Procedural Expected Utility*

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Abstract

This paper studies procedures a decision maker adopts to evaluate two-dimensional risk. She might either follow the standard expected utility model and treat risk in different dimensions as a whole, or apply simplification heuristics to evaluate risk in different dimensions in isolation or sequentially. We axiomatize those procedures by maintaining the Independence axiom within each dimension and relaxing it across dimensions. Through applications in different choice domains, we demonstrate that our procedures can (i) explain experimental evidence of stochastic dominance violations without inducing such violations in the absence of risk, (ii) separate time and risk preferences without assuming a preference for early resolution of uncertainty, and (iii) accommodate non-trivial risk aversion over time lotteries without violating stochastic impatience.

Keywords: Multi-dimensional risk, narrow bracketing, time and risk preferences, time lotteries.

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1 Introduction

Many economic decisions require an understanding of uncertainty across multiple dimensions. For instance, investors must navigate risk across multiple financial markets, consumers face income risk over time, and home sellers are uncertain about both the sale price and the timing of the sale. In this paper, we investigate various procedures people adopt to evaluate the multi-dimensional risk.

We study preferences over lotteries with two-dimensional outcome profiles. These outcome profiles can represent various scenarios, such as income from two sources, consumption in two periods, and the size and the payment date of a prize. In the standard expected utility (EU) model, the decision maker acts as if she evaluates risk in different dimensions collectively. She takes the expectation over utilities of outcome profiles, and evaluates each lottery P according to

$$V^{EU}(P) = \mathbb{E}_P(w(x_1, x_2)).$$

The above evaluation procedure has been shown to be inconsistent with recent empirical evidence. For example, assuming fungibility of money, EU predicts that when faced with two monetary gambles, the individual will combine them into one over total payments, and will never choose options that are first-order stochastically dominated. However, Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009) find that a significant portion of their subjects make dominated decisions in a pair of simple binary choice problems. Another example is in the context where both the prize amount and the payment date can be uncertain. An individual is considered risk averse over time lotteries (RATL) if she prefers to receive a fixed prize on a certain date rather than on a randomly determined date with the same expected delay. On the other hand, if the preference is the opposite, the individual is considered risk seeking over time lotteries (RSTL). DeJarnette et al. (2020) reveal a fundamental incompatibility between Stochastic Impatience, the risky counterpart of impatience, and even a single violation of RSTL in the EU model. However, using incentivized experiments, DeJarnette et al. (2020) show that most of their subjects are RATL in the majority of questions. These findings cast doubt on the predictions of EU when it comes to multi-dimensional risk.

In this paper, we present a unified model called *procedural expected utility* that can address the aforementioned challenges across different choice domains. It allows novel evaluation procedures where the two-dimensional risk is not assessed as a whole. With a procedural expected utility preference indexed by $I \subseteq \{1, 2\}$, the decision maker acts as if she first evaluates risk in each dimension $i \in I$ in isolation, and then evaluates risk in other dimensions. When $I = \emptyset$, we are back to the EU model. When $I \neq \emptyset$, the decision maker adopts some simplification heuristic to evaluate two-dimensional risk.

The first heuristic corresponds to the model with $I = \{1, 2\}$, which we refer to as *narrow expected utility (NEU)*. With this procedure, the decision maker acts as if she first assesses risk in each dimension separately by converting it into a deterministic outcome, and then derives the utility of the corresponding outcome profile. The utility of P is given by

$$V^{NEU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big),$$

where $CE_{v_i}(P_i)$ is the certainty equivalent of marginal lottery P_i calculated using the Bernoulli index v_i in dimension $i = 1, 2$. This evaluation procedure is reminiscent of narrow bracketing introduced by Thaler (1985) and Read, Loewenstein, and Rabin (1999).

The second heuristic posits that the decision maker acts as if she evaluates risk in different dimensions sequentially, and we call the corresponding procedure *sequential expected utility (SEU)*. Assume $I = \{2\}$ for illustration. The decision maker first assesses risk in dimension 2 in isolation, conditioning on each outcome in dimension 1, and then assesses risk in dimension 1. The utility of lottery P is

$$V^{SEU}(P) = \mathbb{E}_{P_1}\Big(w(x_1, CE_v(P_{2|x_1}))\Big),$$

where $P_{2|x}$ is the conditional distribution in dimension 2 given x in dimension 1. The sequential procedure is motivated by asymmetry built in many applications. For example, it is consistent with backward induction when outcomes in different dimensions represent consumption in different periods.

Our main result is a representation theorem that characterizes procedural expected utility preferences with five axioms, three of which can be commonly found in the literature on choices under risk: (i) The preference is complete and transitive. (ii) The decision maker prefers more outcome in each dimension when there is no risk. (iii) The preference satisfies specific conditions for continuity. The last two axioms maintain the Independence axiom in the EU theory within each dimension, and allow violations of it across dimensions when the decision maker treats risk in some dimension in isolation.[1]

Our theory of procedural expected utility preferences is valuable in reconciling discrepancies between EU and empirical evidence from various applications. In Section 4.1, we examine preferences regarding two monetary gambles and demonstrate that the NEU model can accommodate experimental findings on violations of first-order stochastic dominance. Moreover, the narrow bracketing model frequently used to interpret experimental evidence may result in a decision maker choosing X dollars for sure over $Y > X$ dollars for sure when these amounts are suitably divided across the two income dimensions. This can lead to a theory that explains certain anomalies in the data, but at the expense of creating others that are unlikely to exist. By contrast, with an NEU preference, the agent evaluates the gambles by taking the sum of their certainty equivalents, and hence she would never choose less money for sure over more money for sure.

Section 4.2 studies the application to preferences over multi-period consumption under uncertainty. We start by noting that the commonly used Expected Discounted Utility (EDU) is consistent with all of our evaluation procedures. With an EDU preference, the decision maker's time preference and risk preference are intertwined, which is inconsistent with macroeconomic and financial data. We show how different evaluation procedures can lead to different generalizations of EDU that separate time and risk preferences. Then, we study a CRRA-CES specification of the SEU preference, which resembles the two-period version of Epstein and Zin (1989), but exhibits indifference to temporal resolution of uncertainty.

In our final application, we study preferences over lotteries in which both the monetary prize and the payment date are uncertain in Section 4.3. As previously noted, EU is unable to accommodate Stochastic Impatience and violations of

[1]As a result, our paper complements the enormous body of literature focused on violations of EU which occur within a single dimension, such as the Allais paradox.

RSTL simultaneously. The EDU model, in particular, requires all subjects to be RSTL. To see this, note that the utility of time lottery (z, p) is $\mathbb{E}_p[\beta^t]u(z)$, where $u(z) > 0$ is the utility of prize z and $\beta \in (0, 1)$ is the discount factor. By comparison, the utility of receiving z at expected time $t' = \mathbb{E}_p(t)$ is $\beta^{t'}u(z)$. Since β^t is convex in t, the former option will always be preferred. However, our SEU model offers a solution to this inconsistency, as the evaluation of risk in the time dimension is evaluated in isolation, allowing for separation of risk attitudes toward time and intertemporal trade-offs.

Our axiomatic approach uncovers a novel connection between three seemingly disparate challenges to the EU model in different choice domains: the violation of first-order stochastic dominance, the conjunction of time and risk preferences, and the incompatibility between Stochastic Impatience and violations of RSTL. We argue that these challenges arise from the collective evaluation of risk in different dimensions in the EU model, and can be addressed by preserving the Independence axiom within each dimension while relaxing it across dimensions.

Related Literature. Our observation that risk in different dimensions might be treated in different ways is related to the literature on source-dependent preferences following Tversky and Fox (1995) and Tversky and Wakker (1995).[2] The closest work to the present paper is Cappelli et al. (2021), where the decision maker faced with multi-source risk first computes source-dependent certainty equivalents, converts them into the unit of account of a numeraire, and then aggregates them into the overall evaluation. This can be interpreted as an extension of our NEU model to a setting with multiple dimensions, subjective uncertainty and certainty equivalents based on non-EU models. Our paper complements Cappelli et al. (2021) by focusing on how this evaluation procedure departs from the EU theory, and proposing a novel procedure where multi-source risk is evaluated sequentially.[3]

Our paper makes a contribution to the literature on narrow bracketing, which

[2] The recent interest in source-dependent preferences was sparked by the Ellsberg paradox and has led to a vast body of literature. See, for instance, Ergin and Gul (2009), Gul and Pesendorfer (2015), Chew and Sagi (2008), Qiu and Ahn (2021) and references therein.

[3] Our focus on evaluation procedures is also reminiscent of the growing literature on bounded rational choice procedures following Kalai, Rubinstein, and Spiegler (2002), Manzini and Mariotti (2007) and Masatlioglu, Nakajima, and Ozbay (2012). However, the decision maker's behavior in our model is consistent with maximizing a well-defined utility function.

has been the subject of growing interest recently, driven by both theoretical results (Barberis, Huang, and Thaler, 2006, Mu et al., 2021a) and experimental findings (Rabin and Weizsäcker, 2009, Ellis and Freeman, 2021). Various models have been proposed to provide rationales for narrow bracketing, such as the model of Kőszegi and Matějka (2020) based on rational inattention, and that of Lian (2020) where the decision maker has different imperfect information in different decisions. Two closely related works are Vorjohann (2021) and Camara (2021), both of which begin with the expected utility paradigm and model narrow bracketing with an additively separable Bernoulli index. In Vorjohann (2021), the decision maker is characterized by a broad and a narrow EU preferences, the connection between which yields testable implications. Camara (2021) focuses on high-dimensional decisions and shows that computational tractability imposes restrictions on the Bernoulli index. By contrast, we model narrow bracketing as a deviation from the EU model. The two approaches yield different implications regarding preferences over two monetary gambles, which will be discussed in Section 4.1.

Two features of our NEU model of narrow bracketing bear a resemblance to the recent literature. First, the decision maker maximizes a sum of certainty equivalents, which is a decision criterion that has been advocated by Myerson and Zambrano (2019) as an effective rule for risk sharing, and axiomatized by Chambers and Echenique (2012) as a social welfare functional. It also appears in the monotone additive statistics characterized by Mu et al. (2021b). Second, the decision maker ignores correlation between risk in different dimensions, which echoes the experimental evidence of correlation neglect in belief formation (Enke and Zimmermann, 2019), portfolio allocation (Eyster and Weizsäcker, 2016, Kallir and Sonsino, 2009) and school choice (Rees-Jones, Shorrer, and Tergiman, 2020).

In a follow-up work, Ke and Zhang (2023) generalize the analysis of the current paper from a two-dimensional setting to a multi-dimensional setting. They characterize how the decision maker orders and brackets different dimensions, based on which she evaluates risk recursively. As a result, each evaluation procedure corresponds to a hierarchical structure (a laminar family) over the set of dimensions. Ke and Zhang (2023) also show how their general framework can accommodate incompatible approaches to studying inequality aversion and the preference for hedging in ambiguity aversion.

2 Evaluation Procedures

Consider a decision maker faced with risk from two dimensions $i = 1, 2$. For each $i = 1, 2$, let $X_i = [\underline{c}_i, \overline{c}_i] \subset \mathbb{R}$ be a nondegenerate interval of outcomes in dimension i. Let $X = X_1 \times X_2$ be the set of outcome profiles. For an arbitrary set Z, let $\Delta(Z)$ denote the set of all simple lotteries (probability measures with a finite support) on Z, endowed with the topology of weak convergence and the standard mixture operation. For each $u : Z \to \mathbb{R}$ and $\mu \in \Delta(Z)$, let $\mathbb{E}_\mu(u)$ be the expected value of u with respect to μ, and let $\text{supp}(\mu)$ be the support of μ, that is, $\text{supp}(\mu) = \{z \in Z \mid \mu(z) > 0\}$. Denote by $\mathcal{P} = \Delta(X)$ and $\mathcal{P}_i = \Delta(X_i)$ for $i = 1, 2$. The primitive of our analysis is a binary relation $\succsim$ on $\mathcal{P}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively.

We denote by $x = (x_1, x_2), y = (y_1, y_2)$ generic elements of X, denote by p_i, q_i, r_i generic elements of $\mathcal{P}_i$ for $i = 1, 2$, and denote by P, Q, R generic elements of $\mathcal{P}$.[4] For each lottery $P \in \mathcal{P}$, its marginal lottery in dimension 1 is denoted by $P_1 \in \mathcal{P}_1$ where $P_1(x_1) = \sum_{x_2 \in X_2} P(x_1, x_2)$ for all $x_1 \in X_1$. The marginal lottery in dimension 2 can be defined similarly. Let δ_x be the degenerate lottery that yields the outcome profile $x \in X$ with probability 1. When there is no confusion, we identify δ_x with x. Similarly, we can define the degenerate marginal lotteries. For each $i = 1, 2$ and continuous and strictly increasing function $u : X_i \to \mathbb{R}$, the *certainty equivalent* of $p_i \in \mathcal{P}_i$ calculated using u is $CE_u(p_i) = u^{-1}(\mathbb{E}_{p_i}(u)) \in X_i$. For each $P \in \mathcal{P}$ and $x_i \in \text{supp}(P_i)$, let $P_{-i|x_i}$ denote the conditional distribution of the outcome in dimension $-i$ given x_i in dimension i.

Several notational conventions will be useful in our model. First, for each $A \subseteq \{1, 2\}$, let $-A$ be the complement of A, i.e., $-A = \{1, 2\} \backslash A$. We identify A with i if $A = \{i\}$ for some $i = 1, 2$. Second, denote $P_A = P$, $X_A = X$, $x_A = x$ if $A = \{1, 2\}$ and $P_A = P_i, X_A = X_i, x_A = x_i$ if $A = \{i\}$ for some $i = 1, 2$. Third, if we encounter x_A in which $A = \emptyset$, then x_A will be ignored from the expression. Consider the following examples. For any $P_{i|x_A}$ in which $A = \emptyset$, we identify $P_{i|x_A}$

[4] When there is no confusion, we use p, q, r to denote elements in $\mathcal{P}_1 \cup \mathcal{P}_2$.

with P_i. For any P_A in which $A = \emptyset$, we ignore the expectation operator $\mathbb{E}_{P_A}$. Any $(CE_{v_j}(P_{j|x_{-A}}))_{j\in A}$ in which $A = \emptyset$ will be ignored from the expression.

Definition 1. A binary relation $\succsim$ is a *procedural expected utility* preference if it can be represented by $V : \mathcal{P} \to \mathbb{R}$ such that for any $P \in \mathcal{P}$,

$$V(P) = \mathbb{E}_{P_{-I}}\Big(w\Big(x_{-I}, (CE_{v_j}(P_{j|x_{-I}}))_{j\in I}\Big)\Big) \tag{1}$$

where $I \subseteq \{1,2\}$, $w : X \to \mathbb{R}$ and $v_j : X_j \to \mathbb{R}$ for each $j \in I$ are continuous and strictly increasing.[5] In the case, $(I, w, (v_j)_{j\in I})$ is called a procedural expected utility representation of $\succsim$.

The procedural expected utility representation of $\succsim$ may suggest the following interpretation. First, the decision maker separately evaluates the certainty equivalent of risk in each dimension $j \in I$ (if $I \neq \emptyset$), conditioning on the realized outcome $x_{-I} \in X_{-I}$ (if $-I \neq \emptyset$). Second, she calculates the utility of the combination of x_{-I} and the corresponding certainty equivalents. Finally, she assesses risk in dimensions $-I$ by taking the expectation of utility with respect to the distribution of x_{-I} if $-I \neq \emptyset$. Hence, in the associated evaluation procedure, the decision maker acts as if she treats risk in each dimension $j \in I$ *in isolation*. Different values of I lead to different procedures in the following definition.

Definition 2. Suppose that a binary relation $\succsim$ admits a procedural expected utility representation $(I, w, (v_j)_{j\in I})$.
(i) We say $\succsim$ is an *expected utility (EU)* preference if $I = \emptyset$. In this case, w is called an EU representation of $\succsim$, and $\succsim$ is represented by

$$V^{EU}(P) = \mathbb{E}_P(w) = \sum_{(x_1,x_2)\in X} w(x_1, x_2)P(x_1, x_2). \tag{2}$$

(ii) We say $\succsim$ is a *narrow expected utility (NEU)* preference if $I = \{1,2\}$. In this case, (w, v_1, v_2) is called an NEU representation of $\succsim$, and $\succsim$ is represented by

$$V^{NEU}(P) = w\Big(CE_{v_1}(P_1), CE_{v_2}(P_2)\Big). \tag{3}$$

[5] A function $V : \mathcal{P} \to \mathbb{R}$ is said to represent the binary relation $\succsim$ when $P \succsim Q$ if and only if $V(P) \geq V(Q)$ for all $P, Q \in \mathcal{P}$. A function $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ for some positive integer n is strictly increasing if $f(x) > f(y)$ whenever $x, y \in A$, $x \geq y$ and $x \neq y$.

(iii) We say $\succsim$ is a *sequential expected utility (SEU)* preference if $I = \{i\}$, $i = 1, 2$. In this case, (i, w, v_i) is called an SEU representation of $\succsim$, and $\succsim$ is represented by

$$V^{SEU}(P) = \sum_{x_{-i} \in X_{-i}} w(x_{-i}, CE_{v_i}(P_{i|x_{-i}}))P_{-i}(x_{-i}). \tag{4}$$

The standard procedure to evaluate a risky two-dimensional alternative is that associated with the EU preference, where the decision maker acts as if she assesses risk in two dimensions collectively. That is, when faced with a lottery, she first calculates the utility of each outcome profile and then calculates its expected value.

Despite the ubiquitous adoption of EU preferences in economic modeling, people in practice often find it demanding to treat the two-dimensional risk as a whole, and hence might employ heuristics to simplify the evaluation procedure. As in Definition 2, our procedural expected utility model allows for two such heuristics, where the decision maker evaluates risk in different dimensions either *separately* if she has an NEU preference, or *sequentially* if she has an SEU preference.

According to an NEU representation, the decision maker evaluates each lottery as if she first calculates the certainty equivalents of marginal lotteries in both dimensions, and then derives the utility of the corresponding outcome profile. As a result, she neglects the correlation between risk in different dimensions and her attitude toward marginal risk in one dimension is independent of the outcome in the other dimension. This evaluation procedure is reminiscent of narrow bracketing introduced by Thaler (1985) and Read, Loewenstein, and Rabin (1999).

By comparison, with an SEU preference, the decision maker acts as if she follows an order to evaluate risk in two dimensions sequentially. For instance, when $I = \{2\}$, the decision maker first assesses risk in dimension 2 in isolation, conditioning on each realized outcome in dimension 1, and then assesses risk in dimension 1. This feature is motivated by asymmetry built in many applications where outcomes in different dimensions represent, for instance, consumption in different periods or different attributes of an alternative.

3 Characterization

3.1 Axioms

In this section, we introduce axioms that behaviorally characterize the procedural expected utility preference. The first one is rationality.

Axiom 1—Weak Order: The relation $\succsim$ is complete and transitive.

The second axiom states that more outcome is better when there is no risk.

Axiom 2—Monotonicity: For any $(x_1, x_2), (y_1, y_2) \in X$, if $x_1 \geq y_1$, $x_2 \geq y_2$ and $(x_1, x_2) \neq (y_1, y_2)$, then $(x_1, x_2) \succ (y_1, y_2)$.

Unlike the Strong Continuity axiom commonly adopted in the expected utility theory with monetary prizes, our continuity axiom requires $\succsim$ to be continuous in both probabilities and outcomes, but not necessarily jointly.[6]

Axiom 3.1—Continuity in Probabilities: For any $P, R, Q \in \mathcal{P}$, the sets $\{\alpha \in [0,1] : \alpha P + (1-\alpha)Q \succsim R\}$ and $\{\alpha \in [0,1] : R \succsim \alpha P + (1-\alpha)Q\}$ are closed.

Axiom 3.1 is introduced by Herstein and Milnor (1953) and is usually referred to as the axiom of Mixture Continuity. Axiom 3.2, below, asserts that if changing every outcome in the support of P by $(\varepsilon_1, \varepsilon_2)$ renders P better (worse) than Q, for any $\varepsilon_1, \varepsilon_2$ sufficiently close to 0, then P must also be better (worse) than Q. Since the outcome space X is bounded, we need to deal with the possibility that changing some of the outcomes in the support of P may not be feasible. Formally, for any $P \in \mathcal{P}$, choose $\eta > 0$ small enough so that $P(x, y) \cdot P(x', y') > 0$, $|x - x'| \leq \eta$ and $|y - y'| \leq \eta$ implies $(x, y) = (x', y')$. Then, for $\varepsilon = (\varepsilon_1, \varepsilon_2)$ such that $\varepsilon_1, \varepsilon_2 \in (-\eta, \eta)$, define $\phi : X \to X$ as follows: $\phi_\varepsilon(x_1, x_2) = (x_1 + \varepsilon_1, x_2 + \varepsilon_2)$ if $(x_1 + \varepsilon_1, x_2 + \varepsilon_2) \in X$; otherwise, $\phi_\varepsilon(x_1, x_2) = (x_1', x_2')$ such that (x_1', x_2') is the element of X closest to (x_1, x_2) with respect to the distance $d((x_1, x_2), (x_1', x_2')) = |x_1 - x_2'| + |x_2 - x_2'|$. Since we have chosen $\varepsilon_1, \varepsilon_2$ sufficiently small, the restriction of ϕ to the support of P is one-to-one. Then, define $P_\varepsilon \in \mathcal{P}$ such that $P_\varepsilon(\phi_\varepsilon(x_1, x_2)) = P(x_1, x_2)$ if $P(x_1, x_2) > 0$ and $P_\varepsilon(x_1', x_2') = 0$ if $(x_1', x_2') \neq \phi_\varepsilon(x_1, x_2)$ for any $(x_1, x_2) \notin \mathrm{supp}(P)$.

[6] Formally, the Strong Continuity axiom states that for any $Q \in \mathcal{P}$, the sets $\{P \in \mathcal{P} : P \succsim Q\}$ and $\{P \in \mathcal{P} : Q \succsim P\}$ are closed.

Axiom 3.2—Continuity in Outcomes: Consider any $P, Q \in \mathcal{P}$ and sequence $(\varepsilon^n)_{n\geq 1}$ that converges to $(0,0)$. If $P_{\varepsilon^n} \succsim Q$ for all n, then $P \succsim Q$. If $Q \succsim P_{\varepsilon^n}$ for all n, then $Q \succsim P$.

We will refer to the conjunction of the two notions above as **Continuity**.

Axiom 3—Continuity: The relation $\succsim$ satisfies Axioms 3.1 and 3.2.

Note that Axiom 3 is implied by the Strong Continuity axiom. However, the reverse is not true. Indeed, although the EU and the NEU preferences satisfy the Strong Continuity axiom, the SEU preference might violate it since a infinitesimal perturbation in the lottery can induce a non-negligible change in the conditional lotteries, which can lead to a noncontinuous change in the utility level.[7]

The above Axioms 1-3 are either the same as, or natural adaptations of standard axioms in the literature on choices under risk. If $\succsim$ further satisfies the standard Independence axiom, which states that the decision maker's ranking between two lotteries is not reversed when they are mixed with the same lottery, then $\succsim$ is an EU preference. Hence, we need to weaken the Independence axiom to allow for the other evaluation procedures in Definition 2.

Our first relaxation states that the Independence axiom holds *within* each dimension. For each $p \in \mathcal{P}_1$ and $r, r' \in \mathcal{P}_2$, define the conditional preference $\succsim_{2|p}$ in dimension 2 such that $r \succsim_{2|p} r'$ if and only if $(p, r) \succsim (p, r')$. We can similarly define $\succsim_{1|q}$ for each $q \in \mathcal{P}_2$.

Axiom 4—Within-Dimension Independence: For any $i = 1, 2$, $\alpha \in (0,1)$, $p \in \mathcal{P}_{-i}$ and $q, r, s \in \mathcal{P}_i$, if $q \succ_{i|p} r$, then $\alpha q + (1-\alpha)s \succ_{i|p} \alpha r + (1-\alpha)s$.

According to Axiom 4, for each fixed marginal lottery in one dimension, the conditional preference over marginal lotteries in the other dimension satisfies the Independence axiom, and hence admits an expected utility representation. This distinguishes our procedure expected utility model from most existing non-expected utility models in the literature, where violations of the Independence

[7]For instance, consider an SEU representation where $i = 2$, $w(x_1, x_2) = \sqrt{x_1 + x_2}$ and $v(x_2) = x_2$ for all $x_1 \in X_1, x_2 \in X_2$. Denote by P^n and P such that $P^n(1 - \frac{1}{n}, 0) = P^n(1, 2) = \frac{1}{2}$ and $P(1, 0) = P(1, 2) = \frac{1}{2}$ for each $n \geq 1$. Easy to see that P^n converges to P in the weak topology as n goes to infinity. However, $U(P^n)$ converges to $\frac{1}{2}(1 + \sqrt{3}) \neq U(P) = \sqrt{2}$.

axiom occur within each dimension.

Before stating our last axiom, we introduce some notation. For $i = 1, 2$, we say P and Q are *comparable* in dimension i if $P_{-i} = Q_{-i}$ and there exists $x_{-i} \in X_{-i}$ such that $P_i \sim_{i|x_{-i}} Q_i$. In addition, we say P and Q are *conditionally comparable* in dimension i if $P_{-i} = Q_{-i}$ and there exists $x_{-i} \in X_{-i}$ such that $P_{i|y_{-i}} \sim_{i|x_{-i}} Q_{i|y_{-i}}$ for all $y_{-i} \in \text{supp}(P_{-i})$. When there is no confusion, we say two lotteries are (conditionally) comparable if they are (conditionally) comparable in either dimension 1 or 2.[8] For $i = 1, 2$, we say two marginal lotteries $r, r' \in \mathcal{P}_i$ are *(mutually) singular*, denoted by $r \perp r'$, if $\text{supp}(r) \cap \text{supp}(r') = \emptyset$. Two lotteries $P, Q \in \mathcal{P}$ are said to be *singular*, denoted by $P \perp Q$, if $P_i \perp Q_i$ for $i = 1, 2$.

Axiom 5—Across-Dimension Independence: Fix any $\alpha \in (0, 1)$, and $P, Q, R, S \in \mathcal{P}$ such that $P \succ Q, R \sim S$, and P and Q are comparable.

(i) If $P \perp R$ and $Q \perp S$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

(ii) If further P and Q are conditionally comparable in some dimension $i \in \{1, 2\}$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$ when $P_i \perp R_i$ and $Q_i \perp S_i$.

To see how Axiom 5 can accommodate different evaluation procedures, we first discuss why the Independence axiom suggests that the decision maker acts as if she evaluates risk in both dimensions collectively. Consider a decision maker who faces two independent 50-50 gambles between winning \$10 and losing \$10. The corresponding lottery can be written as (p, p) where $p(10) = p(-10) = 0.5$. If we assume fungibility of money when there is no risk, that is, $(10, 10) \sim (20, 0)$, $(10, -10) \sim (-10, 10) \sim (0, 0)$ and $(-10, -10) \sim (-20, 0)$, then by the Independence axiom, the decision maker is indifferent between (p, p) and $(q, 0)$, where $q(20) = q(-20) = 0.25$ and $q(0) = 0.5$. In other words, she can readily transform a lottery with two-dimensional risk into one where there is only non-trivial risk in dimension 1, and then use the conditional preference $\succsim_{1|\delta_0}$ to compare different lotteries. This property guarantees that the decision maker treats the two-dimensional risk as a whole.

By contrast, Axiom 5 allows violations of the Independence axiom across di-

[8] Note that a decision maker with an SEU preference indexed by $I = \{i\}$ is indifferent between two lotteries that are conditionally comparable in dimension i. Similarly, a decision maker with an NEU preference is indifferent between any two comparable lotteries.

mensions when the decision maker treats risk in some dimension in isolation. We will illustrate the intuition of Axiom 5 by focusing on its contrapositive. Suppose $P \succ Q, R \sim S$ but $\alpha P + (1-\alpha)R \precsim \alpha Q + (1-\alpha)S$ for some $\alpha \in (0,1)$. If $P_i \perp R_i$ and $Q_i \perp S_i$ for some $i \in \{1,2\}$, then the mixture has not impact on the conditional risk in dimension $-i$. To see this, consider the mixture between P and R as an example. For any outcome $x_i \in \mathrm{supp}(P_i)$, the conditional risk of P in dimension $-i$ is given by $P_{-i|x_i}$. Since $P_i \perp R_i$, we know $x \notin \mathrm{supp}(Q_i)$ and hence the conditional risk of $\alpha P + (1-\alpha)R$ in dimension $-i$ is also $P_{-i|x_i}$, unaffected by the mixture. In this case, the violation of the independence axiom suggests that the decision maker must evaluate risk in dimension i in isolation. As a result, P and Q cannot be conditionally comparable in dimension i, since otherwise, the decision maker would find P and Q equally desirable, leading to a contradiction with $P \succ Q$. This establishes the contrapositive of part (ii) of Axiom 5.

Now suppose further that $P \perp R$ and $Q \perp S$. As noted above, the conditional risks of $\alpha P + (1-\alpha)R$ and $\alpha Q + (1-\alpha)S$ in both dimensions are not affected by the mixture. Then the violation of the Independence axiom implies that the decision maker must evaluate risk in both dimensions in isolation. By $P \succ Q$, we conclude that P and Q are not comparable. Hence, part (i) of Axiom 5 holds.

3.2 Representation Theorem and Uniqueness Results

Theorem 1: *A binary relation $\succsim$ is a procedural expected utility preference if and only if it satisfies the axioms of Weak Order, Monotonicity, Continuity, Within-Dimension Independence, and Across-Dimension Independence.*

Theorem 1 characterizes the common behavioral implications of different evaluation procedures of risky two-dimensional alternatives: If a decision maker adopts some procedure, then, regardless of which one she uses, her behavior must be consistent with Axioms 1-5, and vice versa. Two observations are worth noting. First, as the procedural expected utility preference satisfies the Independence axiom within each dimension, it complements the vast literature on non-expected utility models. Second, Theorem 1 establishes a connection between heuristics used in evaluation procedures and violations of the Independence axiom across dimensions. That is, the Independence axiom holds if mixture has no impact on risk in dimensions evaluated in isolation.

Now we discuss the uniqueness properties of our representation. Suppose that $(I, w, (v_j)_{j\in I})$ is a procedural expected utility representation of $\succsim$. First, we fix I and focus on Bernoulli indices. Consider a set $B \subseteq \mathbb{R}^n$ for some positive integer n and two functions f, g defined on B. We say f is a *monotone* transformation of g if there exists a continuous and strictly increasing function ϕ defined on $g(B)$ such that $f(x) = \phi(g(x))$ for all $x \in B$. We say f is a *positive affine* transformation of g if there exist $b > 0$ and $c \in \mathbb{R}$ such that $f(x) = bg(x) + c$ for all $x \in B$.

Proposition 1: *Let $\succsim$ be a binary relation on $\mathcal{P}$ and I be a subset of $\{1, 2\}$.*

(i) If $I \neq \{1, 2\}$, then $(I, w, (v_j)_{j\in I})$ and $(I, \hat{w}, (\hat{v}_j)_{j\in I})$ are procedural expected utility representations of $\succsim$ if and only if $\hat{w}$ and $\hat{v}_j, j \in I$ are positive affine transformations of w and $v_j, j \in I$ respectively.

(ii) If $I = \{1, 2\}$, then $(I, w, (v_j)_{j\in I})$ and $(I, \hat{w}, (\hat{v}_j)_{j\in I})$ are procedural expected utility representations of $\succsim$ if and only if $\hat{w}$ is a monotone transformation of w, and $\hat{v}_j$ is a positive affine transformation of v_j for $j = 1, 2$.

Second, we study the uniqueness property of I.

Proposition 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that admits two procedural expected utility representations $(I^1, w^1, (v_j^1)_{j\in I^1})$ and $(I^2, w^2, (v_j^2)_{j\in I^2})$ with $I_1 \neq I_2$.*

(i) If $I^k = \{1, 2\}$ for some $k = 1, 2$, then $\succsim$ has an EU representation w where there exist $w_i : X_i \to \mathbb{R}, i = 1, 2$ such that $w(x) = w_1(x_1) + w_2(x_2)$ for all $x \in X$.

(ii) If $I^1 \cup I^2 = \{i\}$ for some $i = 1, 2$, then $\succsim$ has an EU representation w where there exist functions $w_j : X_j \to \mathbb{R}, j = 1, 2$ and $a : X_{-i} \to \mathbb{R}$ such that $w(x) = w_{-i}(x_{-i}) + a(x_{-i})w_i(x_i)$ for all $x \in X$.

(iii) If $I^1 = \{i\}, I^2 = \{-i\}$ for some $i = 1, 2$, then $\succsim$ has an EU representation w where there exist $w_i : X_i \to \mathbb{R}, i = 1, 2$ such that either $w(x) = w_1(x_1) + w_2(x_2)$ for all $x \in X$, or $w(x) = w_1(x_1) \cdot w_2(x_2)$ for all $x \in X$.

According to Proposition 2, if a decision maker's behavior is consistent with two different evaluation procedures as in Definition 2, then it is also consistent with the procedure associated with an EU preference, where the preference over deterministic outcome profiles satisfies certain separability condition. For instance, part (i) of Proposition 2 provides a characterization of the EU representation with an additively separable Bernoulli index. In the application to multi-period consumption where $w_1 = w_2$ and $i = 2$, the representation in part (ii) corresponds to a two-period version of Uzawa preferences (Uzawa, 1968, Epstein, 1983).

3.3 Proof Sketch of Theorem 1

In what follows, we sketch the proof of Theorem 1; a complete proof appears in the appendix. We focus here only on the sufficiency of the axioms.

Step 1. Representation of $\succsim$ when the decision maker is indifferent between conditionally comparable lotteries. We start by observing that each conditional preference must admit an EU representation. If the decision maker is further indifferent between any two comparable lotteries, then she must neglect correlation in the sense that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$. We also show that the conditional preference $\succsim_{i|q_{-i}}$ in dimension i is independent of $q_{-i} \in \mathcal{P}_{-i}$. This guarantees an NEU representation of $\succsim$. If instead $P \succ Q$ for some comparable P and Q, then we argue that $\succsim$ must be an EU preference.

Step 2. Implications if the assumption in Step 1 fails. Suppose that there exist $P, Q \in \mathcal{P}$ such that $P \succ Q$, and P and Q are conditionally comparable in dimension $i = 1$. The analysis when $i = 2$ is symmetric. In this case, we show that Axiom 5 can be strengthened to the following property.

Axiom 5*. For any $\alpha \in (0, 1)$, $P, Q, R, S \in \mathcal{P}$ such that $P_1 \perp R_1$ and $Q_1 \perp S_1$, if $P \succ Q$ and $R \sim S$, then $\alpha P + (1 - \alpha)R \succ \alpha Q + (1 - \alpha)S$.

Step 3. Representation of $\succsim$ if Axiom 5 holds.* Define $U : \mathcal{P} \to [0, 1]$ such that $P \sim U(P)\delta_{(\bar{c}_1,\bar{c}_2)} + (1-U(P))\delta_{(\underline{c}_1,\underline{c}_2)}$ for all $P \in \mathcal{P}$. We show that U is well-defined and represents $\succsim$. Axiom 5* implies that $U(\alpha P+(1-\alpha)R) = \alpha U(P)+(1-\alpha)U(R)$ for all $P, R \in \mathcal{P}$ such that $P_1 \perp R_1$. Hence, $U(P) = \sum_{x_1 \in X_1} U(x_1, P_{2|x_1})P_1(x_1)$ for all $P \in \mathcal{P}$. Since $\succsim_{2|x_1}$ admits an EU representation, we can find Bernoulli indices w and v_{x_1} for each x_1 such that for all $P \in \mathcal{P}$,

$$U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1). \tag{5}$$

Step 4. Characterize when $\succsim$ admits an SEU representation. We show that if

$\succsim_{2|x_1}$ is independent of $x_1 \in X_1$, that is $v_{x_1} \equiv v$ for some Bernoulli index v, then the representation (5) reduces to an SEU representation with $I = \{2\}$.

Step 5. Characterize when $\succsim$ admits an EU representation. Suppose that $\succsim_{2|x_1}$ depends on $x_1 \in X_1$. For any $p, q \in \mathcal{P}_1$, $r, r' \in \mathcal{P}_2$, and $\alpha \in (0, 1)$, we show that if $r \succsim_{2|p} r'$ and $r \succsim_{2|q} r'$, then $r \succsim_{2|\alpha p+(1-\alpha)q} r'$. Then Harsanyi (1955)'s utilitarianism theorem implies that the Bernoulli index of $\succsim_{2|\alpha p+(1-\alpha)q}$ must be a convex combination of those of $\succsim_{2|p}$ and $\succsim_{2|q}$. Moreover, we can show that v_{x_1} must be a positive affine transformation of $w(x_1, \cdot)$ for all $x_1 \in X_1$ in representation (5), and hence $\succsim$ admits an EU representation with Bernoulli index w.

4 Applications

4.1 Multi-source Income

This section considers a decision maker who receives income from two different sources, such as salary and investment returns, or two monetary gambles.

Suppose that $X_1 = X_2 = Z := [-\bar{x}, \bar{x}]$ for some $\bar{x} > 0$. An outcome in Z is a monetary prize and represents a loss if it takes a negative value. For each $(x_1, x_2) \in X$, the final wealth is $x_1 + x_2 \in 2Z := [-2\bar{x}, 2\bar{x}]$. Each lottery $P \in \mathcal{P}$ represents a joint distribution of income levels from two sources, and induces a distribution over final wealth denoted by $f[P]$. Formally, $f[P] \in \Delta(2Z)$ is derived by taking the sum of two monetary prizes in each contingency, that is, the probability of final wealth level $z \in 2Z$ is $f[P](z) = \sum_{(x_1,x_2):x_1+x_2=z} P(x_1, x_2)$. For any two distributions $p, q \in \Delta(2Z)$, we say that p *(first-order) stochastically dominates* q, denoted by $p \succ_{FOSD} q$, if $p \neq q$ and $\sum_{x \le z} q(x) \ge \sum_{x \le z} p(x)$ for all $z \in 2Z$. We say a binary relation $\succsim$ satisfies *Dominance* if $P \succ Q$ for any $P, Q \in \mathcal{P}$ such that $f[P] \succ_{FOSD} f[Q]$. This property is naturally satisfied by a decision maker who cares about final wealth and prefers more money to less, and hence is commonly assumed in the literature. However, experimental evidence shows that many subjects violate it in practice. Consider the following experiment of Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009).

Example 1. Suppose you face the following pair of concurrent decisions. All risks are independent from each other and will be resolved simultaneously. First examine both decisions, and then indicate your choices. Both choices will be payoff-relevant, i.e., the gains and losses will be added to your overall payment.

Decision 1: Choose between:

A. A sure gain of \$2.40.

B. A 25 percent chance to gain \$10.00, and a 75 percent chance to gain \$0.

Decision 2: Choose between:

C. A sure loss of \$7.50.

D. A 75 percent chance to lose \$10.00, and a 25 percent chance to lose \$0.

Across different treatments of Tversky and Kahneman (1981) and Rabin and Weizsäcker (2009), a large fraction (ranging from 28 percent to 66 percent) of subjects choose A in decision 1 and D in decision 2. However, the distribution of final wealth resulting from the combination of A and D is stochastically dominated by that resulting from the combination of B and C:

$$f[(B,C)] = \frac{3}{4}\delta_{-7.50} + \frac{1}{4}\delta_{2.50} \succ_{FOSD} \frac{3}{4}\delta_{-7.60} + \frac{1}{4}\delta_{2.40} = f[(A,D)].$$

This violation of Dominance is stark since the combination of B and C is equal to the combination of A and D plus a sure payoff of \$0.10. It is inconsistent with models where only the distribution of final wealth enters the utility function, including those allowing dominance violations.[9]

To study evaluation procedures in the setting with risky multi-source income, we impose two additional properties. First, we note that like other experiments on dominance violations, Example 1 involves non-trivial risk in at least some of the options. Such an experimental design makes sense because when all options are riskless, the decision problem is so simple that one might confidently expect subjects to choose the options delivering the highest final wealth. Hence, we require $\succsim$ to satisfy *Dominance without Risk*, that is, $(x_1, x_2) \succ (y_1, y_2)$ if $x_1 + x_2 > y_1 + y_2$. This can also be interpreted as fungibility of money in the absence of risk. Second, we assume that changing the order of monetary gambles in Example 1

[9]Examples include the disappointment theory of Bell (1985) and Loomes and Sugden (1986), the "choice-acclimating personal equilibrim" in the reference-dependent utility theory of Kőszegi and Rabin (2007), and the preference for simplicity of Mononen (2022) and Puri (2022).

has no impact. We say $\succsim$ satisfies *Symmetry* if $(p,q) \sim (q,p)$ for any $p,q \in \Delta(Z)$.

Fact 1. A procedural expected utility preference $\succsim$ satisfies Dominance without Risk and Symmetry if and only if it is represented by one of the following:

$$V^{EU}(P) = \sum_{x_1,x_2} u(x_1+x_2)P(x_1,x_2) \quad \text{and} \quad V^{NEU}(P) = CE_u(P_1) + CE_u(P_2),$$

for each $P \in \mathcal{P}$, where $u : 2Z \to \mathbb{R}$ is continuous and strictly increasing.

Clearly, if $\succsim$ admits an EU representation, then it satisfies Dominance and cannot accommodate dominance violations in Example 1. By contrast, with an NEU preference, the decision maker acts as if she first narrowly evaluates the marginal distribution of income in each source and then takes the sum of the certainty equivalents. This can explain dominance violations in Example 1.

Example 1 (continued). Suppose that the decision maker's preference is represented by V^{NEU} with

$$u(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, x \in Z, \\ -2\sqrt{-x}, & \text{if } x < 0, x \in Z. \end{cases}$$

Here u is a gain-loss Bernoulli index with CRRA risk preference and loss aversion parameter 2. The decision maker will simultaneously choose A and D since $CE_u(A) = 2.4 > 0.625 = CE_u(B)$ and $CE_u(D) = -5.625 > -7.5 = CE_u(C)$.

The decision maker's choice pattern is consistent with the notion of narrow bracketing studied in Thaler (1985) and Read, Loewenstein, and Rabin (1999). Instead of treating two decision problems as a whole, the decision maker narrowly brackets them by making each decision in isolation as if the other decision problem does not exist. The choice of A in decision 1 can be rationalized by risk aversion over gains, and the choice of B in decision 2 can be rationalized by risk seeking over losses, both of which are standard in the theoretical and experimental literature.[10]

Our paper is not the first attempt to provide a utility theory of narrow bracketing. For instance, the narrow preference in Vorjohann (2021) is represented by an

[10] For instance, this is an important component of the cumulative prospect theory introduced by Tversky and Kahneman (1992). They also assume that the decision maker adopts probability weighting, which is absent in our model as we impose Axiom 4.

EU representation with an additively separable Bernoulli index. Camara (2021) derives the same utility function using symmetry and computational tractability.[11] They suggest the following utility function of narrow bracketing in this setting:

$$V^{NB}(P) = \mathbb{E}_{P_1}[u] + \mathbb{E}_{P_2}[u].$$

Unlike our NEU representation in Fact 1, the decision maker with utility function V^{NB} evaluates and adds the *expected utilities* of marginal distributions of income, instead of their *certainty equivalents*. As a result, she might violate Dominance without Risk. To see this, consider the comparison between two portfolios. Portfolio P delivers \$1 in both assets for sure, while portfolio Q delivers \$2 in asset 1 and \$0 in asset 2 for sure. If the decision maker is risk averse, that is, if u is strictly concave, then she will strictly prefer portfolio P to portfolio Q since $2u(1) > u(0) + u(2)$, although both of them deliver total payoff \$2 with certainty. Building such extreme departures from rationality into agents' behavior might lead to a theory that explains certain anomalies in data at the expense of creating others that are unlikely to be present. In contrast, with an NEU preference in Fact 1, the decision maker will always choose more money when there is no risk.

4.2 Multi-period Consumption

In this section, we study the behavior of a decision maker faced with risky consumption in two periods. She finds it complex and demanding to evaluate intertemporal risk and hence might adopt procedures other than the one associated with EU preferences. Each outcome profile represents a consumption stream in two periods $t = 1, 2$. We call $t = 1$ today and $t = 2$ tomorrow. We assume that the consumption space in each period is a compact interval $X_1 = X_2 = C := [\underline{c}, \overline{c}] \subseteq \mathbb{R}_+$ and focus on preferences that satisfy the following assumption.

Assumption 1—Discounted Utility without Risk: There exist a continuous and strictly increasing function $u : C \to \mathbb{R}$ and $\beta \in (0, 1)$ such that for any $x_1, x_2, y_1, y_2 \in C$, we have $(x_1, x_2) \succsim (y_1, y_2) \iff u(x_1) + \beta u(x_2) \geq u(y_1) + \beta u(y_2)$.

[11] Note that computational tractability in Camara (2021) is an asymptotic notion and requires the decision problem to be high-dimensional.

Assumption 1 is introduced by Dillenberger, Gottlieb, and Ortoleva (2020) and posits that in the absence of risk, the decision maker's preference can be represented by the summation of discounted utilities in different periods. This is true for the vast majority of models of time preferences in the literature, including the commonly used Expected Discounted Utility (EDU) representation:

$$V^{EDU}(P) = \mathbb{E}_{P_1}[u] + \beta \mathbb{E}_{P_2}[u]. \tag{6}$$

With an EDU preference, the decision maker's time preference and risk preference are both determined by the same function u: The reciprocal of the elasticity of intertemporal substitution (EIS) coincides with the coefficient of relative risk aversion (RRA). However, numerous empirical studies in macroeconomics, finance, and behavioral economics have suggested the need to separate time and risk preferences.[12] Note that according to Proposition 2, the decision maker's choice behavior is consistent with all evaluation procedures in Definition 2. Below we discuss how different generalizations of EDU can separate time and risk preferences.

First, under Assumption 1, the EU preference can be represented by

$$V^{EU}(P) = \sum_{x_1, x_2} \phi\Big(u(x_1) + \beta u(x_2)\Big) P(x_1, x_2). \tag{7}$$

where $\phi : (1+\beta)u(C) \to \mathbb{R}$ is continuous and strictly increasing.[13] Fixing the discount factor β, the curvature of u captures the decision maker's time preference, while the combination of u and v determines her risk attitude. Moreover, the risk attitude for consumption in period 2 generically depends on the level of consumption in period 1. Such history dependence limits the applicability of (7).

Second, if $\succsim$ is an NEU preference that satisfies Assumption 1, then it has the following representation:

$$V^{NEU}(P) = u(CE_{v_1}(P_1)) + \beta u(CE_{v_2}(P_2)), \tag{8}$$

[12] Bansal and Yaron (2004) and Barro (2009) show that RRA should be much higher than the reciprocal of EIS in order to fit macroeconomic and financial data. See also Andreoni and Sprenger (2012), Nakamura et al. (2017) and references therein.

[13] As noted by Dillenberger, Gottlieb, and Ortoleva (2020), one can interpret (7) as applying the multi-attribute utility function in Kihlstrom and Mirman (1974) to the context of time.

which corresponds to the Dynamic Ordinal Certainty Equivalent (DOCE) model of Selden (1978), Selden and Stux (1978) and Kubler, Selden, and Wei (2020). The decision maker first evaluate risky consumption in each period in separation, and then aggregates the certainty equivalents using discounted utility. The model achieves the separation of time and risk preferences, as the former is captured by u and the latter is captured by v_1 and v_2. Moreover, the decision neglects the correlation between risk across different periods, as she only cares about marginal distributions. This is in contrast with the strong experimental support of correlation aversion in this setting (Andersen et al., 2018, Lanier et al., 2020).

Third, we consider the SEU preference with $I = \{2\}$, where the decision maker acts as if she adopts backward induction and first evaluates tomorrow's risky consumption in isolation. The representation is given by

$$V^{SEU}(P) = \sum_{x_1} \phi\Big(u(x_1) + \beta u(CE_v(P_{2|x_1})\Big) P_1(x_1), \tag{9}$$

which can be interpreted as a two-period adaptation of the recursive preference (Chew and Epstein, 1991) to our domain of lotteries over consumption streams. By comparison, the recursive preference is defined on a richer domain called the space of temporal lotteries introduced by Kreps and Porteus (1978). For the remainder of this section, we will focus on the following specific parameterization of representation (9) with constant coefficients of EIS and RRA:

$$V^{SEU}(P) = \sum_{x_1} \frac{1}{\alpha} \Big\{ x_1^\rho + \beta [\mathbb{E}_{P_{2|x_1}}(x_2^\alpha)]^{\rho/\alpha} \Big\}^{\alpha/\rho} P_1(x_1), \tag{10}$$

where $\gamma := 1 - \alpha \in \mathbb{R}_+ \backslash \{1\}$ is the coefficient of RRA, and $\psi := \frac{1}{1-\rho} \in \mathbb{R}_+ \backslash \{1\}$ is the coefficient of EIS. Hence, the representation (10) separates time and risk preferences. It is reminiscent of a two-period CRRA-CES version of the Epstein-Zin (EZ) preference of Epstein and Zin (1989, 1991) and Weil (1990), which has the following recursive formulation of continuation utility U_t for each period t:

$$U_t = \Big\{ c_t^\rho + \beta \Big[\mathbb{E}_t \Big(U_{t+1}^\alpha \Big) \Big]^{\rho/\alpha} \Big\}^{1/\rho}. \tag{11}$$

Despite the similarity between (10) and (11), the two representations are defined on different domains.[14] In our setting of two periods, the domain of an SEU preference is the set of lotteries $\mathcal{P}$, while the domain of an EZ preference is the set of temporal lotteries $\mathcal{D} := \Delta(C \times \Delta(C))$. The difference between a temporal lottery and a lottery can be illustrated by the following simple example.

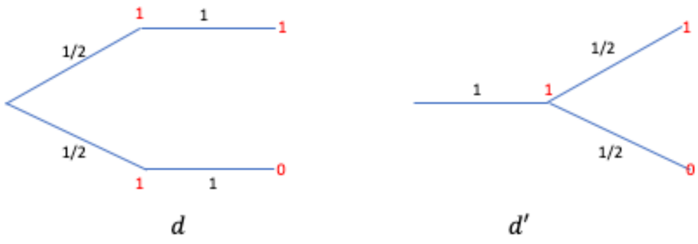

Figure 1: Example of two temporal lotteries that induce the same lottery.

In Figure 1, two temporal lotteries d and d' deliver consumption 1 for sure in period 1 and have equal probability to deliver either consumption 1 or consumption 0 in period 2, determined by a coin flip. They induce the same lottery over consumption streams, but differ in the timing of risk resolution. In d, the coin is flipped in period 1 and the consumer knows the realization her future consumption in advance. In d', the coin is flipped in period 2 and the risk regarding consumption in period 2 is only resolved then. To separate time and risk preferences, the EZ representation (11) entails a non-trivial attitude toward the above difference in timing of risk resolution. Indeed, in most empirical applications of EZ, the decision maker has a preference for early resolution of risk, i.e., she strictly prefers d to d' in Figure 1. For instance, the main estimation results of the long-run risks model of Bansal and Yaron (2004) are based on RRA $=10$ and EIS $=1.5$. Through introspection, Epstein, Farhi, and Strzalecki (2014) show that these parameter values imply that the representative agent is willing to give up around 25 percent of her lifetime consumption in order to have all risk about future consumption resolved in the next period. This timing premium is unrealistically high since the

[14]In the online appendix, we show that an infinite-horizon extension of (10) can induce the same asset pricing implications as the standard EZ model. This is consistent with the observation of Stanca (2021) that the attitude toward timing of risk resolution is not the key behavioral feature driving the results in applications of recursive utility. He highlights the importance of correlation aversion, which describes a decision maker who dislikes positive autocorrelation in the consumption streams, and is equivalent to assuming $\rho > \alpha$ in (10).

risk is about future consumption instead of future income or asset returns, and hence such information has no apparent instrumental value. In other words, the representative agent has no need to reoptimize her contingent consumption plans given early resolution of risk.[15]

By contrast, the SEU representation (10) is defined on a domain where there is no timing of risk resolution, and it achieves the the separation of time and risk preferences because of the sequential evaluation procedure adopted by the decision maker. Indeed, if we extend the SEU representation to the set of temporal lotteries by assuming indifference to temporal resolution of risk, then the decision maker would find d to d' in Figure 1 equally desirable. As a result, she attaches no value to non-instrumental information and the timing premium is always zero. Hence, our SEU preference can accommodate indifference to temporal resolution of risk and the separation of time and risk preferences, which is impossible under the EZ preference.[16] Our theory also suggests a novel connection between the EZ-type behavior, a relaxation of the Independence axiom across periods, and a behavioral heuristic to simplify the evaluation of intertemporal risk.[17]

4.3 Time Lotteries

In this section, we study decisions involving uncertainty about both which and when outcomes will be received. Suppose that $X_1 = Z = [w, b] \subset \mathbb{R}_{++}$ and $X_2 = T = [0, \bar{t}] \subset \mathbb{R}_+$. Each outcome profile $(z, t) \in Z \times T$ denotes a *dated prize* where the monetary prize z is received in period t.[18] Each lottery $P \in \mathcal{P}$ denotes a distribution of dated prizes, where both the prize and the payment date can be random. In particular, a *time lottery* $(z, p) \in Z \times \Delta(T)$ is a lottery where the monetary prize z is fixed.

[15] Meissner and Pfeiffer (2022) show that on average subjects give up 5% of their total consumption to resolve all uncertainty immediately in an experiment.

[16] To achieve a full separation of the three components, one needs to allow more general extensions of the SEU representation to temporal lotteries. We leave it for further research.

[17] By interpreting each dimension as a period when risk is resolved, instead of when consumption happens, one can study the effects of evaluation procedures on the attitudes toward temporal resolution of uncertainty (Ergin and Gul, 2009, Artstein-Avidan and Dillenberger, 2015).

[18] Following DeJarnette et al. (2020), we interpret the outcome profile (z, t) as representing z being consumed in period t, ruling out the possibility of z being saved for future consumption. Our results are not affected by alternative interpretations.

Since early delivery of the prize is more desirable for the decision maker due to impatience, we assume that for any $(x,t),(y,s) \in Z \times T$, if $x \geq y$, $t \leq s$ and $(x,t) \neq (y,s)$, then $(x,t) \succ (y,s)$. By replacing Axiom 2 with this property in Theorem 1, we can characterize the procedural expected utility preference represented by $(I, w, (v_j)_{j \in I})$ where w is strictly increasing in the first argument and strictly decreasing in the second argument, and v_2 is strictly decreasing if $2 \in I$. Moreover, we assume that in the absence of risk, the decision maker's behavior follows the standard exponentially discounted utility model.[19] Hence, we assume that

$$w(z,t) = \phi(e^{-rt}u(z)), \tag{12}$$

where $r > 0$ and $u : Z \to \mathbb{R}_{++}$ and $\phi : [e^{-r\bar{t}}u(w), u(b)] \to \mathbb{R}$ are strictly increasing and continuous.

We will study two properties introduced by DeJarnette et al. (2020). The first one is the risky counterpart of impatience and posits that if the decision maker can choose pair monetary prizes with payment dates in the presence of risk, then she would prefer to receive the highest prize at the earliest time. Formally, we say a binary relation $\succsim$ satisfies *Stochastic Impatience* if for any $t_1, t_2 \in T$ and $x_1, x_2 \in Z$ with $t_1 < t_2$ and $z_1 > z_2$,

$$\frac{1}{2}\delta_{(z_1,t_1)} + \frac{1}{2}\delta_{(z_2,t_2)} \succsim \frac{1}{2}\delta_{(z_2,t_1)} + \frac{1}{2}\delta_{(z_1,t_2)}.$$

We say $\succsim$ satisfies *Strict Stochastic Impatience* if the above holds with $\succ$.

The second property concerns risk attitudes toward time, instead of monetary prizes. The decision maker is said to be *risk averse over time lotteries* if she prefers receiving a prize on a sure date than on a random date with the same mean, that is, for any time lottery $(z,p) \in Z \times \Delta(T)$,

$$(z, \mathbb{E}_p(t)) \succsim (z,p).$$

Analogously, the decision maker is *risk seeking over time lotteries (RSTL)* or *risk*

[19] Fishburn and Rubinstein (1982) show that exponentially discounted utility can be characterized by the Stationarity axiom: For any $z, z' \in Z$, $s,t \in T$, and $\tau \in \mathbb{R}$ with $s+\tau, t+\tau \in T$, if $(z,t) \sim (z', t+\tau)$, then $(z,s) \sim (z', s+\tau)$.

neutral over time lotteries (RNTL) if the above holds with $\precsim$ or $\sim$, respectively.

When $I = \{1, 2\}$ and ϕ in (12) is affine, we derive the EDU model in this setting, that is, $\succsim$ is represented by $V(P) = \mathbb{E}_P(e^{-rt}u(z))$. As is well known, with an EDU preference, the decision maker satisfies Stochastic Impatience and must be RSTL, since the exponential function e^{-rt} is convex in t.[20] However, the experimental evidence of DeJarnette et al. (2020) reveals that the majority of their subjects are RATL in most questions. They also show that such incompatibility between Stochastic Impatience and any violation of RSTL persists in the general EU model and a large class of non-EU models, including those allowing probability weighting. They suggest that one potential solution is to maintain the Independence axiom within each dimension, and weakening it across dimensions. The following result demonstrates this point.

Proposition 3: *Suppose that $\succsim$ has a procedural expected utility representation $(I, w, (v_j)_{j\in I})$ with w given in (12).*

(i) The binary relation $\succsim$ satisfies Stochastic Impatience and violates RSTL if and only if either $I = \{1, 2\}$ and v_2 is not convex, or $I = \{2\}$, v_2 is not convex and ϕ is a convex transformation of the logarithmic function log.[21]

(ii) The binary relation $\succsim$ satisfies Strict Stochastic Impatience and violates RSTL if and only if $I = \{2\}$, v_2 is not convex and ϕ is a strictly convex transformation of the logarithmic function log.

Proposition 3 characterizes when the procedural expected utility preference can accommodate Stochastic Impatience and violations of RSTL simultaneously. The fact that EU preferences and SEU preferences with $I = \{1\}$ are ruled out follows from Theorem 1 of DeJarnette et al. (2020).[22] By comparison, with the other two evaluation procedures, the decision maker acts as if she evaluates risk in time in isolation and her risk attitude toward time is isolated from intertemporal trade-

[20]This feature of EDU has implications in many applications, including dynamic moral hazard (Ely and Szydlowski, 2020), dynamic information acquisition (Zhong, 2022, Chen and Zhong, 2022), and dynamic contract theory (Madsen, 2022).

[21]A function f defined on $B \subseteq \mathbb{R}_{++}$ is a convex transformation of log if there exists a convex function g with $f(x) = g(\log(x))$ for all $x \in B$. We say the transformation is strictly convex if g is strictly convex.

[22]Stochastic Impatience and RSTL only involve lotteries with degenerate conditional lotteries in dimension 1. Hence, EU and SEU preferences with $I = \{1\}$ share the same predictions.

offs. As a result, her preference $\succsim$ violates RSTL if and only if v_2 is not convex. If $\succsim$ is an SEU preference with $I = \{2\}$, then (Strict) Stochastic Impatience requires ϕ to be "more (strictly) convex" than the logarithmic function, as implied by Propositions 2 and 4 of DeJarnette et al. (2020). If instead $\succsim$ is an NEU preference, then the decision maker only cares about marginal distributions and deems irrelevant the paring between prizes and payment dates. Accordingly, she cannot satisfy Strict Stochastic Impatience.[23] We end this section with an example that satisfies the conditions in Proposition 3.

Example 2. Consider the following SEU representation:

$$V^{SEU}(P) = \sum_{z} P_1(z)u(z) \cdot e^{-rK_a(P_{2|z})}, \tag{13}$$

where $u : Z \to \mathbb{R}_{++}$ is continuous and strictly increasing, $K_a(p) = \frac{1}{a} \log \mathbb{E}_p[e^{at}]$ if $a \neq 0$, and $K_0(p) = \mathbb{E}_p(t)$. If $a > 0$ and $r > 0$, then the corresponding preference $\succsim$ satisfies RATL and Strict Stochastic Impatience. Note that K_a is a monotone additive statistic introduced by Mu et al. (2021b), which is a function over random variables that is monotone with respect to stochastic dominance, and additive for sums of independent random variables. Mu et al. (2021b) characterize monotone additive statistics with weighted averages over the family $\{K_a\}$.[24] They study a preference over time lotteries $Z \times \Delta(T)$ which can be represented by $V(z, p) = u(z)e^{-r\Psi(p)}$ with a monotone additive statistic Ψ. Since there is no risk in the monetary prize, Stochastic Impatience is not relevant. By comparison, our procedural expected utility preference is defined on $\Delta(Z \times T)$ and hence we can study the interaction between Stochastic Impatience and the risk attitude toward time. Moreover, the axiom of Within-Dimension Independence implies that K_a in representation (13) cannot be replaced with a general monotone additive statistic.

[23]The NEU preference in this section is different from the one discussed in Section 4.2 (and hence the DOCE model) due to different interpretations of outcome profiles. More detailed discussion can be found in the online appendix.

[24]Mu et al. (2021b) allow $a = \infty$ and $-\infty$ such that K_∞ and $K_{-\infty}$ represent the essential maximum and the essential minimum respectively.

Appendix: Proofs

Proof of Theorem 1. We focus on the "if" part as the "only if" part can be verified routinely. Assume that Axioms 1-5 hold throughout the proof.

Lemma 1: *For each $i = 1, 2$ and $q \in \mathcal{P}_{-i}$, the conditional preference $\succsim_{i|q}$ admits an EU representation with Bernoulli index $v_{i|q}$, which is continuous and unique up to a positive affine transformation. Moreover, if $q \in X_{-i}$, then $v_{i|q}$ can be chosen to be strictly increasing.*

Proof of Lemma 1 . Without loss of generality, fix $i = 1$ and $q \in \mathcal{P}_2$. By Axioms 3.1 and 4, the conditional preference $\succsim_{1|q}$ admits an EU representation with a Bernoulli index $v_{1|q}$ defined on X_1, which is unique up to a positive affine transformation. To see why $v_{1|q}$ is continuous, suppose by contradiction that there exists a sequence (x_1^n) in X_1 such that $x_1^n \to x_1 \in X_1$ and $v_{1|q}(x_1^n) \not\to v_{1|q}(x_1)$. Without loss of generality and passing to a subsequence if necessary, suppose $v_{1|q}(x_1^n) \to a < b = v_{1|q}(x_1)$ and $v_{1|q}(x_1^n) < (a + b)/2$ for all n. Since $\succsim_{1|q}$ admits an EU representation, we can find $r \in \mathcal{P}_1$ with $\sum_{y_1 \in X_1} v_{1|q}(y_1)r(y_1) = (a + b)/2$, that is, $(x_1^n, q) \prec (r, q) \prec (x_1, q)$ for all n. Since $(x_1^n, q) = (x_1, q)_{(x_1^n - x_1, 0)}$ and $(x_1^n - x_1, 0) \to (0, 0)$, Axiom 3.2 implies $(x, q) \precsim (r, q) \prec (x, q)$, a contradiction. Hence $v_{1|q}$ is continuous for each $q \in \mathcal{P}_2$. Moreover, if $q \in X_2$, that is, $q = y_2$ for some $y_2 \in X_2$, then by Axiom 2, the function $v_{1|q}$ must be strictly increasing. $\square$

Step 1: Representation of $\succsim$ when the decision maker is indifferent between (conditionally) comparable lotteries.

Recall that P and Q are comparable if there exists $i \in \{1, 2\}$ such that $P_{-i} = Q_{-i}$ and $P_i \sim_{i|y_{-i}} Q_i$ for some $y_{-i} \in X_{-i}$. Moreover, P and Q are *conditionally comparable* if there exist $i \in \{1, 2\}$ and $x_{-i} \in X_{-i}$ such that $P_{-i} = Q_{-i}$ and $P_{i|y_{-i}} \sim_{i|x_{-i}} Q_{i|y_{-i}}$ for all $y_{-i} \in \text{supp}(P_{-i})$. Consider the following axiom.

Axiom 6—Comparability Indifference: For each $P, Q \in \mathcal{P}$, if P and Q are comparable, then $P \sim Q$.

Lemma 2: *Let $\succsim$ be a binary relation on $\mathcal{P}$. Then $\succsim$ satisfies Axioms 1-6 if and only if it is an NEU preference.*

Proof of Lemma 2. It is easy to verify that an NEU preference satisfies Axioms 1-6. Now suppose Axioms 1-6 hold. Axiom 6 implies that $P \sim (P_1, P_2)$ for all $P \in \mathcal{P}$, and hence we can focus on the restriction of $\succsim$ to $\mathcal{P}_1 \times \mathcal{P}_2$. For each $x_1, y_1 \in X_1$ and $p_2, q_2 \in \mathcal{P}_2$, again by Axiom 6, we have $(x_1, p_2) \sim (x_1, q_2) \iff (y_1, p_2) \sim (y_1, q_2)$.s By Lemma 1, the conditional preferences $\succsim_{2|x_1}$ and $\succsim_{2|y_1}$ must be identical for all $x_1, y_1 \in X_1$. Denote the common conditional preference by $\succsim_2$ and the continuous and strictly increasing Bernoulli index by v_2. The conditional preference in dimension 1 is $\succsim_1$ with the Bernoulli index v_1. For each $p_1 \in \mathcal{P}_1$ and $p_2, q_2 \in \mathcal{P}_2$, if $p_2 \sim_2 q_2$, then p_2 and q_2 are comparable in dimension 2. Axiom 6 implies $(p_1, p_2) \sim (p_1, q_2)$. Hence, for each $(p_1, p_2) \in \mathcal{P}_1 \times \mathcal{P}_2$, we have $(p_1, p_2) \sim (p_1, CE_{v_2}(p_2)) \sim (CE_{v_1}(p_1), CE_{v_2}(p_2))$. The second indifference relation holds since the conditional preference $\succsim_{1|CE_{v_2}(p_2)}$ admits an EU representation v_1.

Now we define $\hat{\succsim}$ as the restriction of $\succsim$ to X. By Axiom 3.1, the binary relation $\hat{\succsim}$ is continuous. Then Debreu's Theorem implies that $\hat{\succsim}$ is represented a continuous utility function w. Axiom 2 guarantees that w is strictly increasing. Therefore (w, v_1, v_2) is an NEU representation of $\succsim$. $\square$

From now on, we assume that $\succsim$ violates Axiom 6. That is, there exist $P, \tilde{P} \in \mathcal{P}$ such that $\tilde{P}$ and P are comparable, and $P \succ \tilde{P}$. We first introduce some additional notation. We say two finite set of lotteries $\mathcal{M}, \mathcal{M}' \subset \mathcal{P}$ are singular, denoted by $M \perp M'$, if $P \perp P'$ for all $P \in M$ and $P' \in M'$. A singleton set $M = \{P\}$ is simply written as P. Similar notation can be defined for sets of marginal lotteries in each dimension $i = 1, 2$. For each set $A \subseteq \mathbb{R}^n$ for some positive integer n, we denote by A^o its interior. Denote by $\bar{c} = (\bar{c}_1, \bar{c}_2)$ and $\underline{c} = (\underline{c}_1, \underline{c}_2)$. Given our relaxation of the Independence axiom, for $R \succ S$ and $\lambda \in (0, 1)$, it is *not* guaranteed that $R \succ \lambda R + (1-\lambda)S \succ S$. Instead, we have the following property.

Lemma 3: *For any $Q \succ Q'$, there exist $\lambda^* \in [0, 1]$ and $Q^* = \lambda^* Q + (1 - \lambda^*)Q'$ such that for any $\varepsilon > 0$, we can find $\lambda_\varepsilon \in (\lambda^* - \varepsilon, \lambda^* + \varepsilon) \cap [0, 1]$ with $Q^* \not\sim \lambda_\varepsilon Q + (1 - \lambda_\varepsilon)Q'$.*

Proof of Lemma 3. Suppose the result fails. Then for any $\lambda \in [0, 1]$, there exists $\varepsilon_\lambda > 0$ such that for any $\lambda' \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda) \cap [0, 1]$, we have $\lambda Q + (1 - \lambda)Q' \sim \lambda' Q + (1 - \lambda')Q'$. Notice that $\{(\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)\}_{\lambda \in [0,1]}$ forms an open cover of the compact set $[0,1]$. We can find a finite subcover of $[0,1]$. By transitivity of $\succsim$, we know $\lambda Q + (1-\lambda)Q' \sim \lambda' Q + (1-\lambda')Q'$ for all $\lambda, \lambda' \in [0,1]$, which leads to $Q \sim Q'$ and a contradiction. $\square$

Lemma 4: *For any $\alpha \in (0,1)$ and $Q, R, S \in \mathcal{P}$ with $Q \perp \{R, S\}$, if $R \sim S$, then $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.*

Proof of Lemma 4. Recall that $X_1 = [\underline{c}_1, \bar{c}_1]$ and there exist $P, \tilde{P} \in \mathcal{P}$ such that $\tilde{P}$ and P are comparable, and $P \succ \tilde{P}$. Without loss of generality, assume that $\tilde{P}$ and P are comparable in dimension 2 and $P_1 = \tilde{P}_1$. Moreover, we can assume $\operatorname{supp}(P_i) \cup \operatorname{supp}(\tilde{P}_i) \in X_i^o = (\underline{c}_i, \bar{c}_i)$ for both $i = 1, 2$. To see this, first note that by Axiom 3.2, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $\varepsilon' = (\varepsilon_1, \varepsilon_2')$ with $0 < \varepsilon_1, \varepsilon_2, \varepsilon_2' < \bar{\varepsilon}$, we have $P_\varepsilon \succ \tilde{P}_{\varepsilon'}$. Since P and $\tilde{P}$ are comparable in dimension 2, we can find $x_1 \in X_1$ with $P_2 \sim_{2|x_1} \tilde{P}_2$. By Lemma 1, there exist $\hat{\varepsilon} = (\hat{\varepsilon}_1, \hat{\varepsilon}_2)$ and $\hat{\varepsilon}' = (\hat{\varepsilon}_1, \hat{\varepsilon}_2')$ such that $0 < \hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_2' < \bar{\varepsilon}$, $P_{\hat{\varepsilon},1} = \tilde{P}_{\hat{\varepsilon},1}$, and $P_{\hat{\varepsilon},2} \sim_{2|x_1} \tilde{P}_{\hat{\varepsilon}',2}$. That is, $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ are comparable in dimension 2 and $P_{\hat{\varepsilon},i}(\underline{c}_i) = \tilde{P}_{\hat{\varepsilon}',i}(\underline{c}_i) = 0$ for $i = 1, 2$. Repeat the above argument by considering negative values of ε.

Denote by $P^* = \lambda^* P + (1-\lambda^*)\tilde{P}$ the lottery in Lemma 3. Clearly, either $P^* \not\sim P$ or $P^* \not\sim \tilde{P}$. By Lemma 3, for any $n > 0$, there exists $\lambda_n \in (\lambda^* - 1/n, \lambda^* + 1/n) \cap [0,1]$ with $P^* \not\sim \lambda_n P + (1-\lambda_n)\tilde{P} := P^n$. Since $\succsim$ is complete by Axiom 1, for each n, either $P^n \succ P^*$ or $P^* \succ P^n$. Then we can find a subsequence of $(P^n)_{n\geq 1}$, still denoted by $(P^n)_{n\geq 1}$, such that either $P^n \succ P^*$ for all n or $P^* \succ P^n$ for all n. Suppose that the former holds. Take any $R, S \in \mathcal{P}$ such that $R \sim S$ and $\{R, S\} \perp \{P, \tilde{P}\}$. Then $\{R, S\} \perp \{P^*, P^n\}$ for all $n \geq 1$. By Lemma 1, P^* and P^n are comparable in dimension 2 for all $n \geq 1$. Axiom 5 implies that for all $\alpha \in (0,1)$ and $n \geq 1$, we have $\alpha P^n + (1-\alpha)R \succ \alpha P^* + (1-\alpha)S$. By Axiom 3.1, take n to infinity and we have $\alpha P^* + (1-\alpha)R \succsim \alpha P^* + (1-\alpha)S$. This holds for all $R \sim S$ with $\{R, S\} \perp \{P, \tilde{P}\}$. By switching the roles of R and S, we get $\alpha P^* + (1-\alpha)S \sim \alpha P^* + (1-\alpha)R$ for all $\alpha \in (0,1)$ and $R, S \in \mathcal{P}$ such that $\{R, S\} \perp \{P, \tilde{P}\}$. The same result holds if $P^* \succ P^n$ for all n. Without loss of generality, we assume that $P^n \succ P^*$ for all n and $P^* \succ \tilde{P}$ from now on.

Fix any $Q \in \mathcal{P}$ such that $Q \perp \{P, \tilde{P}\}$. Then $Q \perp \{P^*, P^n\}$ for each n. By Axiom 5 and Lemma 1, for any $\beta \in (0,1)$, we know $\beta P^* + (1-\beta)Q \succ \beta\tilde{P} + (1-\beta)Q$, and $\beta P^* + (1-\beta)Q$ and $\beta\tilde{P} + (1-\beta)Q$ are comparable in dimension 2. Similarly, as $P^n \succ P^*$ for all n, for any $\beta \in (0,1)$, we know $\beta P^n + (1-\beta)Q \succ \beta P^* + (1-\beta)Q$ and $\beta P^n + (1-\beta)Q$ and $\beta P^* + (1-\beta)Q$ are comparable in dimension 2. For any $R, S \in \mathcal{P}$ such that $R \sim S$ and $\{R, S\} \perp \{P, \tilde{P}, Q\}$, we know $\{R, S\} \perp \{\beta P^n + (1-\beta)Q, \beta P^* + (1-\beta)Q\}$ for all $n \geq 1$. Repeat the previous arguments and we can show that for any $\alpha, \beta \in (0,1)$,

$$\alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)R \sim \alpha[\beta P^* + (1-\beta)Q] + (1-\alpha)S.$$

The above indifference relation can be rearranged as

$$\beta[\alpha P^* + (1-\alpha)R] + (1-\beta)[\alpha Q + (1-\alpha)R] \sim \beta[\alpha P^* + (1-\alpha)S] + (1-\beta)[\alpha Q + (1-\alpha)S].$$

Again by Axiom 3.1, let $\beta \to 0^+$ and we have

$$\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S, \tag{14}$$

for any $\alpha \in (0,1)$, $R \sim S$, $Q \perp \{R, S\}$ and $\{P, \tilde{P}\} \perp \{Q, R, S\}$.

Fix any $Q, R, S \in \mathcal{P}$ with $R \sim S$, and $Q \perp \{R, S\}$. By Axiom 3.2, as $P \succ \tilde{P}$ and all lotteries have finite supports, we can use the same construction as the first paragraph of the proof of Lemma 4 to construct $P_{\hat{\varepsilon}}$, $\tilde{P}_{\hat{\varepsilon}'} \in \mathcal{P}$ such that (i) $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, (ii) $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ are comparable in dimension 2, and (iii) $\{P_{\hat{\varepsilon}}, \tilde{P}_{\hat{\varepsilon}'}\} \perp \{Q, R, S\}$. Then we can derive a counterpart of (14) for $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$. Hence, $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$ for all $\alpha \in (0,1)$. Since Q, R, S are arbitrary, this holds so long as $R \sim S$, and $Q \perp \{R, S\}$. This completes the proof. $\square$

For a fixed $y_2 \in X_2$, we focus on the set of lotteries whose utilities are strictly bounded by two lotteries in $\mathcal{P}_1 \times \{y\}$, that is,

$$\Phi_{2,y_2} = \Big\{P \in \mathcal{P} : \exists\, T, T' \in \mathcal{P} \text{ with } T_2 = T_2' = y_2 \text{ s.t. } T \succ P \succ T'\Big\}.$$

Similarly, for each $x_1 \in X_1$, we can define Φ_{1,x_1}.

Lemma 5: *(i) For each $P, Q, R \in \mathcal{P}$ with $P \succ Q \succ R$, there exists $\lambda \in (0,1)$ such that $\lambda P + (1-\lambda)R \sim Q$.*
(ii) For each $P \in \Phi_{2,y_2}$ with $y_2 \in X_2$, there exists $y_1 \in X_1$ such that $P \sim (y_1, y_2)$.

For each $P \in \Phi_{1,x_1}$ with $x_1 \in X_1$, there exists $x_2 \in X_2$ such that $P \sim (x_1, x_2)$.

Proof of Lemma 5. (i). Denote $A = \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \succ Q\}$ and $\lambda = \inf A$. Suppose by contradiction that $\lambda P + (1-\lambda)R \not\sim Q$. If $\lambda P + (1-\lambda)R \succ Q$, then $\lambda \in A$, which is open by Axiom 3.1. Hence, there exists $\lambda' < \lambda$ with $\lambda' \in A$, which contradicts with the definition of λ. If $\lambda P + (1-\lambda)R \prec Q$, then $\lambda \in \{\alpha \in (0,1) : \alpha P + (1-\alpha)R \prec Q\}$, which is also open. We can find $\varepsilon > 0$ such that $[\lambda, \lambda + \varepsilon) \subseteq (0,1) \backslash A$. Again a contradiction with the definition of λ.

(ii). If $P \in \Phi_{2,y_2}$ for some $y_2 \in X_2$, then we can find $p_1, p_1' \in \mathcal{P}_1$ with $(p_1, y_2) \succ P \succ (p_1', y_2)$. By Lemma 1, we can find a unique $\lambda \in [0,1]$ such that $P \sim (\lambda p_1 + (1-\lambda)p_1', y)$. Again by Lemma 1, there exists $y_1 \in X_1$ such that $P \sim (y_1, y_2)$. The proof for $P \in \Phi_{1,x_1}$ is symmetric. $\square$

Note that comparable lotteries are not necessarily conditionally comparable. The following axiom is weaker than Axiom 6.

Axiom 7—Conditional Comparability Indifference: For each $P, Q \in \mathcal{P}$, if P and Q are conditionally comparable, then $P \sim Q$.

Lemma 6: *Let $\succsim$ be a binary relation on $\mathcal{P}$. If $\succsim$ satisfies Axioms 1-5 and Axiom 7, and violates 6, then it is an EU preference with w.*

Proof of Lemma 6. Fix $\alpha \in (0,1)$ and $Q, R, S \in \mathcal{P}$ such that $R \sim S$, we want to show that $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$. First, assume that for each $i = 1, 2$ and $x_i \in X_i$, either $Q_i(x_i) = 0$ or $Q_{-i|x_i} \neq \delta_{\underline{c}_{-i}}, \delta_{\bar{c}_{-i}}$. Fix $z_1 \in X_1$. For each $x_1 \in \text{supp}(Q_1)$, since $Q_{2|x_1} \neq \delta_{\underline{c}_2}, \delta_{\bar{c}_2}$, by Lemma 1, we can find $q^{x_1} \in \mathcal{P}_2$ such that $q^{x_1} \sim_{2|z_1} Q_{2|x_1}$ and $q^{x_1} \perp \{R_2, S_2\}$. Let $Q' \in \mathcal{P}$ with $Q_1' = Q_1$ and $Q'_{2|x_1} = q^{x_1}$ for all $x_1 \in \text{supp}(Q_1)$. Easy to see that Q and Q' are conditionally comparable in dimension 2. Moreover, we claim that $\beta Q + (1-\beta)P$ and $Q' + (1-\beta)P$ are conditionally comparable in dimension 2 for all $\beta \in (0,1)$ and $P \in \mathcal{P}$. To see this, for any $x_1 \in \text{supp}(P_1) - \text{supp}(Q_1)$, we have $(\beta Q + (1-\beta)P)_{2|x_1} = P_{2|x_1} = (\beta Q' + (1-\beta)P)_{2|x_1}$; for any $x_1 \in \text{supp}(Q_1) - \text{supp}(P_1)$, we have $(\beta Q + (1-\beta)P)_{2|x_1} = Q_{2|x_1} \sim_{2|z_1} Q'_{2|x_1} = (\beta Q' + (1-\beta)P)_{2|x_1}$; for any $x_1 \in \text{supp}(Q_1) \cap \text{supp}(P_1)$, we have $(\beta Q + (1-\beta)P)_{2|x_1} = \gamma Q_{2|x_1} + (1-\gamma)P_{2|x_1} \sim_{2|z_1} \gamma Q'_{2|x_1} + (1-\gamma)P_{2|x_1} = (\beta Q' + (1-\beta)P)_{2|x_1}$, where $\gamma = Q_1(x_1)/(Q_1(x_1) + P_1(x_1))$. Axiom 7 implies that

$Q \sim Q'$ and $\beta Q + (1-\beta)P \sim Q' + (1-\beta)P$ for all $\beta \in (0,1)$ and $P \in \mathcal{P}$. Hence, it suffices to show that $\alpha Q' + (1-\alpha)R \sim \alpha Q' + (1-\alpha)S$, where $Q'_2 \perp \{R_2, S_2\}$. Using a similar argument, we can construct $Q'' \in \mathcal{P}$ such that Q'' and Q' are conditionally comparable in dimension 1. Then we know that $Q'' \sim Q'$ and $\beta Q'' + (1-\beta)P \sim Q' + (1-\beta)P$ for all $\beta \in (0,1)$ and $P \in \mathcal{P}$. Hence, to show that $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$, it suffices to show that $\alpha Q'' + (1-\alpha)R \sim \alpha Q'' + (1-\alpha)S$, where $Q'' \perp \{R, S\}$. The last statement holds due to Lemma 4. Second, we consider an arbitrary Q. Then there exists $\hat{Q}$ such that for any $\lambda \in (0,1)$, for each $i = 1,2$ and $x_i \in X_i$, either $(\lambda Q + (1-\lambda)\hat{Q})_i(x_i) = 0$ or $(\lambda Q + (1-\lambda)\hat{Q})_{-i|x_i} \neq \delta_{\underline{c}_{-i}}, \delta_{\overline{c}_{-i}}$. By the previous case, $\alpha(\lambda Q + (1-\lambda)\hat{Q}) + (1-\alpha)R \sim \alpha(\lambda Q + (1-\lambda)\hat{Q}) + (1-\alpha)S$. Let λ approach 1 and Axiom 3.1 implies $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

Now suppose $R \succ S$. We claim that $R \succ \alpha R + (1-\alpha)S \succ S$. First, note that the claim holds if $R, S \succsim (\underline{c}_1, \overline{c}_2)$ or $R, S \precsim (\underline{c}_1, \overline{c}_2)$. To see this, suppose that the former holds. Then by part (ii) of Lemma 5, we can find $x_1^R, x_1^S \in X_1$ with $x_1^R > x_1^S$, $(x_1^R, \overline{c}_2) \sim R$ and $(x_1^S, \overline{c}_2) \sim S$. By Lemma 1 and the previous result,

$$R \sim (x_1^R, \overline{c}_2) \succ \alpha(x_1^R, \overline{c}_2) + (1-\alpha)(x_1^S, \overline{c}_2) \sim \alpha R + (1-\alpha)S \succ (x_1^S, \overline{c}_2) \sim S.$$

Next, suppose $R \succ (\underline{c}_1, \overline{c}_2) \succ S$. By part (i) of Lemma 5, there exists $\lambda \in (0,1)$ with $\lambda R + (1-\lambda)S \sim (\underline{c}_1, \overline{c}_2)$. Then for any $\alpha \in (0,1)$, if $\alpha > \lambda$, then we can find $\beta \in (0,1)$ such that $\beta + (1-\beta)\lambda = \alpha$ and $R \succ \beta R + (1-\beta)(\underline{c}_1, \overline{c}_2) \sim \alpha R + (1-\alpha)S \succ \lambda R + (1-\lambda)S \succ S$. If $\alpha < \lambda$, then we can find $\beta \in (0,1)$ such that $\beta\lambda = \alpha$ and $R \succ \lambda R + (1-\lambda)S \succ \beta(\underline{c}_1, \overline{c}_2) + (1-\beta)S \sim \alpha R + (1-\alpha)S$. Thus, the claim holds. As an implication, $\alpha R + (1-\alpha)S \succ \beta R + (1-\beta)S$ if $\alpha > \beta$.

For any $R \succ S$, $Q \in \mathcal{P}$ and $\alpha \in (0,1)$, we consider three cases. (i) If $R \succsim Q \succsim S$, then $\alpha R + (1-\alpha)Q \succsim Q \succsim \alpha S + (1-\alpha)Q$ with at least one strict ranking. This implies $\alpha R + (1-\alpha)Q \succ \alpha S + (1-\alpha)Q$. (ii) If $Q \succ R \succ S$, then there exists a some $\lambda \in (0,1)$ with $\lambda Q + (1-\lambda)S \sim R$, and hence $\alpha R + (1-\alpha)Q \sim \alpha(1-\lambda)S + (\alpha\lambda + 1 - \alpha)Q \succ \alpha S + (1-\alpha)Q$. (iii) If $R \succ S \succ Q$, then there exists a some $\lambda \in (0,1)$ with $\lambda R + (1-\lambda)Q \sim S$, and hence $\alpha S + (1-\alpha)Q \sim \alpha\lambda R + (1-\alpha\lambda)Q \prec \alpha R + (1-\alpha)Q$. As a result, $\succsim$ satisfies the Independence axiom and is an EU preference. □

Step 2: Implications if Axiom 7 fails.

For the rest of the proof, we maintain the assumption that $\succsim$ does not satisfy Axiom 7. That is, there exist $P, \tilde{P} \in \mathcal{P}$ and $i \in \{1,2\}$ such that P and $\tilde{P}$ are conditionally comparable in dimension i and $P \succ \tilde{P}$. We further assume that $i = 1$. The other case is symmetric and will be studied in Step 6.

First, we argue that it is without loss of generality to assume that there exists a unique $x_2^* \in \text{supp}(P_2)$ such that $P_{1|y_2} = \tilde{P}_{1|y_2}$ for all $y_2 \in \text{supp}(P_2)\backslash\{x_2^*\}$. To see this, denote $\text{supp}(P_2) = \{x_2^1, ..., x_2^n\}$. Define a sequence of lotteries such that $Q^0 = P$, $Q^n = \tilde{P}$ and for each $i = 1, ..., n-1$, we have $Q^i_{1|x_2^j} = \tilde{P}_{1|x_2^j}$ for all $j \leq i$ and $Q^i_{1|x_2^j} = P_{1|x_2^j}$ for all $j \geq i$. Easy to see that any two lotteries in the sequence are conditionally comparable in dimension 2. Since $Q^0 = P \succ \tilde{P} = Q^n$, there exists some $i \in \{0, ..., n-1\}$ with $Q^i \not\sim Q^{i+1}$ and $Q^i_{1|x_2^j} = Q^{i+1}_{1|x_2^j}$ for all $j \neq i+1$. Hence, we can replace the original P and $\tilde{P}$ with Q^i and Q^{i+1}. We say such P and $\tilde{P}$ are *strongly* conditionally comparable in dimension 1.

The next result is a counterpart of Lemma 4 when Axiom 7 is not satisfied.

Lemma 7: *For any $\alpha \in (0,1)$ and $Q, R, S \in \mathcal{P}$ with $Q_1 \perp \{R_1, S_1\}$, if $R \sim S$, then $\alpha Q + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.*

Proof of Lemma 7. By assumption, $P \succ \tilde{P}$, and P and $\tilde{P}$ are strongly conditionally comparable in dimension 1. That is, $P_{1|y_2} = \tilde{P}_{1|y_2}$ for all $y_2 \in \text{supp}(P_2)\backslash\{x_2^*\}$. Then P and $\alpha P + (1-\alpha)\tilde{P}$, $\alpha P + (1-\alpha)Q$ and $\alpha\tilde{P} + (1-\alpha)Q$ are also strongly conditionally comparable in dimension 1 respectively for all $\alpha \in (0,1)$ and $Q \in \mathcal{P}$. We claim that it is without loss to assume $\text{supp}(P_1) \cup \text{supp}(\tilde{P}_1) \in X_1^o = (\underline{c}_1, \bar{c}_1)$. To see this, first note that by Axiom 3.2, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon = (\varepsilon_1, 0)$ and $\varepsilon' = (\varepsilon_1', 0)$ with $0 < \varepsilon_1, \varepsilon_1' < \bar{\varepsilon}$, we have $P_\varepsilon \succ \tilde{P}_{\varepsilon'}$. Since P and $\tilde{P}$ are strongly conditionally comparable in dimension 1, we can find $x_2 \in X_2$ with $P_{1|x_2^*} \sim_{1|x_2} \tilde{P}_{1|x_2^*}$. By Lemma 1, the conditional preference $\succsim_{1|x_2}$ admits an EU representation with an strictly increasing and continuous Bernoulli index. There exist $\hat{\varepsilon} = (\hat{\varepsilon}_1, 0)$ and $\hat{\varepsilon}' = (\hat{\varepsilon}_1', 0)$ such that $0 < \hat{\varepsilon}_1, \hat{\varepsilon}_1' < \bar{\varepsilon}$ and $P_{\hat{\varepsilon},1|x_2^*} \sim_{1|x_2} \tilde{P}_{\hat{\varepsilon}',1|x_2^*}$. Notice that for any $y_2 \in \text{supp}(P_2)\backslash\{x^*\}$, we have $P_{1|y_2} = \tilde{P}_{1|y_2}$ and hence $P_{\hat{\varepsilon},1|y_2} = \tilde{P}_{\hat{\varepsilon}',1|y_2}$. Hence, $P_{\hat{\varepsilon}} \succ \tilde{P}_{\hat{\varepsilon}'}$, $P_{\hat{\varepsilon}}$ and $\tilde{P}_{\hat{\varepsilon}'}$ are strongly conditionally comparable in dimension 1 and $P_{\hat{\varepsilon},1}(\underline{c}_1) = \tilde{P}_{\hat{\varepsilon}',1}(\underline{c}_1) = 0$. Repeat the argument by considering negative changes and we are done. The result of the proof follows from that of Lemma 4. $\square$

The next lemma generalizes Lemma 7 and shows that the Independence axiom holds subject to certain conditions on supports in dimension 1. The proof is presented in the online appendix.

Lemma 8: *For any $\alpha \in (0,1)$ and $P, Q, R, S \in \mathcal{P}$, the following properties hold:*

(i) If $P \sim Q$ and $P_1 \perp Q_1$, then $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_1 \perp Q_1$, then $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$;

(v) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $\mathrm{supp}(Q_1) \cup \mathrm{supp}(S_1) \subseteq \{\bar{c}, \underline{c}\}$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

The final auxiliary result strengthens Lemma 5 and is key to our representation.

Lemma 9: *For each $P \in \mathcal{P}$, there exists a unique $\alpha \in [0,1]$ such that $P \sim \alpha\delta_{\bar{c}} + (1-\alpha)\delta_{\underline{c}}$. Moreover, if $P \sim \alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}}$ and $Q \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$, then $P \succsim Q$ if and only if $\alpha_1 \geq \alpha_2$.*

Proof of Lemma 9. Since $\underline{c} \precsim P \precsim \bar{c}$ for all $P \in \mathcal{P}$, by Lemma 5, it suffices to show that for any $\alpha_1, \alpha_2 \in (0,1)$, if $\alpha_1 > \alpha_2$, then $\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} \succ \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. By Lemma 5, Lemma 8 and Axiom 3.2, there exists $(x_1, x_2) \in X$ such that $x_1 \neq \bar{c}_1$ and $(x_1, x_2) \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}$. Then part (v) of Lemma 8 implies

$$\begin{aligned}
\alpha_1\delta_{\bar{c}} + (1-\alpha_1)\delta_{\underline{c}} &= (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}(\alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}) \\
&\sim (1 - \frac{1-\alpha_1}{1-\alpha_2})\delta_{\bar{c}} + \frac{1-\alpha_1}{1-\alpha_2}\delta_{(x_1,x_2)} \\
&\succ \delta_{(x_1,x_2)} \sim \alpha_2\delta_{\bar{c}} + (1-\alpha_2)\delta_{\underline{c}}.
\end{aligned}$$

The strict ranking follows from part (ii) in Lemma 8. □

Step 3: Representation of $\succsim$.

Lemma 10: *The relation $\succsim$ is represented by $U : \mathcal{P} \to \mathbb{R}$ where for each $P \in \mathcal{P}$,*

$$U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1), \tag{15}$$

where w and v_{x_1} for all $x_1 \in X_1$ are continuous and strictly increasing. Moreover, w and v_{x_1} are unique up to a positive affine transformation for all $x_1 \in X_1$.

Proof of Lemma 10. By Lemma 9, for each $P \in \mathcal{P}$, there exists a unique $\alpha(P) \in [0,1]$ such that $P \sim \alpha(P)\delta_{\bar{c}} + (1-\alpha(P))\delta_{\underline{c}}$. Define $U : \mathcal{P} \to [0,1]$ such that $U(P) = \alpha(P)$. Then $U[\bar{c}] = 1, U[\underline{c}] = 0$ and U represents $\succsim$.

Fix any $P, Q \in \mathcal{P}$ and $\alpha \in (0,1)$ such that $P_1 \perp Q_1$. By (v) of Lemma 8, we have

$$\begin{aligned}\alpha P + (1-\alpha)Q &\sim \alpha(U(P)\delta_{\bar{c}} + (1-U(P))\delta_{\underline{c}}) + (1-\alpha)(U(Q)\delta_{\bar{c}} + (1-U(Q))\delta_{\underline{c}}) \\ &= (\alpha U(P) + (1-\alpha)U(Q))\delta_{\bar{c}} + (1 - \alpha U(P) - (1-\alpha)U(Q))\delta_{\underline{c}}\end{aligned}$$

By definition of U, we know $\alpha P + (1-\alpha)Q \sim U(\alpha P + (1-\alpha)Q)\delta_{\bar{c}} + (1 - U(\alpha P + (1-\alpha)Q))\delta_{\underline{c}}$. Then Lemma 9 implies

$$U(\alpha P + (1-\alpha)Q) = \alpha U(P) + (1-\alpha)U(Q), \tag{16}$$

for any $\alpha \in (0,1)$ and $P, Q \in \mathcal{P}$ such that $P_1 \perp Q_1$. By applying (16) multiple times for each $P \in \mathcal{P}$, we get

$$U(P) = U(\sum_{x_1 \in X_1} P_1(x_1)(\delta_{x_1}, P_{2|x_1})) = \sum_{x_1 \in X_1} U(\delta_{x_1}, P_{2|x_1}))P_1(x_1). \tag{17}$$

By Lemma 1, for each $x_1 \in X_1$, the conditional preference $\succsim_{2|x_1}$ admits an EU representation with some continuous and strictly increasing Bernoulli index v_{x_1}. Then there exists a function ϕ_{x_1} such that $U(\delta_{x_1}, P_{2|x_1}) = \phi_{x_1}(CE_{v_{x_1}}(p))$ for all $p \in \mathcal{P}_2$. Define $w : X \to \mathbb{R}$ as $w(x_1, x_2) = \phi_{x_1}(x_2) = U(\delta_{x_1}, \delta_{x_2})$ for all $(x_1, x_2) \in X$. Then the utility function (17) can be rewritten (15). By Axiom 2, we know that w is strictly increasing. Note that w is bounded since $w(\underline{c}) \leq w(x) \leq w(\bar{c})$ for all $x \in X$. Moreover, w is unique up to a positive affine transformation.

The final step is to verify that w is continuous. Suppose by contradiction that w is not continuous, then we can find $x \in X$ and a sequence $x^n \to x$ such that $\lim_{n\to\infty} w(x^n) \neq w(x)$. As w is bounded, we can find a subsequence of $(x^n)_{n\geq 1}$ (still denoted by itself) such that $\lim_{n\to\infty} w(x^n) = a \neq b = w(x)$. Without loss of generality, assume that $a < b$. By the representation (15), we can find some

$P \in \mathcal{P}_1 \times \mathcal{P}_2$ with $V(P) \in (a, b)$. As $\lim_{n\to\infty} w(x^n) = a < V(P)$, for n large enough, we have $w(x^n) < V(P) < b$, that is, $x^n \prec P \prec x$. When n goes to infinity, by Axiom 3.2, we must have $x \precsim P$, which leads to a contradiction. $\square$

Step 4: Characterize when $\succsim$ admits an SEU representation.

In general, (15) is not a procedural expected utility preference, as the conditional preference v_{x_1} in dimension 2 can arbitrarily depend on $x_1 \in X_1$. Indeed, if v_{x_1} is independent of x_1, then (15) reduces to an SEU preference with $I = \{2\}$.

Axiom 8—Taste Separability in Dimension 2: For any $x_1, y_1 \in X_1$ and $p, q \in \mathcal{P}_2$, we have $p \succsim_{2|x_1} q$ if and only if $p \succsim_{2|y_1} q$.

Lemma 11: *Suppose that $\succsim$ is represented by (15). If $\succsim$ satisfies Axiom 8, then it is an SEU preference with $I = \{2\}$.*

Proof of Lemma 11. Axiom 8 and Lemma 1 imply that $CE_{v_{x_1}}$ is independent of x_1. Hence (15) reduces to an SEU representation with $I = \{2\}$. $\square$

Step 5. Characterize when $\succsim$ an EU preference.

We also note that (15) is more general than the EU representation: If v_{x_1} is a positive affine transformation of $w(x_1, \cdot)$ for all $x_1 \in X_1$, then (15) reduces to the EU representation. We will show that this is the only case if $\succsim$ violates Axiom 8.

Lemma 12: *Suppose that $\succsim$ is represented by (15). For any $x_1, y_1 \in X_1$ and $\alpha \in (0, 1)$, if $\succsim_{2|x_1}$ and $\succsim_{2|y_1}$ are not identical, then there exist $p, q \in \mathcal{P}_2$ such that $p \succ_{2|x_1} q$, $q \succ_{2|y_1} p$, and $p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. Moreover, we can choose p or q to be the same across all $\alpha \in (0, 1)$.*

Proof of Lemma 12. By assumption, there exist $p, q \in \mathcal{P}_2$ such that $p \succsim_{2|x_1} q$ and $q \succ_{2|y_1} p$, or $p \succsim_{2|y_1} q$ and $q \succ_{2|x_1} p$. We claim that p, q can be chosen such that both relations are strict. Suppose $p \sim_{2|x_1} q$ and $q \succ_{2|y_1} p$. The case where $p \sim_{2|y_1} q$ and $q \succ_{2|x_1} p$ is symmetric. By Lemma 1, we know $q \neq \underline{c}_2$ and $p \neq \underline{c}_2$. Then we can find $\beta \in (0, 1)$ sufficiently close to 1 such that $p \succ_{2|x_1} \beta q + (1 - \beta)\delta_{\underline{c}_2}$ and $\beta q + (1 - \beta)\delta_{\underline{c}_2} \succ_{2|y_1} p$. Hence, there exist $p, q \in \mathcal{P}_2$ with $p \succ_{2|x} q$ and $q \succ_{2|y} p$. Fix any $\alpha \in (0, 1)$. If $p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$, then we are done. If $p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$, then there exists a unique $\beta \in (0, 1)$ such that $\beta p + (1 - \beta)\delta_{\underline{c}_2} \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. Clearly,

$q \succ_{2|y_1} \beta p + (1-\beta)\delta_{\underline{c}_2}$. We claim that $\beta p + (1-\beta)\delta_{\underline{c}_2} \succ_{2|x_1} q$, since otherwise, by (15), we must have $q \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \beta p + (1-\beta)\delta_{\underline{c}_2}$, leading to a contradiction. Hence, the results hold for $\beta p + (1-\beta)\delta_{\underline{c}_2}$ and q. If $q \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} p$, then there exists a unique $\beta' \in (0,1)$ such that the results hold for $\beta' p + (1-\beta')\delta_{\bar{c}_2}$ and q. Note that we have chosen q to be the same across all $\alpha \in (0,1)$. A symmetric proof works when we choose p to be the same across all $\alpha \in (0,1)$. $\square$

For each $x_1, y_1 \in X_1$ and $\alpha \in (0,1)$, define

$$\Gamma_{x_1,y_1}(\alpha) = \Big\{(p,q) \in (\mathcal{P}_2)^2 \mid p \succ_{2|x_1} q, q \succ_{2|y_1} p \text{ and } p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q\Big\}.$$

Endow $(\mathcal{P}_2)^2$ with the product topology. We claim that Γ_{x_1,y_1} satisfies the following properties: (i) If $\Gamma_{x_1,y_1}(\alpha_0) \neq \emptyset$ for some $\alpha_0 \in (0,1)$, then $\Gamma_{x_1,y_1}(\alpha) \neq \emptyset$ for all $\alpha \in (0,1)$; (ii) If $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$ for some $\alpha \in (0,1)$, then for any $\beta \in (0,1)$ and $r \in \mathcal{P}_2$, we have $(\beta p + (1-\beta)r, \beta q + (1-\beta)r) \in \Gamma_{x_1,y_1}(\alpha)$; (iii) The set $\bigcup_{\alpha\in(0,1)} \Gamma_{x_1,y_1}(\alpha)$ is open. The first two properties are direct corollaries of Lemma 12 and Lemma 1. For (iii), suppose that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$ for some $\alpha \in (0,1)$. Then $x_1 \neq y_1$. By Lemma 1, there exists an open neighborhood of (p,q), denoted by $\mathcal{M} \subset (\mathcal{P}_2)^2$, such that for each $(p',q') \in \mathcal{M}$, we have $p' \succ_{2|x_1} q'$ and $q' \succ_{2|y_1} p'$. By the utility representation (15), there exists a unique $\alpha' \in (0,1)$ such that $p' \sim_{2|\alpha'\delta_{x_1}+(1-\alpha')\delta_{y_1}} q'$. Hence, $(p',q') \in \bigcup_{\alpha\in(0,1)} \Gamma_{x_1,y_1}(\alpha)$ for each $(p',q') \in \mathcal{M}$.

Lemma 13: *Suppose that $\succsim$ is represented by (15). If $\succsim$ violates Axiom 8, then it is an EU preference.*

The proof of Lemma 13 can be found in the online appendix. Here we briefly discuss the proof sketch. First, Lemma 1 and (15) imply that for any $p, q \in \mathcal{P}_1$, $r, r' \in \mathcal{P}_2$, and $\alpha \in (0,1)$, if $r \succsim_{2|p} r'$ and $r \succsim_{2|q} r'$, then $r \succsim_{2|\alpha p+(1-\alpha)q} r'$. Second, Harsanyi (1955)'s utilitarianism theorem suggests that the Bernoulli index of $\succsim_{2|\alpha p+(1-\alpha)q}$ must be a convex combination of those of $\succsim_{2|p}$ and $\succsim_{2|q}$. Third, we show that $\succsim_{2|p}$ must be independent of $p \in \mathcal{P}_1$.

Step 6: Repeat Steps 2-5 if there exist P and $\tilde{P}$ that are conditionally comparable in dimension 2 and $P \succ \tilde{P}$.

To summarize, if $\succsim$ satisfies Axioms 1-5, then it admits either an NEU, or an

SEU, or an EU representation. This completes the proof. $\square$

Proof of Proposition 1. The arguments for uniqueness of Bernoulli indices are already contained in the proof of Theorem 1. $\square$

Proof of Proposition 3. First, since Stochastic Impatience and RSTL only involve lotteries with degenerate conditional lotteries in dimension 1. Hence, EU preferences and SEU preferences with $I = \{1\}$ share the same predictions: By proposition 4 of DeJarnette et al. (2020), $\succsim$ satisfies Stochastic Impatience if and only if it is RSTL. hence, it suffices to check the other two procedures. If $\succsim$ admits an NEU representation, then it trivially satisfies Stochastic Impatience as the decision maker is always indifferent between $\frac{1}{2}\delta_{(x_1,t_1)} + \frac{1}{2}\delta_{(x_2,t_2)}$ and $\frac{1}{2}\delta_{(x_2,t_1)} + \frac{1}{2}\delta_{(x_1,t_2)}$. Also, $\succsim$ is RSTL if and only if v_2 is convex.

If $\succsim$ admits an SEU representation with $I = \{2\}$, then again $\succsim$ is RSTL if and only if v is convex. By linearity of the SEU representation for marginal lotteries in dimension 1, $\succsim$ satisfies Stochastic Impatience if and only if the corresponding EU representation with the same parameters satisfies (Strict) Stochastic Impatience, which, by Propositions 2 and 4 of DeJarnette et al. (2020), is equivalent to ϕ being a (strictly) convex transformation of log. $\square$

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Online Appendix

The online appendix to *"Procedural Expected Utility"* is organized by follows. Section A presents axioms for each evaluation procedure in Definition 2. Section B studies the generalization of the SEU preference to a setting with an infinite horizon. Section C considers a restricted domain where the outcomes in different dimensions can be either interpreted as rewards in different periods, or the reward size and payment date respectively, as mentioned in footnote 23. Section D contains omitted proofs.

Online Appendix A: Axioms for Each Evaluation Procedure

Our main result Theorem 1 characterizes the common behavioral properties of different evaluation procedures. In this section, we discuss what additional axioms are needed to identify each of them. First, as is well-known, the EU preference features the standard Independence axiom.

Axiom 9—Independence: For any $P, Q, R \in \mathcal{P}$, if $P \succ Q$, then for any $\alpha \in (0,1)$, we have $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R$.

Second, only marginal lotteries enter the NEU representation.

Axiom 10—Correlation Neutrality: For any $P \in \mathcal{P}$, we have $P \sim (P_1, P_2)$.

Finally, in the SEU representation with $I = \{i\}$, the attitude toward risk in dimension i is independent of the outcome in dimension $-i$, and the decision maker satisfies a stronger version of Axiom 5.

Axiom 11—Strong Cross-Dimension Independence in Dimension $-i$: For any $\alpha \in (0,1)$ and $P, Q, R, S \in \mathcal{P}$ such that $P_{-i} \perp R_{-i}$, $Q_{-i} \perp S_{-i}$, if $P \succ Q$ and $R \sim S$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$.

Axiom 12—Taste Separability in Dimension i: For any $x_{-i}, y_{-i} \in X_{-i}$ and $p, q \in \mathcal{P}_i$, we have $p \succsim_{i|x_{-i}} q$ if and only if $p \succsim_{i|y_{-i}} q$.

The following results characterize each evaluation procedure.

Proposition 4: *Let $\succsim$ be a binary relation on $\mathcal{P}$ that satisfies Axioms 1-5.*

(i) The relation $\succsim$ satisfies Axiom 9 if and only it is an EU preference.
(ii) The relation $\succsim$ satisfies Axiom 10 if and only it is an NEU preference.
(iii) For each $i = 1, 2$, the relation $\succsim$ satisfies Axioms 11 and 12 if and only it is an SEU preference with $I = \{i\}$.

Proof of Proposition 4. Part (i) is the standard von Neumann–Morgenstern EU theorem. For part (ii), suppose $\succsim$ has an SEU representation $(2, w, v_2)$ with $w(\underline{c}) = 0$. By Axiom 10, for any $(x_1, x_2) \in X$ with $x_1 \neq \underline{c}_1$, we have $\frac{1}{2}\delta_{(x_1,x_2)} + \frac{1}{2}\delta_{(\underline{c}_1,\underline{c}_2)} \sim \frac{1}{2}\delta_{(x_1,\underline{c}_2)} + \frac{1}{2}\delta_{(\underline{c}_1,x_2)}$, which leads to $w(x_1, x_2) = w(x_1, \underline{c}_2) + w(\underline{c}_1, x_2)$. Define $u_1 : X_1 \to \mathbb{R}$ and $u_2 : X_2 \to \mathbb{R}$ where $u_1(y_1) = w(y_1, \underline{c}_2)$ for all $y_1 > \underline{c}_1$ and $u_2(y_2) = w(\underline{c}_1, y_2)$ for all $y \in X_2$. By continuity, we know $u_1(\underline{c}_1) = 0$ and $w(x_1, x_2) = u_1(x_1) + u_2(x_2)$. Moreover, for each P, we must have $u_2(CE_{v_2}(P_2)) = \sum_{x_1} u_2(CE_{v_2}(P_{2|x_1}))P_1(x_1)$, which implies v_2 must be a positive affine transformation of u_2. Hence $V^{SEU}(P) = \mathbb{E}_{P_1}(u_1) + \mathbb{E}_{P_2}(u_2)$. The same argument applies if $\succsim$ is an EU preference or an SEU preference with $I = \{1\}$.

To prove part (iii), we assume $i = 2$ and the case with $i = 1$ is symmetric. Note that Axiom 11 is exactly part (iii) of Lemma 8 in the proof of Theorem 1. With the help of other axioms, we can prove the other parts of Lemma 8 as well. Indeed, we can show that Lemma 10 holds and $\succsim$ admits a representation in (15). That is, the utility of $P \in \mathcal{P}$ is $U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1)$. Axiom 12 implies that v_{x_1} is independent of x_1 and hence $\succsim$ admits an SEU representation. $\square$

Online Appendix B: SEU with an Infinite Horizon

In this section, we briefly discuss how to extend the SEU model in (10) to one with multiple periods. Assume that the consumption space in each period $t = 1, ..., T$ is a compact interval $C \subset \mathbb{R}_+$, where T can be $+\infty$. The set of deterministic consumption streams is C^T with a generic element $\mathbf{c} = (c_t)_{t=1}^T$. For each consumption stream $\mathbf{c} \in C^T$, we denote the subsequence of consumption in the first t periods as $\mathbf{c}^t = (c_\tau)_{\tau=1}^t$. The preference is defined on the lottery space $\mathcal{P} = \mathcal{L}(C^T)$. Here we allow for lotteries with infinite supports to accommodate applications in finance. For each lottery P, denote by $P_{[t]}$ the marginal lottery in the first t periods, $1 \leq t < T$. For each subsequence of consumption $\mathbf{c}^t$ in the support of $P_{[t]}$, we define $\phi(P|\mathbf{c}^t)$ as the conditional lottery starting from period $t+1$, given that consumption in the first t periods is $\mathbf{c}^t$. When $T < +\infty$, we have $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^{T-t})$ and when $T = +\infty$, we have $\phi(P|\mathbf{c}^t) \in \mathcal{L}(C^\infty)$. For each finite T, the set $\mathcal{L}(C^{T-t})$ is homeomorphic to a subset of $\mathcal{L}(C^\infty)$ where the consumption levels are always 0 from period $t+1$ on. So we will focus on the case with an infinite horizon.

The following notions are adapted from recursive preferences on temporal lotteries (Chew and Epstein, 1991, Bommier, Kochov, and Le Grand, 2017) to our framework. For each $V : \mathcal{P} := \mathcal{L}(C^\infty) \to \mathbb{R}$ and $p \in \mathcal{P}$, denote

$$m_V(P)(B) \equiv P_1\Big\{c \in C : V(c, \phi(P|c) \in B)\Big\}, \forall B \in \mathcal{B}(V(\mathcal{P}))$$

where $V(\mathcal{P}) \subset \mathbb{R}$ is the image of V on $\mathcal{P}$ and $\mathcal{B}(V(\mathcal{P}))$ is the set of all Borel subsets of $V(\mathcal{P})$. Then $m_V(P)$ is a probability measure over utilities conditional on the current consumption. Now we define the *recursive preference over lotteries* as $V : \mathcal{P} \to \mathbb{R}$ with

$$V(P) = I(m_V(P)),$$
$$V(c, q) = W(c, V(q)),$$

where $m_V(P)$ is defined as above, $I : \mathcal{L}(\mathbb{R}) \to \mathbb{R}$ is a *certainty equivalent*, that is, I is continuous, increasing with respect to first-order stochastic dominance and $I(x) = x$ for each $x \in \mathbb{R}$, and $W : C \times \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing in the second argument. Unlike the recursive preferences Chew and Epstein (1991) and Bommier, Kochov, and Le Grand (2017), V is defined on a different domain and can be discontinuous.

In order to get the CRRA-CES functional form, we can set $I = \phi^{-1} \circ \mathbb{E} \circ \phi$ with $\phi(x) = x^{\alpha/\rho}$ and $W(c, v) = (1-\delta)c^\rho + \delta v$, where $\rho < 1, 0 \neq \alpha < 1$ and $0 < \delta < 1$. The recursive preference is equivalent to the following recursion of value functions (up to a monotonic transformation):

$$V^{SEU}(P) = \frac{1}{\alpha}\mathbb{E}_{P_1}\Big(U_1^\alpha\Big) \tag{18}$$

$$U_t^\rho = (1-\delta)c_t^\rho + \delta\Big[\mathbb{E}_{\phi(P|\mathbf{c}^t)_1}\Big(U_{t+1}^\alpha\Big)\Big]^{\frac{\rho}{\alpha}} \tag{19}$$

where U_t is the value in period t and the expectation is computed with respect to $\phi(P|\mathbf{c}^t)_1$, which is the probability distribution of consumption levels in period $t+1$ conditional on the consumption stream in the first t periods $\mathbf{c}^t$.

Then we explore the implications of SEU in a standard asset pricing problem. The consumer is endowed with initial wealth $W_1 > 0$ in period 1 and chooses the consumption level and saving level in each period. Let S denote the finite state space in each period $t \geq 2$ and $\Omega = S^\infty$ denote the space of state sequences. The consumer has a prior belief over Ω. For each $s^\infty \in \Omega$, we denote by s^t the history of states from period 2 to period t for each $t \geq 2$. Let S^t be the set of all histories till period t.

The consumer's preference is represented by V^{SEU} in (18). She chooses a *consumption plan* $(c_t)_{t\geq 1}$ to maximizes her utility, where $c_1 \in X$ and $c_t : S^t \to X$ for each $t \geq 2$. Given the history of states s^t, the gross return on wealth from period $t-1$ to period t is $R_{w,t}(s^t) > 0$. Then the wealth dynamics are represented by the following equation:

$$W_{t+1}(s^{t+1}) = R_{w,t+1}(s^{t+1})(W_t(s^t) - c_t(s^t)).$$

We assume that the wealth level always lies in C. We say that a consumption plan $(c_t)_{t\geq 1}$ is feasible given initial wealth W_1 if if $c_t(s^t) \leq W_t(s^t)$ for all s^t, t. Each feasible consumption plan $(c_t)_{t\geq 1}$ induces a lottery $P \in \mathcal{P}$ where the consumption in the first period is deterministic. Using V^{SEU} in (18), we can define a utility function over feasible consumption plans as $\hat{V}^{SEU}((c_t)_{t\geq 1})$. Then the optimization problem of the consumer is

$$J^{SEU}(W_1) = \sup \left\{ \hat{V}^{SEU}((c_t)_{t\geq 1}) : (c_t)_{t\geq 1} \text{ is feasible given } W_1 \right\}.$$

To facilitate the comparison of SEU and EZ, we can similarly consider a consumer with a CRRA-CES EZ recursive utility function $\hat{V}^{EZ}$ over feasible consumption plans and the optimal value $J^{EZ}(W_1)$. We assume that RRA>1/EIS, then the EZ consumer has a preference over early resolution of uncertainty, while the SEU consumer exhibits indifference to temporal resolution of uncertainty. The following result shows that the two utility functions lead to the same optimal value.

Proposition 5: *Assume RRA>1/EIS, i.e., $\rho > \alpha$. Then for each $W_1 > 0$, there exist consumption plans $(c_t)_{t\geq 1}$ and $(c_t^n)_{t\geq 1, n\geq 1}$ feasible given W_1 such that $c_1^n \to c_1, c_t^n(s^t) \to c_t(s^t)$ as $n \to \infty$ for each $t \geq 2$, and*

(i). $J^{SEU}(W_1) = \lim_{n\to\infty} \hat{V}^{SEU}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1}) = J^{EZ}(W_1)$;

(ii). For each $n \geq 1, t \geq 2$, c_t^n is injective on Ω_t, that is, $c_t^n(s^t) \neq c_t^n(\hat{s}^t)$ if $s^t \neq \hat{s}^t$.

Proof of Proposition 5. First, by Theorem 5.1 in Epstein and Zin (1989), we can find an optimal consumption plan $(c_t^*)_{t\geq 1}$ for the EZ consumer with $J^{EZ}(W_1) = \hat{V}^{EZ}((c_t^*)_{t\geq 1})$. On the one hand, since $\rho > \alpha$, the EZ consumer has a preference for early resolution of risk. We know that $\hat{V}^{SEU}((c_t)_{t\geq 1}) \leq \hat{V}^{EZ}((c_t)_{t\geq 1})$ for each feasible consumption plan $(c_t)_{t\geq 1}$ and $J^{SEU}(W_1) \leq J^{EZ}(W_1)$. Moreover, for a consumption plan $(c_t)_{t\geq 1}$ where c_t is injective on Ω_t for each $t \geq 2$, the consumption history contains the same information as state history. In this case, $\hat{V}^{SEU}((c_t)_{t\geq 1}) = \hat{V}^{EZ}((c_t)_{t\geq 1})$.

If c_t^* is injective for each $t \geq 2$, we can set $c_t^n \equiv c_t^*$ for each n and the results hold. Otherwise, we want to construct a sequence of consumption plans $(c_t^n)_{t\geq 1, n\geq 1}$ with injective consumption functions such that $c_1^n \to c_1^*, c_t^n(s^t) \to c_t^*(s^t)$ as n goes to infinity for each $t \geq 2$. By continuity of V^{EZ} and hence $\hat{V}^{EZ}$, we know

$$J^{SEU}(W_1) \geq \lim_{n\to\infty} \hat{V}^{SEU}((c_t^n)_{t\geq 1}) = \hat{V}^{EZ}((c_t^*)_{t\geq 1}) = J^{EZ}(W_1).$$

Hence $J^{SEU}(W_1) = J^{EZ}(W_1)$. It remains to construct the sequence of consumption plans. This is easy since $\bigcup_{t\geq 2} c_t^*(S^t)$ is countable, and the space of consumption is a continuum. $\square$

A directly corollary of Proposition 5 is that our SEU model has the same implications in asset pricing as the EZ model when RRA>1/EIS, which is the common parametric assumption in most applications of EZ preferences in finance and macroeconomics. Following the asset pricing literature, let $\gamma := 1 - \alpha$ be the RRA, $\psi := \frac{1}{1-\rho}$ be the EIS and $\theta := \frac{\alpha}{\rho}$.

Corollary 1: *Assume RRA>1/EIS, i.e., $\rho > \alpha$. Denote $\mathbb{E}_t$ as the expectation with respect to s^t. Then for each $W_1 > 0$,* *(i). Euler equation:*

$$\lim_{n\to\infty} \mathbb{E}_t\Big[\delta^\theta(\frac{c^n_{t+1}}{c^n_t})^{-\frac{\theta}{\psi}} R^\theta_{w,t+1}\Big] = \mathbb{E}_t\Big[\delta^\theta(\frac{c_{t+1}}{c_t})^{-\frac{\theta}{\psi}} R^\theta_{w,t+1}\Big] = 1;$$

(ii). Asset pricing formula: for each asset i with gross return $R_{i,t}$,

$$\lim_{n\to\infty} \mathbb{E}_t\Big[\delta^\theta(\frac{c^n_{t+1}}{c^n_t})^{-\frac{\theta}{\psi}} R^{-(1-\theta)}_{w,t+1} R_{i,t+1}\Big] = \mathbb{E}_t\Big[\delta^\theta(\frac{c_{t+1}}{c_t})^{-\frac{\theta}{\psi}} R^{-(1-\theta)}_{w,t+1} R_{i,t+1}\Big] = 1;$$

Proof of Corollary 1. The derivation of the Euler equation and the asset pricing formula for $(c_t)_{t\geq 1}$ can be found in Epstein and Zin (1991). The rest follows from the continuity of the two equations and $c^n_1 \to c_1, c^n_t(s^t) \to c_t(s^t)$ as n goes to infinity for each $t \geq 2$. □

We end this section with a brief discussion about the case with RRA<1/EIS, i.e., when the EZ consumer prefers late solution of risk. In this case, $J^{SEU}(W_1)$ might be strictly higher than $J^{EZ}(W_1)$ due to the discontinuity of V^{SEU} on $\mathcal{P}$, which invalidates our proof of Proposition 5. Interestingly, the SEU consumer might have excessive demand for consumption smoothing across different states of the world in the same period. However, the derivation of the optimal value and Euler equation is much less tractable and we leave it for future study.

Online Appendix C: Restricted Domain

In this section, we consider a restricted domain where the outcomes in different dimensions can be either interpreted as rewards in different periods, or the reward size and payment date respectively. Concretely, let $Z = C = [0, b] \subset \mathbb{R}_+$ and $T = \{0, 1\}$. Each dated reward $(z, t) \in Z \times T$ corresponds to a reward stream $(z, 0) \in C^2$ if $t = 0$ and $(0, z) \in C^2$ if $t = 1$. Applying this operation to each outcome profile in the support of a lottery, we can define a mapping $\chi : \Delta(Z\times T) \to \Delta(C^2)$. Each lottery over dated rewards P and the corresponding lottery over reward streams χ^P represent the same risky alternative.

First, we study the procedural expected utility preference over $\Delta(Z \times T)$, where dimension 1 represents the reward size and dimension 2 represents the payment date. Assume that w is given in (12) and let $\beta = e^{-r}$. Utilities of lottery $P \in \Delta(Z \times T)$ in the EU preference and the NEU preference are given by

$$V^{EU}(P) = \mathbb{E}_P\Big(\phi(\beta^t u(z))\Big), \tag{20}$$

$$V^{NEU}(P) = \beta^{CE_{v_2}(P_2)} \cdot u(CE_{v_1}(P_1)). \tag{21}$$

Second, we consider the the procedural expected utility preference over $\Delta(C^2)$, where dimension 1 represents the reward in period $t = 0$ and dimension 1 represents the reward in period $t = 1$. Assume that Assumption 1 holds and $u(0) = 0$. Utilities of lottery $P \in \Delta(Z \times T)$ in the EU preference and the NEU preference are given by

$$\hat{V}^{EU}(\chi^P) = \mathbb{E}_{\chi^P}\Big(\phi(u(c_1) + \beta u(c_2))\Big) = \mathbb{E}_P\Big(\phi(\beta^t u(z))\Big), \tag{22}$$

$$\hat{V}^{NEU}(\chi^P) = u(CE_{\hat{v}_1}(\chi_1^P)) + \beta u(CE_{\hat{v}_2}(\chi_2^P)). \tag{23}$$

Notice that the EU representations (20) and (22) are identical no matter whether each outcome profile is interpreted as a reward stream or a dated reward. By contrast, the NEU representations (21) and (23) are distinct from each other. To see this, consider a lottery $P \in \Delta(Z \times T)$ where $P(z, 0) = P(z', 1) = 0.5$ with $z \neq z'$. Assume that $v_1 = \hat{v}_1 = \hat{v}_2 = v$ and $v_2(t) = t$ for all $t \in T$. When each outcome profile represents a dated reward, with an NEU preference (21), the decision maker separately evaluates the marginal risk in reward size $P_1 = p \in \Delta(Z)$ with $p(z) = p(z') = 0.5$, and the marginal risk in payment date $P_2 \in \Delta(T)$ with $P_2(0) = P_2(1) = 0.5$. The corresponding utility is given by

$$V^{NEU}(P) = \beta^{\frac{1}{2}} \cdot u(CE_v(p)).$$

When each outcome profile represents a reward stream, with an NEU preference (23), the decision maker separately evaluates the marginal risk in period-0 reward $\chi_1^P = q_1 \in \Delta(C)$ with $q_1(z) = q_1(0) = 0.5$, and the marginal risk in period-0 reward $\chi_2^P = q_2 \in \Delta(C)$ with $q_2(z') = q_2(0) = 0.5$. The corresponding utility is

$$\hat{V}^{NEU}(P) = u(CE_v(q_1)) + \beta u(CE_v(q_2)).$$

The NEU preference (23) is the DOCE model discussed in Section 4.2. As noted by DeJarnette et al. (2020), this model can also accommodate Stochastic Impatience and violations of RSTL, like our SEU model in Section 4.3.

Online Appendix D: Omitted Proofs

1. Proof of Lemma 8

We first show that (i)-(iv) hold for lotteries in Φ_{2,y_2} for each $y_2 \in X_2$.

Lemma 14: *For any $\alpha \in (0,1)$, $y_2 \in X_2$ and $P, Q, R, S \in \Phi_{2,y_2}$, the following properties hold:*

(i) If $P \sim Q$ and $P_1 \perp Q_1$, then $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_1 \perp Q_1$, then $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \succ \alpha Q+(1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P+(1-\alpha)R \sim \alpha Q+(1-\alpha)S$.

Proof of Lemma 14. Suppose $P, Q \in \Phi_{2,y_2}$ for some $y_2 \in X_2$ and $P_1 \perp Q_1$. By Lemma 5 and the definition of Φ_{2,y_2}, there exist $x_P, x_Q \in X_1^o$ such that $P \sim (x_P, y_2)$ and $Q \sim (x_Q, y_2)$. By Lemma 1, we can find $\varepsilon > 0$ such that for all $z_P \in [x_P - \varepsilon, x_P], z_Q \in [x_Q - \varepsilon, x_Q]$, there exist $z'_P \geq x_P, z'_Q \geq x_Q$ with $P \sim (1/2\delta_{z_P} + 1/2\delta_{z'_P}, y_2)$ and $Q \sim (1/2\delta_{z_Q} + 1/2\delta_{z'_Q}, y_2)$. Moreover, as z_P, z_Q increases, z'_P, z'_Q will be decreasing continuously. Since $\mathrm{supp}(P_1) \cup \mathrm{supp}(Q_1)$ is finite, we can construct $z^*_P \neq z^*_Q$, $z^{*\prime}_P \neq z^{*\prime}_Q$ and $z^*_P, z^*_Q, z^{*\prime}_P, z^{*\prime}_Q \notin \mathrm{supp}(P_1) \cup \mathrm{supp}(Q_1)$. Denote $P' = (1/2\delta_{z^*_P} + 1/2\delta_{z^{*\prime}_P}, y_2)$, $Q' = (1/2\delta_{z^*_Q} + 1/2\delta_{z^{*\prime}_Q}, y_2)$. Then $P \sim P', Q \sim Q'$ and P_1, Q_1, P'_1, Q'_1 are singular with respect to each other. Apply Lemma 7 twice and for any $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)Q \sim \alpha P + (1-\alpha)Q' \sim \alpha P' + (1-\alpha)Q'.$$

By Lemma 1 given y_2 as the marginal lottery in dimension 2, we have

$$P \sim Q \Longrightarrow P' \sim Q' \Longrightarrow \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \sim Q' \sim Q,$$
$$P \succ Q \Longrightarrow P' \succ Q' \Longrightarrow P \sim P' \sim \alpha P + (1-\alpha)Q \sim \alpha P' + (1-\alpha)Q' \succ Q' \sim Q.$$

This finishes the proof of (i) and (ii). The other results can be shown similarly. $\square$

The next result states that if the independence property holds on Φ_{2,y_2} for each y_2, then it also holds on their union.

Lemma 15: *For any $\alpha \in (0,1)$ and $P, Q, R, S \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$, the following properties hold:*

(i) If $P \sim Q$ and $P_1 \perp Q_1$, then $\alpha P + (1-\alpha)Q \sim P \sim Q$;

(ii) If $P \succ Q$ and $P_1 \perp Q_1$, then $P \succ \alpha P + (1-\alpha)Q \succ Q$;

(iii) If $P \succ Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)S$;

(iv) If $P \sim Q$, $R \sim S$, $P_1 \perp R_1$ and $Q_1 \perp S_1$, then $\alpha P + (1-\alpha)R \sim \alpha Q + (1-\alpha)S$.

Proof of Lemma 15. First, for any $P, Q, R, S \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$, choose $P^1, P^2 \in \{P, Q, R, S\}$ such that $P^1 \succsim P, Q, R, S \succsim P^2$. We claim that there exist a positive integer K and $z_k \in X_2, k = 1, ..., K$ such that $z_1 < z_2 < ... < z_K$ and $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. To see this, suppose that $P^1 \in \Phi_{2,x_2}$ and $P^2 \in \Phi_{2,y_2}$ with $x_2 \geq y_2$. If $x_2 = y_2$, then $P, Q, R, S \in \Phi_{2,x_2}$ and we are done. Now suppose that $x_2 > y_2$ and by Lemma 1, we can find $t, t' \in X_1$ with $(t, x_2) \succ P^1 \succ (t', x_2)$ and $(t, y_2) \succ P^2 \succ (t', y_2)$. For each $y \in [y_2, x_2]$, denote $H(y) := \{P \in \mathcal{P} : (t, y) \succ P \succ (t', y)\}$. Note that $H(y) \subseteq \Phi_{2,y}$. By Axiom 3.2, for any $y \in [y_2, x_2]$, there exists $\varepsilon_y > 0$ such that $H(y) \cap H(y') \neq \emptyset$ for all $y' \in [y - \varepsilon_y, y + \varepsilon_y] \cap [y_2, x_2]$. Also, $\{P \in \mathcal{P} : P^1 \succsim P \succsim P^2\} \subseteq \cup_{y_2 \leq y \leq x_2} H(y)$. By the Heine–Borel theorem, since $[y_2, x_2]$ is compact and $\big((z - \varepsilon_z, z + \varepsilon_z)\big)_{y_2 \leq z \leq x_2}$ is an open cover of $[y_2, x_2]$, we can find finitely many $z_1 < z_2 < ... < z_K \in [y_2, x_2]$ with $[y_2, x_2] \subseteq \cup_{k=1}^K [z_k - \varepsilon_{z_k}, z_k + \varepsilon_{z_k}]$. Hence,

$$P, Q, R, S \in \{P \in \mathcal{P} : P^1 \succsim P \succsim P^2\} \subseteq \cup_{y_2 \leq y \leq x_2} H(y) = \cup_{k=1}^K H(z_k) \subseteq \cup_{k=1}^K \Phi_{2,z_k}.$$

Then we use induction to show that the properties (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^K \Phi_{2,z_k}$. By Lemma 14, for each $k = 1, ..., K$, those properties hold for $P, Q, R, S \in \Phi_{2,z_k}$. Suppose by induction that they also hold for $P, Q, R, S \in \cup_{k=1}^t \Phi_{2,z_k}$ for some $1 \leq t < K$. By construction, we can find $T^1, T^2 \in \Phi_{2,z_t} \cap \Phi_{2,z_{t+1}}$ with $T^1 \succ T^2$. By Lemma 5 and Lemma 1, since P_1, Q_1, R_1, S_1 have finite supports, we can also find $p_1, p_2, q_1, q_2 \in \mathcal{P}_1$ such that $(p_1, z_{t+1}) \sim (q_1, z_t) \sim T^1$, $(p_2, z_{t+1}) \sim (q_2, z_t) \sim T^2$ and $\{p_1, p_2, q_1, q_2\} \perp \{P_1, Q_1, R_1, S_1\}$.

Suppose $P \succsim Q$, $P_1 \perp Q_1$ and $P, Q \in \cup_{k=1}^{t+1} \Phi_{2,z_k}$. If $P \sim Q$, then $P, Q \in \Phi_{2,z_k}$ for some $k = 1, ..., t+1$ and hence (i) holds by the inductive hypothesis.

Now we check (ii). If $P \succ Q$, then it suffices to consider $P \in \Phi_{2,z_{t+1}} \backslash (\cup_{k=1}^{t} \Phi_{2,z_k})$ and $Q \in (\cup_{k=1}^{t} \Phi_{2,z_k}) \backslash \Phi_{2,z_{t+1}}$. This implies $P \succ T^1 \succ T^2 \succ Q$. By Lemma 5, there exist $\lambda_1 \neq \lambda_2 \in (0,1)$ such that $T^1 \sim \lambda_1 P + (1-\lambda_1) Q$ and $T^2 \sim \lambda_2 P + (1-\lambda_2) Q$. Then (ii) holds for $\lambda = \lambda_1, \lambda_2$. Notice that at the moment we cannot conclude that $\lambda_1 > \lambda_2$. Suppose that $\lambda_i > \lambda_{-i}$ for some $i = 1, 2$. By Lemma 5 and Lemma 1, we can find $P', Q' \in \hat{\mathcal{P}}$ such that $Q' \sim Q$, $P' \sim P$ and marginal lotteries $P_1, P_1', Q_1, Q_1', p_1, q_1, p_2, q_2$ are singular with respect to each other. This guarantees

$$T^1 \sim \lambda_1 P + (1-\lambda_1) Q \sim \lambda_1 P' + (1-\lambda_1) Q \sim \lambda_1 P + (1-\lambda_1) Q' \sim \lambda_1 P' + (1-\lambda_1) Q',$$

$$T^2 \sim \lambda_2 P + (1-\lambda_2) Q \sim \lambda_2 P' + (1-\lambda_2) Q \sim \lambda_2 P + (1-\lambda_2) Q' \sim \lambda_2 P' + (1-\lambda_2) Q'.$$

By (i), for all $\beta, \beta' \in (0,1)$, we have $\beta P + (1-\beta) P' \sim P$ and $\beta' Q + (1-\beta') Q' \sim Q$. Apply Lemma 7 twice and we derive that for each $\lambda, \beta, \beta' \in (0,1)$,

$$\lambda P + (1-\lambda) Q \sim \lambda(\beta P + (1-\beta) P') + (1-\lambda)(\beta' Q + (1-\beta') Q'). \tag{24}$$

For any $\lambda \in (\lambda_{-i}, \lambda_i)$, let $\beta = 1$, $\beta' = \frac{\lambda_i - \lambda}{\lambda_i(1-\lambda)}$, and (24) becomes

$$\begin{aligned} \lambda P + (1-\lambda) Q \sim & \frac{\lambda}{\lambda_i}(\lambda_i P + (1-\lambda_i) Q') + (1 - \frac{\lambda}{\lambda_i}) Q \\ \sim & \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i}) Q. \end{aligned}$$

The second indifference relation holds due to $\lambda_i P + (1-\lambda_i) Q' \sim T^i \sim (q_i, z_t)$ and Lemma 7. Then by the inductive hypothesis on $\cup_{k=1}^{t} \Phi_{2,z_k}$, we have

$$P \succ (q_i, z_t) \succ \lambda P + (1-\lambda) Q \sim \frac{\lambda}{\lambda_i}(q_i, z_t) + (1 - \frac{\lambda}{\lambda_i}) Q \succ Q.$$

If $\lambda > \lambda_i$, then let $\beta = \frac{\lambda-\lambda_i}{\lambda(1-\lambda_i)}$, $\beta' = 0$ and (24) becomes

$$\begin{aligned}\lambda P + (1-\lambda)Q \sim & \frac{\lambda-\lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda-\lambda_i}{1-\lambda_i})(\lambda_i P' + (1-\lambda_i)Q) \\ \sim & \frac{\lambda-\lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda-\lambda_i}{1-\lambda_i})(p_i, z_{t+1})\end{aligned}$$

The second indifference holds due to $\lambda_i P' + (1-\lambda_i)Q \sim T^i \sim (p_i, z_{t+1})$ and Lemma 7. Then by Lemma 14 on $\Phi_{2,z_{t+1}}$, we have

$$P \succ \lambda P + (1-\lambda)Q \sim \frac{\lambda-\lambda_i}{1-\lambda_i}P + (1 - \frac{\lambda-\lambda_i}{1-\lambda_i})(p_i, z_{t+1}) \succ (p_i, z_{t+1}) \succ Q.$$

A symmetric argument works for $\lambda < \lambda_{-i}$. Hence property (ii) holds on $\cup_{k=1}^{t+1}\Phi_{2,z_k}$.

We claim that for $P, Q \in \cup_{k=1}^{t+1}\Phi_{2,z_k}$ with $P \succ Q$, $P_1 \perp Q_1$ and $1 > \lambda_1 > \lambda_2 > 0$, we have $\lambda_1 P + (1-\lambda_1)Q \succ \lambda_2 P + (1-\lambda_2)Q$. To see this, by (24), we can find $P' \sim P$ where P' is singular with respect to both P and Q such that

$$\lambda_1 P + (1-\lambda_1)Q \sim \frac{\lambda_1-\lambda_2}{1-\lambda_2}P' + \frac{1-\lambda_1}{1-\lambda_2}[\lambda_2 P + (1-\lambda_2)Q] \succ \lambda_2 P + (1-\lambda_2)Q.$$

The second strict ranking follows from (ii) since $P \sim P' \succ \lambda_2 P + (1-\lambda_2)Q$.

Given this claim, the proof for (iii) and (iv) on $\cup_{k=1}^{t+1}\Phi_{2,z_k}$ is similar to the proof of (ii). By induction, (i)-(iv) hold for $P, Q, R, S \in \cup_{k=1}^{K}\Phi_{2,z_k}$ and hence arbitrary $P, Q, R, S \in \cup_{y_2 \in X_2}\Phi_{2,y_2}$. □

It is worth noting that $\cup_{y_2 \in X_2}\Phi_{2,y_2}$ is a strict subset of $\mathcal{P}$. The next lemma shows that it omits the worst and the best (degenerate) lotteries.

Lemma 16: $\mathcal{P} \backslash (\cup_{y_2 \in X_2}\Phi_{2,y_2}) = \{\bar{c}, \underline{c}\}$.

Proof of Lemma 16. For each $P \in \mathcal{P}$ with $P \notin \{\bar{c}, \underline{c}\}$, we claim that $\bar{c} \succ P \succ \underline{c}$. If $|\operatorname{supp}(P)| = 1$, then the result follows from Lemma 1. Now we suppose that $|\operatorname{supp}(P_1)| \geq 2$. We can write P as $\sum_{x_1}(x_1, P_{2|x_1})P_1(x_1)$. If $(x_1, P_{2|x_1}) \notin \{\bar{c}, \underline{c}\}$ for all $x_1 \in \operatorname{supp}(P_1)$, then apply part (i) or (ii) in Lemma 15 repeatedly and we can conclude that $P \in \cup_{y \in X_2}\Phi_{2,y}$ and the result holds. Hence it suffices to consider the case where $(x_1, P_{2|x_1}) \in \{\bar{c}, \underline{c}\}$ for some $x_1 \in \operatorname{supp}(P_1)$.

Denote $P = P_1(\bar{c}_1)\delta_{\bar{c}} + P_1(\underline{c}_1)\delta_{\underline{c}} + (1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P'$ where $P' \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$, $P_1(\bar{c}_1) < 1$, $P_1(\underline{c}_1) < 1$ and $P_1(\bar{c}_1) + P_1(\underline{c}_1) > 0$.[25] By Axioms 2 and 3.2, we can find $\varepsilon^1 = (\varepsilon_1, 0), \varepsilon^2 = (0, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small such that $\bar{c} \succ \bar{c} - \varepsilon^1 \sim \bar{c} - \varepsilon^2 \succ P' \succ \underline{c} + \varepsilon^1 \sim \underline{c} + \varepsilon^2 \succ \underline{c}$. For each $\beta \in (0,1)$, denote $P^\beta = \beta P + (1-\beta)/2\delta_{\bar{c}-\varepsilon^2} + (1-\beta)/2\delta_{\underline{c}+\varepsilon^2}$. Notice that $(x_1, P^\beta_{2|x_1}) \notin \{\bar{c}, \underline{c}\}$ for all $x_1 \in \mathrm{supp}(P^\beta_1) = \mathrm{supp}(P_1)$. Hence, we can apply Lemma 1 and Lemma 15 and derive

$$\begin{aligned} P^\beta &= \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + \beta P_1(\underline{c}_1)\delta_{\underline{c}} + \frac{1}{2}(1-\beta)\delta_{\underline{c}+\varepsilon^2} \\ &\quad + \beta(1 - P_1(\bar{c}_1) - P_1(\underline{c}_1))P' \\ &\prec \beta P_1(\bar{c}_1)\delta_{\bar{c}} + \frac{1}{2}(1-\beta)\delta_{\bar{c}-\varepsilon^2} + (1 - \beta P_1(\bar{c}_1) - \frac{1}{2}(1-\beta))\delta_{\bar{c}-\varepsilon^1} \end{aligned}$$

Let $\beta \to 1$ and by Axiom 3.1, we have

$$P \precsim P_1(\bar{c}_1)\delta_{\bar{c}} + (1 - P_1(\bar{c}_1))\delta_{\bar{c}-\varepsilon^1} \prec \bar{c}.$$

The last strict ranking follows from Lemma 1 for conditional preference $\succsim_{1|\bar{c}_2}$. A similar argument can be adopted to show that $P \succ \underline{c}$. By Axiom 3.2 and Axiom 2, we conclude that $P \in \cup_{y_2 \in X_2} \Phi_{2,y_2}$. $\square$

As a direct corollary of Lemma 5 and Lemma 16, for any $P \in \mathcal{P}$, we can find some $x \in X$ such that $P \sim x$. Since $P \succ \underline{c}$ for any $P \neq \underline{c}$ and $P \prec \bar{c}$ for any $P \neq \bar{c}$, we can easily use the arguments in Lemma 14 to show that the independence property holds for $P, Q, R, S \in \Phi_{2,\underline{c}_2} \cup \{\underline{c}\}$ or $P, Q, R, S \in \Phi_{2,\bar{c}_2} \cup \{\bar{c}\}$. Hence, (i)-(iv) of Lemma 8 hold.

Now we prove (v). If $P, R \in \{\bar{c}, \underline{c}\}$, then $P \sim Q$ and $R \sim S$ implies $P = Q$ and $R = S$. The result trivially holds. Without loss of generality, suppose $\bar{c} \succ P \succ \underline{c}$. By Axiom 3.2, Lemma 5 and Lemma 16, there exist $(y_1, y_2) \in X$ and $\varepsilon = (\varepsilon_1, 0)$ with $\varepsilon_1 > 0$ sufficiently small such that $\bar{c} - \varepsilon \succ P \sim Q \sim (y_1, y_2) \succ \underline{c} + \varepsilon$ and $y_1 \notin \{\bar{c}_1, \underline{c}_1\}$. Since $P \sim (y_1, y_2)$, $R \sim S$, $P_1 \perp R_1$ and $y_1 \perp S_1$, by part (iv) in Lemma 8, we have $\alpha P + (1-\alpha)R \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$. Hence it suffices to show that $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$ for all $\alpha \in (0,1)$.

[25] If $P_1(\bar{c}_1) + P_1(\underline{c}_1) = 1$, then P' can be arbitrarily chosen.

By Lemma 5, there exists $\gamma \in (0,1)$ such that $\hat{Q} := \gamma\delta_{\bar{c}-\varepsilon} + (1-\gamma)\delta_{\underline{c}+\varepsilon} \sim P \sim Q$. Since $Q_1 \perp \hat{Q}_1$, for any $\beta \in (0,1)$, part (i) in Lemma 8 implies $Q^\beta := \beta Q + (1-\beta)\hat{Q} \sim Q$. We claim that for any $\alpha, \beta \in (0,1)$, we have $\alpha Q^\beta + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$. To prove the claim, first note that $Q(\bar{c}) \in (0,1)$ as $\bar{c} \succ Q \succ \underline{c}$ and $\mathrm{supp}(Q) \subseteq \{\bar{c}, \underline{c}\}$. Then

$$\begin{aligned} Q^\beta &= \beta Q + (1-\beta)\hat{Q} \\ &= \Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + \Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big). \end{aligned}$$

By Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2 respectively, we can find x_1, x_1' such that $x_1 \neq x_1'$, $\underline{c}_1 < x_1, x_1' < \bar{c}_1$ and

$$\begin{aligned} (x_1, \bar{c}_2) &\sim \frac{\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}}{\beta Q(\bar{c}) + (1-\beta)\gamma}, \\ (x_1', \underline{c}_2) &\sim \frac{\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}}{\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)}. \end{aligned}$$

Part (iv) in Lemma 8 implies

$$Q^\beta \sim \Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + \Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)}.$$

Denote by $\tilde{Q}^\beta$ the right-hand side of the above relation. Then we have

$$\begin{aligned} \alpha Q^\beta + (1-\alpha)S =& \alpha\Big(\beta Q(\bar{c})\delta_{\bar{c}} + (1-\beta)\gamma\delta_{\bar{c}-\varepsilon}\Big) + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\ &+ \alpha\Big(\beta(1-Q(\bar{c}))\delta_{\underline{c}} + (1-\beta)(1-\gamma)\delta_{\underline{c}+\varepsilon}\Big) + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\ \sim& \alpha\Big(\beta Q(\bar{c}) + (1-\beta)\gamma\Big)\delta_{(x_1,\bar{c}_2)} + (1-\alpha)S(\bar{c})\delta_{\bar{c}} \\ &+ \alpha\Big(\beta(1-Q(\bar{c})) + (1-\beta)(1-\gamma)\Big)\delta_{(x_1',\underline{c}_2)} + (1-\alpha)(1-S(\bar{c}))\delta_{\underline{c}} \\ =& \alpha\tilde{Q}^\beta + (1-\alpha)S. \end{aligned}$$

The indifference relation follows from applying Lemma 1 given $\bar{c}_2$ and $\underline{c}_2$ in dimension 2, and part (iv) in Lemma 8 sequentially. Since $(y_1, y_2) \sim Q^\beta \sim \tilde{Q}^\beta$ and $S_1 \perp \{y_1, \tilde{Q}_1^\beta\}$, again by part (iv) in Lemma 8, we have

$$\alpha Q^\beta + (1-\alpha)S \sim \alpha\tilde{Q}^\beta + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S.$$

This holds for all $\alpha, \beta \in (0,1)$. By Axiom 3.1, for any $\alpha \in (0,1)$, let $\beta \to 1$ and we have $\alpha Q + (1-\alpha)S \sim \alpha\delta_{(y_1,y_2)} + (1-\alpha)S$. This completes the proof for (v).

2. Proof of Lemma 13

For two functions f_1 and f_2, we denote by $f_1 \propto f_2$ (or equivalently, $f_2 \propto f_1$) if f_1 is a positive affine transformation of f_2. By assumption, $\succsim$ admits a representation (15), that is, the utility of each $P \in \mathcal{P}$ is $U(P) = \sum_{x_1} w(x_1, CE_{v_{x_1}}(P_{2|x_1}))P_1(x_1)$. Since $\succsim$ violates Axiom 8, there exist $z_1, z_1' \in X_1$ such that $v_{z_1} \not\propto v_{z_1'}$.

Fix any $x_1, y_1 \in X_1$ with $v_{x_1} \not\propto v_{y_1}$ and $\alpha \in [0,1]$. Consider three conditional preferences $\succsim_{2|x_1}, \succsim_{2|y_1}$ and $\succsim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}}$ in dimension 2. We can interpret $\succsim_{2|x_1}$ and $\succsim_{2|y_1}$ as individual preferences, and $\succsim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}}$ as the group preference. By Lemma 1, all three conditional preferences admit EU representations on $\mathcal{P}_2$. Moreover, by linearity of (15), for any $p, q \in \mathcal{P}_2$,

$$\begin{aligned}
p \succ_{2|x_1} q, p \succ_{2|y_1} q \Longrightarrow & w(x_1, CE_{v_{x_1}}(p)) > w(x_1, CE_{v_{x_1}}(q)) \text{ and} \\
& w(y_1, CE_{v_{y_1}}(p)) > w(y_1, CE_{v_{y_1}}(q)) \\
\Longrightarrow & U(\alpha(x_1,p) + (1-\alpha)(y_1,p)) > U(\alpha(x_1,p) + (1-\alpha)(y_1,p)) \\
\Longrightarrow & p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q.
\end{aligned}$$

Similarly, we can show that if $p \sim_{2|x_1} q, p \sim_{2|y_1} q$, then $p \sim_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. Hence, by Harsanyi (1955)'s utilitarianism theorem, there exists a function $\tau : [0,1] \to [0,1]$ such that for each $\alpha \in [0,1]$, we have $v_{\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \propto \tau(\alpha)v_{x_1} + (1-\tau(\alpha))v_{y_1}$.

We claim that τ is strictly increasing. To see this, first note that we can set $\tau(0) = 0$ and $\tau(1) = 1$. Consider $\alpha, \alpha' \in (0,1)$ with $\alpha > \alpha'$. By Lemma 12, we can find $p, q \in \mathcal{P}_2$ such that $p \succ_{2|x_1} q$, $q \succ_{2|y_1} p$, and $p \sim_{2|\alpha'\delta_{x_1}+(1-\alpha')\delta_{y_1}} q$. By (15) and $\alpha > \alpha'$, we have $p \succ_{2|\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} q$. This implies

$$\begin{aligned}
\tau(\alpha)\mathbb{E}_p(v_{x_1}) + (1-\tau(\alpha))\mathbb{E}_p(v_{y_1}) > \tau(\alpha)\mathbb{E}_q(v_{x_1}) + (1-\tau(\alpha))\mathbb{E}_q(v_{y_1}), \\
\tau(\alpha')\mathbb{E}_p(v_{x_1}) + (1-\tau(\alpha'))\mathbb{E}_p(v_{y_1}) = \tau(\alpha')\mathbb{E}_q(v_{x_1}) + (1-\tau(\alpha'))\mathbb{E}_q(v_{y_1}).
\end{aligned}$$

Since $\mathbb{E}_p(v_{x_1}) > \mathbb{E}_q(v_{x_1})$ and $\mathbb{E}_p(v_{y_1}) < \mathbb{E}_q(v_{y_1})$, we conclude that $\tau(\alpha) > \tau(\alpha')$.

For each $\alpha \in (0,1)$, by Lemma 12, we can find $p, q \in \mathcal{P}_2$ such that $(p,q) \in$

$\Gamma_{x_1,y_1}(\alpha)$. Since $v_{\alpha\delta_{x_1}+(1-\alpha)\delta_{y_1}} \propto \tau(\alpha)v_{x_1} + (1-\tau(\alpha))v_{y_1}$, we have

$$\frac{\tau(\alpha)}{1-\tau(\alpha)} = \frac{\mathbb{E}_q(v_{y_1}) - \mathbb{E}_p(v_{y_1})}{\mathbb{E}_p(v_{x_1}) - \mathbb{E}_q(v_{x_1})}, \tag{25}$$

$$\frac{\alpha}{1-\alpha} = \frac{w(y_1, CE_{v_{y_1}}(q)) - w(y_1, CE_{v_{y_1}}(p))}{w(x_1, CE_{v_{x_1}}(p)) - w(x_1, CE_{v_{x_1}}(q))}. \tag{26}$$

For any $\beta \in (0,1)$ and $r \in \mathcal{P}_2$, by Lemma 1, we have $\beta p + (1-\beta)r \succ_{2|x_1} \beta q + (1-\beta)r$ and $\beta q + (1-\beta)r \succ_{2|y_1} \beta p + (1-\beta)r$. By the representation (15), there exists a unique $\alpha' \in (0,1)$ such that $\beta p + (1-\beta)r \sim_{2|\alpha'\delta_{x_1}+(1-\alpha')\delta_{y_1}} \beta q + (1-\beta)r$. By (25), we have

$$\begin{aligned}\frac{\tau(\alpha')}{1-\tau(\alpha')} &= \frac{\mathbb{E}_{\beta q+(1-\beta)r}(v_{y_1}) - \mathbb{E}_{\beta p+(1-\beta)r}(v_{y_1})}{\mathbb{E}_{\beta p+(1-\beta)r}(v_{x_1}) - \mathbb{E}_{\beta q+(1-\beta)r}(v_{x_1})} \\ &= \frac{\mathbb{E}_q(v_{y_1}) - \mathbb{E}_p(v_{y_1})}{\mathbb{E}_p(v_{x_1}) - \mathbb{E}_q(v_{x_1})} = \frac{\tau(\alpha)}{1-\tau(\alpha)}.\end{aligned}$$

Since τ is strictly increasing, we have $\alpha = \alpha'$. By (26), this suggests

$$\frac{w(y_1, CE_{v_{y_1}}(\beta q + (1-\beta)r)) - w(y_1, CE_{v_{y_1}}(\beta p + (1-\beta)r))}{w(x_1, CE_{v_{x_1}}(\beta p + (1-\beta)r)) - w(x_1, CE_{v_{x_1}}(\beta q + (1-\beta)r))} = \frac{\alpha}{1-\alpha} \tag{27}$$

for all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$.

For each $x_1 \in X_1$, since v_{x_1} is unique up to a positive affine transformation, we can normalize that $v_{x_1}(\underline{c}_2) = w(x_1, \underline{c}_2)$ and $v_{x_1}(\overline{c}_2) = w(x_1, \overline{c}_2)$. Define a function $\zeta_{x_1} : [v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2)] \to \mathbb{R}$ such that $\zeta_{x_1}(z) = w(x_1, v_{x_1}^{-1}(z))$ for all $z \in [v_{x_1}(\underline{c}_2), v_{x_1}(\overline{c}_2)]$. Then $\zeta_{x_1}(v_{x_1}(\underline{c}_2)) = v_{x_1}(\underline{c}_2)$ and $\zeta_{x_1}(v_{x_1}(\overline{c}_2)) = v_{x_1}(\overline{c}_2)$. Also, by Lemma 10, the function ζ_{x_1} is continuous and strictly increasing. Rewrite (27) and we derive

$$\frac{\zeta_{y_1}(\beta\mathbb{E}_q(v_{y_1}) + (1-\beta)\mathbb{E}_r(v_{y_1})) - \zeta_{y_1}(\beta\mathbb{E}_p(v_{y_1}) + (1-\beta)\mathbb{E}_r(v_{y_1}))}{\zeta_{x_1}(\beta\mathbb{E}_p(v_{x_1}) + (1-\beta)\mathbb{E}_r(v_{x_1})) - \zeta_{x_1}(\beta\mathbb{E}_q(v_{x_1}) + (1-\beta)\mathbb{E}_r(v_{x_1}))} = \frac{\alpha}{1-\alpha} \tag{28}$$

all $\beta \in (0,1)$ and $r \in \mathcal{P}_2$. This holds for all $\alpha \in (0,1)$, $x_1, y_1 \in X_1$ and $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$.

Let $r = p$, by equations (25) and (28), we have

$$\frac{\dfrac{\zeta_{y_1}(\beta \mathbb{E}_q(v_{y_1}) + (1-\beta)\mathbb{E}_p(v_{y_1})) - \zeta_{y_1}(\mathbb{E}_p(v_{y_1}))}{\beta \mathbb{E}_q(v_{y_1}) - \beta \mathbb{E}_p(v_{y_1})}}{\dfrac{\zeta_{x_1}(\beta \mathbb{E}_q(v_{x_1}) + (1-\beta)\mathbb{E}_p(v_{x_1})) - \zeta_{x_1}(\mathbb{E}_p(v_{x_1}))}{\beta \mathbb{E}_q(v_{x_1}) - \beta \mathbb{E}_p(v_{x_1})}} = \frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{29}$$

As β goes to 0, equation (29) becomes

$$\frac{\lim\limits_{b \to \mathbb{E}_q(v_{y_1})^+} \Big(\zeta_{y_1}(b) - \zeta_{y_1}(\mathbb{E}_q(v_{y_1}))\Big)\Big/\Big(b - \mathbb{E}_q(v_{y_1})\Big)}{\lim\limits_{c \to \mathbb{E}_q(v_{x_1})^-} \Big(\zeta_{x_1}(c) - \zeta_{x_1}(\mathbb{E}_q(v_{x_1}))\Big)\Big/\Big(c - \mathbb{E}_q(v_{x_1})\Big)} = \frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{30}$$

We claim that the two limits on the left-hand side of equation (30) exist as real numbers. If they exist, then the numerator is called the right derivative of ζ_{y_1} at $\mathbb{E}_q(v_{y_1})$, denoted by $\partial_+\zeta_{y_1}(\mathbb{E}_q(v_{y_1}))$, and the denominator is called the left derivative of ζ_{x_1} at $\mathbb{E}_q(v_{x_1})$, denoted by $\partial_-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))$.

To prove the claim, since ζ_{x_1} and ζ_{y_1} are strictly increasing and continuous, they are differentiable almost everywhere on their domains. Hence, the two one-sided derivatives are well-defined almost everywhere. Note that since $v_{x_1} \not\propto v_{y_1}$, we can find $r \in \mathcal{P}_2$ such that $r \sim_{2|x} q$ and $r \not\sim_{2|y} q$. Then, if we change the value of $\beta \in (0,1)$, the value of $\mathbb{E}_{\beta q+(1-\beta)r}(v_{x_1})$ remains unchanged, while the value of $\mathbb{E}_{\beta q+(1-\beta)r}(v_{y_1})$ will form an open interval. Suppose that $\lim\limits_{c \to \mathbb{E}_q(v_{x_1})^-} \Big(\zeta_{x_1}(c) - \zeta_{x_1}(\mathbb{E}_q(v_{x_1}))\Big)\Big/\Big(c - \mathbb{E}_q(v_{x_1})\Big)$ does not exist, then we know that $\lim\limits_{b \to a^+} \Big(\zeta_{y_1}(b) - \zeta_{y_1}(a)\Big)\Big/\Big(b-a\Big)$ does not exist for a contained in an open interval. This contradicts with the condition that $\partial_+\zeta_{y_1}$ is well-defined almost everywhere. Using a similar argument, we conclude that the two limits on the left-hand side of equation (30) exist as real numbers. Hence, equation (30) can be rewritten as

$$\frac{\partial_+\zeta_{y_1}(\mathbb{E}_q(v_{y_1}))}{\partial_-\zeta_{x_1}(\mathbb{E}_q(v_{x_1}))} = \frac{\alpha(1-\tau(\alpha))}{(1-\alpha)\tau(\alpha)}. \tag{31}$$

This holds for all $\alpha \in (0,1)$, $x_1, y_1 \in X_1$ and $p, q \in \mathcal{P}_2$ such that $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$. By Lemma 12, for each $\alpha' \in (0,1)$, we can choose $p^{\alpha'} \in \mathcal{P}_2$ such that $(p^{\alpha'}, q) \in \Gamma_{x_1,y_1}(\alpha')$. As a result, the right hand side of (31) is a constant for all $\alpha \in (0,1)$.

By the properties of Γ_{x_1,y_1}, for any $b \in (v_{y_1}(\underline{c}_2), v_{y_1}(\bar{c}_2))$, we can find some $\alpha \in (0,1)$ and $(p,q) \in \Gamma_{x_1,y_1}(\alpha)$ such that $b = \mathbb{E}_q(v_{y_1})$. Again by $v_{x_1} \not\propto v_{y_1}$, there exists an open interval I_b that contains b and the right derivative $\partial_+\zeta_{y_1}$ is a constant on I_b. Since b can be arbitrary in $(v_{y_1}(\underline{c}_2), v_{y_1}(\bar{c}_2))$, we know that $\partial_+\zeta_{y_1}$ must be a constant on $(v_{y_1}(\underline{c}_2), v_{y_1}(\bar{c}_2))$. Similarly, $\partial_-\zeta_{x_1}$ must be a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2))$.

Recall that the above results are derived by letting $r = p$ in equation (28). Now let $r = q$ and repeat the argument. We can derive that $\partial_-\zeta_{y_1}$ must be a constant on $(v_{y_1}(\underline{c}_2), v_{y_1}(\bar{c}_2))$, and $\partial_+\zeta_{x_1}$ must be a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2))$. Since ζ_{y_1} and ζ_{x_1} are differentiable almost everywhere, $\partial_-\zeta_{y_1} = \partial_+\zeta_{y_1}$ and $\partial_-\zeta_{x_1} = \partial_+\zeta_{x_1}$ almost everywhere on their domains. Hence, we conclude that ζ_{y_1} and ζ_{x_1} are differentiable on $(v_{y_1}(\underline{c}_2), v_{y_1}(\bar{c}_2))$ and $(v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2))$ respectively, and their derivatives remain constant. Take ζ_{x_1} as an example. Since ζ_{x_1} is continuous, $\zeta_{x_1}(a) = a$ for $a = v_{x_1}(\underline{c}_2)$ and $a = v_{x_1}(\bar{c}_2)$, and the derivative of ζ_{x_1} is a constant on $(v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2))$, we have $\zeta_{x_1}(a) = a$ for all $a \in [v_{x_1}(\underline{c}_2), v_{x_1}(\bar{c}_2)]$. By definition of ζ_{x_1}, this implies $w(x_1, x_2) = v_{x_1}(x_2)$ for all $x_2 \in X_2$. Also, we have $w(y_1, x_2) = v_{y_1}(x_2)$ for all $x_2 \in X_2$.

Finally, consider any $x_1 \in X_1$. Since $v_{z_1} \not\propto v_{z_1'}$, either $v_{x_1} \not\propto v_{z_1}$ or $v_{x_1} \not\propto v_{z_1'}$. The above result applies for either the pair of v_{x_1} and v_{z_1} or the pair of v_{x_1} and $v_{z_1'}$. Hence, $w(x_1, x_2) = v_{x_1}(x_2)$ for all $(x_1, x_2) \in X$, and the representation (15) reduces to an EU representation with Bernoulli index w.

3. Proof of Proposition 2

By symmetry, assume that $|I^1| \geq |I^2|$ throughout the proof.

For (i), as $I^1 = \{1, 2\}$, the binary relation $\succsim$ has an NEU representation and hence satisfies Axiom 10. As $I^2 \neq I^1$, we know $I^2 = \emptyset, \{1\}$, or $\{2\}$. We can apply the proof of part (ii) in Proposition 4 to show that $\succsim$ has an EU representation with an additively separable Bernoulli index w.

For (ii), without loss, assume that $I^1 = \{2\}$ and $I^2 = \emptyset$. Then $\succsim$ admits both an EU representation w and an SEU representation $(2, w', v_2)$. Note that $\succsim_{2|x_1}$ can be represented by both $w(x_1, \cdot)$ and v_2 for all $x_1 \in X_1$. Hence, for each $x_1 \in X_1$, there exists $a(x_1) > 0$ and $b(x_1) \in \mathbb{R}$ such that $w(x_1, x_2) = a(x_1)v_2(x_2) + b(x_1)$ for all $x_2 \in X_2$. This finishes the proof with functions $w_1 = b$ and $w_2 = v_2$.

For (iii), suppose $\succsim$ admits two SEU representations $(1, w^1, v_1)$ and $(2, w^2, v_2)$. First, we can normalize $w^1(\bar{c}) = w^2(\bar{c}) = 1$ and $w^1(\underline{c}) = w^2(\underline{c}) = 0$. Then for any $x \in X$, there exists a unique $\lambda \in [0, 1]$ such that $x \sim \lambda\delta_{\bar{c}} + (1-\lambda)\delta_{\underline{c}}$. By linearity of the SEU representation, $w^1(x) = \lambda = w^2(x)$. Hence, we can simply denote $w^1 = w^2 = w$. Using the same argument in (ii), we can find $a_i : X_i \to \mathbb{R}_{++}$ and $b_i : X_i \to \mathbb{R}$ for $i = 1, 2$ such that for each $(x_1, x_2) \in X$,

$$w(x_1, x_2) = a_2(x_2)v_1(x_1) + b_2(x_2) = a_1(x_1)v_2(x_2) + b_1(x_1). \tag{32}$$

Fix $x_2 \in X_2$ and consider (32) for $x_1 = \underline{c}_1$ and $x_1 = \bar{c}_1$ respectively. We can solve for $a_2(x_2)$ and $b_2(x_2)$ as linear functions of $v_2(x_2)$. Hence, we can find real numbers α, β, γ such that $w(x, y) = \alpha v_1(x_1)v_2(x_2) + \beta v_1(x_1) + \gamma v_2(x_2)$. Note that the Bernoulli index w is additively separable if $\alpha = 0$ and multiplicatively separable if $\alpha \neq 0$. In both cases, the SEU representation is indeed an EU representation.

References for Online Appendix

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