Optimal Insurance: Dual Utility, Random Losses and Adverse Selection

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Abstract

We study a generalization of the classical monopoly insurance problem under adverse selection (see Stiglitz [1977]) where we allow for a random distribution of losses, possibly correlated with the agent's risk parameter that is private information. Our model explains patterns of observed customer behavior and predicts insurance contracts most often observed in practice: these consist of menus of several deductible-premium pairs, or menus of insurance with coverage limits-premium pairs. The main departure from the classical insurance literature is obtained here by endowing the agents with risk-averse preferences that can be represented by a dual utility functional (Yaari [1987]).

1 Introduction

A robust empirical finding in various insurance markets is that even moderate risks are often insured via contracts with low deductibles, or with full coverage up to high limits. In a famous early study, Mossin [1968] (page 558) observed:

> "...the conclusion that full coverage is never optimal seems quite plausible, at least when considered as a normative guideline. Casual empirical evidence seems to contradict the conclusion, however; some of our best friends take full coverage."

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Customers purchase insurance contracts with high coverage despite premium costs that are significantly above the value of the expected loss (see Barseghyan et al. [2011], [2016], [2021], Cohen and Einav [2007] and Sydnor [2010], among others). For example, Barseghyan et al. [2021] describe a data set of 111,890 households that choose among deductibles $\{\$100, \$200, \$250, \$500, \$1000\}$ for auto collision insurance. About 65.4% chose the \$500 deductible paying an average premium of \$217. This is, on average, \$49 higher than the premium paid for the \$1000 deductible that was chosen by only 4.6% of the households. Another 30% of the households chose even lower deductibles paying (on average) at least another \$65, i.e., an extra of \$114 over the premium for the \$1000 deductible. Interestingly enough, the average claim rate in the data is only 8.8%! Thus, a reduction in deductible worth, say, $\$800 \cdot 0.088 = \$70.4$ in expected terms is purchased by a large number of households who pay \$114 on average for it. With another large data set of more than 50.000 households, Sydnor [2010] investigated deductible choice (from the same set) for house insurance. Households choosing a \$500 deductible pay an average premium of \$715 per year, yet these all rejected a $\$1,000$ deductible whose average premium was just \$615. As the claim rate is about 5%, these households were willing to pay \$100 to protect against a 5% possibility of paying an additional \$500!

Behavior as described above - for which very large degrees of risk aversion can be inferred - is hardly consistent with postulating that agents are expected utility maximizers: plausible calibrations of expected utility theory generally lead to risk-neutral behavior over small stakes (see, e.g., Rabin [2000]).[1] On the other hand, several authors, e.g., Barseghyan et al. [2016] [2013] have shown that the insurance patterns in their data sets are consistent with alternative explanations based on theories of non-expected utility that involve probability weighting.

Our main purpose in this paper is to provide a convenient analytic model that explains both the patterns of observed customer behavior as above, and the pattern of insurance contracts most often observed in practice: these consist of simple menus of several deductible-premium pairs or menus of full insurance with coverage limits-premium pairs.

The main departure from the classical insurance literature is obtained here by endowing the agents with risk-averse preferences that can be represented by a dual utility functional (Yaari [1987]). An alternative interpretation is that agents have a distorted belief that overweights more adverse events, leading to non-linear probability weighting.[2]

[1]Besides empirical findings, there is ample laboratory evidence that expected utility theory does not perform well in explaining agents' risk attitudes and over small or modest stakes.

[2]There is ample evidence about such phenomena in insurance contexts. See for example Johnson et al [1993].

In addition, we allow for a much more general version of the classical monopoly insurance problem under adverse selection (see Stiglitz [1977]):[3] our framework allows for a random distribution of losses that may be correlated with the agent's risk parameter, the latter being the agent's private information.

Our main results characterize the incentive compatible, individually rational and profit maximizing menus of insurance contracts. Each contract consists of an indemnity in case of loss (which depends on the loss and induces a retention share for the agent) and of a corresponding premium that must be paid up-front. A main assumption underlying the analysis (and often made in the finance and insurance literatures) is that both indemnity and retention functions are increasing in the value of the loss (double-monotonicity). This assumption corresponds to so called *ex-post moral hazard* condition ensuring that the agent benefits neither from increasing the loss (arson) nor from hiding part of the loss.

Under a regularity condition, the optimal scheme is a layer contract: for each risk type, it consists of alternating intervals of losses where the agent's retention function has either slope zero or slope one. We also offer sufficient conditions under which the optimal contract consists either of a menu of deductibles or a menu of coverage limits with different premia, one for each risk type.

It is well known that a deductible contract is welfare-maximizing for any risk-averse agent in the class of contracts with a fixed expected cost for the insurer.[4] Hence an insurance contract with a deductible is, in principle, consistent with the idea that the insurer needs to generate high welfare in order to extract a high revenue. In contrast, the optimality of contracts with coverage limits is somewhat striking since we show that such a contract is worst for any risk averse agent within the set of doubly monotonic contracts that have the same expected cost to the insurer.[5] In other words, the benefits of screening via coverage limits can be higher than the welfare loss from choosing an extremely sub-optimal allocation. As an example, US medical malpractice insurance typlically offers doctors a choice between coverage with limits of US $100K, 200K, 500K, 1000K, 2000K, 3000K. Payments at the limits are often seen in practice.

The analysis allows us to distinguish structural differences between optimal contracts for the case where private information is about the probability of a loss (where we get deductibles) and the case where private information is about the loss magnitudes (where we get coverage limits). Other commonly used instruments such as

[3]A large empirical evidence suggest that market power is prevalent in the insurance industry (see Dafny [2010], Robinson [2004] and Trish and Herring [2015], among others). For example, India's largest life insurer has a market share of 64%.

[4]See for example Van Heerwarden et al [1989]. This result generalizes famous early results by Arrow [1963] and Borch[1960] about variances of such contracts.

[5]This should not be confused with the well-known "live-or-die" contract analyzed by Innes [1990]. Such contracts are not doubly-monotonic.

coinsurance are not optimal in pure adverse selection frameworks.

As an illustration, consider the special case where the distribution of losses is independent of the probability of accident, and where the losses can take a finite number of values. Then, contracts with deductibles are optimal under some regularity conditions and optimal deductibles can only take one of the loss values or zero (which corresponds to full insurance), and lower deductibles are accompanied by higher premia. Thus the optimal menu can be seen as consisting of a basic high deductible - low premium pair complemented by a ladder of additional fees that gradually reduce the deductible until, possibly, full insurance. As illustrated above, such a menu structure is ubiquitous in practice and studied in most of the empirical literature.

Technically, we study a principal-agent problem with interdependent valuations and with type-dependent outside options. For each type of the agent, the allocation is an entire retention function (i.e., for each possible loss, the part of the loss that remains to be covered by the agent) rather than a scalar. Dual utility yields here, for each risk type, a linear optimization problem. Hence, for each type, the optimum is achieved at an extreme point of the set of feasible retention functions that satisfy the ex-post moral hazard constraints. We then offer sufficient conditions that render the obtained collection of retention functions, one for each risk type, incentive compatible.

Yaari's functional is a *rank-dependent* utility functional a la Quiggin [1982]. It replaces the classical von Neumann-Morgenstern independence axiom behind the expected utility (EU) functional with another axiom about mixtures of comonotonic random variables. The mixture is along the payoff axis instead of the probability axis - hence the name dual - and the resulting utility functional uses a non-linear function to distort probabilities rather than payoffs: it weights each payoff by a weight that is decreasing in the size of the payoff. Among other desirable properties, it disentangles attitudes towards risk from the marginal utility of money, that is constant. This property makes it appealing for settings where stakes are moderate: linearity of the agents' utilities in monetary transfers can then coexist with any degree of risk aversion.

In the special case where agents face a binary lottery (e.g., in the classical framework where there is a unique, fixed level of insured loss), our results are more general and can be applied to the class of non-expected utility displaying Constant Risk Aversion (CRA) (see Safra and Segal [1998]) with a convex risk-premium function.[6]

One of the main features that distinguish dual utility (or its variants) from expected utility is first-order risk aversion: in the limit where the stakes become small,

[6] Examples include Yaari's dual utility itself, Gul's [1991] and Loomes and Sugden's [1986] disappointment aversion theories with a linear utility over outcomes, Köszegi and Rabin's [2006] loss-averse utility with a linear utility over outcomes, mean-dispersion utility of the type used in the macro and finance literature (e.g., Rockafellar et al [2006]).

the risk premium vanishes linearly in the size of the risk.[7] This is in stark contrast to any EU preference represented by a twice differentiable utility function that exhibits second-order risk aversion: in the small stakes limit, EU agents become risk neutral and the risk premium they demand vanishes quadratically in the size of the risk.[8] This difference can have far-reaching implications for behavior.[9] For example, under expected utility, full insurance is Pareto-optimal if and only if the premium is actuarially fair (Mossin [1968]). In contrast, full insurance may be optimal even when there is a mark-up if the agent is endowed with an utility function that exhibits first-order risk aversion (Segal and Spivak [1990]).

Finally, we note that Yaari's dual utility functionals correspond to the so-called distortion (or spectral) risk measures, often used in the finance and the insurance literature. These basically consist of weighted sums of the *average values at risk (AVaR)* for each quantile.[10] Thus, our methodology could be applied to an insurer-reinsurer relation where the insurer (agent) uses such a risk measure to assess its portfolio.

The structure of the paper is as follows: We conclude this Section with a literature review. In Section 2 we describe the risk environment, the agent's preferences and the insurance contracts. Section 3 describes the set of feasible mechanisms that satisfy incentive compatibility and individual rationality constraints. In Section 4 we solve the optimal insurance problem within the general class of deterministic insurance contracts that respect two ex-post moral hazard conditions. We also offer conditions under which simple contracts that consist either of menus of deductibles or menus of coverage limits are optimal. Appendix 1 offers an example showing that random insurance mechanisms can improve profit (even though agents are here risk averse!). All proofs are in Appendix 2.

1.1 Related Literature

A large literature following Borch [1960] and Arrow [1963] focuses on models without adverse selection and studies the welfare maximizing insurance policy under a pricing formula where the premium for each policy is proportional to its cost. If the insurer

[7] Guriev [2001] offers a "micro-foundation" for dual utility: a risk neutral agent who faces a bid-ask spread in the credit market will behave as if he were dual risk averse. This can be directly applied to insurance markets if credit is needed to cover accidental losses. The same happens if gains are taxed but losses are not.

[8] See Segal and Spivak [1990] for definitions and a discussion of the various orders of risk aversion.

[9] For example, Epstein and Zin [1990] argue that dual utility can resolve the *equity premium puzzle*: faced with small-stakes lotteries, a dual risk-averse (EU risk-averse) agent requires a risk premium proportional to the standard deviation (variance) of the lottery. Since the standard deviation for small risks is considerably larger than the variance it generates a higher equity premium.

[10] For a good expositions, see for example Rüschendorf [2013], Chapter 7.

is willing to offer any insurance policy desired by the buyer at a premium that only depends on the policy's actuarial value, then the main finding is that the policy chosen by a risk-averse buyer will take the form of full coverage above a deductible minimum.

Raviv [1979] and Huberman et al. [1983] obtain optimality of policies using additional instruments such as upper limits on coverage only under a bankruptcy constraint on the agent or a regulatory constraint on the insurer. Townsend [1979] assumes that the loss can only be verified at a cost in order to obtain the optimality of simple deductibles.

The above mentioned strand of the literature does not treat adverse selection. Chade and Schlee [2012] offer a comprehensive and up-to-date study of monopolistic profit maximization in an insurance market subject to adverse selection where the agents are expected utility maximizers. Their model follows the pioneering work of Rotshchild and Stiglitz [1976] and Stiglitz [1977] by assuming that the private information pertains to the probability of an accident such that all types face the same fixed loss in case of an accident. This also holds for Szalay's [2008] alternative analytic approach to the same problem. Contrasting our framework, the allocation function for each risk type is then a scalar (the share of loss that is insured or retained). In such models, every deterministic feasible policy can be described in terms of a menu of deductibles - hence deductibles cannot be derived endogenously. Chade and Schlee [2012] and Szalay [2008] derive some interesting properties of the profit maximizing mechanism but they are able to analytically solve for the optimum only for agents equipped with either exponential (CARA) or square-root utility functions.

Cohen and Einav [2007] and Sydnor [2010], among others, argue that assuming EU yields implausibly large measures of risk parameters for a range of moderate risks in real insurance markets. Looking at households that purchase property insurance, Barseghyan et al. [2011] reject the hypothesis that subjects have stable expected utility preferences for more than 3/4 of the households. In contrast, Barseghyan et al. [2016] find that stable Yaari and rank-dependent utility preferences cannot be rejected for the majority of households in a large data set of car and home-insurance choices. This finding is confirmed for insurance coverage and 401(k) investment decisions by Einav et al. [2012]. Barseghyan [2021] attempt to explain the observed patterns of behavior by keeping the expected utility hypothesis but assuming that customers have heterogeneous choice sets, i.e., observing a dominated choice of deductibles could be explained by the absence of better choices in the particular set offered to that customer.

In addtion to insurance markets (Barseghyan et al. [2013]), there is empirical evidence also from stock markets (Kliger and Levy [2009]) and sport bets studies (see Snowberg and Wolfers [2010]) that agents do use non-linear probability weighting. Finally, several laboratory experiments illustrated similar findings (see Bruhin, Fehr-

Duda and Epper [2010] and Goeree, Holt and Palfrey [2002] who find support for a quadratic probability weighting).

The idea of theoretically studying insurance markets while equipping agents with some type of rank-dependent utility is not new. Most of the relevant finance/insurance literature uses then *distortion risk measures,* such as those derived from the average value at risk ($AVaR$).

In early work, Doherty and Eeckhoud [1995] study a model without adverse selection. Following Arrow [1963], they are interested in maximizing the agent's welfare under actuarial fair pricing plus a markup, and focus solely on simple (not necessarily optimal) mechanisms such as proportional insurance (coinsurance) or deductibles.

Bernard et al. [2015] and Xu et al. [2019] focus on optimization of the agent's welfare under a random loss, but without adverse selection, i.e., there is a unique risk type. Thus, premia are basically "exogenous" as there is no incentive constraint binding them to insurance level. In the same model without adverse selection, Xu et al. [2019] impose conditions that constrain the agent's ability to manipulate the loss ex-post - we impose analogous conditions here.

Assuming an agent equipped with Yaari's dual utility, Hindriks and De Donder [2003] introduce adverse selection a la Stiglitz [1977]: the private information is about the probability of an accident and the loss in case of accident has a fixed magnitude, independently of the probability of having an accident. They show that a profit-maximizing monopolistic insurer offers full insurance to relatively high risk types while leaving relatively low risk types uninsured. Liang et al. [2022] show that this result is not robust to the presence of a random losses: in their model there are two risk types and the lower risk type is only partially insured.

Finally, Gershkov et al. [2022] analyze optimal auctions in a framework where bidders are equipped with a non-expected utility functional that exhibits constant risk aversion. Contrasting the present framework, their bidders face binary lotteries and the optimal mechanism offers full insurance (while distorting the allocation via endogenous randomization).

2 The Insurance Environment

An agent faces a random loss L distributed on $[0, \overline{L})$, where the maximal loss $\bar{L}$ can be finite or infinite. The agent's private information, his type $\theta \in [\underline{\theta}, \overline{\theta}] = \Theta$, parametrizes the distribution of losses $H_\theta : \mathbb{R}_+ \to [0, 1]$ he faces. The distribution H_θ is increasing in first-order stochastic dominance such that higher types face a stochastically larger loss. We assume that H is uniformly Lipschitz continuous in θ. We denote by $F : \Theta \to [0, 1]$ the distribution of types and by $f : \Theta \to (0, \infty)$ its density.

Finally, we denote by $m(\theta) = \mathbb{E}[L(\theta)]$ the expected loss of type θ, and we assume that this is finite.[11] To illustrate the generality of the setup, consider two important examples:

Asymmetric Information about Loss Probabilities Here the type θ represents the probability of an accident, and the distribution of losses conditional on an accident is given by a fixed distribution Q, independently of type. We obtain that

$$H_\theta(l) = (1-\theta) + \theta\, Q(l)\,. \tag{1}$$

This specification naturally captures health insurance where some agents face a greater risk of requiring certain medical procedures. Almost all of the insurance literature following Stiglitz's [1977] adverse selection model has focused on the special case where $Q(l) = \mathbf{1}_{l \geq l^*}$ puts probability 1 on a single (deterministic) loss $l^* > 0$.

Asymmetric Information about Loss Size Alternatively, the agent's type could influence the size of the loss, but not its probability. For example consider $L = \theta K$, where K is a random variable with support $[0, \infty)$, so that the agent's type multiplies an exogenous damage K distributed according to Q, independently of the agent's type. For this example, we obtain that

$$H_\theta(l) = Q(l/\theta).$$

Here all types face the same probability of accident, but some face higher losses in case of accident. For example, the probability of an earthquake is the same for all agents, but an agent with a higher house value may face a higher damage should his house be destroyed.

2.1 The Agent's Utility Function

Let X be the set of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cumulative distribution function of a random variable $x \leq 0$ is denoted by φ_x.

Definition 1 *Let $g : [0,1] \to [0,1]$ be increasing and onto. The utility given by $\mathcal{U}(x) = -\int_{-\infty}^{0} 1 - g(1 - \varphi_x(s))ds$ is called Yaari's* dual utility.[12]

[11] Note that, in view of the stochastic dominance assumption it is enough to assume that the expected loss of the highest type $m(\overline{\theta})$ is finite.

[12] Yaari only considered bounded random variables. For extensions to integrable random variables in $\boldsymbol{L}_p$ see, for example, Baüerle and Müller [2006] and Rüschendorf [2013].

We assume that the agent is endowed with a Yaari (dual) utility determined by a probability distortion function $g : [0, 1] \to [0, 1]$, where g is increasing, absolutely continuous and satisfies $g(p) \leq p$ with $g(0) = 0$ and $g(1) = 1$.

Our agent is risk-averse in a weak sense: the certainty equivalent of any lottery is less than the lottery's expected value. This is so because

$$\mathcal{U}(x) = \int_{-\infty}^{0} [g(1 - \varphi_x(s)) - 1]ds \leq \int_{-\infty}^{0} -\varphi_x(s)ds = \mathbb{E}[x]$$

While in the EU framework the above definition is equivalent to aversion to mean preserving spreads, here the latter notion of risk aversion is stronger, and is equivalent to g being convex. We shall explicitly assume the stronger convexity assumption in some of our illustrations.

We assume that the agent's willingess-to-pay for insurance is finite, which is equivalent to requiring that $\int_0^\infty 1 - g(H_{\overline{\theta}}(s))ds < \infty$. This is a weak assumption: for example, it is satisfied if the loss L is bounded or if g is bounded from below by a power function (as in the illustration below).[13] Without such a restriction the principal can obtain unbounded profits (without any screening) by offering insurance that covers all losses above a threshold. Finally, as a technical condition, we also assume that $\int_0^{\overline{L}} g'(H_{\underline{\theta}}(l))dl < \infty$ which ensures that the uninsured agent's utility is differentiable in type.

Because any Yaari dual utility is expressed via a Choquet integral, we note that it is additive on comonotonic random variables:[14]

$$\mathcal{U}(x + y) = \mathcal{U}(x) + \mathcal{U}(y)$$

for any comonotonic random variables $x, y \in X$ such that $x + y \in X$ (see Yaari [1987]).

As an illustration, consider Köszegi and Rabin's [2006] loss-averse preferences with linear utility over outcomes. These are given by:[15]

$$\mathcal{U}(x) = \mathbb{E}[x] + \int\int \mu\,(x - y)\, d\varphi_x\,(x)\, d\varphi_x\,(y)$$

where

$$\mu(z) = \begin{cases} z & \text{if } z \geq 0 \\ \lambda z & \text{if } z < 0 \end{cases}.$$

As established by Masatioglu and Raymond [2016] in their Proposition 4, the above

[13] To see this note that if $g(p) \geq p^\kappa$ then $\int_0^\infty 1 - g(H_{\overline{\theta}}(s))ds \leq \int_0^\infty 1 - (H_{\overline{\theta}}(s))^\kappa ds = \int_0^\infty \kappa(H_{\overline{\theta}}(s))^{\kappa-1} x dH_{\overline{\theta}}(s) \leq \kappa \int_0^\infty x dH_{\overline{\theta}}(s) = \kappa m(\overline{\theta}) < \infty$.

[14] Two random variables x, y are comonotonic if there exist another random variable z and two increasing functions h_1 and h_2 such that $x = h_1(z)$ and $y = h_2(z)$.

[15] See also the theory of disappointment without prior due to Delquié and Cillo [2006]).

functional form is a special case of Yaari's dual utility with the distortion

$$g(p) = (2 - \lambda)p + (\lambda - 1)p^2 \,.$$

The agent is risk-averse if and only if the agent is loss averse $\lambda > 1$ which ensures that g is strictly convex.[16]

2.2 The Insurance Contracts

There is a single, risk-neutral monopolistic insurance provider (she) who offers an insurance mechanism to the risk-averse agent (he). In order to rule out cases where the insurance provider exploits the agent's time-inconsistency through a dynamic mechanism, we restrict attention to direct (static) mechanisms.[17] We furthermore restrict attention to non-randomized mechanisms. In Appendix 1 we illustrate through an example that this restriction is not without loss of generality. It is, however, suitable in order to address applications: explicit randomized insurance contracts offered to risk averse agents are, to the best of our knowledge, never observed.

The insurer offers a menu of contracts of the form $(I(\cdot,\theta), t(\theta))_\theta$ where, for every type θ, $I(l,\theta) \in [0,l]$ is the amount covered if loss l occurs, and where $t(\theta)$ is the associated premium. Equivalently, the insurer can be seen as offering a menu of *retention* functions $(R(\cdot,\theta), t(\theta))_\theta$ where $R(l,\theta) = l - I(l,\theta)$ is the part of the loss l that remains to be covered by the agent of type θ.

Assumption 1 *We impose two natural monotonicity conditions (or* ex-post moral hazard *conditions) on the retention R for any θ :*

1. *$R(l,\theta)$ is non-decreasing in l .*
2. *$l - R(l,\theta) = I(l,\theta)$ is non-decreasing in l .*

Part 1 of Assumption 1 ensures that the agent does not benefit from a smaller retention $R(l',\theta) < R(l,\theta)$ by artificially increasing his loss from l to $l' > l$. Part 2 ensures that the agent does not benefit from a higher indemnity $l' - R(l',\theta) = I(l',\theta) > I(l,\theta) = l - R(l,\theta)$ by hiding part of the loss and reporting $l' < l$.

Any function that satisfies assumptions 1 and 2 above is Lipschitz continuous with constant 1, and hence also absolutely continuous. Its derivative exists almost everywhere and satisfies $\partial R(l,\theta)/\partial l \in [0,1]$ for all θ, l.[18]

[16] These preferences are consistent with monotonicity in First-Order Stochastic Dominance (FOSD) if and only if $\lambda \in [0,2]$. When $\lambda = 1$ the model reduces to the standard EU risk neutral preferences.

[17] If the agent has access to commitment or within the class of static mechanisms direct mechanisms are without loss of generality.

[18] In particular, the function $R(\cdot,\theta)$ can be obtained as the integral of its derivative.

Remark: Consider any insurance contract I and retention R_I. If Assumption 1 is not satisfied, then I and R_I are not comonotonic. Then, there exists comonotonic random variables I^* and R_I^* such that $I^* + R_I^* = I + R_I$ and such that $I^* \leq_{cx} I$ and $R_I^* \leq_{cx} R_I$ (see Landsberger and Meilijson [1994]). This implies that the original contract was not Pareto-optimal if the insurer and the agent are strongly risk-averse.

3 Implementable Insurance Contracts

In this section we describe incentive compatible and individually rational insurance contracts, i.e. we delineate the feasible set of contracts among which the insurer looks for the optimal one. For any θ, let $R^{-1}(\cdot, \theta)$ denote the *generalized inverse* of $R(\cdot, \theta)$.[19] For a fixed type θ, the distribution of the random variable $R(\cdot, \theta)$ is given by $H_\theta(R^{-1}(\cdot, \theta))$. The cost of providing insurance to a type θ agent who reports truthfully is given by

$$m(\theta) - \int_0^{R(\overline{L},\theta)} [1 - H_\theta(R^{-1}(l, \theta))dl = m(\theta) - \int_0^{\overline{L}} [1 - H_\theta(l)] \frac{\partial R(l, \theta)}{\partial l} dl,$$

where the equality follows by the change of variable $l = R(z, \theta)$.

3.1 Incentive Compatibility

Fix a mechanism $(R(\cdot, \theta), t(\theta))_\theta$. We denote by $U(\theta, \theta')$ the agent's certainty equivalent assuming that he has type θ, but reports to be type θ'. We slightly abuse notation by using below $U(\theta)$ instead of $U(\theta, \theta)$ for the certainty equivalent the agent obtains when reporting truthfully. We can then use the comonotonic additivity of the dual utility, and obtain that an agent with type θ who reports to be of type θ' obtains a utility of

$$U(\theta, \theta') = -t(\theta') - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l, \theta')}{\partial l} dl.$$

A mechanism $(R(\cdot, \theta), t(\theta))_\theta$ is *incentive compatible* if, for any pair of types θ and θ', it holds that:

$$U(\theta) \equiv U(\theta, \theta) \geq U(\theta, \theta'). \tag{IC}$$

Proposition 1 (Incentive Compatible Mechanisms)

(1) Fix any incentive compatible mechanism $(R(\cdot, \theta), t(\theta))_\theta$. Then the agent's utility

[19] $R^{-1}(l, \theta) = \sup\{z \colon R(z, \theta) \leq l\}$.

is given by

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds.$$

and the seller's profit is given by

$$\pi(R) = \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) - \int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} J(l,\theta) dl \right] f(\theta) d\theta - U(\underline{\theta})$$

where

$$J(l,\theta) = H_\theta(l) - g(H_\theta(l)) + \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta}.$$

(2) *If R is submodular, then the above conditions are also sufficient for the menu of contracts $(R(\cdot,\theta), t(\theta))_\theta$ to be incentive compatible.*

Note that submodularity is a very robust sufficient condition. It does not depend on particular form of the distortion g that determines utility nor on the other features of the environment.

The virtual value J defined above captures the effect of marginally increasing the insurance coverage, or equivalently marginally decreasing the agent's retention, on the insurer's profit. We can split this effect into two parts. The first part measures the effect on the insurer's revenue:

$$(1 - g(H_\theta(l))) + \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta},$$

where $1 - g(H_\theta(l))$ represents the agent's valuation for this marginal increase of insurance coverage and $\frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta}$ is the agent's information rent (note that this term is negative as $\frac{\partial H_\theta(l)}{\partial \theta}$ is negative). The second part

$$-(1 - H_\theta(l))$$

measures the effect on the insurer's costs. The optimal mechanism aims to balance these two effects in deciding the exact amount of insurance to provide to the agent. We also note that the term $H_\theta(l) - g(H_\theta(l))$ measures the size of efficiency gain resulting from providing insurance to a risk averse agent. The more risk-averse the agent is, the larger is the difference between the insurer's cost and the agent's gain.

3.2 The Participation Constraint

We furthermore restrict attention to mechanisms where each agent participates voluntarily. Since the distribution of losses is type dependent, the outside option from non-participation, corresponding to not purchasing insurance, is also type dependent. Nevertheless, we show the participation constraint will be satisfied for all types if and only if it is satisfied for the *lowest* possible type $\theta = \underline{\theta}$ who has here the *highest* utility.

Define first

$$U_{NP}(\theta) = -\int_0^{\overline{L}} [1 - g(H_\theta(l))]\, dl$$

to be type's θ certainty equivalent payoff from non-participation (i.e. not obtaining any insurance). The following *individual rationality (or participation)* constraint needs then to be satisfied

$$U(\theta) \geq U_{NP}(\theta)\,. \tag{PC}$$

Lemma 1 *In an incentive compatible mechanism $(R(\cdot,\theta), t(\theta))_\theta$, the participation constraint is satisfied for all types if and only if it is satisfied for the lowest type $\underline{\theta}$.*

The proof follows because both the utility from non-participation and the equilibrium utility are decreasing in risk type, and because the latter function decreases slower due to the fact that $\partial R(l,\theta)/\partial l \in [0,1]$ for all θ, l.

3.3 Strictly Positive Profit

Our final result in this Section shows that, under rather weak assumptions, a risk neutral insurer necessarily makes a strictly positive profit by offering full insurance to at least some types. In particular, this shows that the insurer makes a strictly positive profit in the optimal mechanism.

Lemma 2 *Assume that $H_{\overline{\theta}}$ is not degenerate*[20] *and that $\frac{\partial H_\theta}{\partial \theta}$ and g' are continuous.*[21] *Then the insurer obtains a strictly positive expected profit in the optimal menu of contracts.*

The proof of Lemma 2 explicitly computes the revenue from a mechanism that provides full insurance for sufficiently high types and no insurance for all lower types. It then shows that this full insurance cut-off can be chosen such that the mechanism generates strictly positive profits and such that all agents have an incentive to participate.

[20] A probability distribution is degenerate if it assigns probability one to single value.

[21] Since H_θ is decreasing in θ for each z, and if g is concave, the derivative of these functions exists almost everywhere, and g is even twice differentiable almost everywhere.

Consider for example the case where

$$H_\theta(l) = (1-\theta) + \theta Q(l),$$

and where θ represents the probability of an accident. We stress that the above result - the optimality of some trade - always holds then even if $\overline{\theta} = 1$ (i.e., even if the highest type incurs some loss with probability one) provided that $H_{\overline{\theta}}$ is not degenerate. This contrasts the case with a unique loss for which Hendren [2013] gives a condition under which no trade is beneficial if probabilities of accident are high enough.

4 Optimal Insurance

We now provide a characterization of the optimal insurance menu under a regularity condition similar to the standard monotonicity condition on the virtual value.

Theorem 1 *Suppose that the virtual value function*

$$J(l,\theta) = H_\theta(l) - g(H_\theta(l)) + \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta}$$

is non-decreasing in θ for all l. Consider the problem

$$\max_R \pi(R) = \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) - \int_0^{\overline{L}} \frac{\partial R(l,\theta)}{\partial l} J(l,\theta) dl \right] f(\theta) d\theta$$

subject to

$$0 \leq \frac{\partial R(l,\theta)}{\partial l} \leq 1 \textit{ for all } \theta \in \Theta.$$

The above problem has a solution that is incentive compatible and thus optimal. In addition, at the optimum $\partial R(l,\theta)/\partial l \in \{0,1\}$ almost everywhere.

When the virtual value function J is non-decreasing, we can find the optimal contract by maximizing the profit functional π. A retention function $R : [0,\bar{L}) \times \Theta \to [0,\bar{L})$ is feasible and satisfies Assumption 1 if and only if $R(0,\theta) = 0$ and $\partial R(l,\theta)/\partial l \in [0,1]$ almost surely.[22] Since any subsequence of such functions $R(\cdot,\theta)$ is uniformly bounded, Lipschitz continuous and hence uniformly continuous, the Arzela-Ascoli Theorem yields that any such sequence has a uniformly convergent subsequence for every θ.[23] Thus, absent incentive constraints, a collection $(R(\cdot,\theta))_\theta$ that maximizes π must

[22]We wish to thank Martin Pollrich and Andreas Kleiner for insightful discussions about this set of functions.

[23]The classical Arzela-Ascoli Theorem assumes a compact support. In order to obtain compactness and existence of extreme points for the case where the support of losses is unbounded, we use instead

exist by continuity of the functional and by the compactness of the set of retention functions that satisfy Assumption 1, combined with Tychonoff's Product Theorem. Furthermore, as the set of policies is convex and as the objective is linear in R, Bauer's principle yields that a maximum is attained at an extreme point. The extreme points of the set of feasible retention functions $R(\theta, l)$ are continuous in l with $R(0, \theta) = 0$ and such that $\partial R(l, \theta)/\partial l \in \{0, 1\}$ almost everywhere.[24] Thus, in the optimal mechanism, an increase in the loss l will be either completely passed on to the agent with type θ ($\partial R(l, \theta)/\partial l = 1$) or not at all ($\partial R(l, \theta)/\partial l = 0$).

Here is one typical example where losses below D and above $C > D$ are passed on to the agent:

$$R(l, \theta) = \begin{cases} l & 0 \leq l \leq D \\ D & D < l \leq C \\ l - C + D & C < l \leq \overline{L} \end{cases} .$$

This retention function R is induced by a contract with a deductible $D \geq 0$ and with a coverage limit $C > D$.

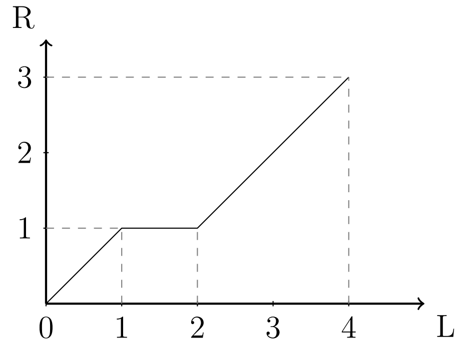

Figure 1: Contract with deductible $D = 1$ and coverage limit $C = 2$

Remark: A commonly observed contract format includes coinsurance, where the agent retains a fraction of the loss and the insurer covers the remaining fraction. By definition, coinsurance contracts have slopes strictly between zero and one. These are not extreme points of our feasible set, and therefore cannot be optimal under our regularity conditions. Coinsurance may be desirable if there is an additional *ex ante* moral hazard concern (e.g., if the agent may take costly care to reduce the risk for loss before it happens) or if the insurer is herself risk-averse so that a less extreme division of risk is desirable.[25]

the extension to a $\sigma-$ compact and locally compact Hausdorff space (see for example Theorem 4.44, page 137 in Folland [1999]).

[24] See also the related characterizations of extreme points of the unit ball of Lipschitz functions (without any monotonicity assumptions) e.g. Smarzewski [1997] and the papers cited there.

[25] Also, if the regularity condition fails, "ironing" may result in some optimal contracts with a co-insurance component.

A natural question stemming from Theorem 1 is: When is virtual value function J non-decreasing in θ? Observe that

$$\frac{\partial J(l,\theta)}{\partial\theta} = -\frac{\partial H_\theta(l)}{\partial\theta}\left(g'(H_\theta(l))\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]' - 1\right) + \frac{1-F(\theta)}{f(\theta)}\frac{\partial[g'(H_\theta(l))\frac{\partial H_\theta(l)}{\partial\theta}]}{\partial\theta}.$$

By assumption, $\frac{\partial H_\theta(l)}{\partial\theta} < 0$ and g is increasing with $g(p) \leq p$. Assume here that g is convex. A monotonically increasing hazard rate together with $g'(H_\theta(l)) \geq 1$ are sufficient for the first term to be non-negative. As g' is increasing and H_θ is decreasing in θ, we get that this condition is equivalent to $g'(H_{\overline{\theta}}(0)) \geq 1$. Because $\lim_{p\to 1} g'(p) > 1$ this condition is satisfied whenever the probability of a loss $1 - H_{\overline{\theta}}(0)$ is sufficiently small even for the type $\overline{\theta}$ who faces the largest risk. In addition, the increasing stochastic concavity[26] (in the usual stochatic order) of the family of random variables with distributions $(H_\theta)_\theta$ is sufficient for the second term to also be non-negative, thus yielding the desired monotonicity. We present below an example which satisfies the above conditions.

Example 1 (Assymetric Information about Loss Probabilities) *Consider an environment with asymmetric information about loss probabilities as defined in* (1), *i.e.*

$$H_\theta(l) = 1 - \theta + \theta Q(l) = 1 + \theta(Q(l) - 1)\,.$$

In this case we have that the derivative $\partial J(l,\theta)/\partial\theta$ equals

$$-(Q(l)-1)g'(H_\theta(l))\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]' + (Q(l)-1) + \frac{1-F(\theta)}{f(\theta)}(Q(l)-1)^2 g''(H_\theta(l)).$$

For J to be non-decreasing in θ we need

$$g'(H_\theta(l))\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right]' - 1 + \frac{1-F(\theta)}{f(\theta)}(1-Q(l))g''(H_\theta(l)) \geq 0.$$

Assuming that the hazard rate is increasing, the above inequality holds if $g'(H_{\overline{\theta}}(0)) = g'(1-\overline{\theta}) > 1$. Intuitively, if the maximal possible probability of a loss $\overline{\theta}$ is sufficiently small, then the inequality will hold for all relevant types θ. For instance, if $g(p) = (1-r)p + rp^2$, then $g'(p) > 1 \Leftrightarrow p > 1/2$, so that $g'(H_\theta(l)) > 1$ holds for any l as long as $\theta < 1/2$. Of course, real-life insurance data present much lower accident probabilities.

We conclude the current Section with the following comparative statics result show-

[26] See Shaked and Shanthikumar, Section 8.C for a definition. Beyond the stochastic monotonicity in θ already assumed above, this means that, for each loss l, $1 - H_\theta(l)$ is concave in θ.

ing that the insurance provider benefits from higher risk aversion of the agent.

Proposition 2 *Assume that $g_2(p) < g_1(p)$ for any $p \in (0,1)$, i.e. g_2 represents a higher risk aversion of the agent. Assume that the optimal retention function if the preferences are represented by g_1 is submodular. Then the insurer's profit in case of the preferences represented by g_2 is higher than the profits in the case the preferences are represented by g_1.*

4.1 Optimality of Deductibles or Coverage Limits

We now display conditions under which it is optimal to restrict attention to two special classes of mechanisms, most often used in practice: the first class consists of menus of contracts of the form $(D,t) = (D(\theta), t(\theta))_\theta$, one for each type θ, where each contract specifies a *deductible* $D(\theta) \in [0, \overline{L}]$ and a *premium* $t(\theta) \in \mathbb{R}$ (i.e., the price of the insurance). For a fixed risk type θ, the associated retention is:

$$R_D(l,\theta) = \begin{cases} D(\theta) & l \geq D(\theta) \\ l & l < D(\theta) \end{cases} .$$

The second class consists of menus of *coverage limits* $(C,t) = (C(\theta), t(\theta))_\theta$ where for type θ all losses up to $C(\theta) \in [0, \overline{L}]$ are covered and $t(\theta)$ is the corresponding premium. For a fixed risk type θ,the associated retention is:

$$R_C(l,\theta) = \begin{cases} 0 & l \leq C(\theta) \\ l - C(\theta) & l > C(\theta) \end{cases}$$

For convenience, we will sometimes refer to the first class as *deductible contracts* and the second class as *cap contracts.* Both types of contracts described above respect the ex-post moral hazard conditions.

To see what is special with these two types of contracts, consider a setting without adverse selection (i.e., there is a single publicly known type θ with loss distribution H_θ). Then, any strongly risk averse agent (even those with non-Yaari utility) prefers the deductible contract to any other contract with the same expected cost, and prefers any contract to the cap contract with the same expected cost to the insurer. The first argument is well-known (see for example Van Heerwaarden et al. [1989]).[27] We reproduce its short proof for completeness, and also because we use it for proving the second, apparently new part about the contract with a coverage limit.[28]

[27]It generalizes famous results by Arrow and by Borch who showed that deductibles lead to the lowest variance among all contracts with the same cost.

[28]We recall here that the "live-or-die" contract studied by Innes [1990] is not doubly monotonic, and is thus different from a contract with a coverage limit.

Denote by $\mathbb{E}[I]$ the expected cost of providing the insurance contract I to a type θ agent.[29]

Theorem 2 *For a given contract $0 \leq I \leq L$ satisfying Assumption 1, let $D \geq 0$ be a solution to $\mathbb{E}[(L-D)_+] = \mathbb{E}[I]$, and let C be a solution to $\mathbb{E}[\min\{L, C\}] = \mathbb{E}[I]$. Then it holds that*

$$R_D(\cdot, \theta) \leq_{cx} R_I(\cdot, \theta) \leq_{cx} R_C(\cdot, \theta)$$

where cx denotes the convex stochastic order.

The above result implies that, if types are observable, then fixing the insurance provision cost, agents are always willing to pay most (least) for a deductible (coverage limit) contract. It follows that any deviation from deductible policies must be driven by the incentive constraints coming from types being unobservable. Note that this observation holds for any risk preference. We show below that, with Yaari utility, in some cases these constraints lead the seller to offer a coverage limit, the worst contract - with a given cost - for the agent.

We present below sufficient conditions under which the two simple forms of insurance discussed above are optimal within the general class of mechanisms. We then provide some examples to illustrate when these conditions hold.

Theorem 3 *Assume that the virtual value $J(l, \theta)$ is non-decreasing in θ for all l.*

(1) Suppose for each θ there exists a unique $l^*(\theta)$ such that $J(l, \theta) \leq 0$ for $l \leq l^*(\theta)$ and $J(l, \theta) \geq 0$ for $l \geq l^*(\theta)$. Then the profit maximizing mechanism consists of a menu of deductible-premium pairs $(D, t) = (D(\theta), t(\theta))_\theta$.

(2) Suppose for each θ there exists a unique $l^*(\theta)$ such that $J(l, \theta) \geq 0$ for $l \leq l^*(\theta)$ and $J(l, \theta) \leq 0$ for $l \geq l^*(\theta)$ Then the profit maximizing mechanism consists of a menu of cap-premium pairs $(C, t) = (C(\theta), t(\theta))_\theta$.

The proof (see Appendix 2) shows that the extreme points where profit is maximized for each type have the above structure, and that submodularity - and hence incentive compatibility - is satisfied. To see what the conditions mean in words: $J(l, \theta)$ is single crossing from above means the designer finds it profitable to cover small losses but not profitable to cover large ones, and therefore a menu of compensation limits becomes optimal. If $J(l, \theta)$ is single crossing from below, then the designer finds it profitable to cover large losses but not small ones. Therefore a menu of deductibles become

[29] The classical literature following Arrow assumes that the premium is given by $P(I) = (1+\delta)\mathbb{E}[I]$ where $\delta \geq 0$ is the load factor (or mark-up): thus, the premium is proportional to the expected cost of providing insurance.

optimal. The assumption that $J(l,\theta)$ is non-decreasing in θ for all l implies that the virtual value of providing extra insurance at any loss level l is always higher for high types. As a result, the designer always prefers to sell more insurance to high types. Thus the incentive compatibility condition in Proposition 1 is naturally satisfied.

Example 2 *Consider the case where the probability of a loss is the agent's private information $H_\theta(l) = 1-\theta+\theta Q(l)$. Let $r \in [0,1]$ and*[30]

$$g(p) = rp^2 + (1-r)p\,.$$

Suppose that $\bar{\theta} < \frac{1}{2}$ and that F has a monotonically increasing hazard rate. We obtain

$$\begin{aligned} J(l,\theta) =& H_\theta(l) - g(H_\theta(l)) + \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta} \\ =& r\left[1-Q(l)\right]\left[\left(\theta - \frac{1+r}{r}\frac{1-F(\theta)}{f(\theta)}\right) - \theta\left(1-Q(l)\right)\right)\left(\theta - 2\frac{1-F(\theta)}{f(\theta)}\right)\right] \end{aligned}$$

is non-decreasing in θ and for any θ, it crosses 0 at most once from below.

Let θ^* denote the solution to

$$\theta - \frac{1+r}{r}\frac{1-F(\theta)}{f(\theta)} = 0$$

and let θ^{**} denote the solution to

$$\theta - \frac{1+r}{r}\frac{1-F(\theta)}{f(\theta)} = \theta\left(\theta - 2\frac{1-F(\theta)}{f(\theta)}\right)$$

Then the profit maximizing mechanism consists of a menu of deductible-premium pairs with a single deductible per risk type, denoted $(D^*(\theta), t^*(\theta))$, which offers no-insurance to agents with accident probabilities $\theta < \theta^*$, full-insurance to agents with accident probabilities $\theta > \theta^{**}$, and a deductible

$$D^*(\theta) = Q^{-1}\left(1 - \frac{\theta - \frac{1+r}{r}\frac{1-F(\theta)}{f(\theta)}}{\theta\left(\theta - 2\frac{1-F(\theta)}{f(\theta)}\right)}\right)$$

for $\theta \in [\theta^*, \theta^{**}]$. Moreover, for all θ, $D^*(\theta)$ is non-increasing in r, the agent's degree of loss aversion.

Note that in the above example, both cutoff points defined above are independent of the loss distribution Q, and are thus solely determined by the agent's type distribution,

[30] This specification of g corresponds to Köszegi and Rabin's [2006] preferences with linear utility over outcomes (see Section 1 above) where we set $r = \lambda - 1$. A higher level of r indicates that the agent is more risk averse.

i.e. the distribution of loss-probabilities. In contrast, the optimal deductibles for types between the cutoff points are jointly determined by both loss distribution and type distribution.

For the special case $r = 1$, i.e. for $g(p) = p^2$, the two cutoff points defined above coincide, and the optimal menu offers either full or no insurance. For all other cases, $\theta^* < \theta^{**}$. This suggests that, with a random loss, the full-or-no insurance policy can be optimal if the agents are sufficiently risk averse. This contrasts the finding of Chade and Schlee [2012] who, in a framework with expected utility, show that full insurance is never optimal even if the loss is deterministic.

Example 3 *Assume that $g(p) = p^2$ and*

$$H_\theta(l) = 1 - e^{-\frac{l}{\theta}}$$

Here the agent's private information θ is the mean of the (exponential) distribution of losses. We obtain that

$$\begin{aligned} J(l,\theta) =& H_\theta(l) - g(H_\theta(l)) + \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta} \\ =& H_\theta(l) \left[1 - H_\theta(l) + \frac{2(1-F(\theta))}{f(\theta)} \times \frac{\partial H_\theta(l)}{\partial \theta} \right] \\ =& H_\theta(l) e^{-\frac{l}{\theta}} \left[1 - \frac{2(1-F(\theta))}{f(\theta)} \times \frac{l}{\theta^2} \right] \end{aligned}$$

As the function

$$1 - \frac{2(1-F(\theta))}{f(\theta)} \times \frac{l}{\theta^2}$$

is decreasing in l, we obtain that $J(l,\theta)$ always crosses 0 from above. Moreover, the solution to $J(l,\theta) = 0$ for all θ is given by[31]

$$C^*(\theta) = \frac{\theta^2 f(\theta)}{2(1-F(\theta))}.$$

Note that $C^*(\theta)$ is increasing in θ. Thus, a contract with coverage limits $(C^*(\theta))_\theta$ is optimal.

Remark: If $\int l dQ(l) = 1$ and if F is supported on $[0, \bar{\theta}]$ with $\bar{\theta} < 0.5$ then $\theta = \mathbb{E}[L(\theta)]$ equals the expected loss both in Example 2 where the agent's private information is about the probability of an accident, and in Example 3 where the private information is about the mean size of a loss. Despite the fact that the cost of providing full insurance

[31] In this example $J(l,\theta)$ is not necessarily non-increasing in θ for all l. But, the ensuing solution is nevertheless incentive compatible.

to an agent of type θ is exactly the same in both examples, the respective optimal incentive compatible contracts look fundamentally different.

In the former case (private information on accident probability), the profit-maximizing contract provides insurance in an optimal way for the agent: as we showed in Theorem 2 a deductible contract minimizes the expected cost to the principal of providing a given utility level to the agent. In contrast, in the case where the agent's private information is about loss size, the realized loss is informative about the agent's risk type: a higher loss is indicative of a higher risk type. By introducing a coverage limit - recall that, keeping the cost fixed, this is the worst contract for the agent - the insurer most effectively discourages high risk types from claiming to be low risk-types, as they would then suffer from the reduction in coverage limit. The revenue gain from this reduction in information rents dominates the efficiency loss due to the inefficient provision of insurance.

4.2 An Illustration: A Finite Number of Possible Losses

In this section we specialize our model to the case where type θ represents the probability of an accident and where the distribution of losses is independent of type and can only take a finite number of values. Thus

$$H_\theta(l) = 1 - \theta + \theta Q(l),$$

where Q is a given distribution with discrete support. This finite-loss case is relevant in practice since the definition/verification of a loss cannot be too refined without incurring extra costs. We restrict attention here to contracts with deductibles. In Theorem 3 and in Example 2 we illustrated when this restriction is without loss of generality.

The optimal mechanism in the class of contracts with deductibles takes a commonly seen form (see above for conditions under which deductibles are overall optimal for this specification of the loss function H): a basic deductible/premium contract, supplemented by a finite ladder of additional fees that, if added to the basic premium, gradually reduce the deductible until possibly reaching full insurance.[32]

Proposition 3 *Assume that the probability of an accident is θ and that, conditional on an accident, there are n different levels of loss $l_1 < l_2 < ... < l_n$ with probabilities $p_1, ..., p_n$, respectively, where $p_i \geq 0,\ \forall i$, and $\sum p_i = 1$. Then, there exists an optimal contract in the class of contracts with deductibles that offers at most $n+1$ of deductibles*

[32]See Hoppe et al. [2011] for an alternative explanation of coarse menus.

such that, for each offered deductible D, it holds either that $D = 0$ (full insurance) or there exists $1 \leq i \leq n$ such that $D_i = l_i$.

A simple corollary can be now obtained for the focal case studied in almost the entire theoretical literature where the loss is deterministic, i.e., there is only one possible loss equal to l: by the above result, we obtain that there exists an optimal insurance contract that offers either no insurance or full insurance (zero deductible).[33] This has been previously shown by DeFeo and Hindriks [2014].

Corollary 1 (Single Loss Case) *Consider the case of a single loss level $H_\theta(z) = 1 - \theta + \theta \mathbf{1}_{z \geq l}$. Assume that the virtual value*

$$J(\theta) = (1-\theta) - g(1-\theta) - \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)$$

crosses zero once from negative to positive at $\theta = \theta^*$.[34] Then the optimal contract offers no insurance (i.e., $D(\theta) = l$) to types $\theta \leq \theta^*$ and full insurance (i.e., $D(\theta) = 0$) to types $\theta \geq \theta^*$. Moreover, the expected profit is given by

$$l \left(\int_{\theta^*}^{1} (1-\theta) f(\theta) \, d\theta - g(1-\theta^*) (1 - F(\theta^*)) \right).$$

The insurer makes higher profits from lower types that are buying this contract, while she makes lower profits (or even losses) from higher types. Yet, the only possibility to attract lower types to acquire such contract is to reduce its price.

Example 4 *Consider loss levels $l_1 = 1$ and $l_2 = 2$ with probabilities $p_1 = p_2 = 1/2$, respectively. Assume that $g(p) = p^{3/2}$ and that types θ distribute uniformly on $[0,1]$. The optimal full insurance contract sells to all types above 0.803. Let:*

$$\begin{aligned} \theta^* &= \frac{1}{25 p_2} \left(15 p_2 - 2\sqrt{-15 p_2 + 16} + 8 \right) = 0.774 \\ \theta^{**} &= 0.84. \end{aligned}$$

The insurer obtains a higher profit with a basic insurance contract with deductible l_1 and premium $(l_2 - l_1)\left(1 - g\left(1 - \frac{1}{2}\theta^*\right)\right)$, combined with an option to reduce the deductible to zero at the extra price of $l_1 (1 - g(1 - \theta^{**}))$. Then types below θ^* obtain no insurance and pay zero, types in the interval $[\theta^*, \theta^{**})$ obtain partial insurance ($D =$

[33] In particular, full insurance for all insured types is optimal independently of the degree of risk aversion and independently of the distribution of accident probabilities. These last two model primitives only determine the set of insured types.

[34] The result is correct even if the virtual value crosses zero from negative to positive several times. Then θ^* must be one of the crossing values.

l_1) and pay $(l_2 - l_1)\left(1 - g\left(1 - \frac{1}{2}\theta^*\right)\right)$ and types above θ^{**} are fully insured ($D = 0$) and pay $(l_2 - l_1)\left(1 - g\left(1 - \frac{1}{2}\theta^*\right)\right) + l_1\left(1 - g\left(1 - \theta^{**}\right)\right)$.

Remark: Whenever there is a unique loss level, our agent faces only binary lotteries. Then, the above analysis holds for a wider class of utility functions that coincide with a Yaari utility for the class of binary lotteries, such as well-known utilities displaying constant risk aversion (see Safra and Segal [1998]). Examples including Gul's [1991] disappointment-averse preferences with linear utility over outcomes,[35] versions of the disappointment aversion theories due to Loomes and Sugden [1986], and Jia et al. [2001] with linear utility over outcomes, and modified Mean-Variance preferences (see Rockafellar et al. [2006]) with linear utility over outcomes are described in Appendix 3.

5 Conclusion

We have analyzed an insurance model with adverse selection where the loss distribution depends on the risk type (that is private information) in a very general form. The insured agents have a dual utility function. In a reinsurance context this means that the primary insurer uses a coherent and convex risk measure in order to assess its risk (while the reinsurer is risk neutral).

A main difference between our model and most of the literature without adverse selection is the pricing formula: here this is endogenously derived from the incentive compatibility and individual rationality constraints instead of assuming a premium that is, say, equal to cost plus a mark-up.

We have shown the optimality of layer contracts under a regularity conditions. We also focused on menus consisting of very simply contracts involving either deductibles or upper coverage limits, and we exhibited conditions under which such menus are optimal in the general class of insurance contracts where, for each risk type, higher losses lead both to higher coverage and to higher retention.

6 Appendix 1: Stochastic Mechanisms

We provide an example showing that a stochastic mechanism may be more profitable than the optimal deterministic mechanism. Consider an agent with risk preference represented by $g(x) = x^2$ and assume $\theta \sim U[0, 1]$. Suppose that there is a single deterministic loss level l, and let the agent's type θ be the probability that a loss occurs. The optimal deterministic mechanism consists of full insurance to types $\theta \geq \frac{2}{3}$

[35] See also Cerreia-Vioglio et al [2020]

at a price $\frac{8}{9}l$, and no insurance for lower types. Using this mechanism, the insurer's profit is $\frac{1}{54}l \approx 0.185l$.

Consider now a stochastic direct mechanism of the form $(t(\theta), p(\theta), l)$ such that: type θ pays a premium $t(\theta)$; in exchange, when a loss occurs, the insurer fully reimburses the agent's loss with (conditional) probability $1 - p(\theta)$. Note that the above class of mechanisms includes the optimal deterministic mechanism.

If type θ reports to be type θ' he will receive $-t(\theta') - l$ with probability $p(\theta')\theta$ and receive $-t(\theta')$ otherwise. Thus, this type of agent has a payoff of

$$\tilde{U}(\theta, \theta') = -l - t(\theta') + g\left(1 - p(\theta')\theta\right) l$$

in the proposed mechanism. We write $\tilde{U}(\theta) = \tilde{U}(\theta, \theta)$ for short.

By using very similar arguments to that of Proposition **??**, one can verify that $(t(\theta), p(\theta), l)$ is incentive compatible if and only if p is non-increasing and it holds that

$$\tilde{U}(\theta) = \tilde{U}(\underline{\theta}) - l \int_{\underline{\theta}}^{\theta} p(z) g\left(1 - p(z)z\right) dz \tag{2}$$

The above imply that

$$t(\theta) = -l - \tilde{U}(\underline{\theta}) + g\left(1 - p(\theta)\theta\right) l + l \int_{\underline{\theta}}^{\theta} p(z) g'\left(1 - p(z)z\right) dz$$

By using similar arguments to that of Lemma 1, it can be shown that the individual rationality constraint holds if and only if

$$\tilde{U}(\underline{\theta}) \geq -l(1 - g(1 - \underline{\theta}) = 0.$$

From now onward, we will only consider mechanism for which $\tilde{U}(\underline{\theta}) = 0$. The insurer's profit is

$$\begin{aligned}
\pi(p,t) = & \int_{\underline{\theta}}^{\bar{\theta}} \left[t(\theta) - (1 - p(\theta))\theta l\right] f(\theta) d\theta \\
& = -l + \int_{\underline{\theta}}^{\bar{\theta}} \left[g\left(1 - p(\theta)\theta\right) l + l \int_{\underline{\theta}}^{\theta} p(z) g'\left(1 - p(z)z\right) dz - (1 - p(\theta))\theta l\right] f(\theta) d\theta \\
=& l \int_{\underline{\theta}}^{\bar{\theta}} \left[g\left(1 - p(\theta)\theta\right) - (1 - p(\theta))\theta + \frac{1 - F(\theta)}{f(\theta)} p(\theta) g'\left(1 - p(\theta)\theta\right)\right] f(\theta) d\theta - l \\
=& l \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta(3\theta - 2)p^2 - (3\theta - 2)p + 1 - \theta\right] d\theta - l
\end{aligned}$$

To obtain the third equality, we used integration by parts:

$$\int_{\underline{\theta}}^{\bar{\theta}} [1 - F(\theta)] \, p(\theta) g' (1 - p(\theta)\theta)) \, d\theta - \int_{\underline{\theta}}^{\bar{\theta}} f(\theta) \left[\int_{\underline{\theta}}^{\theta} p(z) g' (1 - p(z)z)) \, dz \right] d\theta = 0$$

The optimal p is then given by by

$$p^* (\theta) = \begin{cases} 1 & \text{if } \theta \le \frac{1}{2} \\ \frac{1}{2\theta} & \text{if } \frac{1}{2} < \theta < \frac{2}{3} \\ 0 & \text{if } \theta \ge \frac{2}{3} \end{cases}$$

That is, within the above described class of potentially stochastic mechanisms, it is optimal to offer no insurance to agents with type below $\frac{1}{2}$, offers unconditional full insurance to those with type above $\frac{2}{3}$, and offer to reimburse the loss with (conditional) probability $1 - \frac{1}{2\theta}$ to intermediate types in $(\frac{1}{2}, \frac{2}{3})$. This mechanism yields approximately an expected profit of $0.188l > 0.185l$, and is thus superior to the optimal deterministic mechanism.

7 Appendix 2: Proofs

Proof for Proposition 1. (1) We note that $U(\theta, \theta')$ is absolutely continuous in θ with derivative

$$\frac{\partial U(\theta, \theta')}{\partial \theta} = \int_0^{\overline{L}} \frac{\partial R(l, \theta')}{\partial l} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta} dl \, .$$

The above equality follows as

$$\left| \frac{\partial R(l, \theta')}{\partial l} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta} \right| \le c g'(H_\theta(l)) \le c g'(H_{\underline{\theta}}(l)) \, .$$

The bound holds becaue $|\partial R / \partial l| \le 1$ and because $|\frac{\partial H_\theta}{\partial \theta}| \le c$. Also note that $\int_0^{\overline{L}} g'(H_{\underline{\theta}}(l)) dl$ is finite by assumption. By the Envelope Theorem (see e.g. Theorem 2 in Milgrom and Segal [2002]), in any incentive compatible mechanism, the agent's utility is absolutely continuous and is given by:

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l, s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds.$$

It follows that

$$
\begin{aligned}
-t(\theta) - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl =& U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds \\
\Rightarrow t(\theta) =& - U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds \\
& - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl
\end{aligned}
$$

The designer's expected revenue equals

$$
\int_{\underline{\theta}}^{\overline{\theta}} t(\theta) f(\theta) d\theta = - \int_{\underline{\theta}}^{\overline{\theta}} \int_0^{\overline{L}} \frac{\partial R(l,\theta)}{\partial l} \left[1 - g(H_\theta(l)) + \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta} \right] f(\theta) dl d\theta - U(\underline{\theta})
$$

where we used integration by parts to obtain the equality. Her profit then equals

$$
\begin{aligned}
\pi(R) =& \int_{\underline{\theta}}^{\overline{\theta}} \left[t(\theta) - m(\theta) + \int_0^{\overline{L}} [1 - H_\theta(l)] \frac{\partial R(l,\theta)}{\partial l} dl \right] f(\theta) d\theta \\
=& \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) - \int_0^{\overline{L}} \frac{\partial R(l,\theta)}{\partial l} J(l,\theta) dl \right] f(\theta) d\theta - U(\underline{\theta})
\end{aligned}
$$

where

$$
J(l,\theta) = H_\theta(l) - g(H_\theta(l)) + \frac{1 - F(\theta)}{f(\theta)} g'(H_\theta(l)) \frac{\partial H_\theta(l)}{\partial \theta}.
$$

(2) Suppose that $R(l,\theta)$ is submodular. Taking any $\theta < \theta'$, we obtain

$$
\begin{aligned}
U(\theta,\theta') =& - t(\theta') - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta')}{\partial l} dl \\
=& U(\theta') - \int_0^{\overline{L}} \left[\int_\theta^{\theta'} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} ds \right] \frac{\partial R(l,\theta')}{\partial l} dl \\
=& U(\theta) + \int_\theta^{\theta'} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds \\
& - \int_\theta^{\theta'} \left[\int_0^{\overline{L}} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} ds \right] \frac{\partial R(l,\theta')}{\partial l} dl
\end{aligned}
$$

As $R(l,\theta)$ is submodular, we obtain that $\frac{\partial R(l,\theta')}{\partial l} < \frac{\partial R(l,s)}{\partial l}$ for any $s \in (\theta,\theta')$. Also, we know that $g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} < 0$ by the assumption that $\frac{\partial H_s(l)}{\partial s} < 0$. It follows that

$$
\int_\theta^{\theta'} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds \leq \int_\theta^{\theta'} \left[\int_0^{\overline{L}} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} ds \right] \frac{\partial R(l,\theta')}{\partial l} dl
$$

which further implies $U(\theta,\theta') \leq U(\theta)$. Similarly, we can show that $U(\theta',\theta) \leq U(\theta')$ also holds, and conclude that the mechanism $(R(\cdot,\theta), t(\theta))_\theta$ is incentive compatible, as desired. ■

Proof of Lemma 1. The condition is clearly necessary. For sufficiently, observe that both $U_{NP}(\theta)$ and $U(\theta)$ are decreasing in θ. Moreover, for all θ it holds that

$$U'_{NP}(\theta) = \int_0^{\overline{L}} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} dz \leq \int_0^{\overline{L}} \frac{\partial R(z,\theta)}{\partial z} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} dz = U'(\theta)$$

because $0 \leq \frac{\partial R(z,\theta)}{\partial z} \leq 1$ and because $g'(H_\theta(z))\frac{\partial H_\theta(z)}{\partial \theta} \leq 0$ for all θ. Hence, we obtain that

$$U_{NP}(\theta) = U_{NP}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} U'_{NP}(z)dz \leq U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} U'(z)dz = U(\theta)$$

as desired. ■

Proof of Lemma 2. Consider the following simple mechanism: there exists a type θ^* such that

$$R(\theta, l) = \begin{cases} 0 & \theta \geq \theta^* \\ l & \theta < \theta^* \end{cases}.$$

This corresponds to all types $\theta \geq \theta^*$ being offered full insurance (or a deductible zero), while types $\theta < \theta^*$ are offered no insurance at all (or a deductible $\overline{L}$). The expected profit from this mechanism is given by

$$\begin{aligned}
\pi(R) = &- \int_{\underline{\theta}}^{\overline{\theta}} m(\theta)f(\theta)d\theta - \int_{\underline{\theta}}^{\theta^*}\int_0^{\bar{L}} J(\theta,z)dzf(\theta)d\theta - U(\underline{\theta}) \\
= &\int_{\underline{\theta}}^{\theta^*}\int_0^{\bar{L}} \left[g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta} \right] dzf(\theta)d\theta \\
&- \int_{\underline{\theta}}^{\overline{\theta}} m(\theta)f(\theta)d\theta - U(\underline{\theta}).
\end{aligned}$$

Since $m(\theta) = \int_0^{\bar{L}} (1 - H_\theta(z))\, dz$ we can rewrite

$$\begin{aligned}
\pi(R) = &-\bar{L} + \int_{\theta^*}^{\overline{\theta}}\int_0^{\bar{L}} H_\theta(z)dzf(\theta)d\theta - U(\underline{\theta}) - \int_0^{\bar{L}} \left[(1-F(\theta))\, g(H_\theta(z))\right]_{\underline{\theta}}^{\theta^*} dz \\
= &-\bar{L} - U(\underline{\theta}) + \int_{\theta^*}^{\overline{\theta}}\int_0^{\bar{L}} H_\theta(z)dzf(\theta)d\theta + \int_0^{\bar{L}} g(H_{\underline{\theta}}(z))dz - \int_0^{\bar{L}} (1-F(\theta^*))\, g(H_{\theta^*}(z))dz.
\end{aligned}$$

At $\theta^* = \overline{\theta}$,

$$\pi(R) = -\bar{L} - U(\underline{\theta}) + \int_0^{\bar{L}} g(H_0(z))dz = -\bar{L} + \int_0^{\bar{L}} [1 - g(H_\theta(z))]\, dz + \int_0^{\bar{L}} g(H_0(z))dz = 0.$$

Moreover, we have that

$$\begin{aligned}
\frac{\partial \pi}{\partial \theta^*} &= -\int_0^{\bar{L}} f(\theta^*) H_{\theta^*}(z)dz + \int_0^{\bar{L}} f(\theta^*) g\left(H_{\theta^*}(z)\right)dz - \int_0^{\bar{L}} \left(1 - F(\theta^*)\right) g'(H_{\theta^*}(z)) \frac{\partial H_{\theta^*}(z)}{\partial \theta} dz \\
&= -\int_0^{\bar{L}} f(\theta^*) \left[H_{\theta^*}(z) - g\left(H_{\theta^*}(z)\right]\right. dz - \int_0^{\bar{L}} \left(1 - F(\theta^*)\right) g'(H_{\theta^*}(z)) \frac{\partial H_{\theta^*}(z)}{\partial \theta} dz.
\end{aligned}$$

At $\theta^* = \overline{\theta}$,

$$\frac{\partial \pi}{\partial \theta^*} < 0$$

whenever $H_{\overline{\theta}}$ is not degenerate. Continuity guarantees that the derivative remains negative in some interval to the left of $\theta^* = \overline{\theta}$. Hence, such a simple mechanism where sufficiently high types are fully insured while all other types remain uninsured generates a strictly positive expected profit for the insurer. The optimal contract generates expected profits not lower than this simple contract. ■

Proof for Theorem 1. In order to show that the solution to

$$\max \pi(R) = \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) - \int_0^{\bar{L}} \frac{\partial R(l,\theta)}{\partial l} J(l,\theta) dl \right] f(\theta) d\theta$$

subject to

$$0 \leq \frac{\partial R(l,\theta)}{\partial l} \leq 1$$

for all $\theta \in \Theta$ is optimal, we just need to show the resulting retention function is submodular, and therefore incentive compatible.

Take any $\theta < \theta'$. The assumption that $J(l,\theta)$ is non-decreasing in θ for all l ensures that the pointwise maximization solution to the above problem satisfies

$$\frac{\partial R(l,\theta')}{\partial l} \leq \frac{\partial R(l,\theta)}{\partial l}$$

for all l. That is, R is submodular, as desired. ■

Proof of Proposition 2. Let R_{g_1} be the optimal retention function if preferences are represented by g_1. Observe first that as $R_{g_1}(l,\theta)$ is submodular it is also implementable if the preferences are represented by g_2. We show that for every θ if we use the retention function $R_{g_1}(l,\theta)$ also for the preferences represented by g_2 then

$$t_{g_2}(\theta) > t_{g_1}(\theta)$$

where t_{g_i} is the premium in case the preferences are represented by g_i and the retention function is $R_{g_1}(l,\theta)$. Recall that the premium is given by

$$t(\theta) = -U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl\,.$$

As $R(l,s)$ is submodular it follows that $\partial R(l,\theta)/\partial l$ is decreasing in θ. Furthermore, as $\partial R(l,\theta)/\partial l \in [0,1]$ for every fixed value of l the function $\phi_l(\theta) = -\partial R(l,\theta)/\partial l$ defines a positive measure over Θ. We can rewrite the transfer of type θ as follows

$$\begin{aligned}
t(\theta) &= -U(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[\int_0^{\overline{L}} \frac{\partial R(l,s)}{\partial l} g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} dl \right] ds - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl \\
&= -U(\underline{\theta}) + \int_0^{\overline{L}} \int_{\underline{\theta}}^{\theta} \phi_l(s) g'(H_s(l)) \frac{\partial H_s(l)}{\partial s} ds dl - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl \\
&= \int_0^{\overline{L}} [1 - g(H_{\underline{\theta}}(l))]\, dl - \int_0^{\overline{L}} [\phi_l(s)\,(1 - g(H_s(l)))]_{s=\underline{\theta}}^{\theta}\, dl \\
&\qquad + \int_0^{\overline{L}} \int_{\underline{\theta}}^{\theta} (1 - g(H_s(l)))\, d\phi_l(s) dl - \int_0^{\overline{L}} [1 - g(H_\theta(l))] \frac{\partial R(l,\theta)}{\partial l} dl \\
&= \int_0^{\overline{L}} \left(1 - \frac{\partial R(l,\underline{\theta})}{\partial l}\right) [1 - g(H_{\underline{\theta}}(l))]\, dl + \int_0^{\overline{L}} \int_{\underline{\theta}}^{\theta} (1 - g(H_s(l)))\, d\phi_l(s) dl
\end{aligned}$$

where the third line follows from integration by parts and from the property that in the optimal mechanism type $\underline{\theta}$ is indifferent whether to participate or not. As $\partial R(l,\theta)/\partial l \in [0,1]$ and as ϕ is a positive measure, the above term is decreasing in g. Hence, for every fixed retention function, the premium is higher if the agent becomes more risk averse, while the expected cost (given the same retention function) is the same. Adjusting further to optimal retention for g_2 yields the desired result. ■

Proof for Theorem 2.

1. Since $0 \le I \le X$, it follows that $R_I \le X$ and hence that $F_{R_I}(l) \ge F_L(l)$ for all $l \ge 0$ where F denotes here the distribution of the respective random variable. Moreover, $F_{R_d}(l) = F_L(l)$ for $l < D$ and $F_{R_D}(l) = 1$ for $l \ge D$. Therefore F_{R_I} and F_{R_D} cross exactly once and the result follows by Theorem 3.A.44 in Shaked and Shanthikumar (point 3.A.59)

2. Note that I_C has exactly the same structure as R_D. Hence, the argument above

yields $I_C \leq_{cx} I$. By assumption, I and R_I are co-monotone random variables. Let $V(I)$ denote the agent's dual utility when he faces lottery given by I, where the utility function is *arbitrary.* By the comonotonic additivity of the dual utility, we have

$$V(L) = V(I) + V(R_I) = V(I_C) + V(R_{I_C})$$

If V represents a risk-averse agent, we obtain that that $V(I_C) \geq V(I)$ and hence that $V(R_{I_C}) \leq V(R_I)$. Since, by assumption, $EI = EI_C$, we obtain that $ER_I = ER_{I_C}$ Since the risk averse Yaari utility V was arbitrary, Theorem 3.A.7 in Shaked and Shanthikumar (due to Chateauneuf et al. [2004]) yields that $R_{I_C} \geq_{cx} R_I$.

■

Proof of Theorem 3.

1. Fix a type θ and consider the term

$$\int_0^{\overline{L}} J(\theta, l) \frac{\partial R(l, \theta)}{\partial l} dl = - \int_0^{\overline{L}} \frac{\partial J(\theta, l)}{\partial l} R(l, \theta) dl$$

   that is linear in R. The optimal $R^*(\cdot, \theta)$ must be an extreme point of the feasible set. In particular $\frac{\partial R^*(l,\theta)}{\partial l}$ exists almost everywhere and equals either 0 or 1. By the single-crossing assumption, we obtain that a maximum is obtained by setting $\frac{\partial R^*(l,\theta)}{\partial l} = 1$ for $l \leq l^*(\theta)$ and $\frac{\partial R^*(l,\theta)}{\partial l} = 0$ for $l \geq l^*(\theta)$. This yields the extreme point

$$R(l, \theta) = \begin{cases} l & \text{if } l < l^*(\theta) \\ l^*(\theta) & \text{otherwise} \end{cases}$$

   Note that this is equivalent to setting a deductible $D^*(\theta) = l^*(\theta)$. If the virtual value satisfies the monotonicity condition in the Theorem, then the overall obtained menu $\{D^*(\theta)\}_\theta$ is decreasing in θ. In particular, R is submodular and hence incentive compatible.

2. The proof follows as above by first observing that the relevant extreme point satisfies

$$R(l, \theta) = \begin{cases} 0 & \text{if } l < l^*(\theta) \\ l - l^*(\theta) & \text{otherwise} \end{cases}$$

   and hence

$$\frac{\partial}{\partial l} R(l, \theta) = \begin{cases} 0 & \text{if } l < l^*(\theta) \\ 1 & \text{if } l > l^*(\theta) \end{cases}.$$

   By the monotonicity assumption, we obtain that $l^*(\theta') \leq l^*(\theta)$ if $\theta' \leq \theta$. In particular, R is submodular, and hence incentive compatible.

$\blacksquare$

Proof of Proposition 3. It holds that

$$H_\theta(z) = \begin{cases} 1-\theta & \text{if } z < l_1 \\ 1-\theta+\theta\sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} \le z < l_k \text{ and } k \in \{2,...,n\} \\ 1 & \text{if } z \ge l_n \end{cases}$$

$$\text{and } \frac{\partial H_\theta(z)}{\partial\theta} = \begin{cases} -1 & \text{if } z < l_1 \\ -1+\sum_{i=1}^{k-1} p_i & \text{if } l_{k-1} < z < l_k \text{ and } k \in \{2,...,n\} \\ 0 & \text{if } z > l_n \end{cases}$$

In any incentive compatible mechanism, the menu of deductibles $D(\theta)$ is non-increasing in the probability of accident θ. In particular, $D(\theta)$ is continuous almost everywhere.

Fix such a non-increasing menu, and let $\theta_0 = \overline{\theta}$. Denote by $\theta_1 = \inf\{\theta : D(\theta) \le l_1\}$. If this set is empty, define $\theta_1 = \theta_0 = \overline{\theta}$. Similarly, for $i \in \{2,...,n\}$ define $\theta_i = \inf\{\theta : D(\theta) \le l_i\}$ with $\theta_i := \theta_{i-1}$ if the set is empty.

By the monotonicity of $D(\theta)$, it holds that $\underline{\theta} = \theta_n \le \theta_{n-1} \le ... \le \theta_1 \le \theta_0 = \overline{\theta}$. The insurer's profit becomes then

$$\begin{aligned}
\pi &= \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) + \int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} m(\theta) f(\theta) d\theta + \int_{\theta_1}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta \\
&\quad + \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta \\
&\quad + \int_{\underline{\theta}}^{\theta_n} \left[\int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial\theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} m(\theta) f(\theta) d\theta - U(\underline{\theta}) + \int_{\theta_1}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] dz \right] f(\theta) d\theta \\
&\quad + \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} \int_0^{D(\theta)} \left[\begin{array}{c} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i)\left(1-\sum_{i=1}^{k-1} p_i\right) \end{array} \right] dz\ f(\theta) d\theta \\
&= -\int_{\underline{\theta}}^{\overline{\theta}} m(\theta) f(\theta) d\theta - U(\underline{\theta}) + \int_{\theta_1}^{1} D(\theta)[g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] f(\theta) d\theta \\
&\quad + \sum_{k=2}^{n} \int_{\theta_k}^{\theta_{k-1}} D(\theta) \left[\begin{array}{c} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i)\left(1-\sum_{i=1}^{k-1} p_i\right)] \end{array} \right] f(\theta) d\theta
\end{aligned}$$

By definition, in each interval $[\theta_k, \theta_{k-1}]$, the given deductible $D(\theta)$ belongs to the interval $[l_{k-1}, l_k]$, where we denote $l_0 = 0$. Note that, on each interval $[\theta_k, \theta_{k-1}]$, the obtained expression for profit is linear in D :

$$\int_{\theta_k}^{\theta_{k-1}} D(\theta) \left[\begin{array}{l} g(1-\theta+\theta\sum_{i=1}^{k-1} p_i) - \left(1-\theta+\theta\sum_{i=1}^{k-1} p_i\right) \\ +\frac{1-F(\theta)}{f(\theta)} g'(1-\theta+\theta\sum_{i=1}^{k-1} p_i)\left(1-\sum_{i=1}^{k-1} p_i\right) \end{array} \right] f(\theta) d\theta$$

Depending on the sign of the integrand, the above expression is maximized with respect to D at an extreme point of the respective feasible set, i.e., either at $D^*(\theta) = l_{k-1}$ or at $D^*(\theta) = l_k$. Thus, the profit from the given mechanism can be increased by changing all deductibles $D(\theta)$ on the interval $[\theta_k, \theta_{k-1}]$ to the value of $D^*(\theta)$ that maximizes the above expression. The obtained D^* is non-increasing by construction, and thus also implementable. Hence, we have shown that the search for an optimal mechanism can be confined to menus consisting of at most $n+1$ deductibles, where each deductible equals either zero or one of the possible losses. ■

Proof of Corollary 1. Here

$$H_\theta(z) = \begin{cases} 1-\theta \text{ if } z < l \\ 1 \quad\quad \text{if } \; z \geq l \end{cases} \quad \text{and} \quad \frac{\partial H_\theta(z)}{\partial \theta} = \begin{cases} -1 \text{ if } z \leq l \\ 0 \;\text{ if } \; z \geq l \end{cases}$$

The insurer's profit becomes:

$$\begin{aligned} \pi &= \int_{\underline{\theta}}^{\overline{\theta}} \left[-m(\theta) + \int_0^{D(\theta)} [g(H_\theta(z)) - H_\theta(z) - \frac{1-F(\theta)}{f(\theta)} g'(H_\theta(z)) \frac{\partial H_\theta(z)}{\partial \theta}] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\ &= -l + \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_0^{D(\theta)} [g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta)] dz \right] f(\theta) d\theta - U(\underline{\theta}) \\ &= -l + \int_{\underline{\theta}}^{\overline{\theta}} D(\theta) \left[g(1-\theta) - (1-\theta) + \frac{1-F(\theta)}{f(\theta)} g'(1-\theta) \right] f(\theta) d\theta - U(\underline{\theta}) \end{aligned}$$

The above expression is linear in D, and hence the pointwise maximum in the above expression is attained at an extreme point of the feasible set: it can be either at $D = l$ or at $D = 0$, depending on the sign of the virtual value. ■

8 Appendix 3: Binary Lotteries

1. Gul's [1991] disappointment-averse preferences with linear utility over outcomes:

$$\mathcal{U}(x) = \frac{\alpha}{1+(1-\alpha)\beta} \mathbb{E}[x | x \geq CE(x)] + \frac{(1-\alpha)(1+\beta)}{1+(1-\alpha)\beta} \mathbb{E}[x | x < CE(x)]$$

where $CE(x)$ is a certainty equivalent of lottery $x \in X$, α is the probability that the outcome of the lottery is above its certainty equivalent, and β is a parameter. For binary lotteries, the above functional form is a special case of Yaari's dual utility with[36]

$$g\left(p\right) = \frac{p}{1+(1-p)\beta}$$

2. Versions of the disappointment aversion theories due to Loomes and Sugden [1986], and Jia et al. [2001] with linear utility over outcomes:

$$\mathcal{U}(x) = \mathbb{E}(x) + (e-d)\mathbb{E}\left[\max\left\{x - \mathbb{E}(x), 0\right\}\right]$$

where $e > 0$, $d > 0$. For binary lotteries, this is a special case of Yaari's dual utility with

$$g\left(p\right) = p(1+e-d) + (d-e)p^2$$

3. The modified Mean-Variance preferences (see Rockafellar et al. [2006]) with linear utility over outcomes are given by[37]:

$$\mathcal{U}(x) = \mathbb{E}(x) - \frac{1}{2}r\mathbb{E}\left[\mid x - \mathbb{E}(x)\mid\right]$$

where $r \in [0,1]$. For binary lotteries this is again a special case of Yaari's preferences where

$$g(p) = p - rp(1-p)$$

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[36] (Weak) risk aversion corresponds then to $\beta > \frac{1}{2}$ and aversion to mean-preserving spreads corresponds to $\beta > 0$.

[37] The modification relative to the standard mean-variance preferences is needed in order to ensure consistency with FOSD.

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