Procedural Expected Utility*

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March 14, 2023

Abstract

Individuals facing multi-dimensional risk needs to complete two steps in the decision-making process: aggregating deterministic outcomes and assessing risk in each dimension. The classical model, Expected Utility (EU), assumes that outcome aggregation occurs before risk assessment. In this paper, we provide a new axiomatic framework to examine the various procedures used to complete these two steps in different orders. Our framework maintains the Independence axiom within each dimension but allows for relaxation of it cross dimensions, leading to novel procedures that reverse or partially reverse the order of EU. We demonstrate the usefulness of these procedures through different applications.

Keywords: Multi-dimensional risk, narrow bracketing, time and risk preferences, time lotteries.

*This paper is a revised version of Chapter 1 of my dissertation at Princeton University and was previously circulated under the title "A Theory of Choice Bracketing under Risk". I am deeply indebted to my advisor, Faruk Gul, for his continued encouragement and support. I am grateful to Wolfgang Pesendorfer, Pietro Ortoleva and Xiaosheng Mu for many discussions which resulted in significant improvements of the paper. I would also like to thank David Ahn, Roland Bénabou, Tilman Börgers, Modibo Camara, Simone Cerreia-Vioglio, Sylvain Chassang, Xiaoyu Cheng, Joyee Deb, David Dillenberger, Andrew Ellis, Francesco Fabbri, Shaowei Ke, Shengwu Li, Annie Liang, Alessandro Lizzeri, Jay Lu, Yusufcan Masatlioglu, Dan McGee, Lasse Mononen, Evgenii Safonov, Ludvig Sinander, Rui Tang, Dmitry Taubinsky, João Thereze, Can Urgan, Leeat Yariv, Chen Zhao and participants at various seminars and conferences for useful comments and suggestions. All remaining mistakes are my own.

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1 Introduction

Many economic decisions require an understanding of uncertainty across multiple dimensions. For instance, investors must navigate risk across multiple financial markets, consumers face income risk over time, and home sellers are uncertain about both the sale price and the timing of the sale. The evaluation of multi-dimensional risk is a complex task, involving two distinct steps: first, aggregating deterministic outcomes, and second, assessing the risk present in each dimension. In this paper, we investigate the various procedures used to complete these two steps in different orders.

We study preferences over lotteries that have two-dimensional outcome profiles. These outcome profiles can represent various scenarios, such as income from two sources, consumption in two periods, and the size and the payment date of a prize. Our starting point is the observation that the standard model of decision making under risk, Expected Utility (EU), operates under the implicit assumption that the decision maker *first* evaluates the utility of each deterministic outcome profile, and *then* calculates its expectation based on the distribution of outcome profiles. Each lottery P is evaluated according to

$$V^{EU}(P) = \mathbb{E}_P(w(x, y)),$$

where w is a utility function over outcome profiles.

The implicit assumption of outcome aggregation and risk evaluation order in EU has important implications in various applications, which are being challenged by recent experimental evidence. For example, EU predicts that when faced with two independent monetary gambles, the individual will first combine them into one overall gamble and will never choose options that are first-order stochastically dominated. However, Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009) find that a significant portion of their subjects (between 28% and 66%) make dominated decisions in a pair of simple binary choice problems. Another example is in the context of time lotteries, where the prize amount is fixed but the payment date is uncertain. The individual is considered Risk Averse over Time Lotteries (RATL) if she prefers to receive the prize on a certain date rather than on a randomly determined date with the same expected delay. On the other hand, if the preference is the opposite, the individual is considered Risk Seeking over Time Lotteries (RSTL). DeJarnette et al. (2020) reveal a fundamental incompatibility between Stochastic Impatience, the risky counterpart of impatience, and even a single violation of RSTL in the EU model. However, using incentivized experiments, DeJarnette et al. (2020) show that most subjects exhibit RATL behavior in the majority of questions. These findings cast doubt on the predictions of EU when it comes to multi-dimensional risks.

In this paper, we present two novel evaluation procedures that address the aforementioned challenges. The first procedure, referred to as *Reverse Expected Utility* (REU), reverses the order of outcome aggregation and risk evaluation used in EU. For each given lottery P, let P_i denote its marginal lottery in dimension $i = 1, 2$. In REU, the decision maker first assesses risk in each dimension in isolation by converting it into a deterministic outcome, and then aggregates the two resulting outcomes. The utility of P is given by

$$V^{REU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big),$$

where $CE_{v_i}(P_i)$ is the certainty equivalent of marginal lottery P_i calculated using the utility function v_i for dimension $i = 1, 2$.

In the other new procedure, referred to as *Partially Reverse Expected Utility* (PREU), the order used in EU is partially reversed. In PREU, the decision maker first evaluates risk in dimension 2, then aggregates deterministic outcomes, and finally evaluates risk in dimension 1. Each lottery P is evaluated according to

$$V^{PREU}(P) = \mathbb{E}_{P_1}\Big(w(x, CE_v(P_{2|x}))\Big),$$

where $P_{2|x}$ is the conditional distribution in dimension 2 given x in dimension 1.

We say an agent has a *procedural risk preference* if she follows one of the above three evaluation procedures. Our main result is a representation theorem that characterizes procedural risk preferences with five axioms, three of which can be commonly found in the literature on choices under risk: (i) The preference is complete and transitive. (ii) The agent prefers more outcome in each dimension when there is no risk. (iii) The preference satisfies specific conditions for continuity.

The final two axioms relax the Independence axiom in the EU theory. The first one preserves Independence within each dimension, which, along with the other axioms, leads to the conclusion that for a fixed outcome in one dimension, the conditional preference defined over marginal lotteries in the other dimension can be represented by EU. As such, our paper complements the enormous body of literature focused on violations of EU *within* a single dimension, such as the Allais paradox. The last axiom weakens the Independence axiom cross dimensions, stating that the individual's ranking between two lotteries P and Q is not altered when they are mixed with equally appealing lotteries, as long as the mixture has no effect on the risk in P and Q that is assessed prior to outcome aggregation. We will discuss this axiom in detail in Section 2.3.

Our theory of procedural risk preferences is valuable in reconciling discrepancies between EU and evidence from various applications. In Section 3.1, we examine preferences regarding two monetary gambles and demonstrate that the PREU model can accommodate experimental findings on violations of first-order stochastic dominance (Tversky and Kahneman, 1981, Rabin and Weizsäcker, 2009). In PREU, the agent evaluates the gambles by taking the sum of their certainty equivalents. The narrow bracketing model frequently used to interpret experimental evidence may result in a decision maker choosing X dollars for sure over $Y > X$ dollars for sure when these amounts are suitably divided across the two income dimensions. This can lead to a theory that explains certain anomalies in the data, but at the expense of creating others that are unlikely to exist. By contrast, the agent in PREU would never choose less money for sure over more money for sure.

Section 3.2 studies the application to preferences over multi-period consumption under uncertainty. We first note that our three evaluation procedures correspond to three widely studied classes of models in the literature: (i) the generalized Expected Discounted Utility (EDU) model (Kihlstrom and Mirman, 1974, Dillenberger, Gottlieb, and Ortoleva, 2020), (ii) the Dynamic Ordinal Certainty Equivalent (DOCE) model (Selden, 1978, Selden and Stux, 1978), and (iii) the model of Kreps and Porteus (1978). Then we show that a modified two-period model of Epstein and Zin (1989) (EZ) can emerge from the PREU model, which separates time and risk preferences and exhibits indifference to temporal resolution of uncertainty. By contrast, Epstein, Farhi, and Strzalecki (2014) point out that fitting the EZ model to data requires an overly strong preference for early resolution of risk.

In our final application, we study preferences over lotteries in which both the monetary prize and the payment date are uncertain in Section 3.3. As previously noted, EU is unable to accommodate Stochastic Impatience and violations of RSTL simultaneously. The EDU model, in particular, requires all subjects to have RSTL preferences. To see this, note that the utility of time lottery (x, p) is $\mathbb{E}_p[\beta^t]u(x)$, where $u(x) > 0$ is the utility of prize x and $\beta \in (0, 1)$ is the discount factor, whereas the utility of receiving x at expected time $t' = \mathbb{E}_p(t)$ is $\beta^{t'}u(x)$. Since β^t is convex in t, the former option will always be preferred. However, our PREU model offers a solution to this inconsistency, as the evaluation of risk in the time dimension is done before aggregating the size and payment time of the prize, allowing for separation of risk attitudes toward time and intertemporal trade-offs.

Our axiomatic approach uncovers a novel connection between three seemingly disparate challenges to the EU theory in different choice domains: the violation of first-order stochastic dominance, the conjunction of time and risk preferences, and the incompatibility between Stochastic Impatience and violations of RSTL. We show that these challenges arise from the implicit order of outcome aggregation and risk evaluation in EU, and can be addressed by preserving the Independence axiom within each dimension while relaxing it across dimensions

Related Literature. Our paper highlights the importance of the order in which decision makers aggregate outcomes and assess risk when facing multi-dimensional uncertainty, a topic that has garnered increasing attention in both theoretical and experimental literature. For instance, DeJarnette et al. (2020) suggest that switching the order of aggregation in the EU model can resolve the inconsistency between Stochastic Impatience and violations of RSTL. Andreoni, Feldman, and Sprenger (2017) experimentally test predictions for behavior generated by different orders of evaluation in the Cumulative Prospect Theory. In these papers, the order is given as a primitive, while our paper provides a axiomatic theory that endogenizes the order adopted in the decision-making process.[1]

Our observation that risk in different dimensions are treated in different ways is related to the literature on *source-dependent* preferences following Tversky and Fox (1995) and Tversky and Wakker (1995).[2] The closest work to the present paper is Cappelli et al. (2021), where the decision maker faced with multi-source risk first computes source-dependent certainty equivalents, converts them into the unit of account of a numeraire, and then aggregates them into the overall evaluation. This can be interpreted as an extension of our REU model to a setting with multiple dimensions, subjective uncertainty and certainty equivalents based on non-EU models. Our paper complements Cappelli et al. (2021) by focusing on how this evaluation procedure departs from the EU theory by reversing the order of outcome aggregation and risk assessment, and by proposing a novel procedure where the order is partially reversed.

Our paper makes a contribution to the literature on narrow bracketing, which has been the subject of growing interest in recent years, driven by both theoretical results (Barberis, Huang, and Thaler, 2006, Mu et al., 2021a) and experimental findings (Rabin and Weizsäcker, 2009, Ellis and Freeman, 2021). Various models have been proposed to provide rationales for narrow bracketing, such as the model of Kőszegi and Matějka (2020) based on rational inattention, or Lian (2020), which assumes that the decision maker has different, imperfect information in different decisions. Two closely related works are Vorjohann (2021) and Camara (2021), both of which begin with the expected utility paradigm and model narrow bracketing with an additively separable utility function. In Vorjohann (2021), the decision maker is characterized by broad and narrow EU preferences, the connection between which yields testable implications. Camara (2021) focuses on high-dimensional decisions and shows that computational tractability imposes restrictions on the utility function. In contrast, we model narrow bracketing as a deviation from the EU theory, in which the decision maker adds the certainty

[1]Our focus on evaluation procedures is also reminiscent of the growing literature on bounded rational choice procedures following Kalai, Rubinstein, and Spiegler (2002), Manzini and Mariotti (2007) and Masatlioglu, Nakajima, and Ozbay (2012).

[2]The recent interest in source-dependent preferences was sparked by the Ellsberg paradox and has led to a vast body of literature. See, for instance, Ergin and Gul (2009), Gul and Pesendorfer (2015), Chew and Sagi (2008), Qiu and Ahn (2021) and references therein.

equivalents, instead of the expected utilities, of two monetary gambles.[3] We will discuss this distinction in detail in Section 3.1.

Two features of our PREU model of narrow bracketing bear resemblance to the recent literature. First, the decision maker maximizes a sum of certainty equivalents, which is a decision criterion that has been advocated by Myerson and Zambrano (2019) as an effective rule for risk sharing, and axiomatized by Chambers and Echenique (2012) as a social welfare functional. It also appears in the monotone additive statistics characterized by Mu et al. (2021b). Second, the decision maker ignores correlation between risk in different dimensions, which echoes the experimental evidence of correlation neglect in belief formation (Enke and Zimmermann, 2019), portfolio allocation (Eyster and Weizsäcker, 2016, Kallir and Sonsino, 2009) and school choice (Rees-Jones, Shorrer, and Tergiman, 2020).

2 The Model

2.1 Framework

Consider a decision maker faced with risk from two dimensions $i = 1, 2$. Let X_i denote

Let $X = X_1 \times X_2$, where X_i is the space of outcomes in dimension $i \in \{1, 2\}$. For $i \in \{1, 2\}$, we denote by $-i \neq i$ and $-i \in \{1, 2\}$. We assume that X_i is a nontrivial compact interval $[\underline{c}_i, \overline{c}_i] \subset \mathbb{R}$. We call $(x_1, x_2) \in X$ an *outcome profile*, where x_i is the outcome in dimension i for $i \in \{1, 2\}$. A *(joint) lottery*, P, is a probability distribution over X with a finite support. We endow the set of all lotteries, $\mathcal{P} := \Delta(X)$, with the topology of weak convergence and the standard mixture operation.[4] For each lottery $P \in \mathcal{P}$, the *marginal lottery* of P in dimension 1 is denoted $P_1 \in \mathcal{P}_1 := \Delta(X_1)$. That is, $P_1(x) = \sum_{y \in X_2} P(x, y)$ for all $x \in X_1$. The marginal lottery $P_2 \in \mathcal{P}_2 := \Delta(X_2)$ in dimension 2 is defined similarly. Let $\mathcal{P}_1 \times \mathcal{P}_2 \subset \mathcal{P}$ denote the set of *product lotteries*, that is, lotteries

[3] Camara (2021) also notes that using approximation algorithms inconsistent with the EU theory can make the decision maker objectively better off. This can serve as an alternative justification of our approach.

[4] For lotteries $P, Q \in \mathcal{P}$ and $\alpha \in [0, 1]$, the α-mixture of P and Q, denoted by $\alpha P + (1 - \alpha)Q$, is a lottery where $(\alpha P + (1 - \alpha)Q)(x, y) = \alpha P(x, y) + 1 - \alpha)Q(x, y)$ for all $(x, y) \in X$.

of which the two marginal lotteries are independent. For each lottery $P \in \mathcal{P}$, the pair $(P_1, P_2) \in \mathcal{P}$ is a product lottery with the same marginal lotteries as P. We denote by $(x_1, x_2), (y_1, y_2), (x, y)$ generic elements of X, denote by p, q, r generic elements of $\mathcal{P}_1 \cup \mathcal{P}_2$, and denote by P, Q, R generic elements of $\mathcal{P}$. Let $\delta_{(x_1,x_2)}$ be the degenerate lottery (Dirac measure at (x_1, x_2)) that yields the outcome profile $(x_1, x_2) \in X$ with probability 1. When there is no confusion, we also use (x_1, x_2) to denote the degenerate lottery $\delta_{(x_1,x_2)}$ and interpret X as a subset of $\mathcal{P}$.

The primitive of our analysis is a binary relation $\succsim$ on $\mathcal{P}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively.

2.2 Three Evaluation Procedures

In this section, we introduce three procedures adopted by an individual to evaluate lotteries. Since the outcome space is two-dimensional, the individual needs to complete two tasks: (i) aggregation of deterministic outcomes and (ii) evaluation of risk in two dimensions. Different orders of completing the two tasks lead to different utility representations. We start with the standard expected utility model.

Definition 1 (EU). A binary relation $\succsim$ admits an *expected utility (EU)* representation if it can be represented by V^{EU} such that for any $P \in \mathcal{P}$,

$$V^{EU}(P) = \sum_{(x,y)\in X} w(x, y)P(x, y) \tag{1}$$

where w is a continuous and strictly increasing utility index on X.[5] We say w is an EU representation of $\succsim$.

In the EU model, the decision maker is assumed to first evaluate the utility of each deterministic outcome profile and then compute its expectation according to the distribution of outcome profiles. While the expected utility model is widely recognized as the normative benchmark when risk is one-dimensional, it entails a

[5] A function $V : \mathcal{P} \to \mathbb{R}$ is said to represent the binary relation $\succsim$ when $P \succsim Q$ if and only if $V(P) \geq V(Q)$ for all $P, Q \in \mathcal{P}$. A function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ for some positive integer n is strictly increasing if $f(x) > f(y)$ whenever $x, y \in A$, $x \geq y$ and $x \neq y$.

specific order of aggregating deterministic outcomes and evaluating risk when risk is multi-dimensional. As we will discuss in detail in Section 3, such an order cannot accommodate narrow bracketing and dominated choices observed in experiments with monetary gambles, fails to separate time and risk preferences, and implies that stochastically impatient agents must be risk seeking over time lotteries. In the rest of this section, we will introduce two novel evaluation procedures where the order is either completely or partially reversed compared to the one used in the expected utility model.

First, we consider the case where the order is completely inverted; that is, the decision maker first assesses risk in each dimension in isolation by reducing it to a deterministic outcome, and then aggregates the two deterministic outcomes.

Definition 2 (REU)**.** A binary relation $\succsim$ admits a *Reverse Expected Utility (REU)* representation and can be represented by V^{REU} such that for any $P \in \mathcal{P}$,

$$V^{REU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big) \tag{2}$$

where w, v_1 and v_2 are continuous and strictly increasing utility indices on X, X_1 and X_2 respectively. We say (w, v_1, v_2) is a REU representation of $\succsim$.

For a given lottery P, the decision maker calculates the certainty equivalent $CE_{v_i}(P_i)$ of the marginal lottery P_i in dimension $i = 1, 2$ using some dimension-specific expected utility index v_i. Here the certainty equivalent $CE_{v_i}(P_i)$ is the unique outcome $x_i \in X_i$ that solves $v_i(x) = \sum_z v_i(z)P_i(z)$. In other words, the decision maker is indifferent between the deterministic outcome $CE_{v_i}(P_i)$ and the risky prospect P_i in dimension i. In the REU model, since the decision maker only takes into account the interaction between two dimensions after evaluating the risk, she ignores the correlation between risks in different dimensions and her attitude toward marginal risk in one dimension does not depend on the outcome or marginal risk in the other dimension. This is in sharp contrast with the EU model and has strong testable implications.

Then, we consider the case where the order is only partially reversed. The decision maker first assesses risk in one dimension, then aggregates deterministic outcomes, and finally assesses risk in the other dimension. The implied model is inherently asymmetric. For the main part of this paper, we assume that the decision maker evaluates risk in dimension 2 before that in dimension 1. This is motivated by asymmetry built in many applications where there is a natural order on dimensions such as time. In Section 4.2, we will briefly discuss the other case where risk in dimension 2 is evaluated first and how our characterization results can be generalized. For each $P \in \mathcal{P}$ and $x \in X_1$ such that $P(x, y) > 0$ for some $y \in X_2$, let $P_{2|x}$ denote the conditional distribution of the outcome in dimension 2 given outcome x in dimension 1.

Definition 3 (PREU)**.** A binary relation $\succsim$ admits a *Partially Reverse Expected Utility (PREU)* representation if it is represented by V^{PREU} such that for any $P \in \mathcal{P}$,

$$V^{PREU}(P) = \sum_{x \in X_1} w(x, CE_v(P_{2|x}))P_1(x) \tag{3}$$

where w and v are continuous and strictly increasing utility indices on X and X_2 respectively. We say (w, v) is a PREU representation of $\succsim$

In the PREU model, the decision maker's evaluation of a lottery P can be decomposed into three steps. In the first step, she calculates the certainty equivalent $CE_v(P_{2|x})$ of distribution of the outcome in dimensional 2 conditional on each possible outcome x in dimension 1. Then, she aggregates each pair of the outcome in dimension 1 and the corresponding conditional certainty equivalent in dimension 2. In the final step, she evaluates the risk in dimension 1 using expected utility. It is worthwhile to note that the decision maker's assessment of conditional risk in dimension 2 is independent of the outcome in dimension 1, unlike in a generic EU model. We elaborate this point in Remark 2 below.

As a summary, we say that a binary relation $\succsim$ is a *procedural risk preference* if $\succsim$ admits one of the three representations in Definitions 1-3: EU, REU and PREU. We end this section with several remarks.

Remark 1 (*Alternative formulation of the REU representation*)*.* One can rewrite the REU representation (2) as

$$V^{REU}(P) = \hat{w}\Big(\mathbb{E}_{v_1}[P_1], \mathbb{E}_{v_2}[P_2]\Big) \tag{4}$$

for all $P \in \mathcal{P}$, where $\mathbb{E}_{v_i}[P_i] = \sum_{z \in X_i} v_i(z) P_i(z)$ is the expected utility of marginal lottery P_i under index v_i in dimension $i = 1, 2$, and $\hat{w}$ is defined on $v_1[X_1] \times v_2[X_2]$ such that

$$\hat{w}(a_1, a_2) = w(v_1^{-1}(a_1), v_2^{-1}(a_2))$$

for all $(a_1, a_2) \in v_1[X_1] \times v_2[X_2]$.[6] When there is no confusion, we also write $\mathbb{E}_{P_i}(v_i) = \mathbb{E}_{v_i}[P_i]$. The formulation (2) is adopted in this paper since w can be interpreted as the utility function when there is no risk and it captures how the decision maker evaluates deterministic outcomes in two dimensions, independent of her attitude toward risk. As we will argue in Section 3, this interpretation is helpful in applications where there is a natural functional form of w. Examples include summing up money from different income sources and additive utility with exponential discounting when there are multiple periods.

Remark 2 (*Comparison between the PREU and the EU representations*). The EU representation (1) can be rewritten in a way comparable to the PREU representation (3):

$$V^{EU}(P) = \sum_{x \in X_1} w(x, CE_{v_x}(P_{2|x})) P_1(x) \tag{5}$$

for all $P \in \mathcal{P}$, where for each $x \in X_1$, the utility index v_x is a function defined on X_2 such that $v_x(y) = w(x, y)$ for all $y \in X_2$. The only difference between equations (3) and (5) is whether the evaluation of conditional risk in dimension 2 depends on the outcome in dimension 1. In the formulation (5), utility index v_x generically depends on x since risk is evaluated *after* aggregation of deterministic outcomes in two dimensions. By contrast, a decision maker whose preference admits a PREU representation assesses risk in dimension 2 *before* aggregating outcomes. Hence, she does not take into account the effect of the outcome in dimension 1 on her attitude toward risk in dimension 2.

Remark 3 (*Comparison between the PREU and the REU representations*). Unlike the REU model, the decision maker whose preference admits a PREU represen-

[6] For a function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ for some positive integer n, we denote its image by

$$f[X] := \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in X\}.$$

tation takes into account the fact that different realizations of the outcome in dimension 1 imply different risky prospects in dimension 2 and she does not ignore correlation. This proves to be useful in the application to multi-period consumption, where correlation aversion has strong experimental support (Andersen et al., 2018, Lanier et al., 2020).

Remark 4 (*Connection with narrow bracketing*). If one interprets the EU model as a rational benchmark, then the other two models can be interpreted as behavioral heuristics adopted by the decision maker to simplify the evaluation of multi-dimensional risk, in the sense that risk in either one or both dimensions is evaluated in isolation. This is reminiscent of the notion of narrow bracketing in psychology and behavioral economics. Narrow bracketing, introduced and formalized by Tversky and Kahneman (1981), Thaler (1985) and Read, Loewenstein, and Rabin (1999), describes a situation where a decision maker faced with multiple choice problems attempts to solve each in isolation, without taking account of the interactions between the problems. Following this notion, one can argue that a decision maker narrowly brackets risk in both dimensions if her preference admits a REU representation, and in one dimension if her preference admits a PREU representation. We will discuss the connection with narrowly bracketing in more detail in Section 3.1.

2.3 Axioms and the Representation Theorem

The main result of this paper is a representation theorem that behaviorally characterizes the three evaluation procedures in Definitions 1-3 with axioms. The decision maker's preference $\succsim$ satisfies those axioms if and only if it is a procedural risk preference

The first basic axiom we impose is rationality.

Axiom 1—Weak Order: The relation $\succsim$ is complete and transitive.

The next axiom states the more outcome is better than less when there is no risk.

Axiom 2—Monotonicity: For any $(x_1, x_2), (y_1, y_2) \in X$, if $x_1 \geq y_1$, $x_2 \geq y_2$

and $(x_1, x_2) \neq (y_1, y_2)$, then $(x_1, x_2) \succ (y_1, y_2)$.

Our continuity axiom combines two weaker notions of the Strong Continuity axiom commonly adopted in the expected utility theory with monetary prizes: Axiom 3.1 states that $\succsim$ is continuous in probabilities, and Axiom 3.2 states that $\succsim$ is continuous in outcomes. By comparison, the Strong Continuity axiom would require $\succsim$ to be *jointly* continuous in both probabilities and outcomes.[7]

Axiom 3.1—Mixture Continuity: For any $P, R, Q \in \mathcal{P}$, the sets $\{\alpha \in [0,1] : \alpha P + (1-\alpha)Q \succsim R\}$ and $\{\alpha \in [0,1] : R \succsim \alpha P + (1-\alpha)Q\}$ are closed.

Axiom 3.2, below, asserts that if changing every outcome in the support of P by $(\varepsilon_1, \varepsilon_2)$ renders P better (worse) than Q, for any $\varepsilon_1, \varepsilon_2$ sufficiently close to 0, then P must also be better (worse) than Q. Since the outcome space X is bounded, we need to deal with the possibility that increasing (decreasing) some of the outcomes in the support of P may not be feasible. Formally, for any $P \in \mathcal{P}$, choose $\eta > 0$ small enough so that $P(x,y) \cdot P(x', y') > 0$, $|x - x'| \leq \eta$ and $|y - y'| \leq \eta$ implies $(x, y) = (x', y')$. Then, for $\varepsilon = (\varepsilon_1, \varepsilon_2)$ such that $\varepsilon_1, \varepsilon_2 \in (-\eta, \eta)$, define $\phi : X \to X$ as follows: $\phi_\varepsilon(x,y) = (x + \varepsilon_1, y + \varepsilon_2)$ if $(x + \varepsilon_1, y + \varepsilon_2) \in X$; otherwise, $\phi_\varepsilon(x, y) = (x', y')$ such that (x', y') is the element of X closest to (x, y) with respect to the distance $d((x,y),(x',y')) = |x - x'| + |y - y'|$. Since we have chosen $\varepsilon_1, \varepsilon_2$ sufficiently small, the restriction of ϕ to the support of P is one-to-one. Then, define $P_\varepsilon \in \mathcal{P}$ as follows

$$P_\varepsilon(\phi_\varepsilon(x,y)) = P(x,y)$$

if $P(x,y) > 0$ and set $P_\varepsilon(x', y') = 0$ if $(x', y') \neq \phi_\varepsilon(x, y)$ for any (x, y) in the support of P.

Axiom 3.2—Continuity in Outcomes: For any $P, Q \in \mathcal{P}$ and sequence $(\varepsilon^n)_{n \geq 1}$ in $\mathbb{R}^2$ that converges to $(0,0)$, if $P_{\varepsilon^n} \succsim Q$ for all n, then $P \succsim Q$. If $Q \succsim P_{\varepsilon^n}$ for all n, then $Q \succsim P$.

We will refer to the conjunction of the two notions above as **Continuity**.

[7] Formally, the Strong Continuity axiom states that for any $Q \in \mathcal{P}$, the sets $\{P \in \mathcal{P} : P \succsim Q\}$ and $\{P \in \mathcal{P} : Q \succsim P\}$ are closed.

Axiom 3—Continuity: The relation $\succsim$ satisfies Axioms 3.1 and 3.2.

Note that Axiom 3 is implied by the Strong Continuity axiom. However, the reverse is not true. Although the EU and the REU representations satisfy the Strong Continuity axiom, the PREU representation might violate it since a infinitesimal perturbation in the lottery can induce a non-negligible change in the conditional lotteries in dimension 2, which can lead to a noncontinuous change in the utility level.[8]

The above Axioms 1-3 are either the same as, or natural adaptations of standard axioms in the literature on choices under risk. To establish the representation theorem, we need two more axioms which relax the Independence axiom in the expected utility theory.

The Independence axiom states that the decision maker's ranking between two lotteries is not reversed when they are mixed with the same lottery. Our first relaxation asserts that this continues to hold within each dimension.

Axiom 4—Within-Dimension Independence: For any $P, Q, R \in \mathcal{P}_1 \times \mathcal{P}_2$ such that $P_i = Q_i = R_i$ for some $i \in \{1, 2\}$, if $P \succ Q$, then for any $\alpha \in (0, 1)$, we have $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)R$.

To interpret Axiom 4, we first denote by $\succsim_{i|p}$ the conditional preference over marginal lotteries $\mathcal{P}_i$ in dimension i given marginal lottery $p \in \mathcal{P}_{-i}$ in the other dimension. Formally, for each $p \in \mathcal{P}_1$ and $r, r' \in \mathcal{P}_2$, we define the conditional preference $\succsim_{2|p}$ such that $r \succsim_{2|p} r'$ if and only if $(p, r) \succsim (p, r')$. The conditional preference $\succsim_{1|q}$ in dimension 1 can be similarly defined for each $q \in \mathcal{P}_2$. Then a preference $\succsim$ satisfies Within-Dimension Independence if and only if the conditional preference $\succsim_{i|p}$ satisfies the standard Independence axiom on the set of marginal lotteries $\mathcal{P}_{-i}$ for all $p \in \mathcal{P}_i$ and $i = 1, 2$. Along with other axioms, this implies each conditional preference $\succsim_{i|p}$ admits an expected utility representation. Hence, unlike most existing non-expected utility models in the literature, we

[8]For instance, consider a PREU representation where $X_1 = X_2 = [0, 10]$, $w(x, y) = \sqrt{x+y}$ and $v(x) = x$ for all $x, y \in [0, 10]$. Denote by P^n and P where $P^n(1 - \frac{1}{n}, 0) = P^n(1, 2) = \frac{1}{2}$ and $P(1, 0) = P(1, 2) = \frac{1}{2}$ for each $n \geq 1$. Easy to see that P^n converges to P in the weak topology as n goes to infinity. However, the utility of P^n is $U(P^n) = \frac{1}{2}\sqrt{1 - \frac{1}{n}} + \frac{1}{2}\sqrt{3}$, which converges to $\frac{1}{2}(1 + \sqrt{3}) \neq U(P) = \sqrt{2}$.

maintain the Independence axiom within each dimension. This allows us to focus on the behavioral implications of different orders of risk evaluation and outcome aggregation.[9]

Before stating our last axiom, we introduce some notation. For $i = 1, 2$, we say two marginal lotteries $r, r' \in \mathcal{P}_i$ are *comparable in dimension i* if there exists some deterministic outcome $x \in X_{-i}$ such that $r \sim_{i|x} r'$. In other words, marginal lotteries r and r' are comparable in dimension i if and only if r is neither always strictly better nor always strictly worse than r' in dimension i. For instance, two marginal lotteries are never comparable if one first-order stochastically dominates the other. Two lotteries $P, Q \in \mathcal{P}$ are *comparable* if P_i and Q_i are comparable in dimension i for both $i = 1, 2$. For $i = 1, 2$, we say two marginal lotteries $r, r' \in \mathcal{P}_i$ are *(mutually) singular*, denoted by $r \perp r'$, if $\text{supp}(r) \cap \text{supp}(r') = \emptyset$.[10] Sometimes we also say that r is singular with respect to r'.

Axiom 5—Cross-Dimension Independence: For any $P, Q, R, S \in \mathcal{P}$ such that $P_1 \perp R_1$, $Q_1 \perp S_1$, and P and Q are comparable, if $P \succ Q$ and $R \sim S$, then for any $\alpha \in (0, 1)$, we have $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)S$.

Axiom 5 states that the decision maker's ranking between two risky prospects P and Q is not reversed when they are mixed with equally attractive lotteries, so long as the mixture has no impact on the risk in P and Q that is evaluated before the decision maker aggregates outcomes across two dimensions. This additional qualification is guaranteed by two conditions. The first one requires P and Q to be comparable. If the decision maker assesses risk in both dimensions before aggregating outcomes, then the primitive that the decision maker strictly prefers P to Q does not hold, since the comparability of P and Q implies she must deem the two lotteries equally attractive. Hence, the evaluation procedure associated with the REU representation trivially satisfies Axiom 5.

The second condition demands that the marginal lotteries of two mixed lot-

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[9] It would be interesting to further relax Axiom 4 to accommodate behavioral factors like the certainty effect and probability weighting. We leave it for future research.

[10] In measure theory, two measures μ and ν on the same measure space Ω are singular if there are two measurable sets A and B such that $A \cup B = X$, $A \cap B = \emptyset$ and $\mu(A) = \nu(B) = 0$. When μ and ν are finitely supported (like marginal lotteries in our setting), it is equivalent to requiring their supports to be disjoint.

teries (i.e., P and R, Q and S) in dimension 1 must have disjoint supports. It guarantees that the mixture has no effect on risk in dimension 2, which is evaluated before aggregation of outcomes if the preferences admits either an EU or a PREU representation. To see this, consider the mixture of P and R. For any outcome $x \in \text{supp}(P_1)$, the conditional risk of P in dimension 2 is given by $P_{2|x}$. Since $x \notin \text{supp}(Q_1)$ by assumption, the conditional risk of $\alpha P + (1-\alpha)R$ in dimension 2 is still $P_{2|x}$, unaffected by the mixture. As a result, Axiom 5 imposes an additional independence property across dimensions when the decision maker does not evaluates risk in dimension 1 before aggregating outcomes.

Our representation theorem states the decision maker adopts one of the three evaluation procedures if and only if her preference satisfies the above five axioms.

Theorem 1: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-5 if and only if $\succsim$ is a procedural risk preference.*

Theorem 1 characterizes the common behavioral implications of three distinct evaluation procedures that the decision maker might adopt when faced with two-dimensional risk. Each procedure features a different order of completing two tasks: aggregation of deterministic outcomes and assessment of risk in two dimensions. To test whether the decision maker's choice behavior is consistent with one of the three procedures without knowing the specific order, an outside observer just needs to check the five axioms. In Section 4.1, we impose the additional axioms to identify each evaluation procedure and discuss the connection between them.

2.4 Proof Sketch of Theorem 1

In what follows, we sketch the proof of Theorem 1; a complete proof appears in Appendix B. We focus here only on the sufficiency of the axioms for the representations.

Step 1. Characterize when $\succsim$ admits a REU representation. We start by observing that each conditional preference must admit an EU representation. If the decision maker is indifferent between any two comparable lotteries, then she must neglect correlation in the sense that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$, as P and

(P_1, P_2) are comparable. We also show that the conditional preference $\succsim_{i|q_{-i}}$ in dimension i is independent of $q_{-i} \in cP_{-i}$. This guarantees a REU representation of $\succsim$.

Step 2. Implications of axioms when the decision maker has a strict ranking between some comparable lotteries. We show that the "local" independence property in Axiom 5 can be strengthened to the following stronger version by dropping the comparability requirements:

Axiom 5*. For any $P, Q, R, S \in \mathcal{P}$ such that $P_1 \perp R_1$ and $Q_1 \perp S_1$, if $P \succ Q$ and $R \sim S$, then for any $\alpha \in (0, 1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Step 3. Representation of $\succsim$. Define $U : \mathcal{P} \to [0, 1]$ such that $P \sim U(P)\delta_{(\overline{c}_1,\overline{c}_2)}+ (1 - U(P))\delta_{(\underline{c}_1,\underline{c}_2)}$ for all $P \in \mathcal{P}$. We show that U is well-defined and represents $\succsim$. As an implication of Axiom 5*, the utility function U satisfies the linearity condition that $U(\alpha P + (1-\alpha)R) = \alpha U(P) + (1-\alpha)U(R)$ for all $P, R \in \mathcal{P}$ such that $P_1 \perp R_1$. Hence, $U(P) = \sum_{x \in X_1} U(x, P_{2|x})P_1(x)$ for all $P \in \mathcal{P}$. Since $\succsim_{2|x}$ admits an EU representation, we can find utility indices w and v_x for each x such that for all $P \in \mathcal{P}$,

$$U(P) = \sum_x w(x, CE_{v_x}(P_{2|x}))P_1(x).$$

We note that this is exactly equation (5) where v_x can arbitrarily depend on x.

Step 4. Characterize when $\succsim$ admits a PREU representation. We show that if $\succsim_{2|x}$ is independent of $x \in X_1$, that is $v_x \equiv v$ for some utility index v, then the representation derived in Step 3 reduces to a PREU representation.

Step 5. Show that if $\succsim$ does not admit a REU or a PREU representation, then it must admit an EU representation. Our Axiom 4 turns out to entail strong implications for the representation in Step 3. For instance, for any $p, q \in \mathcal{P}_1$ and $\alpha \in (0, 1)$, we can adopt Harsanyi (1955)'s utilitarianism theorem to show that the utility index of conditional preference $\succsim_{2|\alpha p+(1-\alpha)q}$ must be a convex combination of those of conditional preference $\succsim_{2|p}$ and $\succsim_{2|q}$. Indeed, we establish that v_x must be a positive affine transformation of $w(x, \cdot)$ for all $x \in X_1$ and hence $\succsim$ admits an EU representation with utility index w.

2.5 Uniqueness

We now discuss the uniqueness properties of our representations. Consider an arbitrary set $A \subseteq \mathbb{R}^n$ for some positive integer n and two functions f, g defined on A. We say f is a *monotone* transformation of g, denoted by $f \propto^m g$, if there exists a continuous and strictly increasing function ϕ defined on $g(A)$ such that $f(x) = \phi(g(x))$ for all $x \in A$. We say f is a *positive affine* transformation of g, denoted by $f \propto^a g$, if there exist $b > 0$ and $c \in \mathbb{R}$ such that $f(x) = bg(x) + c$ for all $x \in A$. Our uniqueness results directly follow from those in the expected utility theory.

Proposition 1: *Let $\succsim$ be a binary relation on $\mathcal{P}$.*
(i) Two functions w and w' are EU representations of $\succsim$ if and only if $w \propto^a w'$.
(ii) Two tuples (w, v_1, v_2) and (w', v'_1, v'_2) are REU representations of $\succsim$ if and only if $w \propto^m w'$ and $v_i \propto^a v'_i$ for both $i = 1, 2$.
(iii) Two tuples (w, v) and (w', v') are PREU representations of $\succsim$ if and only if $w \propto^a w'$ and $v \propto^a v'$.

3 Applications

3.1 Multi-source Income

This section considers a decision maker who receives income from two different sources, such as salary and investment returns. or two monetary gambles. We characterize the procedural risk preference in this setting, and show that the decision maker might make dominated decisions as observed by Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009). Notably, unlike models of narrow bracketing in Vorjohann (2021) and Camara (2021), our decision maker never chooses less money over more money when there is no risk.

Suppose that $X_1 = X_2 = Z := [-\bar{x}, \bar{x}]$ for some $\bar{x} > 0$. An outcome $x \in Z$ is a monetary prize and represents a loss if it takes a negative value. For each pair of income levels $(x_1, x_2) \in X$, the final wealth is $x_1 + x_2 \in 2Z := [-2\bar{x}, 2\bar{x}]$. Each lottery $P \in \mathcal{P}$ represents a joint distribution of income levels from two sources, and induces a distribution over final wealth, denoted by $f[P]$. Formally, $f[P] \in \Delta(2Z)$

is derived by taking the sum of two monetary prizes in each realization, that is, the probability of each final wealth level $z \in 2Z$ is $f[P](z) = \sum_{(x,y):x+y=z} P(x,y)$. For any two distributions $p, q \in \Delta(2Z)$, we say that p *(first-order) stochastically dominates* q, denoted by $p \succ_{FOSD} q$, if $p \neq q$ and $\sum_{x \leq z} q(x) \geq \sum_{x \leq z} p(x)$ for all $z \in 2Z$. The following axiom states that the decision maker's preference should respect stochastic dominance in distributions over final wealth.

Axiom 6—Dominance: For any $P, Q \in \mathcal{P}$, if $f[P] \succ_{FOSD} f[Q]$, then $P \succ Q$.

Intuitively, a decision maker who only cares about final wealth and prefers more money to less would choose P over Q if $f[P] \succ_{FOSD} f[Q]$, since P assigns higher probability to higher levels of final wealth than Q. Hence, Axiom 6 is a desirable normative property commonly assumed in the literature. However, experimental evidence shows that many subjects violate it. Consider the following experiment studied in Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009).

Example 1. Suppose you face the following pair of concurrent decisions. All risks are independent from each other and will be resolved simultaneously. First examine both decisions, and then indicate your choices. Both choices will be payoff-relevant, i.e., the gains and losses will be added to your overall payment.

Decision 1: Choose between:

A. A sure gain of \$2.40.

B. A 25 percent chance to gain \$10.00, and a 75 percent chance to gain \$0.

Decision 2: Choose between:

C. A sure loss of \$7.50.

D. A 75 percent chance to lose \$10.00, and a 25 percent chance to lose \$0.

Across different treatments in Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009), a large fraction (ranging from 28 percent to 66 percent) of subjects choose A in decision 1 and D in decision 2. However, the distribution of final wealth resulting from the combination of A and D is stochastically dominated by that resulting from the combination of B and C:

$$f[(B,C)] = \frac{3}{4}\delta_{-7.50} + \frac{1}{4}\delta_{2.50} \succ_{FOSD} \frac{3}{4}\delta_{-7.60} + \frac{1}{4}\delta_{2.40} = f[(A,D)].$$

Such violations of Axiom 6 are stark since the combination of B and C is equal to the combination of A and D plus a sure payoff of \$0.10. Rabin and Weizsäcker (2009) also find that when subjects are asked to make a single decision by choosing from $f[(A,C)]$, $f[(A,D)]$, $f[(B,C)]$ and $f[(A,D)]$, the violation rates (i.e., the fraction of people choosing $f[(A,D)]$) are reduced to 0 percent and 6 percent, respectively, in the laboratory and the survey. This choice pattern is inconsistent with models where only the distribution of final wealth enters the utility function, including those which allow dominance violations.[11]

Before showing how the procedural risk preference can explain dominance violations in Example 1, we argue that two additional axioms are reasonable in the setting with multi-source income. First, like other experiments on dominance violations, Example 1 involves non-trivial risk in at least some of the options. Such an experimental design makes sense because when all options are riskless, the decision problem is so simple that one can confidently believe almost all subjects will choose the options delivering the highest final wealth. Hence, we require the preference to satisfy the following relaxation of Axiom 6.

Axiom 6*—Dominance without Risk: For any $(x_1, x_2), (y_1, y_2) \in X$, if $x_1 + x_2 > y_1 + y_2$, then $(x_1, x_2) \succ (y_1, y_2)$.

Second, based on the symmetric nature of the decision problem, we assume that changing the order of monetary gambles in Example 1 has no impact on the decision maker's choices.[12]

Axiom 7—Symmetry: For any $p, q \in \Delta(Z)$, we have $(p, q) \sim (q, p)$.

The following proposition characterizes the procedural risk preference that satisfies the above two axioms.

[11] Examples include the disappointment theory of Bell (1985) and Loomes and Sugden (1986), the solution concept called "choice-acclimating personal equilibrim" in the reference-dependent utility theory of Kőszegi and Rabin (2007), the gambling preferences in Diecidue et al. (2004), and the preference for simplicity in Mononen (2022) and Puri (2022).

[12] Asymmetry might be built in other applications with multi-source income, such as insurance decision with background income risk. In those applications, we can weaken Axiom 7 by only requiring the conditional preferences $\succsim_{1|\delta_0}$ and $\succsim_{2|\delta_0}$ are identical. Then we can derive a variant of Proposition 2 where the PREU representation is also allowed.

Proposition 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-5, 6* and 7 if and only if $\succsim$ admits either an EU representation V^{EU} or a REU representation V^{REU}:*

$$V^{EU}(P) = \sum_{x,y} u(x+y)P(x,y) \quad \textit{and} \quad V^{REU}(P) = CE_u(P_1) + CE_u(P_2),$$

for each $P \in \mathcal{P}$, where $u : 2Z \to \mathbb{R}$ is continuous and strictly increasing.

Clearly, if $\succsim$ admits an EU representation, then it satisfies Axiom 6 and cannot accommodate dominance violations in Example 1. Hence, we focus on the case where $\succsim$ admits a REU representation. That is, the decision maker first reduces the marginal income risk in each source to a certainty equivalent, and then adds the two certainty equivalents. With standard parametric assumptions, we can explain dominance violations in Example 1.

Example 1 (continued). Suppose that the decision maker's preference is represented by V^{REU} with

$$u(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, x \in Z, \\ -2\sqrt{-x}, & \text{if } x < 0, x \in Z. \end{cases} \tag{6}$$

Here u is a gain-loss utility index with CRRA risk preference and loss aversion parameter 2.[13] Then the decision maker will simultaneously choose A in decision 1 and D in decision 2 since

$$CE_u(A) = 2.4 > 0.625 = CE_u(B) \text{ and } CE_u(D) = -5.625 > -7.5 = CE_u(C).$$

The decision maker's choice pattern is consistent with the notion of narrow bracketing studied in Thaler (1985) and Read, Loewenstein, and Rabin (1999). Instead of treating the two decision problems as a whole, the decision maker narrowly brackets them by making each decision in isolation as if the other decision problem does not exist. The choice of A in decision 1 can be rationalized by risk aversion over gains, and the choice of B in decision 2 can be rationalized by risk

[13] This is consistent with the empirical estimates of loss aversion. See Brown et al. (2022) for a meta-analysis on this topic.

seeking over losses, both of which are standard in the theoretical and experimental literature.[14] It is worth noting that the the decision maker in our model satisfies Axiom 6* and hence will never choose less total money over more total money when there is no risk.

Our paper is not the first attempt to provide a utility theory of narrow bracketing. For instance, the narrow preference in Vorjohann (2021) is represented by an EU representation with an additively separable utility index. Camara (2021) derives the same utility function using symmetry and computational tractability.[15] Both papers assume the expected utility paradigm and suggest the following utility function of narrow bracketing: For each $P \in \mathcal{P}$,

$$V^{Narrow}(P) = \mathbb{E}_u[P_1] + \mathbb{E}_u[P_2].$$

Unlike our REU representation in Proposition 2, the decision maker with utility function V^{Narrow} evaluates and adds the *expected utility* of the two marginal distributions of income, instead of the *certainty equivalents*. As a result, the decision maker might violate dominance even without risk. To see this intuitively, consider the decision maker needs to choose from different portfolios of two assets. Portfolio P delivers \$1 in both assets for sure, while portfolio Q delivers \$2 in asset 1 and \$0 in asset 2 for sure. If the decision maker is risk averse, that is, if u is strictly concave, then she will strictly prefer portfolio P to portfolio Q since $2u(1) > u(0) + u(2)$, although both of them deliver total payoff \$2 with certainty. Building such extreme departures from rationality into agents' behavior might lead to a theory that explains certain anomalies in data at the expense of creating others that are not likely to be present. Hence, our new REU representation in Proposition 2 can be useful in this case.[16]

We end this section with a brief discussion of the *partial-narrow bracketing* model studied in Barberis, Huang, and Thaler (2006), Rabin and Weizsäcker

[14]For instance, this is an important component in the cumulative prospect theory introduced by Tversky and Kahneman (1992). They also assume that the decision maker adopts probability weighting, which is absent in our model as we impose Axiom 4. See also footnote 9.

[15]Note that computational tractability in Camara (2021) is an asymptotic notion and requires the decision problem to be high-dimensional.

[16]The preference represented by V^{Narrow} is a procedural risk preference characterized in Theorem 1. Indeed, it admits all three representations: EU, REU and PREU.

(2009) and Ellis and Freeman (2021). Without predicting dominance violations when there is no risk, one adaptation of the model to our setting can be written as $V^{Partial}$ such that for each $P \in \mathcal{P}$,

$$V^{Partial}(P) = \alpha \mathbb{E}_u[f[P]] + (1-\alpha) u(CE_u(P_1) + CE_u(P_2)),$$

where $\alpha \in [0,1]$ and $1-\alpha$ determines the degree of narrow bracketing.[17] In general, this partial-narrow bracketing model violates certain axioms in Theorem 1 such as Axiom 4, and hence does not represent some procedural risk preference. We can justify such exclusion in two ways. Theoretically, evaluating a risky prospect using the weighted average of two utility functions is more complex and demanding than using either of them. Hence, we might expect the decision maker to reject these hybrid heuristics. Empirically, Ellis and Freeman (2021) show that allowing for such intermediate levels of narrow bracketing does not significantly help explain the data in their experiments.

3.2 Multi-period Consumption

In this section, we study the preference of a decision maker faced with risky consumption in two periods. She finds it complex and demanding to evaluate intertemporal risk and hence might adopt procedures other than expected utility. Each outcome profile represents a consumption stream in two periods $t = 1, 2$. We call $t = 1$ today and $t = 2$ tomorrow. For simplicity, we assume that the consumption space in each period is a compact interval $X_1 = X_2 = C := [\underline{c}, \overline{c}] \subseteq \mathbb{R}_+$ and focus on preferences that satisfy the following assumption.

Assumption 1—Discounted Utility without Risk: There exists a continuous and strictly increasing function $u : C \to \mathbb{R}$ and $\beta \in (0,1)$ such that for any $x_1, x_2, y_1, y_2 \in C$, we have $(x_1, x_2) \succsim (y_1, y_2)$ if and only if $u(x_1) + \beta u(x_2) \geq u(y_1) + \beta u(y_2)$.

Assumption 1 is introduced by Dillenberger, Gottlieb, and Ortoleva (2020)

[17]The partial-narrow bracketing representation in Rabin and Weizsäcker (2009) is $V'(P) = \alpha \mathbb{E}_u[f[P]] + (1-\alpha)(\mathbb{E}_u[P_1] + \mathbb{E}_u[P_2])$. Like the narrow bracketing model V^{Narrow}, it is a procedural risk preference that generically violates Axiom 6*.

and posits that in the absence of risk, the decision maker's preference can be represented by the summation of discounted utilities in different periods. This is true for the vast majority of models of time preferences in the literature. Adding Assumption 1 to Theorem 1 leads to the following characterization of procedural risk preferences in the setting with multi-period consumption.

Proposition 3: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-5 and Assumption 1 if and only if $\succsim$ admits either an EU representation w, or a REU representation (w, v_1, v_2), or a PREU representation (w, v), where there exist $\beta \in (0,1)$ and two continuous and strictly increasing functions $u : C \to \mathbb{R}$ and $\phi : u(C) \to \mathbb{R}$ such that for any $x_1, x_2 \in C$,*

$$w(x_1, x_2) = \phi\Big(\frac{u(x_1) + \beta u(x_2)}{1+\beta}\Big).$$

Below we discuss the three different utility representations in Proposition 3 in more detail and show how they relate to commonly used models in the literature. First, suppose that $\succsim$ is represented by

$$V^{EU}(P) = \sum_{x_1, x_2} \phi\Big(\frac{u(x_1) + \beta u(x_2)}{1+\beta}\Big) P(x_1, x_2). \tag{7}$$

Following Dillenberger, Gottlieb, and Ortoleva (2020), we call the EU representation (7) an Kihlstrom-Mirman (KM) representation as it can be interpreted as an application of the multi-attribute utility function in Kihlstrom and Mirman (1974) to the context of time. Fixing the discount factor β, the curvature of u captures the decision maker's time preference measured by the elasticity of intertemporal substitution (EIS), while both u and v determine the decision maker's risk attitude. When ϕ is affine, $\succsim$ can be represented by the standard Expected Discounted Utility (EDU) model:

$$V^{EDU}(P) = \mathbb{E}_u[P_1] + \beta \mathbb{E}_u[P_2]. \tag{8}$$

Second, assume that $\succsim$ admits a REU representation in Proposition 3. By Proposition 1, the utility index w is unique up to a monotone transformation.

Hence, we can take ϕ to be affine and $\succsim$ is represented by

$$V^{REU}(P) = u(CE_{v_1}(P_1)) + \beta u(CE_{v_2}(P_2)). \tag{9}$$

The representation (9) corresponds to the Dynamic Ordinal Certainty Equivalent (DOCE) model studied in Selden (1978), Selden and Stux (1978) and Kubler, Selden, and Wei (2020b). The decision maker first evaluates the marginal distribution of consumption in each period and then aggregates the certainty equivalents using discounted utility. The time preference is capture by u, and the risk preference is captured by v_1 and v_2.

Finally, we can write the PREU representation in Proposition 3 as

$$V^{PREU}(P) = \sum_{x_1} \phi\Big(\frac{u(x_1) + \beta u(CE_v(P_{2|x})}{1+\beta}\Big) P_1(x_1). \tag{10}$$

The representation (10) can be interpreted as a two-period version of most recursive preferences built upon Kreps and Porteus (1978).[18] The decision maker evaluates risk in different periods recursively. Given the current consumption, she first reduces the conditional future risk to its certainty equivalent. Then, she aggregates this certainty equivalent and the corresponding current consumption using discounted utility. Finally, she evaluates the risk in current consumption. It is worth noting that the recursive preference is typically defined on the domain of temporal lotteries introduced by Kreps and Porteus (1978), which is richer than our domain of distribution over consumption streams. This leads to different axiomatic foundations and behavioral implications. In Section 3.2.1, we will illustrate this point in detail by drawing a comparison between our PREU representation (10) and a two-period version of the Epstein-Zin (EZ) preferences in Epstein and Zin (1989, 1991) and Weil (1990).

Before moving on, we revisit our interpretation of the procedural risk preference in the setting with multi-period consumption. Since different dimensions represent different time periods, the representations characterized in Proposition 3 differ

[18] To understand how our two-period model is related to the infinite-horizon recursive preference, we can simply fix the consumption levels for all periods other than $t = 1, 2$. See Chew and Epstein (1991) for a detailed discussion of recursive preferences.

in the order in which time and risk are aggregated. In the EU representation (7), the decision maker first aggregates time and then aggregates risk. In the REU representation (9), the decision maker completely reverses the order and first aggregates risk within each period and then aggregates the certainty equivalents as in discounted utility. In the PREU representation (10), the decision maker's evaluation is consistent with backward induction: she first aggregates tomorrow's risk, then aggregates time, and finally aggregates today's risk.[19] Note that in EDU, the order of aggregation does not matter.[20]

3.2.1 Epstein-Zin-Type Preferences

In the EDU model (8), both time and risk preferences are determined by the same utility index u. As a result, the reciprocal of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion (RRA). However, numerous empirical studies in macroeconomics, finance, and behavioral economics have suggested the need to separate time and risk preferences.[21] One of the most commonly used models that achieve such separation is the CRRA-CES version of EZ studied in Epstein and Zin (1991) and Weil (1990), where the coefficients of both EIS and RRA are constant. In this section, we focus on the following specification of the PREU representation (10) and compare it with EZ.[22]

[19]DeJarnette et al. (2020) make a similar observation in the context of time lotteries, which we will discuss in Section 3.3. Andreoni, Feldman, and Sprenger (2017) experimentally test predictions for behavior generated by different orderings of evaluation in the cumulative prospect theory.

[20]EDU is also a special case of the REU representation (9) by setting $u = v_1 = v_2$, and of the PREU representation (10) by assuming that $v = u$ and ϕ is the identity function.

[21]For instance, Bansal and Yaron (2004) and Barro (2009) show that the RRA coefficient should be much higher than the reciprocal of the EIS coefficient in order to fit macroeconomic and financial data. See also Barsky et al. (1997), Andreoni and Sprenger (2012), Nakamura et al. (2017) and references therein.

[22]The other two representations in Proposition 3 can also achieve a separation of time and risk preferences. For instance, in the EU representation (7), fixing the discount factor β, the time preference is determined by u solely, while the risk preference is determined by both u and ϕ. However, this model is not commonly used as it lacks time consistency in the sense that the risk preference in period 2 depends on the consumption in period 1. Similarly, the REU representation is less popular than EZ since it requires the decision maker to ignore intertemporal correlation.

Definition 4. A binary relation $\succsim$ admits an *EZ-PREU* representation if it is represented by (10) where there exist $\rho, \alpha \in (-\infty, 1)\backslash\{0\}$ such that $u(x) = \frac{x^\rho}{\rho}$, $v(x) = \frac{x^\alpha}{\alpha}$ for all $x \in C$, and $\phi(b) = v \circ u^{-1}(b)$ for all $b \in u(C)$. That is, $\succsim$ is represented by

$$V^{EZ-PREU}(P) = \sum_{x_1} \frac{1}{\alpha} \Big\{ c_1^\rho + \beta [\mathbb{E}_{P_{2|x_1}}(c_2^\alpha)]^{\rho/\alpha} \Big\}^{\alpha/\rho} P_1(x_1). \tag{11}$$

We say $\succsim$ is an *EZ-type Preference.*

In the EZ-PREU representation, $\gamma := 1 - \alpha$ is the coefficient of RRA, and $\psi := \frac{1}{1-\rho}$ is the coefficient of EIS. Hence, it separates time and risk preferences. When $\alpha = \rho$, the EZ-PREU representation reduces to EDU.

To facilitate the comparison between EZ-type preferences and the standard EZ preferences, we assume that the consumption in the first period is deterministic and focus on the restriction of $\succsim$ to $C \times \Delta(C)$.[23] Using a monotone transformation, (11) can be rewritten as

$$V^{EZ-PREU}(c_1, p) = \Big\{ c_1^\rho + \beta [\mathbb{E}_p(c_2^\alpha)]^{\rho/\alpha} \Big\}^{1/\rho}, \tag{12}$$

where $(c_1, q) \in C \times \Delta(C)$. By comparison, the standard CRRA-CES EZ preference entails the following recursive formulation of continuation utility U_t for each period t:

$$U_t = \Big\{ c_t^\rho + \beta \Big[\mathbb{E}_t \Big(U_{t+1}^\alpha \Big) \Big]^{\rho/\alpha} \Big\}^{1/\rho}. \tag{13}$$

Despite the similarity between the EZ-PREU representation (12) and the recursive EZ representation (13), we cannot simply compare them since EZ is defined on a space which involves temporal resolution of uncertainty and is richer than the set of lotteries over consumption streams we consider in this paper. Following Kreps and Porteus (1978), we call it the set of *temporal lotteries* $\mathcal{D}$ and denote by d a generic element in $\mathcal{D}$. The difference between a temporal lottery and a lottery can be illustrated by the following simple example.

In Figure 1, the two temporal lotteries d and d' deliver consumption 1 for

[23] As noted by Kubler et al. (2020a), the restriction of $\succsim$ to $C \times \Delta(C)$ also admits an REU representation (9). In the appendix, we briefly discuss how to extend the EZ-PREU representation to the infinite horizon setting.

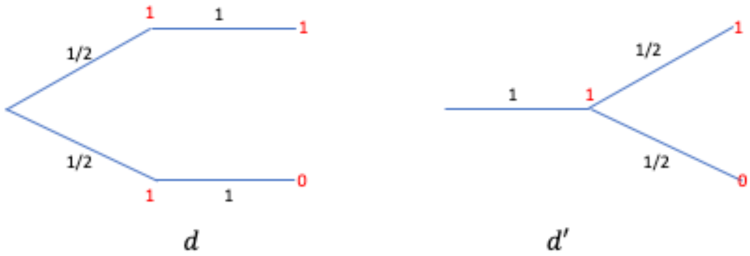

Figure 1: Example of two temporal lotteries that induce the same lottery.

sure in period 1 and have equal probability to deliver either consumption 1 or consumption 0 in period 2, determined by a coin flip. They induce the same lottery over consumption streams, but differ in the timing of risk resolution. In d, the coin is flipped in period 1 and the consumer knows the realization her future consumption in advance. In d', the coin is flipped in period 2 and the risk regarding consumption in period 2 is only resolved then. Note that such information is about future consumption, instead of future income, and hence there is no apparent planning advantage to flip the coin early.

The separation of time and risk preferences in EZ builds upon a non-trivial attitude toward the above difference in timing of risk resolution. Formally, for any lottery $(c_1, p) \in C \times \Delta(C)$, we denote by d^E the temporal lottery that induces lottery (c_1, p) and has early resolution of risk, like d in Figure 1. Similarly, we denote by d^L the temporal lottery that induces lottery (c_1, p) and has late resolution of risk, like d' in Figure 1. Based on the EZ representation (13), the utilities of d^E and d^L can be written as

$$V^{EZ}(d^E) = \left\{\mathbb{E}_p\left[(c_1^\rho + \beta c_2^\rho)^{\alpha/\rho}\right]\right\}^{1/\alpha} \quad \text{and} \quad V^{EZ}(d^L) = \left\{c_1^\rho + \beta[\mathbb{E}_p(c_2^\alpha)]^{\rho/\alpha}\right\}^{1/\rho}.$$

Since $V^{EZ}(d^L) = V^{EZ-PREU}(c_1, p)$, the EZ-PREU representation (12) can be interpreted as the restriction of the EZ representation on temporal lotteries where there is no early resolution of uncertainty.[24] It is well-known that when $\rho > \alpha$, that is, when RRA $> 1/\text{EIS}$, a consumer with the EZ preference exhibits a preference for early resolution of risk and strictly prefers d to d'. Symmetrically, she has a

[24]Similarly, $V^{EZ}(d^E)$ agrees with a monotone transformation of $V^{EU}(c, p)$ with the utility index $w(c_1, c_2) = \frac{1}{\alpha}(c_1^\rho + \beta c_2^\rho)^{\alpha/\rho}$.

preference for late resolution of risk when $\rho < \alpha$. When $\alpha = \rho$, the consumer is indifferent to timing of risk resolution and EZ reduces to EDU. Hence, the EZ model achieves a non-trivial separation of the time and risk preferences only if it entails a strict preference for either early or late resolution of risk.[25]

By contrast, our EZ-type preference is defined on the standard domain of distributions over consumption streams. As a result, the separation of time and risk preferences is based on a partially reversed evaluation procedure, instead of a non-neutral attitude toward timing of risk resolution. If we extend the EZ-type preference to the domain of temporal lotteries by assuming indifference to temporal resolution of risk, then our results reveal the compatibility between such indifference and the separation of time and risk preferences, which is impossible under EZ.[26] Moreover, our theory suggests a novel connection between the EZ-type preference, a relaxation of the Independence axiom for intertemporal risk, and a behavioral heuristic to simplify the evaluation of intertemporal risk.

The difference between the EZ preference and the EZ-type preference is also empirically relevant and important. Thanks to the separation of time and risk preferences and the convenient functional form, EZ has been widely applied in macroeconomics and finance. For instance, the long-run risks model of Bansal and Yaron (2004) considers a representative agent with CRRA-CES EZ preferences and delivers a unified explanation for several long-standing puzzles of asset markets, including the equity premium puzzle (Mehra and Prescott, 1985), the excessive asset price volatility, and the large cross-sectional differences in average returns across equity portfolios. Their main empirical results are based on parameters with RRA = 10 and EIS = 1.5, but have ignored the quantitative implications of the strong preference for early resolution of risk. Through introspection, Epstein, Farhi, and Strzalecki (2014) show that the preference parameters and calibrated endowment process imply that the representative agent in the long-run risks model is willing to give up 25 or 30 percent of her lifetime consumption in

[25] In most empirical applications of EZ, the decision maker has a preference for early resolution of risk. For instance, the main estimation results of Bansal and Yaron (2004) are based on RRA = 10 and EIS = 1.5.

[26] To achieve a full separation of the attitude toward timing of risk resolution, the time preference, and the risk preference, one needs to allow more general extension of the EZ-type preference. We leave it for further research.

order to have all risk about future consumption resolved in the next period. This timing premium is unrealistically high since the risk is about future consumption instead of future income or asset returns, and hence such information has no apparent instrumental value. In other words, the representative agent has no need to reoptimize her contingent consumption plans given early resolution of risk. Meissner and Pfeiffer (2022) show that on average subjects give up 5% of their total consumption to resolve all uncertainty immediately in an experiment.[27] Epstein, Farhi, and Strzalecki (2014) also show that the high timing premium is robust to other models using EZ preferences (such as the rare disasters model in Barro (2009)), different preference parameter values and more general risk preferences. By contrast, the representative agent with an EZ-type preference attaches no value to non-instrumental information and the timing premium is always zero.[28]

3.3 Time Lotteries

In this section, we study the procedural risk preference in decision problems involving uncertainty in both which and when outcomes will be received. Suppose that $X_1 = Z = [w, b] \subset \mathbb{R}_{++}$ and $X_2 = T = [0, \bar{t}] \subset \mathbb{R}_{+}$. Each outcome profile $(x, t) \in Z \times T$ denotes a *dated prize* where the monetary prize x is received in period t.[29] Each lottery $P \in \mathcal{P}$ denotes a distribution of dated prizes, where both the prize and the payment date can be uncertain. In particular, a *time lottery* $(x, p) \in X \times \Delta(T)$ is a lottery where the monetary prize x is fixed but the date is random. To apply Theorem 1, we need to modify the Axiom 2 since early delivery of the prize is more desirable for the decision maker due to impatience.

[27] Meissner and Pfeiffer (2022) also show a negative correlation between predicted and elicited timing premia in EZ, which questions the structural connection between the preference for early resolution of uncertainty and the preference parameters in EZ.

[28] In the appendix, we show that an infinite-horizon extension of the EZ-PREU representation can induce the same asset pricing implications as the standard EZ model. This is consistent with the observation of Stanca (2021) that the attitude toward timing of risk resolution is not the key behavioral feature driving the results in applications of recursive utility. He highlights the importance of correlation aversion, which describes a decision maker who dislikes positive autocorrelation in the consumption streams, and is equivalent to assuming $\rho > \alpha$ in representation (11).

[29] Following DeJarnette et al. (2020), we interpret the outcome profile (x, t) as representing x being consumed in period t, ruling out the possibility of x being saved for future consumption. Our results are not affected by alternative interpretations.

Axiom 2*—**Monotonicity and Impatience:** For any $(x, t), (y, s) \in Z \times T$, if $x \geq y$, $t \leq s$ and $(x, t) \neq (y, s)$, then $(x, t) \succ (y, s)$.

Moreover, we assume that in the absence of risk, the decision maker's behavior follows the standard exponentially discounted utility, that is, $U(x, t) = e^{-rt}u(x)$ for some positive, continuous and strictly increasing function u over prizes and a discount factor $r > 0$. Fishburn and Rubinstein (1982) show that this functional form is characterized by Axiom 2* and the following Stationarity axiom.

Axiom 8—Stationarity: For any $x, y \in Z$, $s, t \in T$, and $\tau \in \mathbb{R}$ with $s+\tau, t+\tau \in T$, if $(x, t) \sim (y, t + \tau)$, then $(x, s) \sim (y, s + \tau)$.

The next result characterizes the procedural risk preference over dated prizes.

Proposition 4: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1, 2*, 3-5 and Axiom 8 if and only if $\succsim$ admits either an EU representation w, or a REU representation (w, v_1, v_2), or a PREU representation (w, v), where there exist $r > 0$ and two continuous and strictly increasing function $u : Z \to \mathbb{R}_{++}$ and $\phi : [e^{-r\bar{t}}u(w), u(b)] \to \mathbb{R}$ such that for any $(x, t) \in Z \times T$,*

$$w(x, t) = \phi(e^{-rt}u(x)).$$

In Proposition 4, we identify EU with the triple (u, r, ϕ), REU with the tuple (u, r, v_1, v_2), and PREU with the tuple (u, r, ϕ, v). We do not need to keep track of ϕ in the REU representation because it is unique up to a monotone transformation.

We will study two properties introduced by DeJarnette et al. (2020). The first one, called *Stochastic Impatience*, is the risky counterpart of impatience and posits that if the decision maker can choose pair monetary prizes with payment dates in the presence of risk, then she would prefer to receive the highest prize at the earliest time. The second property concerns risk attitudes toward time, instead of monetary prizes. The decision maker is said to be *risk averse over time lotteries* if she prefers receiving a prize on a sure date than on a random date with the same mean. The formal definitions are as follows.

Definition 5 (Stochastic Impatience)**.** A binary relation $\succsim$ satisfies *Stochastic*

Impatience if for any $t_1, t_2 \in T$ and $x_1, x_2 \in Z$ with $t_1 < t_2$ and $x_1 > x_2$,

$$\frac{1}{2}\delta_{(x_1,t_1)} + \frac{1}{2}\delta_{(x_2,t_2)} \succsim \frac{1}{2}\delta_{(x_2,t_1)} + \frac{1}{2}\delta_{(x_1,t_2)}.$$

We say $\succsim$ satisfies *Strict Stochastic Impatience* if the above holds with $\succ$.

Definition 6 (Risk Attitudes toward Time)**.** A binary relation $\succsim$ is *risk averse over time lotteries (RATL)* if for any time lottery $(x, p) \in X \times \Delta(T)$,

$$(x, \mathbb{E}_p(t)) \succsim (x, p).$$

Analogously, $\succsim$ is *risk seeking over time lotteries (RSTL)* or *risk neutral over time lotteries (RNTL)* if the above holds with $\precsim$ or $\sim$, respectively.

When ϕ in Proposition 4 is affine, we get the EDU model in this setting. As is well known, the decision maker satisfies Stochastic Impatience and must be RSTL, since the exponential function is convex.[30] However, the experimental evidence of DeJarnette et al. (2020) suggests that the majority of their subjects are RATL. They show that such incompatibility between Stochastic Impatience and any violation of RSTL persists in the general EU model and a large class of non-EU models, including those allowing probability weighting. Below we argue that our procedural risk preference can provide a solution.

Proposition 5: *Let $\succsim$ be a binary relation that satisfies the axioms in Proposition 4.*
(i) The relation $\succsim$ satisfies Stochastic Impatience and violates RSTL if and only if it admits either a REU representation (u, r, v_1, v_2) or a PREU representation (u, r, ϕ, v) where v_2 and v are not convex and ϕ is a convex transformation of $\ln$.[31]
(ii) The relation $\succsim$ satisfies Strict Stochastic Impatience and violates RSTL if and only if it admits a PREU representation (u, r, ϕ, v) where v is not convex and ϕ is a strictly convex transformation of $\ln$.

[30] This feature of EDU has important implications in many applications, like dynamic moral hazard (Ely and Szydlowski, 2020) and dynamic information acquisition (Zhong, 2022, Chen and Zhong, 2022).

[31] A function f defined on $A \subseteq \mathbb{R}_{++}$ is a convex transformation of ln if there exists a convex function g with $f(x) = g(\ln(x))$ for all $x \in A$. We say the transformation is strictly convex if g is strictly convex.

We now illustrate the intuition behind Proposition 5. The fact that the EU representation is ruled out directly follows from Theorem 1 of DeJarnette et al. (2020). To see why the other two representations can accommodate Stochastic Impatience and violations of RSTL simultaneously, note that with corresponding decision procedures, the decision maker will evaluate risk in time before aggregating the two dimensions using discounted utility. This separates the risk attitude toward time from intertemporal trade-offs. As a result, the decision maker violates RSTL if and only if v and v_2 are not convex respectively. If $\succsim$ admits an PREU representation, Stochastic Impatience requires ϕ to be "more convex" than the logarithmic function.[32] Moreover, if $\succsim$ admits an REU representation, then the decision maker only cares about marginal distributions and deems irrelevant the paring between prizes and payment dates. Hence, the decision maker cannot satisfy Strict Stochastic Impatience.[33]

Example 2. Consider the following example of PREU representation:

$$V^{PREU}(P) = \sum_x P_1(x)u(x)\frac{1}{\mathbb{E}_{P_{2|x}}[e^{rt}]}.$$

where $r > 0$ and $u : Z \to \mathbb{R}_{++}$ is continuous and strictly increasing. Easy to verify that it represents a preference $\succsim$ that is RATL and satisfies Strict Stochastic Impatience. Moreover, we can rewrite the utility function as $V^{PREU}(P) = \sum_x P_1(x)u(x)e^{-r\psi(P_{2|x})}$, where $\psi(p) = \frac{1}{r}\log \mathbb{E}_p[e^{rt}]$ for each $p \in \Delta(T)$ is a monotone additive statistic introduced by Mu, Pomatto, Strack, and Tamuz (2021b).[34]

[32] This follows from Propositions 2 and 4 of DeJarnette et al. (2020), since Stochastic Impatience imposes the same restriction on ϕ in the EU and the PREU representations.

[33] The REU representation in this section is different from the one discussed in Section 3.2 (and hence the DOCE model) due to different interpretations of outcome profiles. This can be seen by embedding $Z \times T$ in Z^T, the space of consumption streams with continuous time, and comparing the (extensions of) two models.

[34] As an application of their main results, Mu, Pomatto, Strack, and Tamuz (2021b) studies a preference over time lotteries $Z \times \Delta(T)$ without imposing the Independence axiom in the time dimension. They derive a similar utility function $V(x, p) = u(x)e^{-r\Psi(p)}$, where Ψ is more general than the certainty equivalent of an expected utility function. By comparison, we consider a preference over lotteries of dated prizes $\Delta(Z \times T)$ that satisfies Axiom 4, the axiom of Within-Dimension Independence.

4 Discussion

4.1 Characterization of Each Evaluation Procedure

We have studied three procedures the decision maker uses to evaluate two-dimensional risk. Our main result, Theorem 1, characterizes their common behavioral properties. In this section, we discuss what additional axioms are needed to identify each of them.

First, as is well-known, the EU representation features the standard Independence axiom.

Axiom 9—Independence: For any $P, Q, R \in \mathcal{P}$, if $P \succ Q$, then for any $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$.

Second, for each lottery $P \in \mathcal{P}$, only its marginal lotteries enter the REU representation.

Axiom 10—Correlation Neutrality: For any $P \in \mathcal{P}$, we have $P \sim (P_1, P_2)$.

Finally, in the PREU representation, the attitude toward risk in dimension 2 is independent of the outcome in dimension 1, and the decision maker satisfies a stronger version of Axiom 5 where the comparability constraint is dropped.

Axiom 11—Strong Cross-Dimension Independence: For any $P, Q, R, S \in \mathcal{P}$ such that $P_1 \perp R_1$, $Q_1 \perp S_1$, if $P \succ Q$ and $R \sim S$, then for any $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Axiom 12—Taste Separability in Dimension 2: For any $x, y \in X_1$ and $p, q \in \mathcal{P}_2$, we have $p \succsim_{2|x} q$ if and only if $p \succsim_{2|y} q$.

The following results establish the sufficiency of the above axioms for each representation.

Proposition 6: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that satisfies Axioms 1-5.*
(i) The relation $\succsim$ satisfies Axiom 9 if and only it admits an EU representation.
(ii) The relation $\succsim$ satisfies Axiom 10 if and only it admits a REU representation.
(iii) The relation $\succsim$ satisfies Axioms 11 and 12 if and only it admits a PREU

representation.

As a direct corollary, we can also identify the intersection of each pair of representations.

Corollary 1: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that satisfies Axioms 1-5.*
(i) The following statements are equivalent:
(a) The relation $\succsim$ admits both an EU and a REU representations.
(b) The relation $\succsim$ admits both a PREU and a REU representations.
(c) The relation $\succsim$ admits an EU, a PREU and a REU representations.
(d) The relation $\succsim$ admits both an EU representation w where w is additively separable.
(ii) The relation $\succsim$ admits both an EU and a PREU representations if and only if it admits an admits both an EU representation w where there exist functions $w_1, b : X_1 \to \mathbb{R}$ and $w_2 : X_2 \to \mathbb{R}$ such that for each $(x_1, x_2) \in X$, we have $w(x_1, x_2) = w_1(x_1) + b(x_1)w_2(x_2)$.

Part (i) of Corollary 1 provides a characterization of the EU representation with an additively separable utility index, the narrow bracketing model studied in Vorjohann (2021) and Camara (2021). In the application to multi-period consumption where $w_1 = w_2$, the representation in part (ii) corresponds to a two-period version of Uzawa preferences (Uzawa, 1968, Epstein, 1983). Note that the continuity and strict monotonicity of w imposes joint restrictions on w_1, b and w_2, which we did not specify in Corollary 1. For instance, if $b(x_1) > 0$ for all $x_1 \in X_1$, then w_2 must be strictly increasing.

4.2 Extensions

As a final remark, we discuss possible extensions of the procedural risk preference. First, as mentioned in Section 2.2, we have assumed that the decision maker first evaluates risk in dimension 2 in the PREU representation. Such asymmetry is captured by Axiom 5 since only marginal lotteries in dimension 1 are required to be mutually singular. If we instead impose the singularity condition on dimension 2, then in Theorem 1, we can derive a variant of the PREU representation where risk in dimension 1 is evaluated first, that is, $V(P) = \sum_{y \in X_2} w(CE_v(P_{1|y}), y)P_2(y)$. We call this utility function a PREU* representation.

One way to accommodate both PREU and PREU* in the representation theorem is to relax Axiom 5. Alternatively, one can first identify the evaluation order of the decision maker from her choices, and then impose behavioral axioms accordingly. In a follow-up project, Ke and Zhang (2023) adopt this approach and extend representations studied in this paper to a setting with more than two dimensions. They characterize how the decision maker orders and brackets different dimensions, based on which she evaluates risk. For instance, in the EU representation, she brackets the two dimensions together; in the REU representation, she brackets the two dimensions separately; in the PREU or the PREU* representation, she has a strict ranking between the two dimensions. The main complexity there is the exponentially increasing number of evaluation procedures.

Second, recall that in the REU representation, the decision maker satisfies Axiom 10 and neglects correlation between risk in different dimensions. One can incorporate this idea of correlation neglect into the other two evaluation procedures by replacing the joint probability $P(x, y)$ with the product of marginal probabilities $P_1(x)P_2(y)$ in the EU representation (1), and replacing the conditional distribution $P_{2|x}$ with the marginal distribution P_2 in the PREU representation (3). An early version of this paper (Zhang, 2021) studies these two representations. A further extension is to consider general correlation-sensitive preferences, like those of Ellis and Piccione (2017), Ellis (2021), Lanzani (2022) and Stanca (2021).

Finally, it would be interesting to extend the analysis in the present paper to other applications. For instance, one can study social preferences by interpreting each outcome profile as an allocation of payoffs between to subjects. Ellis and Freeman (2021) and Exley and Kessler (2018) find that the fairness concerns are narrowly bracketed. Similarly, by interpreting each dimension as a period when risk is resolved, instead of when consumption happens, one can study the effects of ordering on the attitudes toward temporal resolution of uncertainty (Ergin and Gul, 2009, Artstein-Avidan and Dillenberger, 2015).

Appendix A: PREU in Infinite Horizon

In this section, we briefly discuss how to extend the PREU model in (12) to one with multiple periods. Assume that the consumption space in each period $t = 1, ..., T$ is a compact interval $C \subset \mathbb{R}_+$, where T can be $+\infty$. The set of deterministic consumption streams is C^T with a generic element $\mathbf{c} = (c_t)_{t=1}^T$. For each consumption stream $\mathbf{c} \in C^T$, we denote the subsequence of consumption in the first t periods as $\mathbf{c}^t = (c_\tau)_{\tau=1}^t$.

The preference is defined on the lottery space $\mathcal{P} = \mathcal{L}(C^T)$. Here we allow for lotteries with infinite supports to accommodate applications in finance. For each lottery P, denote $P_{[t]}$ as the marginal lottery in the first t periods, $1 \leq t < T$. For each subsequence of consumption $\mathbf{c}^t$ in the support of $P_{[t]}$, we define $\phi(P|\mathbf{c}^t)$ as the conditional lottery starting from period $t+1$, given that consumption in the first t periods are $\mathbf{c}^t$. When $T < +\infty$, $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^{T-t})$ and when $T = +\infty$, $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^\infty)$. Note that for each finite T, $\mathcal{L}(C^{T-t})$ is homeomorphic to a subset of $\mathcal{L}(C^\infty)$ where the consumption levels are always 0 from period $t+1$ on. So we will focus on the case with an infinite horizon.

The following notions are adapted from recursive preferences on temporal lotteries (Chew and Epstein, 1991, Bommier, Kochov, and Le Grand, 2017) to our framework. For each $V : \mathcal{P} := \mathcal{L}(C^\infty) \to \mathbb{R}$ and $p \in \mathcal{P}$, denote

$$m_V(P)(B) \equiv P_1\Big\{c \in C : V(c, \phi(P|c) \in B)\Big\}, \forall B \in \mathcal{B}(V(\mathcal{P}))$$

where $V(\mathcal{P}) \subset \mathbb{R}$ is the image of V on $\mathcal{P}$ and $\mathcal{B}(V(\mathcal{P}))$ is the set of all Borel subsets of $V(\mathcal{P})$. Then $m_V(P)$ is a probability measure over utilities conditional on the current consumption. Now we define the *recursive preference over lotteries* as $V : \mathcal{P} \to \mathbb{R}$ with

$$V(P) = I(m_V(P)),$$
$$V(c, q) = W(c, V(q)),$$

where $m_V(P)$ is defined as above, $I : \mathcal{L}(\mathbb{R}) \to \mathbb{R}$ is a *certainty equivalent*, that is, I is continuous, increasing with respect to first order stochastic dominance and

$I(x) = x$ for each $x \in \mathbb{R}$, $W : C \times \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing in the second argument. Unlike the recursive preferences Chew and Epstein (1991) and Bommier, Kochov, and Le Grand (2017), V is defined on a different domain and can be discontinuous.

In order to get the CRRA-CES functional form, we can set $I = \phi^{-1} \circ \mathbb{E} \circ \phi$ with $\phi(x) = x^{\alpha/\rho}$ and $W(c, v) = (1-\delta)c^\rho + \delta v$, where $\rho < 1, 0 \neq \alpha < 1$ and $0 < \delta < 1$. The recursive preference is equivalent to the following recursion of value functions (up to a monotonic transformation):

$$V^{PREU}(P) = \frac{1}{\alpha}\mathbb{E}_{P_1}\Big(U_1^\alpha\Big) \tag{14}$$

$$U_t^\rho = (1-\delta)c_t^\rho + \delta\Big[\mathbb{E}_{\phi(P|\mathbf{c}^t)_1}\Big(U_{t+1}^\alpha\Big)\Big]^{\frac{\rho}{\alpha}} \tag{15}$$

where U_t is the value in period t and the expectation is computed with respect to $\phi(P|\mathbf{c}^t)_1$, which is the probability distribution of consumption levels in period $t+1$ conditional on the consumption stream in the first t periods $\mathbf{c}^t$.

Then we explore the implications of PREU in a standard asset pricing problem. The consumer is endowed with initial wealth $W_1 > 0$ in period 1 and chooses the consumption level and saving level in each period. Let S denote the finite state space in each period $t \geq 2$ and $\Omega = S^\infty$ denote the space of state sequences. The consumer has a prior belief over Ω. For each $s^\infty \in \Omega$, we denote by s^t the history of states from period 2 to period t for each $t \geq 2$. Let S^t be the set of all histories till period t.

The consumer's preference is represented by V^{PREU} in (14). She chooses a *consumption plan* $(c_t)_{t\geq 1}$ to maximizes her utility, where $c_1 \in X$ and $c_t : S^t \to X$ for each $t \geq 2$. Given the history of states s^t, the gross return on wealth from period $t-1$ to period t is $R_{w,t}(s^t) > 0$. Then the wealth dynamics are represented by the following equation:

$$W_{t+1}(s^{t+1}) = R_{w,t+1}(s^{t+1})(W_t(s^t) - c_t(s^t)).$$

We assume that the wealth level always lies in C. We say that a consumption plan $(c_t)_{t\geq 1}$ is feasible given initial wealth W_1 if if $c_t(s^t) \leq W_t(s^t)$ for all s^t, t. Each feasible consumption plan $(c_t)_{t\geq 1}$ induces a lottery $P \in \mathcal{P}$ where the consumption in the first period is deterministic. Using V^{PREU} in (14), we can define a utility function over feasible consumption plans as $\hat{V}^{PREU}((c_t)_{t\geq 1})$. Then the optimization problem of the consumer is

$$J^{PREU}(W_1) = \sup \left\{ \hat{V}^{PREU}((c_t)_{t\geq 1}) : (c_t)_{t\geq 1} \text{ is feasible given } W_1 \right\}.$$

To facilitate the comparison of PREU and EZ, we can similarly consider a consumer with a CRRA-CES EZ recursive utility function $\hat{V}^{EZ}$ over feasible consumption plans and the optimal value $J^{EZ}(W_1)$. We assume that RRA>1/EIS, then the EZ consumer has a preference over early resolution of uncertainty, while the PREU consumer exhibits indifference to temporal resolution of uncertainty. The following proposition shows that the two utility functions lead to the same optimal value.

Proposition 7: *Assume RRA>1/EIS, i.e., $\rho > \alpha$. Then for each $W_1 > 0$, $J^{PREU}(W_1) = J^{EZ}(W_1)$ exists and there exist consumption plans $(c_t)_{t\geq 1}$ and $(c_t^n)_{t\geq 1, n\geq 1}$ feasible given W_1 such that $c_1^n \to c_1, c_t^n(s^t) \to c_t(s^t)$ as n goes to infinity for each $t \geq 2$, and*

(i). $J^{PREU}(W_1) = \lim_{n\to\infty} \hat{V}^{PREU}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1}) = J^{EZ}(W_1)$;

(ii). For each $n \geq 1, t \geq 2$, c_t^n is injective on Ω_t, that is, $c_t^n(s^t) \neq c_t^n(\hat{s}^t)$ if $s^t \neq \hat{s}^t$.

We sketch the proof of Proposition 7. Recall that in Section 3.2.1, we observe that, in the setup with two periods, V^{PREU} agrees with V^{EZ} when risk does not resolve early. This observation remains valid with an infinite horizon. On the one hand, since the EZ consumer has a preference for early resolution of risk, we know that $\hat{V}^{PREU}((c_t)_{t\geq 1}) \leq \hat{V}^{EZ}((c_t)_{t\geq 1})$ for each feasible consumption plan $(c_t)_{t\geq 1}$ and $J^{PREU}(W_1) \leq J^{EZ}(W_1)$. On the other hand, for a consumption plan $(c_t)_{t\geq 1}$ where c_t is injective on Ω_t for each t, the consumption history contains the same information as state history and there is no early resolution of risk. In this case, $\hat{V}^{PREU}((c_t)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1})$. By continuity of V^{EZ} and hence $\hat{V}^{EZ}$, we can find a sequence of consumption plans $(c_t^n)_{t\geq 1, n\geq 1}$ with injective consumption functions to approximate the optimal value of the EZ consumer. This implies $J^{PREU}(W_1) = J^{EZ}(W_1)$.

A directly corollary of Proposition 7 is that our PREU model has the same implications in asset pricing as the EZ model when RRA>1/EIS, which is the common parametric assumption in most applications of EZ preferences in finance and macroeconomics. Following the asset pricing literature, let $\gamma := 1-\alpha$ be the RRA, $\psi := \frac{1}{1-\rho}$ be the EIS and $\theta := \frac{\alpha}{\rho}$.

Corollary 2: *Assume RRA>1/EIS, i.e., $\rho > \alpha$. Denote $\mathbb{E}_t$ as the expectation with respect to s^t. Then for each $W_1 > 0$,*

(i). Euler equation:

$$\lim_{n\to\infty} \mathbb{E}_t\Big[\delta^\theta (\frac{c^n_{t+1}}{c^n_t})^{-\frac{\theta}{\psi}} R^\theta_{w,t+1}\Big] = \mathbb{E}_t\Big[\delta^\theta (\frac{c_{t+1}}{c_t})^{-\frac{\theta}{\psi}} R^\theta_{w,t+1}\Big] = 1;$$

(ii). Asset pricing formula: for each asset i with gross return $R_{i,t}$,

$$\lim_{n\to\infty} \mathbb{E}_t\Big[\delta^\theta (\frac{c^n_{t+1}}{c^n_t})^{-\frac{\theta}{\psi}} R^{-(1-\theta)}_{w,t+1} R_{i,t+1}\Big] = \mathbb{E}_t\Big[\delta^\theta (\frac{c_{t+1}}{c_t})^{-\frac{\theta}{\psi}} R^{-(1-\theta)}_{w,t+1} R_{i,t+1}\Big] = 1;$$

We end this section with a brief discussion about the case with RRA<1/EIS, i.e., when the EZ consumer prefers late solution of risk. In this case, $J^{PREU}(W_1)$ might be strictly higher than $J^{EZ}(W_1)$ due to the discontinuity of V^{PREU} on $\mathcal{P}$, which invalidates our proof of Proposition 7. Interestingly, the PREU consumer might have excessive demand for consumption smoothing across different states of the world in the same period. However, the derivation of the optimal value and Euler equation is much less tractable and we leave a formal treatment of this case for future study.

Appendix B: Proofs

Proof of Theorem 1.

Necessity: Suppose that the preference $\succsim$ admits one of the three representations: EU, REU and PREU. It is easy to verify that $\succsim$ satisfies Axiom 1, Axiom 2 and Axiom 4. If $\succsim$ admits either an EU representation or a REU representation, then it satisfies Axiom 5 and the strong continuity axiom, which is stronger than Axiom

3. Now suppose that (w, v) is a PREU representation of $\succsim$. To verify that $\succsim$ satisfies Axiom 3.1 (Mixture Continuity), consider any $P, Q \in \mathcal{P}$ and $\alpha \in [0,1]$. The utility of $\alpha P + (1-\alpha)Q \in \mathcal{P}$ is given by

$$\begin{aligned}V(\alpha P + (1-\alpha)Q) =& \alpha \sum_{\substack{x:P_1(x)>0,\\ Q_1(x)=0}} w(x, CE_v(P_{2|x}))P_1(x) \\ &+(1-\alpha) \sum_{\substack{x:Q_1(x)>0,\\ P_1(x)=0}} w(x, CE_v(Q_{2|x}))Q_1(x) \\ &+ \sum_{\substack{x:Q_1(x)>0,\\ P_1(x)>0}} w(x, CE_v(\lambda_x P_{2|x} + (1-\lambda_x)Q_{2|x}))[\alpha P_1(x) + (1-\alpha)Q_1(x)]\end{aligned}$$

where for each $x \in X_1$ such that $P_1(x) > 0$ and $Q_1(x) > 0$,

$$\lambda_x = \frac{\alpha P_1(x)}{\alpha P_1(x) + (1-\alpha)Q_1(x)}.$$

Clearly, $V(\alpha P + (1-\alpha)Q)$ is continuous in α and $\succsim$ satisfies Axiom 3.1. Axiom 3.2 can be similarly verified.

Finally we check Axiom 5 for the PREU representation. For each $P, Q, R, S \in \mathcal{P}$ and $\alpha \in (0,1)$, if $\text{supp}(P_1) \cap \text{supp}(R_1) = \text{supp}(Q_1) \cap \text{supp}(S_1) = \emptyset$, then

$$\begin{aligned}V(\alpha P + (1-\alpha)R) =& \alpha \sum_{x:P_1(x)>0} w(x, CE_v(P_{2|x}))P_1(x) \\ &+ (1-\alpha) \sum_{x:R_1(x)>0} w(x, CE_v(R_{2|x}))R_1(x) \\ =& \alpha V(P) + (1-\alpha)V(R). \\ V(\alpha Q + (1-\alpha)S) =& \alpha V(Q) + (1-\alpha)V(S).\end{aligned}$$

Hence $P \succ Q, R \sim S$ implies $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Sufficiency: We first introduce some notation. For any set of lotteries $\mathcal{A} \subseteq \mathcal{P}$, denote by $\max_{\succsim} \mathcal{A} = \{P \in \mathcal{A} : P \succsim Q, \forall Q \in \mathcal{A}\}$ the set of most preferred lotteries in $\mathcal{A}$ under $\succsim$ whenever it is well-defined. For any set $A \subseteq \mathbb{R}^n$ for some positive integer n, we denote by A^o its interior.

Assume that Axioms 1-5 hold throughout the proof. For each dimension $i = 1, 2$ and marginal lottery $q \in \mathcal{P}_{-i}$, the following lemma guarantees that the conditional preference $\succsim_{i|q}$ defined on $\mathcal{P}_i$ admits an EU representation.

Lemma 1: *For each $i = 1, 2$ and $q \in \mathcal{P}_{-i}$, the conditional preference $\succsim_{i|q}$ admits an EU representation with utility index $v_{i|q}$, which is continuous and unique up to a positive affine transformation. Moreover, if $q \in X_{-i}$, then $v_{i|q}$ can be chosen to be strictly increasing.*

Proof of Lemma 1 . Without loss of generality, fix $i = 1$ and $q \in \mathcal{P}_2$. The case where $i = 2$ can be proved similarly. By Axioms 3.1 and 4, $\succsim_{1|q}$ admits an EU representation with a utility index $v_{1|q}$ defined on X_1, which is unique up to a positive affine transformation. To see why $v_{1|q}$ is continuous, suppose by contradiction that there exists a sequence (x^n) in X_1 such that $x^n \to x \in X_1$ and $v_{1|q}(x^n) \not\to v_{1|q}(x)$. Without loss of generality and passing to a subsequence if necessary, suppose $v_{1|q}(x^n) \to a < b = v_{1|q}(x)$ and $v_{1|q}(x^n) < (a+b)/2$ for all n. Since $\succsim_{1|q}$ admits an EU representation, we can find $r \in \mathcal{P}_1$ with $\sum_{y \in X_1} v_{1|q}(y) r(y) = (a+b)/2$, that is, $(x^n, q) \prec (r, q) \prec (x, q)$ for all n. Since $(x^n, q) = (x, q)_{(x^n - x, 0)}$ and $(x^n - x, 0) \to (0, 0)$, Axiom 3.2 implies $(x, q) \precsim (r, q) \prec (x, q)$, a contradiction. Hence $v_{1|q}$ is continuous for each $q \in \mathcal{P}_2$. Moreover, if $q \in X_2$, that is, $q = y$ for some $y \in X_2$, then by Axiom 2, the function $v_{1|q}$ must be strictly increasing. $\square$

A direct corollary of Lemma 1 guarantees the existence of "certainty equivalents" of marginal lotteries.

Corollary 3: *For each $P \in \mathcal{P}_1 \times \mathcal{P}_2$, there exist $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$ such that $P \sim (P_1, x_2) \sim (x_1, P_2) \sim (y_1, y_2)$.*

Proof of Corollary 3 . We will prove the result for the case where $P_1 \notin X_1, P_2 \notin X_2$. The proof for other cases is similar. By Lemma 1, we can find $x_2, a, a' \in X_2$ such that either $\sum_x v_{2|P_1}(x) P_2(x) = v_{2|P_1}(x_2)$ and hence $P \sim (P_1, x_2)$, or $v_{2|P_1}(a) > \sum_x v_{2|P_1}(x) P_2(x) > v_{2|P_1}(a')$. In the latter case, since $v_{2|P_1}$ is continuous and X_2 is a closed interval, there exists $x_2 \in X_2$ such that $v_{2|P_1}(x_2) = \sum_x v_{2|P_1}(x) P_2(x)$, which implies $P \sim (P_1, x_2)$. Similarly, we can find $x_1 \in X_1$ with $P \sim (x_1, P_2)$. Now let $y_2 = x_2$. Repeat the above argument for the product lottery (P_1, x_2) and we can find $y_1 \in X_1$ with $(y_1, y_2) \sim (P_1, x_2) \sim P$. $\square$

Step 1: Characterize when $\succsim$ admits a REU representation.

Recall that two lotteries P and Q are comparable if P_i and Q_i are comparable for both $i = 1, 2$, i.e., there exist $x \in X_1$ and $y \in X_2$ such that $(x, P_2) \sim (x, Q_2)$ and $(P_1, y) \sim (Q_1, y)$. The following axiom states that the decision maker is indifferent between two comparable lotteries.

Axiom 13—Comparability Indifference: For each $P, Q \in \mathcal{P}$, if P and Q are comparable, then $P \sim Q$.

If the preference $\succsim$ admits a REU representation (w, v_1, v_2), then, in addition to Axioms 1-5, it satisfies Axiom 13. To see why, consider a pair of comparable lotteries P and Q. By assumption, $CE_{v_1}(P_1) = CE_{v_1}(Q_1)$ and $CE_{v_2}(P_2) = CE_{v_2}(Q_2)$. Then

$$V^{REU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big) = w\Big(CE_{v_1}(Q_1), CE_{v_2}(Q_2)\Big) = V^{REU}(Q),$$

which means $P \sim Q$. The next lemma further claims that Axioms 1-13 are also sufficient for $\succsim$ to admit a REU representation.

Lemma 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-5 and Axiom 13 if and only if it admits a REU representation.*

Proof of Lemma 2. We focus on the sufficiency of Axioms 1-13, since the necessity has been established above. By Axiom 13, the decision maker ignores correlation in the sense that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$. Hence it suffices to study the restriction of $\succsim$ to product lotteries $\mathcal{P}_1 \times \mathcal{P}_2$. For each $x_1, y_1 \in X_1$ and $p_2, q_2 \in \mathcal{P}_2$, again by Axiom 13, we have

$$(x_1, p_2) \sim (x_1, q_2) \Longleftrightarrow p_2 \text{ and } q_2 \text{ are comparable in dimension } 2 \Longleftrightarrow (y_1, p_2) \sim (y_1, q_2).$$

By Lemma 1 and uniqueness of the EU representation, the conditional preferences $\succsim_{2|x}$ and $\succsim_{2|x'}$ must be identical for all $x, x' \in X_1$. Denote the common conditional preference by $\succsim_2$ and the continuous and strictly increasing EU index by v_2. Similarly, one can show that the conditional preferences $\succsim_{1|y}$ and $\succsim_{1|y'}$ must be identical for all $y, y' \in X_2$. Denote it by $\succsim_1$ and the corresponding EU index by v_1. For each $p_1 \in \mathcal{P}_1$ and $p_2, q_2 \in \mathcal{P}_2$, if $p_2 \sim_2 q_2$, then p_2 and q_2 are comparable in dimension 2. Axiom 13 implies $(p_1, p_2) \sim (p_1, q_2)$. Hence, for each $(p_1, p_2) \in \mathcal{P}_1 \times \mathcal{P}_2$, we have $(p_1, p_2) \sim (p_1, CE_{v_2}(p_2)) \sim (CE_{v_1}(p_1), CE_{v_2}(p_2))$. The second indifference relation holds since the conditional preference $\succsim_{1|CE_{v_2}(p_2)}$ admits an EU representation v_1.

Now we define $\hat{\succsim}$ as the restriction of $\succsim$ to X. By Axiom 3.1, the binary relation $\hat{\succsim}$ is continuous. Then Debreu's Theorem implies that $\hat{\succsim}$ is represented a continuous utility function w. Axiom 2 guarantees that w is strictly increasing. Therefore w, v_1 and v_2 are continuous and strictly increasing, and the tuple (w, v_1, v_2) is a REU representation of $\succsim$. $\square$

Step 2: Auxiliary results.

For the rest of the proof, we maintain the assumption that $\succsim$ does not satisfy Axiom 13. That is, there exists a pair of comparable lotteries P and $\tilde{P}$ such that $P \succ \tilde{P}$.

We will prove some auxiliary results which will be used in later steps. We first introduce some additional notation. For any lottery $P \in \mathcal{P}$, denote by $M(P)$ the set of lotteries the set of lotteries whose marginal lotteries are the same as those of P. Recall that two marginal lotteries r, r' are singular, i.e., $r \perp r'$ if $\mathrm{supp}(r) \cap \mathrm{supp}(r') = \emptyset$. Easy to see that for $P, Q, R \in \mathcal{P}$, if $R_1 \perp P_1$ and $R_1 \perp Q_1$, then $R_1 \perp \lambda P_1 + (1-\lambda) Q_1$ for all $\lambda \in (0,1)$. Also, if $P_1 \perp Q_1$, then $P_1 \perp Q'_1$ for all $Q' \in M(Q)$. When there is no confusion, for $i = 1, 2$, we say two finite sets of marginal lotteries $M, M' \subset \mathcal{P}_i$ are singular, denoted by $M \perp M'$, if $p \perp p'$ for all $p \in M$ and $p' \in M'$. A singleton set $M = \{q\}$ is simply written as q.

Given our relaxation of the Independence axiom, for $P \succ \tilde{P}$ and $\lambda \in (0,1)$, it is *not* guaranteed that $P \succ \lambda P + (1-\lambda)\tilde{P} \succ \tilde{P}$. Instead, we have the following weaker property.

Lemma 3: *For any $Q \succ Q'$, there exist $\lambda^* \in [0,1]$ and $Q^* = \lambda^* Q + (1-\lambda^*) Q'$ such that for any $\varepsilon > 0$, we can find $\lambda_\varepsilon \in (\lambda^* - \varepsilon, \lambda^* + \varepsilon) \cap [0,1]$ with $Q^* \not\sim \lambda_\varepsilon Q + (1-\lambda_\varepsilon) Q'$.*

Proof of Lemma 3. Suppose the result fails. Then for any $\lambda \in [0,1]$, there exists $\varepsilon_\lambda > 0$ such that for any $\lambda' \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda) \cap [0,1]$, we have $\lambda Q + (1-\lambda)Q' \sim \lambda' Q + (1-\lambda')Q'$. Notice that $\{(\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)\}_{\lambda \in [0,1]}$ forms an open cover of the compact set $[0,1]$. We can find a finite subcover of $[0,1]$. By transitivity of $\succsim$, we know $\lambda Q + (1-\lambda)Q' \sim \lambda' Q + (1-\lambda')Q'$ for all $\lambda, \lambda' \in [0,1]$, which leads to $Q \sim Q'$ and a contradiction. $\square$

The next Lemma shows that if $R \sim S$ and both of them are singular with respect to Q, then the decision maker still finds them equally attractive when they are mixed with Q.

Lemma 4: *For any $\alpha \in (0,1)$ and $Q, R, S \in \mathcal{P}$ with $Q_1 \perp \{R_1, S_1\}$, if $R \sim S$, then $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.*

Proof of Lemma 4. Recall that $X_1 = [\underline{c}_1, \overline{c}_1]$ and there exist $P, \tilde{P} \in \mathcal{P}$ such that $\tilde{P} \in M(P)$ and $P \succ \tilde{P}$. Without loss of generality, we can assume $\text{supp}(P_1) \cup \text{supp}(\tilde{P}_1) \in X_1^o = (\underline{c}_1, \overline{c}_1)$. To see why, suppose $P_1(\underline{c}_1) > 0$ or $\tilde{P}_1(\underline{c}_1) > 0$. By Axiom 3.2, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon = (\varepsilon_1, 0)$ and $\varepsilon' = (\varepsilon_1', 0)$ with $0 < \varepsilon_1, \varepsilon_1' < \bar{\varepsilon}$, we have $P_\varepsilon \succ \tilde{P}_{\varepsilon'}$. Since $\tilde{P} \in M(P)$, we can find $x_2 \in X_2$ with $P_1 \sim_{1|x_2} \tilde{P}_1$. By Lemma 1, the conditional preference $\succsim_{1|x_2}$ admits an EU representation with an strictly increasing and continuous index $v_{1|x_2}$. There exist $\hat{\varepsilon} = (\hat{\varepsilon}_1, 0)$ and $\hat{\varepsilon}' = (\hat{\varepsilon}_1', 0)$ such that $0 < \hat{\varepsilon}_1, \hat{\varepsilon}_1' < \bar{\varepsilon}$ and $P_{\hat{\varepsilon},1} \sim_{1|x_2} \tilde{P}_{\hat{\varepsilon}',1}$, that is, $P_{\hat{\varepsilon},1}$ and $\tilde{P}_{\hat{\varepsilon}',1}$ are comparable in dimension 1. Hence, $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, $\tilde{P}_{\hat{\varepsilon}'} \in M(P_{\hat{\varepsilon}})$, and $P_{\hat{\varepsilon},1}(\underline{c}_1) = \tilde{P}_{\hat{\varepsilon}',1}(\underline{c}_1) = 0$. Repeat the argument if $P_{\hat{\varepsilon},1}(\overline{c}_1) > 0$ or $\tilde{P}_{\hat{\varepsilon}',1}(\overline{c}_1) > 0$ and we are done.

Denote by $P^* = \lambda^* P + (1-\lambda^*)\tilde{P}$ the lottery found in Lemma 3. Clearly, either $P^* \not\sim P$ or $P^* \not\sim \tilde{P}$. By Lemma 3, for any $n > 0$, there exists $\lambda_n \in (\lambda^* - 1/n, \lambda^* + 1/n) \cap [0,1]$ with $P^* \not\sim \lambda_n P + (1-\lambda_n)\tilde{P} := P^n$. Since $\succsim$ is complete by Axiom 1, for each n, either $P^n \succ P^*$ or $P^* \succ P^n$. Then we can find a subsequence of $(P^n)_{n \geq 1}$, still denoted by $(P^n)_{n \geq 1}$, such that either $P^n \succ P^*$ for all n or $P^* \succ P^n$ for all n. Suppose that the former case holds. Take any $R, S \in \mathcal{P}$ such that $R \sim S$ and both R_1 and S_1 are singular with respect to both P_1 and $\tilde{P}_1$. Then both R_1 and S_1 are also singular with respect to both P_1^* and P_1^n for all $n \geq 1$. By Lemma 1, $P^*, P^n \in M(P)$ for all $n \geq 1$. Axiom 5 implies that for all

$\alpha \in (0,1)$ and $n \geq 1$, we have $\alpha P^n + (1-\alpha)R \succ \alpha P^* + (1-\alpha)S$. By Axiom 3.1, take n to infinity and we have $\alpha P^* + (1-\alpha)R \succsim \alpha P^* + (1-\alpha)S$. This holds for all $R \sim S$ with $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1\}$. By symmetry, for all $\alpha \in (0,1)$ and $R, S \in \mathcal{P}$ such that $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1\}$, we have

$$\alpha P^* + (1-\alpha)S \sim \alpha P^* + (1-\alpha)R.$$

The same result holds if $P^* \succ P^n$ for all n. Without loss of generality, we assume that $P^n \succ P^*$ for all n and $P^* \succ \tilde{P}$ from now on.

Fix any $Q \in \mathcal{P}$ such that $Q_1 \perp \{P_1, \tilde{P}_1\}$. Then $Q_1 \perp \{P_1^*, P_1^n\}$ for each n. By Axiom 5 and Lemma 1, for any $\beta \in (0,1)$, we have $\beta P^* + (1-\beta)Q \succ \beta\tilde{P} + (1-\beta)Q$ and $\beta P^* + (1-\beta)Q, \beta\tilde{P} + (1-\beta)Q \in M(\beta P + (1-\beta)Q)$. Similarly, as $P^n \succ P^*$ for all n, for any $\beta \in (0,1)$, we have $\beta P^n + (1-\beta)Q \succ \beta P^* + (1-\beta)Q$ and $\beta P^n + (1-\beta)Q \in M(\beta P + (1-\beta)Q)$. For any $R, S \in \mathcal{P}$ such that $R \sim S$ and $\{R_1, S_1\} \perp \{P_1, \tilde{P}_1, Q_1\}$, we know $\{R_1, S_1\} \perp \{\beta P_1^n + (1-\beta)Q_1, \beta P_1^* + (1-\beta)Q_1\}$ for all $n \geq 1$.

Repeat the previous arguments and we can show that for any $\alpha, \beta \in (0,1)$,

$$\alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)R \sim \alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)S.$$

The above indifference relation can be rearranged as

$$\beta[\alpha P^* + (1-\alpha)R] + (1-\beta)[\alpha Q + (1-\alpha)R] \sim \beta[\alpha P^* + (1-\alpha)S] + (1-\beta)[\alpha Q + (1-\alpha)S].$$

Again by Axiom 3.1, let $\beta \to 0^+$ and we have

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S, \tag{16}$$

for any $\alpha \in (0,1)$, $R \sim S$, $Q_1 \perp \{R_1, S_1\}$ and $\{P_1, \tilde{P}_1\} \perp \{Q_1, R_1, S_1\}$.

Fix $P, \tilde{P}$ and Q such that $Q_1 \perp \{P_1, \tilde{P}_1\}$. We want to strengthen property (16) by discarding the constraint that $\{P_1, \tilde{P}_1\} \perp \{R_1, S_1\}$. By Axiom 3.2, as $P \succ \tilde{P}$, we can find $\bar{\varepsilon} > 0$ such that for all $\varepsilon = (\varepsilon_1, 0)$ and $\varepsilon' = (\varepsilon_1', 0)$ with $0 < \varepsilon_1, \varepsilon_1' < \bar{\varepsilon}$, we have $P_\varepsilon \succ \tilde{P}_{\varepsilon'}$. Note that $\tilde{P}_\varepsilon \in M(P_\varepsilon)$. Since $\text{supp}(P_1) \cup \text{supp}(Q_1)$ is finite, $Q_1 \perp \{P_1, \tilde{P}_1\}$, and $\text{supp}(P_1) \cup \text{supp}(\tilde{P}_1) \subset X_1^o$, we can make $\bar{\varepsilon}$ small enough such that for all $\varepsilon_1, \varepsilon_1' \in (0, \bar{\varepsilon})$, we have $\text{supp}(P_{\varepsilon,1}) \cup \text{supp}(\tilde{P}_{\varepsilon',1}) \subset X_1^o$, $\text{supp}(P_{\varepsilon,1}) \cap \text{supp}(Q_1) = \emptyset$, $\text{supp}(\tilde{P}_{\varepsilon',1}) \cap \text{supp}(Q_1) = \emptyset$, and $\{P_{\varepsilon,1}, \tilde{P}_{\varepsilon',1}\} \perp Q_1$. Then

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S,$$

for any $\varepsilon_1, \varepsilon_1' \in (0, \bar{\varepsilon})$, $\alpha \in (0,1)$, $R \sim S$, $Q_1 \perp \{P_1, \tilde{P}_1, R_1, S_1\}$ and $\{P_{\varepsilon,1}, \tilde{P}_{\varepsilon',1}\} \perp \{R_1, S_1\}$. Then we argue that by varying ε, we can further get rid of the constraint that R_1, S_1 are singular with respect to $P_{\varepsilon,1}, \tilde{P}_{\varepsilon',1}$ for some $\varepsilon_1, \varepsilon_1' \in (0, \bar{\varepsilon})$. This is guaranteed by the fact that each lottery in $\mathcal{P}$ has a finite support and $\text{supp}(P_{\varepsilon,1}) \cup \text{supp}(\tilde{P}_{\varepsilon',1}) \subset X_1^o$ for each $\varepsilon_1, \varepsilon_1' \in (0, \bar{\varepsilon})$. Concretely, for any $R \sim S$ with $Q_1 \perp \{R_1, S_1\}$, we can always find $\varepsilon_1^*, \varepsilon_1^{*'} \in (0, \bar{\varepsilon})$ such that $\{P_{\varepsilon^*,1}, \tilde{P}_{\varepsilon^{*'},1}\} \perp \{R_1, S_1\}$. Thus,

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S,$$

for any $\alpha \in (0,1)$, $R \sim S$, and $Q_1 \perp \{P_1, \tilde{P}_1, R_1, S_1\}$. The same argument can be applied to further relax the requirement that $Q_1 \perp P_1$ and $Q_1 \perp \tilde{P}_1$. Hence, for any $Q \in \mathcal{P}$, we have

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S,$$

for any $\alpha \in (0,1)$, $R \sim S$, and $Q_1 \perp \{R_1, S_1\}$. □

Note that Lemma 4 can be interpreted as a version of the independence property which requires $R \sim S$ and certain singularity conditions. The remaining auxiliary results aim to generalize Lemma 4. First, for a fixed $y \in X_2$, we focus on the set of lotteries whose utilities are strictly bounded by two lotteries in $\mathcal{P}_1 \times \{y\}$, that is,

$$\Phi_{2,y} = \Big\{P \in \mathcal{P} : \exists\, T, T' \in \mathcal{P} \text{ with } T_2 = T_2' = y \text{ s.t. } T \succ P \succ T'\Big\}.$$

Lemma 5 shows that for each $P \in \Phi_{2,y}$, we can find some $x \in X_1$ such that $P \sim (x, y)$.

Lemma 5: *(i). For each $P, Q, R \in \mathcal{P}$ with $P \succ Q \succ R$, there exists $\lambda \in (0,1)$ such that $\lambda P + (1-\lambda)R \sim Q$.*

(ii). For each $P \in \Phi_{2,y}$ for some $y \in X_2$, there exists $x \in X_1$ such that $P \sim (x, y)$.

Proof of Lemma 5. (i). Denote $A = \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \succ Q\}$ and $\lambda = inf A$. We claim that $\lambda P + (1-\lambda)R \sim Q$. Suppose by contradiction that $\lambda P + (1-\lambda)R \not\sim Q$. If $\lambda P + (1-\lambda)R \succ Q$, then $\lambda \in A$, which is open by Axiom 3.1. Hence there exists $\lambda' < \lambda$ with $\lambda' \in A$, which contradicts with the definition of λ. If $\lambda P + (1-\lambda)R \prec Q$, then $\lambda \in \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \prec Q\}$, which is also open. We can find $\varepsilon > 0$ such that $[\lambda, \lambda+\varepsilon) \subseteq (0,1)\backslash A$. Again a contradiction with the definition of λ.

(ii). If $P \in \Phi_{2,y}$ for some $y \in X_2$, then we can find $p_1, p_1' \in \mathcal{P}_1$ with $(p_1, y) \succ P \succ (p_1', y)$. By Lemma 1, we can find a unique $\lambda \in [0,1]$ such that $P \sim (\lambda p_1 + (1-\lambda)p_1', y)$. Again by Lemma 1, there exists $x \in X_1$ such that $P \sim (x, y)$. $\square$

The next lemma generalizes Lemma 4 and shows that the independence property holds on $\Phi_{2,y}$ for each $y \in X_2$ subject to the singularity constraint.

Lemma 6: *For each $y \in X_2$ and $P, Q, R, S \in \Phi_{2,y}$, the following properties hold:*

(i). $P \sim Q$ and $P_1 \perp Q_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii). $P \succ Q$ and $P_1 \perp Q_1 \implies$ For each $\alpha \in (0,1)$, we have $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii). $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv). $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

Proof of Lemma 6. We will prove (i) and (ii). The other two results can be shown similarly. Suppose $P, Q \in \Phi_{2,y}$ for some $y \in X_2$ and $P_1 \perp Q_1$. By Lemma 5 and the definition of $\Phi_{2,y}$, there exist $x_P, x_Q \in X_1^o$ such that $P \sim (x_P, y)$ and $Q \sim (x_Q, y)$. By Lemma 1, we can find $\varepsilon > 0$ such that for all $z_P \in [x_P - \varepsilon, x_P]$, $z_Q \in [x_Q - \varepsilon, x_Q]$, there exist $z_P' \geq x_P, z_Q' \geq x_Q$ with $P \sim (1/2\delta_{z_P} + 1/2\delta_{z_P'}, y)$ and $Q \sim (1/2\delta_{z_Q} + 1/2\delta_{z_Q'}, y)$. Moreover, as z_P, z_Q increases, z_P', z_Q' will be decreasing continuously. Since $\text{supp}(P_1) \cup \text{supp}(Q_1)$ is finite, we can construct $z_P^* \neq z_Q^*$, $z_P^{*'} \neq z_Q^{*'}$ and $z_P^*, z_Q^*, z_P^{*'}, z_Q^{*'} \notin \text{supp}(P_1) \cup \text{supp}(Q_1)$. Denote $P' = (1/2\delta_{z_P^*} + 1/2\delta_{z_P^{*'}}, y)$, $Q' =$

$(1/2\delta_{z_Q^*} + 1/2\delta_{z_Q^{*\prime}}, y)$. Then $P \sim P', Q \sim Q'$ and P_1, Q_1, P_1', Q_1' are singular with respect to each other. Apply Lemma 4 twice and we get for any $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)Q \sim \alpha P + (1-\alpha)Q' \sim \alpha P' + (1-\alpha)Q'.$$

By Lemma 1 given y as the marginal lottery in dimension 2, we have

$$P \sim Q \Longrightarrow P' \sim Q' \Longrightarrow \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \sim Q' \sim Q,$$
$$P \succ Q \Longrightarrow P' \succ Q' \Longrightarrow P \sim P' \sim \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \succ Q' \sim Q.$$

□

The next result states that if the independence property holds on $\Phi_{2,y}$ for each y, then it also holds on their union.

Lemma 7: *For each $y \in X_2$ and $P, Q, R, S \in \cup_{y \in X_2}\Phi_{2,y}$, the following properties hold:*

(i). $P \sim Q$ and $P_1 \perp Q_1 \Longrightarrow$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii). $P \succ Q$ and $P_1 \perp Q_1 \Longrightarrow$ For each $\alpha \in (0,1)$, we have $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii). $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \Longrightarrow$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv). $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \Longrightarrow$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

Proof of Lemma 7. First, for any $P, Q, R, S \in \cup_{y \in X_2}\Phi_{2,y}$, we claim that there exist a positive integer K and $z_k \in X_2, k = 1, ..., K$ such that $z_1 < z_2 < ... < z_K$ and $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. Choose $P^1, P^2 \in \{P, Q, R, S\}$ such that $P^1 \succsim P, Q, R, S \succsim P^2$. To see this, suppose that $P^1 \in \Phi_{2,y_1}$ and $P^2 \in \Phi_{2,y_2}$ with $y_1 \geq y_2$. If $y_1 = y_2$, then $P, Q, R, S \in \Phi_{2,y_1}$ and we are done. Now suppose that $y_1 > y_2$ and by Lemma 1, we can find $t, t' \in X_1$ with $(t, y_1) \succ P^1 \succ (t', y_1)$ and $(t, y_2) \succ P^2 \succ (t', y_2)$. For each $y \in [y_2, y_1]$, denote $H(y) := \{P \in \mathcal{P} : (t, y) \succ P \succ (t', y)\}$. Note that $H(y) \subseteq \Phi_{2,y}$. By Axiom 3.2, for any $y \in [y_2, y_1]$, there exists $\varepsilon_y > 0$ such that $H(y) \cap H(y') \neq \emptyset$ for all $y' \in [y - \varepsilon_y, y + \varepsilon_y] \cap [y_2, y_1]$. Also, $\{P \in \mathcal{P} : P^1 \succsim$

$P \succsim P^2\} \subseteq \cup_{y_2 \le y \le y_1} H(y)$. By the Heine–Borel theorem, since $[y_2, y_1]$ is compact and $\big((z-\varepsilon_z, z+\varepsilon_z)\big)_{y_2 \le z \le y_1}$ is an open cover of $[y_2, y_1]$, we can find finitely many $z_1 < z_2 < ... < z_K \in [y_2, y_1]$ with $[y_2, y_1] \subseteq \cup_{k=1}^K [z_k - \varepsilon_{z_k}, z_k + \varepsilon_{z_k}]$. Hence,

$$P, Q, R, S \in \{P \in \mathcal{P} : P^1 \succsim P \succsim P^2\} \subseteq \cup_{y_2 \le y \le y_1} H(y) = \cup_{k=1}^K H(z_k) \subseteq \cup_{k=1}^K \Phi_{2,z_k}.$$

Then we use induction to show that the properties (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. By Lemma 6, for each $k = 1, ..., K$, those properties hold for $P, Q, R, S \in \Phi_{2,z_k}$. Suppose by induction that they also hold for $P, Q, R, S \in \cup_{k=1}^t \Phi_{2,z_k}$ for some $1 \le t < K$. By construction, we can find $T^1, T^2 \in \Phi_{2,z_t} \cap \Phi_{2,z_{t+1}}$ with $T^1 \succ T^2$. By Lemma 5 and Lemma 1, since P_1, Q_1, R_1, S_1 have finite supports, we can also find $p_1, p_2, q_1, q_2 \in \mathcal{P}_1$ such that $(p_1, z_{t+1}) \sim (q_1, z_t) \sim T^1$, $(p_2, z_{t+1}) \sim (q_2, z_t) \sim T^2$ and $\{p_1, p_2, q_1, q_2\} \perp \{P_1, Q_1, R_1, S_1\}$.

For (i), suppose $P \succsim Q$, $P_1 \perp Q_1$ and $P, Q \in \cup_{k=1}^{t+1} \Phi_{2,z_k}$. If $P \sim Q$, then $P, Q \in \Phi_{2,z_k}$ for some $k = 1, ..., t+1$ and hence (i) holds by the inductive hypothesis.

Now we check (ii). If $P \succ Q$, then it suffices to consider the case where $P \in \Phi_{2,z_{t+1}} \backslash (\cup_{k=1}^t \Phi_{2,z_k})$ and $Q \in (\cup_{k=1}^t \Phi_{2,z_k}) \backslash \Phi_{2,z_{t+1}}$. This implies $P \succ T^1 \succ T^2 \succ Q$. By Lemma 5, there exist $\lambda_1 \neq \lambda_2 \in (0,1)$ such that $T^1 \sim \lambda_1 P + (1-\lambda_1) Q$ and $T^2 \sim \lambda_2 P + (1-\lambda_2) Q$. Then (ii) holds for $\lambda = \lambda_1, \lambda_2$. Notice that at the moment we cannot conclude that $\lambda_1 > \lambda_2$. Suppose that $\lambda_i > \lambda_{-i}$ for some $i = 1, 2$. By Lemma 5 and Lemma 1, we can find $P', Q' \in \hat{\mathcal{P}}$ such that $Q' \sim Q, P' \sim P$ and marginal lotteries $P_1, P_1', Q_1, Q_1', p_1, q_1, p_2, q_2$ are singular with respect to each other. This guarantees

$$T^1 \sim \lambda_1 P + (1-\lambda_1) Q \sim \lambda_1 P' + (1-\lambda_1) Q \sim \lambda_1 P + (1-\lambda_1) Q' \sim \lambda_1 P' + (1-\lambda_1) Q',$$

$$T^2 \sim \lambda_2 P + (1-\lambda_2) Q \sim \lambda_2 P' + (1-\lambda_2) Q \sim \lambda_2 P + (1-\lambda_2) Q' \sim \lambda_2 P' + (1-\lambda_2) Q'.$$

By (i), for all $\beta, \beta' \in (0,1)$, we have $\beta P + (1-\beta) P' \sim P$ and $\beta' Q + (1-\beta') Q' \sim Q$. Apply Lemma 4 twice and we derive that for each $\lambda, \beta, \beta' \in (0,1)$,

$$\lambda P + (1-\lambda) Q \sim \lambda(\beta P + (1-\beta) P') + (1-\lambda)(\beta' Q + (1-\beta') Q'). \tag{17}$$

For any $\lambda \in (\lambda_{-i}, \lambda_i)$, let $\beta = 1$, $\beta' = \frac{\lambda_i - \lambda}{\lambda_i(1-\lambda)}$, and (17) becomes

$$\begin{aligned}\lambda P + (1-\lambda)Q \sim & \frac{\lambda}{\lambda_i}(\lambda_i P + (1-\lambda_i)Q') + (1 - \frac{\lambda}{\lambda_i})Q \\ \sim & \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i})Q.\end{aligned}$$

The second indifference relation holds due to $\lambda_i P + (1-\lambda_i)Q' \sim T^i \sim (q_i, z_t)$ and Lemma 4. Then by the inductive hypothesis on $\cup_{k=1}^{t} \Phi_{2,z_k}$, we have

$$P \succ (q_i, z_t) \succ \lambda P + (1-\lambda)Q \sim \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i})Q \succ Q.$$

If $\lambda > \lambda_i$, then let $\beta = \frac{\lambda - \lambda_i}{\lambda(1-\lambda_i)}$, $\beta' = 0$ and (17) becomes

$$\begin{aligned}\lambda P + (1-\lambda)Q \sim & \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(\lambda_i P' + (1-\lambda_i)Q) \\ \sim & \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(p_i, z_{t+1})\end{aligned}$$

The second indifference holds due to $\lambda_i P' + (1-\lambda_i)Q \sim T^i \sim (p_i, z_{t+1})$ and Lemma 4. Then by Lemma 6 on $\Phi_{2,z_{t+1}}$, we have

$$P \succ \lambda P + (1-\lambda)Q \sim \frac{\lambda - \lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda - \lambda_i}{1-\lambda_i})(p_i, z_{t+1}) \succ (p_i, z_{t+1}) \succ Q.$$

A symmetric argument works for $\lambda < \lambda_{-i}$. Hence property (ii) holds on $\cup_{k=1}^{t+1} \Phi_{2,z_k}$.

We claim that for $P, Q \in \cup_{k=1}^{t+1} \Phi_{2,z_k}$ with $P \succ Q$, $P_1 \perp Q_1$ and $1 > \lambda_1 > \lambda_2 > 0$, we have $\lambda_1 P + (1-\lambda_1)Q \succ \lambda_2 P + (1-\lambda_2)Q$. To see this, by (17), we can find $P' \sim P$ where P' is singular with respect to both P and Q such that

$$\lambda_1 P + (1-\lambda_1)Q \sim \frac{\lambda_1 - \lambda_2}{1-\lambda_2}P' + \frac{1-\lambda_1}{1-\lambda_2}[\lambda_2 P + (1-\lambda_2)Q] \succ \lambda_2 P + (1-\lambda_2)Q.$$

The second strict ranking follows from property (ii) since $P \sim P' \succ \lambda_2 P + (1 - \lambda_2)Q$.

Given this claim, the proof for (iii) and (iv) on $\cup_{k=1}^{t+1} \Phi_{2,z_k}$ is similar to the proof of (ii). By induction, (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^{K} \Phi_{2,z_k}$ and hence arbitrary

$P, Q, R, S \in \cup_{y \in X_2} \Phi_{2,y}$. $\square$

It is worthwhile to notice that $\cup_{y \in X_2} \Phi_{2,y}$ is a strict subset of $\mathcal{P}$. The next lemma shows that it omits the worst and the best (degenerate) lotteries. Denote $\bar{c} = (\bar{c}_1, \bar{c}_2)$ and $\underline{c} = (\underline{c}_1, \underline{c}_2)$.

Lemma 8: $\mathcal{P} \backslash (\cup_{y \in X_2} \Phi_{2,y}) = \{\bar{c}, \underline{c}\}$.

Proof of Lemma 8. For each $P \in \mathcal{P}$ with $P \notin \{\bar{c}, \underline{c}\}$, we claim that $\bar{c} \succ P \succ \underline{c}$. By Axiom 3.2 and Axiom 2, this implies $P \in \cup_{y \in X_2} \Phi_{2,y}$. If $|\operatorname{supp}(P)| = 1$, then the result follows from Lemma 1 and $P \in \cup_{y \in X_2} \Phi_{2,y}$. Now we suppose that $|\operatorname{supp}(P_1)| \geq 2$. We can write P as $\sum_x (x, P_{2|x}) P_1(x)$. If $(x, P_{2|x}) \notin \{\bar{c}, \underline{c}\}$ for all $x \in \operatorname{supp}(P_1)$, then apply part (i) or (ii) in Lemma 7 repeatedly and we can conclude that $P \in \cup_{y \in X_2} \Phi_{2,y}$ and the result holds. Hence it suffices to prove the case where $(x, P_{2|x}) \in \{\bar{c}, \underline{c}\}$ for some $x \in \operatorname{supp}(P_1)$.

Denote $P = P_1(\bar{c}_1)\delta_{\bar{c}} + P_1(\underline{c}_1)\delta_{\underline{c}} + (1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P'$ where $P' \in \cup_{y \in X_2} \Phi_{2,y}$, $P_1(\bar{c}_1) < 1$, $P_1(\underline{c}_1) < 1$ and $P_1(\bar{c}_1) + P_1(\underline{c}_1) > 0$.[35] By Axioms 2 and 3.2, we can find $\varepsilon^1 = (\varepsilon_1, 0), \varepsilon^2 = (0, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that $\bar{c} \succ \bar{c} - \varepsilon^1 \sim \bar{c} - \varepsilon^2 \succ P' \succ \underline{c} + \varepsilon^1 \sim \underline{c} + \varepsilon^2 \succ \underline{c}$. For each $\beta \in (0,1)$, denote $P^\beta = \beta P + (1-\beta)/2\delta_{\bar{c}-\varepsilon^2} + (1-\beta)/2\delta_{\underline{c}+\varepsilon^2}$. Notice that $(x, P^\beta_{2|x}) \notin \{\bar{c}, \underline{c}\}$ for all $x \in \operatorname{supp}(P^\beta_1) = \operatorname{supp}(P_1)$. Hence we can apply Lemma 1 and Lemma 7 and derive

$$
\begin{aligned}
P^\beta &= \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + \beta P_1(\underline{c}_1)\delta_{\underline{c}} + \frac{1}{2}(1-\beta)\delta_{\underline{c}+\varepsilon^2} + \beta(1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P' \\
&\prec \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + (1 - \beta P_1(\bar{c}_1) - \frac{1}{2}(1-\beta))\delta_{\bar{c}-\varepsilon^1}
\end{aligned}
$$

Let $\beta \to 1$ and by Axiom 3.1, we have

$$
P \precsim P_1(\bar{c}_1)\delta_{\bar{c}} + (1 - P_1(\bar{c}_1))\delta_{\bar{c}-\varepsilon^1} \prec \bar{c}.
$$

The last strict ranking follows from Lemma 1 for conditional preference $\succsim_{1|\bar{c}_2}$. A similar argument can be adopted to show that $P \succ \underline{c}$. This completes the proof. $\square$

[35] If $P_1(\bar{c}_1) + P_1(\underline{c}_1) = 1$, then P' can be arbitrarily chosen.

As a direct corollary of Lemma 5 and Lemma 8, for any $P \in \mathcal{P}$, we can find some $x \in X$ such that $P \sim x$.

Since $P \succ (\underline{c}_1, \underline{c}_2)$ for any $P \neq (\underline{c}_1, \underline{c}_2)$ and $P \prec (\bar{c}_1, \bar{c}_2)$ for any $P \neq (\bar{c}_1, \bar{c}_2)$, we can easily use the arguments in Lemma 6 to show that the independence property holds for $P, Q, R, S \in \Phi_{2,\underline{c}_2} \cup \{(\underline{c}_1, \underline{c}_2)\}$ or $P, Q, R, S \in \Phi_{2,\bar{c}_2} \cup \{(\bar{c}_1, \bar{c}_2)\}$. Hence we can remove the restriction that $P, Q, R, S \in \cup_{y \in X_2} \Phi_{2,y}$ in Lemma 7 and derive the following corollary.

Corollary 4: *The following properties hold:*

(i). $P \sim Q$ and $P_1 \perp Q_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii). $P \succ Q$ and $P_1 \perp Q_1 \implies$ For each $\alpha \in (0,1)$, we have $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii). $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv). $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1 \implies$ For each $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

The next lemma proves a variant of part (iv) in Corollary 4.

Lemma 9: *For any $P, Q, R, S \in \mathcal{P}$ such that $P_1 \perp R_1$ and $\operatorname{supp}(Q_1) \cup \operatorname{supp}(S_1) \subseteq \{\bar{c}, \underline{c}\}$. If $P \sim Q$ and $R \sim S$, then for all $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.*

Proof of Lemma 9. If $P, R \in \{\bar{c}, \underline{c}\}$, then $P \sim Q$ and $R \sim S$ implies $P = Q$ and $R = S$. The result trivially holds. Without loss of generality, suppose $\bar{c} \succ P \succ \underline{c}$. By Axiom 3.2, Lemma 5 and Lemma 8, there exist $(y_1, y_2) \in X$ and $\varepsilon = (\varepsilon_1, 0)$ with $\varepsilon_1 > 0$ sufficiently small such that $\bar{c} - \varepsilon \succ P \sim Q \sim (y_1, y_2) \succ \underline{c} + \varepsilon$ and $y_1 \notin \{\bar{c}_1, \underline{c}_1\}$. Since $P \sim (y_1, y_2)$, $R \sim S$, $P_1 \perp R_1$ and $y_1 \perp S_1$, by part (iv) in Corollary 4, we have $\alpha P + (1-\alpha)R \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$. Hence it suffices to show that $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$.

By Lemma 5, there exists $\gamma \in (0,1)$ such that $\hat{Q} := \gamma\delta_{\bar{c}-\varepsilon} + (1-\gamma)\delta_{\underline{c}+\varepsilon} \sim P \sim Q$. Since $Q_1 \perp \hat{Q}_1$, for any $\beta \in (0,1)$, part (i) in Corollary 4 implies $Q^\beta := \beta Q + (1-\beta)\hat{Q} \sim Q$. We claim that for any $\alpha, \beta \in (0,1)$, we have $\alpha Q^\beta +$

$(1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$.

To prove the claim, first note that $Q(\bar{c}) \in (0,1)$ as $\bar{c} \succ Q \succ \underline{c}$ and $\text{supp}(Q) \subseteq \{\bar{c}, \underline{c}\}$. Then

$$Q^\beta = \beta Q + (1-\beta)\hat{Q} = \Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + \Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big).$$

By Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2 respectively, we can find $x_1, x_1' \in (\underline{c}_1, \bar{c}_1)$ such that $x_1 \neq x_1'$ and

$$(x_1, \bar{c}_2) \sim \frac{\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}}{\beta Q(\bar{c}) + (1-\beta)\gamma} \quad , \quad (x_1', \underline{c}_2) \sim \frac{\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}}{\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)}.$$

Part (iv) in Corollary 4 implies

$$Q^\beta \sim \Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + \Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)}.$$

Denote by $\tilde{Q}^\beta$ the right-hand side of the above relation. Then we have

$$\begin{aligned}
\alpha Q^\beta + (1-\alpha)S =& \alpha\Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\
&+ \alpha\Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big) + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\
\sim& \alpha\Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\
&+ \alpha\Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)} + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\
=& \alpha\tilde{Q}^\beta + (1-\alpha)S.
\end{aligned}$$

The indifference relation follows from applying Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2, and part (iv) in Corollary 4 sequentially. Since $(y_1, y_2) \sim Q^\beta \sim \tilde{Q}^\beta$ and $S_1 \perp \{y_1, \tilde{Q}_1^\beta\}$, again by part (iv) in Corollary 4, we have

$$\alpha Q^\beta + (1-\alpha)S \sim \alpha\tilde{Q}^\beta + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S.$$

This holds for all $\alpha, \beta \in (0,1)$. By Axiom 3.1, for any $\alpha \in (0,1)$, let $\beta \to 1$ and we have $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$. This completes the proof. $\square$

The final auxiliary result strengthens Lemma 5 and is key to our representation.

Lemma 10: *For each $P \in \mathcal{P}$, there exists a unique $\alpha \in [0,1]$ such that $P \sim \alpha\delta_{\bar{c}} + (1-\alpha)\delta_{\underline{c}}$. Moreover, if $P \sim \alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}}$ and $Q \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$, then $P \succsim Q$ if and only if $\alpha_1 \geq \alpha_2$.*

Proof of Lemma 10. Since $\underline{c} \precsim P \precsim \bar{c}$ for all $P \in \mathcal{P}$, by Lemma 5, it suffices to show that for any $\alpha_1, \alpha_2 \in (0,1)$, if $\alpha_1 > \alpha_2$, then $\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} \succ \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. By Lemma 5, Lemma 8 and Axiom 3.2, there exists $(x_1, x_2) \in X$ such that $x_1 \neq \bar{c}_1$ and $(x_1, x_2) \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. Then Lemma 9 implies

$$\begin{aligned}
\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} &= (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}(\alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}) \\
&\sim (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}\delta_{(x_1,x_2)} \\
&\succ \delta_{(x_1,x_2)} \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}.
\end{aligned}$$

The strict ranking follows from part (ii) in Corollary 4. □

Step 3: Representation of $\succsim$.

Lemma 11: *The binary relation $\succsim$ is represented by $U : \mathcal{P} \to \mathbb{R}$ such that for each $P \in \mathcal{P}$,*

$$U(P) = \sum_x w(x, CE_{v_x}(P_{2|x}))P_1(x) \tag{18}$$

where w and v_x for all $x \in X_1$ are continuous and strictly increasing. Moreover, w and v_x are unique up to a positive affine transformation for all $x \in X_1$.

Proof of Lemma 11. By Lemma 10, for each $P \in \mathcal{P}$, there exists a unique $\alpha(P) \in [0,1]$ such that $P \sim \alpha(P)\delta_{\bar{c}} + (1-\alpha(P))\delta_{\underline{c}}$. Define $U : \mathcal{P} \to [0,1]$ such that $U(P) = \alpha(P)$. Then $U[\bar{c}] = 1, U[\underline{c}] = 0$ and U represents $\succsim$.

Fix any $P, Q \in \mathcal{P}$ and $\alpha \in (0,1)$ such that $P_1 \perp Q_1$. By Lemma 9, we have

$$\begin{aligned}
\alpha P + (1-\alpha)Q &\sim \alpha(U(P)\delta_{\bar{c}} + (1-U(P))\delta_{\underline{c}}) + (1-\alpha)(U(Q)\delta_{\bar{c}} + (1-U(Q))\delta_{\underline{c}}) \\
&= (\alpha U(P) + (1-\alpha)U(Q))\delta_{\bar{c}} + (1 - \alpha U(P) - (1-\alpha)U(Q))\delta_{\underline{c}}
\end{aligned}$$

By definition of U, we know $\alpha P+(1-\alpha)Q \sim U(\alpha P+(1-\alpha)Q)\delta_{\bar{c}}+(1-U(\alpha P+(1-\alpha)Q))\delta_{\underline{c}}$. Then Lemma 10 implies

$$U(\alpha P+(1-\alpha)Q)=\alpha U(P)+(1-\alpha)U(Q), \tag{19}$$

for any $\alpha \in (0,1)$ and $P, Q \in \mathcal{P}$ such that $P_1 \perp Q_1$. By applying (19) multiple times for each $P \in \mathcal{P}$, we get

$$U(P)=U(\sum_{x\in X_1} P_1(x)(\delta_x, P_{2|x}))=\sum_{x\in X_1} U(\delta_x, P_{2|x}))P_1(x). \tag{20}$$

By Lemma 1, for each $x \in X_1$, the conditional preference $\succsim_{2|x}$ admits an EU representation with some continuous and strictly increasing utility index v_x. Then there exists a continuous and strictly increasing function ϕ_x such that for all $p \in \mathcal{P}_2$,

$$U(\delta_x, P_{2|x})=\phi_x(CE_{v_x}(p)).$$

Define $w : X \to \mathbb{R}$ as $w(x,y)=\phi_x(y)=U(\delta_x,\delta_y)$ for all $(x,y)\in X$. Then the utility function (20) can be rewritten as for all $P \in \mathcal{P}$

$$U(P)=\sum_x w(x, CE_{v_x}(P_{2|x}))P_1(x).$$

This is exactly (18). By Axiom 2, we know that w is strictly increasing. Note that w is bounded since $w(\underline{c}_1,\underline{c}_2)\le w(x,y)\le w(\bar{c}_1,\bar{c}_2)$ for all $(x,y)\in X$. Moreover, w is unique up to a positive affine transformation.

The final step is to verify that w is continuous. Suppose by contradiction that w is not continuous, then we can find $(x,y)\in X$ and a sequence $(x_n,y_n)\to(x,y)$ such that $\lim_{n\to\infty} w(x_n,y_n)\neq w(x,y)$. As w is bounded, we can find a subsequence of $\big((x_n,y_n)\big)_{n\ge1}$ (still denoted by $\big((x_n,y_n)\big)_{n\ge1}$) such that $\lim_{n\to\infty} w(x_n,y_n)=a\neq b=w(x,y)$. Without loss of generality, assume that $a<b$. Note that for $x_1\neq x_2$ and $\eta\in(0,1)$, we have $V(\eta(x_1,y_1)+(1-\eta)(x_2,y_2))=\eta V(x_1,y_1)+(1-\eta)V(x_2,y_2)$. We can find some $P\in\mathcal{P}_1\times\mathcal{P}_2$ with $V(P)\in(a,b)$. As $\lim_{n\to\infty} w(x_n,y_n)=a<V(P)$, for n large enough, we have $w(x_n,y_n)<V(P)<b$, that is, $(x_n,y_n)\prec P\prec(x,y)$. When n goes to infinity, $(x_n,y_n)\to(x,y)$. However, by Axiom 3.2, we must have $(x,y)\precsim P$, which leads to a contradiction. Hence w is continuous. $\square$

Step 4: Characterize when $\succsim$ admits a PREU representation.

We note that the utility representation (18) in Lemma 11 is more general than the PREU representation, since the conditional preference in dimension 2 is represented by v_x, which can depend on the outcome x in dimension 1. Indeed, if v_x is independent of x, then (18) reduces to the PREU representation. This property is captured by the Axiom 12 introduced in the main text, which states that the decision maker's evaluation of risk in dimension 2 does not depend on the outcome in dimension 1. This makes sense when the decision maker evaluates risk in dimension 2 before aggregating outcomes. The next result characterizes the PREU representation.

Lemma 12: *Let $\succsim$ be a binary relation on $\mathcal{P}$. If $\succsim$ satisfies Axioms 1-5 and Axiom 12, and violates Axiom 13, then it admits a PREU representation.*

Proof of Lemma 12. By Lemma 11, the binary relation $\succsim$ admits a representation (18). Axiom 12 and Lemma 1 then imply that the certainty equivalent function CE_{v_x} is independent of x. Hence reduces to the PREU representation. $\square$

Step 5: Show that if $\succsim$ does not admit a REU or a PREU representation, then it must admit an EU representation.

We also note that the utility representation (18) in Lemma 11 is more general than the EU representation. If v_x is a positive affine transformation of $w(x,\cdot)$ for all $x \in X_1$, then (18) reduces to the EU representation. The final step of the proof is to show that this is the only possible case if $\succsim$ does not satisfy Axioms 13 and 12. We start with an implication if Axiom 12 fails.

Lemma 13: *Assume that $\succsim$ admits a representation given by (18). For any $x, y \in X_1$ and $\alpha \in (0,1)$, if $\succsim_{2|x}$ and $\succsim_{2|y}$ are not identical, then there exist $p, q \in \mathcal{P}_2$ such that $p \succ_{2|x} q$, $q \succ_{2|x} p$, and $p \sim_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$. Moreover, we can choose p to be the same across all $\alpha \in (0,1)$, or choose q to be the same across all $\alpha \in (0,1)$.*

Proof of Lemma 13. By assumption, there exist $p, q \in \mathcal{P}_2$ such that $p \succsim_{2|x} q$ and $q \succ_{2|y} p$, or $p \succsim_{2|y} q$ and $q \succ_{2|x} p$. We claim that p, q can be chosen such that both relations are strict. Suppose $p \sim_{2|x} q$ and $q \succ_{2|y} p$. The case where $p \sim_{2|y} q$ and $q \succ_{2|x} p$ is symmetric. By Lemma 1, we know $q \neq \underline{c}_2$ and $p \neq \underline{c}_2$. Then we can find $\beta \in (0,1)$ sufficiently close to 1 such that $p \succ_{2|x} \beta q + (1-\beta)\delta_{\underline{c}_2}$ and $\beta q + (1-\beta)\delta_{\underline{c}_2} \succ_{2|y} p$. Hence, there exist $p, q \in \mathcal{P}_2$ with $p \succ_{2|x} q$ and $q \succ_{2|y} p$. Fix any $\alpha \in (0,1)$. If $p \sim_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$, then we are done. If $p \succ_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$, then there exists a unique $\beta \in (0,1)$ such that $\beta p + (1-\beta)\delta_{\underline{c}_2} \sim_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$. Clearly, $q \succ_{2|y} \beta p + (1-\beta)\delta_{\underline{c}_2}$. We claim that $\beta p + (1-\beta)\delta_{\underline{c}_2} \succ_{2|x} q$, since otherwise, by the utility representation (18), we must have $q \succ_{2|\alpha\delta_x+(1-\alpha)\delta_y} \beta p + (1-\beta)\delta_{\underline{c}_2}$, leading to a contradiction. Hence, the results hold for $\beta p + (1-\beta)\delta_{\underline{c}_2}$ and q. If $q \succ_{2|\alpha\delta_x+(1-\alpha)\delta_y} p$, then there exists a unique $\beta' \in (0,1)$ such that the results hold for $\beta' p + (1-\beta')\delta_{\bar{c}_2}$ and q. Note that we have chosen q to be the same across all $\alpha \in (0,1)$. A symmetric proof works when we choose p to be the same across all $\alpha \in (0,1)$. $\square$

Endow $(\mathcal{P}_2)^2$ with the product topology. For each $x, y \in X_1$ and $\alpha \in (0,1)$, define

$$\Gamma_{x,y}(\alpha) = \Big\{(p,q) \in (\mathcal{P}_2)^2 \mid p \succ_{2|x} q, q \succ_{2|x} p \text{ and } p \sim_{2|\alpha\delta_x+(1-\alpha)\delta_y} q\Big\}.$$

We claim that $\Gamma_{x,y}$ satisfies the following properties: (i) If $\Gamma_{x,y}(\alpha_0) \neq \emptyset$ for some $\alpha_0 \in (0,1)$, then $\Gamma_{x,y}(\alpha) \neq \emptyset$ for all $\alpha \in (0,1)$; (ii) If $(p,q) \in \Gamma_{x,y}(\alpha)$ for some $\alpha \in (0,1)$, then for any $\beta \in (0,1)$ and $r \in \mathcal{P}_2$, we have $(\beta p+(1-\beta)r, \beta q+(1-\beta)r) \in \Gamma_{x,y}(\alpha)$; (iii) The set $\bigcup_{\alpha\in(0,1)} \Gamma_{x,y}(\alpha)$ is open. The first two properties are direct corollaries of Lemma 13 and Lemma 1. For (iii), suppose that $(p,q) \in \Gamma_{x,y}(\alpha)$ for some $\alpha \in (0,1)$. Then $x \neq y$. By Lemma 1, there exists an open neighborhood of (p,q), denoted by $\mathcal{M} \subset (\mathcal{P}_2)^2$, such that for each $(p',q') \in \mathcal{M}$, we have $p' \succ_{2|x} q'$ and $q' \succ_{2|y} p'$. By the utility representation (18), there exists a unique $\alpha' \in (0,1)$ such that $p' \sim_{2|\alpha'\delta_x+(1-\alpha')\delta_y} q'$. Hence, $(p',q') \in \bigcup_{\alpha\in(0,1)} \Gamma_{x,y}(\alpha)$ for each $(p',q') \in \mathcal{M}$.

The following lemma characterizes when $\succsim$ admits an EU representation.

Lemma 14: *Let $\succsim$ be a binary relation on $\mathcal{P}$. If $\succsim$ satisfies Axioms 1-5 and violates Axioms 13 and 12, then it admits an EU representation.*

Proof of Lemma 14. By Lemma 11, the binary relation $\succsim$ admits a representation (18), that is, the utility of each $P \in \mathcal{P}$ is $U(P) = \sum_x w(x, CE_{v_x}(P_{2|x}))P_1(x)$. Since $\succsim$ violates Axiom 12, there exist $x_0, x_1 \in X_1$ such that $v_{x_0} \not\propto^a v_{x_1}$.

Fix any $x, y \in X_1$ with $v_x \not\propto^a v_y$ and $\alpha \in [0, 1]$. Consider three conditional preferences $\succsim_{2|x}, \succsim_{2|y}$ and $\succsim_{2|\alpha\delta_x+(1-\alpha)\delta_y}$ in dimension 2. We can interpret $\succsim_{2|x}$ and $\succsim_{2|y}$ as individual preferences, and $\succsim_{2|\alpha\delta_x+(1-\alpha)\delta_y}$ as the group preference. By Lemma 1, all three conditional preferences admit EU representations on $\mathcal{P}_2$. Moreover, by linearity of (18), for any $p, q \in \mathcal{P}_2$,

$$\begin{aligned}
p \succ_{2|x} q, p \succ_{2|y} q &\Longrightarrow w(x, CE_{v_x}(p)) > w(x, CE_{v_x}(q)), w(y, CE_{v_y}(p)) > w(y, CE_{v_y}(q)) \\
&\Longrightarrow U(\alpha(x, p) + (1-\alpha)(y, p)) > U(\alpha(x, p) + (1-\alpha)(y, p)) \\
&\Longrightarrow p \succ_{2|\alpha\delta_x+(1-\alpha)\delta_y} q.
\end{aligned}$$

Similarly, we can show that if $p \sim_{2|x} q, p \sim_{2|y} q$, then $p \sim_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$. Hence, by Harsanyi (1955)'s utilitarianism theorem, there exists a function $\tau : [0, 1] \to [0, 1]$ such that for each $\alpha \in [0, 1]$, we have $v_{\alpha\delta_x+(1-\alpha)\delta_y} \propto^a \tau(\alpha)v_x + (1 - \tau(\alpha))v_y$.

We claim that τ is strictly increasing. To see this, first note that we can set $\tau(0) = 0$ and $\tau(1) = 1$. Consider $\alpha, \alpha' \in (0, 1)$ with $\alpha > \alpha'$. By Lemma 13, we can find $p, q \in \mathcal{P}_2$ such that $p \succ_{2|x} q$, $q \succ_{2|y} p$, and $p \sim_{2|\alpha'\delta_x+(1-\alpha')\delta_y} q$. By (18) and $\alpha > \alpha'$, we have $p \succ_{2|\alpha\delta_x+(1-\alpha)\delta_y} q$. This implies

$$\begin{aligned}
\tau(\alpha)\mathbb{E}_{v_x}[p] + (1 - \tau(\alpha))\mathbb{E}_{v_y}[p] > \tau(\alpha)\mathbb{E}_{v_x}[q] + (1 - \tau(\alpha))\mathbb{E}_{v_y}[q], \\
\tau(\alpha')\mathbb{E}_{v_x}[p] + (1 - \tau(\alpha'))\mathbb{E}_{v_y}[p] = \tau(\alpha')\mathbb{E}_{v_x}[q] + (1 - \tau(\alpha'))\mathbb{E}_{v_y}[q].
\end{aligned}$$

Since $\mathbb{E}_{v_x}[p] > \mathbb{E}_{v_x}[q]$ and $\mathbb{E}_{v_y}[p] < \mathbb{E}_{v_y}[q]$, we conclude that $\tau(\alpha) > \tau(\alpha')$.

For each $\alpha \in (0, 1)$, by Lemma 13, we can find $p, q \in \mathcal{P}_2$ such that $(p, q) \in \Gamma_{x,y}(\alpha)$. Since $v_{\alpha\delta_x+(1-\alpha)\delta_y} \propto^a \tau(\alpha)v_x + (1 - \tau(\alpha))v_y$, we have

$$\frac{\tau(\alpha)}{1 - \tau(\alpha)} = \frac{\mathbb{E}_{v_y}[q] - \mathbb{E}_{v_y}[p]}{\mathbb{E}_{v_x}[p] - \mathbb{E}_{v_x}[q]}, \tag{21}$$

$$\frac{\alpha}{1 - \alpha} = \frac{w(y, CE_{v_y}(q)) - w(y, CE_{v_y}(p))}{w(x, CE_{v_x}(p)) - w(x, CE_{v_x}(q))}. \tag{22}$$

For any $\beta \in (0,1)$ and $r \in \mathcal{P}_2$, by Lemma 1, we have $\beta p + (1-\beta)r \succ_{2|x} \beta q + (1-\beta)r$ and $\beta q + (1-\beta)r \succ_{2|y} \beta p + (1-\beta)r$. By the representation (18), there exists a unique $\alpha' \in (0,1)$ such that $\beta p+(1-\beta)r \sim_{2|\alpha'\delta_x+(1-\alpha')\delta_y} \beta q+(1-\beta)r$. By (21), we have

$$\begin{aligned}\frac{\tau(\alpha')}{1-\tau(\alpha')} &= \frac{\mathbb{E}_{v_y}[\beta q + (1-\beta)r] - \mathbb{E}_{v_y}[\beta p + (1-\beta)r]}{\mathbb{E}_{v_x}[\beta p + (1-\beta)r] - \mathbb{E}_{v_x}[\beta q + (1-\beta)r]} \\ &= \frac{\mathbb{E}_{v_y}[q] - \mathbb{E}_{v_y}[p]}{\mathbb{E}_{v_x}[p] - \mathbb{E}_{v_x}[q]} = \frac{\tau(\alpha)}{1-\tau(\alpha)}.\end{aligned}$$

Since τ is strictly increasing, we have $\alpha = \alpha'$. By (22), this suggests

$$\frac{w(y, CE_{v_y}(\beta q + (1-\beta)r)) - w(y, CE_{v_y}(\beta p + (1-\beta)r))}{w(x, CE_{v_x}(\beta p + (1-\beta)r)) - w(x, CE_{v_x}(\beta q + (1-\beta)r))} = \frac{\alpha}{1-\alpha} \tag{23}$$

for all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$.

For each $x \in X_1$, since v_x is unique up to a positive affine transformation, we can normalize that $v_x(\underline{c}_2) = w(x, \underline{c}_2)$ and $v_x(\overline{c}_2) = w(x, \overline{c}_2)$. Define a function $\zeta_x : [v_x(\underline{c}_2), v_x(\overline{c}_2)] \to \mathbb{R}$ such that $\zeta_x(z) = w(x, v_x^{-1}(z))$ for all $z \in [v_x(\underline{c}_2), v_x(\overline{c}_2)]$. Then $\zeta_x(v_x(\underline{c}_2)) = v_x(\underline{c}_2)$ and $\zeta_x(v_x(\overline{c}_2)) = v_x(\overline{c}_2)$. Also, by Lemma 11, the function ζ_x is continuous and strictly increasing. Rewrite (23) and we derive

$$\frac{\zeta_y(\beta\mathbb{E}_{v_y}[q] + (1-\beta)\mathbb{E}_{v_y}[r]) - \zeta_y(\beta\mathbb{E}_{v_y}[p] + (1-\beta)\mathbb{E}_{v_y}[r])}{\zeta_x(\beta\mathbb{E}_{v_x}[p] + (1-\beta)\mathbb{E}_{v_x}[r]) - \zeta_x(\beta\mathbb{E}_{v_x}[q] + (1-\beta)\mathbb{E}_{v_x}[r])} = \frac{\alpha}{1-\alpha} \tag{24}$$

all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$. This holds for all $\alpha \in (0,1)$, $x, y \in X_1$ and $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x,y}(\alpha)$.

Let $r = p$, by equations (21) and (24), we have

$$\frac{\Big(\zeta_y(\beta\mathbb{E}_{v_y}[q] + (1-\beta)\mathbb{E}_{v_y}[p]) - \zeta_y(\mathbb{E}_{v_y}[p])\Big)\Big/\Big(\beta\mathbb{E}_{v_y}[q] - \beta\mathbb{E}_{v_y}[p]\Big)}{\Big(\zeta_x(\beta\mathbb{E}_{v_x}[q] + (1-\beta)\mathbb{E}_{v_x}[p]) - \zeta_x(\mathbb{E}_{v_x}[p])\Big)\Big/\Big(\beta\mathbb{E}_{v_x}[q] - \beta\mathbb{E}_{v_x}[p]\Big)} = \frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{25}$$

As β goes to 0, equation (25) becomes

$$\frac{\lim\limits_{b\to E_{v_y}[q]^+}\Big(\zeta_y(b)-\zeta_y(\mathbb{E}_{v_y}[q])\Big)\Big/\Big(b-\mathbb{E}_{v_y}[q]\Big)}{\lim\limits_{c\to E_{v_x}[q]^-}\Big(\zeta_x(c)-\zeta_x(\mathbb{E}_{v_x}[q])\Big)\Big/\Big(c-\mathbb{E}_{v_x}[q]\Big)}=\frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{26}$$

We claim that the two limits on the left-hand side of equation (26) exist as real numbers. If they exist, then the numerator is called the right derivative of ζ_y at $\mathbb{E}_{v_y}[q]$, denoted by $\partial_+\zeta_y(E_{v_y}[q])$, and the denominator is called the left derivative of ζ_x at $\mathbb{E}_{v_x}[q]$, denoted by $\partial_-\zeta_x(E_{v_x}[q])$. To prove the claim, since ζ_x and ζ_y are strictly increasing and continuous, they are differentiable almost everywhere on their domains. Hence, the two one-sided derivatives are well-defined almost everywhere. Note that since $v_x \not\propto v_y$, we can find $r\in\mathcal{P}_2$ such that $r\sim_{2|x} q$ and $r\not\sim_{2|y} q$. Then, if we change the value of $\beta\in(0,1)$, the value of $\mathbb{E}_{v_x}[\beta q+(1-\beta)r]$ remains unchanged, while the value of $\mathbb{E}_{v_y}[\beta q+(1-\beta)r]$ will form an open interval. Suppose that $\lim\limits_{c\to E_{v_x}[q]^-}\Big(\zeta_x(c)-\zeta_x(\mathbb{E}_{v_x}[q])\Big)\Big/\Big(c-\mathbb{E}_{v_x}[q]\Big)$ does not exist, then we know that $\lim\limits_{b\to a^+}\Big(\zeta_y(b)-\zeta_y(a)\Big)\Big/\Big(b-a\Big)$ does not exist for a contained in an open interval. This contradicts with the condition that $\partial_+\zeta_y$ is well-defined almost everywhere. Using a similar argument, we conclude that the two limits on the left-hand side of equation (26) exist as real numbers. Hence, equation (26) can be rewritten as

$$\frac{\partial_+\zeta_y(E_{v_y}[q])}{\partial_-\zeta_x(E_{v_x}[q])}=\frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{27}$$

This holds for all $\alpha\in(0,1)$, $x,y\in X_1$ and $p,q\in\mathcal{P}_2$ such that $(p,q)\in\Gamma_{x,y}(\alpha)$. By Lemma 13, for each $\alpha'\in(0,1)$, we can choose $p^{\alpha'}\in\mathcal{P}_2$ such that $(p^{\alpha'},q)\in\Gamma_{x,y}(\alpha')$. As a result, the right hand side of equation (27) is a constant for all $\alpha\in(0,1)$.

By the properties of $\Gamma_{x,y}$, for any $b\in(v_y(\underline{c}_2),v_y(\overline{c}_2))$, we can find some $\alpha\in(0,1)$ and $(p,q)\in\Gamma_{x,y}(\alpha)$ such that $b=E_{v_y}[q]$. Again by $v_x\not\propto v_y$, there exists an open interval I_b that contains b and the right derivative $\partial_+\zeta_y$ is a constant on I_b. Since b can be arbitrary in $(v_y(\underline{c}_2),v_y(\overline{c}_2))$, we know that $\partial_+\zeta_y$ must be a constant on $(v_y(\underline{c}_2),v_y(\overline{c}_2))$. Similarly, $\partial_-\zeta_x$ must be a constant on $(v_x(\underline{c}_2),v_x(\overline{c}_2))$.

Recall that the above results are derived by letting $r=p$ in equation (24). Now let $r = q$ and repeat the argument. We can derive that $\partial_- \zeta_y$ must be a constant on $(v_y(\underline{c}_2), v_y(\overline{c}_2))$, and $\partial_+ \zeta_x$ must be a constant on $(v_x(\underline{c}_2), v_x(\overline{c}_2))$. Since ζ_y and ζ_x are differentiable almost everywhere, $\partial_- \zeta_y = \partial_+ \zeta_y$ and $\partial_- \zeta_x = \partial_+ \zeta_x$ almost everywhere on their domains. Hence, we conclude that ζ_y and ζ_x are differentiable on $(v_y(\underline{c}_2), v_y(\overline{c}_2))$ and $(v_x(\underline{c}_2), v_x(\overline{c}_2))$ respectively, and their derivatives remain constant. Take ζ_x as an example. Since ζ_x is continuous, $\zeta_x(a) = a$ for $a = v_x(\underline{c}_2)$ and $a = v_x(\overline{c}_2)$, and the derivative of ζ_x is a constant on $(v_x(\underline{c}_2), v_x(\overline{c}_2))$, we have $\zeta_x(a) = a$ for all $a \in [v_x(\underline{c}_2), v_x(\overline{c}_2)]$. By definition of ζ_x, this implies $w(x, z) = v_x(z)$ for all $z \in X_2$. Also, we have $w(y, z) = v_y(z)$ for all $z \in X_2$.

Finally, consider any $x' \in X_1$. Since $v_x \not\approx v_y$, either $v_{x'} \not\approx v_x$ or either $v_{x'} \not\approx v_y$. The above result applies for either the pair of $v_{x'}$ and v_x or the pair of $v_{x'}$ and v_y. Hence, $w(x', z) = v_{x'}(z)$ for all $(x', z) \in X$, and the representation (18) reduces to an EU representation with utility index w. $\square$

To summarize, by Lemma 2, Lemma 12 and Lemma 14, if $\succsim$ satisfies Axioms 1-5, then it admits either a REU, or a PREU, or an EU representation. This completes the proof. $\square$

Proof of Proposition 1. The arguments for uniqueness of utility indices are already contained in the proof of Theorem 1. $\square$

Proof of Proposition 2. The necessity of axioms is easy to show. For sufficiency, by Theorem 1, the relation $\succsim$ admits either an EU representation w, or a REU representation (w, v_1, v_2), or a PREU representation (w, v). By Axiom 6*, we can find an increasing and continuous function u with $w(x, y) = u(x + y)$ for all $x, y \in Z$. By Axiom 7, in the REU representation, v_1 must be a positive affine transformation of v_2. It remains to be shown that PREU reduces to one of the other two representations. Suppose that $\succsim$ admits a PREU representation, that is, the utility of lottery P is $V^{PREU}(P) = \sum_{x \in X_1} u(x + CE_v(P_{2|x}))P_1(x)$. Since $(x, p) \sim (p, x)$ by Axiom 7, we know that v must be a positive affine transformation of $u(x + \cdot)$ for all $x \in Z$. In this case, the PREU representation reduces to an EU representation. $\square$

Proof of Proposition 3. The result is an immediate corollary of Theorem 1 since Assumption 1 implies that w must be a monotone transformation of $u(x_1) + \beta u(x_2)$. $\square$

Proof of Proposition 4. Since $Z = [w, b] \subset \mathbb{R}_{++}$ and $T = [0, \bar{t}] \subset \mathbb{R}_+$, the restriction of $\succsim$ to deterministic dated prizes $Z \times T$ satisfies A0-A5 of Fishburn and Rubinstein (1982) and hence by their Theorem 2, we can find $r > 0$ and a continuous and strictly increasing function $u : Z \to \mathbb{R}_{++}$ such that $(x, t) \succsim (y, s)$ if and only if $e^{-rt}u(x) \geq e^{-rs}u(y)$. The rest of the proof follows from Theorem 1 since Assumption 1 implies that w must be a monotone transformation of $e^{-rt}u(x)$. $\square$

Proof of Proposition 5. First, note that by Proposition 4, the EU representation is indeed a Generalized Expected Discounted Utility model (GEDU) introduced by DeJarnette et al. (2020). By their Proposition 4, in EU, $\succsim$ satisfies Stochastic Impatience if and only if it is RSTL. It suffices to check the other two procedures. If $\succsim$ admits a REU representation, then it trivially satisfies Stochastic Impatience as the decision maker neglects the correlation between the prize and the payment date and is always indifferent between $\frac{1}{2}\delta_{(x_1,t_1)} + \frac{1}{2}\delta_{(x_2,t_2)}$ and $\frac{1}{2}\delta_{(x_2,t_1)} + \frac{1}{2}\delta_{(x_1,t_2)}$. The risk attitude toward time is determined by v_2, and $\succsim$ is RSTL if and only if v_2 is convex.

Suppose $\succsim$ admits a PREU representation. The risk attitude toward time is determined by v, and $\succsim$ is RSTL if and only if v is convex. Note that

$$V^{PREU}(\frac{1}{2}\delta_{(x_1,t_1)}+\frac{1}{2}\delta_{(x_2,t_2)}) = \frac{1}{2}\phi(e^{-rt_1}u(x_1))+\frac{1}{2}\phi(e^{-rt_2}u(x_2)) = V^{EU}(\frac{1}{2}\delta_{(x_1,t_1)}+\frac{1}{2}\delta_{(x_2,t_2)}),$$

$$V^{PREU}(\frac{1}{2}\delta_{(x_2,t_1)}+\frac{1}{2}\delta_{(x_1,t_2)}) = \frac{1}{2}\phi(e^{-rt_1}u(x_2))+\frac{1}{2}\phi(e^{-rt_2}u(x_1)) = V^{EU}(\frac{1}{2}\delta_{(x_2,t_1)}+\frac{1}{2}\delta_{(x_1,t_2)}).$$

Hence, $\succsim$ satisfies Stochastic Impatience if and only if the corresponding EU representation with the same parameters satisfies Stochastic Impatience, which, by Propositions 2 and 4 of DeJarnette et al. (2020), is equivalent to ϕ being a convex transformation of ln. This completes the proof for part (i). Part (ii) holds since Strict Stochastic Impatience rules out REU and requires ϕ in PREU to be a strictly convex transformation of ln. $\square$

Proof of Proposition 6. Part (i) is the standard von Neumann–Morgenstern EU theorem. For part (ii), suppose that $\succsim$ admits an PREU representation w, where $w(\underline{c}_1, \underline{c}_2)$ is normalized to 0. By Axiom 10, for any $(x, y) \in X$ with $x \neq \underline{c}_1$, we have $\frac{1}{2}\delta_{(x,y)} + \frac{1}{2}\delta_{(\underline{c}_1,\underline{c}_2)} \sim \frac{1}{2}\delta_{(x,\underline{c}_2)} + \frac{1}{2}\delta_{(\underline{c}_1,y)}$, which leads to $w(x, y) = w(x, \underline{c}_2) + w(\underline{c}_1, y)$. Define continuous functions $u_1 : X_1 \to \mathbb{R}$ and $u_2 : X_2 \to \mathbb{R}$ where $u_1(x) = w(x, \underline{c}_2)$ for all $x > \underline{c}_1$ and $u_2(y) = w(\underline{c}_1, y)$ for all $y \in X_2$. By continuity, we know $u_1(\underline{c}_1) = 0$ and $w(x, y) = u_1(x) + u_2(y)$. Then the utility of lottery $P \in \mathcal{P}$ is $V^{PREU}(P) = V^{PREU}(P_1, P_2) = \mathbb{E}_{P_1}(u_1) + u_2(CE_v(P_2))$, which is a REU representation. Indeed, this requires $u_2(CE_v(P_2)) = \sum_x u_2(CE_v(P_{2|x}))P_1(x)$, which implies v must be a positive affine transformation of u_2. Hence $V^{PREU}(P) = \mathbb{E}_{P_1}(u_1) + \mathbb{E}_{P_2}(u_2)$. The same argument can show that if $\succsim$ admits an EU representation w, then w is additively separable.

To prove part (iii), note that Axiom 11 is exactly part (iii) of Corollary 4 in the proof of Theorem 1. With the help of other axioms, we can prove the other parts of Corollary 4 as well. Indeed, we can show that Lemma 11 holds and $\succsim$ admits a representation in (18). That is, the utility of $P \in \mathcal{P}$ is $U(P) = \sum_x w(x, CE_{v_x}(P_{2|x}))P_1(x)$. Axiom 12 implies that v_x is independent of x and hence $\succsim$ admits a PREU representation. $\square$

Proof of Corollary 1. To prove part (i), first note that if (d) holds, then $\succsim$ admits an EU, a PREU and a REU representations and hence the other three statements hold. If any of (a), (b) and (c) holds, then $\succsim$ admits either an EU or a REU representation and satisfies Axiom 10. According to the proof of Proposition 6, $\succsim$ must admit an EU representation with an additively separable utility index.

For part (ii), suppose that $\succsim$ admits both an EU representation w and a PREU representation. Then there exists a continuous and strictly increasing utility function $w_2 : X_2 \to \mathbb{R}$ such that $w(x, y) = (x)v(y) + w_1(x)$ where $a(x) > 0, b(x) \in \mathbb{R}$ for all $(x, y) \in X$. The reverse can be easily verified. $\square$

Proof of Proposition 7. First, by Theorem 5.1 in Epstein and Zin (1989), we can find an optimal consumption plan $(c_t^*)_{t\geq 1}$ for the EZ consumer with $J^{EZ}(W_1) =$

$\hat{V}^{EZ}((c_t^*)_{t\geq 1})$. On the one hand, since $\rho > \alpha$, the EZ consumer has a preference for early resolution of risk. We know that $\hat{V}^{PREU}((c_t)_{t\geq 1}) \leq \hat{V}^{EZ}((c_t)_{t\geq 1})$ for each feasible consumption plan $(c_t)_{t\geq 1}$ and $J^{PREU}(W_1) \leq J^{EZ}(W_1)$. Moreover, for a consumption plan $(c_t)_{t\geq 1}$ where c_t is injective on Ω_t for each t, the consumption history contains the same information as state history. In this case, $\hat{V}^{PREU}((c_t)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1})$.

If c_t^* is injective for each $t \geq 2$, we can set $c_t^n \equiv c_t^*$ for each n and the results holds. Otherwise, we want to construct a sequence of consumption plans $(c_t^n)_{t\geq 1, n\geq 1}$ with injective consumption functions such that $c_1^n \to c_1^*, c_t^n(s^t) \to c_t^*(s^t)$ as n goes to infinity for each $t \geq 2$. By continuity of V^{EZ} and hence $\hat{V}^{EZ}$, we know

$$J^{PREU}(W_1) \geq \lim_{n\to\infty} \hat{V}^{PREU}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t^*)_{t\geq 1}) = J^{EZ}(W_1).$$

Hence $J^{PREU}(W_1) = J^{EZ}(W_1)$. It remains to construct the sequence of consumption plans. This is easy since $\bigcup_{t\geq 2} c_t^*(S^t)$ is countable (as the union of a countable number of countable sets is countable), and the space of consumption is a continuum. $\square$

Proof of Corollary 2. The derivation of the Euler equation and the asset pricing formula for $(c_t)_{t\geq 1}$ can be found in Epstein and Zin (1991). The rest follows from the continuity of the two equations and $c_1^n \to c_1, c_t^n(s^t) \to c_t(s^t)$ as n goes to infinity for each $t \geq 2$. $\square$

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