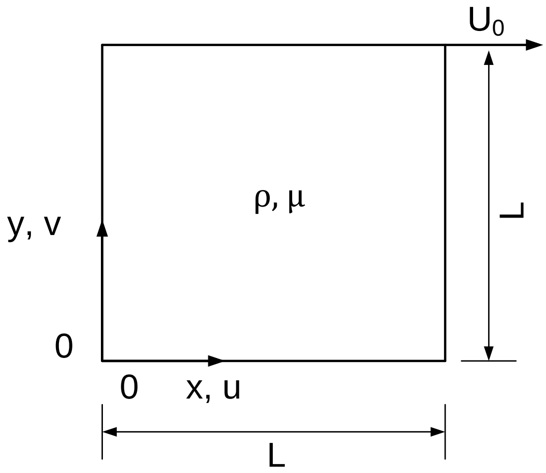
Spatial discretization error of steady, laminar lid driven cavity flow

Aim: Estimation of the spatial discretization error in the simulation of the steady, laminar lid driven 2D cavity flow with the method of generalized Richardson extrapolation. The cavity is quadratic with the top wall (lid) moving with constant velocity U_0. The other three walls defining the boundary of the cavity are at rest.



$$\mathrm{Re}_L = \frac{\rho U_0 L}{\mu} = 100 \quad \text{with } L = 1.0 \text{ m and } U_0 = 1.0 \text{ m/s}$$

→ $$\nu = \frac{\mu}{\rho} = 0.01$$

→ $$\rho = 1.0 \frac{kg}{m^3} \quad , \quad \mu = 0.01 \frac{kg}{ms}$$

The spatial discretization error has to be estimated by generalized Richardson extrapolation for the following four quantities:

- $$\underline{u}_1 \equiv \underline{u}\left(x = \frac{L}{2}, y = 0.9L \right)$$
- $$\underline{u}_2 \equiv \underline{u}\left(x = 0.9L, y = \frac{L}{2} \right)$$
- $$\overline{p}_{left} = \frac{1}{A_{left}} \int_{y=0}^{L} p(0, y) dy \cdot t$$ (depth: t = 1 m)
- $$\overline{p}_{right} = \frac{1}{A_{right}} \int_{y=0}^{L} p(L, y) dy \cdot t$$ (depth: t = 1 m)

The error estimation is used to determine the spatial discretization uncertainty U_D for all six scalar quantities (u_1, v_1, u_2, v_2, $\overline{p}_{left}$, $\overline{p}_{right}$) to present the results as $f \pm U_D$, where f is one of the six scalar quantities.

Grids:

The uncertainty due to the spatial discretization has to be estimated by generalized Richardson extrapolation. Therefore the solution for the six scalar quantities has to be calculated on a minimum of three systematically refined grids. Here the following four grid types have to be used.

- Structured, Cartesian, equidistant

  3 grids with 40 (coarse), 80 (medium) und 160 (fine) intervals per *edge*, using the block structure shown in Figure 1.

  Complete the following Table with your data.

Grid	First interval spacing [m]	Number of cells	Ratio
Coarse: G1_3			1.0
Medium: G1_2			1.0
Fine: G1_1			1.0

- Structured, Cartesian, non equidistant I

  3 grids with increase of the interval width away from the walls. Maximum ratio is q = 1.06774. The ratio q (= q_i, i=1,2,3) varies from grid to grid, as the number of intervals is always doubled while the interval spacing at the walls is reduced from L/80 over L/160 to L/320.

  The following block structure has to be used, with geometric series *Geometric 1* and *Geometric 2*.

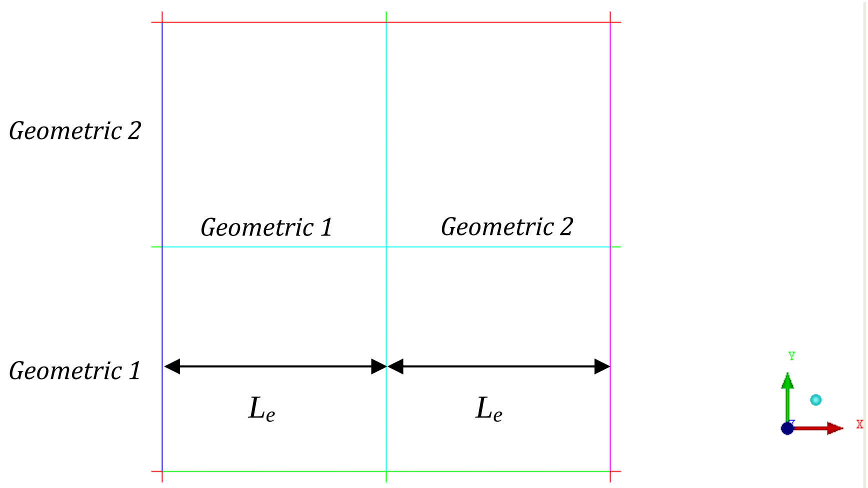


Figure 1: Block structure (to be used for all block structured grids).

Complete the following Table with your data.

Grid	First interval spacing [m]	Number of cells	Ratio
Coarse: G2_3	L/80		
Medium: G2_2	L/160		
Fine: G2_1	L/320		

- Structured, Cartesian, non equidistant II

Here the finest grid is the same as for the structured, Cartesian, non equidistant I grids (so only 5 non equidistant grids have to be created in total). Starting from the finest grid, the medium and coarse grids have to be created by always halving the number of intervals while keeping the ratio constant, q = 1.016. Therefore the interval spacing on the walls is not halved when going from one grid to the next coarser. Again the block structure shown in Figure 1 has to be used.

Complete the following Table with your data.

Grid	First interval spacing [m]	Number of cells	Ratio
Coarse: G3_3			q = 1.016
Medium: G3_2			q = 1.016
Fine: G3_1	L/320		q = 1.016

- Unstructured, triangles

  The coarsest grid has in *Global Element Seed Size* a value of 0.025 for *Max Element*. The refined grids (medium, fine) are generated by *Adjust Mesh Density* in *Edit Mesh*.

Grid	Number of cells
Coarse: G4_3	
Medium: G4_2	
Fine: G4_1	

For the structured, Cartesian, non equidistant II the mesh parameters have to be determined analytically in advance. The geometric series is defined as

$$L_e = \Delta_I \frac{q^n - 1}{q - 1},$$

where the length of each *edge* is L_e, the first spacing Δ_I, the ratio q and the number of intervals n.

- Solve for n.
- How large is L_e?
- Determine the first spacing $\Delta_{I,i}$ for the two grids i=2,3, if q = 1.016 = constant and $\Delta_{I,1}=L/320$ is given.

Richardson Extrapolation:

The solutions f_i on each grid triple i=1 (fine), i=2 (medium) und i=3 (coarse) are used to determine the solution f_{ext}, extrapolated towards h=0, if certain conditions are fulfilled. Here f is one of the six variables to be analyzed (u_1, v_1, u_2, v_2, $\overline{p}_{left}$, $\overline{p}_{right}$) and h is a characteristic grid width. Extrapolation is only possible when the solutions converge towards h=0 and do not diverge. Convergence or divergence are determined by the parameter *R*,

$$R = \frac{f_2 - f_1}{f_3 - f_2}. \tag{1}$$

Convergence is obtained for $|R| < 1$, i.e. when the change between the medium grid and fine grid solution is smaller than the change between the coarse grid and medium grid solution. If the sign of the aforementioned changes differs, then oscillating behavior is obtained, for -1<R<0 oscillating convergence and for R<-1 oscillating divergence.

Only in case of monotonic convergence, 0<R<1, the spatial discretization errors can be estimated in the following way:

The fine grid has index 1, the medium grid index 2 and the coarse grid index 3. The following ratios of the characteristic grid widths are defined:

$$r_{21} = h_2 / h_1, \quad r_{32} = h_3 / h_2 \tag{2}$$

In 2D the characteristic grid with is obtained by dividing the total area of the computational domain by the number of cells and taking the square root of the result.

$$h_i = \sqrt{\frac{A_{\text{cavity}}}{N_i}}, \quad N_i : \text{number of cells of grid } i$$

If the solution on each grid is developed in a truncated power series (only retaining the first terms), the following system of equations is obtained,

$$\begin{array}{lllll} f_1 & = & f_{ext} + g_p h_1^p & = & f_{ext} + g_p h_1^p \\ f_2 & = & f_{ext} + g_p h_2^p & = & f_{ext} + g_p (r_{21} h_1)^p \\ f_3 & = & f_{ext} + g_p h_3^p & = & f_{ext} + g_p (r_{21} r_{32} h_1)^p \end{array}, \tag{3}$$

with the three unknowns

- f_{ext} : extrapolated solution
- g_p : coefficient in the power series
- p : observed order of the solution

For the special case of a constant ratio of characteristic grid widths, $r = r_{21} = r_{32}$, the solution of (3) yields

$$p = \frac{ln\left[\left(f_3 - f_2\right)/\left(f_2 - f_1\right)\right]}{ln(r)} \tag{4}$$

$$f_{ext} = f_1 + \frac{f_1 - f_2}{r^p - 1} \tag{5}$$

With (5) the extrapolated solution (towards grid width equal to zero) is estimated. Furthermore estimates of the spatial discretization error are obtained on each grid.

$$DE_1 = f_1 - f_{ex} = \frac{f_2 - f_1}{r^p - 1} \tag{6}$$

$$\begin{aligned} DE_2 &= f_2 - f_{ex} = \frac{r^p\left(f_2 - f_1\right)}{r^p - 1} = r^p DE_1 \\ DE_3 &= f_3 - f_{ex} = \frac{r^{2p}\left(f_2 - f_1\right)}{r^p - 1} = r^{2p} DE_1 \end{aligned} \tag{7}$$

The magnitude of the estimated relative spatial discretization error on the fine grid is defined as

$$\left|E_1\right| = \frac{1}{r_{21}^p - 1}\left|\frac{f_2 - f_1}{f_1}\right|. \tag{8}$$

Multiplication of this expression with a factor of safety (F_s = 1.25) yields finally the uncertainty of the solution with regards to spatial discretization on the fine grid

$$\left|U_{D,1}\right| = \frac{F_s}{r_{21}^p - 1}\left|\frac{f_2 - f_1}{f_1}\right|. \tag{9}$$

These uncertainties on the fine grid have to be determined for all six scalar variables, u_1, v_1, u_2, v_2, $\overline{p}_{left}$ and $\overline{p}_{right}$.

Deliverables

1. The results of the six scalar variables on each of the 12 grids has to be delivered in ASCII format (unformatted text).

2. All necessary steps for the determination of the uncertainties, eqn. (9), have to be delivered as tables. These are:

   - Determination of characteristic grid width.
   - Determination of *R* by eqn. (1).
   - For variables and grids showing **monotonic convergence (0 < R < 1)**: determine p (eqn. (4)), DE_1 (eqn. (6)), E_1 (eqn. (8)) and finally $U_{D,1}$ (eqn. (9)).
   - For variables and grids showing **no monotonic convergence**: determine $U_{D,1}$ from

     $$\left|U_{D,1}\right| = 3\frac{\Delta_{\max}}{|f_1|} \tag{10}$$

     where Δ_{max} is the magnitude of the maximum difference of the solutions for f on all three grids,

     $$\Delta_{\max} = \max\left(|f_1 - f_2|, |f_1 - f_3|, |f_2 - f_3|\right) \tag{11}$$

3. All the tables from 2. have to be included into a short report. Ideally figures of the convergence behavior with grid width are added. The short report should also contain

   - Information on the grids: copy and paste the tables from the first part of the homework (grid generation), extend by one more column for the characteristic grid width h for each grid, fill in the numbers. In addition add a figure of each grid.
   - Information on your numerical solutions: material properties (copy & paste from the first part of the task), boundary conditions (copy & paste from the first part of the task), all employed numerical approximations, iterative convergence criteria and convergence behavior (figures; from Fluent: File -> Save Picture ...).

   All tables and figures should be numbered consecutively and have a caption, e.g.

Table 1: Parameters of Cartesian grids

Figure 1: Small VGU logo

Do not write a novel, but just enough text to explain what is shown in the tables and figures and why you choose the numerical solution parameters that you employed.