MAXIMA / MINIMA but for graphs.

Example question:

Q. $y = 4x + \frac{9}{x}$

Find the coordinates of the turning point and determine if each of these are maximum or minimum.

$y = 4x + \frac{9}{x}$

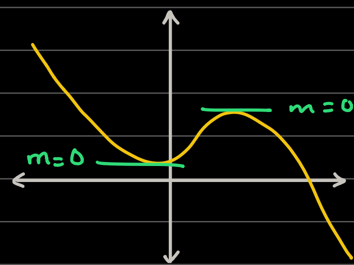
$\frac{d}{dx} \cdot y = 4 - \frac{9}{x^2}$

$4 - \frac{9}{x^2} = 0$ → This gives us the points at which the gradient of the tangent is 0, ie. stationary points

$4x^2 = 9$

$x^2 = \frac{9}{4}$

$x = \pm\frac{3}{2}$ → The two x coordinates at which the graph turns



Just an example, not representative of the actual function in this case.

when $x = 3/2$

$y = 4x + \frac{9}{x}$

$y = 4\left(\frac{3}{2}\right) + \frac{2}{3}(9)$

$y = 6 + 6$

$y = 12$

$\left(\frac{3}{2}, 12\right)$ → turning point # 1

when $x = -3/2$

$y = 4x + \frac{9}{x}$

$y = 4\left(-\frac{3}{2}\right) + \frac{-2}{3}(9)$

$y = -6 - 6$

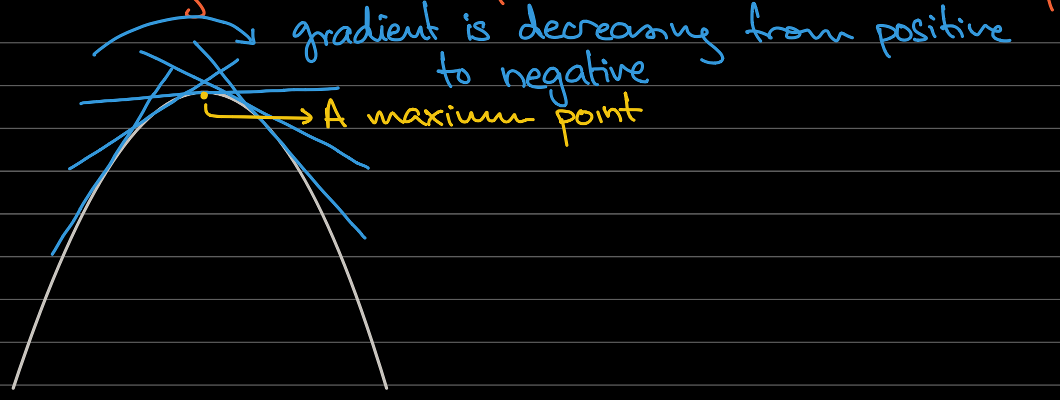
$y = -12$

$\left(-\frac{3}{2}, -12\right)$ → turning point #2

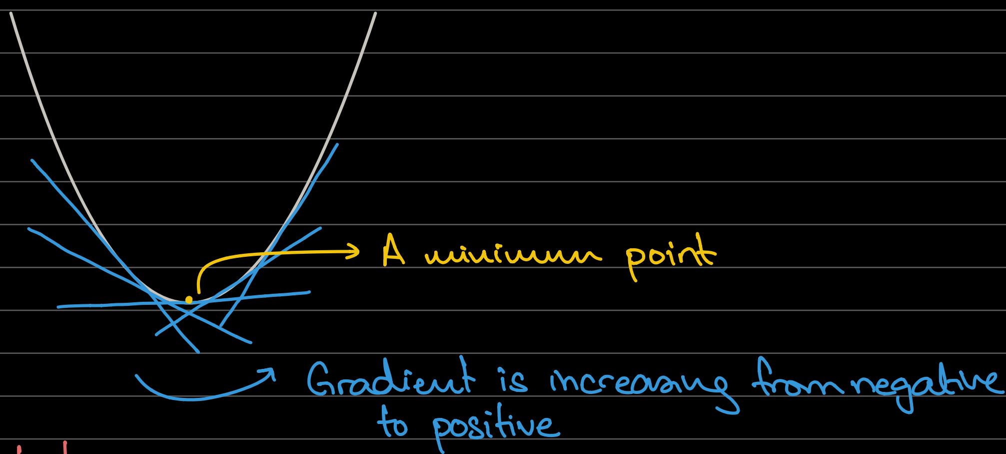
To determine whether a point is a maximum or a minimum point, we must do the second derivative test.

↳ The second derivative at a point will give us the rate of change of the gradient at that certain point

If the second derivative returns a negative value, then that means that the gradient is decreasing, hence the point must be a maximum point.



On the other hand, if the second derivative returns a positive value, that means the gradient is going from negative to positive and the graph is turning upwards, and hence, the turning point is a minimum point.



The second derivative test:

$$\frac{dy}{dx} = 4 - \frac{9}{x^2}$$

$$\left.\frac{d^2}{dx^2} y\right|_{x=3/2} = \frac{18}{x^3} = \left(\frac{2}{3}\right)^3 18 = \frac{8}{27}(18) = \frac{144}{27} = \frac{16}{3}$$

→ positive result, therefore the stationary point $\left(\frac{3}{2}, 12\right)$ is a minimum point

For $x = -3/2$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-3/2} = \frac{18}{x^3} = \left(-\frac{2}{3}\right)^3 18 = \frac{-8}{27}(18) = -\frac{16}{3}$$

→ negative result, therefore the stationary point $\left(-\frac{3}{2}, -12\right)$ is a maximum point

Questions to attempt:

From worksheets (NG) slide 72 onwards:

~~1 to 10 from slide 72~~ Done

1, 3, 5, 7, 9 from slide 73

From Slide 72:

1. Already done above

2. $y = x^2 - 2x^4$. Find stationary points and their natures

$$\frac{dy}{dx} = 2x - 8x^3$$

$$2x - 8x^3 = 0$$
$$x(-8x^2 + 2) = 0$$

$x = 0$ or $-8x^2 + 2 = 0$

$$\frac{2}{8} = \frac{8x^2}{8}$$
$$\sqrt{\frac{1}{4}} = \sqrt{x^2}$$
$$\pm\frac{1}{2} = x$$

$\therefore, x = 0, \pm\frac{1}{2}$

For $x = 0$
$$y = x^2 - 2x^4$$
$$= 0^2 - 2(0)^4$$
$$= 0$$

$(0, 0)$ Minimum

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 - 24x^2 = 2 - 0$$
$$= 2$$
↳ positive

For $x = \frac{1}{2}$

$$y = \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^4$$
$$= \frac{1}{4} - 2\left(\frac{1}{16}\right)$$
$$= \frac{2}{8} - \frac{1}{8}$$
$$= \frac{1}{8}$$

$\left(\frac{1}{2}, \frac{1}{8}\right)$ Maximum

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pm\frac{1}{2}} = 2 - 24x^2 = 2 - 24\left(\pm\frac{1}{2}\right)^2$$
$$= 2 - 6$$
$$= -4$$
↳ negative

For $x = -\frac{1}{2}$

Same value as $x = \frac{1}{2}$ because even powers result in positive numbers

3. $\left(-\frac{1}{2}, \frac{1}{8}\right)$ Maximum

3. $y = ax^2 + bx + c$ has max point at $(2, 18)$
passes through $(0, 10)$

Find a, b, and c

$\frac{dy}{dx} = 2ax + b$

$a(0)^2 + b(0) + c = 10$ → given point
$c = 10$

$2a(2) + b = 0$
$4a + b = 0$ (1)

↳ grad at max point is 0

$ax^2 + bx + c = y$
$a(2)^2 + 2b + c = 18$
$4a + 2b + c = 18$
$4a + 2b = 8$ (2)

$4a + b = 0$
$b = -4a$
$b = -4(-2)$
$b = 8$

$4a + 2(-4a) = 8$
$4a - 8a = 8$
$-4a = 8$
$a = -2$

$\therefore$, $a = -2$, $b = 8$, and $c = 10$ → Ans

4. $y = x^3 - 6x^2 + 9x$ Find turning points + Sketch graph

$\frac{dy}{dx} = 3x^2 - 12x + 9 = 0$

$3x^2 - 9x - 3x + 9 = 0$
$3x(x-3) - 3(x-3) = 0$
$(3x-3)(x-3) = 0$
$3x - 3 = 0$ or $x - 3 = 0$
$x = 1$ $x = 3$

$y = (1)^3 - 6(1)^2 + 9(1)$
$= 1 - 6 + 9$
$y = 4$

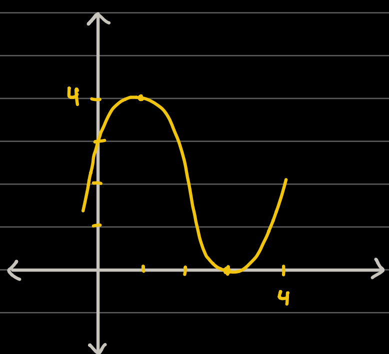
$y = (3)^3 - 6(3)^2 + 9(3)$
$= 27 - 54 + 27$
$y = 0$

$(1, 4)$ Max.

$(3, 0)$ Min.

$\frac{d^2y}{dx^2} = 6x - 12 = 6 - 12$
$= -6$
$x = 1$

$\frac{d^2y}{dx^2} = 6x - 12 = 18 - 12$
$= 6$
$x = 3$



5. $y = 8x + \frac{1}{2x^2}$ Find turning point and its nature

$= 8x + \frac{1}{2}x^{-2}$

$\frac{dy}{dx} = 8 + -1x^{-3}$

$= 8 - \frac{1}{x^3}$

$8 - \frac{1}{x^3} = 0$

$8 = \frac{1}{x^3}$

$x^3 = \frac{1}{8}$

$x = \frac{1}{2}$

For $x = \frac{1}{2}$

$y = 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^2}$

$= 4 + \frac{1}{2}(4)$

$= 4 + 2$

$y = 6$

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = \frac{3}{x^4} = 3 \cdot \left(\frac{2}{1}\right)^4$

$= 3 \cdot 16$

$= 48$

$\left(\frac{1}{2}, 6\right)$ Minimum $\rightarrow$ Ans

6. Illegible

7. $y = x(x-1)^2$ Find turning points and their natures

$y = x(x^2 - 2x + 1)$

$y = x^3 - 2x^2 + x$

$\frac{dy}{dx} = 3x^2 - 4x + 1$

$3x^2 - 4x + 1 = 0$

$3x^2 - 3x - x + 1 = 0$

$3x(x-1) - 1(x-1) = 0$

$(3x-1)(x-1) = 0$

$x = \frac{1}{3}$ or $x = 1$

$y = \frac{1}{3}\left(\frac{1}{3} - 1\right)^2$

$= \frac{1}{3}\left(-\frac{2}{3}\right)^2$

$= \frac{1}{3} \cdot \frac{4}{9}$

$y = 1(1-1)^2$

$= 1(0)^2$

$= 0$

$(1, 0)$ Minimum

$\frac{d^2y}{dx^2} = 6x - 4$

$= \frac{4}{27}$ $\quad\quad$ $= 6 - 4$

$= 2$

$\left(\frac{1}{3}, \frac{4}{27}\right)$ Maximum

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{3}} = 6x - 4 = 6\left(\frac{1}{3}\right) - 4$

$= 2 - 4$

$= -2$

8. $y = 2x^3 - 3x^2 - 12x + 18$ Find stationary points and natures

$\frac{dy}{dx} = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$6x^2 - 12x + 6x - 12 = 0$

$6x(x-2) + 6(x-2) = 0$

$(6x+6)(x-2) = 0$

$x = -1$ or $x = 2$

$y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 18$

$= 2(-1) - 3(1) + 12 + 18$

$= -2 - 3 + 12 + 18$

$= -5 + 12 + 18$

$= 13 + 12$

$y = 25$

$(-1, 25)$ Maximum

$y = 2(2)^3 - 3(2)^2 - 12(2) + 18$

$= 2(8) - 3(4) - 12(2) + 18$

$= 16 - 12 - 24 + 18$

$= 34 - 36$

$= -2$

$(2, -2)$ Minimum

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 12x - 6 = 12(-1) - 6$

$= -12 - 6$

$= -18$

$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 12x - 6 = 12(2) - 6$

$= 24 - 6$

$= 18$

9. $y = x^2 + \frac{16}{x}$ Find stationary point and its nature

$= x^2 + 16x^{-1}$

$\frac{dy}{dx} = 2x - \frac{16}{x^2}$

$2x - \frac{16}{x^2} = 0$

$2x = \frac{16}{x^2}$

For $x = 2$

$y = x^2 + \frac{16}{x}$

$= (2)^2 + \frac{16}{2}$

$= 4 + 8$

$= 12$

$(2, 12)$ Minimum

$$2x^3 = 16$$
$$x^3 = 8$$
$$x = 2$$

$$\left.\frac{d^2y}{dx}\right|_{x=2} = 2 + \frac{32}{x^3} = 2 + \frac{32}{2^3}$$
$$= 2 + \frac{32}{8}$$
$$= 2 + 4$$
$$= 6$$

10. $y = (x^2 - 4x)^{-3}$ Find stationary point and it's nature

$$\frac{dy}{dx} = -3(x^2-4x)^{-4}(2x-4)$$

$$0 = (2x-4) \cdot \frac{-3}{(x^2-4x)^4}$$
$$0 = 3(2x-4)$$
$$0 = 6x-12$$
$$12 = 6x$$
$$2 = x$$

For $x = 2$

$$y = \frac{1}{(2^2-4(2))^3}$$
$$= \frac{1}{(4-8)^3}$$
$$= \frac{1}{(-4)^3}$$
$$= \frac{1}{-64}$$
$$= \frac{-1}{64}$$

$$\left(2, \frac{-1}{64}\right)$$

✓

11. $y = \dfrac{16x^3 + 4x^2 + 1}{2x^2}$ Find stationary point and its nature

$$= \underbrace{(16x^3 + 4x^2 + 1)}_{f}\underbrace{\left(\frac{1}{2}x^{-2}\right)}_{g}$$

Product Rule

$$(f \cdot g)' = f' \cdot g + f + g'$$

$$\frac{dy}{dx} = (48x^2 + 8x)\left(\frac{1}{2x^2}\right) + (16x^3 + 4x^2 + 1)\left(\frac{-1}{x^3}\right)$$

$$= \left(\frac{48x^2 + 8x}{2x^2}\right) + \left(\frac{-16x^3 - 4x^2 - 1}{x^3}\right)$$

$$= \frac{(48x^2 + 8x)(x^3) + (-16x^3 - 4x^2 - 1)(2x^2)}{(2x^2)(x^3)}$$

$$= \frac{48x^5 + \cancel{8x^4} - 32x^5 - \cancel{8x^4} - 2x^2}{}$$

$2x^5$

$$= \frac{16x^5 - 2x^2}{2x^5}$$

$$= \frac{8x^3 - x^2}{x^5}$$

$$= \frac{\cancel{x^2}(8x^3 - 1)}{x^5}$$

$$= \frac{8x^3 - 1}{x^3}$$

$$0 = \frac{8x^3 - 1}{x^3}$$

$$0 = 8x^3 - 1$$

$$\frac{1}{8} = x^3$$

$$\frac{1}{2} = x$$

$$y = \frac{16\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 + 1}{2\left(\frac{1}{2}\right)^2}$$

$$= \frac{16\left(\frac{1}{8}\right) + 4\left(\frac{1}{4}\right) + 1}{2\left(\frac{1}{4}\right)}$$

$$= \frac{2 + 1 + 1}{\frac{1}{2}}$$

$$= \frac{4}{1}(2)$$

$$= 8$$

$$\left(\frac{1}{2}, 8\right)$$

11. Re-attempt

$$y = \frac{16x^3 + 4x^2 + 1}{2x^2}$$

$$= 8x + 2 + \frac{1}{2}x^{-2}$$

$$\frac{dy}{dx} = 8 - \frac{1}{x^3}$$

$\left(\frac{1}{2}, 8\right)$ Minimum

$$8 - \frac{1}{x^3} = 0$$

$$8 = \frac{1}{x^3}$$

$$x^3 = \frac{1}{8}$$

$$x = \frac{1}{2} \rightarrow y = 8$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = \frac{3}{x^4} = \frac{3}{\left(\frac{1}{2}\right)^4} = \frac{3}{\left(\frac{1}{16}\right)} = (3)(16) = 48$$

Slide 73 from WS NG

9. $x^4 y = 8$

$$y = \frac{8}{x^4}$$

$$z = x + y$$

$$z = x + \frac{8}{x^4}$$

$$\frac{dz}{dx} = 1 - \frac{32}{x^5}$$

$$0 = 1 - \frac{32}{x^5}$$

$$1 = \frac{32}{x^5}$$

$$x^5 = 32$$

$$x = 2$$

$$y = 0.5$$

$\rightarrow$ Ans

$$\frac{d^2z}{dx^2} = \frac{160}{x^6}$$

$\rightarrow$ For all values of x this is positive, hence $x = 2$ is a minimum point