11.6 LINES AND PLANES IN THREE DIMENSIONS

In two dimensions the graph of a linear equation, either y=ax+b or parametric (x(t),y(t)), is always a straight line, and a point and slope or two points completely determine the line. In three dimensions the graph of a linear equation, either z=ax+by+c or parametric (x(t), y(t), z(t)), has more freedom, and it can be a line or a plane. In this section we examine the equations of lines and planes and their graphs in 3–dimensional space, discuss how to determine their equations from information known about them, and look at ways to determine intersections, distances, and angles in three dimensions. Lines and planes are the simplest graphs in three dimensions, and they are useful for a variety of geometric and algebraic applications. They are also the building blocks we need to find tangent lines to curves and tangent planes to surfaces (Fig. 1) in three dimensions.

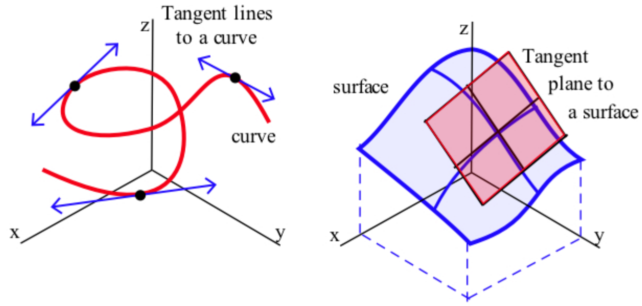


Fig. 1: Tangent lines and planes in three dimensions

Lines in Three Dimensions

Early in calculus we often used the point–slope formula to write the equation of the line through a given point $P = (x_0, y_0)$ with a given slope m: $y - y_0 = m(x - x_0)$. If the point $P = (x_0, y_0)$ is given and the direction of the line is parallel to a given vector $\mathbf{A} = \langle a, b \rangle$, then it is easier to use parametric equations to specify an equation for the line:

a point $Q = (x, y) \neq P$ is on the line if and only if $\dfrac{y - y_0}{x - x_0} = \dfrac{\text{rise}}{\text{run}} = \dfrac{b}{a} = \dfrac{b\cdot t}{a\cdot t}$ for some $t \neq 0$

so the equations $y - y_0 = b\cdot t$, and $x - x_0 = a\cdot t$ describe points on the line and

$x = x_0 + a\cdot t, y = y_0 + b\cdot t$.

> Parametric Equation of a Line in Two Dimensions
>
> Parametric equations for a line through the point $P = (x_0, y_0)$ and parallel to the vector $\mathbf{A} = \langle a, b \rangle$ are
>
> $x = x(t) = x_0 + a\cdot t$ and $y = y(t) = y_0 + b\cdot t$. (Fig. 2)

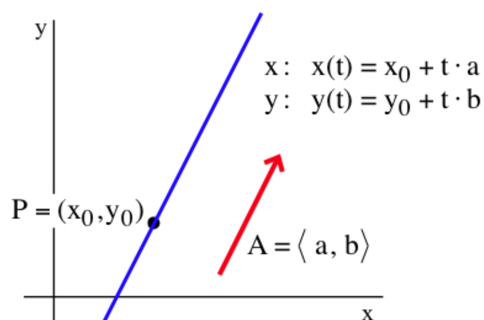


Fig. 2

Notice that the coordinates of the point (x_0, y_0) become the constant terms for x(t) and y(t), and the components of the vector $\langle a, b \rangle$ become the coefficients of the variable terms in the parametric equations. This same pattern is true for lines in three (and more) dimensions.

Example 1: Find parametric equations for the lines through the point $P = (1,2)$ that are (a) parallel to the vector $\mathbf{A} = \langle 3, 5 \rangle$, and (b) parallel to the vector $\mathbf{B} = \langle 6, 10 \rangle$. Then graph the two lines.

Solution: (a) $x(t) = 1 + 3t$, $y(t) = 2 + 5t$. (b) $x(t) = 1 + 6t$, $y(t) = 2 + 10t$.

The graphs of the lines are shown in Fig. 3. In this example, both sets of parametric equations have the same line when graphed. If we interpret t as time, and $(x(t), y(t))$ as the location of an object at time t, then the graph is a picture of all of the points on the path of the object (for all times). The objects in (a) and (b) have the same path, but they are at different points on the path at time t (except when $t = 0$).

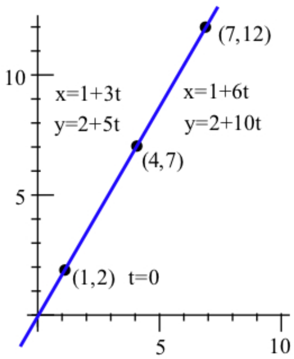


Fig. 3

Practice 1: Find parametric equations for the lines through the point $P = (3,-1)$ that are (a) parallel to the vector $\mathbf{A} = \langle 2, -4 \rangle$, and (b) parallel to the vector $\mathbf{B} = \langle 1, 5 \rangle$. Then graph the two lines.

The parametric pattern works for lines in three dimensions.

Parametric Equation of a Line in Three Dimensions

An equation of a line through the point $P = (x_0, y_0, z_0)$ and parallel to the vector $\mathbf{A} = \langle a, b, c \rangle$ is given by the parametric equations

$$x = x(t) = x_0 + at,$$
$$y = y(t) = y_0 + bt,$$
$$z = z(t) = z_0 + ct. \quad \text{(Fig. 4)}$$

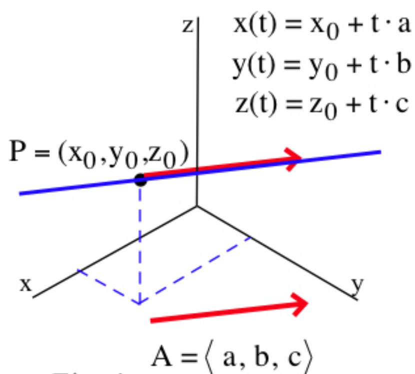


Fig. 4

Proof: To show that $P = (x_0, y_0, z_0)$ is on the line, put $t = 0$ and evaluate the three parametric equations: $x = x(0) = x_0 + 0$, $y = y(0) = y_0 + 0$, and $z = z(0) = z_0 + 0$.

To show that the line described by the parametric equations has the same direction as $\mathbf{A}$, we pick another point Q on the line and show that the vector from P to Q is parallel to $\mathbf{A}$.

Put $t = 1$, and let $Q = (x(1), y(1), z(1)) = (x_0 + a, y_0 + b, z_0 + c)$. Then the vector from P to Q is $\mathbf{V} = \langle (x_0 + a) - x_0, (y_0 + b) - y_0, (z_0 + c) - z_0 \rangle = \langle a, b, c \rangle = \mathbf{A}$ so the line described by the parametric equations is parallel to $\mathbf{A}$.

Example 2: Find parametric equations for the line (a) through the point $P = (3,0,2)$ and parallel to the vector $\mathbf{A} = \langle 1, 2, 0 \rangle$, and (b) through the two points $P = (3,0,2)$ and $Q = (5, -1, 1)$.

Solution: (a) $x(t) = 3 + 1t, y(t) = 0 + 2t, z(t) = 2 + 0t$.

(b) We can use the two points to get a direction for the line. The direction of the line is parallel to the vector from P to Q, $\langle 2, -1, -1 \rangle$, so

$x(t) = 3 + 2t, y(t) = 0 - 1t , z(t) = 2 - 1t$.

The graphs of these two lines are shown in Fig. 5.

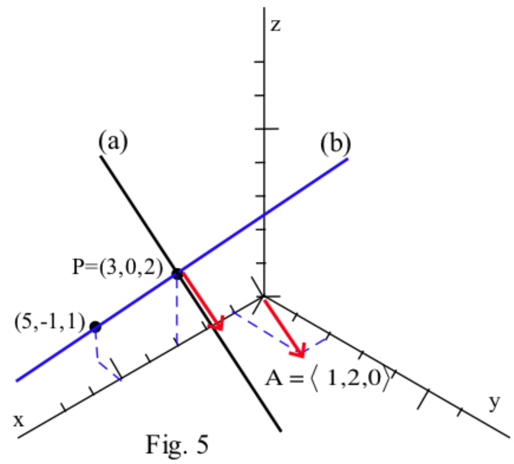


Fig. 5

Practice 2: Find parametric equations for the line

(a) through the point $P = (2, -1, 0)$ and parallel to the vector $\mathbf{A} = \langle 3, -4, 1 \rangle$, and (b) through the two points $P = (3,0,2)$ and $Q = (2, 5, 4)$.

Given a point $P = (x_0, y_0, z_0)$ and a vector $\mathbf{A} = \langle a, b, c \rangle$, the point Q is on the line through P in the direction of $\mathbf{A}$ if and only if $Q = (x_0+at, y_0+bt, z_0+ct)$ for some value of t.

Once we are able to write the equations of lines in three dimensions, it is natural to ask about the points and angles of intersection of these lines.

Example 3: (a) Find the point of intersection of the lines K: $x_K = 2 + t, y_K = 3 + 2t, z_K = 0 + t$ and L: $x_L = 5 - t, y_L = 1 + 2t, z_L = -1 + t$.

(b) Find the angle of intersection of the lines K and L.

(c) Where does line L intersect the xy–plane?

Solution: (a) If we could find a value of t so that $x_K = x_L$, $y_K = y_L$, and $z_K = z_L$, that would say that the lines not only intersect at the point (x_K, y_K, z_K), but that the two objects were at that point at the same time. For these parametric equations, there is no value of t such that $x_K = x_L$, $y_K = y_L$, and $z_K = z_L$, but it is still possible for the lines to intersect, just not at the same "time."

To see if the lines go through a common point, but at different "times," we can change the parameter for the line L to "s" instead of "t," and represent L as $x_L = 5 - s, y_L = 1 + 2s, z_L = -1 + s$. Then the equations $x_K = x_L$, $y_K = y_L$, and $z_K = z_L$ become

$$x:\ 2 + t = 5 - s,\quad y:\ 3 + 2t = 1 + 2s,\ \text{ and }\ z:\ 0 + t = -1 + s .$$

Solving the first two equations for s and t, we get $s = 2$ and $t = 1$. These values also satisfy the equation for the z–coordinate, $0 + t = -1 + s$, so $x = 2 + t = 3, y = 3 + 2t = 5$, and $z = 0 + t = 1$. The point $(3, 5, 1)$ lies on both lines. (If the values of s and t from the x–component and y–component equations do not satisfy the z–component equation, the lines do not intersect.)

(b) Since the lines intersect, we can find their angle of intersection. Line K is parallel to $\mathbf{A} = \langle 1, 2, 1 \rangle$, the coefficients of the t terms, and L is parallel to $\mathbf{B} = \langle -1, 2, 1 \rangle$, and the angle between the lines equals the angle between **A** and **B** :

$$\cos(\theta) = \frac{\mathbf{A} \bullet \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{4}{\sqrt{6}\sqrt{6}} = \frac{2}{3} \quad \text{so} \quad \theta \approx 0.84 \quad (\text{about } 48.2°).$$

(c) Every point on the xy–plane has z–coordinate equal to 0, so we can set $-1 + t = 0$ and solve for $t = 1$. When $t = 1$, then $x = 5 - (1) = 4$ and $y = 1 + 2(1) = 3$ so line L intersects the xy–plane at the point $(4, 3, 0)$.

Practice 3: If the pairs of lines in (a) or (b) intersect, find the point and angle of intersection.

(a) K: $x = 1 + t, y = 1 - 2t, z = -3 + 2t$ and L: $x = 8 + 4t, y = -4 + t, z = -5 - 8t$.

(b) K: $x = 1 + t, y = 1 - 2t, z = 2 + 2t$ and L: $x = 8 + 4t, y = -4 + t, z = 3 + t$.

(c) Where does L: $x = 8 + 4t, y = -4 + t, z = 3 + t$ intersect the yz–plane?

Practice 4: An arrow is shot from the point (1,2,3) and travels in a straight line in the direction $\langle 4, 5, 1 \rangle$. Will the arrow go over a 10 foot high wall built on the xy–plane along the line $y = 20$? (Fig. 6)

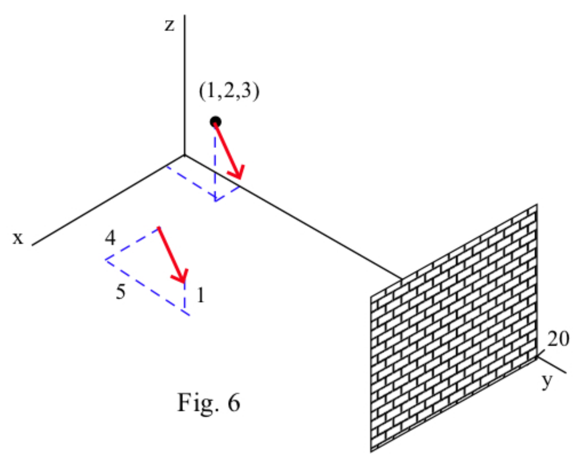


Fig. 6

Planes in Three Dimensions

The vectors in a plane point in infinitely many directions (Fig. 7) so, at first thought, you might think it would be more difficult to find an equation for a plane than for a line. Fortunately, however, there is only one vector (and its scalar multiples) that is perpendicular to the plane (Fig. 8), and this "normal" vector makes the task easy.

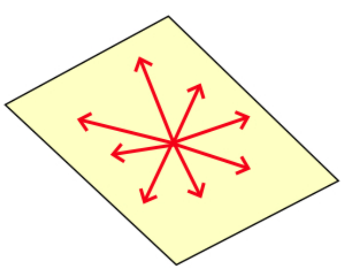


Fig. 7: Vectors in a plane

Suppose $P = (x_0, y_0, z_0)$ is a point on the plane that has normal vector $\mathbf{N} = \langle a, b, c \rangle$. Let $Q = (x, y, z)$ be another point. Since **N** is perpendicular to every vector on the plane, the point Q is on the plane if and only if **N** is perpendicular to the vector from P to Q.

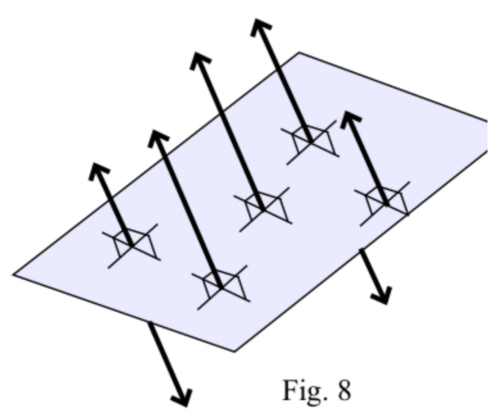


Fig. 8

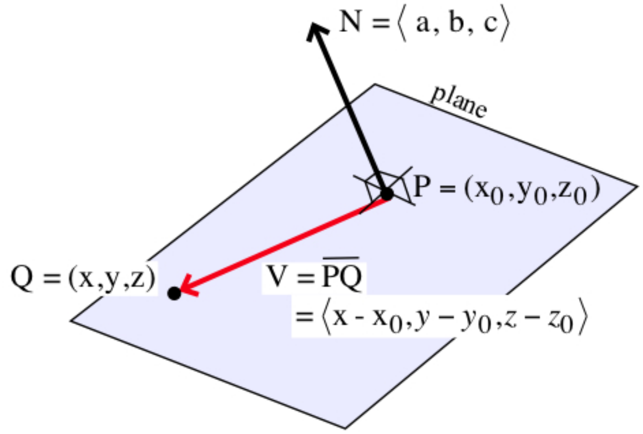
That is the idea that leads to an easy equation for the plane.

Let $\mathbf{V}$ = vector from P to Q = $\langle x - x_0, y - y_0, z - z_0 \rangle$. Then Q is on the plane if and only if $\mathbf{V}$ is perpendicular to $\mathbf{N}$: $\mathbf{V} \bullet \mathbf{N} = 0$. But $\mathbf{V} \bullet \mathbf{N} = a(x - x_0) + b(y - y_0) + c(z - z_0)$, so the point Q is on the plane if and only if $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Equation for a Plane in Three Dimensions:

(Point–Normal Form)

An equation for a plane through the point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{N} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. (Fig. 9)



Equation of the plane
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Fig. 9

The Point–Normal form is the fundamental pattern for the equation of a plane, and other information can usually be translated so the Point–Normal form can be used. If we have point P and two vectors $\mathbf{U}$ and $\mathbf{V}$ in the plane, then we can use the result that the cross product of two vectors is perpendicular to each of them to find a vector perpendicular to the plane: $\mathbf{N} = \mathbf{U} \times \mathbf{V}$. Once we have $\mathbf{N}$, we can use the Point–Normal form for the equation of the plane. If we have three points P, Q, and R, we can form the vectors $\mathbf{U}$ from P to Q and $\mathbf{V}$ from P to R, calculate the normal vector $\mathbf{N} = \mathbf{U} \times \mathbf{V}$ and then use the Point–Normal form for the equation for the plane.

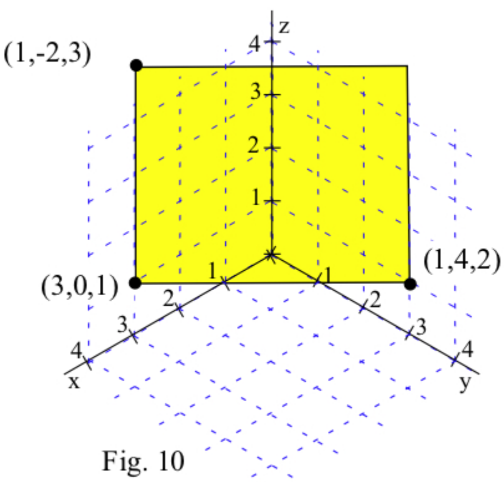


Fig. 10

Example 4: Find the equation of the plane:

(a) through the point $(1, -2, 3)$ and with normal vector $\mathbf{N} = \langle 5, 3, -4 \rangle$,

(b) through the points $P = (1, -2, 3)$, $Q = (3,0,1)$ and $R = (1,4,2)$. (Fig. 10)

Solution: (a) The point (x,y,z) is on the plane if and only if $5(x-1) + 3(y+2) - 4(z-3) = 0$, or, equivalently, $5x + 3y - 4z = -13$.

(b) Let $\mathbf{U}$ = vector from P to Q = $\langle 2, 2, -2 \rangle$ and $\mathbf{V}$ = vector from P to R = $\langle 0, 6, -1 \rangle$.

$$\text{Then } \mathbf{N} = \mathbf{U} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -2 \\ 0 & 6 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -2 \\ 6 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -2 \\ 0 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ 0 & 6 \end{vmatrix} = 10\mathbf{i} + 2\mathbf{j} + 12\mathbf{k}.$$

The equation of the plane is $10(x-1) + 2(y+2) + 12(z-3) = 0$ or $10x + 2y + 12z = 42$ (or $5x+y+6z=21$).

Practice 5: Find the equation of the plane:

(a) through the point $(4, 1, 0)$ and with normal vector $\mathbf{N} = \langle 3, -2, 6 \rangle$,

(b) determined by the lines K: $x = 2 + t$, $y = 2 + 2t$, $z = 4 - t$ and L: $x = 2 - 3t$, $y = 2 + t$, $z = 4 + 4t$. (Fig. 11) (The lines intersect at (2,2,4).)

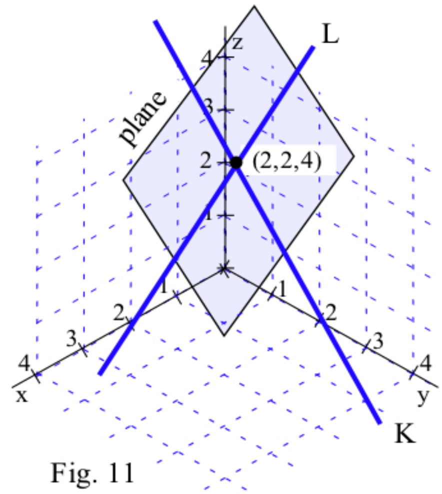


Fig. 11

Example 5: (a) Find a normal vector to the plane $3x - 2y + 4z = 12$.

(b) Where does the plane $3x - 2y + 4z = 12$ intersect each coordinate axis?

(c) Where does the line $x = 3 + 5t$, $y = -4 + 2t$, $z = 1 - 2t$ intersect the plane $3x - 2y + 4z = 12$?

Solution: (a) We can get one normal vector to the plane simply by using the coefficients of the variables of the equation of the plane: $\mathbf{N} = \langle 3, -2, 4 \rangle$. Any other nonzero vector is normal to the plane if and only if it is a nonzero scalar multiple of $\mathbf{N}$.

(b) Every point on the x–axis has y=0 and z=0, so we can find where the plane intersects the x–axis by setting y and z equal to zero in the plane equation and solving for x: $3x - 2(0) + 4(0) = 12$ so $x = 4$. The plane intersects the x–axis at (4,0,0). Similarly, the plane intersects the y–axis at (0,–6,0) and the z–axis at (0,0,3).

(c) Substitute the parametric patterns for x, y, and z from the line into the equation for the plane and then solve for t: $3(3 + 5t) - 2(-4 + 2t) + 4(1 - 2t) = 12$ so

$$9 + 15t + 8 - 4t + 4 - 8t = 12$$

$$21 + 3t = 12 \text{ and } t = -3.$$

Then $x = 3 + 5(-3) = -12$, $y = -10$, and $z = 7$. The point $(-12, -10, 7)$ is on the line and on the plane.

Practice 6: (a) Find a normal vector to the plane $3x + 10y - 4z = 30$.

(b) Where does the plane $3x + 10y - 4z = 30$ intersect each coordinate axis?

(c) Where does the line $x = 4 + t$, $y = -2 + 2t$, $z = 4 - t$ intersect the plane $3x + 10y - 4z = 30$?

Normal vectors also provide us with a way to determine the angle between a line and a plane and the angle between two planes.

We can also find where a parametric line intersects a sphere by substituting the x(t), y(t) and z(t) equations for x, y, and z in the equation of the sphere and then solving for the value of t. The line $x= 3t$, $y= 12 - 8t$, $z= 5 + 7t$ intersects the sphere $x^2 + y^2 + z^2 = 13^2$ when t=0 and t=1: at the points (0, 12, 5) and (3, 4, 12).

Angles of Intersection: Line & Plane and Plane & Plane

The angle between a line and a plane (Fig. 12) is

$\pi/2$ – (angle between the line and the normal vector of the plane) .

The angle between two planes (Fig. 13) is the angle between the normal vectors of the planes.

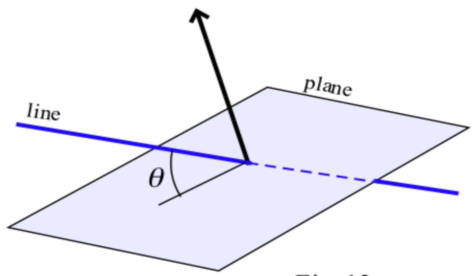


Fig. 12

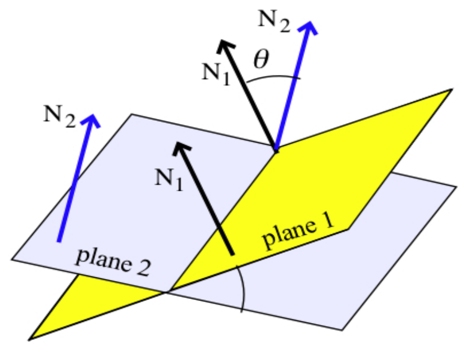


Fig. 13

If the normal vectors of two planes have different directions, then the planes intersect in a line (the set of points common to both planes is a line), and we can determine parametric equations for that line. One method is to eliminate the y variable from the equations of the planes and write x in terms of z: $x = x(z)$. Then we eliminate the x variable and write y in terms of z: $y = y(z)$. Finally, we treat the variable z as the parameter "t" in the parametric equations for the line, and we have $x = x(t), y = y(t), z = t$.

Example 6: Let P be the plane $12x + 7y - 3z = 43$ and Q be the plane $4x + 7y + 13z = 19$.

(a) Find a parametric equation representation of the line of intersection of the two planes.

(b) Find the angle between the planes.

Solution: (a) This is basically an algebra problem to use the equations of the two planes to solve for two of the variables in terms of the third variable. Then we can treat that third variable as the parameter t and write the equation of the line of intersection in parametric form. We can eliminate y and solve for x in terms of z: (equation for P) – (equation for Q) is $8x - 16z = 24$ so $x = 3 + 2z$. Then we can eliminate x and solve for y in terms of z: (equation for P) – 3(equation for Q) is $-14y - 42z = -14$ so $y = 1 - 3z$. Treating the variable z as our parameter "t" we have the line $\mathbf{x = 3 + 2t, y = 1 - 3t, \text{and } z = t}$.

As a check, when $t = 0$ then $x = 3, y = 1$, and $z = 0$, and the point (3,1,0) lies on both planes. When $t = 1$, then $x = 5, y = -2$, and $z = 1$, and the point (5,–2,1) also lies on both planes.

(b) The angle between the planes is the angle between the normal vectors of the planes. $\mathbf{N}_P = \langle 12, 7, -3 \rangle$, $\mathbf{N}_Q = \langle 4, 7, 13 \rangle$ so

$$\cos(\theta) = \frac{\mathbf{N}_P \bullet \mathbf{N}_Q}{|\mathbf{N}_P||\mathbf{N}_Q|} = \frac{58}{\sqrt{202}\sqrt{234}} \approx 0.267 \text{ so } \theta \approx 1.30 \text{ (about } 74.5°) .$$

Practice 7: Let R be the plane $2x + 3y - z = 13$ and S be the plane $2x - y + 3z = 1$.
(a) Find the line of intersection of the two planes. (b) Find the angle between the planes.

Sometimes the following alternate method is easier: find two points that lie on the intersection and then write the parametric equations for the line through those two points.

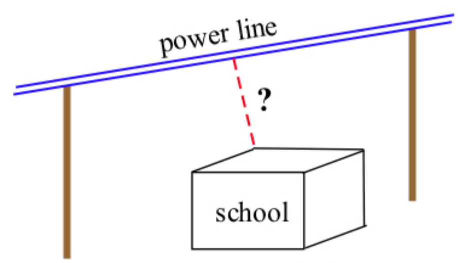
Example 7: Let P be the plane $7x - y - 11z = 10$ and Q be the plane $9x + y - 5z = 22$.
Find a parametric equation representation of the line of intersection of the two planes.

Solution: Set one variable equal to 0, say $z=0$, so we then have $7x - y = 10$ and $9x + y = 22$. Adding these two equations together gives $16x = 32$ so $x=2$ and then $y=4$ so one point on the intersection of the planes is $A = (2, 4, 0)$. Setting $x=0$, we have $-y - 11z = 10$ and $y - 5z = 22$. Adding these together gives $-16z = 32$ so $z=-2$ and then $y=12$ so another point on the intersection is $B = (0, 12, -2)$. A parametric equation representation of the line through A and B is $x(t) = 2 - 2t$, $y(t) = 4 + 8t$, $z(t) = -2t$. (We could have put $y=0$ and then found the point $C = (3, 0, 1)$. Check that point C satisfies the equations we just found.)

Distance Algorithms

Distances between objects are needed in a variety of applications in three dimensions:

- How close does the power line come to the school (Fig. 14)?
- How close can airplanes on two different linear flight paths come to each other (Fig. 15)?
- How far is the charged particle from the planar plate (Fig. 16)?



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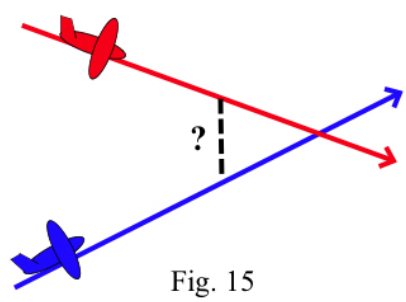

Fig. 15

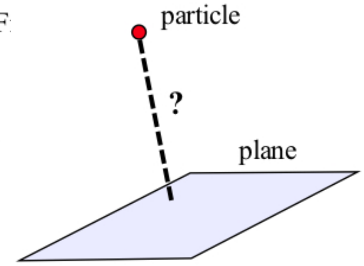

Fig. 16

Below we give a collection of distance algorithms, geometric constructions, and formulas for the distances between different types of objects. At first this may seem to be a large task, but most of the patterns use the idea of vector projection and follow from thinking about the geometry of the situations.

The distance from a point to a line in two dimensions was discussed in Section 11.3 and is presented again here because it uses the type of reasoning we need for three dimensions and because the resulting formula for two dimensions reappears for some distances in three dimensions.

Two dimensions: Distance from a **point** $P = (x_0, y_0)$ to a **line** $ax + by = c$ (Fig.17).

(1) Find **N** perpendicular to the line: $\mathbf{N} = \langle a, b \rangle$ works .

(2) Find a point Q on the line and form the vector **V** from P to Q .

(3) Distance = | Projection of **V** onto **N** |

$$= \frac{|\mathbf{V}\bullet\mathbf{N}|}{|\mathbf{N}|} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} .$$

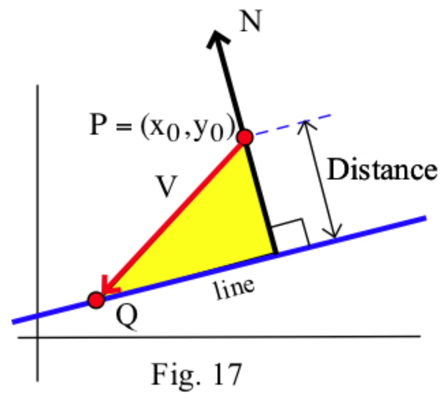

Fig. 17

Three dimensions: Distance from a **point** P to a **plane** with normal vector **N** (Fig. 18).

(1) Find a point Q on the plane and form the vector **V** from P to Q .

(2) Distance = | Projection of **V** onto **N** |

$$= \frac{|\mathbf{V}\bullet\mathbf{N}|}{|\mathbf{N}|} .$$

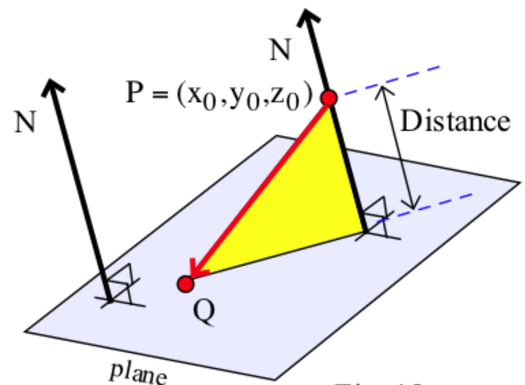

Fig. 18

Three dimensions: Distance from a **line** parallel to **U** to a **line** parallel to **V** (**U** not parallel to **V**) (Fig. 19).

(1) Find a point P on one line and a point Q on the other line .

(2) Form the vector **W** from P to Q .

(3) Calculate $\mathbf{N} = \mathbf{U} \times \mathbf{V}$ **.**

(4) Distance = | projection of **W** onto **N** | = $\frac{|\mathbf{W}\bullet\mathbf{N}|}{|\mathbf{N}|}$.

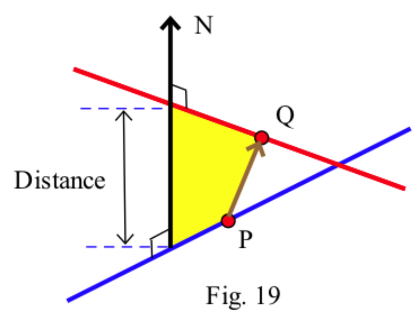

Fig. 19

(If **U** and **V** are parallel, pick a point on one line and then use the "point to line" formula.

In each of the previous situations, the distances were the lengths of suitable projections and the distance patterns involved dot products. In the next case, we also need the length of a projection, but then we use the Pythagorean formula to solve for the distance we want. The simplified result looks similar to the previous patterns, but has a cross product instead of a dot product.

Three dimensions: Distance from a **point** P to a **line** parallel to **U** (Fig. 20).

(1) Find a point Q on the line and form the vector **V** from P to Q .

(2) Put **W** = | projection of **V** onto **U** | = $\dfrac{|\mathbf{V}\cdot\mathbf{U}|}{|\mathbf{U}|} = \dfrac{|\mathbf{V}|\,|\mathbf{U}|\,|\cos(\theta)|}{|\mathbf{U}|} = |\mathbf{V}|\,|\cos(\theta)|$.

(3) Then $(\text{distance})^2 = |\mathbf{V}|^2 - |\mathbf{W}|^2$

$$= |\mathbf{V}|^2 - |\mathbf{V}|^2\cos^2(\theta)$$
$$= |\mathbf{V}|^2(1-\cos^2(\theta))$$
$$= |\mathbf{V}|^2\sin^2(\theta)$$
$$= \frac{|\mathbf{V}|^2\,|\mathbf{U}|^2\sin^2(\theta)}{|\mathbf{U}|^2} = \left(\frac{|\mathbf{V}\times\mathbf{U}|}{|\mathbf{U}|}\right)^2 .$$

(4) Distance = $\dfrac{|\mathbf{V}\times\mathbf{U}|}{|\mathbf{U}|}$.

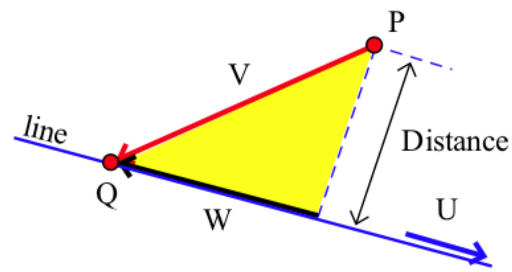

Fig. 20

Example 8: A power line runs in a straight line from the point A = (200, 0, 100) to the point B = (100, 700, 200). How close does the line come to the corner of a school at P = (300, 400, 0)?

Solution: This problem uses the pattern for the distance from a point to a line in three dimensions.

P = (300, 400, 0) and **U** = the vector from A to B = $\langle -100, 700, 100\rangle = 100\langle -1, 7, 1\rangle$:

$|\mathbf{U}| = 100\sqrt{51}$.

Since A is on the line, let **V** = vector from P to A = $\langle -100, -400, 100\rangle = 100\langle -1, -4, 1\rangle$.

$$\mathbf{V}\times\mathbf{U} = (100)(100)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 1 \\ -1 & 7 & 1 \end{vmatrix} = (100)^2(-11\mathbf{i} - 0\mathbf{j} - 11\mathbf{k}) .\ |\mathbf{V}\times\mathbf{U}| = 100^2\sqrt{242} .$$

Finally, distance from power line to school = $\dfrac{|\mathbf{V}\times\mathbf{U}|}{|\mathbf{U}|} = \dfrac{100^2\sqrt{242}}{100\sqrt{51}} \approx 217.8$.

PROBLEMS

In problems 1 – 8, find parametric equations for the lines.

1. The line through the point (2, –3, 1) and parallel to the vector $\langle 3, 4, 2\rangle$.
2. The line through the point (0, 5, –2) and parallel to the vector $\langle -3, 1, -4\rangle$.
3. The line through the point (–2, 1, 4) and parallel to the vector $\langle 5, 0, -3\rangle$.
4. The line through the origin and parallel to the vector $\langle 1, 2, 3\rangle$.
5. The line through the points (2, –1, 3) and (3, 4, –2).
6. The line through the points (7, 3, –4) and (5, 0, –2).
7. The line through the points (3, –2, 1) and (3, 4, –1).

8. The line through the origin and $(3, 4, -2)$.

In problems 9 – 12, the equations for a pair of lines are given. Determine whether the lines intersect, and if they do intersect, find the point and angle of intersection.

9. Line L: $x = 2 + t, y = -1 + t, z = 3 + 2t$. Line K: $x = 2 - t, y = -1 + 2t, z = 3 + 4t$.

10. Line L: $x = 6 + 2t, y = 3 - 2t, z = -2 + 3t$. Line K: $x = 6 + t, y = 3 + 5t, z = -2 - 3t$.

11. Line L: $x = 1 + 3t, y = 5 - t, z = -2 + 2t$. Line K: $x = 9 + 4t, y = 5, z = 4 + 3t$.

12. Line L: $x = 1 + 2t, y = 1 + 3t, z = 3 + 5t$. Line K: $x = 2 + t, y = 3 + t, z = 1 + t$.

In problems 13 – 26, find equations for the planes.

13. The plane through the point $(2, 3, 1)$ and perpendicular to the vector $\langle 5, -2, 4 \rangle$.

14. The plane through the point $(4, 0, -2)$ and perpendicular to the vector $\langle 3, 1, -5 \rangle$.

15. The plane through the point $(-3, 5, 6)$ and perpendicular to the vector $\langle 0, 3, 0 \rangle$.

16. The plane through the origin and perpendicular to the vector $\langle 2, -2, 1 \rangle$.

17. The plane through the points $(1, 2, 3)$, $(5, 2, 1)$, and $(4, -1, 3)$.

18. The plane through the points $(3, 5, 3)$, $(-2, 5, 4)$, and $(1, 5, 6)$.

19. The plane through the points $(-4, 2, 5)$, $(1, 2, 1)$, and $(3, -3, 5)$.

20. The plane through the origin, $(1, 2, 3)$, and $(4, 5, 6)$.

21. The plane through the point $(2, -5, 7)$ and parallel to the xy–plane.

22. The plane through the point $(2, 4, 1)$ and parallel to the yz–plane.

23. The plane through the point $(4, 2, 3)$ and parallel to the plane $3x - 2y + 5z = 15$.

24. The plane through the origin and parallel to the plane $2x + 3y - z = 12$.

25. The plane through the point $(4, 1, 3)$ and perpendicular to the line $x = 2 + 5t, y = 1 - 3t, z = 2t$.

26. The plane through the point $(2, 7, 4)$ and perpendicular to the y–axis.

Try to answer problems 27 – 32 without doing any algebra — just think visually.

27. Where does the plane $x = 0$ intersect the plane $z = 0$?

28. Where does the plane $y = 2$ intersect the plane $z = 3$?

29. Where does the plane $3x + 2y + z = 30$ intersect the x–axis?

30. Where does the plane $3x + 2y + z = 30$ intersect the y–axis?

31. Where do the three planes $x = 4$, $y = 2$, and $z = 1$ intersect?

32. Where does the y–axis intersect the plane $z = 3$?

In problems 33 – 36, two planes are given. Represent their line of intersection using parametric equations and find the angle between the planes.

33. $4x - 2y + 2z = 10$ and $3x - 2y + 3z = 36$.

34. $x + y + 3z = 9$ and $2x + y - 3z = 18$.

35. $5y - z = 10$ and $x + 2y + 4z = 16$.

36. $x = 4$ and $3x - 5y + z = 20$.

In problems 37 – 40, find where the given line intersects the plane and find the angle the line makes with the plane.

37. Line L: $x = 7 + 2t$, $y = 5 + t$, $z = 2 + 4t$ and the plane $5x + y - 2z = 20$.

38. Line L: $x = -2 + t$, $y = 12 - 3t$, $z = -5 - t$ and the plane $5x + y - 2z = 20$.

39. Line L: $x = 4 + 2t$, $y = 2 - 3t$, $z = 7 + t$, and the plane $z = 5$.

40. The x–axis and the plane $4x - 2y + 5z = 12$.

41. Is it possible for three distinct planes to intersect at a single point?

42. Is it possible for three distinct planes to intersect along a line?

43. A bird is flying in a straight line from the feeder (Fig. 21) to the corner of the house. Write an equation for its line of flight.

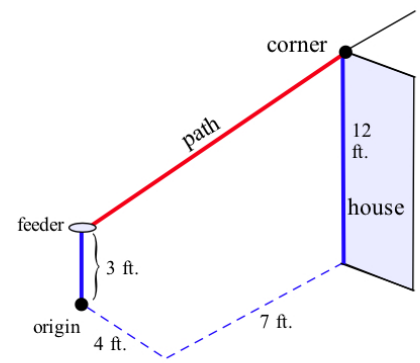

Fig. 21

44. What angle does the plane $3x + 2y + z = 10$ make with
(a) the xy–plane? (b) the xz–plane? (c) the yz–plane?

45. What angle does the plane $ax + by + cz = d$ make with each coordinate plane?

46. The four corners of a mirror are located at A(2,2,0), B(2,6,0), C(0,2,3), and D(0,6,3) (Fig. 22). A laser at L(5,0,0) directs a beam of light along the line $x = 5 - t$, $y = t$, $z = 0.5t$.

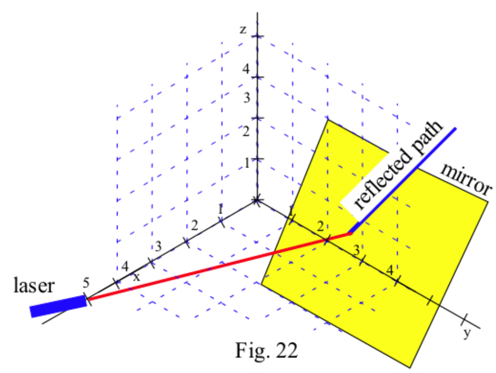

Fig. 22

(a) Write an equation for the plane of the mirror.
(b) Where does the light beam hit the mirror?
(c) What is the angle of incidence of the light with the mirror?

Many of the following applications can be solved in several ways including the methods of vectors and dot products from section 11.4.

47. When it was built, the Great Pyramid at Giza, Egypt was 481 feet tall and had a square base with length 756 feet on each side.
(a) Find the angle each side of the pyramid makes with the base.
(b) Find the angle each side makes with an adjacent side.
(c) Find the angle each edge makes with the base.

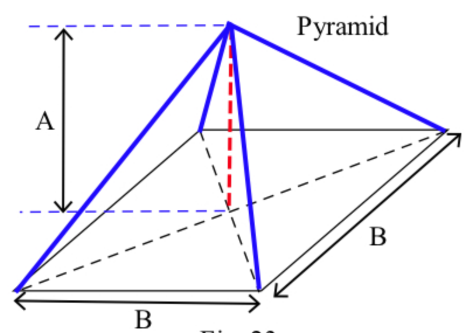

Fig. 23

48. In order to write a program to analyze a general pyramid (Fig. 23) with square base of length B and height A you need to determine the following:

(a) the angle each side makes with the base,

(b) the area of each side,

(c) the angle each edge makes with the base,

(d) the angle each side makes with an adjacent side, and

(e) the volume of the pyramid.

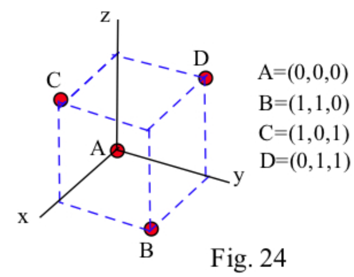

Fig. 24

49. Four molecules are located at the corners of a cube as shown in Fig. 24. The center of the cube is the point O = (0.5, 0.5, 0.5) Show that the angles AOB, AOC, and BOC are equal.

50. The four points (1,0,0), (–0.5,0.866,0), (–0.5,–0.866,0), and (0,0,1.414) are the vertices of an equilateral tetrahedron with center O = (0, 0, 0.354) (Fig. 25), and a molecule is located at each vertex.

(a) Show that the tetrahedron is really equilateral.

(b) Show that the angles AOB, AOC, and BOC are equal.

(c) Find the angle between the plane determined by AOB and the plane determined by AOC.

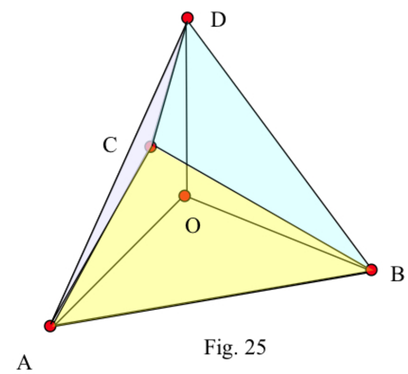

Fig. 25

Distances in Three Dimensions

51. Find the distance from the point (4, –1, 2) to the line x = 2 + t, y = 3 + 2t, z = 4t.

52. Find the distance from the point (4, –1, 2) to the line x = 1 – 3t, y = 2 + t, z = 3.

53. Find the distance from the point (2, 3, 1) to the plane 4x + 3y – z = 10.

54. Find the distance from the point (4, –3, 0) to the plane –2x + 5y + 3z = 15.

55. Find the distance (shortest distance) between the line x = 1 + 3t, y = 2 + 5t, z = 1 – t and the line x = 3 – 2t, y = 5 + t, z = 2 + 2t.

56. Find the distance (shortest distance) between the line x = 5 + 2t, y = 2 + t, z = 2 – 3t and the line x = 3 – 2t, y = 1 – 3t, z = 1.

In problems 57 – 62, the parameterized straight–line paths of two objects are given.

(a) Do the objects "crash" (so they are at the same location at the same time)? If so, at what time?

(b) Do the paths of the objects intersect (so the objects are at the same point but at different times)? If so, how close do the objects get to each other?

(c) Do the objects and their paths miss each other? If so, how close do the objects get to each other and how close do their paths get to each other?

57. Object A is at $x = 9 + t$, $y = 18 + t$, $z = 25 - 2t$ and object B is at $x = 3 + 2t$, $y = 30 - t$, and $z = 7 + t$.

58. Object A is at $x = -1 + t$, $y = -3 + 2t$, $z = 1 + t$ and object B is at $x = 2$, $y = t$, and $z = -2 + 2t$.

59. Object A is at $x = 5 - 5t$, $y = t$, $z = 5t$ and object B is at $x = 6 - 3t$, $y = 5 - 2t$, and $z = -3 + 4t$.

60. Object A is at $x = -4 + 5t$, $y = 3 + 2t$, $z = 16 - 3t$ and object B is at $x = 12 - t$, $y = 6 + 3t$, and $z = 3 + 4t$.

61. Object A is at $x = 5 - 2t$, $y = 0$, $z = 1$ and object B is at $x = 0$, $y = -1 + t$, and $z = 0$.

62. Object A is at $x = 1 + 3t$, $y = 2 + 2t$, $z = 3 + t$ and object B is at $x = 7 - t$, $y = 5 + 2t$, and $z = 3 + 3t$.

63. The bearing of an airplane is due north, and it passes directly above the origin at an altitude of 30,000 feet. How close does the airplane come to a balloon 5,000 feet directly above the point $x = 2{,}000$ and $y = 4{,}000$? (Assume the earth is "almost flat" in this region.)

64. At time t, airplane A is at $(-3 + t, 0, 1)$ and car B is at $(0, -5 + 2t, 0)$.

(a) How close do they come to each other?

(b) How close do their paths come to each other?

65. At time t, car A is at $(-3 + t, 2 + 2t, 0)$ and airplane B is at $(t, -5 + 2t, t)$.

(a) How close do they come to each other?

(b) How close do their paths come to each other?

66. Create your own problem, like problems 59 – 64, so the objects on different paths crash at the point $(3, 4, 5)$ at $t = 2$.

67. Create your own problem, like problems 59 – 64, with two objects on different paths so object A goes through the point $(3, 4, 5)$ at $t = 2$ and object B goes through the point $(3, 4, 5)$ at $t = 3$. (Then the paths intersect, but the objects do not crash.)

Practice Answers

Practice 1: (a) $x(t) = 3 + 2t$, $y(t) = -1 - 4t$.

(b) $x(t) = 3 + t$, $y(t) = -1 + 5t$.

The graphs are shown in Fig. 26.

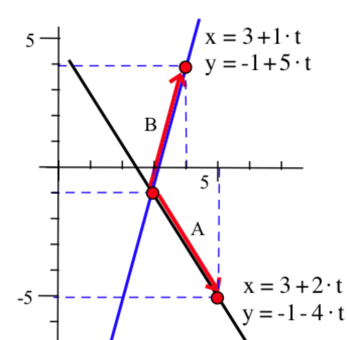

Fig. 26

Practice 2: (a) $x(t) = 2 + 3t, y(t) = -1 - 4t, z(t) = 0 + t$.

(b) The line is parallel to the vector $\mathbf{B} = \langle -1, 5, 2 \rangle$ from P to Q. Using P as the starting point, a parametric equation of the line is $x(t) = 3 - 1t$, $y(t) = 0 + 5t, z(t) = 2 + 2t$. (Using Q as the starting point, $x(t) = 2 - t$, $y(t) = 5 + 5t$, $z(t) = 4 + 2t$.)

Practice 3: Replacing the t parameter with s for line L and then setting the components equal to each other, we get: x: $1 + t = 8 + 4s$, y: $1 - 2t = -4 + s$, and z : $-3 + 2t = -5 - 8s$.
Solving the first two equations for t and s, $t = 3$ and $s = -1$. These values for t and s also satisfy the third equation so $x = 1 + (3) = 4, y = 1 - 2(3) = -5$, and $z = -3 + 2(3) = 3$.
The point $(4, -5, 3)$ lies on both lines.
The lines are parallel to $\mathbf{A} = \langle 1, -2, 2 \rangle$ and $\mathbf{B} = \langle 4, 1, -8 \rangle$ so

$$\cos(\theta) = \frac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{-14}{(3)(9)} \quad \text{and } \theta \approx 2.12 \quad (\text{about } 121.2°).$$

(b) Replacing the t parameter with s for line L and then setting the components equal to each other, we get: x: $1 + t = 8 + 4s$, y: $1 - 2t = -4 + s$, and z : $2 + 2t = 3 + s$.
Solving the first two equations (they are the same as in part (a)) for t and s, $t = 3$ and $s = -1$. But these values do not satisfy the third equation, $2 + 2(3) \neq 3 + (-1)$, so the lines do not intersect.

(c) Every point in the yz–plane has x coordinate 0, so set $8 + 4t = 0$ to get $t = -2$. Then $y = -4 + (-2) = -6$ and $z = 3 + (-2) = 1$ so the line intersects the yz–plane at $(0, -6, 1)$.

Practice 4: The parametric equations for the line of travel of the arrow are $x = 1 + 4t, y = 2 + 5t$, $z = 3 + t$. The arrow reaches the wall when $y = 20$ so $2 + 5t = 20$ and $t = 18/5$. At that time, $t = 18/5$, the height of the arrow is $z = 3 + t = 3 + (18/5) = 33/5 = 6.6$ feet so the arrow does not go over the wall.

Practice 5: (a) The point (x,y,z) is on the plane if and only if $3(x-4) - 2(y-1) + 6(z-0) = 0$, or, equivalently, $3x - 2y + 6z = 10$.

(b) Taking the directions of the lines K and L from their parametric equations, we know line K is parallel to $\mathbf{U} = \langle 1, 2, -1 \rangle$ and line L is parallel to $\mathbf{V} = \langle -3, 1, 4 \rangle$. Then

$$\mathbf{N} = \mathbf{U} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 9\mathbf{i} - 1\mathbf{j} + 7\mathbf{k} .$$

The point (2,2,4) is on both lines so it is on the plane. Using the point (2,2,4) and the normal vector $\mathbf{N} = \langle 9, -1, 7 \rangle$, the equation of the plane is

$$9(x-2) - 1(y-2) + 7(z-4) = 0 \text{ or } 9x - y + 7z = 44.$$

Practice 6: (a) $\mathbf{N} = \langle 3, 10, -4 \rangle$ and all nonzero scalar multiples of $\mathbf{N}$ are normal to the plane.

(b) The plane crosses the x–axis at (10,0,0), the y–axis at (0,3,0), and the z–axis at (0,0,–7.5) .

(c) Solving $3(4 + t) + 10(-2 + 2t) - 4(4 - t) = 30$ for t, we get $12 + 3t - 20 + 20t - 16 + 4t = 30$ so $-24 + 27t = 30$ and $t = 2$. Then $x = 6$, $y = 2$, and $z = 2$. The point (6,2,2) is on the line and on the plane.

Practice 7: (a) We can eliminate y and solve for x in terms of z: (equation for R) + 3(equation for S) is $8x + 8z = 16$ so $x = 2 - z$. Then we can eliminate x and solve for y in terms of z: (equation for R) – (equation for S) is $4y - 4z = 12$ so $y = 3 + z$. Treating the variable z as our parameter "t" we have the line $\mathbf{x = 2 - t, y = 3 + t, \text{and } z = t}$.

As a check, when $t = 0$ then $x = 2$, $y = 3$, and $z = 0$, and the point (2,3,0) lies on both planes. When $t = 1$, then $x = 1$, $y = 4$, and $z = 1$, and the point (1,4,1) also lies on both planes.

(b) The angle between the planes is the angle between the normal vectors of the planes.
$\mathbf{N}_R = \langle 2, 3, -1 \rangle$, $\mathbf{N}_S = \langle 2, -1, 3 \rangle$ so

$$\cos(\theta) = \frac{\mathbf{N}_R \bullet \mathbf{N}_S}{|\mathbf{N}_R||\mathbf{N}_S|} = \frac{-2}{\sqrt{14}\sqrt{14}} = \frac{-1}{7} \approx -0.143 \text{ so } \theta \approx 1.71 \quad (\text{about } 98.2°) .$$