Turbulence

General Nature of Turbulence

In all the problems we have analyzed to date, the fluid elements travel along smooth predictable trajectories. This state of affairs is called:

laminar flow - fluid elements travel along smooth deterministic trajectories

These trajectories are straight parallel lines for simple pipe flows. However, this is not the only solution to the equations of motion. Consider the following experiment

Reynolds Experiment (1882) - inject a thin stream of dye into a fully developed flow in a pipe; observe the dye downstream. (see S:37)

For laminar flow, dye stream appears as a straight colored thread (see Fig. 1 below). As the total flow rate of fluid in the pipe is increased, a sudden change in the appearance of the dye stream occurs. The thread of dye becomes more radially mixed with the fluid and, far enough downstream, its outline becomes blurred. For turbulent flow, irregular radial fluctuations of dye thread

Using a pipe with a sharp-edge entrance, Reynolds determined the critical flow rate for a large number of fluids and pipe sizes. He found in all cases, the transition occurred at a critical value of a dimensionless group:

$$\underbrace{\frac{\rho\langle\bar{v}_z\rangle D}{\mu}}_{\text{Re}} = 2300 \pm 200$$

where $\langle\bar{v}_z\rangle$ is the cross-sectional average velocity (= volumetric flowrate / pipe area). Today, we know this dimensionless group as the ***Reynolds number***.

origin of turbulence - instability of laminar-flow solution to N-S eqns

instability - small perturbations (caused by vibration, etc.) grow rather than decay with time.

That the laminar-flow solution is metastable for Re>2100 can be seen from Reynolds experiment performed with a pipe in which disturbances are minimized:

- reduce vibration
- fluid enters pipe smoothly
- smooth pipe wall

Under such conditions, laminar flow can be seen to persist up to Re = 10^4. However, just adding some vibrations (disturbance) can reduce the critical Re to 2100.

The onset of turbulence causes a number of profound changes in the nature of the flow:

- dye thread breaks up: streamlines appear contorted and random
- sudden increase in $\Delta p/L$
- local v_z fluctuates wildly with time
- similar fluctuations occur in v_r and v_θ

As a consequence of these changes, no simplification of the N-S equation is possible: v_r, v_θ, v_z and p all depend on r, θ, z and t.

Turbulent Flow in Pipes

Velocity profiles are often measured with a ***pitot tube***, which is a device with a very slow response time. As a consequence of this slow response time, the rapid fluctuations with time tend to average out. In the descriptions which follow, we will partition the instantaneous velocity $\mathbf{v}$ into a time-averaged value $\bar{\mathbf{v}}$ (denoted by the overbar) and a fluctuation $\mathbf{v}'$

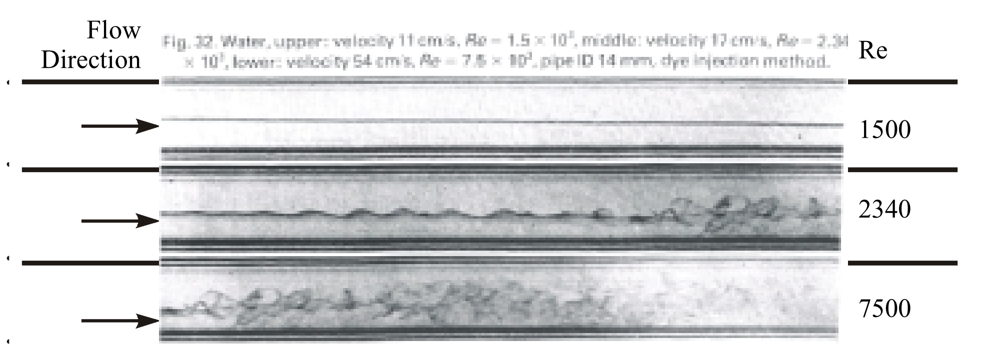


Fig. 1: Reynolds experiment

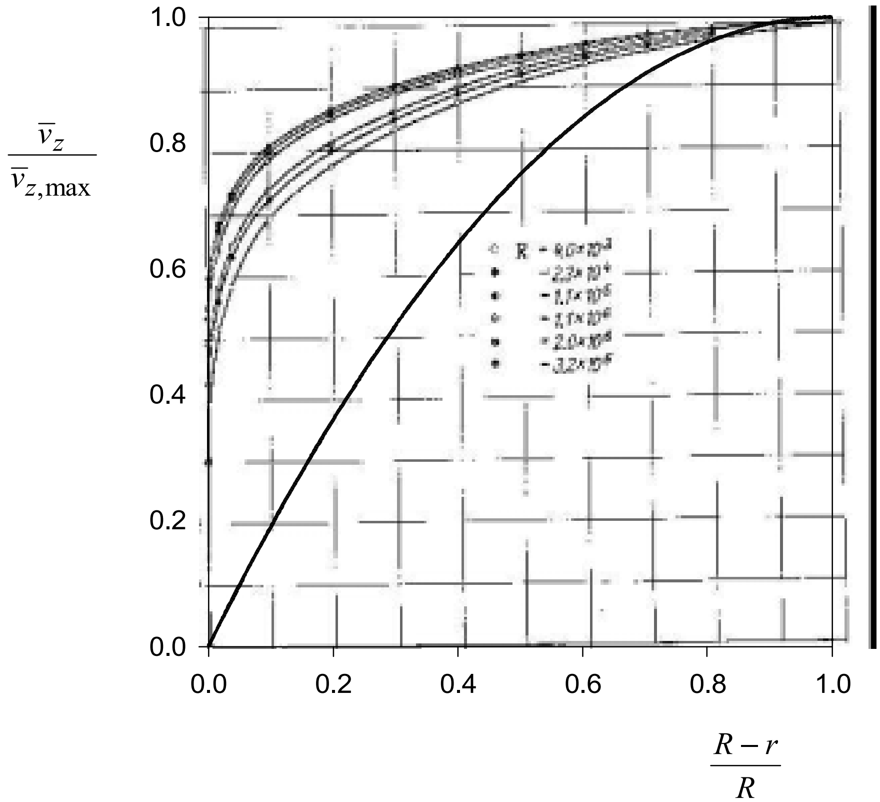


Fig. 2: Shape of turbulent velocity profile for pipe flow depends on Reynolds number.

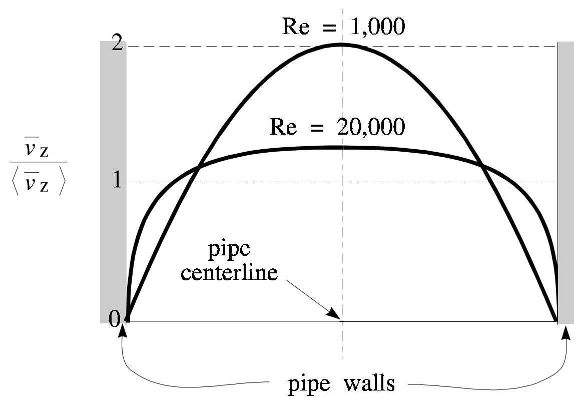
(denoted by the prime):

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$$

Cross-sectional area averages will be denoted by enclosing the symbol inside carets:

$$\langle \bar{v}_z \rangle = \frac{\int_0^R \bar{v}_z(r) 2\pi r\, dr}{\int_0^R 2\pi r\, dr} = \frac{Q}{\pi R^2}$$

In laminar flow, the velocity profile for fully developed flow is parabolic in shape with a maximum velocity occurring at the pipe center that is twice the cross-sectional mean velocity:



In turbulent flow, the *time-averaged* velocity profile has a flatter shape (see Fig. 2 above). Indeed as the Reynolds number increases the shape changes such that the profile becomes even flatter. The profile can be fit to the following *empirical* equation:

$$\bar{v}_z(r) = \bar{v}_{z,\max}\left(\frac{R-r}{R}\right)^{1/n}$$

where the value of the parameter n depends on Re:

Re	n	$v_{max}/\langle v_z\rangle$
4.0×10^3	6	1.26
2.3×10^4	6.6	1.24
1.1×10^5	7	1.22
1.1×10^6	8.8	1.18
2.0×10^6	10	1.16
3.2×10^6	10	1.16

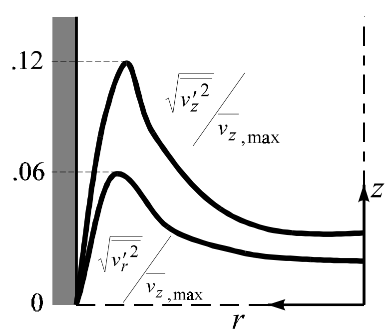
The reduction in the ratio of maximum to average velocity reflects the flattening of the profile as n becomes larger. Of course, this equation gives a "kink" in the profile at r=0 and predicts infinite slope at r=R, so it shouldn't be applied too close to either boundary although it gives a reasonable fit otherwise.

How big are the fluctuations relative to the maximum velocity? Instantaneous speeds can be obtained for air flows using a ***hot-wire anemometer***. This is simply a very thin wire which is electrically heated above ambient by passing a current through it. As a result of electrical heating (I^2R) the temperature of the wire will depend on the heat transfer coefficient, which in turn depends on the velocity of flow over the wire:

$$v_z \uparrow \text{-->}\ h \uparrow \text{-->}\ T_{wire}\text{-}T_{air} \downarrow$$

It's easy to determine the temperature of the wire from its electrical resistance, which generally increases with temperature. The reason for making the wire very thin is to decrease its thermal inertia. Very thin wires can respond rapidly to the rapid turbulent fluctuations in v_z.

Anyway, the instantaneous speed can be measured. Then the time-averaged speed and the fluctuations can be calculated. The root-mean-square fluctuations depend on radial position, as shown at right. Typically the axial fluctuations are less than 10% of the maximum velocity whereas the radial fluctuations are perhaps half of the axial.



Note that the fluctuations tend to vanish at the wall. This is a result of no-slip (applies even in turbulent flow) which requires that the *instantaneous* velocity must vanish at the wall for all time, which implies that the time average and the instantaneous fluctuations must vanish.

Time-Smoothing

As we will see shortly, these fluctuations profoundly increase transport rates for heat, mass, and momentum. However, in many applications (e.g. the velocity profile measured with a pitot tube), we would be content to predict the time-averaged velocity profile. So let's try to time-average the Navier-Stokes equations with the hope that the fluctuations will average to zero.

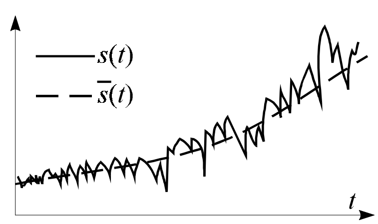
First, we need to define what we mean by a ***time-averaged*** quantity. Suppose we have some property like velocity or pressure which fluctuates with time:

$$s = s(t)$$

We can average over some time interval of half width Δt:

$$\overline{s}(t) \equiv \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} s(t')dt'$$

We allow that the time-averaged quantity might still depend on time, but we have averaged out the rapid fluctuations due to turbulence.



Now let's define another quantity called the ***fluctuation*** about the mean:

$$s'(t) \equiv s(t) - \overline{s}(t)$$

Time-Smoothing of Continuity Equation

The simple functional form of our experimentally measured velocity profile — $\overline{v}_z(r)$ — is exactly the same as for laminar flow. This suggests, that if we are willing to settle for the time-averaged velocity profile, then I might be able to get the result from the NSE. Let's try to time-smooth the equation of motion and see what happens. We will start with the equation of continuity for an incompressible flow:

$$\nabla \cdot \mathbf{v} = 0$$

Integrating the continuity equation for an incompressible fluid and dividing by $2\Delta t$:

$$\frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} \nabla \cdot \mathbf{v} dt' = \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} 0 dt' = 0$$

Thus the right-hand-side of the equation remains zero. Let's take a closer look at the left-hand side. Interchanging the order of differentiation and integration:

$$\frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} \nabla \cdot \mathbf{v} dt' = \nabla \cdot \left\{ \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} \mathbf{v} dt' \right\} = \nabla \cdot \overline{\mathbf{v}}$$

Substituting this result for the left-hand side of the continuity equation, leaves:

$$\nabla \cdot \overline{\mathbf{v}} = 0$$

Thus the form of the continuity equation has not changed as a result of time-smoothing.

Time-Smoothing of the Navier-Stokes Equation

Encouraged by this simplification, we try to time-smooth the Navier-Stokes equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

After integrating both sides with respect to time and dividing by $2\Delta t$, we can break the integral of the sum into the sum of the integrals. Most of the terms transform in much the same way as the left-hand side of the continuity equation. The result is

$$\rho \frac{\partial \overline{\mathbf{v}}}{\partial t} + \rho \overline{\mathbf{v} \cdot \nabla \mathbf{v}} = -\nabla \overline{p} + \mu \nabla^2 \overline{\mathbf{v}} + \rho \mathbf{g}$$

With a little additional massaging (see Whitaker), the remaining term can be expressed as

$$\rho \overline{\mathbf{v} \cdot \nabla \mathbf{v}} = \rho \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} + \nabla \cdot \left(\rho \overline{\mathbf{v}'\mathbf{v}'} \right)$$

If the second term on the right-hand side were zero, then NSE after time-smoothing would have exactly the same form as before time-smoothing. Unfortunately, this term is not zero. Although the average of the fluctuations is zero, the average of the square of the fluctuations is not zero. So this second term cannot be dropped. Thus the time-smoothed Navier-Stokes equation becomes:

$$\rho \frac{\partial \overline{\mathbf{v}}}{\partial t} + \rho \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} = -\nabla \overline{p} + \mu \nabla^2 \overline{\mathbf{v}} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\tau}}^{(t)}$$

where $\qquad \underline{\underline{\tau}}^{(t)} = -\rho \overline{\mathbf{v}'\mathbf{v}'}$

has units of stress or pressure and is called the ***Reynold's stress***. Sometimes it is also called the ***turbulent stress*** to emphasize that arises from the turbulent nature of the flow. The existence of this new term is why even the time-averaged velocity profile inside the pipe is different from that during laminar flow. Of course, our empirical equation for the $\overline{v}_z(r)$ is also different from that for laminar flow.

Although we don't yet know how to evaluate this Reynolds stress, we can add it to the viscous stress and obtain a differential equation for their sum which we can solve for the simple case of pipe flow. Here's how we do it. First, recall that for incompressible Newtonian fluid, the stress is related to the rate of strain by Newton's law of viscosity. Time smoothing this constitutive equation yields:

$$\underline{\underline{\overline{\tau}}} = \mu \left[\nabla \underline{\overline{\mathbf{v}}} + \left(\nabla \underline{\overline{\mathbf{v}}} \right)^t \right]$$

Taking the divergence:

$$\nabla \cdot \underline{\underline{\overline{\tau}}} = \mu \nabla^2 \overline{\mathbf{v}}$$

If we now make this substitution, NSE becomes

$$\begin{aligned} \rho \frac{\partial \overline{\mathbf{v}}}{\partial t} + \rho \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} &= -\nabla \overline{p} + \nabla \cdot \underline{\underline{\overline{\tau}}} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\tau}}^{(t)} \\ &= -\nabla \overline{p} + \rho \mathbf{g} + \nabla \cdot \underline{\underline{\tau}}^{(T)} \end{aligned} \tag{200}$$

where $\qquad \underline{\underline{\tau}}^{(T)} = \underline{\underline{\overline{\tau}}} + \underline{\underline{\tau}}^{(t)}$

is the total stress, i.e., the sum of the time-averaged viscous stress and the Reynolds stress. Thus we see that the Reynolds stress appears in the equations of motion in the same manner as the viscous stress. Indeed the <u>sum of the two contributions plays the same role in turbulent flows that the viscous friction played in laminar flow</u>.

Analysis of Turbulent Flow in Pipes

We can make the same assumptions (i.e. the same guess) about the functional form of the *time-averaged* velocity and pressure profile in turbulent flow that we made for laminar flow: we will assume that the time-averaged velocity profile is axisymmetric (v_θ=0, $\partial/\partial\theta$=0) and fully developed ($\partial/\partial z$=0).

$$\bar{v}_z = \bar{v}_z(r) \qquad \bar{v}_r = \bar{v}_\theta = 0$$
$$\bar{P} = \bar{P}(z)$$

Then the z-component of (200) yields:

$$0 = -\underbrace{\frac{\partial \bar{P}}{\partial z}}_{f(z)} + \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{\tau}_{rz}^{(T)}\right)}_{g(r)} - \frac{1}{r}\underbrace{\frac{\partial \bar{\tau}_{\theta z}^{(T)}}{\partial \theta}}_{0} - \underbrace{\frac{\partial \bar{\tau}_{zz}^{(T)}}{\partial z}}_{0}$$

where $\bar{P}$ is the time-averaged dynamic pressure. This form of this equation was obtained using the tables for Euler's equation in cylindrical coordinates (click here), after replacing the instantaneous quantities by their time averages, except that the instantaneous viscous stresses τ has been replaced by (minus) the *total* stress $\tau^{(T)}$. Expecting the time-averaged flow to be axisymmetric ($\partial/\partial\theta = 0$) and fully developed ($\partial/\partial z = 0$, except for pressure), the last two terms in this equation can be dropped and the second term is a function of r only.

This leaves us with the same equation we had for laminar flow: a function of r only equal to a function of z only. The only way these two terms can sum to zero for all r and z is if both equal a spatial constant:

$$\frac{d\bar{P}}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\bar{\tau}_{rz}^{(T)}\right) = -\frac{\Delta P}{L} < 0$$

This implies that pressure $\bar{P}$ varies linearly with z. Solving for the *total* stress $\bar{\tau}_{rz}^{(T)}$ by integrating:

$$\bar{\tau}_{rz}^{(T)} = -\frac{1}{2}\frac{\Delta P}{L}r = \bar{\tau}_{rz} + \bar{\tau}_{rz}^{(t)} \qquad (201)$$

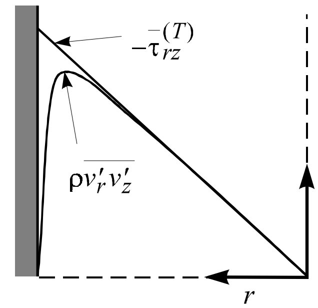
The integration constant was chosen to be zero to avoid having the stress unbounded at r=0. Now this is the *total* stress: the sum of the Reynolds stress

$$\tau_{rz}^{(t)} = -\rho\overline{v_r' v_z'}$$

and a viscous contribution from time-smoothing Newton's law of viscosity:

$$\bar{\tau}_{rz} = \mu\frac{d\bar{v}_z}{dr}$$

The latter can be determined by differentiating the time-averaged velocity profile. If we subtract this from the total we can determine $\bar{\tau}_{rz}^{(t)}$ -- one of the components of the Reynolds stress tensor. The result is:



Notice that the turbulent stress tends to vanish near the wall. This can be explained by noting that at the wall, "no slip" between the fluid and the stationary wall requires that the instanteous velocity, as well as its time average, must be zero:

$$v_z = \bar{v}_z = 0 \Longrightarrow v_z' = 0 \Longrightarrow \bar{\tau}_{rz}^{(t)} = -\rho\overline{v_r' v_z'} = 0$$

In terms of the relative importance of these two contributions to the total, one can define three regions:

8. ***turbulent core***: $\tau^{(t)} >> \tau$. This covers most of the cross section of the pipe.

9. ***laminar sublayer***: $\tau^{(t)} << \tau$. Very near the wall, the fluctuations must vanish (along with the Reynolds stress) but the viscous stress are largest.

10. ***transition zone***: $\tau^{(t)} \approx \tau$. Neither completely dominates the other.

When applied to the situation of fully developed pipe flows, continuity is automatically satisfied and the time-smoothed Navier-Stokes equations yields only one equation in 2 unknowns:

2 unknowns: $\quad \bar{v}_z(r)$ and $\rho\overline{v_r' v_z'}$

Clearly another relationship is needed to complete the model. This missing relationship is the ***constitutive equation*** relating the Reynolds stress to the time-smoothed velocity profile. One might be tempted to define a quantity like the viscosity to relate stress to the time-averaged velocity.

$$\bar{\tau}_{rz}^{(t)} \stackrel{?}{=} \mu^{(t)}\frac{d\bar{v}_z}{dr}$$

But if you define the "turbulent viscosity" this way, its value turns out to depend strongly on position.

$$\frac{\mu^{(t)}}{\mu} = \begin{cases} \approx 100 & \text{near pipe centerline} \\ 0 & \text{at pipe wall} \end{cases}$$

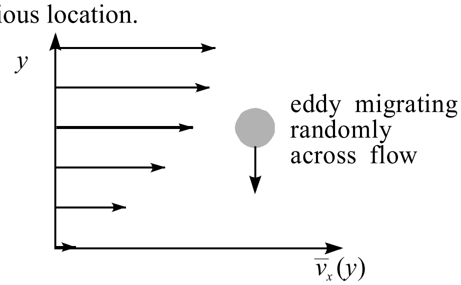
So unlike the usual viscosity, $\mu^{(t)}$ is not a material property (since it depends on position rather than just the material).

Prandtl's Mixing Length Theory

The first successful constitutive equation for turbulence was posed by Prandtl in 1925. Prandtl imagined that the fluctuations in instantaneous fluid velocity at some fixed point were caused by ***eddies*** of fluid which migrate across the flow from regions having higher or lower time-averaged velocity.

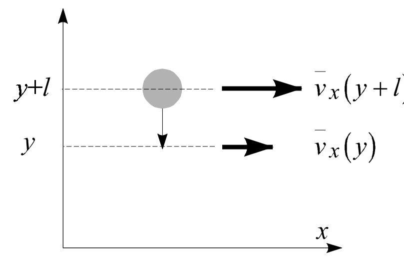
eddy - a packet of fluid (much larger than a fluid element) which can undergo random migration across streamlines of the time-smoothed velocity field.

These eddies have a longitudinal velocity which corresponds to the time-average velocity at their previous location.



As this eddy moves across the streamlines, it gradually exchanges momentum with the surrounding fluid which is moving at a different longitudinal velocity. But this exchange does not occur instantaneously. The eddy retains its original velocity for a brief period of time. We might call this the ***mixing time***. During this time, the eddy migrates laterally a distance l called the mixing length:

mixing length (l) - characteristic distance an eddy migrates normal to the main flow before mixing



Although momentum exchange between eddies occurs continuously in actual turbulent flow, as an idealization, Prandtl imagined that a migrating eddy keeps all of its original velocity until it migrated a distance l and then suddenly it exchanges it. This is like a molecule of gas retaining its momentum until it collides with another gas molecule, which causes a sudden exchange of momentum. Indeed, you might find it helpful to think of the mixing length as being the analog of ***mean-free-path*** in the kinetic theory of gases. Recall that:

mean-free path - average distance a gas molecule travels before colliding with another gas molecule.

Now suppose we are monitoring the instantaneous velocity at a distance y from the wall when an eddy drifts into our location from $y+l$. Because this migrating eddy has a higher velocity than the average fluid at y, we will observe an positive fluctuation when the eddy arrives. To estimate the magnitude of the fluctuation, we can expand the time-smoothed velocity profile in Taylor series about $y=y$:

$$\overline{v}_x(y+l) = \overline{v}_x(y) + \left.\frac{d\overline{v}_x}{dy}\right|_y l + \frac{1}{2}\left.\frac{d^2\overline{v}_x}{dx^2}\right|_y l^2 + \cdots$$

Assuming that l is sufficiently small that we can truncate this series without introducing significant error:

$$\left(v'_x\right)_{above} = \overline{v}_x(y+l) - \overline{v}_x(y) \approx l\frac{d\overline{v}_x}{dy}$$

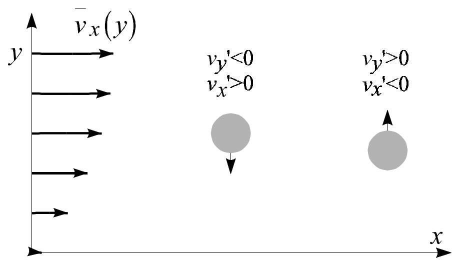
where the subscript "above" is appended to remind us that is the fluctuation resulting from an eddy migrating from above. At some later time, another eddy might migrate to our location from below, producing a negative fluctuation in velocity:

$$\left(v'_x\right)_{below} = \overline{v}_x(y-l) - \overline{v}_x(y) \approx -l\frac{d\overline{v}_x}{dy}$$

Of course the average fluctuation is zero: $\overline{v_x'} = 0$, but the average of the squares is not:

$$\overline{(v_x')^2} \approx \tfrac{1}{2}\left\{(v_x')^2_{above} + (v_x')^2_{below}\right\} \approx l^2\left(\frac{d\bar{v}_x}{dy}\right)^2 \qquad (202)$$

Now let's turn our attention to v_y'. This is related to how fast the eddies migrate, and the sign depends on whether they are migrating upward or downward.



If the eddy migrates from above, it represents a negative y-fluctuation (it is moving in the $-y$ direction). Such an eddy will have a greater x-velocity than the fluid receiving it, consequently generating a positive x-fluctuation:

$$v_y'<0 \rightarrow v_x'>0 \quad \rightarrow \quad v_x'v_y'<0$$

On the other hand, if the eddy migrates from below, it represents a positive y-fluctuation but has less x-velocity than the fluid receiving it, generating a negative x-fluctuation:

$$v_y'>0 \rightarrow v_x'<0 \quad \rightarrow \quad v_x'v_y'<0$$

Finally, if there is no vertical migration of eddies, there is no reason for the x-velocity to fluctuate:

$$v_y' = 0 \quad \rightarrow \quad v_x' = 0$$

These three statements suggest that the y-fluctuations are proportional to the x-fluctuations, with a negative proportionality constant:

$$v_y' \approx -\alpha v_x'$$

where $\alpha>0$. Alternatively, we can write:

$$v_x'v_y' = -\alpha(v_x')^2$$

Time averaging and then substituting (202):

$$\overline{v_x'v_y'} = -\alpha\overline{(v_x')^2} = -\alpha l^2\left(\frac{d\bar{v}_x}{dy}\right)^2$$

Absorbing the unknown α into the (still unknown) mixing length parameter:

$$\bar{\tau}_{xy}^{(t)} = -\rho\overline{v_x'v_y'} = \rho l^2\left(\frac{d\bar{v}_x}{dy}\right)^2 \qquad (203)$$

which serves as a constitutive equation for turbulent flow. Comparing this result with Newton's law of viscosity:

$$\bar{\tau}_{xy} = \mu\frac{d\bar{v}_x}{dy}$$

we could conclude that an apparent ***turbulent viscosity*** is given by:

$$\mu^{(t)} = \rho l^2\left|\frac{d\bar{v}_x}{dy}\right|$$

Of course, this viscosity is not a true fluid property, because it depends strongly on the velocity profile.

For this theory to be useful, we need a value for the "mixing length" l. There are two properties of l which we can easily deduce. First of all, l was defined as the distance normal to the wall which the eddy travels before becoming mixed with local fluid. Clearly, this mixing must occur before the eddy "bumps" into the wall, so:

Property #1: $l < y$

where y is the distance from the wall. Secondly, we know from no-slip that the fluctuations all vanish at the wall. Consequently, the Reynolds stress must vanish at the wall. Since the velocity gradient does not vanish, we must require that the mixing length vanish at the wall:

Property #2: $l=0$ at $y=0$

If it's not a constant, the next simplest functional relationship between l and y which satisfies both these properties is:

$$l = ay \qquad (204)$$

where a is some constant and $0<a<1$.

Prandtl's "Universal" Velocity Profile

The velocity profile in turbulent flow is essentially flat, except near the wall where the velocity gradients are steep. Focussing attention on this region near the flow, Prandtl tried to deduce the form for the velocity profile in turbulent flow. Recall from (201) that in pipe flow, the total stress varies linearly from 0 at the center line to a maximum value at the wall:

$$\bar{\tau}_{rz}^{(T)} = -\frac{1}{2}\frac{\Delta P}{L}r = \tau_0 \frac{r}{R} < 0 \qquad (205)$$

where we have defined

$$\tau_0 \equiv -(1/2)(R/L)\Delta P > 0$$

which represents the stress on the wall. In the "turbulent core", the Reynolds stress dominates the "laminar" stress; then substituting (203) through (205):

$$\bar{\tau}_{xy}^{(t)} \approx \bar{\tau}_{rz}^{(T)}$$

$$\rho \underbrace{l^2}_{a^2y^2} \left(\frac{d\bar{v}_x}{dy}\right)^2 \approx \tau_0\left(1-\frac{y}{R}\right) \qquad (206)$$

Dividing through by ρ and substituting (204):

$$a^2y^2\left(\frac{d\bar{v}_x}{dy}\right)^2 \approx \underbrace{\frac{\tau_0}{\rho}}_{v^{*2}}\left(1-\frac{y}{R}\right) \qquad (207)$$

The ratio τ_0/ρ has units of velocity-squared, which serves as a convenient choice for a characteristic turbulent velocity:

$$v^* \equiv \sqrt{\frac{\tau_0}{\rho}}$$

is called the ***friction velocity***. The dimensionless turbulent velocity will be denoted as

$$v^+ = \frac{\bar{v}_x}{v^*}$$

Taking the square-root of (207):

$$ay\frac{d\bar{v}_x}{dy} = \underbrace{\sqrt{\frac{\tau_0}{\rho}}}_{v^*}\sqrt{\left(1-\frac{y}{R}\right)} = v^*\sqrt{\left(1-\frac{y}{R}\right)}$$

The general solution to this 1st order ODE is

$$v^+\left(y^+\right) = C + \frac{2}{a}\sqrt{1-\frac{y^+}{R^+}} - \frac{2}{a}\tanh^{-1}\sqrt{1-\frac{y^+}{R^+}} \qquad (208)$$

where C is the integration constant, v^* is called the ***friction velocity*** and where we have introduced dimensionless variables:

$$v^+ = \frac{\bar{v}_x}{v^*}$$

$$y^+ = \frac{v^*}{\nu}y \qquad \text{and} \qquad R^+ = \frac{v^*}{\nu}R$$

Near the wall (i.e. for $y<<R$ or $y^+<<R^+$), we can simplify (208):

for $y^+<<R^+$:

$$\sqrt{1-\frac{y^+}{R^+}} = 1 - \frac{y^+}{2R^+} + O\left(y^{+2}\right) \qquad (209)$$

$$\tanh^{-1}\sqrt{1-\frac{y^+}{R^+}} = \frac{1}{2}\ln\frac{4R^+}{y^+} + O\left(y^+\right)$$

Dropping the higher-order terms:

$$v^+\left(y^+\right) = \underbrace{\frac{2-\ln\left(4R^+\right)}{a} + C}_{c} + \frac{1}{a}\ln y^+ \qquad (210)$$

where c is a collection of constants.

This result can be derived more easily by starting over with a simplified (207) which applies when $y<<R$:

$$ay\frac{d\bar{v}_x}{dy} = v^*$$

or

$$\underbrace{\frac{d\bar{v}_x}{v^*}}_{dv^+} = \frac{1}{a}\frac{dy}{y} = \frac{1}{a}\frac{dy^+}{y^+} \qquad (211)$$

which integrates to

$$v^+ = \frac{1}{a}\ln y^+ + c \qquad (212)$$

where c is some integration constant. y^+ is a dimensionless distance from the wall:

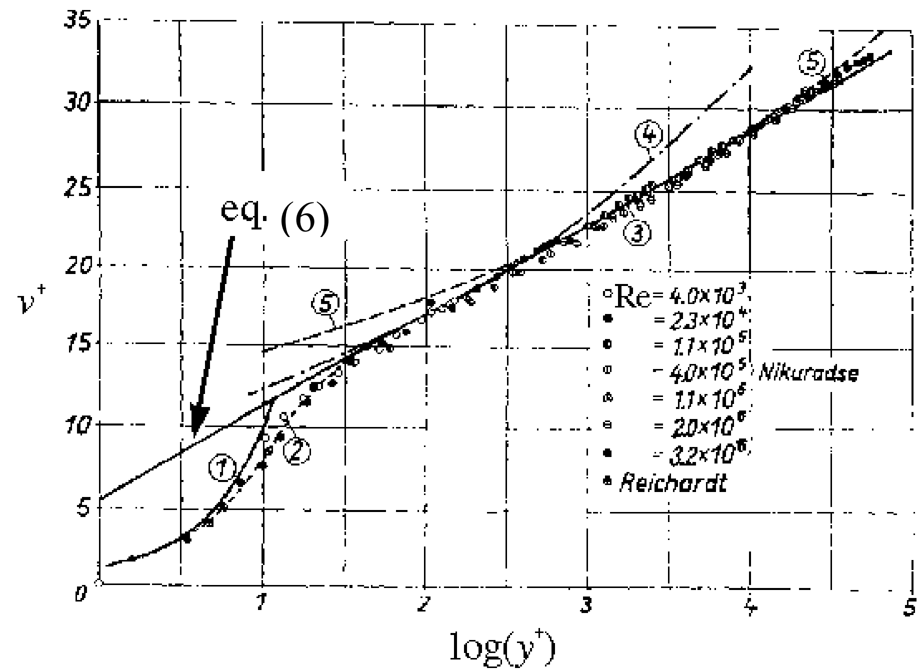


Fig. 3: Time-average turbulent velocity profile near wall. Nondimensionalization removes the effect of Reynolds number from the shape, resulting in a "universal velocity profile"

$$y^+ = \frac{v^*}{\nu} y$$

The choice of ν/v^* as a characteristic turbulent length scale is not apparent from (211), but will become apparent when we nondimensionalize (214) below.

When plotted on semi-log coordinates (see Fig. 3), as suggested by (212), experimental velocity profiles do indeed show a linear region which extends over a couple of decades of y^+ values. Moreover, the slope and intercept of this straight line don't seem to depend on the Reynolds number. Indeed, the slope and intercept also don't seem to depend on the shape of the conduit. Rectangular conduits yields the same velocity profile on these coordinates. This is called ***Prandtl's Universal Velocity Profile***:

$y^+>26$: $$v^+ = 2.5 \ln y^+ + 5.5 \tag{213}$$

which applies for $y^+>26$ (the ***turbulent core***). This coefficient of $\ln y^+$ corresponds to a=0.4. so (204) becomes:

$$l = 0.4 y$$

Recall that we reasoned that a had to be in the range of 0 to 1 to be physically realistic. Of course (213) also does not apply near the center of the pipe, since $y^+ \approx R^+$ there, whereas (213) was derived by assuming that $y^+ << R^+$ (see (209)).

Laminar Sublayer

In the ***laminar sublayer***, Reynolds stress can be totally neglected, leaving just viscous stress. This close to the wall, the total stress is practically a constant equal to the wall shear stress τ_0:

$y << R$: $$\begin{cases} \bar{\tau}_{xy} \approx \bar{\tau}_{rz}^{(T)} \\ \mu \dfrac{d\bar{v}_x}{dy} = \tau_0 \end{cases}$$

Then we can integrate the above ODE for $\bar{v}_x$, subject to $\bar{v}_x = 0$ at y=0 (i.e. no slip):

$$\bar{v}_x = \frac{\tau_0}{\mu} y \tag{214}$$

Dividing both sides by v^*, then dividing numerator and denominator of the right-hand side by ρ, we can make the result dimensionless:

$$\underbrace{\frac{\bar{v}_x}{v^*}}_{v^+} = \frac{1}{v^*}\frac{\tau_0/\rho}{\mu/\rho} y = \frac{1}{v^*}\frac{v^{*2}}{\nu} y = \underbrace{\frac{v^*}{\nu} y}_{y^+}$$

or $$v^+ = y^+ \tag{215}$$

which applies for $0<y^+<5$ (the ***laminar sublayer***).

Prandtl's Universal Law of Friction

Let's try to figure deduce the analog of Poisueille's Formula (see page 54) for turbulent flow. Poisueille's Formula is the relationship between volumetric flowrate through the pipe and pressure drop. Volumetric flowrate Q is calculated by integrating the axial component of fluid velocity of the cross section of the pipe:

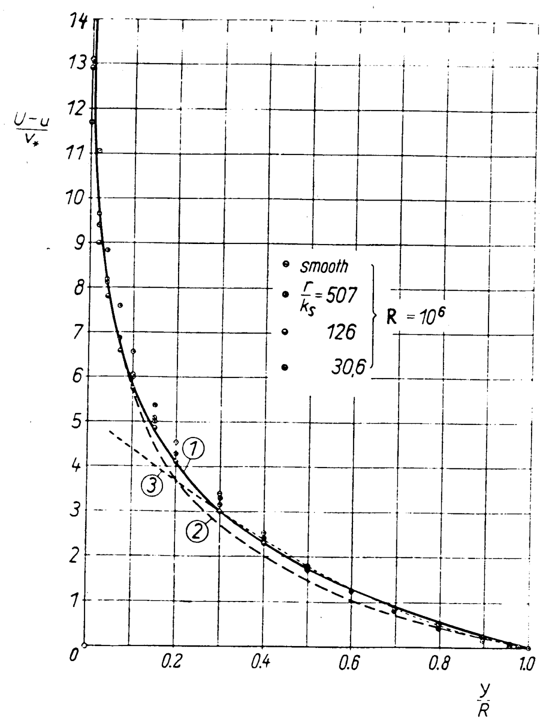
$$\left\langle \bar{v}_z \right\rangle = \frac{Q}{\pi R^2} = \frac{2}{R^2}\int_0^R r\,\bar{v}_z(r)\,dr$$

Now we are going to use (213) for the velocity profile, although we assumed in (211) that $y<<R$ (where $y = R-r$).

Although the derivation of (213) assumed that we are very close to the wall ($y^+<<R^+$, see (209)), (213) works remarkly well right out to the centerline $y^+=R^+$. The plot at right shows the velocity profiles (with different wall roughness on the walls) compared with predictions based on (213). The ordinate is

$$\begin{aligned}\frac{\bar{v}_z(R)-\bar{v}_z(r)}{v^*} &= v^+\left(R^+\right)-v^+\left(y^+\right)\\ &= \left[2.5\ln R^+ + 5.5\right]-\left[2.5\ln y^+ + 5.5\right]\\ &= 2.5\left(\ln R^+ - \ln y^+\right) = 2.5\ln\frac{R^+}{y^+}\\ &= 2.5\ln\frac{R}{y} = 2.5\ln\frac{R}{R-r}\end{aligned}$$

This equation (represented by Curve ①) is compared with experimental data in the following figure:



Note that (213) predicts an infinite velocity-difference at y=0, whereas the actual velocity must be finite. Of course, (213) does not apply right up to the wall because very near the wall the Reynolds stresses are <u>not</u> dominant.

Substituting (213) and integrating:

$$\left\langle \bar{v}_z \right\rangle = v^*\left[2.5\ln\left(\frac{v^* R}{\nu}\right)+1.75\right] \tag{216}$$

Now the friction velocity can be related to the ***friction factor***, whose usual definition can be expressed in terms of the variables in this analysis:

$$f \equiv \frac{\tau_0}{\frac{1}{2}\rho\left\langle \bar{v}_z \right\rangle^2} = 2\left(\frac{v^*}{\left\langle \bar{v}_z \right\rangle}\right)^2$$

Thus $$\frac{\left\langle \bar{v}_z \right\rangle}{v^*} = \sqrt{\frac{2}{f}}$$

Likewise, the usual definition of Reynolds number yields

$$\mathrm{Re} \equiv \frac{2\left\langle \bar{v}_z \right\rangle R}{\nu}$$

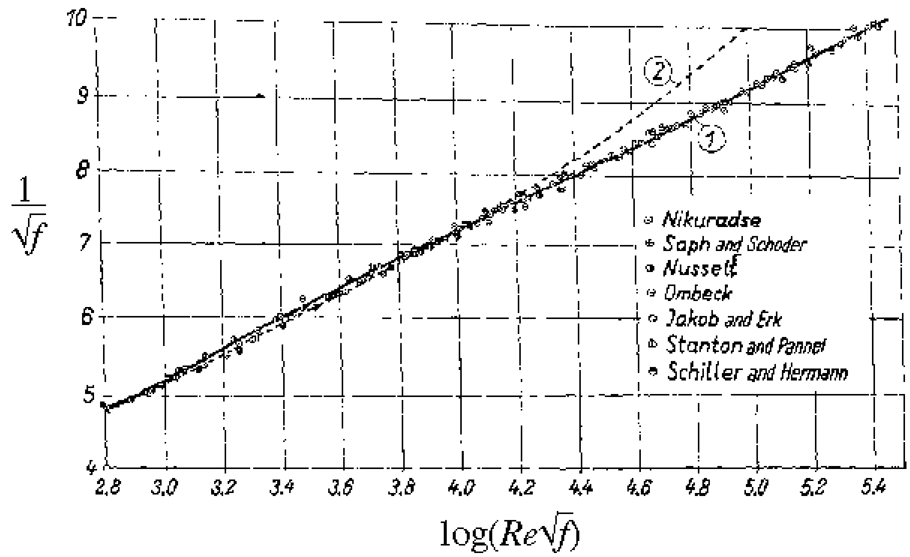


Fig. 4: Relationship between friction factor *f* and Reynolds number implied by Prandtl's law of fraction

Thus $$\frac{v^* R}{\nu} = \underbrace{\frac{\langle\bar{v}_z\rangle R}{\nu}}_{\mathrm{Re}/2} \times \underbrace{\frac{v^*}{\langle\bar{v}_z\rangle}}_{\sqrt{f/2}} = \frac{\mathrm{Re}\sqrt{f}}{2\sqrt{2}}$$

Relating v^* to f and $\langle\bar{v}_z\rangle$ to Re, (216) can be written as:

$$\frac{1}{\sqrt{f}} = 1.77\ln\left(\mathrm{Re}\sqrt{f}\right) - 0.60$$

or $$\frac{1}{\sqrt{f}} = 4.07\log_{10}\left(\mathrm{Re}\sqrt{f}\right) - 0.60$$

Fig. 4 shows that experimental data plotted as $1/\sqrt{f}$ versus $\log_{10}\left(\mathrm{Re}\sqrt{f}\right)$ does indeed produce a linear relationship. The solid line in Fig. 4 has slightly different values for the coefficients:

$$\frac{1}{\sqrt{f}} = 4.0\log_{10}\left(\mathrm{Re}\sqrt{f}\right) - 0.40 \qquad (217)$$

which is called ***Prandtl's (universal) law of friction***. It applies virtually over the entire range of Reynolds numbers normally encountered for turbulent pipe flow: $2100 < \mathrm{Re} < 5\times10^6$.

Although Fig. 4 is useful in establishing the functional relationship, it awkward to use: given a particular value for Re, determining the corresponding f is a trial-and-error problem. The same experimental data and the corresponding prediction from (217) is replotted in Fig. 5 in the more conventional form of friction factor (λ) versis Reynolds number (R). Curve ③ is the prediction from (217).

Besides being "universal" in terms of any Reynolds number, (217) is also universal in terms of geometry. Fig. 6 is a plot of friction factor (λ) versis Reynolds number (R) for long cylindrical ducts of different cross-sectional shapes. The friction factors for different shapes have been offset by factors of 2 or 4 for ease of viewing. The characteristic length used in these correlations if the hydraulic diameter:

$$d_h \equiv \frac{4A}{C}$$

where A is the cross-sectional area and C is the wetted perimeter. For a circular duct, we have

$$A = \pi R^2$$

and $$C = 2\pi R$$

so that $$d_h \equiv \frac{4A}{C} = \frac{4\times\pi R^2}{2\pi R} = 2R$$

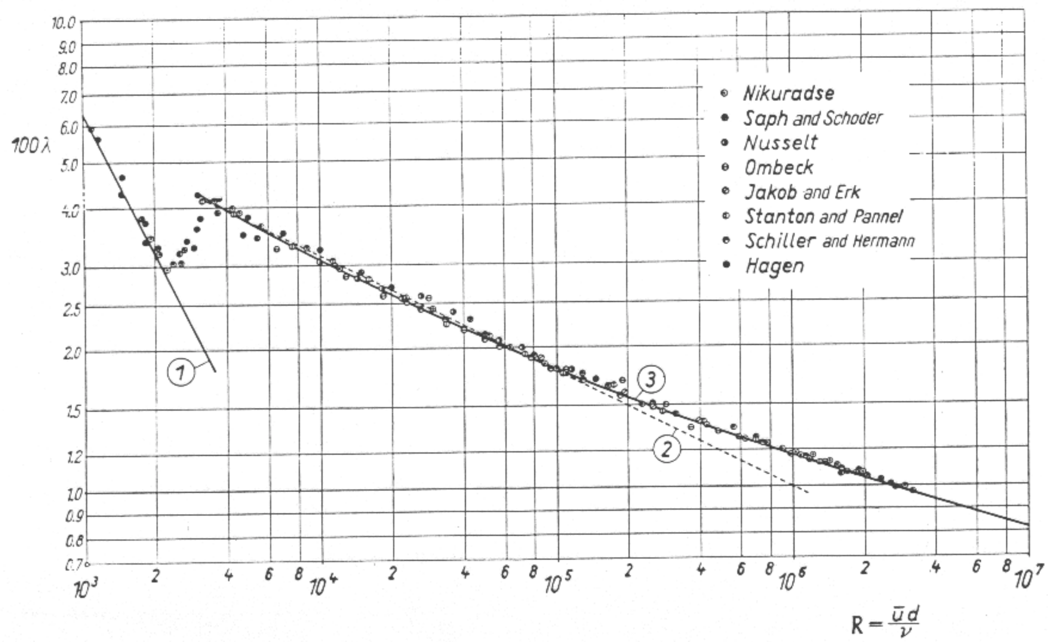


Fig. 5: A replot of the data in Fig. 4.

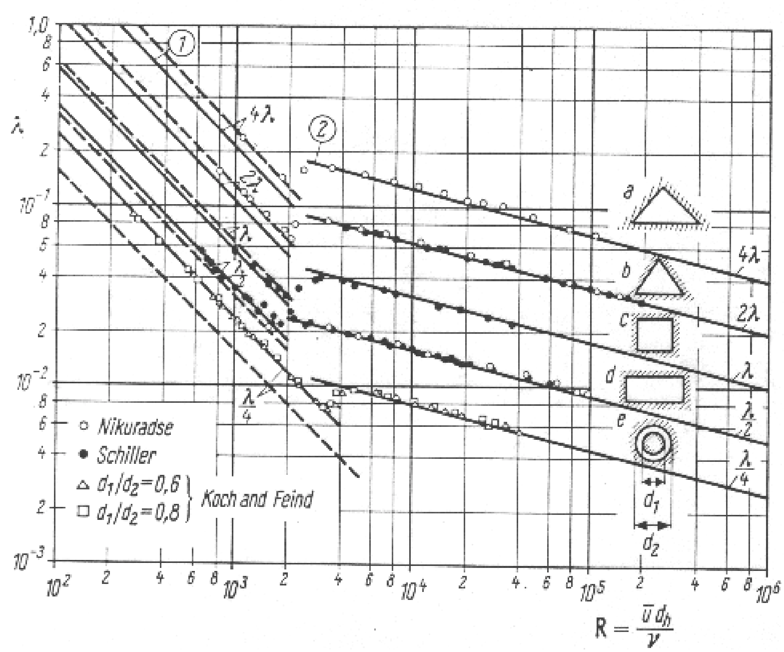


Fig. 6: Friction factors measured for different geometries also follow Prandtl's law of friction (217).