Continuous-Time Costly Reversibility in Mean Field: A KS-Free Master-Equation Formulation, Derivations, and Computation

Self-contained derivation and implementation notes

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Abstract

This paper derives and explains a continuous-time, mean-field (master-equation) formulation of Zhang's costly-reversibility model. The approach is *Krusell–Smith (KS)-free*: aggregation enters through a single, explicit price-externality term generated by inverse demand, while strategic interaction across firms is encoded via the Lions derivative in the master equation. We fix primitives and state minimal boundary and regularity conditions; we then present two computational routes: (i) a stationary HJB–FP fixed point, and (ii) direct collocation of the stationary master PDE. Both routes are implementable with standard, monotone PDE schemes or modern function approximation (e.g., kernel/DeepSets representations for measures).

A central message is that the mean-field structure clarifies aggregation: the only economy-wide wedge in the firm problem is the product of the firm's own output and the slope of inverse demand evaluated at aggregate output. Under isoelastic demand, this wedge reduces to a scalar multiple of the firm's output. This provides a clean decomposition between *private marginal value of capital* (through the Hamiltonian) and *general-equilibrium feedback* (through the price externality). We work *conditional on the aggregate state x*, which removes common-noise second-order measure terms in the stationary master equation; Appendix C briefly outlines how those terms arise in the full common-noise setting.

We provide compact verification diagnostics (Euler and distributional residuals), explicit boundary conditions at $k = 0$ (reflecting), and growth/integrability conditions that guarantee all terms are finite. A small pseudo-JAX template illustrates how to evaluate the master-equation residual with an empirical measure. Throughout, we connect the construction to the canonical MFG literature for existence, uniqueness, and equivalence of the HJB–FP and master formulations.

Contents

Executive Summary / Cheat-Sheet (One Page)

Pedagogical Insight: Economic Intuition & Context

Primitives. Firms hold capital $k \geq 0$ and idiosyncratic productivity z. The aggregate state x shifts demand and marginal revenue. Technology is $q = e^{x+z}k^{\alpha}$ with $\alpha \in (0, 1)$. Inverse demand is $P(Y)$ with slope $P'(Y) < 0$, where $Y = \int e^{x+z}k^{\alpha}\, m(, \mathrm{d}k, , \mathrm{d}z)$. Capital follows $dk = (i - \delta k), \mathrm{d}t$ with asymmetric, convex costs $h(i, k)$. Dividends are $\pi = P(Y)\, e^{x+z}k^{\alpha} - i - h(i, k) - f$. Shocks evolve in z and x with generators L_z, L_x. Discounting uses $r(x)$ (or constant ρ).

Core equations. Value $V(k, z, x; m)$, master value $U(k, z, x, m)$.

- **Stationary HJB**: $r(x)V = \max_i\{\pi + V_k(i - \delta k) + L\_zV + L\_xV\}$.
- **Kolmogorov–Forward (FP)**: $\partial_t m = -\partial_k[(i^* - \delta k)m] + L\_z^* m$. Stationary: $\partial_t m = 0$.
- **Stationary Master Equation**: own-firm HJB terms + population-transport integrals of δ_mU *+ explicit price externality*

$$\int \delta_m \pi \,, \mathrm{d}m = e^{x+z} k^{\alpha}\, Y(m,x)\, P'(Y(m,x)).$$

Isoelastic simplification. For $P(Y) = Y^{-\eta}$, we have

$$Y\, P'(Y) = -\eta\, P(Y),$$

and therefore

$$\int \delta_m \pi \,, \mathrm{d}m = -\eta\, P(Y)\, e^{x+z} k^{\alpha}.$$

Two solution routes.

A. HJB–FP fixed point (robust):

0.1. Fix x (grid/invariant law). Guess m.
0.2. Compute $Y, P(Y)$. Solve HJB $\Rightarrow i^*$.
0.3. Solve stationary FP for m'. Update $m \leftarrow m'$.

B. Direct master-PDE collocation (KS-free):

0.1. Parameterize U and δ_mU (DeepSets/kernel for measures).
0.2. Build (ME) residual on empirical m, *including* $e^{x+z} k^{\alpha} Y P'(Y)$.
0.3. Penalize KKT/boundaries; recover i^* from the Hamiltonian; validate by Route A.

Diagnostics. Euler residuals for HJB, mass-balance for FP, and full ME residual. Use monotone stencils in k (upwinding) and conservative fluxes at $k = 0$.

1 Notation and Acronyms

Acronyms used in text: HJB, FP, ME, MFG, SDF, KKT, KS, RCE, TFP, CES, W2, FVM, SL.

2 Primitives and Assumptions

Assumption 2.1: Model specification; used verbatim

(i) **Firm states:** $k \in \mathbb{R}_+$, $z \in \mathbb{R}$. **Aggregate state:** $x \in \mathbb{R}$. **Population law:** $m \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R})$.

(ii) **Technology:** $q(k,z,x) = e^{x+z} k^{\alpha}$, $\alpha \in (0,1)$.

Symbol	Type	Meaning
k	state	Capital (≥ 0); reflecting boundary at $k = 0$
i	control	Net investment; $dk = (i - \delta k), \mathrm{d}t$
z	state	Idiosyncratic productivity; diffusion with generator L_z
x	state	Aggregate (business-cycle) shock; generator L_x
m	measure	Cross-sectional law on $\mathbb{R}_+ \times \mathbb{R}$ for (k, z)
$\xi = (\kappa, \zeta)$	point	Generic element in support of m ("marginal firm")
$q(k, z, x)$	output	$e^{x+z}k^{\alpha}$, $\alpha \in (0, 1)$
$Y(m, x)$	scalar	Aggregate quantity $\int e^{x+z}k^{\alpha}\, m(, \mathrm{d}k, , \mathrm{d}z)$
$P(\cdot)$	function	Inverse demand; $P' = P'(Y) < 0$
η	parameter	Demand elasticity for isoelastic $P(Y) = Y^{-\eta}$
α	parameter	Capital elasticity in production
δ	parameter	Depreciation rate
$\phi_{\pm}$	parameters	Adjustment-cost curvatures for $i \gtrless 0$
$h(i, k)$	function	Irreversible adjustment cost (convex, asymmetric)
f	parameter	Fixed operating cost
σ_z, σ_x	parameters	Diffusion volatilities of z and x
μ_z, μ_x	functions	Drift coefficients in L_z, L_x
$r(x)$	function	Short rate (or constant ρ) under pricing measure
$\pi(\cdot)$	function	Dividends $P(Y)e^{x+z}k^{\alpha} - i - h(i, k) - f$
$V(k, z, x; m)$	function	Stationary value function (HJB)
$U(k, z, x, m)$	function	Master value function (ME)
$\delta\_mU(\xi; k, z, x, m)$	function	Lions derivative wṙ.t. m in direction $\xi = (\kappa, \zeta)$
D_m	operator	Lions derivative operator (measure Fréchet derivative)
L_z, L_x	operators	Generators in z and x; L_z^* is the adjoint of L_z
$i^*(\cdot)$	policy	Optimal net investment from HJB/KKT
$\bar{\iota}(k)$	function	Lower bound on disinvestment (optional)
e_k, e_z	vectors	Canonical unit vectors in k and z directions
W, B	processes	Brownian motions for z and x (independent)
$b(\xi, x, m)$	vector	Drift at ξ: $(i^*(\xi, x, m) - \delta\kappa)e_k + \mu_z(\zeta)e_z$

Table 1: Notation used throughout.

(iii) **Product market:** $P = P(Y)$ with $Y(m, x) = \int e^{x+z}k^{\alpha}\, m(, \mathrm{d}k, , \mathrm{d}z)$, $P'(\cdot) < 0$.

(iv) **Capital law:** $dk_t = (i_t - \delta k_t), \mathrm{d}t$, $i \in \mathbb{R}$.

(v) **Irreversibility/adjustment:** h convex and asymmetric,

$$h(i, k) = \begin{cases} \frac{\phi_+}{2}\dfrac{i^2}{k}, & i \geq 0, \\ \frac{\phi_-}{2}\dfrac{i^2}{k}, & i < 0,\ \phi_- > \phi_+. \end{cases}$$

(vi) **Dividends:** $\pi(k, i, z, x, m) = P(Y(m, x))\, e^{x+z}k^{\alpha} - i - h(i, k) - f$.

(vii) **Shocks:** $dz_t = \mu_z(z_t), \mathrm{d}t + \sigma_z, \mathrm{d}W_t$, $dx_t = \mu_x(x_t), \mathrm{d}t + \sigma_x, \mathrm{d}B_t$ (independent).

(viii) **Discounting:** short rate $r(x)$ (or constant ρ).

(ix) **Generators:** for smooth u,

$$L\_zu = \mu_z(z)\,u_z + \tfrac{1}{2}\sigma_z^2 u_{zz}, \qquad L\_xu = \mu_x(x)\,u_x + \tfrac{1}{2}\sigma_x^2 u_{xx}.$$

Assumption 2.2: Minimal regularity/boundary

(a) $h(\cdot, k)$ convex, lower semicontinuous; $k \mapsto h(i, k)$ measurable with $h(i, k) \geq 0$ and $h(i, k) \geq c\, i^2/k$ for some $c > 0$ on $k > 0$. The asymmetry $\phi_- > \phi_+$ holds.

(b) P Lipschitz on compact sets with $P' < 0$; $P(Y)$ and $Y(m, x)$ finite for admissible m.

(c) μ_z, μ_x locally Lipschitz; $\sigma_z, \sigma_x \geq 0$ constants.

(d) *Boundary at $k = 0$:* reflecting; feasible controls satisfy $i^*(0, \cdot) \geq 0$; and $U_k(0, \cdot) \leq 1$.

(e) *Growth:* $U(k, z, x, m) = O(k)$ as $k \to \infty$.

(f) *Integrability:* m integrates k^α and $1/k$ wherever they appear.

Pedagogical Insight: Economic Intuition & Context

Economic reading. The convex asymmetry $\phi_- > \phi_+$ produces *investment bands*: small changes in the shadow value V_k around the frictionless cutoff 1 generate very different investment responses on the two sides of the kink. Aggregation operates through Y only, and the inverse-demand slope $P'(Y)$ is the sole channel through which the cross-section affects an individual firm's HJB. The reflecting boundary at $k = 0$ formalizes limited liability and the irreversibility of capital.

Connections to the Literature

Where this sits. Zhang (2005) emphasizes how costly reversibility shapes asset prices. The present mean-field formulation adds an equilibrium price mapping and a master PDE that makes the cross-sectional feedback explicit and computational. For master equations and Lions derivatives, see Lasry & Lions (2007), Cardaliaguet–Delarue–Lasry–Lions (2019), and Carmona & Delarue (2018).

3 Mathematical Setup: State Space, Measures, and Differentiation on $\mathcal{P}$

3.1 State space and probability metrics

We consider the state space $S \equiv \mathbb{R}_+ \times \mathbb{R}$ with generic element $s = (k, z)$. The population law m is a Borel probability measure on S. For well-posedness of the measure terms in the master equation (ME), we tacitly restrict to the W_2-finite set

$$\mathcal{P}_2(S) \equiv \left\{ m \in \mathcal{P}(S) : \int (\kappa^2 + \zeta^2)\, m(, \mathrm{d}\kappa, , \mathrm{d}\zeta) < \infty \right\}.$$

The quadratic Wasserstein distance W_2 metrizes weak convergence plus convergence of second moments. It is natural for diffusions and for the functional Itô calculus on $\mathcal{P}_2$.

Definition 3.1: Lions derivative

Let $F : \mathcal{P}_2(S) \to \mathbb{R}$. The *Lions derivative* $D\_mF(m) : S \to \mathbb{R}^{d_s}$ (here $d_s = 2$) is defined by lifting: pick a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a square-integrable random variable $X : \Omega \to S$ with law m. If there exists a Fréchet-derivative $D\tilde{F}(X)$ of the lifted map $\tilde{F} : L^2(\Omega; S) \to \mathbb{R}$, then $D\_mF(m)(\xi)$ is any measurable version that satisfies

$$D\tilde{F}(X) \cdot H = \mathbb{E}\left[\langle D\_mF(m)(X), H\rangle\right] \quad \text{for all } H \in L^2(\Omega; S).$$

When we write $\delta\_mU(\xi; k, z, x, m)$, we identify the derivative of $m \mapsto U(k, z, x, m)$ at point $\xi \in S$.

Lemma 3.1: Chain rule for composite functionals

Let $F(m) = G(\Phi(m))$ with $G : \mathbb{R} \to \mathbb{R}$ differentiable and $\Phi(m) = \int \varphi(\xi)\, m(, \mathrm{d}\xi)$ for some integrable $\varphi : S \to \mathbb{R}$. Then $D\_mF(m)(\xi) = G'(\Phi(m))\, \varphi(\xi)$.

Proof. The lift of Φ is $\tilde{\Phi}(X) = \mathbb{E}[\varphi(X)]$. The Gâteaux derivative is $\delta\tilde{\Phi}(X) \cdot H = \mathbb{E}[\varphi'(X) \cdot H]$ when φ is differentiable or, for integral functionals, φ itself plays the role of a density; composing with G gives the stated direction derivative. □

Mathematical Insight: Rigor & Implications

Application to the price externality. With $\varphi(\xi) = e^{x+\zeta}\kappa^\alpha$ and $G = P$, Lemma **??** yields $D\_m\big(P(\Phi(m))\big)(\xi) = P'(Y)\, e^{x+\zeta}\kappa^\alpha$. Multiplying by the *this-firm* factor $e^{x+z}k^\alpha$ produces the integrand of the last line in the ME.

3.2 Generators, domains, and adjoints

The generator L_z acts on $C_b^2(\mathbb{R})$ functions of z. The adjoint L_z^* acts on densities $m(k, z)$ (when they exist) as

$$L\_z^* m = -\partial_z(\mu_z m) + \tfrac{1}{2}\sigma_z^2 \partial_{zz} m.$$

The transport in k is first-order; the adjoint contributes $-\partial_k\big[(i^* - \delta k)m\big]$. No diffusion in k implies a degenerate (hyperbolic) structure in that dimension; numerical schemes must upwind in k.

4 Firm Problem and the Stationary HJB

Let $V(k, z, x; m)$ denote the value of a firm at (k, z) given aggregate (x, m). The stationary HJB is

$$\boxed{r(x)\, V = \max\_i \in \mathbb{R}\Big\{\pi(k, i, z, x, m) + V\_k\,(i - \delta k) + L\_zV + L\_xV\Big\}} \tag{HJB}$$

The interior first-order condition reads

$$0 = \partial_i \pi + V_k = -(1 + h_i(i, k)) + V_k \quad \Longrightarrow \quad i^*(k, z, x, m) = h_i^{-1}\big(V_k - 1\big),$$

with complementarity if $i \geq -\bar{\imath}(k)$ is imposed.[1]

[1] A practical and economically natural choice is to encode a no-scrap constraint $i \geq -\delta k$, which ensures non-negativity of capital along admissible paths.

Proposition 4.1: Explicit policy under asymmetric quadratic cost

For $h(i,k) = \frac{\phi_+}{2}\frac{i^2}{k}\,\mathbf{1}\_i \geq 0 + \frac{\phi_-}{2}\frac{i^2}{k}\,\mathbf{1}\_i < 0$ with $\phi_- > \phi_+$, the optimal policy is

$$i^*(k,z,x,m) = \begin{cases} \dfrac{k}{\phi_+}\big(V_k - 1\big), & V_k \geq 1, \\ \dfrac{k}{\phi_-}\big(V_k - 1\big), & V_k < 1, \end{cases}$$

plus complementarity if a bound $i \geq -\bar{\iota}(k)$ applies.

Proof. On each half-line, $h_i(i,k) = \phi_\pm\, i/k$. The FOC $1 + h_i(i,k) = V_k$ gives $i = (k/\phi_\pm)(V_k - 1)$. Strict convexity in i ensures a unique maximizer; the kink at $i = 0$ maps to $V_k = 1$. Lower bounds are handled by KKT complementarity. □

Proposition 4.2: Convex Hamiltonian and well-posed policy map

Define the Hamiltonian

$$\mathcal{H}(k,z,x,m,p) \equiv \max_{i\in\mathbb{R}}\{\pi(k,i,z,x,m) + p\,(i - \delta k)\}.$$

Then $\mathcal{H}$ is convex in $p = V_k$. The optimizer $i^*(k,z,x,m;p)$ is single-valued, piecewise linear with slope $k/\phi_\pm$, and globally Lipschitz on compact k-sets. Hence the feedback map $p \mapsto i^*(\cdot;p)$ is well-posed and stable to perturbations of p.

Pedagogical Insight: Economic Intuition & Context

	Intuition	The firm compares marginal V_k to the frict less unit price of investment. If $V_k > 1$, inv with slope controlled by ϕ_+; if $V_k < 1$, di vest, with slope dampened by ϕ_- (costli The kink at $V_k = 1$ generates inaction bar
Intuition vs. math.	**Mathematics**	The Hamiltonian is a convex conjugate c (after shifting by $p-1$). KKT conditions p duce a piecewise-affine policy with a chang slope at $p = 1$. Global well-posedness foll from coercivity of h in i and measurability k.

5 Kolmogorov–Forward (FP) Equation

Given x and the policy i^*, the population law m_t on (k,z) satisfies

$$\partial\_tm = -\frac{\partial}{\partial k}\left(\left(i^*(k,z,x,m) - \delta k\right) m\right) + L\_z^* m \tag{FP}$$

where L_z^* is the adjoint of L_z. In stationary equilibrium conditional on x: $\partial_t m = 0$.

5.1 Boundary and integrability

Reflecting at $k = 0$ implies zero probability flux through the boundary: $\big[(i^* - \delta k)m\big]\big|_{k=0} = 0$, and feasibility requires $i^*(0, \cdot) \geq 0$. Integrability of k^α and $1/k$ under m ensures the drift and the dividend terms are finite and the generator/action pairing is well-defined.

Mathematical Insight: Rigor & Implications

Degenerate transport in k. The k-direction is purely hyperbolic. Schemes must be *upwind* in k and *conservative* to maintain $\int m = 1$. A monotone FVM with Godunov fluxes provides stability and positivity. The lack of diffusion in k also means that corners in policy (from irreversibility) do not smooth out via second-order terms; numerical filters should not smear the kink.

6 Market Clearing and Price Mapping

Aggregate quantity and price are

$$Y(m, x) = \int e^{x+z} k^\alpha \, m(, \mathrm{d}k, , \mathrm{d}z), \qquad P = P(Y(m, x)), \quad P' < 0.$$

In the isoelastic case $P(Y) = Y^{-\eta}$ with $\eta > 0$,

$$Y \, P'(Y) = -\eta \, P(Y). \tag{6.1}$$

Pedagogical Insight: Economic Intuition & Context

Economic content. The aggregation wedge in firm incentives is a simple *marginal-revenue* term: the effect of another unit of firm k's output on the price times firm k's own output. Under isoelastic demand this becomes a proportional tax on revenue with rate η, varying over the business cycle through $P(Y)$.

7 Master Equation (Stationary, Conditional on x)

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