Topics in Macroeconomics

Endogeneous grid-point method (EGM)

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Motivation

Solving HH problems is often slow – Why?

- Consider standard infinite-horizon consumption-savings problem with states (a, y):
  - a Beginning-of-period assets
  - y Risky labour income following first-order Markov chain
- At each point (a_i, y_j) we maximise the objective

$$f(a') = u\left(x_{ij} - a'\right) + \beta \mathbf{E}\left[V(a', y') \,\middle|\, y_j \right]$$

  where x_{ij} is the cash at hand.
- Any numerical maximiser will call $f(\bullet)$ repeatedly to
  1. Determine the objective's value at some candidate point
  2. Determine the derivative at some candidate point
  3. Numerically differentiate the objective function
- This quickly adds up to numerous calls, which can be computationally expensive, depending on how difficult it is to compute expectations, etc.

Endogenous grid method

- The insight behind EGM (due to Carroll, 2006): Compute expectation only once!
- How can we do that if we don't know the optimal solution?
  - Exogenously impose the optimal solution (in the above case: a')
  - Determine implied beginning-of-period assets a
  - This gives rise to endogenous grid of beginning-of-period asset levels!

Endogenous grid-point method

Advantages

- Considerably faster than any other known method in this class of models
- No need for a maximiser or root-finder
- Works very well with *linear* interpolation, no need for splines, etc.

Disadvantages

- Does not always work
- Does not scale well to multiple continuous state or control variables (see Druedahl and Jørgensen (2017) for one attempted solution)
- Tricky (but possible) to combine with discrete choices, e.g. due to extensive-margin labour supply, fixed costs (see Iskhakov et al. (2017), Fella (2014))

Example: HH problem with risky labour

- Consider infinite-horizon consumption-savings problem

$$
\begin{aligned}
V(a, y) = \max_{c,\, a'} & \left\{ u(c) + \beta \mathbf{E}\left[V(a', y') \,\middle|\, y \right] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}
$$

  where

  y Labour income process on state space $\mathcal{G}_y$ with transition probability $\Pr\left(y' = y_j \,|\, y = y_i \right) = \pi_{ij}$

- Preferences are CRRA:

$$
u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}
$$

Illustration of “standard” approach

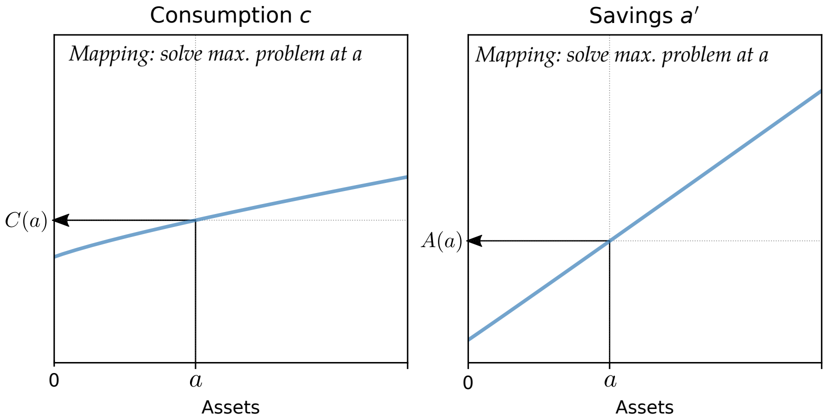

Figure 1: Mapping from exogenous assets to consumption and savings.

Illustration of EGM approach

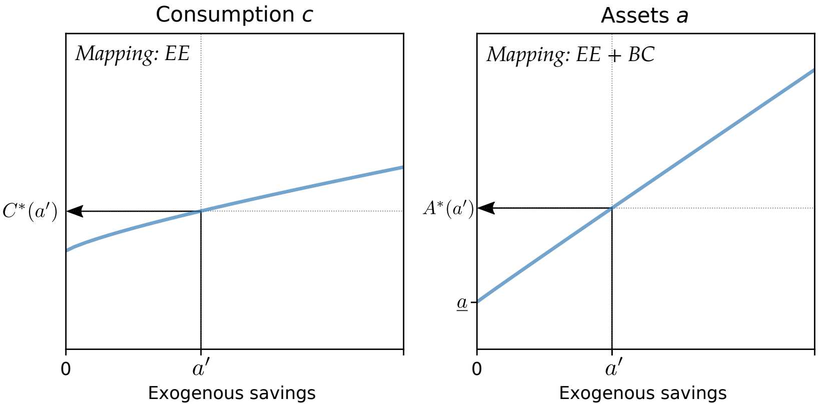

Figure 2: Mapping from exogenous savings to consumption and assets.

Deriving the Euler equation

- Combining the FOCs for c and a' yields the Euler equation

$$u'(c) = \beta \mathbf{E} \left[\partial V(a', y') / \partial a' \,\middle|\, y \right] \tag{1}$$

- For this problem, the envelope condition (see (5)) is

$$\frac{\partial V(a, y)}{\partial a} = (1 + r) u'\big(C(a, y)\big) \tag{2}$$

  where $C(a, y)$ is the consumption policy function.

- Combine (1) and (2) to get the more "familiar" variant of the Euler equation:

$$u'(c) = \beta (1 + r) \mathbf{E} \left[u'\big(C(a', y')\big) \,\middle|\, y \right]$$

Using the Euler equation

- Assume we know or have guessed $C(a', y')$
- We can exogenously fix a' and use $u'(c) = c^{-\gamma}$ to get an equation in a single unknown, c:

$$c = \left(\beta(1+r)\mathbf{E}\left[C(a', y')^{-\gamma} \,\middle|\, y \right] \right)^{-\frac{1}{\gamma}} \tag{3}$$

- From the BC, we can recover the implied beginning-of-period asset level a:

$$a = \frac{1}{1+r}\left[c + a' - y\right] \tag{4}$$

Solution to household problem

- To summarise, we found

  $c = C^*(a', y)$ Optimal consumption as a function of a'

  $a = A^*(a', y)$ Beginning-of-period assets as a function of a'

- Each a'_i gives us a tuple (a_i, c_i):
  - Use $(a_i, c_i)_{i=1}^{N_{a'}}$ to interpolate consumption policy onto exogenous beginning-of-period asset grid, $c = C(a, y)$
  - Use $\left(a_i, a'_i\right)_{i=1}^{N_{a'}}$ to interpolate savings policy onto exogenous beginning-of-period asset grid, $a' = A(a, y)$
- Important: using the Euler eq. implies that HH is at interior solution!
  - Implication: $\underline{a} = A^*(0, y)$ for $a' = 0$ is the highest asset level for which household does *not* save anything.
  - HH consumes everything for lower asset levels:

$$C(a, y) = (1 + r)a + y \qquad \forall\, a \leq \underline{a}$$

Solution algorithm (infinite horizon)

1. Fix exogenous savings grid $a' \in \mathcal{G}_{a'} = \left(a'_1, \ldots, a'_{N_{a'}}\right)$
2. Fix initial guess for consumption policy, $C_1(a, y)$. Usually the guess is to consume all resources.
3. In iteration n, proceed as follows:
   1. For each point (a'_i, y_j), compute the expectation
   $$m'_{ij} = \mathbf{E}\left[\, C_{n-1}(a'_i, y')^{-\gamma} \,\middle|\, y_j \,\right]$$
   2. Inver the Euler eq. as in (3) to get consumption today:
   $$c_{ij} = \left[\beta(1+r)m'_{ij}\right]^{-\frac{1}{\gamma}}$$
   3. Use the budget constraint as in (4) to find beginning-of-period assets:
   $$a_{ij} = \frac{1}{1+r}\left[c_{ij} + a'_i - y_j\right]$$
   4. Use the points (a_{ij}, c_{ij}) to get the updated consumption policy $C_n(\bullet, y_j)$ for each j. Set $C_n(a, y_j) = (1+r)a + y_j$ for all $a \leq \underline{a}_j$
4. Terminate iteration when C_{n-1} and C_n are close.

Solution algorithm (finite horizon)

1. Fix exogenous savings grid $a' \in \mathcal{G}_{a'} = \left(a'_1, \ldots, a'_{N_{a'}}\right)$
2. Compute consumption policy in terminal period T: this is usually $C_T(a, y) = (1 + r)a + y$, unless there is a bequest motive.
3. For each period $t = T - 1, T - 2, \ldots, 1$, proceed as follows:
   1. For each point (a'_i, y_j), compute the expectation

   $$m'_{ij} = \mathbf{E}\left[\, C_{t+1}(a'_i, y')^{-\gamma} \,\middle|\, y_j \,\right]$$

   2. Inver the Euler eq. as in (3) to get consumption today:

   $$c_{ij} = \left[\beta(1 + r)m'_{ij}\right]^{-\frac{1}{\gamma}}$$

   3. Use the budget constraint as in (4) to find beginning-of-period assets:

   $$a_{ij} = \frac{1}{1 + r}\left[c_{ij} + a'_i - y_j\right]$$

   4. Use the points (a_{ij}, c_{ij}) to get consumption policy $C_t(\bullet, y_j)$ for each j. Set $C_t(a, y_j) = (1 + r)a + y_j$ for all $a \leq \underline{a}_j$

Parametrisation for HH problem with risky labour

EGM with linear interpolation

- Assume labour process follows AR(1),

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad\qquad \varepsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma^2\right)$$

  which we discretise as a Markov chain using the Rouwenhorst (1995) or Tauchen (1986) methods.

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
ρ	Autocorrelation of AR(1) process	0.95
σ	Conditional std. dev. of AR(1) process	0.20
N_y	Number of states for Markov chain	3

Table 1: Parameters for HH problem with risky labour income

Policy functions

EGM with linear interpolation

Solution for different income levels: low, middle, high

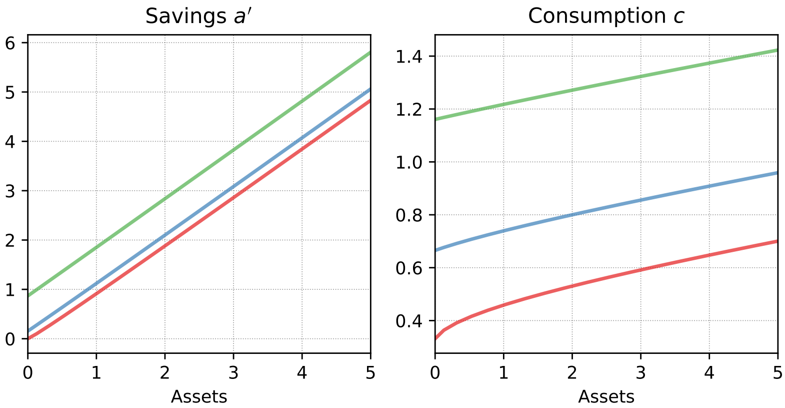

Figure 3: Solution with approx. 100 points on savings grid.

Functions of exogenous savings grid

EGM with linear interpolation

Solution for different income levels: low, middle, high

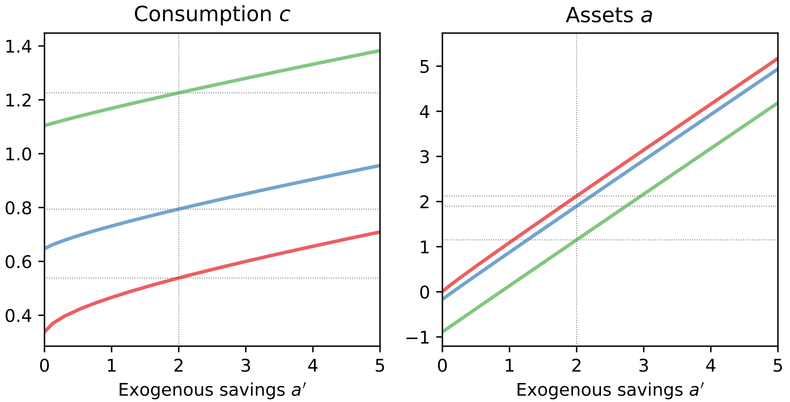

Figure 4: Solution with approx. 100 points on savings grid.

Relative run times

Run times for solving the above problem with $N_a = N_{a'} = 1000$ and $N_y = 3$:

Method	Time (seconds)	Rel. time
VFI – grid search	12.8	1.00
VFI – linear interpolation	170.6	13.32
EGM	0.4	0.03

When plain EGM fails

- Whenever we cannot determine where we “came from” (e.g. models with default)
- Discrete choices introduce jumps in policy functions:

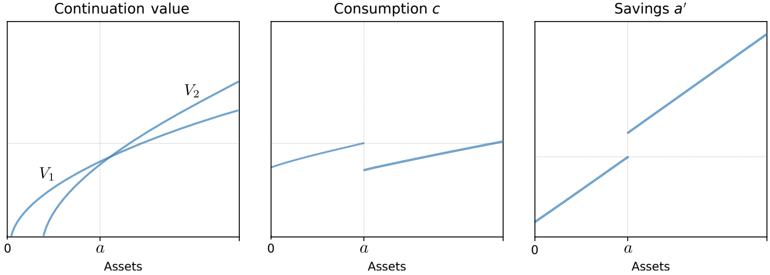

Figure 5: Jumps due to discrete choice variables.

Main take-aways

Use EGM whenever you can!

- With only one continuous state, no discrete choices:
  - Straightforward application of plain EGM, potentially with minor extensions
  - Also includes models with portfolio choice, intensive-margin labour supply
- With additional discrete choice variables:
  - Probably works, but more tedious (e.g. Iskhakov et al. (2017))
  - Still considerably faster than VFI
- With multiple continuous state variables:
  - Probably not worth the effort

Appendix

Envelope condition

- Consider the following value function, where $a^\star$ are optimal savings $a^\star = A(a, y)$:

$$V(a, y) = u\left((1+r)a + y - a^\star\right) + \beta \mathbf{E}\left[V(a^\star, y) \middle| y \right]$$

We used the BC to substitute for $c^\star = (1+r)a + y - a^\star$

- Take derivatives w.r.t. a:

$$\begin{aligned} \frac{\partial V(a,y)}{\partial a} &= u'\left((1+r)a + y - a^\star\right)\left[(1+r) - \frac{\partial a^\star}{\partial a}\right] + \beta \mathbf{E}\left[\frac{\partial V(a^\star, y)}{\partial a^\star} \frac{\partial a^\star}{\partial a} \middle| y \right] \\ &= u'\left(c^\star\right)(1+r) + \frac{\partial a^\star}{\partial a} \underbrace{\left\{ -u'\left(c^\star\right) + \beta \mathbf{E}\left[\frac{\partial V(a^\star, y)}{\partial a^\star} \middle| y \right] \right\}}_{=0} \end{aligned} \quad (5)$$

- The FOC implies that the second term on the r.h.s. is zero!

References I

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