2.1

$$\underset{2k\times 1}{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

prior

$$\pi(\beta) \propto \exp\left(-\frac{1}{2}(\beta-\beta_0)' B_0^{-1} (\beta-\beta_0)\right) \quad (*)$$

likelihood

$$p(y_g \mid w_g, x_g, \lambda, \beta, \alpha_g)$$

$$\propto \exp\left(-\frac{1}{2}(\varepsilon_g - 0)' (\delta^2 I_{n_g})^{-1} (\varepsilon_g - 0)\right)$$

$$\Rightarrow p(\{y_g\} \mid \{w_g\}, \{x_g\}, \lambda, \beta, \{\alpha_g\})$$

$$\propto \prod_{g=1}^{G} \exp\left(-\frac{1}{2} \varepsilon_g' (\delta^2 I_{n_g})^{-1} \varepsilon_g\right)$$

$$= \exp\left(-\frac{1}{2} \sum_{g=1}^{G} \varepsilon_g' (\delta^2 I_{n_g})^{-1} \varepsilon_g\right) \quad (**)$$

posterior

$$p(\beta \mid \{y_g\}, \{w_g\}, \{x_g\}, \lambda, \{\alpha_g\})$$

$$\propto \pi(\beta)\, p(\{y_g\} \mid \{w_g\}, \{x_g\}, \lambda, \beta, \{\alpha_g\})$$

$$= \exp\left(-\frac{1}{2}\left\{(\beta-\beta_0)' B_0^{-1} (\beta-\beta_0) + \sum_{g=1}^{G} \varepsilon_g' (\delta^2 I_{n_g})^{-1} \varepsilon_g\right\}\right)$$

Expand $(*)$ and $(**)$:

$$(*) = -\frac{1}{2}\beta' B_0^{-1}\beta + \frac{1}{2}\beta_0' B_0^{-1}\beta + \frac{1}{2}\beta' B_0^{-1}\beta - \underbrace{\frac{1}{2}\beta_0' B_0^{-1}\beta_0}$$

$$= \underbrace{-\frac{1}{2}\beta' B_0^{-1}\beta}_{-\frac{1}{2}\beta' B^{-1}\beta \text{ term}} + \underbrace{\beta_0' B_0^{-1}\beta}_{\hat{\beta}' B^{-1}\beta \text{ term}} - \underbrace{\frac{1}{2}\beta' B_0^{-1}\beta_0}_{\text{other term}}$$

$$(**) = \underbrace{-\frac{1}{2}\beta' \sum_{g=1}^{G} X_g' (\delta^2 I_{n_g})^{-1} X_g}_{-\frac{1}{2}\beta' B^{-1}\beta \text{ term}}\, \beta + \underbrace{\sum_{g=1}^{G} (y_g' S_g' - \alpha_g' \alpha_g')(\delta^2 I_{n_g})^{-1} X_g \beta}_{\hat{\beta}' B^{-1}\beta \text{ term}} + \text{other term}$$

Re-arrange (*) + (**) into $-\frac{1}{2}\beta' B^{-1}\beta$ term $+\ \hat{\beta}' B^{-1}\beta$ term + other term

$$
\begin{aligned}
(*) + (**) = & -\underbrace{\frac{1}{2}\beta'\left(\sum_{g=1}^{G} X_g'(\delta^2 I_{n_g})^{-1} X_g + B_0^{-1}\right)\beta}_{-\frac{1}{2}\beta' B^{-1}\beta} \\
& + \underbrace{\left(\beta_0' B_0^{-1} + \sum_{g=1}^{G}(y_g' s_g' - l_g' \alpha_g')(\delta^2 I_{n_g})^{-1} X_g\right)}_{\hat{\beta}' B^{-1}\beta}\beta \\
& + \text{other term}
\end{aligned}
$$

$$\propto \mathcal{N}_{2k}(\beta; \hat{\beta}, B)$$

where
$$\boxed{B^{-1} = \left(\sum_{g=1}^{G} X_g'(\delta^2 I_{n_g})^{-1} X_g + B_0^{-1}\right)}$$

$$\hat{\beta}' B^{-1} = \beta_0' B_0^{-1} + \sum_{g=1}^{G}(y_g' s_g' - l_g' \alpha_g')(\delta^2 I_{n_g})^{-1} X_g$$

$$
\begin{aligned}
\hat{\beta}' &= \hat{\beta}' B^{-1} B \\
&= \left(\beta_0' B_0^{-1} + \sum_{g=1}^{G}(y_g' s_g' - l_g' \alpha_g')(\delta^2 I_{n_g})^{-1} X_g\right) B
\end{aligned}
$$

$$\boxed{\hat{\beta} = B\left(B_0^{-1}\beta_0 + \sum_{g=1}^{G} X_g'(\delta^2 I_{n_g})^{-1}(s_g y_g - \alpha_g l_g)\right)}$$

2-2

prior: $\pi(\alpha_g) \propto \exp\left(-\frac{1}{2}(\alpha_g-\alpha_0)' A_0^{-1}(\alpha_g-\alpha_0)\right)$

likelihood: $p(y_g | W_g, x_g, \lambda, \beta, \alpha_g) \propto \exp\left(-\frac{1}{2}\, \varepsilon_g' (\delta^2 I_{n_g})^{-1} \varepsilon_g\right)$

posterior:

$$p(\alpha_g | y_g, W_g, x_g, \lambda, \beta) = \pi(\alpha_g) \cdot p(y_g | W_g, X_g, \lambda, \beta, \alpha_g)$$

$$\propto \exp\left(-\frac{1}{2}\left\{\underset{(*)}{(\alpha_g-\alpha_0)' A_0^{-1}(\alpha_g-\alpha_0)} + \underset{(**)}{\varepsilon_g'(\delta^2 I_{n_g})^{-1}\varepsilon_g}\right\}\right)$$

$$\propto N(\alpha_g ; \hat{\alpha}_g, A_n)$$

wher

$$(*) = -\frac{1}{2}\alpha_g' A_0^{-1}\alpha_g + \alpha_0' A_0^{-1}\alpha_g - \frac{1}{2}\alpha_0' A_0^{-1}\alpha_0$$

$$-\frac{1}{2}\varepsilon_g' (\delta^2 I_n)^{-1}\varepsilon_g$$

$$= -\frac{1}{2}\left[y_g' S_g' - \beta' X_g' - \underline{l_g' \alpha_g'}\right](\delta^2 I_{n_g})^{-1}\left[S_g y_g - X_g \beta - \underline{\alpha_g l_g}\right]$$

$$= -\frac{1}{2} l_g' \alpha_g' (\delta^2 I_{n_g})^{-1} \alpha_g l_g$$

$$+\frac{1}{2}(y_g' S_g' - \beta' X_g')(\delta^2 I_{n_g})^{-1}\alpha_g l_g + \frac{1}{2} l_g' \alpha_g' (\delta^2 I_{n_g})^{-1}(S_g y_g - X_g \beta)$$

$$-\frac{1}{2}(y_g' S_g' - \beta' X_g')(\delta^2 I_{n_g})^{-1}(S_g y_g - X_g \beta)$$

Re-arrange into $-\frac{1}{2}\alpha_g' A_n^{-1}\alpha_g$ term + $\hat{\alpha}' A_n^{-1} \alpha_g$ term + other

$$= -\frac{1}{2}\alpha_g' l_g' (\delta^2 I_{n_g})^{-1} l_g \alpha_g + (y_g' S_g' - \beta' X_g')(\delta^2 I_{n_g})^{-1} l_g \alpha_g + \text{other}$$

And then, re-arrange $(*)+(**)$ into $-\frac{1}{2}\alpha_g' A_n^{-1}\alpha_g + \hat{\alpha}_g' A_n^{-1}\alpha_g$ + other

$$(*)+(**) = -\frac{1}{2}\alpha_g'\left(A_0^{-1}+l_g'(\delta^2 I_{n_g})^{-1}l_g\right)\alpha_g$$

$$+\left(\alpha_0' A_0^{-1} + (y_g' S_g' - \beta' X_g'(\delta^2 I_{n_g})^{-1}l_g)\right)\alpha_g + \text{other}$$

$$\Rightarrow \boxed{A_n = \left(A_0^{-1}+l_g'(\delta^2 I_{n_g})^{-1}l_g\right)^{-1}}$$

$$\hat{\alpha}_g' A_n^{-1} = \left(\alpha_0' A_0^{-1} + (y_g' S_g' - \beta' X_g'(\delta^2 I_{n_g})^{-1}l_g\right)$$

$$\Rightarrow \boxed{\hat{\alpha}_g = A_n\left(A_0^{-1}\alpha_0 + l_g'(\delta^2 I_{n_g})^{-1}(S_g Y_g - X_g\beta)\right)}$$