1. Simple Differentiation ✓
2. Chain rule ✓
3. Grad / tan / normals → Doing
4. Increasing / decreasing functions
5. Maxima / Minima
6. Related Rates

Gradients / Tangents / Normals

Attempt Questions from WS NG slide 67 onwards

1, ~~2~~, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~8~~, ??10, ~~11~~, ~~13~~, ~~16~~

2. $y = ax^2 + bx$

$(2,4) \rightarrow m = -8$

$\frac{dy}{dx} = 2ax + b$

$-8 = 2a(2) + b$

$-8 = 4a + b$

$-8 - b = 4a$

$-8 - 12 = 4a$

$-20 = 4a$

$-5 = a \rightarrow$ Ans

$4 = a(2)^2 + b(2)$

$4 = 4a + 2b$

$4 = 2b - 8 - b$

$4 = b - 8$

$12 = b \rightarrow$ Ans

3. $y = (4 - 2x)^5$

$\frac{dy}{dx} = 5(4 - 2x^2)^4(-6x)$

$= 5(4 - 2(-1)^2)^4(-6(-1))$

$= 5(4 - 2)^4(6)$

$= 5(2)^4(6)$

$= 5 \times 16 \times 6$

$m = 480 \longrightarrow$ Ans

4. $y = \frac{4}{x^3}$ where $x = 2$

$\left.\frac{dy}{dx}\right| = -12x^{-4}$

$= \frac{-12}{x^4} \longrightarrow \frac{-12}{(2)^4} = \frac{-12}{16} = -\frac{3}{4}$

$y = mx + c$

$y = \frac{-3}{4}x + c$

$\frac{1}{2} = \frac{-3}{4}(2) + c$

$\frac{1}{2} = \frac{-6}{4} + c$

$\frac{2}{4} + \frac{6}{4} = c$

$\frac{8}{4} = c$

$2 = c$

$\frac{4}{x^3} = y$

$\frac{4}{2^3} = y$

$\frac{4}{8} = y$

$\frac{1}{2} = y \quad (2, \frac{1}{2})$

$y = \frac{-3}{4}x + 2$

$4y + 3x = 2 \rightarrow$ Ans 1.

$0 = \frac{-3}{4}x + 2$

$\frac{8}{3} = x$

when $x = 0$, $y = 2 \rightarrow (0, 2)$

when $y = 0$, $x = \frac{8}{3} \quad (\frac{8}{3}, 0)$

↳ R is midpoint

$\frac{0 + \frac{8}{3}}{2}, \frac{2 + 0}{2}$

$(\frac{8}{6} \frac{4}{3}, 1)$

$(\frac{4}{3}, 1) \rightarrow$ R Ans

$x \quad y$

$1 = \frac{4}{(\frac{4}{3})^3}$

$1 = \frac{4}{\frac{64}{27}}$

$1 \neq \frac{108}{64} \rightarrow$ Ans.

Hence, R does NOT lie on the curve

5. $y = x^3 - 7x^2 + 14x - 8$

$\left.\frac{dy}{dx}\right|_{x=1} = 3x^2 - 14x + 14$

$= 3 - 14 + 14 = 3$

gm

$y = 3x + c$

$y = (1)^3 - 7(1)^2 + 14(1) - 8$

$= 1 - 7 + 14 - 8$

$= 0$

$$0 = 3(1) + C$$
$$-3 = C$$
$$y = 3x - 3 \rightarrow \underline{\text{Ans}}$$

$$3 = 3x^2 - 14x + 14$$
$$0 = 3x^2 - 14x + 11$$
$$= 3x^2 - 11x - 3x + 11$$
$$= x(3x - 11) - 1(3x - 11)$$
$$= (x-1)(3x-11)$$

$x = 1$ or $x = \frac{11}{3} \rightarrow \underline{\text{Ans}}$ the "other" x-value

$\downarrow$
Already have this

6. $y = \dfrac{a}{x} + bx^3$

$$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{a}{x^2} + 3bx^2 = 7$$

$$-\frac{a}{x^2} + 3bx^2 = 7$$
$$-\frac{a}{(3)^2} + 3b(3)^2 = 7$$
$$-\frac{a}{9} + 3b(9) = 7$$
$$-\frac{a}{9} + 27b = 7 \quad (1)$$

$$6 = \frac{a}{3} + b(3)^3$$
$$18 = a + 81b \quad (2)$$
$$18 - 81b = a$$

(1) $-\dfrac{a}{9} + 27b = 7$

(2) $18 - 81b = a$

$$-\frac{(18 - 81b)}{9} + 27b = 7$$

$$\frac{81b - 18}{9} + 27b = 7$$

$$9b - 2 + 27b = 7$$

$$\frac{36b}{36} = \frac{9}{36}$$
$$b = \frac{1}{4} \rightarrow \underline{\text{Ans}}$$

$18 - 81b = a$

$18 - 81\left(\frac{1}{4}\right) = a$

$18 - \frac{81}{4} = a$

$18 - 20.25 = a$

$-2.25 = a \rightarrow$ Ans

8. $y = \frac{8}{x} - 6\sqrt{x}$

$= 8x^{-1} - 6x^{0.5}$

$\frac{dy}{dx} = -8x^{-2} - 3x^{-0.5}$

$= \frac{-8}{x^2} - \frac{3}{\sqrt{x}}$ $\quad = \frac{-8}{4^2} - \frac{3}{\sqrt{4}}$

$x = 4$ $\quad = \frac{-8}{16} - \frac{3}{2}$

$= \frac{-8}{16} - \frac{24}{16}$

$= \frac{-32}{16}$

$m = -2 \rightarrow$ Ans

10. Illegible

11. $y = \frac{x^2 - 1}{x}$

$= \frac{x^2}{x} - \frac{1}{x}$

$= x - \frac{1}{x}$

$\frac{dy}{dx} = 1 + \frac{1}{x^2}$

$1 + \frac{1}{x^2} = 5$

$x^2 + 1 = 5x^2$

$1 = 4x^2$

$\sqrt{\frac{1}{4}} = \sqrt{x^2}$

$\pm\frac{1}{2} = x \rightarrow$ Ans

13. $y = \frac{a}{x} + bx$

$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{-a}{x^2} + b \quad = \frac{-a}{(-2)^2} + b$

$= b - \frac{a}{4} = -8$ (1)

$y = \frac{a}{-2} + b(-2)$

$4 = -\frac{1}{2}a - 2b$ (2)

$4 = -\frac{1}{2}a - 2\left(\frac{a}{4} - 8\right)$

$4 = -\frac{1}{2}a - \frac{1}{2}a + 16$

$4 = 16 - a$

$a = 16 - 4$

$a = 12 \rightarrow$ Ans

$b = \frac{a}{4} - 8$

$b = \frac{12}{4} - 8$

$b = 3 - 8$

$b = -5 \rightarrow$ Ans

16. $y = x^3 - 8x^2 + 15x$

$\left.\frac{dy}{dx}\right|_{x=4} = 3x^2 - 16x + 15 \quad = 3(4)^2 - 16(4) + 15$

$= 3(16) - 64 + 15$

$= 48 - 64 + 15$

$m = -1 \rightarrow$ Ans

gradient of tangent at given point

$y = mx + c$

$y = -1x + c$

$-4 = -1(4) + c$

$-4 = -4 + c$

$0 = c$

eq $\rightarrow$ $y = -x$

$-x = x^3 - 8x^2 + 15x$

$0 = x^3 - 8x^2 + 16x$

$0 = x(x^2 - 8x + 16)$

$x = 0$ or $x^2 - 8x + 16 = 0$

$x^2 - 4x - 4x + 16 = 0$

$x(x-4) - 4(x-4) = 0$

$(x-4)(x-4) = 0$

$y = -1(0) + c$

$y = 0$

$(0,0) \rightarrow$ Ans $\quad x = 4$
intersection of tangent and graph

Questions to attempt:

From WS NG Q18 – 28 Slide 67 onwards

18. $y = 3x - 8$ is tangent $y = ax^2 + bx$ at $x = 2$

$\left.\frac{dy}{dx}\right|_{x=2} = 2ax + b \rightarrow 2ax + b = 3x - 8$

$2a = 3 \quad b = -8$

$a = \frac{3}{2}$

$a = 1.5 \quad b = -8$

$\therefore a = 1.5, \; b = -8 \rightarrow$ Ans

19. $y = x^2 - 4x + 5$, Find normal to curve at $(3, 2)$
↳ perpendicular to tangent

$\left.\frac{dy}{dx}\right|_{x=3} = 2x - 4 = 2(3) - 4$

$= 6 - 4$

$= 2 \rightarrow$ grad. at $(3, 2)$

$m_\perp \times 2 = -1$

$m_\perp = -\frac{1}{2} \rightarrow$ grad of normal

$y = -\frac{1}{2}x + C$

$2 = -\frac{1}{2}(3) + C$

$2 + \frac{3}{2} = C$

$\frac{7}{2} = C$

$y = -\frac{1}{2}x + \frac{7}{2}$

$y = \frac{7 - x}{2}$

$2y = 7 - x$

$2y + x = 7 \rightarrow$ Ans Eq. of normal to curve at $(3, 2)$

20. $y = 2x^3 - 5x^2 \rightarrow$ normal (at $(2, -4)$) crosses x-axis at A, Find A

$$\left.\frac{dy}{dx}\right|_{x=2} = 6x^2 - 10x = 6(2)^2 - 10(2)$$
$$= 6(4) - 20$$
$$= 24 - 20$$
$$= 4 \rightarrow \text{grad of tangent at } x = 2$$

$$m_\perp \times 4 = -1$$
$$m_\perp = \frac{-1}{4} \rightarrow \text{grad of normal at } x = 2$$

$$y = -\frac{1}{4}x + c$$
$$-4 = -\frac{1}{2}(2) + c$$
$$-4 = -1 + c$$
$$-3 = c$$

$$y = -\frac{1}{4}x - 3$$
$$4y = -x - 12$$
$$4y + x = -12 \rightarrow \text{eq. of normal at } x = 2$$

$$4y + x = -12$$
$$4(0) + x = -12$$
$$x = -12$$
$$y = 0$$

$\therefore$, $A = (-12, 0)$

↳ Ans coordinates of A

21. Illegible $\longrightarrow$ $y = ax^2 - \frac{b}{x}$ [Attempt later]

22. $y = 2x - \frac{8}{x^4}$

$$= 2x - 8x^{-4}$$

$$\frac{dy}{dx} = 2 + 32x^{-5}$$
$$= 2 + \frac{32}{x^5}$$

→ for all positive values of x, the fraction will get smaller as x increases but it will never be 0 and will always be positive

Since the gradient function always gives a value > 2, then it proves that in the original function, y will increase as x increases for all positive values of x.

23. $A \rightarrow x = 2$, $B \rightarrow x = 4$ on $y = 3x - \frac{8}{x}$

Find ~~point~~ x-coordinate of... point of intersection of tangents to curve at A and B

$y = 3x - \frac{8}{x}$

$= 3x - 8x^{-1}$

$\frac{dy}{dx} = 3 + 8x^{-2}$

$= 3 + \frac{8}{x^2}$

$x = 2$ and $x = 4$

$x = 2$

$m = 3 + \frac{8}{2^2}$

$= 3 + \frac{8}{4}$

$= 3 + 2$

$= 5 \rightarrow$ grad of tangent at A

$x = 4$

$m = 3 + \frac{8}{4^2}$

$= 3 + \frac{8}{16}$

$= 3 + 0.5$

$= 3.5$

$\hookrightarrow$ grad of tangent at B

$y = mx + c$

$y = 5x + c$

$2 = 5(2) + c$

$-8 = c$

$y = 5x - 8$

$\hookrightarrow$ tan at A

$y = mx + c$

$y = 3.5x + c$

$10 = 3.5(4) + c$

$10 = 14 + c$

$-4 = c$

$y = 3.5x - 4$

$\hookrightarrow$ tan at B

$y = 3x - \frac{8}{x}$

$y = 3(2) - \frac{8}{2}$

$= 6 - 4$

$= 2$

$A = (2, 2)$

$y = 3x - \frac{8}{x}$

$= 3(4) - \frac{8}{(4)}$

$= 12 - 2$

$= 10$

$B = (4, 10)$

$5x - 8 = 3.5x - 4$

$5x - 3.5x = 8 - 4$

$1.5x = 4$

$\frac{3}{2}x = 4$

$x = \frac{8}{3} \rightarrow$ Ans

x-coordinate of point of intersection

24. $y = 2x + \frac{6}{x} \rightarrow$ find eq. of normal at $(2, 7)$

$= 2x + 6x^{-1}$

$\frac{dy}{dx} = 2 - 6x^{-2}$

$= 2 - \frac{6}{x^2}$

$x = 2$

$= 2 - \frac{6}{2^2}$

$= 2 - \frac{6}{4}$

$= 2 - 1.5$

$= 0.5 \rightarrow$ grad of tangent

$m_\perp \times 0.5 = -1$

$m_\perp \times \frac{1}{2} = -1$

$m_\perp = -2 \rightarrow$ grad of normal

$y = mx + C$

$y = -2x + C$

$7 = -2(2) + C$

$7 = -4 + C$

$11 = C$

$y = 11 - 2x$

$y + 2x = 11 \rightarrow$ shown, eq. of normal at $(2,7)$

$$11 - 2x = 2x + \frac{6}{x}$$

$$11 = 4x + \frac{6}{x}$$

$$11x = 4x^2 + 6$$

$$0 = 4x^2 - 11x + 6$$

$$= 4x^2 - 8x - 3x + 6$$

$$= 4x(x-2) - 3(x-2)$$

$$= (4x-3)(x-2)$$

$x = \frac{3}{4}$ or $x = 2$ ↳ Already given

Ans
x-coordinate of second point of intersection of curve and tangent at $x = 2$

25. $y = 2x^2 - 1$, find eq. of normal at $\left(\frac{1}{2}, -\frac{1}{2}\right)$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 4x = 4\left(\frac{1}{2}\right)$$

$= 2 \rightarrow$ grad of tangent at $x = \frac{1}{2}$

$m_\perp \times 2 = -1$

$m_\perp = -\frac{1}{2} \rightarrow$ grad of normal at $x = \frac{1}{2}$

$y = mx + C$

$y = -\frac{1}{2}x + C$

$-\frac{1}{2} = -\frac{1}{2}\left(\frac{1}{2}\right) + C$

$y = -\frac{1}{2}x - \frac{1}{4}$

$4y = -2x - 1$

$4y + 2x = -1 \rightarrow$ Ans eq. of normal

$-\frac{1}{2} = -\frac{1}{4} + C$ at $x = \frac{1}{2}$

$\frac{-2}{4} + \frac{1}{4} = C$

$-\frac{1}{4} = C$

26. $3x - 4y + 7 = 0 \rightarrow$ line
$8y = 3x^2 + 5 \rightarrow$ curve
$\rightarrow \cap$ at A and B

$3x - 4y + 7 = 0$

$\frac{3x+7}{4} = y$

$\frac{6x+14}{8} = y$

$y = \frac{3x^2+5}{8}$

$\frac{6x+14}{8} = \frac{3x^2+5}{8}$

$6x + 14 = 3x^2 + 5$

$0 = 3x^2 - 6x - 14 + 5$

$0 = 3x^2 - 6x - 9$

$0 = 3x^2 - 9x + 3x - 9$

$0 = 3x(x-3) + 3(x-3)$

$0 = (3x+3)(x-3)$

$x = -1$ (A) $\quad x = 3$ (B)

$\left.\frac{dy}{dx}\right| = 6x$

$x = -1$ and $x = 3$

$x = -1$
$m = 6(-1)$
$= -6 \rightarrow m$ at A

$x = 3$
$m = 6(3)$
$= 18 \rightarrow m$ at B

grad at... A = 6, B = 18 $\rightarrow$ Ans

27. $y = 3x^2 + 5x - 12$, $m = 17$ at P, find P

$\left.\frac{dy}{dx}\right| = 6x + 5$

$17 = 6x + 5$

$12 = 6x$

$2 = x$

$\hookrightarrow y = 3(2^2) + 5(2) - 12$

$= 3(4) + 5(2) - 12$

$= 12 + 10 - 12$

$= 22 - 12$

$y = 10$

$P = (2, 10) \rightarrow$ Ans P

$0 = 3x^2 + 5x - 12$

$= 3x^2 + 9x - 4x - 12$

$= 3x(x+3) - 4(x+3)$

$= (3x-4)(x+3)$

(Q) $x = \frac{4}{3}$ or $x = -3$ (R)

$m = 6x + 5$

$= 6\left(\frac{4}{3}\right) + 5$

$= \frac{24}{3} + 5$

$m = 8 + 5$

$m = 13$

$m = 6x + 5$

$= 6(-3) + 5$

$= -18 + 5$

$m = -13$

grad of tangent at ... Q $= 13$, R $= -13$ → Ans

28. $P = (3, 4)$

$y = 3x^2 - 12x + 13$

Normal to curve at P

$\left.\frac{dy}{dx}\right|_{x=3} = 6x - 12 = 6(3) - 12$

$= 18 - 12$

$= 6 \rightarrow$ grad of tang. at $x = 3$

$m_\perp \times 6 = -1$

$m_\perp = -\frac{1}{6} \rightarrow$ grad of normal at $x = 3$

$y = -\frac{1}{6}x + c$

$4 = -\frac{1}{6}(3) + c$

$4 = -\frac{1}{2} + c$

$4\frac{1}{2} = c$

$\frac{9}{2} = c$

$y = -\frac{1}{6}x + \frac{9}{2}$

$y = \frac{27 - x}{6}$

$y = \frac{27 - (-3)}{6}$

$= \frac{30}{6}$

$y = 5$

$x + 3 = 0$

$x = -3$

Ans coordinates of normal at P and $x+3=0 \longrightarrow (-3, 5)$