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Two Lucas Trees with Log Utility: Structured Continuous-Time Notes

Technical Appendix

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Abstract

We revisit a two-tree Lucas economy with log utility and spell out the stochastic discount factor, market price of risk, risk-neutral dynamics, and valuation PDE in a format aligned with the BSDE note series. The presentation pairs economic intuition with compact symbolic checks (SymPy) and a Lean projection lemma to mirror the rigor of BSDE_12 while keeping the model minimal.

1 Primitives and Notation

Assumption 1 (Two-Tree Lucas Environment)**.** *1. Two dividend processes $(D_{1,t}, D_{2,t})$ satisfy geometric diffusion dynamics*

$$\frac{\mathrm{d}D_{i,t}}{D_{i,t}} = \mu_i \,\mathrm{d}t + \boldsymbol{\sigma}_i^\top \mathrm{d}\boldsymbol{W}_t, \quad i \in \{1,2\}, \tag{1}$$

where $\boldsymbol{W}$ is a d-dimensional Brownian motion with identity covariance.

2. *Aggregate consumption equals total dividends: $C_t = D_{1,t} + D_{2,t}$. Log utility and discount rate $\rho > 0$ yield lifetime utility $\mathbb{E}\big[\int_0^\infty e^{-\rho t} \log C_t \,\mathrm{d}t\big]$.*

3. *Volatility vectors $\boldsymbol{\sigma}_i \in \mathbb{R}^d$ and drifts μ_i are constants with bounded magnitude ensuring positive dividend paths.*

Definition 1 (Consumption Shares and Aggregates)**.** Let $s_i \equiv D_i/C \in (0,1)$ with $s_1 + s_2 = 1$. Under **??**, consumption growth obeys

$$\frac{\mathrm{d}C_t}{C_t} = \mu_C \,\mathrm{d}t + \boldsymbol{\sigma}_C^\top \mathrm{d}\boldsymbol{W}_t, \quad \mu_C \equiv s_1\mu_1 + s_2\mu_2, \quad \boldsymbol{\sigma}_C \equiv s_1\boldsymbol{\sigma}_1 + s_2\boldsymbol{\sigma}_2. \tag{2}$$

Notation. Inner products use $\langle u, v\rangle$ and $\|u\|^2 = \langle u, u\rangle$. All stochastic integrals are in the Itô sense, and expectations condition on information at time t.

2 Stochastic Discount Factor and CAPM

Proposition 1 (Log-Utility SDF Dynamics)**.** *Under **??**, the stochastic discount factor (SDF)*

$$\Lambda_t = e^{-\rho t} C_t^{-1} \tag{3}$$

solves

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = -r_t\,\mathrm{d}t - \boldsymbol{\lambda}_t^\top \mathrm{d}\boldsymbol{W}_t, \quad r_t = \rho + \mu_C - \|\boldsymbol{\sigma}_C\|^2, \quad \boldsymbol{\lambda}_t = \boldsymbol{\sigma}_C. \tag{4}$$

Moreover, any traded return with diffusion $\boldsymbol{\sigma}_R$ satisfies the instantaneous CAPM relation

$$\mathbb{E}_t[R] - r_t = \langle \boldsymbol{\lambda}_t, \boldsymbol{\sigma}_R \rangle. \tag{5}$$

Proof. Apply Itô's lemma to Λ_t with $\mathrm{d}C_t/C_t$ from (**??**). The diffusion term equals $-\langle \boldsymbol{\sigma}_C, \mathrm{d}\boldsymbol{W}_t\rangle$, so the instantaneous covariance with any asset return R of diffusion $\boldsymbol{\sigma}_R$ produces (**??**). □

Economic reading. Log utility fixes the market price of risk at consumption volatility. Precautionary savings lowers the short rate by $\|\boldsymbol{\sigma}_C\|^2$, reinforcing how aggregate risk tightens discounting.

References (minimal)

Cochrane (2005), Duffie (2001), Lucas (1978).