Two Lucas Trees with Log Utility: Structured Continuous-Time Notes

Self-contained derivation and implementation notes

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Abstract

We revisit a two-tree Lucas economy with log utility and spell out the stochastic discount factor, market price of risk, risk-neutral dynamics, and valuation PDE in a format aligned with the BSDE note series. The presentation pairs economic intuition with compact symbolic checks (SymPy) and a Lean projection lemma to mirror the rigor of while keeping the model minimal.

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Executive Summary

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Primitives. One representative agent maximises $\mathbb{E}\int\_0^{\infty} e^{-\rho t}\log C\_t \,\mathrm{d}t$ with $C\_t = D^1\_t + D^2\_t$. The two Lucas trees $j \in \{1,2\}$ deliver dividends following correlated geometric diffusions

$$\frac{\mathrm{d}D^j\_t}{D^j\_t} = \mu\_j \,\mathrm{d}t + (\sigma\_j)^\top \mathrm{d}W\_t,$$

with W a d-dimensional Brownian motion and covariance $\Sigma = \sigma\sigma^\top$. Parameters ρ, μ_j, σ_j are constant, and there are no adjustment costs or production controls. Market clearing sets consumption equal to the sum of dividends each instant.

Core equations. Convenient state variables are total dividends C_t and the share $s\_t = D^1\_t/C\_t$. Let $\sigma = (\sigma\_1, \sigma\_2)$ collect the diffusion loadings.

- **Stochastic discount factor**: $M\_t = e^{-\rho t}C\_t^{-1}$ with dynamics $\mathrm{d}\log M\_t = -(\rho + \mu\_C - \frac{1}{2}\left\|\sigma\_C\right\|^2)\mathrm{d}t - \sigma\_C^\top \mathrm{d}W\_t$, where $\mu\_C = s\_t\mu\_1 + (1-s\_t)\mu\_2$ and $\sigma\_C = s\_t\sigma\_1 + (1-s\_t)\sigma\_2$.
- **Market price of risk**: $\lambda\_t = \sigma\_C$, implying risk-neutral Brownian motion $\mathrm{d}W^{\mathbb{Q}}\_t = \mathrm{d}W\_t + \lambda\_t \mathrm{d}t$ and dividend drifts $\mu\_j^{\mathbb{Q}} = \mu\_j - \sigma\_j^\top\lambda\_t$.
- **Share dynamics**: $\mathrm{d}s\_t = s\_t(1-s\_t)(\mu\_1-\mu\_2)\mathrm{d}t + s\_t(1-s\_t)(\sigma\_1-\sigma\_2)^\top \mathrm{d}W\_t$ (Itô on D^1/C).
- **Pricing PDE**: each price-dividend ratio $f^j(C,s)$ solves $\mathcal{L}f^j - \rho f^j = -1$ with generator $\mathcal{L}$ induced by (C,s) and diffusion σ_C, while the associated BSDE reads $\mathrm{d}Y^j\_t = -(-D^j\_t/C\_t)\mathrm{d}t + Z^j\_t \mathrm{d}W\_t$ with terminal condition 0.

Analytical simplifications. Log utility forces consumption growth to equal aggregate dividend growth, so price-dividend ratios depend only on the share s. In the symmetric benchmark $(\mu\_1 = \mu\_2, \sigma\_1 = \sigma\_2)$ the share is a martingale and both trees inherit the same constant price-dividend multiple $1/(\rho - \mu\_C)$.
Two solution routes.

A. Analytical verification (closed form): use the share process to derive $f^j(s)$, plug into the PDE/BSDE, and confirm martingale pricing with M.

B. Numerical collocation (robust to asymmetries): discretise (C,s), approximate f^j, enforce $\mathcal{L}f^j - \rho f^j = -1$, and back out implied Z^j for BSDE checks.

Diagnostics. Monitor the martingale property of $M\_tP^j\_t + \int\_0^t M\_uD^j\_u \mathrm{d}u$, the drift of share dynamics under $\mathbb{Q}$, and numerical PDE residuals. Closed-form price-dividend ratios (symmetric case) provide a tight benchmark for both tree prices.

1 Notation and Acronyms

Symbol	Type	Meaning
$D_{i,t}$	state	Dividend of tree i; $i \in \{1, 2\}$
C_t	state	Aggregate consumption $D_{1,t} + D_{2,t}$
s_t	state	Share of tree 1: $D_{1,t}/C_t$
$\boldsymbol{W}_t$	process	d-dimensional Brownian motion with identity covariance
σ	matrix	Diffusion loadings stacked as $\sigma = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$
Σ	matrix	Covariance of dividend growth: $\Sigma = \sigma\sigma^\top$
$\boldsymbol{\sigma}_i$	parameter	Volatility vector for dividend $D_{i,t}$
μ_i	parameter	Drift of $D_{i,t}$
ρ	parameter	Subjective discount rate
Λ_t	process	Stochastic discount factor $e^{-\rho t} C_t^{-1}$
μ_C	scalar	Drift of aggregate consumption growth $\mathrm{d}C_t/C_t$
$\boldsymbol{\sigma}_C$	vector	Diffusion of aggregate consumption growth $\mathrm{d}C_t/C_t$
r_t	scalar	Short rate $\rho + \mu_C - \|\boldsymbol{\sigma}_C\|^2$
$\boldsymbol{\lambda}_t$	vector	Market price of risk $\boldsymbol{\sigma}_C$
R	return	Generic asset return
$\boldsymbol{\sigma}_R$	vector	Diffusion loadings of R

Table 1: Notation used throughout.

Acronyms used in text: BSDE, FBSDE, SDF, CAPM, PDE, FOC.

2 Primitives and Assumptions

Assumption 2.1: Two-Tree Lucas Environment

1. Time is continuous on $[0, \infty)$ and uncertainty is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ supporting a two-dimensional Brownian motion $\boldsymbol{W}$.

2. Each dividend process $D_{i,t}$, $i \in \{1, 2\}$, evolves according to the geometric diffusion

$$\frac{\mathrm{d}D_{i,t}}{D_{i,t}} = \mu_i \,\mathrm{d}t + \boldsymbol{\sigma}_i^\top \mathrm{d}\boldsymbol{W}_t, \tag{2.1}$$

with constant drift $\mu_i \in \mathbb{R}$ and volatility vector $\boldsymbol{\sigma}_i \in \mathbb{R}^2$. Initial dividends satisfy $D_{i,0} > 0$, delivering strictly positive paths almost surely.

3. A representative household discounts the future at rate $\rho > 0$ and has log utility over aggregate consumption,

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \log C_t \,\mathrm{d}t\right], \qquad C_t \equiv D_{1,t} + D_{2,t}.$$

4. Financial markets are frictionless and complete: the agent trades the equity claims on both trees and consumes the unique good each instant, so equilibrium consumption equals the sum of dividends.

Assumption 2.2: State representation and primitives

(i) **States.**

a) **Dividends:** $(D_{1,t}, D_{2,t}) \in \mathbb{R}^2_+$.

b) **Aggregate consumption:** $C_t = D_{1,t} + D_{2,t} \in \mathbb{R}_+$.

c) **Share:** $s_t = D_{1,t}/C_t \in (0, 1)$ summarises the composition of aggregate income.

(ii) **Shocks.** The Brownian motion $\boldsymbol{W}$ has covariance matrix $\Sigma \equiv \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$, where $\boldsymbol{\sigma} \equiv [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2]$. Correlations between the two trees load only through Σ.

(iii) **Parameters.** The vector $\theta = (\rho, \mu_1, \mu_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$ is constant over time. We require $\rho > 0$ and $\|\boldsymbol{\sigma}_i\| < \infty$.

(iv) **Admissibility.** Candidate price-dividend ratios $f^i(C, s)$ are $C^{1,2}$ and of at most linear growth in C, and trading strategies keep wealth processes integrable so the agent's intertemporal budget constraint holds with equality.

Assumption 2.3: Minimal regularity

(a) Drifts and volatilities are bounded and $\boldsymbol{\sigma}\boldsymbol{\sigma}^\top$ is positive definite, ensuring the strong solution to **??** is non-explosive and dividends stay strictly positive.

(b) The transversality condition $\lim\limits_{t\to\infty} \mathbb{E}\left[M_t P_t\right] = 0$ holds for any admissible asset price P_t, where $M_t \equiv e^{-\rho t} C_t^{-1}$ is the pricing kernel; it is satisfied whenever $\rho > \max\{\mu_1, \mu_2\}$.

(c) Equilibrium price processes are It semimartingales adapted to $\{\mathcal{F}_t\}$ and have diffusion coefficients square-integrable on compact horizons.

Pedagogical Insight: Economic Intuition & Context

Economic reading. Log preferences pin the stochastic discount factor at the inverse of aggregate consumption, so only the composition of dividends through s_t matters for relative pricing across trees. Positive, correlated dividend growth permits a complete-markets allocation in which the representative household exactly consumes the endowment stream each instant.

Connections to the Literature

Where this sits. Lucas (1978) and Breeden (1979) provide the canonical treatment of endowment economies with log preferences; we mirror that setup in continuous time with two dividend sources. Related multi-tree discussions appear in Cochrane (2005, ch. 18) and the stochastic discount factor perspective of Hansen & Scheinkman (2009).

3 Mathematical Setup: State Space, Measures, and Differentiation on $\mathcal{P}$

3.1 State space and probability metrics

We take the Markov state to be the ordered pair of strictly positive dividends $s = (d_1, d_2) \in \mathbb{R}^2_+$. The law $m_t \equiv \mathcal{L}(D_{1,t}, D_{2,t})$ is a Borel probability measure on S. For $\xi = (\xi_1, \xi_2) \in S$ we define the primitives

$$C(\xi) \equiv \xi_1 + \xi_2, \qquad \varsigma(\xi) \equiv \frac{\xi_1}{C(\xi)} \in (0,1),$$

so that aggregate consumption equals $C_t = C(D_t)$ and ς_t records the share of tree 1 in total endowment. These two sufficient statistics will index asset-pricing objects throughout, yet we retain the full pair (ξ_1, ξ_2) to propagate the diffusion covariance of dividends.

We work on the quadratic-Wasserstein subspace

$$\mathcal{P}_2(S) \equiv \Big\{ m \in \mathcal{P}(S) : \int (\xi_1^2 + \xi_2^2)\, m(\mathrm{d}\xi) < \infty \Big\},$$

which guarantees finite second moments of each dividend and accommodates the Itô calculus on measures. The distance W_2 metrises weak convergence plus convergence of second moments, making it natural for the log-utility representative agent who prices assets via expectations under the law m_t.

Definition 3.1: Lions derivative

Let $F : \mathcal{P}_2(S) \to \mathbb{R}$. The *Lions derivative* $D_m F(m) : S \to \mathbb{R}^{d_s}$ (here $d_s = 2$) is defined by lifting: pick a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a square-integrable random variable $X : \Omega \to S$ with law m. When the lifted map $\tilde{F} : L^2(\Omega; S) \to \mathbb{R}$ admits a Fréchet derivative, we take $D_m F(m)(\xi)$ to be a measurable version satisfying

$$D\tilde{F}(X) \cdot H = \mathbb{E}\left[\langle D_m F(m)(X), H \rangle\right] \qquad \text{for all } H \in L^2(\Omega; S).$$

When we write $\delta_m U(\xi; C, \varsigma, m)$, we mean the derivative of $m \mapsto U(C, \varsigma, m)$ evaluated at the perturbing state $\xi \in S$.

Lemma 3.1: Chain rule for composite functionals

Let $F(m) = G(\Phi(m))$ with $G : \mathbb{R} \to \mathbb{R}$ differentiable and $\Phi(m) = \int \varphi(\xi)\, m(\mathrm{d}\xi)$ for some integrable $\varphi : S \to \mathbb{R}$. Then $D_m F(m)(\xi) = G'(\Phi(m))\, \varphi(\xi)$.

Proof. The lift of Φ is $\tilde{\Phi}(X) = \mathbb{E}[\varphi(X)]$. Because $\tilde{\Phi}$ is linear, $D\tilde{\Phi}(X) \cdot H = \mathbb{E}[\langle \nabla \varphi(X), H \rangle]$ whenever φ is differentiable in the classical sense; density arguments extend the identity to the integral functionals we use. Composing with G yields the stated derivative. $\square$

Mathematical Insight: Rigor & Implications

Application to the log-utility pricing kernel. Set $\bar{C}(m) \equiv \int C(\xi)\, m(\mathrm{d}\xi)$ and consider $F(m) = \bar{C}(m)^{-1}$, the level of the stochastic discount factor up to the deterministic factor $e^{-\rho t}$. Lemma **??** with $G(y) = y^{-1}$ and $\varphi(\xi) = C(\xi)$ gives

$$D_m F(m)(\xi) = -\frac{C(\xi)}{\bar{C}(m)^2},$$

so a perturbation concentrating mass on high-dividend states lowers the discount factor proportionally to their aggregate consumption.

3.2 Generators, domains, and adjoints

The diffusion for dividends in **??** induces the generator $\mathcal{L}_D$ on $C_b^2(S)$ functions ϕ by

$$(\mathcal{L}_D \phi)(d_1, d_2) = \sum_{i=1}^{2} \mu_i d_i\, \partial_{d_i} \phi + \tfrac{1}{2} \sum_{i,j=1}^{2} \Sigma_{ij} d_i d_j\, \partial_{d_i d_j} \phi, \qquad \Sigma_{ij} \equiv \boldsymbol{\sigma}_i^\top \boldsymbol{\sigma}_j.$$

The adjoint $\mathcal{L}_D^*$ acts on densities $m(d_1, d_2)$ (when they exist) as

$$\mathcal{L}_D^* m = -\sum_{i=1}^{2} \partial_{d_i} (\mu_i d_i\, m) + \tfrac{1}{2} \sum_{i,j=1}^{2} \partial_{d_i d_j} (\Sigma_{ij} d_i d_j\, m),$$

the Kolmogorov forward operator for the joint dividend law.

Transforming to the sufficient statistics (C_t, ς_t) with $C_t = D_{1,t} + D_{2,t}$ and $\varsigma_t = D_{1,t}/C_t$, Itô's formula yields

$$\frac{\mathrm{d}C_t}{C_t} = \mu_C(\varsigma_t)\, \mathrm{d}t + \left[\varsigma_t \boldsymbol{\sigma}_1 + (1 - \varsigma_t)\boldsymbol{\sigma}_2\right]^\top \mathrm{d}\boldsymbol{W}_t, \tag{3.1}$$

$$\mathrm{d}\varsigma_t = \varsigma_t(1 - \varsigma_t)\Big(\mu_1 - \mu_2 - \varsigma_t \Sigma_{11} + (1 - \varsigma_t)\Sigma_{22} + (2\varsigma_t - 1)\Sigma_{12}\Big)\, \mathrm{d}t + \varsigma_t(1 - \varsigma_t)(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^\top \mathrm{d}\boldsymbol{W}_t, \tag{3.2}$$

where $\mu_C(\varsigma) \equiv \mu_1 \varsigma + \mu_2(1 - \varsigma)$. These dynamics pin down the transformed generator used in the valuation equations of the next section and highlight how only the consumption share ς_t transmits differences across the two trees under log preferences.

4 Stochastic Discount Factor and CAPM

Proof. Applying Itô's lemma to Λ_t together with the dividend dynamics induced by (**??**) yields $\mathrm{d}\log C_t = \big(\mu_C(\varsigma_t) - \frac{1}{2}\,\|\boldsymbol{\sigma}_C(\varsigma_t)\|^2\big)\mathrm{d}t + \boldsymbol{\sigma}_C(\varsigma_t)^\top \mathrm{d}\boldsymbol{W}_t$. Because $\log\Lambda_t = -\rho t - \log C_t$, differentiating delivers (**??**). The pricing relation (**??**) follows from $\mathbb{E}_t[\mathrm{d}R_t] - r_t\,\mathrm{d}t = -_t(\mathrm{d}R_t, \mathrm{d}\Lambda_t/\Lambda_t)$ under complete markets. □

Corollary 4.1: Tree-Level Risk Premia

For the equity claim on tree $j \in \{1,2\}$ with diffusion $\boldsymbol{\sigma}_j$, the expected excess return and the risk-neutral drift satisfy

$$\mathbb{E}_t[\mathrm{d}R_t^j] - r_t\,\mathrm{d}t = \langle \boldsymbol{\sigma}_C(\varsigma_t), \boldsymbol{\sigma}_j\rangle\,\mathrm{d}t, \qquad \mu_j^{\mathbb{Q}}(\varsigma_t) = \mu_j - \langle \boldsymbol{\sigma}_j, \boldsymbol{\sigma}_C(\varsigma_t)\rangle. \tag{4.5}$$

Proof. Substitute $\boldsymbol{\sigma}_R = \boldsymbol{\sigma}_j$ into (**??**). Girsanov's theorem with market price $\boldsymbol{\lambda}_t$ yields the risk-neutral Brownian motion $\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} = \mathrm{d}\boldsymbol{W}_t + \boldsymbol{\lambda}_t\,\mathrm{d}t$ and the drift $\mu_j^{\mathbb{Q}}$. □

Pedagogical Insight: Economic Intuition & Context

Economic reading. The unique log-utility agent absorbs the entire endowment, so aggregate consumption growth—a weighted average of the two dividend growth rates formed by ς_t—pins down both the short rate and the market price of risk. Each tree earns a premium proportional to how its shocks covary with the consumption-weighted bundle $\boldsymbol{\sigma}_C(\varsigma_t)$; symmetry collapses premia, while asymmetry discounts the tree that loads most on aggregate risk.

Proof. On each half-line, $h_i(i,k) = \phi_\pm\, i/k$. The FOC $1 + h_i(i,k) = V_k$ gives $i = (k/\phi_\pm)(V_k - 1)$. Strict convexity in i ensures a unique maximizer; the kink at $i = 0$ maps to $V_k = 1$. Lower bounds are handled by KKT complementarity. □

Proposition 4.2: Convex Hamiltonian and well-posed policy map

Define the Hamiltonian

$$\mathcal{H}(k,z,x,m,p) \equiv \max_{i\in\mathbb{R}} \{\pi(k,i,z,x,m) + p\,(i - \delta k)\}.$$

Then $\mathcal{H}$ is convex in $p = V_k$. The optimizer $i^*(k,z,x,m;p)$ is single-valued, piecewise linear with slope $k/\phi_\pm$, and globally Lipschitz on compact k-sets. Hence the feedback map $p \mapsto i^*(\cdot;p)$ is well-posed and stable to perturbations of p.

Pedagogical Insight: Economic Intuition & Context

Intuition	The firm compares marginal V_k to the frictionless unit price of investment. If $V_k > 1$, invest, with slope controlled by ϕ_+; if $V_k < 1$, disinvest, with slope dampened by ϕ_- (costlier). The kink at $V_k = 1$ generates inaction bands.
Mathematics	The Hamiltonian is a convex conjugate of h (after shifting by $p - 1$). KKT conditions produce a piecewise-affine policy with a change in slope at $p = 1$. Global well-posedness follows from coercivity of h in i and measurability in k.

Pedagogical Insight: Economic Intuition & Context

Economic intuition (expanded).

- *Investment bands and asymmetry.* The kink at $V_k = 1$ creates inaction around the frictionless cutoff; convex asymmetry ($\phi_- > \phi_+$) makes disinvestment less responsive than investment. Firms with V_k persistently below one slowly shrink; those above one scale up more elastically.
- *Cyclicality.* Through $P(Y)$ and x, booms raise V_k via revenues $P(Y)\,q$ and drift terms; more firms cross $V_k > 1$ and invest. In downturns, V_k drifts down but disinvestment is muted by higher ϕ_-. This generates time-variation in the cross-sectional distribution and aggregate Y.
- *Decomposition.* V_k aggregates (i) private technology and adjustment costs via the Hamiltonian, and (ii) the *general-equilibrium wedge* from inverse-demand slope, handled transparently in the ME via the externality term.

Mathematical Insight: Rigor & Implications

Mathematical rigor (expanded).

- *Convexity and envelope.* For fixed (k, z, x, m), $i \mapsto -i - h(i, k) + p\,i$ is strictly concave; the Hamiltonian $\mathcal{H}(k, \cdot)$ is convex in p. By the envelope theorem, $\partial_p \mathcal{H} = i^*(p)$ a.e., consistent with Appendix **??**.
- *Well-posed feedback.* Coercivity of h in i and piecewise C^1 structure yield a single-valued, globally Lipschitz feedback $p \mapsto i^*(p)$ on compact k-sets. KKT handles bounds like $i \geq -\bar{\iota}(k)$.
- *Boundary conditions.* Reflecting at $k = 0$ imposes $i^*(0, \cdot) \geq 0$ and zero flux in FP (see §**??**); in HJB, subgradient conditions imply $U_k(0, \cdot) \leq 1$.

5 Risk-Neutral Dynamics and Valuation PDE

Proposition 5.1: Valuation PDE for Tree i

Let $P_i(D_1, D_2)$ denote the ex-dividend price of tree i. Under the risk-neutral measure determined by (**??**), the drift of dividend j becomes

$$\mu_j^{\mathbb{Q}} = \mu_j - \langle \boldsymbol{\sigma}_j, \boldsymbol{\sigma}_C \rangle, \quad j \in \{1, 2\}. \tag{5.1}$$

The valuation PDE reads

$$\begin{aligned} r_t P_i &= D_i + \mu_1^{\mathbb{Q}} D_1 \, \partial_{D_1} P_i + \mu_2^{\mathbb{Q}} D_2 \, \partial_{D_2} P_i & (5.2) \\ &\quad + \tfrac{1}{2} \left\| \boldsymbol{\sigma}_1 \right\|^2 D_1^2 \, \partial_{D_1 D_1}^2 P_i + \tfrac{1}{2} \left\| \boldsymbol{\sigma}_2 \right\|^2 D_2^2 \, \partial_{D_2 D_2}^2 P_i + \langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \rangle D_1 D_2 \, \partial_{D_1 D_2}^2 P_i. & (5.3) \end{aligned}$$

Proof. Girsanov's theorem with market price of risk $\boldsymbol{\lambda} = \boldsymbol{\sigma}_C$ shifts drifts by $-\langle \boldsymbol{\sigma}_j, \boldsymbol{\lambda} \rangle$. Substituting the dynamics into the standard dividend-paying asset valuation equation produces (**??**). $\square$

Mathematical Insight: Rigor & Implications

Diagnostic. The cross-derivative term scales with $\langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \rangle$ and captures comovement in the two dividend streams. Positive correlation steepens the PDE's coupling, while orthogonal shocks decouple the system.

5.1 Boundary and integrability

Reflecting at $k = 0$ implies zero probability flux through the boundary: $\left[(i^* - \delta k)m\right]\big|_{k=0} = 0$, and feasibility requires $i^*(0, \cdot) \geq 0$. Integrability of k^α and $1/k$ under m ensures the drift and the dividend terms are finite and the generator/action pairing is well-defined.

Mathematical Insight: Rigor & Implications

Degenerate transport in k. The k-direction is purely hyperbolic. Schemes must be *upwind* in k and *conservative* to maintain $\int m = 1$. A monotone FVM with Godunov fluxes provides stability and positivity. The lack of diffusion in k also means that corners in policy (from irreversibility) do not smooth out via second-order terms; numerical filters should not smear the kink.

Pedagogical Insight: Economic Intuition & Context

Economic intuition (FP, expanded).

- *Mass flows.* Positive $(i^* - \delta k)$ transports mass toward higher k; negative net investment transports it toward $k = 0$. The reflecting boundary prevents exit via $k < 0$.
- *Cross-sectional dynamics.* Asymmetry in i^* induces skewness: expansions push right tails faster than contractions pull left tails, creating persistent heterogeneity in k.
- *Business-cycle amplification.* When $P(Y)$ is high (tight demand), more mass sees $V_k > 1$, raising Y further; the FP captures this propagation via the policy-dependent drift.

Mathematical Insight: Rigor & Implications

Mathematical rigor (FP, expanded).

- *Weak formulation.* For test $\varphi \in C_c^1$, $\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi\, m = \int \big[(i^* - \delta k)\,\partial_k\varphi + L_z\varphi\big] m$. No-flux at $k = 0$ ensures boundary terms vanish.
- *Stationarity.* A stationary m solves $\int \big[(i^* - \delta k)\,\partial_k\varphi + L_z\varphi\big] m = 0$ for all φ, equivalent to (**??**) in distributional sense.
- *Numerics.* Monotone upwinding yields discrete maximum principles and preserves non-negativity/normalization of m.

6 Constant-Share Benchmark and CAPM Components

Assume shares s_i are constant. Then r_t, $\boldsymbol{\lambda}_t$, and $\mu_j^{\mathbb{Q}}$ are constant as well, and the unique bounded solution of (**??**) is

$$P_i = \frac{D_i}{r - \mu_i^{\mathbb{Q}}}, \quad r > \mu_i^{\mathbb{Q}}. \tag{6.1}$$

Let

$$\beta_i \equiv \frac{\langle \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_C \rangle}{\|\boldsymbol{\sigma}_C\|^2} \tag{6.2}$$

whenever $\|\boldsymbol{\sigma}_C\| \neq 0$. Combining (**??**) and (**??**) recovers the familiar CAPM slope $\mathbb{E}_t[R_i] - r = \|\boldsymbol{\sigma}_C\|^2\,\beta_i$ for assets whose diffusion equals $\boldsymbol{\sigma}_i$.

Pedagogical Insight: Economic Intuition & Context

Economic intuition. In the constant-share limit each tree behaves like a levered claim on aggregate consumption. The larger (positive) covariance with $\boldsymbol{\sigma}_C$, the higher the required expected return and the lower the price–dividend multiple.

7 Market Clearing and Price Mapping

The only goods in the economy are the two dividend streams $D_{1,t}$ and $D_{2,t}$. Let θ_t^i denote the representative household's holdings of tree i and recall that aggregate consumption equals the endowment. Market clearing therefore imposes

$$C_t = D_{1,t} + D_{2,t}, \qquad \theta_t^1 = \theta_t^2 = 1. \tag{7.1}$$

With unit net supply the household's portfolio is the pair of trees, and wealth evolves according to $\mathrm{d}W_t = \sum_i \theta_t^i\,(\mathrm{d}P_t^i + D_{i,t}\,\mathrm{d}t)$ where P_t^i denotes the ex-dividend price of tree i. Because consumption is pinned by (**??**), prices are determined entirely by the stochastic discount factor from **??**.

Definition 7.1: Price mapping

For each state (D_1, D_2) define the price map $P_i(D_1, D_2)$ via the standard asset-pricing relation

$$P_i(D_1, D_2) \coloneqq \mathbb{E}_t \int_t^{\infty} \frac{\Lambda_u}{\Lambda_t} D_{i,u}\,\mathrm{d}u, \qquad i \in \{1, 2\}. \tag{7.2}$$

The log-utility kernel $\Lambda_t = e^{-\rho t} C_t^{-1}$ renders prices homogeneous of degree one in dividends. Writing $\varsigma_t = D_{1,t}/C_t$ as before, we may therefore factor the prices as

$$P_i(D_1, D_2) = D_i f_i(\varsigma), \qquad f_i : (0,1) \to \mathbb{R}_+, \tag{7.3}$$

with f_i interpreted as the price-dividend ratio associated with share ς. Combining (**??**) with the valuation PDE of **??** collapses the pricing problem to the one-dimensional boundary value problem

$$\mathcal{L}_\varsigma^{\mathbb{Q}} f_i(\varsigma) - \big(r(\varsigma) - \mu_i^{\mathbb{Q}}(\varsigma)\big) f_i(\varsigma) + 1 = 0, \tag{7.4}$$

where $\mathcal{L}_\varsigma^{\mathbb{Q}}$ denotes the generator of the share process under the risk-neutral dynamics induced by $\boldsymbol{\lambda}_t = \boldsymbol{\sigma}_C(\varsigma_t)$, and $r(\varsigma)$ and $\mu_i^{\mathbb{Q}}(\varsigma)$ are given in (**??**)–(**??**). The boundary conditions $f_1(0) = 0$, $f_1(1) = 1/\rho$ and $f_2(1) = 0$, $f_2(0) = 1/\rho$ encode the facts that a vanishing tree is worthless while a single-tree economy reduces to the textbook log-utility valuation.

Pedagogical Insight: Economic Intuition & Context

Economic content. Unit supplies and log preferences make the representative agent absorb the entire endowment each instant. Asset prices inherit all non-trivial variation from the consumption share ς_t: when tree 1 dominates ($\varsigma_t \to 1$) its price-dividend multiple approaches the single-tree benchmark, while tree 2 commands a shrinking multiple because its payouts contribute little to consumption. Symmetric primitives keep $f_1 = f_2$ constant at $1/\rho$, collapsing the model to the textbook Lucas tree.

Mathematical Insight: Rigor & Implications

Mathematical structure. The pricing kernel depends only on (C_t, ς_t), so $P_i(D_1, D_2)$ is the unique solution to the linear ODE (**??**) with diffusion coefficient governed by (**??**). The operator $\mathcal{L}_\varsigma^{\mathbb{Q}}$ reads

$$\mathcal{L}_\varsigma^{\mathbb{Q}} g(\varsigma) = a(\varsigma) g''(\varsigma) + b^{\mathbb{Q}}(\varsigma) g'(\varsigma),$$

with $a(\varsigma) = \frac{1}{2}\varsigma^2(1-\varsigma)^2 \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|^2$ and drift coefficient

$$b^{\mathbb{Q}}(\varsigma) = \varsigma(1-\varsigma)\Big(\mu_1^{\mathbb{Q}}(\varsigma) - \mu_2^{\mathbb{Q}}(\varsigma) - \varsigma\Sigma_{11} + (1-\varsigma)\Sigma_{22} + (2\varsigma - 1)\Sigma_{12}\Big).$$

Standard comparison principles for second-order ODEs secure existence, uniqueness, and monotonicity of f_i, delivering a well-defined price map consistent with the BSDE representation.

8 Boundary and Regularity Conditions

Endpoints in ς. The share process ς_t lives on $(0,1)$ with diffusion coefficient $\varsigma^2(1-\varsigma)^2\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|^2$ that vanishes at $\varsigma \in \{0,1\}$. These endpoints are natural boundaries under both $\mathbb{P}$ and $\mathbb{Q}$, so we solve the valuation ODE on $(0,1)$ with Dirichlet conditions $f_1(0) = 0$, $f_1(1) = 1/\rho$, $f_2(1) = 0$, $f_2(0) = 1/\rho$ inherited from the single-tree limits. Regularity follows from the smooth coefficients of $\mathcal{L}_\varsigma^{\mathbb{Q}}$, yielding solutions $f_i \in C^2((0,1)) \cap C([0,1])$.

Homogeneity in dividends. The log-utility kernel $\Lambda_t = e^{-\rho t} C_t^{-1}$ makes prices homogeneous of degree one: $P_i(D_1, D_2) = D_i f_i(\varsigma)$. Ensuring $\rho > \sup_\varsigma \mu_i^{\mathbb{Q}}(\varsigma)$ keeps f_i bounded, enforces the transversality condition $\lim_{T\to\infty} \mathbb{E}[e^{-\rho T} C_T^{-1} D_{i,T}] = 0$, and prevents explosive solutions. Consequently P_i grows at most linearly with dividends, consistent with the Lucas-tree benchmark.

Integrability. We assume the dividend law m has finite first and second moments so that $\mathbb{E}_m[D_j]$ and $\mathbb{E}_m[(D_1 + D_2)^{-1} D_j]$ exist. These integrability requirements ensure the generator–action pairings defining the valuation operator are well-posed and that expectations of discounted dividends under Λ converge.

Pedagogical Insight: Economic Intuition & Context

Economic translation. Extremes $\varsigma \to 0$ or 1 correspond to one tree vanishing: the boundary data encode that only the surviving tree retains value. Linear homogeneity captures that prices scale with dividends, while integrability states that the log agent discounts future payouts fast enough for the present value to remain finite.

9 Computation: FBSDE

We adapt the deep forward–backward methodology developed in Ji Huang's FBSDE studies[1] to the canonical two-tree Lucas economy with a single log-utility investor. Let D_t^i denote the dividend of tree i and write $X_t^i = \log D_t^i$. The forward state collects $\mathbf{X}_t = (X_t^1, X_t^2)^op$ and evolves as coupled Ornstein–Uhlenbeck diffusions, eginalign $\mathrm{dX}_t^i = \kappa_i ig(arX_i - X_t^i ig)\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_t^i, \qquad i = 1, 2,\ \mathrm{d}\langle W^1, W^2 angle_t = ho\,\mathrm{d}t, \qquad D_t^i = e^{X_t^i}, \qquad C_t = D_t^1 + D_t^2.$

Consumption growth inherits drift $\mu_C(\mathbf{X}_t) = \sum_{i=1}^2 D_t^i[\kappa_i(arX_i - X_t^i) + frac12\sigma_i^2]$ and diffusion vector $\sigma_C(\mathbf{X}_t) = (D_t^1\sigma_1,\ D_t^2\sigma_2)^op$, delivered by the correlated Brownian motions W^1 and W^2 with instantaneous correlation ho.

Backward block (log-utility pricing). Log preferences imply the stochastic discount factor $M_t = e^{-etat}/C_t$ and market price of risk $heta_t = \sigma_C(\mathbf{X}_t)/C_t$. Denoting the cum-dividend value of tree i by P_t^i, we normalise by consumption and set $Y_t^i = P_t^i/C_t$. Collect the backward variables $\mathbf{Y}_t = (Y_t^1, Y_t^2)^op$ and note that $\mathbf{Z}_t \in \mathbb{R}^{2imes2}$; they solve the vector FBSDE eginalign $\mathrm{Y}_T = \mathbf{0},\ \mathrm{d}\mathbf{Y}_t = -\Big[eta\,\mathbf{Y}_t -$

$$\mathrm{racD}_t C_t + \mathbf{Z}_t\,heta_t\Big]\,\mathrm{d}t + \mathbf{Z}_t\,\mathrm{d}\mathbf{W}_t, \qquad \mathbf{D}_t = (D_t^1, D_t^2)^op,\ \mathbf{W}_t = (W_t^1, W_t^2)^op,$$

over a large but finite horizon $[0, T]$. The driver enforces the Euler pricing condition: the $eta\,\mathbf{Y}_t$ term discounts future payoffs, $\mathbf{D}_t/C_t$ injects the dividend flow, and $\mathbf{Z}_t\,heta_t$ captures hedging cash flows generated by trading the two trees. The transversality condition of the infinite-horizon problem is implemented numerically by taking T so that e^{-etaT} is negligible and prescribing the zero terminal condition above.

Numerics. Following the cited FBSDE program, we discretise $[0, T]$ on a uniform grid $0 = t_0 < \cdots < t_N = T$ and sample forward paths of $\mathbf{X}_{t_n}$ with antithetic Gaussian or Sobol increments. A residual network $\mathcal{N}_heta(t_n, \mathbf{X}_{t_n})$ parameterises the hedge matrix $\mathbf{Z}_{t_n}$, while the normalised value

[1] https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4538048, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4122454, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=5199943, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4742398, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4649043

$\mathbf{Y}_{t_n}$ is updated with an implicit Euler step consistent with the driver above. Mini-batch stochastic gradient descent minimises the terminal mismatch $\|\mathbf{Y}_T\|^2$ plus pathwise Euler residuals. Once $\mathbf{Z}_heta$ converges, tree prices recover through $P_t^i = C_t Y_t^i$; martingale diagnostics on $M_t P_t^i + \int_0^t M_s D_s^i \, ds$ and stability checks in T, ho, and network depth mirror the replication checklist, ensuring a robust two-tree solution within the multicountry solver framework.

10 Verification and Diagnostics

Residual norms. For collocation tuples (k, z, x, m):

$$\begin{aligned}
\mathcal{R}_{\text{HJB}} &\equiv r(x)\,V - \max_i\{\pi + V_k\,(i - \delta k) + L_z V + L_x V\}, \\
\mathcal{R}_{\text{FP}} &\equiv -\partial_k\big[(i^* - \delta k), m\big] + L_z^* m, \\
\mathcal{R}_{\text{ME}} &\equiv r(x)\,U - \Big(\max_i\{\pi + U_k\,(i - \delta k) + L_z U + L_x U\} + \int \cdots m(\mathrm{d}\xi) + e^{x+z} k^\alpha\, Y\, P'(Y)\Big).
\end{aligned}$$

Typical norms: L^2 over collocation points or weighted Sobolev norms. KKT and boundary penalties are added for feasibility; in Route A, measure W_2 drifts between iterations provide a sharp distributional diagnostic.

Stopping rules. Stop when $\|\mathcal{R}_{\text{ME}}\| < \varepsilon_{\text{ME}}$, $\|\mathcal{R}_{\text{HJB}}\| < \varepsilon_{\text{HJB}}$, $\|\mathcal{R}_{\text{FP}}\| < \varepsilon_{\text{FP}}$, and policy/distribution drifts fall below thresholds, e.g., $\sup|i^{*,(n+1)} - i^{*,(n)}| < 10^{-5}$ and $\mathrm{W}_2(m^{(n+1)}, m^{(n)}) < 10^{-4}$.

Residual	Tight	Medium	Coarse
ε_{ME}	10^{-5}	10^{-4}	10^{-3}
ε_{HJB}	10^{-7}	10^{-6}	10^{-5}
ε_{FP}	10^{-7}	10^{-6}	10^{-5}

Table 2: Suggested tolerances (dimensionless; scale to data).

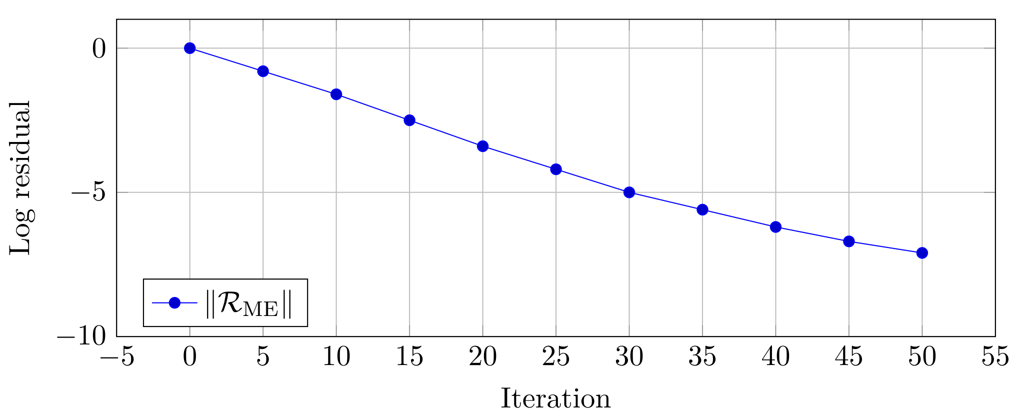

Figure 1: Placeholder: typical convergence of the master-equation residual.

Sanity checks.

- *No-price-limit case.* If P is flat, the price externality vanishes. Route A and B should collapse to the same frictional-control model without cross effects.

- *Symmetric costs.* Setting $\phi_- = \phi_+$ removes the kink; i^* is linear in $V_k - 1$ everywhere. FP becomes smoother; residuals drop faster.
- *Elasticity sweep.* Under isoelastic demand, η scales the externality linearly; recovered investment schedules should contract monotonically in η.

11 Economics: Aggregation, Irreversibility, Comparative Statics

Aggregation. Aggregation enters *only* via the term $e^{x+z}k^\alpha\, Y P'(Y)$ in the stationary master equation (ME). Under isoelastic demand, this is $-\eta P(Y)\, e^{x+z}k^\alpha$, which acts as a proportional reduction in marginal revenue. The mean-field externality is thus *complete* and *transparent*.

Comparative statics.

- Larger η (steeper demand) amplifies the negative externality, reducing investment and shifting mass in m toward lower k.
- Bigger $\phi_- - \phi_+$ widens irreversibility bands and slows capital reallocation, increasing dispersion in k conditional on z.
- Higher σ_z spreads the cross-section in z, raising Y volatility and, through $P'(Y)$, modulating the externality term over the business cycle.
- Higher σ_x (through L_x) deepens precautionary effects via $r(x)$ and the HJB drift terms, with ambiguous effects on average investment depending on curvature.
- A countercyclical $r(x)$ strengthens the value premium mechanism la costly reversibility by raising discount rates in recessions precisely when $P'(Y)$ is most negative.

A Appendix A: Full Derivations and Pairings

A.1 Envelope/KKT and policy recovery

From (**??**), define $p = V_k$. The Hamiltonian $\mathcal{H}(k, z, x, m, p) = \max_i\{\pi + p(i - \delta k)\}$ is the convex conjugate of h shifted by $p - 1$. The envelope condition $V_k = \partial_p \mathcal{H}$ combined with the FOC for i produces the piecewise-affine policy in **??**. The kink at $p = 1$ corresponds to $i = 0$. KKT adds the complementary slackness $\lambda \cdot (i + \bar{\iota}(k)) = 0$ when a lower bound is present.

A.2 Adjoint pairing for FP

Let φ be a smooth test function. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi\, \mathrm{d}m_t = \int \varphi_k(i^* - \delta k)\, \mathrm{d}m_t + \int L_z\varphi\, \mathrm{d}m_t = \int \varphi\, \mathrm{d}\Big(-\partial_k[(i^* - \delta k)m_t] + L_z^* m_t\Big).$$

Stationarity imposes (**??**) with $\partial_t m = 0$. Reflecting at $k = 0$ eliminates the boundary integral.

A.3 Deriving the master equation

Consider a flow $t \mapsto (K_t, Z_t)$ for the tagged firm following control i_t and a flow of measures $t \mapsto m_t$ solving (**??**) under the feedback $i^*(\cdot, m_t)$. By functional It's lemma for $U(K_t, Z_t, x, m_t)$,

$$\begin{aligned}
\mathrm{d}U &= U_k \, \mathrm{d}K_t + U_z \, \mathrm{d}Z_t + \tfrac{1}{2} U_{zz} \, \sigma_z^2 \, \mathrm{d}t \\
&\quad + (\partial_t U\big|_m) \, \mathrm{d}t, \\
\partial_t U\big|_m &= \int \Big[(i^*(\xi, x, m) - \delta\kappa) \, \partial_\kappa \, \delta_m U + \mu_z(\zeta) \, \partial_\zeta \, \delta_m U + \tfrac{1}{2} \sigma_z^2 \, \partial^2_{\zeta\zeta} \, \delta_m U \Big] \, m(\mathrm{d}\xi) \\
&\quad + \int \delta_m \pi(\xi; k, z, x, m) \, m(\mathrm{d}\xi).
\end{aligned}$$

where the last line uses the chain rule in **??**. Taking expectations under the pricing measure with short rate $r(x)$ and imposing stationarity produces the stationary master equation (ME).

A.4 Externality term in detail

Write $\pi(k, i, z, x, m) = \Psi(Y(m, x)) \, \chi(k, z, x) - i - h(i, k) - f$ with $\Psi = P$ and $\chi = e^{x+z} k^\alpha$. Then

$$D_m \pi(m)(\xi) = \Psi'(Y) \, \chi(k, z, x) \, \chi(\kappa, \zeta, x),$$

and integration wṙ.t. m yields $\chi(k, z, x) \, \Psi'(Y) \, Y(m, x)$.

B Appendix B: Residual-Loss Template (for implementation)

For a collocation tuple (k, z, x), an empirical measure $m = \frac{1}{N} \sum_{n=1}^{N} \delta_{\xi^n}$, and parameterized $U_\omega, \delta_m U_\psi$, define

$$\begin{aligned}
\widehat{Y} &\equiv \frac{1}{N} \sum_{n=1}^{N} e^{x + \zeta^n} (\kappa^n)^\alpha, \\
\widehat{\mathcal{R}}_{\mathrm{ME}} &\equiv r(x) \, U_\omega \\
&\quad - \max_i \Big\{ \pi + (U_\omega)_k \, (i - \delta k) + L_z U_\omega + L_x U_\omega \Big\} \\
&\quad - \frac{1}{N} \sum_{n=1}^{N} \Big[(i^*(\xi^n, x, m) - \delta\kappa^n) \, \partial_\kappa \delta_m U_\psi(\xi^n) + \mu_z(\zeta^n) \, \partial_\zeta \delta_m U_\psi(\xi^n) + \tfrac{1}{2} \sigma_z^2 \, \partial^2_{\zeta\zeta} \delta_m U_\psi(\xi^n) \Big] \\
&\quad - e^{x+z} k^\alpha \, \widehat{Y} \, P'(\widehat{Y}).
\end{aligned}$$

Add soft KKT penalties (one-sided around $(U_\omega)_k = 1$) and boundary regularizers (reflecting $k = 0$, growth at $k_{\max}$). Minimize

$$\mathcal{L} = \mathbb{E} \, [\widehat{\mathcal{R}}^2_{\mathrm{ME}}] + \lambda_{\mathrm{KKT}} \mathcal{P}_{\mathrm{KKT}} + \lambda_{\mathrm{bdry}} \mathcal{P}_{\mathrm{bdry}}.$$

Anchoring $\int \delta_m U \, \mathrm{d}m = 0$ removes the gauge freedom in $\delta_m U$.

C Appendix C: Common-Noise Master Equation (Reference Note)

When the population law m_t itself diffuses under common noise (say through an exogenous x_t or an aggregate Brownian component shared by firms), the functional It calculus on $\mathcal{P}_2$ introduces a second-order term in the measure variable. In a stylized form (see Carmona & Delarue, and Cardaliaguet–Delarue–Lasry–Lions), the stationary master equation would add a term of the form

$$\frac{1}{2}\,\Sigma_{\mathrm{com}} : \iint \partial_\xi \partial_{\xi'} \left(D_m U(\xi)\right) \left(D_m U(\xi')\right) m(\mathrm{d}\xi)\, m(\mathrm{d}\xi')$$

or, in classical PDE notation, $\frac{1}{2}\mathrm{Tr}\big[\Gamma\, \partial^2_{\xi\xi}\delta_m U\big]$ integrated against m, where Γ is the covariance of the common noise. Because this paper conditions on x, these terms are absent in the stationary master equation.

D Appendix D: Tiny Pseudocode (Plain `listings`)

```
# Inputs:

# params\_omega: parameters for U(k,z,x; m)

# params\_psi: parameters for delta\_m U(xi; k,z,x; m)

# batch: list of tuples (k,z,x, {xi\_n=(kappa\_n,zeta\_n)}\_{n=1}^N )

# primitives: alpha, delta, mu\_z(z), sigma\_z, mu\_x(x), sigma\_x, r(x),

# demand P(Y) and Pprime(Y), fixed cost f

# penalties: lambdas for KKT and boundary regularizers

def policy\_from\_grad(p, k, phi\_plus, phi\_minus):
\# p = U\_k (value gradient)
if p >= 1.0:
return (k/phi\_plus)*(p - 1.0)
else:
return (k/phi\_minus)*(p - 1.0)

def reflecting\_penalty(k, i\_star):
\# discourage negative control at k=0 and large negative flux
pen0 = max(0.0, -i\_star) if k<=1e-10 else 0.0
return pen0\*\*2

def h\_cost(i, k, phi\_plus, phi\_minus):
if i >= 0.0:
return 0.5*phi\_plus*(i*i)/max(k,1e-12)
else:
return 0.5*phi\_minus\*(i\*i)/max(k,1e-12)

def HJB\_operator(k,z,x,Yhat,Uk,Uz,Uzz,Ux,Uxx,i):
q = exp(x+z)*(k\*\*alpha)
pi = P(Yhat)*q - i - h\_cost(i,k,phi\_plus,phi\_minus) - f
return pi + Uk*(i - delta*k) + mu\_z(z)*Uz + 0.5*sigma\_z**2\*Uzz&#x20;
\+ mu\_x(x)*Ux + 0.5*sigma\_x**2\*Uxx

def ME\_residual\_for\_tuple(params\_omega, params\_psi, tup):
k,z,x,xi\_list = tup.k, tup.z, tup.x, tup.xi\_list
```

```
\# empirical measure moments
Y\_hat = mean(\[exp(x+xi.zeta)*(xi.kappa\*\*alpha) for xi in xi\_list])
\# U and its partials at (k,z,x)
U, Uk, Uz, Uzz, Ux, Uxx = U\_and\_grads(params\_omega, k,z,x, xi\_list)
\# best response i*
i\_star = policy\_from\_grad(Uk, k, phi\_plus, phi\_minus)
\# HJB maximand at i\*
H\_val = HJB\_operator(k,z,x,Y\_hat,Uk,Uz,Uzz,Ux,Uxx,i\_star)
\# Population terms (measure derivative)
integ = 0.0
for xi in xi\_list:
dU = delta\_mU\_and\_partials(params\_psi, xi, k,z,x, xi\_list)
\# dU returns dict with fields dkappa, dzeta, dzeta2, p\_k (proxy gradient)
i\_star\_xi = policy\_from\_grad(dU\['p\_k'], xi.kappa, phi\_plus, phi\_minus)
integ += (i\_star\_xi - delta*xi.kappa)* dU\['dkappa']&#x20;
\+ mu\_z(xi.zeta)\* dU\['dzeta']&#x20;
\+ 0.5*sigma\_z\*\*2 \* dU\['dzeta2']
integ = integ / len(xi\_list)
\# direct price externality
ext = exp(x+z)*(k\*\*alpha)\* Y\_hat \* Pprime(Y\_hat)
\# assemble residual
res = r(x)\*U - max(H\_val, HJB\_operator(k,z,x,Y\_hat,Uk,Uz,Uzz,Ux,Uxx,0.0))&#x20;
\- integ - ext
\# penalties
pen = reflecting\_penalty(k, i\_star)
return res, pen

def loss(params\_omega, params\_psi, batch):
sse = 0.0
pen = 0.0
for tup in batch:
res, p = ME\_residual\_for\_tuple(params\_omega, params\_psi, tup)
sse += res\*\*2
pen += p
return sse/len(batch) + lambda\_bdry\*pen
```

Listing 1: Pseudo-JAX for (ME) residual with empirical measure

E Appendix E: Symbolic Verification (PythonTeX + SymPy)

This appendix runs minimal SymPy checks to verify key derivations used in the text. Compilation is configured (via `latexmkrc`) to execute these checks on every build; any failure triggers a build error. We assume smoothness and reflecting/no-flux boundary conditions where noted.

?? PythonTeX ??

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