technique. First plug in the second FOC of firm (equation 39) to substitute away η_t, and then combine the two expectation terms

$$e^{z_t} + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_{t+1})}{u'(c_t)} f(\theta_{t+1})\eta_{t+1}\big] = \frac{1}{\zeta}(w_t + (1-\zeta)b) \tag{46}$$

Not surprisingly, by the FOC of the firm (equation 38) and the definition of job-filling possibility (equation 42):

$$f(\theta_{t+1})\eta_{t+1} = \kappa\theta_{t+1} \tag{47}$$

Finally, we reach the desired result

$$w_t = \zeta\big(e^{z_t} + \beta(1-\delta)\mathbb{E}_t\big[\frac{u'(c_t)}{u'(c_0)}\kappa\theta_{t+1}\big]\big) + (1-\zeta)b \tag{48}$$

■

7 Codes Link

Please click the following link for the codes and the graphs, or copy and paste:

https://gla-my.sharepoint.com/:f:/g/personal/2495262w_student_gla_ac_uk/EnvagU6H6k5KrKfbQ2G7-UgBu6BcXZWRgTNZMKvyxJ3Bgw?e=idtraC

References

[1] David N. DeJong, Chetan Dave, and Dawson Books. *Structural macroeconometrics*. Princeton University Press, Princeton, NJ, 2nd edition, 2011.

[2] Jordi Galí. *Monetary policy, inflation, and the business cycle: an introduction to the new Keynesian framework and its applications*. Princeton University Press, Princeton;Oxford;, second edition, 2015.

[3] Arthur J. Hosios. On the efficiency of matching and related models of search and unemployment. *The Review of economic studies*, 57(2):279–298, 1990.

[4] Paul Klein. Using the generalized schur form to solve a multivariate linear rational expectations model. *Journal of economic dynamics control*, 24(10):1405–1423, 2000.

[5] Nancy L. Stokey, Robert E. Lucas, Edward C. Prescott, and Askews Holts Library Services. *Recursive methods in economic dynamics*. Harvard University Press, Cambridge, Massachusetts, 1989.