A Appendix A: Detailed Derivations

A.1 HJB Analysis: Hamiltonian, KKT, and Convexity

We analyze the maximization problem embedded in the HJB equation eq:HJB. Define the optimized Hamiltonian $\mathcal{H}(k, z, x, m, p)$ with $p = V_k$:

$$\mathcal{H}(k, z, x, m, p) = \max_{i\in} \Big\{ \pi(k, i, z, x, m) + p\,(i - \delta k) \Big\}. \qquad (1)$$

Substituting the profit function from ass:primitives yields

$$\mathcal{H} = P(Y)e^{x+z}k^{\alpha} - f - p\delta k + \max_{i\in} \Big\{ (p-1)i - h(i, k) \Big\},$$

so the maximization term is the Legendre–Fenchel transform of the convex adjustment cost h evaluated at $p - 1$.

Optimal Policy Derivation (KKT/FOC). Because h is convex in i, the objective is concave. The first-order condition (FOC) is

$$p - 1 - h_i(i, k) = 0 \qquad i^* = h_i^{-1}(p - 1).$$

For the asymmetric quadratic cost $h(i, k) = \phi_{\pm}2i^2/k$ from ass:primitives we obtain

[leftmargin=1.25em]

- $i \geq 0$: $h_i = \phi_+ i/k$. The FOC gives $p - 1 = \phi_+ i/k$ so $i = k(p-1)/\phi_+$, which is feasible only when $p \geq 1$.
- $i < 0$: $h_i = \phi_- i/k$. The FOC gives $p - 1 = \phi_- i/k$ so $i = k(p-1)/\phi_-$, which requires $p < 1$.

This recovers the piecewise-affine policy in prop:policy.

[sympycheckstyle] extbfVerification: FOCs for the optimal policy. See Appendix E, Check 2, for a symbolic confirmation of the formulas above.

[leanproofstyle] extbfFormalization: Optimization structure in Lean 4. lean import Mathlib.Analysis.Convex.Basic

– Parameters: k ¿ 0, phi$_{p}lus > 0, phi_{m}inus > 0. variables k phi_{p}lus phi_{m}inus p : Real (hk : k > 0) (hp_{p} : phi_{p}lus > 0) (hp_{m} : phi_{m}inus > 0)$

– Asymmetric quadratic adjustment cost. def adjustment$_{c}ost (i : Real) : Real := if i >= 0 then (phi_{p}lus/2) * (i^2/k) else (phi_{m}inus/2) * (i^2/k)$

– Objective maximized inside the Hamiltonian. def objective (i : Real) : Real := (p - 1) * i - adjustment$_{c}osti$

– Candidate optimal policy i*(p). def optimal$_{p}olicy : Real := if p >= 1 then k * (p-1)/phi_{p}lus else k * (p-1)/phi_{m}inus$

– TODO: Show that ‘optimal$_{p}olicy‘ maximizes ‘objective‘ using convexity + FOCs$. Status: structure encoded; maximality proof pending.

Convexity of the Hamiltonian. The map $p \mapsto \mathcal{H}(k, z, x, m, p)$ is convex: the supremum of affine functions of p is convex. The envelope theorem implies $\partial_p \mathcal{H} = i^*(p)$, and the policy is monotone in p because $\phi_{\pm} > 0$, so the second derivative is non-negative.

[sympycheckstyle] extbfVerification: Envelope property. Appendix E, Check 3, confirms $\partial_p \mathcal{H} = i^*(p)$.

Integration by parts. Assume boundary terms vanish (compact support / zero-flux). The transport component with $v = i^* - \delta k$ satisfies

$$\int v\,\partial_k \varphi\, m\, k = -\int \varphi\,\partial_k(vm)\,k,$$

while diffusion terms satisfy $\int (\varphi)m = \int \varphi(m)$. Consequently

$$\mathcal{L}^* m = -\partial_k((i^* - \delta k)m) + m,$$

and the FP equation follows from $\partial_t m = \mathcal{L}^* m$.

[sympycheckstyle] extbfVerification: Adjoint identity for the transport term. import sympy as sp k = sp.symbols('k', positive=True) phi = sp.Function('phi')(k) m = sp.Function('m')(k) v = sp.Function('v')(k)

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$$\mathcal{H}(k, z, x, m, p) = \max_{i \in} \Big\{ \pi(k, i, z, x, m) + p\,(i - \delta k) \Big\}. \tag{2}$$

Substituting the profit function from ass:primitives yields

$$\mathcal{H} = P(Y)e^{x+z}k^{\alpha} - f - p\,\delta k + \max_{i \in} \Big\{ (p-1)i - h(i,k) \Big\},$$

so the maximization term is the Legendre–Fenchel transform of the convex adjustment cost h evaluated at $p - 1$.

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[sympycheckstyle] extbfVerification: Envelope property. Appendix E, Check 3, confirms $\partial_p \mathcal{H} = i^*(p)$.

A.2 Adjoint Pairing and the FP Equation

We derive the Kolmogorov–Forward (FP) equation eq:FP via the weak formulation. For a smooth test function $\varphi(k, z)$ with compact support,

$$[\varphi(k_t, z_t)] = \int \varphi(k, z)\, m_t(k, z).$$

Let $\mathcal{L}$ denote the generator of (k_t, z_t) under P, conditional on x and the policy i^*:

$$\mathcal{L}\varphi = (i^*(k, z) - \delta k)\, \partial_k \varphi + \varphi.$$

The weak formulation yields $t \int \varphi\, m_t = \int (\mathcal{L}\varphi)\, m_t$. To obtain the strong form we identify the adjoint operator $\mathcal{L}^*$ defined by $\langle \mathcal{L}\varphi, m\rangle = \langle \varphi, \mathcal{L}^* m\rangle$.

Integration by parts. Assume boundary terms vanish (compact support / zero-flux). The transport component with $v = i^* - \delta k$ satisfies

$$\int v\, \partial_k \varphi\, m\, k = -\int \varphi\, \partial_k (vm)\, k,$$

while diffusion terms satisfy $\int (\varphi) m = \int \varphi (m)$. Consequently

$$\mathcal{L}^* m = -\partial_k((i^* - \delta k)m) + m,$$

and the FP equation follows from $\partial_t m = \mathcal{L}^* m$.

[sympycheckstyle] extbfVerification: Adjoint identity for the transport term. python import sympy as sp k = sp.symbols('k', positive=True) phi = sp.Function('phi')(k) m = sp.Function('m')(k) v = sp.Function('v')(k)

L$_{p}hi = v * sp.diff(phi, k) L_{s}tar_{m} = -sp.diff(v * m, k)$

LHS = sp.diff(phi * v * m, k) RHS = (L$_{p}hi * m) + (phi * L_{s}tar_{m}) assert sp.simplify(LHS - RHS) == 0 print("Adjoint pairing for transport verified.")$

A.3 Deriving the generalized master equation

We outline the stationary Generalized Master Equation (ME) eq:ME. When the HJB coefficients $r(m)$ and (m) depend on m, the standard derivation must be enhanced to accommodate interactions in the coefficients (Bensoussan et al., 2013).

Functional Itô calculus. Consider $U(k_t, z_t, x, m_t)$. The generalized functional Itô lemma tracks the evolution of U under changes in both the state (k_t, z_t) and the law m_t. Conditioning on x and using the pricing measure Q, stationarity requires $r(m)U$ to balance all drift contributions.

Decomposing the m-dependence. Two channels link the law m to valuations:

[label=(iii), leftmargin=1.5em]

1. **Transport.** Variations of m feed through the Lions derivative even if coefficients are fixed.
2. **Externalities.** Variations of m alter π, r, and , feeding back into the HJB residual itself.

Applying the generalized chain rule produces $\mathrm{E}_{transport} = \int \Big[(i^*(\xi, m) - \delta\kappa)\, \partial_\kappa(\xi; \cdot) + (\xi; \cdot) \Big]\, m(\xi)$,
$E_{price} = \int \delta_m \pi(\xi; k, z, x, m)\, m(\xi)$,
$E_{pricing} = \int \Big[-(\delta_m r(\xi; m))U + (\delta_m(\xi; m))U_x \Big]\, m(\xi)$.

[didacticstyle] extbfSign intuition. If a perturbation of m raises the short rate ($\delta_m r > 0$), valuations fall, hence the negative sign. Conversely, if the perturbation raises the risk-neutral drift ($\delta_m > 0$) the asset gains value when $U_x > 0$, giving a positive contribution.

Collecting the HJB drift, transport, and externality terms and imposing the pricing-measure martingale condition delivers the ME eq:ME.

Capital law: $dk_t = (i_t - \delta k_t)t$, $i \in$.

Irreversibility/adjustment: h convex and asymmetric,

$$h(i, k) = \{\ \phi_+ 2\, i^2 k, i \geq 0, \phi_- 2\, i^2 k, i < 0,\ \phi_- > \phi_+.$$

Dividends: $\pi(k, i, z, x, m) = P()\, e^{x+z} k^\alpha - i - h(i, k) - f$.

Shocks (Physical measure P): $dz_t = \mu_z(z_t)t + \sigma_z W_t$, $dx_t = \mu_x(x_t)t + \sigma_x B_t$ (independent).

Consumer Preferences and GE Closure: A representative agent maximizes lifetime utility ${}^P \left[\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right]$. Goods market clearing implies C_t equals aggregate dividends D_t.

Generators (under P): for smooth u,

$$u = \mu_z(z)\, u_z + 12\sigma_z^2 u_{zz}, \qquad u = \mu_x(x)\, u_x + 12\sigma_x^2 u_{xx}.$$

Minimal regularity/boundaryregularity

[label=(c),itemsep=0.2em]

1. $h(\cdot, k)$ convex, lower semicontinuous; $h(i, k) \geq c\, i^2/k$ for some $c > 0$ on $k > 0$.
2. P Lipschitz on compact sets with $P' < 0$.
3. μ_z, μ_x locally Lipschitz; $\sigma_z, \sigma_x \geq 0$ constants.
4. *Boundary at $k = 0$:* reflecting; $i^*(0, \cdot) \geq 0$; and $U_k(0, \cdot) \leq 1$.
5. *Growth:* $U(k, z, x, m) = O(k)$ as $k \to \infty$.
6. *Integrability:* m integrates k^α and $1/k$.
7. *Pricing Regularity:* Equilibrium aggregate consumption $C(x)$ is sufficiently regular (e.g., C^2) to admit well-defined dynamics via Itô's lemma (see sec:pricing).

[didacticstyle] **Economic reading.** The convex asymmetry $\phi_- > \phi_+$ produces *investment bands*. Aggregation operates through the product market (Y) and the financial market (via the SDF). The inverse-demand slope $P'(Y)$ and the endogenous market price of risk $\lambda(x, m)$ are the channels through which the cross-section affects an individual firm's valuation and decisions.

[literaturestyle] **Where this sits.** Zhang (2005) emphasizes how costly reversibility shapes asset prices, using an exogenous SDF. The present mean-field formulation adds an equilibrium product price mapping and a fully endogenous SDF, leading to a master PDE that makes the cross-sectional feedback explicit. For master equations, see Cardaliaguet–Delarue–Lasry–Lions (2019). The generalization for interactions in coefficients (required for the pricing externalities) follows Bensoussan, Frehse, and Yam (2013).

B Mathematical Setup: State Space, Measures, and Differentiation on $\mathcal{P}$P

B.1 State space and probability metrics

We consider the state space $S \equiv_+ \times$. The population law $m \in \mathcal{P}(S)$. We restrict to the W_2-finite set $\mathcal{P}_2(S)$, metrized by the quadratic Wasserstein distance W_2.

Lions derivativelions Let $F : \mathcal{P}_2(S) \to$. The *Lions derivative* $F(m) : S \to^{d_s}$ is defined by lifting F to a function $\tilde{F}$ on $L^2(\Omega; S)$ and taking the Fréchet derivative $D\tilde{F}$.

We use $\delta_m F(\xi; m)$ to denote the Lions derivative of a functional $F(m)$ evaluated at ξ. When we write $(\xi; k, z, x, m)$, we identify the derivative of $m \mapsto U(k, z, x, m)$ at point $\xi \in S$.

Chain rule for composite functionalschain Let $F(m) = G(\Phi(m))$ with $G :\to$ differentiable and $\Phi(m) = \int \varphi(\xi)\, m(\xi)$. Then $F(m)(\xi) = G'(\Phi(m))\, \varphi(\xi)$.

[mathstyle] **Application to the price externality.** With $\varphi(\xi) = e^{x+\zeta}\kappa^{\alpha}$ and $G = P$, Lemma **??** yields $(P(\Phi(m)))(\xi) = P'(Y)\,e^{x+\zeta}\kappa^{\alpha}$. Multiplying by the *this-firm* factor $e^{x+z}k^{\alpha}$ produces the integrand of the price externality term in the ME.

B.2 Generators, domains, and adjoints

The generator acts on $C_b^2()$ functions of z. The adjoint acts on densities $m(k,z)$ as $m = -\partial_z(\mu_z m) + 12\sigma_z^2\partial_{zz}m$. The transport in k is first-order; the adjoint contributes $-\partial_k[(i^* - \delta k)m]$.

C General Equilibrium and Asset Pricing

The economy is closed by the representative consumer (ass:primitives).

Aggregate Dividends and Consumption$agg_c dAggregatedividendsare: D(m,x) = \int \pi(k, i^*(k,z,x,m), z, x, m)\, m(k,z). In equilibrium, the goods market clears:$C(m,x)=D(m,x).

The stochastic discount factor (SDF), Λ_t, is determined by the consumer's marginal utility: $\Lambda_t = e^{-\rho t}C_t^{-\gamma}$.

Equilibrium Consumption Dynamicsass:c$_dynamics We assume that in a stationary equilibrium, the distributio$

$\mu_C(x_t)t + \sigma_C(x_t)B_t. where by Itô's lemma: \mu_C(x) = \frac{1}{C(x)}(C)(x) = \frac{C'(x)}{C(x)}\mu_x(x) + \frac{1}{2}\frac{C''(x)}{C(x)}\sigma_x(x)^2,$

$\sigma_C(x) = \frac{C'(x)}{C(x)}\sigma_x(x).$

Endogenous Short Rate and Market Price of Riskpricing The equilibrium short rate $r(x)$ and the market price of risk $\lambda(x)$ for the aggregate shock B_t are given by the Lucas pricing (CCAPM) formulas: r(x) $= \rho + \gamma\mu_C(x) - \frac{1}{2}\gamma(\gamma+1)\sigma_C(x)^2,$

$\lambda(x) = \gamma\sigma_C(x)$. The SDF dynamics are $\frac{\Lambda_t}{\Lambda_t} = -r(x_t)t - \lambda(x_t)B_t$. Follows from applying Itô's lemma to Λ_t and matching drift and diffusion terms. See Appendix E for verification.

[didacticstyle] **Endogeneity and Feedback.** Crucially, $r(x)$ and $\lambda(x)$ depend on the equilibrium consumption dynamics, which in turn depend on the cross-sectional distribution $m(x)$ and its sensitivity to x. This introduces a general equilibrium feedback channel. We denote this dependence as $r(x,m)$ and $\lambda(x,m)$ when analyzing the general mean-field system before imposing the stationary equilibrium structure $m(x)$.

D Firm Problem and the Stationary HJB

The firm maximizes the present value of dividends discounted by the endogenous SDF Λ. This valuation is performed under the risk-neutral measure Q, obtained via Girsanov's theorem using the market price of risk $\lambda(x,m)$.

The risk-neutral drift of x is:

$$(x,m) = \mu_x(x) - \sigma_x(x)\lambda(x,m). \tag{3}$$

The risk-neutral generator is:

$$V = V_x + \frac{1}{2}\sigma_x^2 V_{xx}. \tag{4}$$

We assume idiosyncratic risk z is diversifiable and its generator L_z remains unchanged under Q.

The stationary HJB equation is

$$r(x,m)\,V = \max_{i\in} \Big\{\pi(k,i,z,x,m) + V_k\,(i-\delta k) + V + V\Big\}\ HJB \tag{5}$$

The interior first-order condition reads

$$0 = \partial_i \pi + V_k = -(1 + h_i(i,k)) + V_k \quad \Longrightarrow \quad i^*(k,z,x,m) = h_i^{-1}(V_k - 1).$$

Explicit policy under asymmetric quadratic costpolicy For the adjustment cost in ass:primitives, the optimal policy is

$$i^*(k,z,x,m) = \{\ k\phi_+\,(V_k-1), V_k \geq 1, k\phi_-\,(V_k-1), V_k < 1.$$

[didacticstyle] **Economic intuition (expanded).**

[leftmargin=1.15em,itemsep=0.25em]

- *Investment bands.* The kink at $V_k = 1$ creates inaction; asymmetry $(\phi_- > \phi_+)$ makes disinvestment less responsive.
- *Cyclicality and Asset Pricing.* Valuation depends on both cash flows (via $P(Y)$) and the endogenous SDF. The countercyclical market price of risk $\lambda(x)$ (Zhang 2005) implies that in recessions (low x), risk premia are high (is low), depressing firm values, particularly for firms constrained by costly reversibility.
- *Decomposition.* V_k aggregates (i) private technology and costs via the Hamiltonian, and (ii) the *general-equilibrium wedges* from inverse demand and asset pricing, made explicit in the ME via externalities.

[mathstyle] **Mathematical rigor (expanded).**

[leftmargin=1.15em,itemsep=0.25em]

- *Convexity.* The Hamiltonian $\mathcal{H}(k,z,x,m,p) \equiv \max_i\{\pi + p\,(i-\delta k)\}$ is convex in $p = V_k$.
- *Risk-Neutral Valuation.* The HJB equation represents the valuation under Q. The dependence of r and on m reflects the GE feedback.
- *Boundary conditions.* Reflecting at $k = 0$ implies $i^*(0,\cdot) \geq 0$ and $U_k(0,\cdot) \leq 1$.

E Kolmogorov–Forward (FP) Equation

The evolution of the population law m_t occurs under the physical measure P. Given x and the policy i^*, m_t satisfies

$$\partial_t m = -\frac{\partial}{\partial k}\left((i^\star(k,z,x,m) - \delta k)\,m\right) + mFP \tag{6}$$

In stationary equilibrium conditional on x: $\partial_t m = 0$.

E.1 Boundary and integrability

Reflecting at $k = 0$ implies zero probability flux: $[(i^* - \delta k)m]|_{k=0} = 0$.

[mathstyle] **Degenerate transport in k.** The k-direction is purely hyperbolic. Schemes must be *upwind* in k and *conservative* to maintain $\int m = 1$.

F Market Clearing

Aggregate quantity and price are

$$Y(m,x) = \int e^{x+z} k^{\alpha}\, m(k,z), \qquad P = P(Y(m,x)), \quad P' < 0.$$

In the isoelastic case $P(Y) = Y^{-\eta}$ with $\eta > 0$,

$$Y\, P'(Y) = -\eta\, P(Y). \tag{7}$$

[didacticstyle] **Economic content.** The aggregation wedge in firm incentives comes from two sources. First, the *marginal-revenue* term (price externality). Second, the *SDF* determined by aggregate consumption (pricing externality).

G Master Equation (Stationary, Conditional on x)

Define the master value $U(k,z,x,m)$ and the Lions derivative $(\xi; k, z, x, m)$ at $\xi = (\kappa, \zeta)$. The stationary master equation characterizes the equilibrium value U. Because the coefficients of the HJB ($r(m)$ and (m)) depend on m, we must use the generalized formulation (Bensoussan et al., 2013).

$$r(x,m)\, U(k,z,x,m) = \underbrace{\max_i \{\pi(k,i,z,x,m) + U_k(i - \delta k) + U + U\}}_{Own-firm HJB terms (under}$$

Q)

$$+ \underbrace{\int \Big[(i^*(\xi,x,m) - \delta\kappa)\, \partial_\kappa(\xi;\cdot) + (\xi;\cdot)\Big]\, m(\xi)}_{Population transport (under} \text{P)} via$$

$$+ \underbrace{\int \delta_m \pi(\xi;\, k,z,x,m)\, m(\xi)}_{Direct price externality}$$

$$+ \underbrace{\int [-(\delta_m r(\xi;x,m))U + (\delta_m(\xi;x,m))U_x]\, m(\xi)}_{Pricing externality}. \text{ ME } (8)$$

where $(\xi;\cdot)$ abbreviates $(\xi; k, z, x, m)$, and the transport term integrates the action of the physical generators on .

[didacticstyle] extbfInterpreting the Master Equation. eq:ME decomposes the general-equilibrium effects:

[leftmargin=1.25em,label=(vii)]

1. **HJB terms:** Optimization by the representative firm under Q, taking prices (P, r, λ) as given.

2. **Population transport:** The evolution of m under P moves U through .

3. **Price externality:** Impact of m on dividends π via the product price $P(Y)$.

4. **Pricing externality:** Impact of m on the valuation operators (discount rate r and risk-neutral drift) via the SDF.

G.1 Derivation of the General Equilibrium Externalities

General Equilibrium Externalitiesexternality The GE externalities consist of the price externality (E_{price}) and the pricing externality ($E_{pricing}$).

Price Externality. Let $\chi(k, z, x) = e^{x+z}k^{\alpha}$. The integrated price externality is:

$$E_{price} = \chi(k, z, x)\, Y(m, x)\, P'(Y(m, x)).$$

If $P(Y) = Y^{-\eta}$, $E_{price} = -\eta\, P(Y)\, \chi(k, z, x)$.

Pricing Externality. The pricing externality term is:

$$E_{pricing} = \int [-(\delta_m r(\xi; x, m))U + (\delta_m(\xi; x, m))U_x]\, m(\xi).$$

Since $= \mu_x - \sigma_x\lambda$, we have $\delta_m = -\sigma_x\delta_m\lambda$.

The price externality derivation follows from lem:chain. The pricing externality arises from the generalized Master Equation formulation for MFGs with interactions in the coefficients (see Appendix A.3).

[mathstyle] **Complexity of Pricing Externalities.** The terms $\delta_m r$ and $\delta_m\lambda$ depend on the Lions derivatives of the consumption dynamics $(\mu_C(m), \sigma_C(m))$. Calculating these requires understanding how the equilibrium consumption process $C(m)$ responds to infinitesimal changes in the distribution m. This involves differentiating through the fixed-point definition of the equilibrium, which is highly complex.

[sympycheckstyle] **Verification: Proposition ?? (Price Externality Derivation)** import sympy as sp

Symbols Y, eta = sp.symbols('Y eta', positive=True) k, z, x = sp.symbols('k z x', real=True) alpha = sp.symbols('alpha', positive=True)

Production function and price chi = sp.exp(x + z) * k**alpha P = sp.Function('P')(Y)

Integrated externality structure integrated$_e xternality = chi * sp.diff(P, Y) * Y$

Isoelastic demand P$_i so = Y**(-eta) externality_i so = sp.simplify(integrated_e xternality.subs(P, P_i so)) tar$
$-eta * P_i so * chi$

assert sp.simplify(externality$_i so - target_i so) == 0 print("Proposition 8.1 price externality structure verified

G.2 Equivalence and Monotonicity

Equivalence of HJB–FP and ME formulationsequivalence Assume ass:primitives,ass:regularity hold. Under sufficient differentiability of U and , and appropriate generalized monotonicity conditions on the GE feedback (including $P'(Y) < 0$ and the pricing feedback), stationary solutions (V, m) of the coupled HJB–FP system coincide with stationary solutions of eq:ME conditional on x, with $V(k, z, x; m) = U(k, z, x, m)$.

[literaturestyle] **Equivalence and uniqueness.** Establishing uniqueness in this GE setting requires generalizing the Lasry–Lions monotonicity condition to account for the pricing externalities. This depends on the properties of the equilibrium consumption process and how it responds to changes in the distribution m.

[node distance=15mm, box/.style=rectangle, draw=darkgreen, rounded corners, thick, inner sep=6pt, align=center] [box] (hjb) Own-firm HJB (under Q)
$r(m), (m)$; [box, right=of hjb] (pop) Population transport (under P) via $\delta_m U$; [box, below right=of pop, xshift=-1cm] (price) Price externality
$YP'(Y)$; [box, below left=of pop, xshift=1cm] (pricing) Pricing externality $\delta_m r, \delta_m \lambda$; [box, below=of hjb] (me) ME residual;
[-¿, thick] (hjb) – (me); [-¿, thick] (pop) – (me); [-¿, thick] (price) – (me); [-¿, thick] (pricing) – (me);

Figure 1: Schematic composition of the stationary master equation: HJB contributions, population transport, price externality, and the newly introduced pricing externalities.

H Boundary and Regularity Conditions

(This section remains largely unchanged, emphasizing the reflecting boundary at $k = 0$, growth conditions on U, and integrability of m. The regularity of the pricing kernel is now crucial, as noted in ass:regularity.)

I Computation: Two KS-Free Routes

I.1 Route A: HJB–FP Fixed Point (General Equilibrium)

The GE closure requires solving for the equilibrium distribution $m(x)$ simultaneously across all aggregate states x to determine the consumption dynamics and asset prices.

Algorithm (General Equilibrium).

[leftmargin=1.5em,label=**A.1**]

1. **Discretize** x**.** Define a grid for the aggregate state x.
2. **Initialize** $m^{(0)}(x)$**.** Initialize the distribution $m(x)$ for all x on the grid.
3. **Aggregate Dynamics.** Given $m^{(n)}(x)$, compute $C^{(n)}(x) = D(m^{(n)}(x), x)$. Estimate the dynamics $\mu_C^{(n)}(x), \sigma_C^{(n)}(x)$ using numerical differentiation of $C^{(n)}(x)$.
4. **Pricing.** Compute $r^{(n)}(x), \lambda^{(n)}(x)$ using prop:pricing. Construct the risk-neutral generator $^{(n)}$.
5. **HJB step.** Solve the coupled system eq:HJB for $V^{(n)}(x)$ across the x-grid. Recover $i^{*,(n)}(x)$.
6. **FP step.** Solve stationary eq:FP for $\widehat{m}^{(n+1)}(x)$ for each x, using $i^{*,(n)}(x)$.
7. **Update.** Dampen and update $m^{(n+1)}(x)$. Iterate until convergence of $m(x), r(x), \lambda(x)$.

[didacticstyle] **Computational Note.** The GE closure significantly increases computational cost. The HJB must be solved as a coupled system across x. Estimating the consumption dynamics requires accurate derivatives of $C(x)$, capturing how the equilibrium distribution $m(x)$ shifts with x.

I.2 Route B: Direct Master-PDE Collocation (General Equilibrium)

We parameterize $U_\omega(k, z, x, \cdot)$ and $_\psi(\xi; k, z, x, \cdot)$. We must also parameterize the pricing functions $r_\theta(m), \lambda_\theta(m)$ and, crucially, their Lions derivatives $\delta_m r_\theta, \delta_m \lambda_\theta$.

Residual construction. At each collocation tuple $(k, z, x; \{\xi^n\})$, compute the ME residual $\widehat{\mathcal{R}}_{\text{ME}}$ (Appendix B), including the empirical estimate of the pricing externality:

$$\widehat{E}_{pricing} = \frac{1}{N} \sum_{n=1}^{N} \left[-(\delta_m r_\theta(\xi^n)) U_\omega + (-\sigma_x \delta_m \lambda_\theta(\xi^n))(U_\omega)_x \right].$$

Minimize the empirical mean of $\widehat{\mathcal{R}}^2_{\text{ME}}$ plus penalties and consistency conditions ensuring r_θ, λ_θ match the SDF derived from the implied consumption dynamics.

[mathstyle] **Complexity of Route B in GE.** The requirement to compute and integrate the Lions derivatives of the pricing functions $(\delta_m r, \delta_m \lambda)$ makes Route B extremely challenging in the GE setting, as these derivatives depend on the complex response of equilibrium consumption dynamics to the measure m.

J Verification and Diagnostics

Residual norms. For collocation tuples (k, z, x, m): $\mathrm{R}_{\mathrm{HJB}} \equiv r(x,m)\,V - \max_i\{\pi + V_k\,(i - \delta k) + V + V\}$,
$\mathcal{R}_{\mathrm{FP}} \equiv -\partial_k[(i^* - \delta k), m] + m$,
$\mathcal{R}_{Pricing} \equiv \|r(x) - (\rho + \gamma\mu_C(x) - \cdots)\| + \|\lambda(x) - \gamma\sigma_C(x)\|$,
$\mathcal{R}_{\mathrm{ME}} \equiv r(x,m)\,U - \Big(HJBterms + Transport + E_{price} + E_{pricing}\Big)$.

Sanity checks.

[leftmargin=1.25em]

- *Risk-neutral case.* If $\gamma = 0$, then $\lambda = 0$ and $r = \rho$. The pricing externalities vanish, and =. The model must collapse to the partial equilibrium version.
- *Risk aversion sweep.* Increasing γ raises the market price of risk λ (typically), reducing the risk-adjusted drift of x and affecting valuation V.

K Economics: Aggregation, Irreversibility, and Asset Pricing

Aggregation. Aggregation enters via the general equilibrium externalities. The mean-field formulation explicitly decomposes these into the product market feedback (E_{price}) and the financial market feedback ($E_{pricing}$).

Irreversibility and Asset Prices. The asymmetry $\phi_- > \phi_+$ (costly reversibility) interacts with the endogenous SDF to generate the value premium (Zhang, 2005). If the market price of risk $\lambda(x)$ is countercyclical, firms constrained by adjustment costs become riskier in recessions, leading to higher expected returns.

Comparative statics.

[leftmargin=1.25em]

- Larger η (steeper demand) amplifies the negative price externality, reducing investment.
- Higher γ (risk aversion) increases the sensitivity of asset prices to consumption dynamics, amplifying the pricing externality and typically increasing risk premia.
- Higher σ_x increases aggregate risk. This raises λ (if $\gamma > 0$) and deepens precautionary effects, with ambiguous effects on average investment depending on the balance of risk adjustment and discount rate effects.

sectionAppendix A: Detailed Derivations
labelapp:derivations
subsectionHJB Analysis: Hamiltonian, KKT, and Convexity
labelapp:hjb$_analysis$
We analyze the maximization problem embedded in the HJB equation
eqrefeq:HJB. Define the optimized Hamiltonian
$mathcalH(k,z,x,m,p)$ with $p = V_k$:
beginequation
labeleq:hamiltonian
mathcalH(k,z,x,m,p) =
max$_{iinR}$
Big
$pi(k,i,z,x,m) + p,(i - deltak)Big.$
$endequationSubstitutingtheprofitfunctionfrom$
$Crefass:primitivesyields$
mathcalH = P(Y)e$^{x+z}k^{alpha-f-pdeltak+max_{iinR}Big(p-1)i-h(i,k)Big,}$
$sothemaximizationtermistheLegendre--Fencheltransformoftheconvexadjustmentcostthevaluatedat$p-1.
paragraphOptimal Policy Derivation (KKT/FOC). Because h is convex in i, the objective is concave. The first-order condition (FOC) is
p-1 - h$_i(i,k) = 0$
$quad$
$implies$
$quadi^* = h_i^{-1}(p-1).$

For the asymmetric quadratic cost $h(i,k) =$
$tfracphi_{pm}2i^2/k$ from
Crefass:primitives we obtain
beginitemize[leftmargin=1.25em]
item i

$ge0$: $h_i =$
phi_+i/k. The FOC gives $p-1 =$
phi_+i/k so $i = k(p-1)/$
phi_+, which is feasible only when p

$ge1$.
item $i < 0$: $h_i =$
phi_-i/k. The FOC gives $p-1 =$
phi_-i/k so $i = k(p-1)/$
phi_-, which requires $p < 1$.
enditemize This recovers the piecewise-affine policy in
Crefprop:policy.
begintcolorbox[sympycheckstyle]
textbfVerification: FOCs for the optimal policy. See Appendix E, Check 2, for a symbolic confirmation of the formulas above.
endtcolorbox
begintcolorbox[leanproofstyle]

textbfFormalization: Optimization structure in Lean 4.
beginmintedlean import Mathlib.Analysis.Convex.Basic

– Parameters: k ¿ 0, phi$_{plus} > 0, phi_minus > 0. variables k phi_plus phi_minus p : Real (hk : k > 0) (hp_p : phi_plus > 0) (hp_m : phi_minus > 0)$

– Asymmetric quadratic adjustment cost. def adjustment$_{c}ost(i : Real) : Real := if i >= 0 then (phi_plus/2) * (i^2/k) else (phi_minus/2) * (i^2/k)$

– Objective maximized inside the Hamiltonian. def objective (i : Real) : Real := (p - 1) * i - adjustment$_{c}osti$

– Candidate optimal policy i*(p). def optimal$_{p}olicy : Real := if p >= 1 then k * (p - 1)/phi_plus else k * (p - 1)/phi_minus$

– TODO: Show that ‘optimal$_{p}olicy$‘ maximizes‘objective‘ using convexity + FOCs. Status: structure encoded; maximality proof pending.
endtcolorbox

paragraphConvexity of the Hamiltonian. The map p
$mapsto$
$mathcalH(k, z, x, m, p)$ is convex: the supremum of affine functions of p is convex. The envelope theorem implies
$partial_p$
$mathcalH = i^*(p)$, and the policy is monotone in p because
$phi_{pm} > 0$, so the second derivative is non-negative.

begintcolorbox[sympycheckstyle]
textbfVerification: Envelope property. Appendix E, Check 3, confirms
$partial_p$
$mathcalH = i^*(p)$.
endtcolorbox

subsectionAdjoint Pairing and the FP Equation
labelapp:fp$_{d}erivation$

We derive the Kolmogorov–Forward (FP) equation
eqrefeq:FP via the weak formulation. For a smooth test function
$varphi(k, z)$ with compact support,

E[
varphi(k$_t, z_t$)
=
int
varphi(k,z) m$_t$(
$diffk$,
$diffz$).
]Let
mathcalL$denotethegeneratorof$(k$_t, z_t$) under
$mathbbP$, conditional on x and the policy i^*:

mathcalL
varphi = (i$^*(k, z) -$
$deltak$)
,
$partial_k$
$varphi +$
Lz
$varphi$.

The weak formulation yields
$tfracdiffdifft$
int
$varphi\ m_t =$
$int($
$mathcalL$
$varphi)\, m_t$. To obtain the strong form we identify the adjoint operator
$mathcalL^*$ defined by
$langle$
$mathcalL$
$varphi, m$
$rangle =$
$langle$
$varphi,$
$mathcalL^* m$
$rangle.$
paragraphIntegration by parts. Write
$xi = ($
$kappa,$
$zeta)$ for a generic point in state space, and assume boundary terms vanish (compact support / zero-flux). The transport component with $v($
$xi) = i^*($
$xi) -$
$delta$
$kappa$ satisfies
int v
,
partial$_{kappa}$
$varphi$
$, m$
$diff$
$kappa = -$
int
$varphi$
$partial_{kappa}(vm)$
$diff$
$kappa,$

while diffusion terms satisfy
$int($
Lz
$varphi)m =$
int
$varphi($
$Lzadjm)$. Consequently
mathcalL$^* m = -$
$partial_k$

$big((i^* -$
$deltak)m$
$big) +$
$Lzadjm,$

and the FP equation follows from
$partial_t m =$
$mathcalL^* m.$

begintcolorbox[sympycheckstyle]
textbfVerification: Adjoint identity for the transport term.
beginpyconsole import sympy as sp k = sp.symbols('k', positive=True) phi = sp.Function('phi')(k) m = sp.Function('m')(k) v = sp.Function('v')(k)

$\mathrm{L}_p hi = v * sp.diff(phi, k) L_s tar_m = -sp.diff(v * m, k)$

LHS = sp.diff(phi * v * m, k) RHS = $(\mathrm{L}_p hi * m) + (phi * L_s tar_m) assert sp.simplify(LHS - RHS) == 0 print("Adjoint pairing for transport verified.")$
$endpyconsole$
$endtcolorbox$

subsectionDeriving the generalized master equation
labelapp:me$_d erivation$

We outline the stationary Generalized Master Equation (ME) eqrefeq:ME. When the HJB coefficients $r(m)$ and $LxQ(m)$ depend on m, the standard derivation must be enhanced to accommodate interactions in the coefficients (Bensoussan et al., 2013).

paragraphFunctional It
$^o calculus. Consider \mathrm{U}(\mathrm{k}_t, z_t, x, m_t)$. The generalized functional It
$^o lemma tracks the evolution of \mathrm{U} under changes in both the state (\mathrm{k}_t, z_t)$ and the law m_t. Conditioning on x and using the pricing measure
$mathbbQ$, stationarity requires $r(m)U$ to balance all drift contributions.

paragraphDecomposing the m-dependence. Two channels link the law m to valuations:
beginenumerate[label=(
roman*), leftmargin=1.5em]
item
textbfTransport. Variations of m feed through the Lions derivative
dmU even if coefficients are fixed.
item
textbfExternalities. Variations of m alter
pi, r, and
$muxQ$, feeding back into the HJB residual itself.
endenumerate Applying the generalized chain rule produces
beginalign* $\mathrm{E}_{texttransport} =$
int
$Bigl[(i^*($
$xi, m) -$
$delta$
$kappa)$
,
$partial_{kappa}$

$dmU($
$xi;$
$cdot) +$
Lz
$dmU($
$xi;$
$cdot)$
$Bigr]$
$, m($
$diff$
$xi),$

$E_{textprice} =$
int
$delta_m$
$pi($
$xi; k, z, x, m)$
$, m($
$diff$
$xi),$

$E_{textpricing} =$
int
$Bigl[-($
$delta_m r($
$xi; m))U + ($
$delta_m$
$muxQ($
$xi; m))U_x$
$Bigr]$
$, m($
$diff$
$xi).$
$endalign*$

begintcolorbox[didacticstyle]
textbfSign intuition. If a perturbation of m raises the short rate ($delta_m r > 0$), valuations fall, hence the negative sign. Conversely, if the perturbation raises the risk-neutral drift ($delta_m$ $muxQ > 0$) the asset gains value when $U_x > 0$, giving a positive contribution.
endtcolorbox

Collecting the HJB drift, transport, and externality terms and imposing the pricing-measure martingale condition delivers the ME
eqrefeq:ME.

A Appendix B: Residual-Loss Template (for implementation)

For a collocation tuple (k, z, x), an empirical measure $m = 1N \sum_{n=1}^{N} \delta_{\xi^n}$, and parameterized $U_{\omega,\psi}$, r_θ, λ_θ and their Lions derivatives $\delta_m r_\theta, \delta_m \lambda_\theta$.

$\widehat{Y} \equiv \frac{1}{N} \sum_{n=1}^{N} e^{x+\zeta^n} (\kappa^n)^\alpha$,
$\widehat{} \equiv \mu_x(x) - \sigma_x \lambda_\theta(x, m)$,
$\widehat{E}_{pricing} \equiv \frac{1}{N} \sum_{n=1}^{N} [-(\delta_m r_\theta(\xi^n))U_\omega + (-\sigma_x \delta_m \lambda_\theta(\xi^n))(U_\omega)_x]$,
$\widehat{\mathcal{R}}_{\mathrm{ME}} \equiv r_\theta(x, m)\, U_\omega$
$- \max_i \left\{ \pi + (U_\omega)_k\,(i - \delta k) + U_\omega + (\lambda_\theta) U_\omega \right\}$
$- Transport(_\psi)$
$- e^{x+z} k^\alpha\, \widehat{Y}\, P'(\widehat{Y}) - \widehat{E}_{pricing}$. Minimize the loss $\mathcal{L} = [\widehat{\mathcal{R}}^2_{\mathrm{ME}}]$ plus penalties and consistency conditions ensuring r_θ, λ_θ match the SDF derived from the implied consumption dynamics.

B Appendix C: Common-Noise Master Equation (Reference Note)

When the population law m_t itself diffuses under common noise, the functional Itô calculus on $\mathcal{P}_2$ introduces a second-order term in the measure variable. The stationary master equation would add a term related to the covariance of the common noise and second-order measure derivatives of U. Because this paper conditions on x, these terms are absent.

C Appendix D: Tiny Pseudocode (Plain `listings`listings)

(The pseudocode structure remains similar to the original but must incorporate the endogenous pricing functions and the pricing externalities in the residual calculation, as illustrated in Appendix B.)

D Appendix E: Symbolic Verification (PythonTeX + SymPy)

This appendix runs minimal SymPy checks to verify key derivations used in the text. Compilation is configured (via `latexmkrc`) to execute these checks on every build; any failure triggers a build error. We assume smoothness and reflecting/no-flux boundary conditions where noted.

import sympy as sp

1) Isoelastic simplification: Y P'(Y) = -eta P(Y) Y, eta = sp.symbols('Y eta', positive=True) P = Y**(-eta) check1 = sp.simplify(Y*sp.diff(P, Y) + eta*P) assert check1 == 0 print("Isoelastic: Y*P'(Y) = -eta*P(Y) [OK]")

2) KKT/FOC solution for i* with asymmetric quadratic costs i, k, p, phi$_plus, phi_minus = sp.symbols('ikpphi_plusphi_minus', positive = True)h_plus = 0.5*phi_plus*i**2/kFOC_plus = sp.Eq(sp.diff(-i-h_plus+p*i, i), 0)sol_plus = sp.solve(FOC_plus, i)[0]h_minus = 0.5 * phi_minus * i * *2/kFOC_minus =$

$sp.Eq(sp.diff(-i-h_minus+p*i,i),0)sol_minus = sp.solve(FOC_minus,i)[0]assertsp.simplify(sol_plus - k*(p-1)/phi_plus) == 0assertsp.simplify(sol_minus-k*(p-1)/phi_minus) == 0print('KKT/FOCpiecewisei*formulas[OK]')$

3) Envelope property for Hamiltonian in p: dH/dp = i*(p) H$_plus = (-i-h_plus+p*i).subs(i,sol_plus)dHp_dp = sp.simplify(sp.diff(H_plus,p))assertsp.simplify(dHp_dp-sol_plus) == 0print('Envelope : dH/dpequalsi*(p)[OK]')$

4) SDF dynamics and pricing kernel components (Ito's lemma check) Verification of CCAPM short rate and market price of risk t = sp.symbols('t', positive=True) rho, gamma = sp.symbols('rho gamma', positive=True) C = sp.Function('C')(t) mu$_C, sigma_C = sp.symbols('mu_C sigma'_C, real = True)$

SDF: Lambda = f(t, C) = exp(-rho*t) * C**(-gamma) Assume dC = C*mu$_C dt + C * sigma_C dBf = sp.exp(-rho*t)*C**(-gamma)df_dt = sp.diff(f,t)df_dC = sp.diff(f,C)df_dC2 = sp.diff(f,C,2)$

Drift of dLambda (Ito's lemma) drift$_Lambda = df_dt+df_dC*(C*mu_C)+ 0.5*df_dC2*(C*sigma_C)**2DriftofdLambda/Lambda = -rdtdrift_dLambda_Lambda = sp.simplify(drift_Lambda/f)$

Target risk-free rate r (Ramsey rule + precautionary savings) r$_target = rho+gamma*mu_C-0.5*gamma*(gamma+1)*sigma_C**2assertsp.simplify(drift_dLambda_Lambda+ r_target) == 0$

Diffusion of dLambda diff$_Lambda = df_dC*(C*sigma_C)DiffusionofdLambda/Lambda = -lambdadBlambda_actual = sp.simplify(-diff_Lambda/f)lambda_target = gamma*sigma_C assertsp.simplify(lambda_actual - lambda_target) == 0$

print('SDF dynamics (r and lambda) via Ito [OK]')

print('SymPy verification checks passed.')

References

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