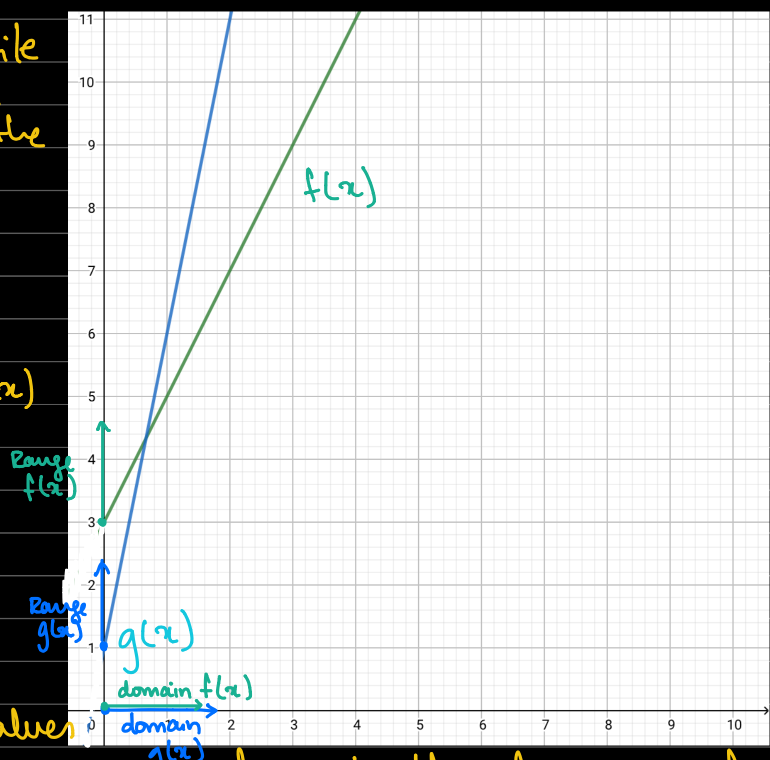
SELF-EXPLANATION OF COMPOSITION OF FUNCTIONS

↳ Domain & Range

Example: $f(x) = 2x + 3,\ x \geq 0$
$g(x) = 5x + 1,\ x \geq 0$

Whenever we make a composite function, we are replacing the x value of a function with the y-equivalent expression of another function.

For example, in $fg(x)$, the value of x in f is being replaced by the expression of $g(x)$ that returns the value of y on the line g for a given value of x.



$g(x) = 5x + 1$

y of $g = 5x + 1$] ↳ The possible values of this expression according to the domain of $g(x)$ is the range of $g(x)$.

$f(x) = 2x + 3$

$f(y \text{ of } g) = 2(y \text{ of } g) + 3$ ⟶ And because of this, it now makes sense that the range of $g(x)$, (ie. possible y values on the line g for any given x value within the domain) forms the domain (possible x values) of $f(x)$ because we replaced the x values in $f(x)$ with $g(x)$.

$= 2(5x + 1) + 3$

$= 10x + 2 + 3$

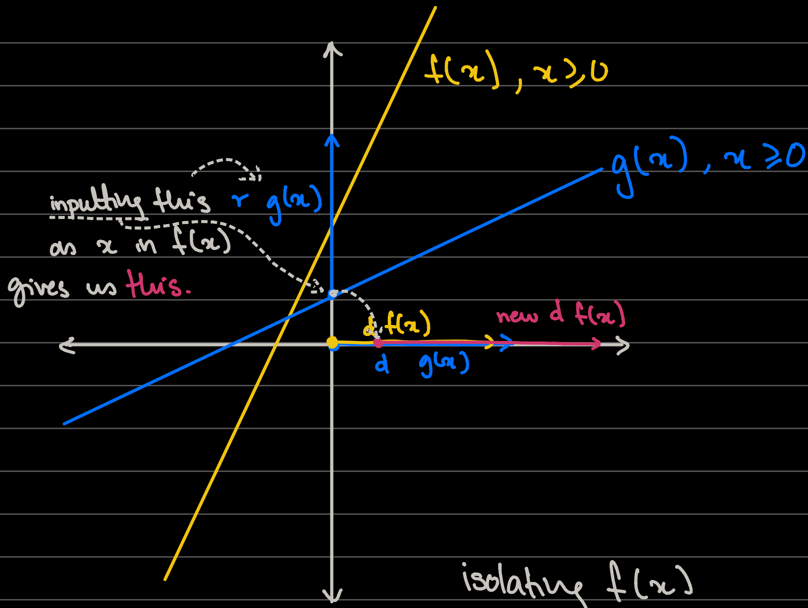
$= 10x + 5$

So, at this point we know ...

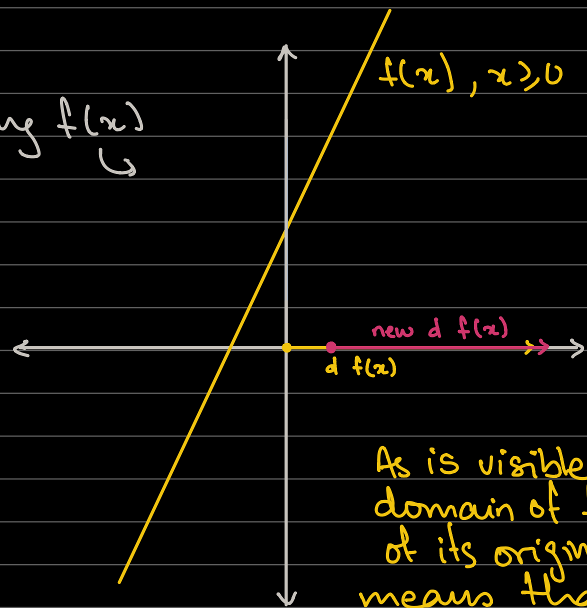
① The range of $g(x)$ is determined by inputting either the maximum or minimum value of the domain (or both) and thus finding the possible range of y-values.

② This range of $g(x)$, when forming $fg(x)$, becomes the new domain of $f(x)$, because we replace x in $f(x)$ with $g(x)$.

Another graphical example

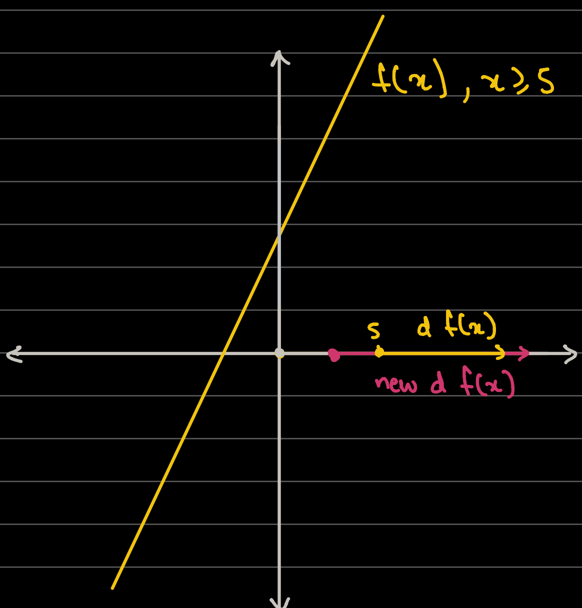


isolating f(x)

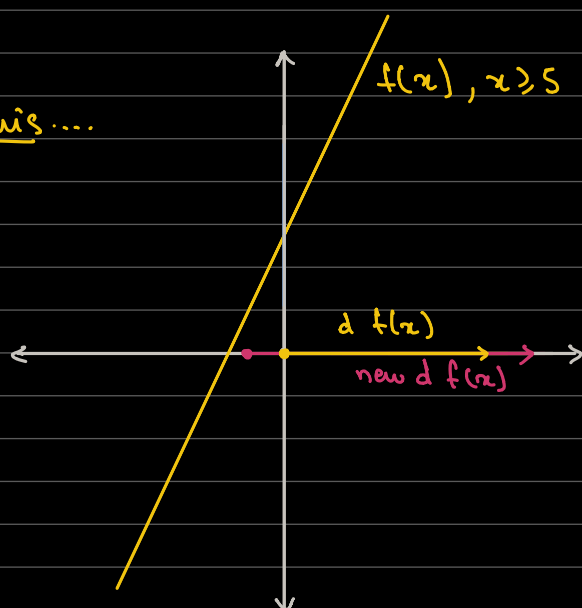


As is visible here, the new domain of f(x) is a subset of its original domain, which means that fg(x) is possible.

If, however, <u>this</u>....

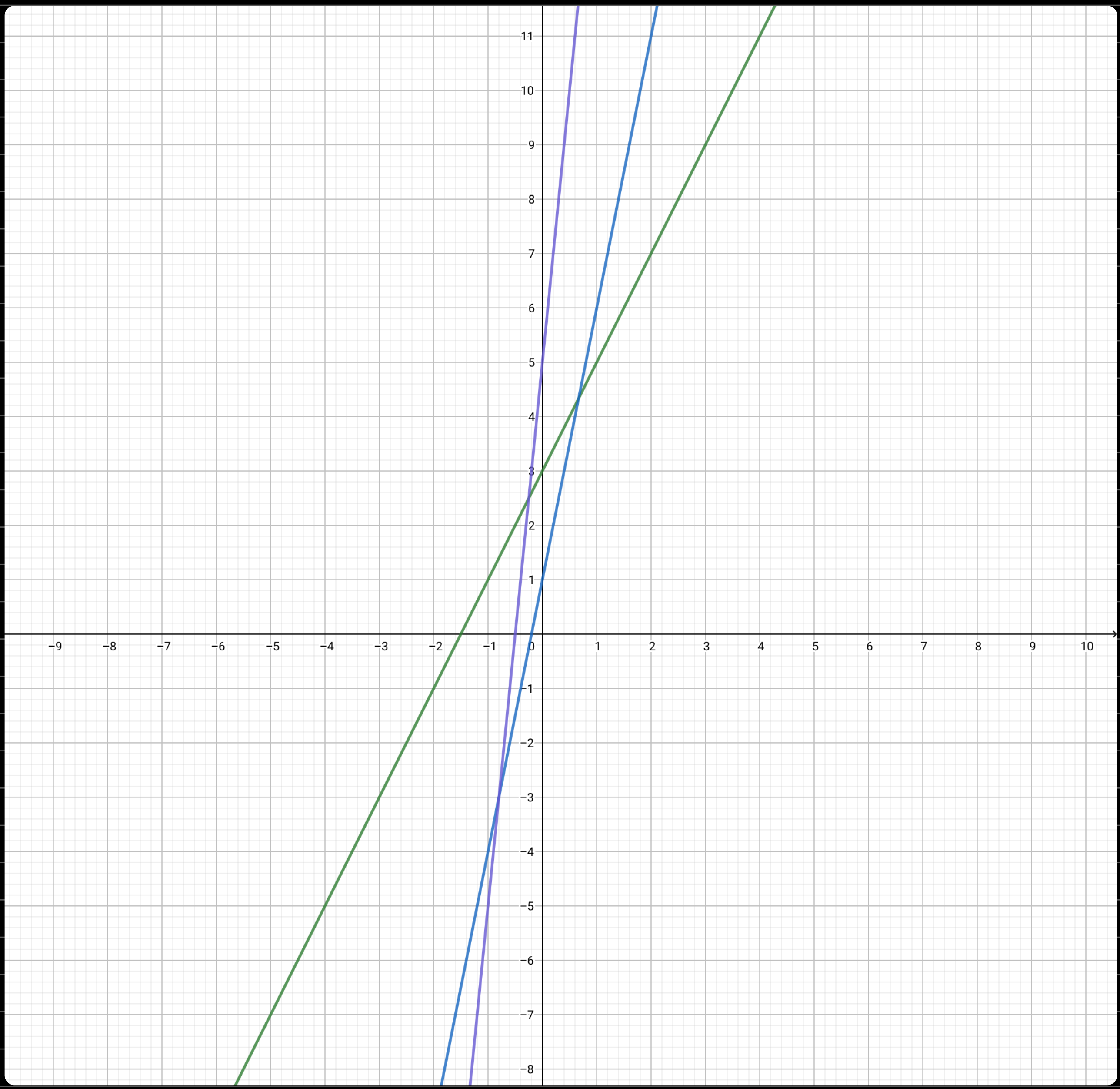


or <u>this</u>....



... was the case, then fg(x) wouldn't be possible

Now, assuming that $fg(x)$ is possible



we take the new domain of $f(x)$ that we just found and plug it into $f(x)$ to find the range of $fg(x)$.

Before any further explanation, it's important to know that:

The domain of $g(x)$ is the same as the domain of $g(x)$.

why?

ie.

$f(x) = 2x_f + 3 \rightarrow$ using x of f / x_f

$g(x) = 5x_g + 1 \rightarrow$ using x of g / x_g

when we compose $fg(x)$...

$f(g(x)) = 2(5x_g+1)+3$

... we end up replacing x_f with $g(x_g)$, therefore, in the final expression of $fg(x)$, we end up using x_g.

↳ And the possible values of x_g are given by the domain of $g(x)$

↳ Thus, the domain of $g(x)$ and $fg(x)$ are the same

Example questions.

$f(x) = 2x+3, \quad x \geq 0$
$g(x) = 5x+1, \quad x \geq 0$

Find the range and domain of $fg(x)$.

Domain of $g(x)$ = $x \geq 0$

↳ plug in x as $0 \rightarrow 5(0)+1$
$= +1$
↳ $g(x)$ must be ≥ 1

Therefore, range of $g(x) \Rightarrow \underline{g(x) \geq 1}$

↳ New domain of $f(x)$

Original domain of $f(x)$: $x \geq 0$
New domain of $f(x)$ $= x \geq 1$

$x \geq 1$ is a subset of $x \geq 0$ so $fg(x)$ is possible

Taking domain as $x \geq 1$ and plugging it into $f(x)$...

$2(1)+3$
$= 2+3$
$= 5 \rightarrow$ Range of $fg(x)$

$fg(x) \geq 5$, $\underline{x \geq 0}$ → to verify this ⇒

$10x+5 \geq 5$
$10x \geq 0$
$\underline{\underline{x \geq 0}}$

$fg(x) = 2(5x+1)+3$
$= 10x+5$

Q. $f(x) = 3x+1, \quad x \leq a$
$g(x) = -x^2-1, \quad x \leq -1$

i) Find the maximum value of a for which gf can be formed.
ii) For the case where $a = -1$ ...
a) Solve the equation $fg(x) + 14 = 0$

b) find the set of values of x which satisfy the following inequality: $gf(x) \leq -50$

i). $gf(x)$

↳ range of this is $g(x)$'s domain.

$f(x)$, $x \leq a$

$f(a) = 3a+1$

$f(a) \leq 3a+1 \rightarrow$ Range.

$g(x)$, $x \leq 3a+1$ → new domain of $g(x)$ must be a subdomain of the original domain

Therefore, $3a+1 \leq -1$ $\qquad x \leq -1$

$3a \leq -2$

$a \leq \frac{-2}{3}$ → a must be lower than or equal to this value for $gf(x)$ to be valid

$a = -\frac{2}{3}$ → The highest value of a for which gf is valid.

ii) a) $f(g(x)) \Rightarrow 3(-x^2-1)+1$, $x \leq -1$

$= -3x^2-3+1$

$= -3x^2-2$

$-3x^2-2+14=0$

$\frac{-3x^2}{-3} = \frac{-12}{-3}$

$x = \pm 2$ → however, the domain was established as $x \leq -1$ here

therefore $\underline{\underline{x = -2}}$

b) $gf(x) \leq -50$

$\Rightarrow -(3x+1)^2-1$, $x \leq -1$

$-1(3x+1)^2-1 \leq -50$

$-(3x+1)^2 \leq -49$

$(3x+1)^2 \geq 49$

$3x+1 \geq \pm 7$

$3x \geq \pm 7-1$

$x \geq \frac{\pm 7-1}{3}$

$x \geq \frac{7-1}{3}$ $\qquad x \leq \frac{-7-1}{3}$

$x \geq 2$ $\qquad x \leq -2.6$

↳ These are the raw values

When we apply the domain, $x \le -1$, there's only one set of values of x

$$= \underline{\underline{x \le -2.6}}$$