Pattern for Induction

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October 20, 2020

1 Introduction

The pattern for induction has the following elements (see Section 5.1.5 of the textbook).

- The predicates used in induction (for this class) are typically equalities or inequalities, like $P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ or like $P(n) : n \geq 2^n$. For fall 2020 and winter 2021, we will only talk about (and test on) equalities.
- We want to know if these predicates are true for an infinite number of values for n.
- Induction has two steps: the *basis step* and the *induction step*.
- The basis step is usually pretty easy. The basis step requires a proof of $P(b)$ for the smallest value of b for which the equality is true.
- The induction step has a **pattern** that can often be followed, with the key part of the pattern going by the name of *the induction hypothesis*. Simply put, the induction step requires you to prove $P(k) \to P(k+1)$. We do this by assuming that $P(k)$ is true and using that assumption to prove $P(k+1)$. You label the step of the proof where use the assumption that $P(k)$ as something like "by the induction hypothesis".
- You need to label the steps of an induction proof, calling out when you begin and end the basis step, when you begin and end the induction step, when you use the induction hypothesis during the induction step, and when you have completed the proof.

We'll look at the *induction step pattern* in detail in this tutorial. The examples include an equality proof (sufficient for Fall 2020 and Winter 2021) and an inequality proof (which can be ignored for Fall 2020 and Winter 2021).

2 Applying the Induction Pattern to an Equality

Consider Example 3 from Section 5.1.7 of the textbook. The predicate that we want to show is true for all $n \geq 0$ is:

$$P(n): \quad \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

The predicate takes two arguments, the left-hand side (LHS) of the equality and the right-hand side (RHS) of the equality. We will have completed the proof by induction for all values of n when we show that the LHS = RHS for all $n \geq 0$. I'll use the color red for the LHS and the color blue for the RHS. Thus,

$$P(n): \quad \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

The Basis Step

We want to show that $P(n)$ is true for all $n \geq 0$. Recall form the pattern that the *basis step* is to prove $P(b)$ fo the smallest value of b. Since we want to show that $P(n)$ is true for all $n \geq 0$, we set $b = 0$. (I use b to emphasize that this is the basis step.) We prove the basis step by substituting $b = 0$ into the LHS and the RHS and seeing if they match. Begin with the LHS.

$$\text{LHS} \quad : \sum_{i=0}^{0} 2^i = 2^0 = 1.$$

And now for the RHS.

$$\text{RHS} \quad : 2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1.$$

The LHS and the RHS match, so $P(0)$ is true.

The Induction Step

The induction step is to prove $P(k) \rightarrow P(k+1)$, where we get to assume that $k \geq b$, that is, k is at least as big as the value used in the basis step.

Some Background

Although our task is to prove that $P(k) \rightarrow P(k+1)$, the actual task is to *prove* $P(k+1)$ *using the assumption that* $P(k)$ *is true*. We could go through a formal justification of why we do this, but intuitively the justification is that an implication $P(k) \rightarrow P(k+1)$ must satisfy the truth table

$P(k)$	$P(k+1)$	$P(k) \to P(k+1)$
T	**T**	**T**
T	F	F
F	T	T
F	F	T

The only "interesting" row in the table is the first row, so the induction step gets to assume $P(k)$ is true and use that to prove $P(k+1)$ is true. 5

What Are P(k) and P$(k+1)$?

I like to start an induction proof by writing down what $P(k)$ and $P(k+1)$ are. I'll do this side-by-side, emphasizing the LHS and RHS.

$P(k)$			$P(k+1)$		
LHS	=	RHS	LHS	=	RHS
$\sum_{i=0}^{k} 2^i$	=	$2^{k+1} - 1$	$\sum_{i=0}^{k+1} 2^i$	=	$2^{k+1+1} - 1$

Notice how $P(k+1)$ is derived from $P(k)$ by replacing every occurrence of k in $P(k)$ by $k+1$ in $P(k+1)$.

What Do I Do with P$(k+1)$?

Our task is to make the parts of $P(k+1)$ look a lot like the parts of $P(k)$ so that I can use the induction hypothesis (the assumption that $P(k)$ is `true`) to show that the LHS and RHS in the $P(k+1)$ column of the table match.

Let's look at the LHS of $P(k+1)$ and try to make it look like the LHS of $P(k)$. For sums like this, it is pretty easy:

$$\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1}.$$

I got this by noting that

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= 2^0 + 2^1 + \ldots 2^k + 2^{k+1} \\
&= \left(2^0 + 2^1 + \ldots 2^k\right) + 2^{k+1} \\
&= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1}
\end{aligned}$$

The formula in the parentheses, $\left(\sum_{i=0}^{k} 2^i\right)$ is the LHS of $P(k)$. The stuff left over, 2^{k+1} is the *difference between* $P(k)$ *and* $P(k+1)$.

Using the Induction Hypothesis

Here's where the induction hypothesis kicks in. Since I assumed that $P(k)$ was true, I assumed that

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

We can plug this hypothesis into the derivation.

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(2^0 + 2^1 + \ldots 2^k\right) + 2^{k+1} \\
&= \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} \\
&= \left(2^{k+1} - 1\right) + 2^{k+1}
\end{aligned}$$

where the step between the second and last lines happened because of the induction hypothesis.

Our goal is to see if we can make the last line look like the RHS of $P(k+1)$, which is $2^{k+1+1} - 1$. Dropping the parentheses (and the colors when they no longer make sense) and regrouping gives

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= \left(2^{k+1} - 1\right) + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \\
&= 2^{k+1} + 2^{k+1} - 1 \\
&= \left(2^{k+1} + 2^{k+1}\right) - 1 \\
&= \left(2 \cdot 2^{k+1}\right) - 1 \\
&= \left(2^{k+1+1}\right) - 1 \\
&= 2^{k+1+1} - 1,
\end{aligned}$$

which is the RHS of $P(k+1)$. Thus, we showed that the LHS $\sum_{i=0}^{k+1} 2^i$ of $P(k+1)$ equalled the RHS $2^{k+1+1} - 1$ of $P(k+1)$ by assuming that $P(k) = T$ and using this assumption to make the red to blue substitution in the line of reasoning above.

This means that the implication $P(k) \to P(k+1)$ is true, and the induction step is complete.

Concluding the Proof

You must include a statement like the following to conclude your proof. The basis step is true, the induction step is true, and therefore by induction the predicate $P(n)$ is true for all $n \geq 0$.

3 Applying the Induction Pattern to an Inequality

For equalities, I pretty much rely on the pattern. For inequalities, I adapt it a bit. Consider example 6 from Section 5.1.7 of the textbook.

$$P(n) \quad 2^n < n!$$

for $n \geq 4$. This is an inequality predicate.

The Basis Step

Let's break $P(n)$ into it's LHS and RHS,

$$P(n) \quad 2^n < n!.$$

Let's plug in $n = 4$ into the LHS,

$$2^4 = 16,$$

and the RHS

$$4! = 24.$$

This plugging and chugging yields

$$P(4) \quad 16 = 2^4 < 4! = 24.$$

The basis step is complete.

The Induction Step

Let's start with the statement of $P(k)$ and $P(k+1)$.

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	$<$	$(k+1)!$

Following the pattern above, let's start with the LHS of $P(k+1)$ and see if we can make it look like the LHS of $P(k)$.

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	?	$(k+1)!$
			$2 \cdot 2^k$	?	$(k+1)!$
			$2 \cdot 2^k$	?	$(k+1)!$

I've changed the colors around so that we can see the LHS of $P(k)$ appearing in the LHS in the $P(k+1)$ column. I've also replaced the $<$ with a ? in the $P(k+1)$ column to call our attention to the fact that we need to prove the inequality. Note that the uncolored 2 in $2 \cdot 2^k$ is the difference between the LHS of $P(k)$ and the LHS of $P(k+1)$.

The induction hypothesis is hanging out in the $P(k)$ column. Let's plug it in.

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	?	$(k+1)!$
			$2 \cdot 2^k$	?	$(k+1)!$
			$2 \cdot 2^k$	?	$(k+1)!$
			$2 \cdot 2^k < 2 \cdot k!$	?	$(k+1)!$

The *induction hypothesis* worked as follows. We (a) plugged the claim made in the $P(k)$ column into (b) the LHS of the $P(k+1)$ column.

Let's take a snapshot of what we know so far:

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	?	$(k+1)!$
			$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!$	?	$(k+1)!$

Here's where we'll deviate from the pattern for the equality that was given in the previous section. It's not uncommon when I do induction proofs on inequalities to reach a point where I'm not sure how to proceed using just the LHS in the $P(k+1)$ column. When that happens, I start playing around with the RHS of the $P(k+1)$ column.

Let's start playing around with the RHS in the $P(k+1)$ column to see if we can make it look like what is happening to the LHS of the $P(k+1)$ column. The goal will be to make the LHS look a lot like the RHS, close enough that we can figure out how to proceed.

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	?	$(k+1)!$
			$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!$	?	$(k+1)k! = (k+1)!$
			$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!$	?	$(k+1)k! = (k+1)!$

The LHS and RHS on the bottom row look a lot alike. Recall that the goal is to make the LHS be less than the RHS. That would happen if I knew that $2 < k+1$.

Recall that we are only claiming $P(n)$ for $n \geq 4$, which means that $k \geq 4$. For $k \geq 4$ we know that $2 < (k+1)$, which means that

$$2 \cdot 2^k < 2 \cdot k! < (k+1)k!$$

Plugging that back into the table and cleaning up gives

$P(k)$			$P(k+1)$		
LHS	$<$	RHS	LHS	$<$	RHS
2^k	$<$	$k!$	2^{k+1}	?	$(k+1)!$
			$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!$	?	$(k+1)k! = (k+1)!$
			$2^{k+1} = 2 \cdot 2^k < 2 \cdot k!$	$<$	$(k+1)k! = (k+1)!$
			2^{k+1}	$<$	$(k+1)!$

This means that when $P(k)$ is true, $P(k+1)$ is true. Thus, $P(k) \rightarrow P(k+1)$, which completes the induction step.

Final Statement

Since we proved the basis step for $n = 4$ and the induction step for $k \geq 4$, by induction it follows that, $\forall n \geq 4$, the formula $2^n < n!$ is true.