Competing to Commit: Markets with Rational Inattention*

Carlo M. Cusumano[†] Francesco Fabbri[‡] Ferdinand Pieroth[§]

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Abstract

Two homogeneous-good firms compete for a consumer's unitary demand. The consumer is rationally inattentive and pays entropy costs to process information about firms' offers. Compared to a collusion benchmark, competition produces two effects. As in standard models, competition puts downward pressure on prices. But, additionally, an *attention effect* arises: Competition acts as a commitment device to keep market offers in check, allowing the consumer to engage in trade more often. For high enough attention costs, this attention effect dominates: Firms' profits are higher under competition than under collusion. Our model can explain why markets with common ownership may remain competitive.

Keywords: Rational inattention; Competition; Common ownership

JEL codes: D21; D43; D83

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[†]Yale University, Department of Economics; carlo.cusumano@yale.edu

[‡]Princeton University, Department of Economics; ffabbri@princeton.edu

[§]Yale University, Department of Economics; ferdinand.pieroth@yale.edu

1 Introduction

Consumers' ability to process information about prices is naturally at the heart of the idea of competition. Standard models of Bertrand competition assume that consumers can perfectly spot different firms' offers and optimally choose the best one. However, in many situations, finding the best offer requires costly attention. Consider, for example, a consumer deciding on which mortgage to apply for, where to purchase life insurance, or which food delivery service to use. Even if all the information needed for optimal decision-making is available, the consumer has to process this information. Mortgage and life insurance contracts can be challenging to understand, and learning which delivery service offers the lowest fees or the best promotions requires time and cognitive effort.

If information processing is costly, rational consumers must decide not only which offer to accept but also how much attention to pay to each offer. Since firms' price-setting decisions depend on how strongly consumers react to price changes, understanding how consumers allocate attention to offers is crucial. In particular, consumers' endogenous attention allocation can be a novel channel through which changes in the market structure shape economic outcomes.

This paper studies the impact of competition in markets with costly information processing. We show that increasing the level of competition has two effects. As in standard settings, competition puts downward pressure on prices, i.e., it has a *pricing effect*. Moreover, an *attention effect* emerges: Since the consumers' information strategy depends on the level of competition, the demand curve increases pointwise as this level rises. For a range of information processing costs, we show that the attention effect dominates: Firms' equilibrium profits are *higher* when they compete than when they collude.

We introduce rational inattention to an otherwise standard Bertrand duopoly setting. Our model consists of a representative consumer and two firms that sell a good of common quality.[1] The firms make take-it-or-leave-it offers to the consumer, which we characterize by their monetary value. The consumer has unitary demand and chooses an information structure to learn about the quality of the good and the firms' offers. Following an extensive literature building on Sims (2003), we take the information processing cost, or *attention cost*, to be proportional to the expected entropy reduction. After processing the information, the consumer decides whether and from which firm to buy the good.

Our model applies particularly well to markets where the monetary value of an offer is difficult to understand or compare across firms. Examples of this kind are contracts for health or life insurance and complex loans. More generally, our model captures settings where consumers need to process information about offers, for example, because they are comprised of several individual prices, fees, and discounts. Focusing on information processing allows us to model consumers' attention continuously.[2] In particular, rational inattention provides a tractable framework to

[1] We extend our results to any number of firms in Section 4.3

[2] While models of consumer search in the spirit of Stahl (1989) and captive consumers in the spirit of Varian (1980) also analyze consumers' limited attention, they typically assume that the consumer's information about a firm's price is binary: The consumer either observes the price or not. In contrast, the rational inattention framework allows the consumer to choose any information structure. As a result, it enables us to study more subtle changes in the consumer's attention strategy.

analyze the resulting trade-off between optimal decision-making and costly information processing.

We characterize the firms' and consumer's behavior by using the solution concept of Bayes Nash Equilibrium. To avoid an infinite multiplicity of equilibrium outcomes, we impose a refinement that requires the consumer's strategy to be *robust to vanishing perturbations* (RVP). Intuitively, because entropy costs ignore off-path events, rational inattention does not place any restriction on the consumer's behavior following a firm's deviation. RVP requires that for all possible deviations in the equilibrium price-setting behavior of the firms, the consumer strategy must be optimal against some (possibly correlated) vanishing belief perturbation consistent with such deviations.[3] This means that the consumer's off-path behavior can be rationalized by some arbitrarily small perturbations in the firms' strategies.

To isolate the effects of competition, we begin by establishing a benchmark where the firms collude. We find that in the collusive setting, the unique RVP trading equilibrium outcome is identical to the unique credible trading equilibrium outcome of Ravid's (2020) ultimatum bargaining game. Using this equivalence and Ravid's results, it follows that trade can be sustained under collusion if and only if the parameter k that governs the consumer's unit cost of information processing is below a threshold k^*. Moreover, trade under collusion is always inefficient, namely it occurs with probability strictly less than one.

In our model of competition, we are interested in RVP equilibria in which both firms trade with positive probability, which we call *competitive trading equilibrium*.[4] We show that such an equilibrium exists if and only if trade can also be sustained under collusion, i.e., if and only if k is below the threshold k^*. Moreover, whenever a competitive trading equilibrium exists, it is unique. As in the collusion setting, no trade can be sustained if attention costs are above k^*: The consumer is unwilling to pay any attention to the firms' offers under such high costs, irrespective of the number of firms in the market and their incentives to compete, and consequently the firms cannot be deterred from overcharging the consumer.

Competition always increases trading efficiency when a competitive trading equilibrium exists. Intuitively, competition acts as a *commitment device* to keep the market's offers in check. Compared to collusion, this allows the consumer to engage in trade more often without risking being overcharged, increasing her endogenous demand pointwise. This *attention effect* of competition emerges because, for any fixed demand, firms' offers improve with competition, which in turn increases the inattentive consumer's willingness to engage in trade. In fact, if the unit cost of information processing is below a second, lower threshold $\bar{k} < k^*$, the attention effect is strong enough that trade efficiency is restored. The consumer in this case buys with probability one, so she disregards any information about the value of firms' offers relative to the no-purchase outside option. Since the consumer only processes information about the price difference between the firms and not their absolute value, under collusion the firms could exploit this situation and coordinate on overcharging the consumer. In contrast, competing firms have incentives to

[3]RVP naturally extends the notion of *credible best response* introduced by Ravid (2020) in a monopoly setting, and is similar in spirit but weaker than Selten's (1975) trembling-hand perfection.

[4]Competitive trading equilibria are the only equilibria in which competitive forces are present. If one of the firms is does not trade with positive probability, the active firm faces the same environment as a monopolist.

undercut each other, resulting in low prices when the consumer is attentive to price differences. Hence, an equilibrium with efficient trade exists as long as information processing is sufficiently inexpensive, i.e., $k \leq \bar{k}$.

The increase in trade efficiency can be strong enough to make firms better off under competition than under collusion. In particular, our main result states that there always exists a region of relatively high attention costs in which firms achieve higher profits if they compete than if they collude. This result is in stark contrast with those of standard models that ignore information processing costs. Since the firms produce perfect substitutes, competition in frictionless models keeps demand unchanged and puts downward pressure on prices, thus reducing firms' profits. In contrast, when the consumer is rationally inattentive, competition provides her with assurance that firms will not overcharge and makes her willing to engage in trade more often. This attention effect always dominates the pricing effect of competition for attention costs close enough to the no-trade threshold k^*. Intuitively, if k is close to k^*, the consumer focuses on deciding *whether* to buy and pays less attention to comparing firms' offers. This implies that competing firms' pricing behavior is similar to that of colluding firms in this parameter region. As a result, the attention effect dominates the pricing effect, and firms benefit from the additional commitment power generated by competition.

We further show that competition can improve the welfare of *both* sides of the market simultaneously. General results about consumer surplus and total welfare are difficult to obtain due to the consumer's information processing cost. However, we establish that consumer surplus is higher under competition whenever trade is efficient or attention costs are relatively high.

We apply our results to the question of how common ownership influences competition. The common ownership hypothesis states that firms in the same market have fewer incentives to compete if common investors own shares of all of them. Contrary to this hypothesis, our main result shows that common investors may want to create incentives for the firms to compete. As an illustration, consider the German consumer electronic store chain MediaMarkt. In 1990, MediaMarkt acquired its direct competitor Saturn. Instead of merging the two chains and their management, MediaMarkt decided to keep the management and branding separate: "The MediaMarkt and Saturn brands operate independently on the German market and are in direct competition with one another".[5] Given their position as Germany's market leader, this strategy seems counter to standard economic intuition. Our framework offers an explanation: When consumers are rationally inattentive about endogenous offers, holdings may benefit from owning actively competing firms.

Recent empirical research finds evidence which is broadly in line with the above illustration, namely showing that some markets with high levels of common ownership display more competition than may be expected. For example, Backus, Conlon, and Sinkinson (2021a) find no significant impact of common ownership on pricing in the cereal industry. Dennis, Gerardi, and Schenone (2022) show that there are no anti-competitive effects due to common ownership in the airline industry. Our results provide a rationale for common owners actively designing incentives to promote competition among commonly owned firms in the presence of rational inattention.

[5] Translated from https://www.mediamarktsaturn.com/mediamarktsaturn-deutschland.

Related Literature. We contribute directly to the literature on rational inattention in markets with one (Martin (2017), Boyacı and Akçay (2018), Yang (2019), Ravid, Roesler, and Szentes (2022), Mensch and Ravid (2022), Thereze (2022b)) or many sellers (Bloedel and Segal (2021), Wu (2022), Thereze (2022a), Janssen and Kasinger (2022)). Following the ultimatum bargaining model of Ravid (2020), we introduce a refinement in the spirit of, but weaker than, Selten (1975) trembling-hand selecting for the equilibria where the consumer's best response follows Matějka and McKay's (2015) solution *everywhere*, determining demand on and off-path. Price competition with rationally inattentive consumers was previously discussed by Matějka and McKay (2012). Within a similar framework, we provide novel insights about the implication of consumer inattention: When information processing costs are high enough, the industry obtains higher profits under competition than under collusion. Furthermore, we do not restrict the attention strategies to coincide with the optimal solution of Matějka and McKay (2015) but derive the consumer's best response on and off-path using a refinement. Inattention to equilibrium play also features in Matějka (2015), which shows that price rigidity is optimal in a monopoly model where consumers are inattentive about prices. However, as the monopolist commits to a public price schedule, the consumer's off-path behavior is constrained by subgame perfection.

Our work connects with the literature on search costs (Burdett and Judd (1983), Stahl (1989)) but differs substantially in how we model consumers. In particular, flexible and endogenous information acquisition allows us to quantify the amount of information processed beyond the binary case of search models, delivering different equilibrium predictions. For example, the seminal work of Diamond (1971) shows that firms exploit the presence of small search costs to charge monopoly prices. In contrast, as attention costs approach zero, the pricing effect dominates the attention effect in our model, and equilibrium prices converge to marginal cost.[6] Furthermore, in the presence of search costs, firms' strategy space must be richer to account for higher profits. Cachon, Terwiesch, and Xu (2008) study parallel and sequential search when firms choose both prices and product assortment: Lower search costs encourage broader assortments, which implies a market expansion as consumers match their preferences more easily. When supported by either higher prices or sufficiently high consumers' outside options, this leads to higher profits. In our framework, lower attention costs may give rise to higher profits driven by higher trade efficiency.

We also relate to the behavioral literature, which builds on Varian (1980), that justifies price unobservability by bounded rationality and regards consumers' cognitive abilities as exogenous. For example, in Spiegler (2006), the consumer evaluates firms' pricing strategies using an exogenous sampling procedure. In this framework, contrary to our main result, profits are unaffected by the industry size. Gabaix and Laibson (2006) show that competitive firms exploit non-sophisticated consumers by shrouding add-on prices. However, competition brings profits to zero, while profits are positive in our model. De Clippel, Eliaz, and Rozen (2014) model attention as the number of markets the consumers can perfectly explore. They find that consumer welfare is higher if the expected level of attention is lower. Our model reverses this conclusion. Armstrong and

[6] This observation was already noted by Matějka and McKay (2012).

Vickers (2022) show that firms' entry may decrease consumer surplus and increase the industry's profits in settings with *exogenous* consideration sets. Our results are driven instead by *endogenous* attention allocation.[7] In a setting with primary and secondary price features, Heidhues, Johnen, and Kőszegi (2021) show that a pro-competitive effect emerges only under regulations on secondary features. This result complements our analysis by providing conditions under which market structure changes can be considered pro-competitive. Hefti (2018) studies the consequences of choice overload when firms manipulate their product's visibility via costly marketing, showing that it prevents equilibrium outcomes from being fully competitive. However, as this framework models attention and demand separately, it cannot deliver our novel prediction that competition expands attention-based demand. Attention shapes demand in Bordalo, Gennaioli, and Shleifer (2016), where the product's most salient attribute captures the consumer's attention. The critical difference with our approach is that the rationally inattentive consumer allocates attention *ex-ante*, while the salient thinkers on which attribute is salient *ex-post*.

The literature on strategic price complexity provides a different rationale for consumers' failure to choose the best offer by allowing firms to compete, besides prices, on the complexity level or "format" of their price structure. Increasing the complexity level makes price comparison more difficult, inducing consumers' mistakes and ultimately leading to larger firms' market power. Works in this area are Carlin (2009), Piccione and Spiegler (2012), Chioveanu and Zhou (2013). Refer to Spiegler (2016) for a survey. By varying price complexity, while fixing consumer behavior, this literature finds that industry profits may increase in the number of competing firms. Our model instead fixes price complexity and derives the consumer's optimal response to changes in the market structure. Moreover, we can explain industry profits above monopoly profits, which cannot occur under strategic complexity.

Outline. The remainder of the paper proceeds as follows. In Section 2, we introduce the model, discuss the equilibrium notion we use for the analysis, and study its implications for the best response of the consumer. In Section 3, we describe the benchmark case of colluding firms and show its main equilibrium features. Section 4 analyzes competition. It characterizes the competitive trading equilibrium where both firms trade, introduces the attention effect and studies the equilibrium effects of competition on industry and consumer surplus. A discussion follows in Section 5. All omitted proofs are in the Appendix.

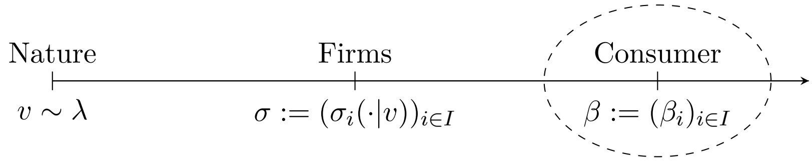
2 Model

Two identical firms with zero marginal costs compete for a consumer's unitary demand. Product quality is common, stochastic, and perfectly observed by the firms. After observing the product's quality, each firm makes a simultaneous offer to the consumer. The consumer (she) does not observe the product quality or the firms' offers directly. Instead, she holds beliefs about their joint distribution and costly processes information to improve her purchasing decision. We

[7] Consumers in Armstrong and Vickers (2022) always trade whenever they observe at least one firm's offer. Therefore, an increase in the industry surplus *implies* a decrease in the consumer surplus. This implication is not valid in our framework, as competition expands the trading surplus of the economy, allowing the coexistence of both higher industry and consumer surplus. See Appendix **??** on consumer surplus for more details.

interpret the consumer's information processing decision as an attention problem and use these terms interchangeably in our analysis. Following the literature on rational inattention initiated by Sims (2003), we call the consumer *rationally inattentive*, and we assume that the attention cost is entropy-based, as we describe below.

Game structure. The following timeline summarizes the game structure of the model. We formalize each element below.



1. Product quality v is drawn according to a probability measure $\lambda \in \Delta(\mathbb{R})$. We assume that λ has strictly positive finite support, i.e., $\operatorname{supp}\lambda =: V \subseteq (0,\infty)$ is finite.
2. After observing the product quality realization v, each firm $i \in I := \{1,2\}$ makes a simultaneous offer to the consumer. Denote firm i's strategy by $\sigma_i : V \to \Delta(\mathbb{R}_+)$. We interpret every $x \in \mathbb{R}_+$ as the *monetary value* associated with an offer, and for simplicity of exposition call x the price a firm charges.
3. We refer to each profile of the exogenous product quality and the endogenous firms' offers (v, x_1, x_2) as a *state*. The consumer does not directly observe the realized state. Rather, she holds beliefs about it and pays attention by selecting an *information structure* to learn more about it. The attention cost is proportional to the mutual information between states and signals, which equals the expected entropy reduction between the consumer's prior belief over states and her posterior beliefs obtained via Bayesian updating. After a signal realizes, the consumer makes a purchasing decision, and the game ends.

   Without loss of generality, we restrict the consumer's strategy space to *recommendation* or *attention strategies*.[8] A recommendation strategy β is a profile (β_1,β_2) such that, for every $i \in I$, $\beta_i : V \times \mathbb{R}^2_+ \to [0,1]$ denotes the *conditional* probability of accepting the offer of firm i. That is, $\beta_i(v,x_1,x_2)$ is the probability of receiving the recommendation "accept i's offer" given the state (v,x_1,x_2) realized. Naturally, for every $(v,x_1,x_2) \in V \times \mathbb{R}^2_+$, it holds that $\sum_{i\in I}\beta_i(v,x_1,x_2) \le 1$.

   Denote the consumer's prior belief over states by $\mu \in \Delta(V \times \mathbb{R}^2_+)$.[9] Exploiting the restriction on recommendation strategies, we can write mutual information as

[8] If the consumer did not have access to an ex-post randomization device, this would be a direct consequence of Matějka and McKay (2015). However, even in the presence of an ex-post randomization device, action recommendation strategies are without loss. Suppose the consumer found it optimal to strictly randomize between two actions a_1 and a_2 upon receiving a signal $\bar{s}$. In that case, she could split $\bar{s}$ into two signals s_1 and s_2 with the same posterior as $\bar{s}$ such that: (i) action a_i is played upon receiving message s_i, and (ii) the overall probability of playing each action a_i is unchanged. Since the resulting new information structure is neither more nor less informative than the original one, the consumer would be completely indifferent. Moreover, any play of the game between the consumer and the firms would unfold in the same way.

[9] We endow $V \times \mathbb{R}^2_+$ with the product σ-algebra between the discrete σ-algebra on V and the standard Borel σ-algebra on $\mathbb{R}^2_+$. We also endow both $\Delta(\mathbb{R}^2_+)$ and $\Delta(V \times \mathbb{R}^2_+)$ with the topology of strong convergence.

$$I(\mu, \beta) := H(\mathbb{E}_\mu[\beta]) - \mathbb{E}_\mu[H(\beta)], \tag{1}$$

where $H(p) = -p_1 \log(p_1) - p_2 \log(p_2) - (1-p_1-p_2) \log(1-p_1-p_2)$ is the Shannon entropy associated with the probability measure $(p_1, p_2, 1-p_1-p_2)$ consistent with $p = (p_1, p_2)$. Mutual information as formalized in equation (1) asserts that attention costs are proportional to the difference in entropy between the conditional and the unconditional distribution of playing each action. In other words, $I(\mu, \beta)$ captures how much the uncertainty about a specified plan of action decreases with information.

Payoffs. Once firms make offers and the consumer selects a recommendation strategy, payoffs are obtained. The consumer's utility given product's quality $v \in V$ is

$$U := \sum_{i \in I} (v - x_i)\beta_i(v, x_1, x_2) - k \cdot I(\beta, \mu).$$

That is, the consumer's utility equals his gains from trade net of the costs of processing information. The parameter $k > 0$ is the unit cost of information processing for the consumer: It represents the monetary value the consumer assigns to each bit of information she processes.

The payoff obtained by each firm $i \in I$ equals

$$\Pi_i^C := \beta_i(v, x_1, x_2) \cdot x_i,$$

where the superscript C stands for "Competition." The competing firms in our model adopt the standard profit-maximizing behavior. Since the consumer uses a recommendation strategy, $\beta_i(v, x_1, x_2)$ represents the endogenous demand firm i faces.

Discussion. In our framework, prices are not directly observable by the consumer. Instead, she chooses an information structure which (stochastically) determines what she learns about offers. As motivated in the Introduction and discussed also in Ravid (2020), this assumption captures situations where offers entail complex contracts or include multiple prices. Additionally, this assumption can be viewed as modeling a consumer at a very early stage of the purchasing decision, when she does not yet have access to prices. Consider, for example, the decision of where to have dinner. While restaurant prices are perfectly observable, one has to physically go to the restaurant or visit their website to actually access the menu. Either of these processes involve costs in time and cognitive effort. If ex-post switching costs are high, this leads to the same trade-off between optimal decision making and costly information processing studied in our model.

Our results rely on the consumer processing costly information about the equilibrium offers of the firms and not on the uncertainty about the product quality. As we further discuss in Section 5, similar findings hold in settings where quality is known and firms' marginal cost is stochastic.

2.1 Equilibrium Refinement

We adopt Bayes Nash Equilibrium (BNE) as the solution concept for our duopoly model with rational inattention. The *assessment* (μ, σ, β) is a BNE if (i) μ is *consistent* with σ,[10] (ii) β is a best response to μ, and (iii) for every $i \in I$, σ_i is a best response to σ_{-i} given β.

As discussed by Ravid (2020), standard BNE is too permissive to make sharp predictions about equilibrium outcomes in games with rational inattention if attention is directed toward endogenous variables. For instance, the following example shows that any division of surplus between the firms and the consumer is attainable as a BNE in our model.

Example 1. *For any $\alpha \in [0,1]$, there exists a BNE where each firm obtains profits equal to $\alpha \cdot \mathbb{E}_\lambda[v]/2$, while the consumer's payoff is given by $(1-\alpha) \cdot \mathbb{E}_\lambda[v]$. Consider the assessment $(\mu^\alpha, \sigma^\alpha, \beta^\alpha)$: μ^α is consistent with σ^α, $\sigma_i^\alpha(\alpha v|v) = 1$ for all $i \in I$ and $v \in V$, and $\beta_i^\alpha(v, x_1, x_2) = 1/2 \cdot \mathbf{1}_{\{x_1 = x_2 = \alpha v\}}$ for all $v \in V$ and $x_1, x_2 \geq 0$. In words, $(\mu^\alpha, \sigma^\alpha, \beta^\alpha)$ represents the situation where the consumer holds correct beliefs, the firms offer a price equal to a share α of the product quality, and the consumer trades with certainty when offers equal αv, in which case her purchase is uniform across firms, and never buys the product otherwise. Firms have no incentives to deviate since any deviation would imply zero profits. We now argue that the consumer has no incentives to deviate either. First, the consumer is indifferent between the firms' offers and optimally chooses to buy with probability 1 overall. At the same time, the consumer displays deterministic beliefs about firms' behavior conditional on each $v \in V$. Since the attention strategy β^α implies uniform acceptance of the offers μ^α-almost surely, the consumer incurs no information processing costs. This shows that β^α is optimal given μ^α. As a result, the assessment $(\mu^\alpha, \sigma^\alpha, \beta^\alpha)$ is a BNE and satisfies the desired surplus allocation.*

Example 1 shows that standard BNE yields a multiplicity of equilibria, each associated with a different division of the surplus between the firms and the consumer. Intuitively, the attention cost the consumer bears is prior-dependent and, therefore, is not affected by her off-path strategy. Therefore, despite the firms' optimal strategies depending on the consumer's reaction on and off-path, BNE does not require the consumer's off-path threats to be credible. In the example, the consumer threatens never to buy after any deviation by a firm, even if such deviation lowers prices: A non-credible threat.

The previous example breaks down if we allow the firms to make arbitrarily small mistakes on-path. When this is the case, the consumer's attention strategy β^α is sub-optimal: The attention cost associated with reacting to the firms' price deviations is larger than the related benefit.[11] To address the multiplicity of equilibria, we use this logic and impose an additional property on

[10] The belief $\mu \in \Delta(V \times \mathbb{R}_+^2)$ is consistent with the profile σ if for every $v \in V$, and for every Borel measurable set $E \subseteq \mathbb{R}_+^2$, we have

$$\mu(v, E) = \lambda(v) \cdot \int_E 1 \mathrm{d}\sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v).$$

[11] More formally, the consumer's attention strategy β^α is not robust to ε-perturbations of the consumer's equilibrium belief. To illustrate, consider the prior $\mu^{\alpha,\varepsilon} = (1-\varepsilon)\mu^\alpha + \varepsilon\mu^*$ where $\varepsilon \in (0,1)$, μ^a is defined as before for $\alpha \in [0,1]$, and μ^* is consistent with λ and assigns probability 1 to firms' offers being different from αv. An argument similar to Ravid's (2020) shows that, for small enough values of ε, the consumer prefers attention strategy $\hat{\beta}$ to β^α, where $\hat{\beta}$ accepts all offers for sure randomizing uniformly between firms.

the consumer's best response which we call *robustness to vanishing perturbations* (RVP). RVP requires the consumer's strategy to be justified under some arbitrarily small belief perturbations both on and off the conjectured path of play. It implies that the consumer no longer considers perfectly informative off-path signals to be costless.

Definition 1. *Let μ be consistent with the profile σ and β a best response to μ. We say that β is robust to vanishing perturbations (RVP) if for every $v^* \in V$ and $x_1^*, x_2^* \geq 0$, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ such that*

- $\tilde{\sigma}^n(\cdot|\cdot) \in \Delta(\mathbb{R}_+^2)^V$ *is a vector of (possibly correlated) probability measures on* $\mathbb{R}_+^2$*,*
- $\tilde{\sigma}^n(x_1^*, x_2^*|v^*) > 0$ *for every* $n \in \mathbb{N}$*,*
- $\tilde{\sigma}^n(\cdot|v) \to \sigma_1(\cdot|v) \otimes \sigma_2(\cdot|v)$ *strongly for all* $v \in V$*,*
- μ^n *is consistent with* $\tilde{\sigma}^n$*,*
- β *is a best reply to* μ^n *for every* $n \in \mathbb{N}$*.*

Definition 2. *(μ, σ, β) is an RVP equilibrium if it is a BNE and β is RVP.*

RVP naturally extends the notion of *credible best response* introduced by Ravid (2020) to a multi-firm setting, and is similar in spirit, albeit considerably weaker than, Selten's (1975) trembling-hand perfection.[12] Like Ravid, we allow belief perturbations to vary with off-path deviations while trembling hand perfection does not. Moreover, we also allow for correlated belief perturbations about the firms' offers.

We solve our duopoly model using RVP equilibrium (hereafter, just *equilibrium*). Despite its weakness, this refinement is strong enough to obtain sharp predictions regarding the offers accepted by the consumer on-path and the overall trade probability. These variables are sufficient to characterize the most important economic statistics of our model: The industry profits and the consumer surplus.

2.2 Consumer's Best Response

We study the implications of RVP for the consumer's best response. As Matějka and McKay (2015) show, optimal rational inattentive behavior is characterized by a multinomial logit forumla μ-almost surely. Our refinement extends this feature: Lemma 1 shows that β is an RVP best response to μ if and only if it displays *everywhere* a multinomial logit formula adjusted for the consumer's prior beliefs.

Lemma 1. *β is an RVP best response to μ if and only if for every $v \in V$ and $x_1, x_2 \geq 0$*

$$\beta_i(v, x_1, x_2) = \frac{\pi_i \cdot e^{\frac{v - x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v - x_j}{k}} + 1 - \pi_1 - \pi_2}, \quad (i \in I) \tag{2}$$

[12] Formally, RVP requires that for every state, a sequence of vanishing belief perturbations exists such that (i) the sequence put a positive probability on that state, and (ii) β is a best reply to every element of the sequence. Instead, trembling hand perfection requires that a sequence of vanishing *full-support* belief perturbations exists such that β is a best reply to every element of the sequence.

where (π_1, π_2) *solves*

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_\mu \left[\log \left(\pi_1' \cdot e^{\frac{v - x_1}{k}} + \pi_2' \cdot e^{\frac{v - x_2}{k}} + (1 - \pi_1' - \pi_2') \right) \right] \quad \textit{subject to} \quad \pi_1' + \pi_2' \leq 1. \tag{3}$$

In particular, for every $i \in I$,

$$\pi_i = \mathbb{E}_\mu[\beta_i]. \tag{4}$$

The "only if" direction follows from the fact that for each state (v, x_1, x_2), RVP requires that β is a best response to some vanishing perturbation that places strictly positive probability on that state. The proof for the "if" direction identifies vanishing perturbations for all possible firms' deviations that justify β as the consumer's best response. Finding such sequences of beliefs is possible because our equilibrium refinement is relatively weak and allows for potentially correlated belief perturbations.

Equation (2) identifies the entire class of consumer optimal RVP recommendation strategies. In particular, it describes the consumer's optimal RVP strategy up to the free variables π_1 and π_2. Each π_i describes the consumer's *trade engagement level* with firm $i \in I$. Problem (3) and equation (4) state that the trade engagement level π_i is chosen optimally by the consumer and equals the average probability of trade with firm $i \in I$.

Symmetry. While Lemma 1 describes the consumer's best response in arbitrary assessments, it is convenient to also describe her optimal strategy when we restrict the market participants to behave symmetrically. This restriction is without loss in the class of equilibria we consider and simplifies the characterization of the optimal trade engagement level. To be precise, consider the following definition.

Definition 3. *We say that*

1. $\sigma = (\sigma_1, \sigma_2)$ *is symmetric if* $\sigma_1 = \sigma_2$.
2. β *is symmetric if for each* $v \in V$ *and* $x_1, x_2 \geq 0$, *we have* $\beta_i(v, x_1, x_2) = \beta_{-i}(v, x_2, x_1)$.

The assessment (μ, σ, β) *is symmetric if* μ *is consistent with* σ, *and both* σ *and* β *are symmetric.*

As Appendix B shows, we can restrict the equilibrium analysis of competition to symmetric assessments only. The argument is as follows. First, competition only plays a role in equilibrium when both firms actively trade with the consumer. Otherwise, the unique active firm behaves like a monopolist, making de facto inconsequential the presence of the competitor. Furthermore, firms are ex-ante identical, so the consumer cannot trade with them asymmetrically. Intuitively, if the consumer traded with firm 1 *more often*, i.e., $\pi_1 > \pi_2 > 0$, it would charge higher equilibrium prices. However, this would induce the consumer to trade *less often* with firm 1, a contradiction. Thus, any equilibrium assessment where firms actively compete must feature a consumer's symmetric recommendation strategy. In turn, this implies symmetric equilibrium play from the firms. The following result characterizes the consumer's behavior in any symmetric equilibrium.

Corollary 1. *Let μ be consistent with symmetric strategy profile σ and β be symmetric. Then, β is an RVP best response if and only if for every $v \in V$ and $x_1, x_2 \geq 0$, we have*

$$\beta_i(v, x_1, x_2) = \frac{\pi \cdot e^{\frac{v-x_i}{k}}}{\pi \cdot \sum_{j=1,2} e^{\frac{v-x_j}{k}} + 1 - 2\pi}, \ (i \in I) \tag{5}$$

where

(i) $\pi = \mathbb{E}_\mu [\beta_i] \in [0, 1/2]$ *for every* $i \in I$.

Moreover, exactly one of the following statements is true.

(ii) $\pi = 0$, *and* $\mathbb{E}_\mu \left[e^{\frac{v-x_i}{k}}\right] \leq 1$ *for every* $i \in I$.

(iii) $\pi = 1/2$, *and* $\mathbb{E}_\mu \left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \leq 1/2$.

(iv) $\pi \in (0, 1/2)$, $\mathbb{E}_\mu \left[e^{\frac{v-x_i}{k}}\right] \geq 1$ *for every* $i \in I$, *and* $\mathbb{E}_\mu \left[\left(e^{\frac{v-x_1}{k}} + e^{\frac{v-x_2}{k}}\right)^{-1}\right] \geq 1/2$.

Corollary 1 is a direct specialization of Lemma 1 to the symmetric case. Equation (5) is the symmetric counterpart of equation (2), while points (*i*), (*ii*), (*iii*), and (*iv*) characterize the optimal symmetric trade engagement levels when μ is consistent with a symmetric strategy profile of the firms. In particular, as (*ii*), (*iii*), and (*iv*) show, the optimal trade engagement level depends on the expected exponential value of the gains from trade.

We call $(\psi, \xi) \in [0, 1] \times \mathbb{R}_+^V$ an *equilibrium outcome* whenever $\psi = \pi_1 + \pi_2$ is the consumer's overall trade engagement level, and $(\xi(v))_{v \in V}$ are the symmetric equilibrium offers accepted by the consumer on-path. We say that two assessments are outcome equivalent if they imply the same equilibrium outcome. Abusing notation, the equilibrium outcome associated with a no-trade equilibrium is $(0, \emptyset)$.

3 Benchmark: Collusion

To understand the equilibrium effects of competition, we formulate a benchmark case where the firms collude. Under colusion, the model remains identical except for the firms' incentives, which perfectly internalize each others' profits. For this reason, the industry surplus describes the preferences of each colluding firm:

$$\Pi^M := \sum_{j \in I} \beta_j(v, x_1, x_2) \cdot x_j, \quad \forall i \in I$$

where M stands for "Monopoly."

Our model with collusion is outcome equivalent to the ultimatum bargaining model of Ravid (2020), which we henceforth refer to as the *monopoly model* for simplicity of exposition.[13] The intuition is as follows. If the consumer trades with only one of the two firms in equilibrium, collusion is *de facto* equivalent to a monopoly. As the demand faced by one of the two firms is null,

[13]It turns out that colluding firms submit the same offer when both are active, which allows us to describe the equilibrium outcome under collusion by the overall trade probability and the on-path accepted offers.

the internalization effect plays no role, and the unique active firm has the same incentives as a monopolist. At the same time, the inactive firm's payoff is independent of its price. Therefore, this firm charges a high price in equilibrium to validate the consumer's decision not to engage in trade with them. On the other hand, if both firms are active, the consumer's attention strategy satisfies the multinomial logit of equation (2) adjusted for some $\pi_1, \pi_2 > 0$. In equilibrium, sellers' offers are symmetric and, more importantly, equal to the monopolist's offer of Ravid's (2020) model when facing the *aggregate demand.* Intuitively, when firms perfectly internalize each other's profits, they have no incentive to charge different prices. Moreover, if they charge the same price, they face the same aggregate demand as a monopolist. As a result, they act as if they were serving the consumer in a monopoly market, which implies that the analysis of Ravid (2020) applies verbatim to our collusion benchmark.

Let $k^* > 0$ be the unique solution to the following equation

$$\mathbb{E}_\lambda \left[e^{v/k^* - 1} \right] = 1. \tag{6}$$

The following result characterizes the main equilibrium predictions of the collusive trading equilibrium, i.e., the equilibrium under collusion in which the consumer trades with positive probability.

Theorem 1. *A collusive trading equilibrium outcome exists if and only if $k < k^*$. If a collusive trading equilibrium outcome exists, it is unique and equilibrium trade is inefficient. That is, $\pi_1 + \pi_2 < 1$.*

Theorem 1 emphasizes two main features of the collusion benchmark. First, trade cannot be sustained as an equilibrium outcome if attention costs are too high. Second, equilibrium trade is never efficient: The consumer's probability of buying the product is always smaller than 1. The elemental force at play for both results is that the consumer does not process enough information to sustain efficient trade. If information costs are too high, the consumer does not pay enough attention to prevent firms from overcharging, making it sub-optimal for her to trade with positive probability. To understand why equilibrium trade should be inefficient, suppose the consumer trades with probability one. Always accepting either offer is equivalent to never using the no-trade *outside option*, which, in an RVP equilibrium, implies that the consumer disregards learning about prices in absolute terms. As a result, colluding firms could coordinate on a simultaneous price increase, making equilibrium offers too unappealing to sustain trade.

Two remarks are in order. First, because firms' products are perfectly homogeneous, the equivalence between collusion and monopoly is straightforward without rational inattention. However, this equivalence is not immediate in the presence of information processing costs since colluding firms may use different prices to influence the consumer's attention. Nevertheless, this strategic manipulation of the consumer's attention does not bite in equilibrium: If firm i charges a higher price, the consumer optimally shifts her demand away from i, which makes it optimal for the cartel to lower i's price, a contradiction. Second, the results of this section do not depend on the particular interpretation of collusion we employ. As we show in Appendix C, our results hold whether the two firms set the prices together or independently as long as they internalize each other's profits.

4 Competition

To study the impact of competition, we first identify equilibria in which competitive forces are present. Robustness to vanishing perturbations implies that competition delivers a finite multiplicity of equilibrium outcomes. First, there always exists a trivial no-trade equilibrium outcome, where firms overcharge the consumer, who consequently does not trade. Second, there is a class of equilibria where the consumer only trades with one firm. This class is outcome equivalent to all collusive trading equilibria. Finally, there is an equilibrium where the consumer trades with both firms. Since this is the unique equilibrium class where firms actively compete on-path, we call this equilibrium the *competitive trading equilibrium.* In this section, we characterize its properties and describe the equilibrium effects of competition.

4.1 The Competitive Trading Equilibrium

As argued in Appendix B, every competitive trading equilibrium must be symmetric. Given the consumer's best reply of Corollary 1, firms behave as if they are facing a symmetric downward-sloping multinomial logit demand. The following lemma characterizes the firms' equilibrium strategies.

Lemma 2. *Suppose (μ, σ, β) is a competitive trading equilibrium. For every $v \in V$, each seller $i \in I$ plays a symmetric pure strategy $\sigma_i(\cdot|v) = \delta_{x(v)}$ given by*

$$x(v) = k \cdot (1 + \phi(v)) \tag{7}$$

where $\phi = \phi(v)$ is the unique solution to

$$\left(1 + e^{\phi} \cdot \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi = 1. \tag{8}$$

Lemma 2 follows from the fact that firms are not facing a perfectly elastic demand, even though their products are homogeneous. Notice that the consumer behaves as if products were differentiated because the information she gets is noisy. It is too costly to process information about ex-post gains of trade perfectly and, as a result, the consumer is not certain about the best offer and makes mistakes. This mechanism explains why, according to equation (7), the firms can charge prices above marginal costs in equilibrium.

The relationship between the consumer's trade engagement level and firms' offers is crucial. The function ϕ captures the price-setting incentives of the firms. Note that for fixed v and k, equation (8) shows that $\phi(v)$ is increasing in the consumer's trade engagement level π. This shows that π is at the heart of the equilibrium analysis of our model. If firms submit appealing offers to the consumer, the consumer chooses a high trade engagement level, in line with equation (4). At the same time, if the consumer engages more in trade, demand expands and elasticity declines. As a result, the firms submit worse offers.

The following theorem characterizes the existence and uniqueness of competitive trading equilibria by identifying the region of attention costs where competition sustains trade in equilibrium. Recall from equation (6) that k^* is defined as the unique solution to $\mathbb{E}_\lambda \left[e^{v/k^*-1}\right] = 1$.

Theorem 2. *A competitive trading equilibrium exists if and only if $k < k^*$. If a competitive trading equilibrium exists, it is unique.*

Competition can not sustain trade when attention costs are too high. The consumer is unwilling to process any information, implying that demand does not change with firms' offers. As a result, firms overcharge the consumer, leading to a breakdown of trade. Conversely, a competitive trading equilibrium exists if attention costs are moderately low. The consumer is willing to process some information to find the best offer. Consequently, the firms face downward-sloping demand curves and make appealing offers to the consumer.

The theorem also shows that whenever a competitive trading equilibrium exists, it is unique. For instance, suppose there is a second competitive trading equilibrium in which the consumer's overall trade engagement level is higher. Due to this expansion in demand, firms' marginal revenue is higher everywhere, prompting firms to make less appealing offers compared to the original equilibrium. This induces the consumer to *reduce* the overall trade engagement level, a contradiction.

An immediate implication of Theorem 2 is that competition cannot sustain trade when collusion could not: In both cases, a trading equilibrium exists if and only if $k < k^*$. When the attention costs exceed k^*, the consumer does not process any information. Since this includes information about which offer is better, the downward pressure on prices induced by competition vanishes when $k > k^*$, implying that competition cannot prevent a breakdown in trade.

4.2 Equilibrium Effects of Competition

We investigate the impact of competition in markets with rational inattention by comparing the competitive and the collusive trading equilibrium outcomes. Denote by $2\pi^C := \pi_1^C + \pi_2^C$ and $\pi^M := \pi_1^M + \pi_2^M$ the overall trade engagement level under competition and collusion, respectively.

The attention effect. Proposition 1 describes a novel effect of competition that we name the *attention effect*: The consumer's endogenous demand expands when firms compete. In other words, for any level of attention cost, the overall trade engagement level under competition is strictly higher than under collusion. Competition thus alleviates the efficiency losses that occur under collusion, where trade is always inefficient due to costly information processing.

Proposition 1. *The consumer engages in trade more often under competition than under collusion: For any $k \in (0, k^*)$, $0 < \pi^M < 2\pi^C \leq 1$.*

Intuition is as follows. Suppose both competing firms face half the demand faced by a monopolist. The resulting offers would be more favorable to the consumer due to the *pricing effect*, i.e., the fact that competing firms have stronger incentives to charge low prices since they do not internalize each other's profits. However, the consumer would trade more often at these lower prices. Therefore, the resulting equilibrium trade probabilities have to satisfy $\pi^M < 2\pi^C$.

As Proposition 2 shows, an equilibrium with efficient trade, which we call a *sure-trade equilibrium*, exists when the consumer's unit attention cost is relatively low. To see why, suppose trade is efficient, i.e., the consumer trades with each firm with equal probability $\pi = 0.5$. By Lemma 2, this implies that firms charge a price of $x(v) = 2k$ for all $v \in V$. From Corollary 1, we know that, under this configuration of prices, the consumer wants to trade with certainty if and only if $\mathbb{E}_\lambda\left[e^{2-v/k}\right] \leq 1$. Thus, let $\bar{k} > 0$ be the unique solution to

$$\mathbb{E}_\lambda\left[e^{2-v/\bar{k}}\right] = 1.$$

Notice that $\bar{k}$ is lower than the threshold characterizing the existence of a competitive trading equilibrium, i.e., $\bar{k} < k^*$.[14]

Proposition 2. *Under competition, a sure-trade equilibrium exists if and only if $k \leq \bar{k}$.*

Notice that when trade is efficient, the consumer demand does not react to offers in absolute terms but only to price differences.[15] Under collusion, this attention strategy does not prevent firms from submitting unreasonable offers. In contrast, as competing firms have incentives to undercut the competitor's offer, this strategy is effective in disciplining prices under competition. Moreover, equilibrium prices strictly increase with k. Intuitively, the higher the unit attention cost k, the less the consumer reacts to price changes. This effect leads firms to charge a higher equilibrium price, which explains the existence of the threshold $\bar{k} > 0$ that characterizes equilibrium trade efficiency. If the consumer's attention costs exceed $\bar{k}$, the constant equilibrium price becomes too high relative to the expected quality, and buying with certainty is not optimal for the consumer.

Observe that, in a sure-trade equilibrium attention costs are equal to zero and, therefore, the economy achieves first-best social welfare. When $k \leq \bar{k}$, the consumer uniformly randomizes between the firms' offers and does not pay information processing costs on path. As a result, the (ex-post) social welfare of the economy is maximal and equals v. This prediction contrasts with the collusion benchmark: When the firms collude, the consumer must resort to the no-trade outside option to discipline firms' pricing strategies, which results in welfare losses.

Figure 1 illustrates Proposition 1 and 2. It displays the equilibrium trade engagement level under competition and collusion as a function of $k \in (0, k^*)$. For values of $k \leq \bar{k}$, the unique competitive trading equilibrium features sure trade, i.e., $2\pi^C = 1$. For $k > \bar{k}$, $2\pi^C$ is decreasing in k, but as Proposition 1 shows, it is always strictly above π^M.

The following corollary describes the consumer's observable behavior under competition.

[14] To see why, notice that $\mathbb{E}_\lambda\left[e^{2-v/k^*}\right] > \mathbb{E}_\lambda\left[e^{1-v/k^*}\right] = \mathbb{E}_\lambda\left[\frac{1}{e^{v/k^*-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda\left[e^{v/k^*-1}\right]} = 1$. Therefore, $\bar{k} < k^*$.

[15] From equation (5), if $\pi = 1/2$, the consumer's symmetric best response becomes

$$\beta_i(v, x_1, x_2) = \frac{1}{1 + e^{\frac{x_i - x_j}{k}}}, \quad \forall i \in I.$$

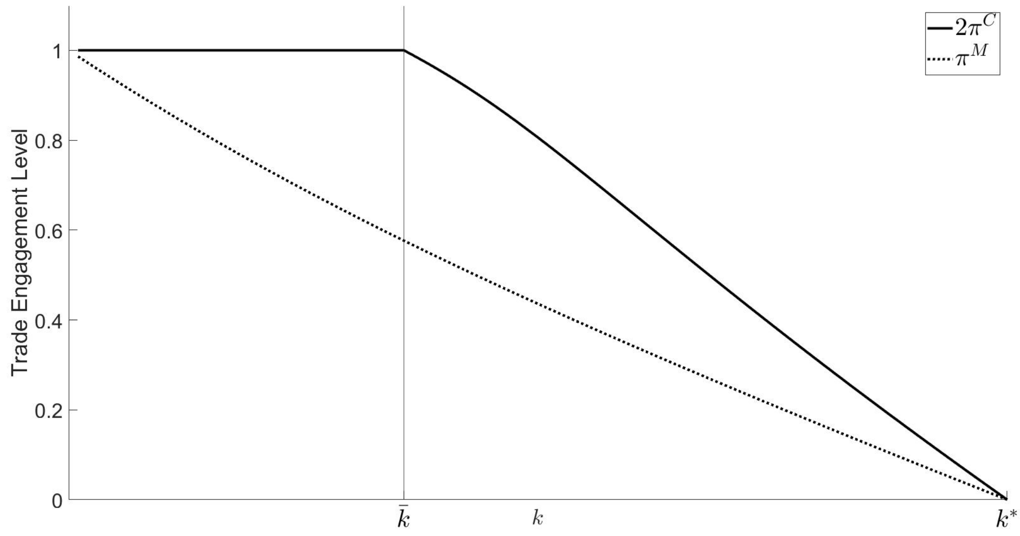


Figure 1: Overall trade engagement level in the competitive and collusive trading equilibrium with a binary quality distribution.

Corollary 2. *In a competitive trading equilibrium, the consumer's attention strategy satisfies*

$$\beta_i(v, x(v), x(v)) = \begin{cases} 1/2 & \textit{if } k \leq \bar{k} \\ 1 - k/x(v) & \textit{if } k \in (\bar{k}, k^*) \end{cases} \quad \textit{for all } v \in V.$$

When the cost of information processing is small, the equilibrium resembles the standard Bertrand competition outcome. As illustrated in Figures 1 and 2, the consumer buys with certainty at a price that does not vary with quality when $k \in (0, \bar{k})$. However, as the consumer's information processing costs grant the industry positive market power, firms successfully submit offers above marginal cost. When k is above $\bar{k}$, we observe an equilibrium outcome resembling Ravid's (2020) monopoly analysis: Trade is inefficient and firms' offers depend on the product's quality (see Figure 2). In the limit, i.e., as $k \uparrow k^*$, the industry's behavior is identical in both settings because the consumer's information processing about relative prices vanishes.[16] This explains why Theorem 2 shows that the range of attention costs that support trade is the same under competition and collusion.

Industry surplus. Proposition 1 and 2 show that trade efficiency increases with competition, which implies that the sum of industry surplus and consumer trade surplus, i.e., without considering attention costs, increases. The remainder of this section studies which entity benefits from competition. Without information processing costs, i.e., at $k = 0$, competition benefits the consumer at the expense of the firms. Since the equilibrium outcomes are continuous in k, this also holds for small information processing costs.

[16] As the overall trade engagement level decreases, the likelihood that the consumer considers the competitor's offer vanishes, and the firms focus on prevailing over the no-trade outside option: Each competing firm's objective approximates the one of a colluding firm.

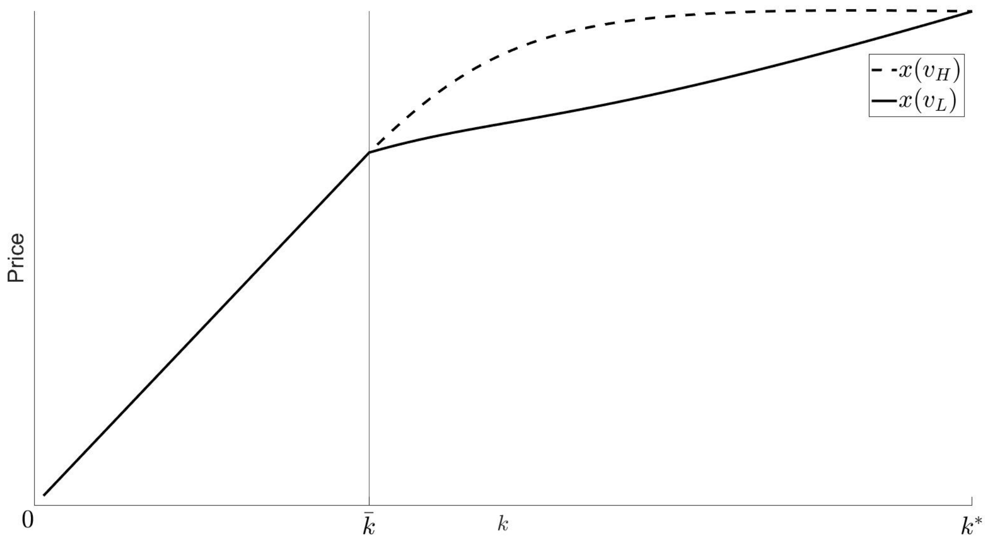


Figure 2: Equilibrium prices in the competitive trading equilibrium with a binary quality distribution.

In contrast, Theorem 3 shows that when attention costs are high enough,[17] the aggregate industry surplus under competition is *higher* than under collusion. For such costs, it adds to Proposition 1 that the positive effect competition has on demand *dominates* the negative effect it has on prices: The attention effect prevails over the pricing effect.

For every attention cost $k \in (0, k^*)$, let $\Pi^M(k)$ and $2\Pi^C(k)$ be the aggregate industry surplus in the collusive and the competitive trading equilibrium, respectively.

Theorem 3. *There exists $\hat{k} \in (0, k^*)$ such that the industry surplus is higher under competition than under collusion for all k between $\hat{k}$ and k^*, i.e., $2\Pi^C(k) > \Pi^M(k) > 0$ for all $k \in (\hat{k}, k^*)$.*

Figure 3 illustrates the content of the theorem for one specific distribution of v. The sum of the competitors' profits is larger than the collusive profits for information costs in the interval $(\hat{k}, k^*)$. Theorem 3 states that such a region exists for any distribution of v.

The proof of Theorem 3 revolves around the use of *de L'Hopital rule* to prove that

$$\lim_{k \uparrow k^*} \frac{\pi^C(k)}{\pi^M(k)} > \frac{1}{2},$$

even though trading probabilities under competition and collusion are both converging to zero as $k \uparrow k^*$. This fact implies that, as k grows large, (i) the behavior of each firm in the competitive trading equilibrium approximates the equilibrium behavior of the colluding firms and, at the same time, (ii) each firm faces strictly more than half the aggregate equilibrium demand under collusion. Given points (i) and (ii), we conclude that for k large enough, the firms' total profits are strictly higher under competition than under collusion, proving the statement.

The driver of our main result is that the consumer rationally chooses the amount of information

[17]With high enough, we mean high within the range of costs that support trade in equilibrium $(0, k^*)$.

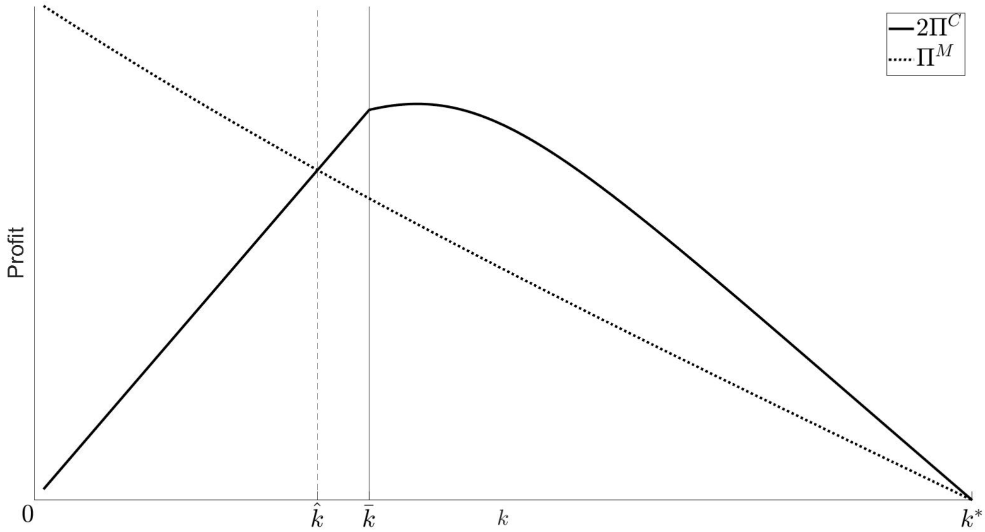


Figure 3: Industry surplus in the competitive and collusive trading equilibrium with a binary quality distribution.

to process and, in particular, the optimal trade engagement level π. If the consumer chose the same attention strategy regardless of the market structure, the firms would be worse off under competition, as in any standard competition model with fixed demand. However, in our model, competition acts as a commitment device not to overcharge the inattentive consumer. As the attention effect shows, competitive pricing supports a higher trade engagement level in equilibrium. When attention costs are high, this outward shift in demand dominates the impact on prices, leading to an increase in total profits.

Consumer surplus. Under collusion, rational inattention allows the consumer to obtain a positive utility in situations where costless information does not. Competition reverses this logic: When multiple firms are active, the consumer's surplus is smaller than the total surplus for any $k > 0$, which implies that it is smaller than the surplus obtained in the frictionless benchmark ($k = 0$).

In general, characterizing how consumer surplus reacts to competition for a fixed unit cost of information processing is difficult.[18] Competition puts downward pressure on prices, making the consumer better off, but at the same time, leads the consumer to engage in trade more often, allowing firms to charge higher prices. Moreover, whether competition makes information processing less or more costly is ambiguous in equilibrium unless $k \leq \bar{k}$.

Lemma 3 shows that if the consumer is uncertain about the quality of the product, she always benefits from competition whenever expected market offers decrease. Denote by U^m the

[18]Consumer surplus is easy to characterize only when quality is commonly known, i.e., V is a singleton. Suppose $V = \{v_o\}$ with $v_o > 0$. Under collusion, offers equal v_o, implying that the consumer has no trading surplus. Under competition, Lemma 2 implies that offers equal v_o if $k > \bar{k} = v_o/2$, and $2k \leq v_o$ otherwise. Additionally, since firms use pure strategies and quality is known, the consumer does not incur any information processing costs on path in both cases. Therefore, the consumer benefits strictly from competition when $k < v_o/2$ and receives the same surplus otherwise.

consumer surplus and by x^m the firms' offers when the market structure is $m \in \{C, M\}$.

Lemma 3. *If quality is uncertain, the consumer surplus is strictly higher under competition whenever, in expectation, competitive prices are lower than under collusion: If* $|V| > 1$*, then* $\mathbb{E}_\lambda[x^C(v)] \leq \mathbb{E}_\lambda[x^M(v)]$ *implies* $\mathbb{E}[U^C] > \mathbb{E}[U^M]$.

To prove Lemma 3, we show that competition implies a more dispersed distribution of the consumer's gains from trade than collusion. Since rationally inattentive agents enjoy risk in (ex-post) utils, we conclude the argument by invoking well-established results in risk theory (Meyer, 1977).

We use Lemma 3 to identify two sufficient conditions under which competition unambiguously helps the consumer: Efficient trade, $k \leq \bar{k}$, and high information processing costs, $k \uparrow k^*$. In light of Theorem 3, this last result implies that when information processing costs are high enough, competition can (Pareto) improve the economic situation of both sides of the market: Industry and consumer altogether.

Proposition 3. *There exists* $\bar{k}' \in [\bar{k}, k^*)$ *such that the expected consumer surplus weakly increases with competition whenever* $k \in (0, \bar{k}] \cup (\bar{k}', k^*)$*. Moreover, if* $\max\{\bar{k}', \hat{k}\} < k < k^*$*, competition increases surplus for both sides of the economy.*

4.3 More than Two Firms

In this subsection, we extend the analysis of Section 4.2 to the presence of multiple firms.[19] When firms perfectly internalize each other profits, they jointly maximize the industry surplus, implying that each firm effectively behaves as if facing the consumer in a monopoly market. As a result, the equilibrium predictions under collusion are independent of the number of firms.

The number of active firms matters under competition, since it impacts the strength of both the pricing and attention effect. We proceed to show that equilibrium trade efficiency expands, implying that the region of attention costs where competition sustains efficient trade becomes larger as more firms compete. Furthermore, the attention effect still dominates the pricing effect when k grows large, ensuring that a version of Theorem 3 holds for arbitrary numbers of firms.

We call *competitive* a trading equilibrium where *all* firms are active. First, we characterize the firms' optimal behavior in equilibrium.

Lemma 4. *Suppose* (μ, σ, β) *is a competitive trading equilibrium with* $N \geq 2$ *firms. For every* $v \in V$*, each seller* $i \in I$ *plays a symmetric pure strategy* $\sigma_i(\cdot|v) = \delta_{x(v,N)}$ *given by*

$$x(v, N) = k \cdot (1 + \phi(v, N))$$

where $\phi = \phi(v, N)$ *is the unique solution to*

$$\left((N-1) + e^{\phi} \cdot \frac{1 - N\pi^N}{\pi^N e^{\frac{v-k}{k}}} \right) \phi = 1. \tag{9}$$

[19] Since the results of Section 2.2 naturally extends to $N \geq 2$ firms, we omit to discuss the consumer's optimal behavior. In particular, RVP characterizes the consumer's best response *everywhere* in the state space $V \times \mathbb{R}_+^N$ by a multinomial logit formula adjusted for the parameters $(\pi_1, ..., \pi_N)$.

Equation (9) is the counterpart to equation (8) with N firms, where $\pi^N \in (0, 1/N]$ represents the symmetric trade engagement level of the consumer with each firm in equilibrium.[20] It shows that for any fixed overall trade engagement level $\bar{\pi}$ of the consumer, the firm's undercutting incentives intensify as the number of active firms increases: The pricing effect becomes stronger as N gets large. To see why, fix $N, M \geq 2$ and suppose that $N\pi^N = M\pi^M = \bar{\pi} \in (0, 1]$. Equation (9) implies that $\phi(v, N) < \phi(v, M)$ if and only if $N > M$.

The following result describes the properties of the competitive trading equilibria with more than two firms. Let $\bar{k}(N) > 0$ be the unique solution to $\mathbb{E}\left[e^{\frac{N}{N-1} - v/\bar{k}(N)}\right] = 1$, and denote by $\Pi^C(N)$ the expected profit of each firm in the competitive trading equilibrium with N active firms.

Proposition 4. *Suppose there are $N \geq 2$ firms.*

(i) *A competitive trading equilibrium exists if and only if $k < k^*$. If a competitive trading equilibrium exists, it is unique.*

(ii) *The consumer's overall trade engagement level in the competitive trading equilibrium increases with N.*

(iii) *Under competition, a sure-trade equilibrium exists if and only if $k \leq \bar{k}(N)$.*

(iv) *There always exists a region of attention costs such that adding competing firms increases industry surplus. Formally, let $N > M \geq 2$. There exists $\hat{k} \in (\bar{k}(M), k^*)$ such that $N \cdot \Pi^C(N) > M \cdot \Pi^C(M)$ for all $k \in (\hat{k}, k^*)$.*

The existence and uniqueness properties of the competitive trading equilibria do not change with the number of active firms. For every N, by Lemma 4, the function $\phi(v, N)$ strictly increases with π, which implies that at most one competitive trading equilibrium exists.[21] Moreover, as $\pi \downarrow 0$, the function $\phi(v, N)$ converges to 0, irrespective of the value of N. This implies that the pricing effect vanishes when it becomes prohibitively costly for the consumer to process information about offers. As a result, competition with $N \geq 2$ firms supports equilibrium trade only when $k < k^*$, like under collusion.

Although the presence of N firms is inconsequential for the existence and uniqueness of competitive trading equilibria, varying the number of active firms affects trade efficiency. The intuition behind parts (*ii*) and (*iii*) of Proposition 4 follows the one of Propositions 1 and 2. As N grows, firms have stronger incentives to undercut their competitors' offers. Anticipating this, the consumer engages in trade more often, implying that the overall trade engagement level increases with N. In other words, the attention effect becomes stronger as the number of active firms increases. As a result, $\bar{k}(N)$ strictly increases with N, and sure-trade becomes easier to sustain when N grows. This feature implies that introducing an additional competing firm restores trade efficiency for a range of attention costs.

Part (*iii*) of Proposition 4 further implies that in the limit, as the number of firms grows, i.e.,

[20] As in the duopoly case, a competitive trading equilibrium with $N \geq 2$ firms must be symmetric.

[21] Otherwise, the equilibrium where the consumer engages in trade more often is the one where, on-path, the terms of trade are worse for the consumer, a contradiction.

$N \uparrow \infty$, competition does not guarantee sure-trade if the quality of the product is uncertain. Formally, if $|V| > 1$, $\lim_{N\uparrow\infty} \bar{k}(N) := \bar{k}(\infty) < k^*$. If prices exceed average quality, the consumer is not willing to buy with certainty. When quality is uncertain, this would be the case in a sure-trade equilibrium for k close to k^*, regardless of the number of competitors.

In Section 4.2, we discussed how competition acts as a commitment device that supports higher equilibrium prices through an expansion in demand. Part *(iv)* of Proposition 4 adds to that discussion: As $k \uparrow k^*$, the attention effect is more substantial the larger the number of active firms. This implies that for any number of competing firms N, there exists a region of attention costs where adding a new competitor strictly increases the total surplus of the industry.

5 Discussion

We conclude by discussing the implication of our main results on the common ownership hypothesis and considering an extension with obfuscation. We also discuss the role of exogenous uncertainty and entropy-based information costs in our analysis.

Common ownership. As shown by Backus, Conlon, and Sinkinson (2021b), there has been a dramatic growth in common ownership in the US stock market over the last forty years. The main driver is the increased concentration in the investment-fund industry and the diffusion of new financial instruments facilitating portfolio diversification. Some economists regard this phenomenon as a threat to competition in important American industries (e.g., Azar, 2011; Azar, Schmalz, and Tecu, 2018; Azar, Raina, and Schmalz, 2022). Others, inspired by these early contributions, challenge the anti-competitive effects of common ownership previously found (e.g., Kennedy et al., 2017; Dennis, Gerardi, and Schenone, 2022).[22]

At the heart of the debate is the *common ownership hypothesis*, a theory advanced by Rotemberg (1984) stating that firms have fewer incentives to compete if they have common owners. If managers act to maximize their value to investors, the presence of common owners may induce firms to internalize competitors' profits, which implies tacit collusion. Skeptics criticize two aspects of the theory. First, they argue that the common ownership hypothesis requires that shareholders have perfect control over firms' business strategies, a presumption which has been empirically challenged.[23] Second, they criticize that the focus on a single market misrepresents the nature of portfolio diversification, centered around multi-market interaction. Spillover effects between markets along a supply chain may imply that anti-competitive pricing does not necessarily benefit common shareholders.[24]

Our framework enriches the theoretical debate around the common ownership hypothesis by providing a third reason why anti-competitive concerns may not be well-grounded. In the presence of rationally inattentive consumers, we show that internalizing competitors' profits may produce counterproductive effects on value-oriented common investors. To illustrate this

[22] Refer to Backus, Conlon, and Sinkinson (2019) for an extensive survey.

[23] See the March 2017 viewpoint of BlackRock: *Index Investing and Common Ownership Theories.* However, Antón et al. (2018) show that common ownership can still affect economic outcomes even if the common owners do not have perfect control.

[24] See *Common Sense Doesn't Support Common Ownership Hypothesis*, January 2022 ICI viewpoint.

point, consider the following thought experiment: Suppose one investor (e.g., a holding) owns both firms in a duopoly market and controls the management incentives of the firms. That is, they can generate incentives to compete by keeping the firms' management separate and paying bonuses based on individual profits or to collude by appointing a joint management that receives payment indexed to the holding's performance.[25] Our model predicts that if attention costs are high enough, the investor maximizes the value of their portfolio when the two firms compete. Therefore, the value-maximizing common investor would design the managers' incentives as if the firms were owned separately.

In Appendix G, we extend the argument above to cases in which common investors only own partial shares of the two firms. Following Azar and Vives (2021), we allow firms to *partially and symmetrically* internalize the competitor's profit. In particular, firm i's incentives are

$$\Pi_i^{\rho} := \beta_i(v, x_1, x_2) \cdot x_i + \rho \cdot (\beta_{-i}(v, x_1, x_2) \cdot x_{-i}),$$

where $\rho \in [0, 1]$ denotes the degree of profits' internalization. When $\rho \uparrow 1$, the industry approximates collusion, and the economy displays the same features of Ravid (2020), i.e., trade failure and inefficiencies. On the other hand, when $\rho \downarrow 0$, we observe the implications of perfect competition discussed in section 4.2, i.e., sure-trade and the attention effect. Our main result is replicated in this setting: If attention costs are high enough, firms obtain higher profits the less they internalize each other's profits. In markets with common ownership, consumer inattention incentivizes firms to remain competitive.

Obfuscation. As the attention cost governs how easily the consumer understands market offers, one can also interpret it as a measure of *industry complexity*. In our model, we assume that k is exogenous to isolate the equilibrium effects of competition under rational inattention. However, in many settings, firms can interfere with how consumers perceive and process information about their offers. For example, by purposely shrouding or complexly framing specific product features, firms could hinder competition and retain market power. Moreover, it is well established that such strategic obfuscation may increase with competition and benefit producers at the expense of the consumers (Carlin, 2009; Chioveanu and Zhou, 2013).

We replicate some of these findings. In Appendix D, we consider an extension of our model by defining an ex-ante stage where the firms directly influence the consumer's information processing cost. We show that, even if strategic obfuscation is feasible, competing firms can still generate higher expected profits than under collusion as long as the industry is sufficiently complex.

The role of exogenous uncertainty. Our model assumes that marginal cost is normalized to zero while quality is potentially random. This assumption enables us to directly compare our results to the ultimatum bargaining model of Ravid (2020), which is equivalent to our collusion benchmark. In Appendix E, we explain that qualitatively identical findings emerge in a model where the consumer knows the product's quality but is uncertain about the firms'

[25] This is in line with evidence from Antón et al. (2018) that common owners influence managerial incentives when this benefits them.

marginal cost. This alternative specification is particularly compelling to analyze economic situations where primary production inputs, e.g., energy, crops, and logistics, exhibit rapid and significant price volatility. Indeed, under these circumstances, consumers may be unsure about the firms' current pricing. For instance, this alternative model fits the utility service market well. Consumer prices move strongly with commodity prices, but consumers might be uncertain about this relationship's strength. Also, utility service contracts can be rather complex: They can feature indexed costs, sign-up bonuses, loyalty rebates, and various switching costs. Thus, even though the quality of the product is known, the consumer still has to process costly information to find the best offer.

The model with random marginal costs highlights that our findings do not depend on the specific source of uncertainty our model features. Instead, the critical assumption is that the consumer pays costly attention to a payoff-relevant feature of the offers chosen endogenously by the firms.[26] In particular, all our main results also hold in a model without uncertainty. To see this, suppose that v is known, i.e., V is a singleton. Since the firms use pure strategies, the consumer does not process information on-path. Nonetheless, RVP requires that the consumer would optimally respond to any small deviations of the firms, uniquely pinning down her optimal attention strategy. In other words, even if the consumer does not process information in equilibrium, information processing about price deviations endogenously determines the demand and, therefore, the equilibrium outcome.

We include random quality in the model for added realism. Typically, consumers are not fully aware of a product's details or a firm's production technology and they actively process information about market offers. Adding uncertainty about the product quality allows us to capture these features while also fostering comparability with the ultimatum bargaining model of Ravid (2020).

Beyond entropy. The intuition behind our main findings can be explained without discussing any specific information cost function. Suppose consumers choose how much attention to allocate to a purchasing decision. In that case, the optimal attention strategy may depend on the firms' incentives to compete, making the consumer's demand endogenous to the market structure. In particular, more competitive markets may induce the consumer to engage more often in trade. If this attention effect is large, it can lead to scenarios where competing firms achieve higher profits than colluding firms.

Entropy-based information costs are tractable and allow us to make sharp economic predictions. However, they are not necessary for our results as we show in Appendix F, where we extend our analysis to more general cost functions. In particular, we show that the attention effect, i.e., the consumer's demand expands when firms compete (Proposition 1), holds for a class of other cost functions. Furthermore, we discuss an example with quadratic costs where the attention effect dominates the pricing effect, leading to higher profits under competition than under collusion.

[26] In fact, our results also extend to an adaptation of the model with a fixed price and the firms choosing the product's quality at an increasing cost, which underlines the relevant ingredients of the model.

References

Antón, Miguel et al. (2018). "Common ownership, competition, and top management incentives".

Armstrong, Mark and John Vickers (2022). "Patterns of competitive interaction". *Econometrica* 90(1), pp. 153–191.

Azar, José (2011). "A new look at oligopoly: Implicit collusion through portfolio diversification". PhD thesis. Princeton University.

Azar, José, Sahil Raina, and Martin Schmalz (2022). "Ultimate ownership and bank competition". *Financial Management* 51(1), pp. 227–269.

Azar, José, Martin Schmalz, and Isabel Tecu (2018). "Anticompetitive effects of common ownership". *The Journal of Finance* 73(4), pp. 1513–1565.

Azar, José and Xavier Vives (2021). "General equilibrium oligopoly and ownership structure". *Econometrica* 89(3), pp. 999–1048.

Backus, Matthew, Christopher Conlon, and Michael Sinkinson (2019). "The common ownership hypothesis: Theory and evidence". *Economic Studies at Brookings*, pp. 1980–2017.

Backus, Matthew, Christopher Conlon, and Michael Sinkinson (2021a). "Common ownership and competition in the ready-to-eat cereal industry". *National Bureau of Economic Research.*

Backus, Matthew, Christopher Conlon, and Michael Sinkinson (2021b). "Common ownership in America: 1980–2017". *American Economic Journal: Microeconomics* 13(3), pp. 273–308.

Bloedel, Alexander W and Ilya R Segal (2021). "Persuating a Rational Inattentive Agent". *Working Paper.*

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer (2016). "Competition for attention". *Review of Economic Studies* 83(2), pp. 481–513.

Boyacı, Tamer and Yalçın Akçay (2018). "Pricing when customers have limited attention". *Management Science* 64(7), pp. 2995–3014.

Burdett, Kenneth and Kenneth L Judd (1983). "Equilibrium price dispersion". *Econometrica*, pp. 955–969.

Cachon, Gérard P, Christian Terwiesch, and Yi Xu (2008). "On the effects of consumer search and firm entry in a multiproduct competitive market". *Marketing Science* 27(3), pp. 461–473.

Carlin, Bruce I (2009). "Strategic price complexity in retail financial markets". *Journal of Financial Economics* 91(3), pp. 278–287.

Chioveanu, Ioana and Jidong Zhou (2013). “Price competition with consumer confusion”. *Management Science* 59(11), pp. 2450–2469.

Cyert, Richard M and Morris H DeGroot (1973). “An analysis of cooperation and learning in a duopoly context”. *American Economic Review* 63(1), pp. 24–37.

De Clippel, Geoffroy, Kfir Eliaz, and Kareen Rozen (2014). “Competing for consumer inattention”. *Journal of Political Economy* 122(6), pp. 1203–1234.

Dennis, Patrick, Kristopher Gerardi, and Carola Schenone (2022). “Common ownership does not have anticompetitive effects in the airline industry”. *The Journal of Finance* 77(5), pp. 2765–2798.

Diamond, Peter A (1971). “A model of price adjustment”. *Journal of Economic Theory* 3(2), pp. 156–168.

Edgeworth, Francis Ysidro (1881). “Mathematical psychics: An essay on the application of mathematics to the moral sciences”. 10. CK Paul.

Flynn, Joel P and Karthik Sastry (2021). “Strategic mistakes”. *Available at SSRN 3663481.*

Fosgerau, Mogens et al. (2020). “Discrete choice and rational inattention: A general equivalence result”. *International Economic Review* 61(4), pp. 1569–1589.

Gabaix, Xavier and David Laibson (2006). “Shrouded attributes, consumer myopia, and information suppression in competitive markets”. *The Quarterly Journal of Economics* 121(2), pp. 505–540.

Hefti, Andreas (2018). “Limited attention, competition and welfare”. *Journal of Economic Theory* 178, pp. 318–359.

Heidhues, Paul, Johannes Johnen, and Botond Kőszegi (2021). “Browsing versus studying: A pro-market case for regulation”. *The Review of Economic Studies* 88(2), pp. 708–729.

Janssen, Aljoscha and Johannes Kasinger (2022). “Obfuscation and Rational Inattention”. *Working Paper.*

Kennedy, Pauline et al. (2017). “The competitive effects of common ownership: Economic foundations and empirical evidence”. *Available at SSRN 3008331.*

Martin, Daniel (2017). “Strategic pricing with rational inattention to quality”. *Games and Economic Behavior* 104, pp. 131–145.

Matějka, Filip (2015). “Rigid pricing and rationally inattentive consumer”. *Journal of Economic Theory* 158, pp. 656–678.

Matějka, Filip and Alisdair McKay (2012). “Simple market equilibria with rationally inattentive consumers”. *American Economic Review* 102(3), pp. 24–29.

Matějka, Filip and Alisdair McKay (2015). “Rational inattention to discrete choices: A new foundation for the multinomial logit model”. *American Economic Review* 105(1), pp. 272–98.

Mensch, Jeffrey and Doron Ravid (2022). “Monopoly, Product Quality, and Flexible Learning”. *arXiv preprint arXiv:2202.09985.*

Meyer, Jack (1977). “Second degree stochastic dominance with respect to a function”. *International Economic Review*, pp. 477–487.

Milgrom, Paul and John Roberts (1990). “Rationalizability, learning, and equilibrium in games with strategic complementarities”. *Econometrica*, pp. 1255–1277.

Piccione, Michele and Ran Spiegler (2012). “Price competition under limited comparability”. *Quarterly Journal of Economics* 127(1), pp. 97–135.

Ravid, Doron (2020). “Ultimatum bargaining with rational inattention”. *American Economic Review* 110(9), pp. 2948–63.

Ravid, Doron, Anne-Katrin Roesler, and Balázs Szentes (2022). “Learning before trading: on the inefficiency of ignoring free information”. *Journal of Political Economy* 130(2), pp. 346–387.

Rotemberg, Julio (1984). “Financial transaction costs and industrial performance”. Cambridge, Massachusetts Institute of Technology.

Selten, Reinhard (1975). “Reexamination of the perfectness concept for equilibrium points in extensive games”. *International Journal of Game Theory* 4, pp. 25–55.

Sims, Christopher A (2003). “Implications of rational inattention”. *Journal of Monetary Economics* 50(3), pp. 665–690.

Spiegler, Ran (2006). “Competition over agents with boundedly rational expectations”. *Theoretical Economics* 1(2), pp. 207–231.

Spiegler, Ran (2016). “Choice complexity and market competition”. *Annual Review of Economics* 8, pp. 1–25.

Stahl, Dale O (1989). “Oligopolistic pricing with sequential consumer search”. *American Economic Review*, pp. 700–712.

Thereze, João (2022a). “Adverse Selection and Endogenous Information”. *Working Paper.*

Thereze, João (2022b). “Screening when Information is Costly”. *Working Paper.*

Varian, Hal R (1980). “A model of sales”. *American Economic Review* 70(4), pp. 651–659.

Wu, Liangjie (2022). “Partially Directed Search in the Labor Market”. *Working Paper.*

Yang, Ming (2019). “Optimality of debt under flexible information acquisition”. *Review of Economic Studies* 87(1), pp. 487–536.

Appendix

The Appendix is structured as follows: Appendix A includes most of the omitted proofs from the main text. Appendix B pertains to our model under competition. It shows that the restriction to pure strategies for the firms is without loss in any competitive trading equilibrium, and that competitive equilibria must be in symmetric assessments only. Appendix C pertains to the collusion benchmark. There, we prove the equilibrium outcome-equivalence between Ravid (2020), perfect price coordination, and perfect profit internalization, i.e., the collusion benchmark introduced in Section 3. The remaining appendices are related to model extensions. Appendix D studies strategic price obfuscation within our framework. Appendix E shows that our results extend to a framework with known product quality and random marginal costs. Appendix F shows that our analysis is robust to using models of costly information processing different from Sims' (2003) rational inattention. Finally, Appendix G extends our analysis to the case of partial and symmetric profits internalization.

A Omitted proofs

Proof of Lemma 1

Proof. ("Only if" direction.) Let μ be consistent with a strategy profile σ of the sellers. Suppose that β is a best response to μ and that is robust to vanishing perturbations. Since β is a best response to μ, we know from Matějka and McKay (2015) that

$$\beta_i(w, y_1, y_2) = \frac{\pi_i \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I)$$

μ-a.s., where each (π_1, π_2) is a solution to problem (3). Now, fix $v \in V$ and $x_1, x_2 \geq 0$ arbitrarily. Then, there exists a sequence $(\mu^n, \tilde{\sigma}^n)$ with the desired properties such that β is a best response to μ^n for every $n \in \mathbb{N}$. Again from Matějka and McKay (2015), we know that β must take the following logit functional form

$$\beta_i(w, y_1, y_2) = \frac{\pi_i^n \cdot e^{\frac{w-y_i}{k}}}{\sum_{j=1,2} \pi_j^n \cdot e^{\frac{w-y_j}{k}} + 1 - \pi_1^n - \pi_2^n}, \ (i \in I) \tag{10}$$

μ^n-a.s. for every $n \in \mathbb{N}$, where each $\pi^n = (\pi_1^n, \pi_2^n)$ is a solution to

$$\max_{\pi_1', \pi_2' \geq 0} \mathbb{E}_{\mu^n}\left[\log\left(\pi_1' \cdot e^{\frac{v-x_1}{k}} + \pi_2' \cdot e^{\frac{v-x_2}{k}} + (1 - \pi_1' - \pi_2')\right)\right] \text{ subject to } \pi_1' + \pi_2' \leq 1. \tag{11}$$

Since β is a best reply to all μ^n, and $\mu^n(v, x_1, x_2) > 0$ for all n, we know that $(\pi_1^n, \pi_2^n) = (\bar{\pi}_1, \bar{\pi}_2)$ for some $\bar{\pi}_1, \bar{\pi}_2 \in [0, 1]$.

Now, let (v', x_1', x_2') be a generic element in the support of μ. Since $\tilde{\sigma}^n \to \sigma$ strongly implies that $\mu^n \to \mu$ strongly, we have that $\mu^n(\text{Supp}(\mu)) > 0$ for large n. Once again, because β has to be a best response at all n, we know that

$$\beta_i(v', x_1', x_2') = \frac{\bar{\pi}_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \bar{\pi}_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \bar{\pi}_1 - \bar{\pi}_2} = \frac{\pi_i \cdot e^{\frac{v'-x_i'}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v'-x_j'}{k}} + 1 - \pi_1 - \pi_2}, \ (i \in I).$$

Therefore, $\bar{\pi}_i = \pi_i$ for all $i \in I$. Finally, equation (4) directly follows from Corollary 2 in Matějka and McKay (2015). This concludes the proof of the only if direction.

("If" direction.) Suppose $\beta = (\beta_1, \beta_2)$ is the profile given by equation (2), where (π_1, π_2) solves (3). From Matějka and McKay (2015), we know that β is *a* best response to μ. In what follows, we show that β is also robust to vanishing perturbations. To do so, we distinguish between five cases.

Case 1: The profile (π_1, π_2) satisfies $\pi_1, \pi_2 > 0$ and $\pi_1 + \pi_2 < 1$, i.e., (π_1, π_2) is an interior solution to (3).

In this case, because of strict concavity, the FOCs associated to the objective function in (3) are both necessary and sufficient to characterize (π_1, π_2). In particular, after standard manipulations, such FOCs can be written as $\pi_i = \mathbb{E}_\mu[\beta_i]$ for every $i \in I$.

Now, fix any $v' \in V$ and $x_1', x_2' \geq 0$. We want to find a vanishing belief perturbation μ^n consistent with some (possibly) correlated strategy $\tilde{\sigma}^n$ such that (i) $\mu^n(v', x_1', x_2') > 0$, (ii) $\mu^n \to \mu$ strongly, and (iii) β is a best response to μ^n for all $n \in \mathbb{N}$.

As a first step, notice that since (π_1, π_2) is interior, to prove (iii) it is sufficient to show that $\mathbb{E}_{\mu^n}[\beta_i] = \pi_i$ for every $n \in \mathbb{N}$ and $i \in I$. Moreover, observe that $\beta_i(v, v, v) = \pi_i$ for all $i \in I$ and $v \in V$.

Consider the following claim.

Claim 1. *There exists an $\varepsilon > 0$ such that for every $y \geq 0$, we have $\beta_1(v', 0, y) > \pi_1 + \varepsilon$ and $\beta_2(v', y, 0) > \pi_2 + \varepsilon$.*

Claim 1 is a direct consequence of the multinomial logit formulation of profile (β_1, β_2). Its proof follows from the fact that each β_i is strictly increasing in x_j with $j \neq i$, and the observation that $\beta_i(w, 0, 0) > \pi_i$ for every $i \in I$ and $w \in V$. Importantly, Claim 1 guarantees the existence of $\alpha^* \in (0, 1)$ such that $(1 - \alpha^*)\beta_1(v', 0, y) > \pi_1$ and $(1 - \alpha^*)\beta_2(v', y, 0) > \pi_2$ for all $y \geq 0$. We will use this observation momentarily.

Fix any $\alpha \in (0, \alpha^*)$ such that $\alpha\beta_i(v', x_1', x_2') < \pi_i$ for every $i \in I$. For every $y \geq 0$, define $x_1^o(y) \geq 0$ as the unique solution to

$$\alpha\beta_1(v', x_1', x_2') + (1 - \alpha)\beta_1(v', x_1^o(y), y) = \pi_1. \tag{12}$$

Observe that such $x_1^o(y)$ is well-defined for all $y \geq 0$ because of Claim 1, our assumptions on α, and the fact that β_1 is strictly decreasing in x_1 and satisfies $\beta_1 \downarrow 0$ as $x_1 \uparrow \infty$. Now, notice that

$$\alpha\beta_2(v', x_1', x_2') + (1 - \alpha)\beta_2(v', x_1^o(0), 0) > \pi_2$$

and

$$\alpha\beta_2(v', x_1', x_2') + (1 - \alpha)\beta_2(v', x_1^o(y), y) < \pi_2,$$

as $y \uparrow \infty$.[27] By continuity, this implies that there exists (x_1^*, x_2^*) such that

$$\alpha\beta_i(v', x_1', x_2') + (1 - \alpha)\beta_1(v', x_1^*, x_2^*) = \pi_i, \quad \forall i \in I. \tag{13}$$

Now, for every $n \in \mathbb{N}$, define $\tilde{\sigma}'(\cdot|\cdot)$ as follows: $\tilde{\sigma}'(\cdot|v) = \delta_{(v,v)}$ if $v \neq v'$ and

$$\tilde{\sigma}'(\cdot|v') = \alpha\delta_{(x_1', x_2')} + (1 - \alpha)\delta_{(x_1^*, x_2^*)}.$$

Given this, set $\tilde{\sigma}^n := \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for every $n \in \mathbb{N}$, and let μ^n be the belief consistent with $\tilde{\sigma}^n$ for every $n \geq 0$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly. Moreover, $\mathbb{E}_{\mu^n}[\beta_i] = \pi_i$ for every $i \in I$ given

[27] This follows because $x_1^o(y)$ is bounded above by $\lim_{y\uparrow\infty} x_1^o(y) < \infty$.

(13). This shows that β is a best response to μ^n for all $n \in \mathbb{N}$, as required.

Case 2:[28] The profile (π_1, π_2) is such that $0 = \pi_1 < \pi_2 < 1$.

In this case, necessary and sufficient conditions for the optimality of (π_1, π_2) are: (a) $\pi_2 = \mathbb{E}_\mu[\beta_2]$, and (b)

$$\mathbb{E}_\mu \left[\frac{e^{\frac{v-x_1}{k}}}{\pi_2 e^{\frac{v-x_2}{k}} + 1 - \pi_2} \right] \leq 1.$$

Once again, fix any $v' \in V$ and $x_1', x_2' \geq 0$. We want to find a vanishing belief perturbation μ^n consistent with some (possibly) correlated strategy $\tilde{\sigma}^n$ such that (i) $\mu^n(v', x_1', x_2') > 0$, (ii) $\mu^n \to \mu$ strongly, and (iii) β is a best response to μ^n for all $n \in \mathbb{N}$.

Let $\alpha \in (0, 1)$ be so small that the following two conditions hold:

1. For every $y \geq 0$, the (unique) offer $x_2^o(y) \geq 0$ such that

$$\alpha \beta_2(v', x_1', x_2') + (1 - \alpha)\beta_2(v', y, x_2^o(y)) = \pi_2$$

   is well-defined.

2. There exists $y^* \geq 0$ sufficiently large such that

$$\alpha \frac{e^{\frac{v'-x_1'}{k}}}{\pi_2 e^{\frac{v'-x_2'}{k}} + 1 - \pi_2} + (1 - \alpha) \frac{e^{\frac{v'-y^*}{k}}}{\pi_2 e^{\frac{v'-x_2^o(y^*)}{k}} + 1 - \pi_2} \leq 1.$$

Define $\tilde{\sigma}'(\cdot|\cdot)$ as follows: $\tilde{\sigma}(\cdot|v) = \delta_{(v,v)}$ for $v \neq v'$, and

$$\tilde{\sigma}(\cdot|v') = \alpha \delta_{(x_1', x_2')} + (1 - \alpha)\delta_{(y^*, x_2^o(y^*))}.$$

Given this, set $\tilde{\sigma}^n := \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for every $n \in \mathbb{N}$, and let μ^n be the belief consistent with $\tilde{\sigma}^n$ for every $n \geq 0$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly. Moreover, $\mathbb{E}_{\mu^n}[\beta_2] = \pi_2$, and

$$\mathbb{E}_{\mu^n} \left[\frac{e^{\frac{v-x_1}{k}}}{\pi_2 e^{\frac{v-x_2}{k}} + 1 - \pi_2} \right] \leq 1,$$

i.e., conditions (a) and (b) hold. This shows that β is a best response to μ^n for all $n \in \mathbb{N}$, as required.

Case 3:[29] The profile (π_1, π_2) is such that $0 = \pi_1 < \pi_2 = 1$.

[28]A symmetric argument handles the case where $0 = \pi_2 < \pi_1 < 1$.
[29]A symmetric argument handles the case where $0 = \pi_2 < \pi_1 = 1$.

In this case, necessary and sufficient conditions for the optimality of (π_1, π_2) are:

$$\mathbb{E}_\mu\left[e^{\frac{x_2-v}{k}}\right] \le 1 \quad \text{and} \quad \mathbb{E}_\mu\left[e^{\frac{x_2-x_1}{k}}\right] \le 1.$$

Once again, fix any $v' \in V$ and $x'_1, x'_2 \ge 0$. We want to find a vanishing belief perturbation μ^n consistent with some (possibly) correlated strategy $\tilde{\sigma}^n$ such that (i) $\mu^n(v', x'_1, x'_2) > 0$, (ii) $\mu^n \to \mu$ strongly, and (iii) β is a best response to μ^n for all $n \in \mathbb{N}$.

Let $\alpha \in (0,1)$ be so small, and $x^*_1 \ge 0$ be so large that the following two conditions hold:

$$\alpha e^{\frac{x'_2-v'}{k}} + (1-\alpha)e^{\frac{-v'}{k}} \le 1,$$

and

$$\alpha e^{\frac{x'_2-x'_1}{k}} + (1-\alpha)e^{\frac{-x^*_1}{k}} \le 1.$$

Given this, define $\tilde{\sigma}'(\cdot|\cdot)$ as follows: $\tilde{\sigma}(\cdot|v) = \delta_{(v,v)}$ for $v \ne v'$, and

$$\tilde{\sigma}(\cdot|v') = \alpha\delta_{(x'_1,x'_2)} + (1-\alpha)\delta_{(x^*_1,0)}.$$

Set $\tilde{\sigma}^n := \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for every $n \in \mathbb{N}$, and let μ^n be the belief consistent with $\tilde{\sigma}^n$ for every $n \ge 0$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly. Moreover,

$$\mathbb{E}_{\mu^n}\left[e^{\frac{x_2-v}{k}}\right] \le 1 \quad \text{and} \quad \mathbb{E}_{\mu^n}\left[e^{\frac{x_2-x_1}{k}}\right] \le 1.$$

This shows that β is a best response to μ^n for all $n \in \mathbb{N}$, as required.

Case 4: The profile (π_1, π_2) is such that $\pi_1, \pi_2 > 0$ and $\pi_1 + \pi_2 = 1$.

In this case, necessary and sufficient conditions for the optimality of (π_1, π_2) are:

$$\mathbb{E}_\mu[\beta_1/\pi_1] = 1 = \mathbb{E}_\mu[\beta_2/\pi_2] \quad \text{and} \quad \mathbb{E}_\mu\left[\frac{1}{\pi_1 e^{\frac{v-x_1}{k}} + \pi_2 e^{\frac{v-x_2}{k}}}\right] \le 1.$$

Once again, fix any $v' \in V$ and $x'_1, x'_2 \ge 0$. We want to find a vanishing belief perturbation μ^n consistent with some (possibly) correlated strategy $\tilde{\sigma}^n$ such that (i) $\mu^n(v', x'_1, x'_2) > 0$, (ii) $\mu^n \to \mu$ strongly, and (iii) β is a best response to μ^n for all $n \in \mathbb{N}$.

As a first step, observe that $\mathbb{E}_\mu[\beta_1/\pi_1] = 1$ *if and only if* $\mathbb{E}_\mu[\beta_2/\pi_2] = 1$. Moreover, β_1/π_1 depends on (x_1, x_2) only through $\Delta = x_2 - x_1$ because $\pi_1 + \pi_2 = 1$, and that $\beta_1/\pi_1 = 1$ if and only if $\Delta = 0$. This implies that there exists $\alpha \in (0,1)$ and $x^*_1, x^*_2 \ge 0$ small enough such that the following two conditions hold:

$$\alpha \frac{e^{\frac{v'-x'_1}{k}}}{\pi_1 e^{\frac{v'-x'_1}{k}} + \pi_2 e^{\frac{v'-x'_2}{k}}} + (1-\alpha)\frac{e^{\frac{v'-x^*_1}{k}}}{\pi_1 e^{\frac{v'-x^*_1}{k}} + \pi_2 e^{\frac{v'-x^*_2}{k}}} = 1,$$

and

$$\alpha \frac{1}{\pi_1 e^{\frac{v'-x_1'}{k}} + \pi_2 e^{\frac{v'-x_2'}{k}}} + (1-\alpha) \frac{1}{\pi_1 e^{\frac{v'-x_1^*}{k}} + \pi_2 e^{\frac{v'-x_2^*}{k}}} \leq 1.$$

Given this, define $\tilde{\sigma}'(\cdot|\cdot)$ as follows: $\tilde{\sigma}(\cdot|v) = \delta_{(v,v)}$ for $v \neq v'$, and

$$\tilde{\sigma}(\cdot|v') = \alpha\delta_{(x_1',x_2')} + (1-\alpha)\delta_{(x_1^*,x_2^*)}.$$

Set $\tilde{\sigma}^n := \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for every $n \in \mathbb{N}$, and let μ^n be the belief consistent with $\tilde{\sigma}^n$ for every $n \geq 0$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly. Moreover,

$$\mathbb{E}_{\mu^n}[\beta_1/\pi_1] = 1 = \mathbb{E}_{\mu^n}[\beta_2/\pi_2] \quad \text{and} \quad \mathbb{E}_{\mu^n}\left[\frac{1}{\pi_1 e^{\frac{v-x_1}{k}} + \pi_2 e^{\frac{v-x_2}{k}}}\right] \leq 1.$$

This shows that β is a best response to μ^n for all $n \in \mathbb{N}$, as required.

Case 5: The profile (π_1, π_2) is such that $\pi_1, \pi_2 = 0$.

In this case, necessary and sufficient conditions for the optimality of (π_1, π_2) are:

$$\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \leq 1, \quad \forall i \in I.$$

Once again, fix any $v' \in V$ and $x_1', x_2' \geq 0$. We want to find a vanishing belief perturbation μ^n consistent with some (possibly) correlated strategy $\tilde{\sigma}^n$ such that (i) $\mu^n(v', x_1', x_2') > 0$, (ii) $\mu^n \to \mu$ strongly, and (iii) β is a best response to μ^n for all $n \in \mathbb{N}$.

There exists an $\alpha \in (0,1)$ and $y_1^*, y_2^* \geq 0$ large enough so that

$$\alpha e^{\frac{v'-x_i'}{k}} + (1-\alpha)e^{\frac{v'-y_i^*}{k}} \leq 1.$$

Given this, define $\tilde{\sigma}'(\cdot|\cdot)$ as follows: $\tilde{\sigma}(\cdot|v) = \delta_{(v,v)}$ for $v \neq v'$, and

$$\tilde{\sigma}(\cdot|v') = \alpha\delta_{(x_1',x_2')} + (1-\alpha)\delta_{(y_1^*,y_2^*)}.$$

Set $\tilde{\sigma}^n := \frac{n-1}{n}\sigma + \frac{1}{n}\tilde{\sigma}'$ for every $n \in \mathbb{N}$, and let μ^n be the belief consistent with $\tilde{\sigma}^n$ for every $n \geq 0$. By construction, $\tilde{\sigma}^n \to \sigma$ strongly. Moreover,

$$\mathbb{E}_{\mu^n}\left[e^{\frac{v-x_i}{k}}\right] \leq 1, \quad \forall i \in I.$$

This shows that β is a best response to μ^n for all $n \in \mathbb{N}$, as required.

Cases 1-5 show that one can always find belief perturbations rationalizing any price deviation of the firms that still justify the strategy β given by (2). Thus, β is indeed an RVP best response to μ. This completes the proof of the Lemma.

□

subsubsection*Proof of Corollary 1

Proof. Consider the following definition.

Definition 4. *We say that μ is symmetric if for every measurable $A \subseteq V \times \mathbb{R}^2_+$, we have $\mu(A) = \mu(A^{sym})$, where*

$$A^{sym} := \bigcup \{(v, x_1, x_2) : (v, x_2, x_1) \in A\}$$

is the symmetric conjugate of A.

The proof of Corollary 1 uses the following lemma, whose proof is omitted.

Lemma 5 (Proof omitted.)**.** *Let σ be symmetric, and suppose that μ is consistent with σ. Then, μ is symmetric.*

(Cor. 1: "Only if" direction.) Since σ is symmetric, Lemma 5 implies the belief μ is symmetric. As a result, given the concavity of the objective function, it is without loss to focus on solutions (π_1, π_2) to (3) such that $\pi_1 = \pi_2$.[30] Points (i), (ii), (iii) and (iv) follows directly from the use of Lagrangian methods. (See also the results in Matějka and McKay (2015).) We omit the details.

(Cor. 1: "If" direction.) Let $\beta = (\beta_1, \beta_2)$ be given by (5). Clearly, β is symmetric. Furthermore, given the symmetry of μ, we know from Matějka and McKay (2012) that β is a best response to μ. To prove that β is indeed robust to vanishing perturbations we need to find appropriate belief perturbations. To this goal, the belief perturbations found in the proof of Lemma 1 suffice. □

Proof of Theorem 1

See Appendix C.

Proof of Lemma 2

Proof. A profile (μ, β, σ) is a competitive trading equilibrium if and only if it is symmetric trading equilibrium. That is, (a) μ is consistent with σ, (b) β is given by (5) with $\pi > 0$, and (c) σ is symmetric and σ_i is a best response to σ_{-i} given β.

We focus on the equilibrium behavior of the firms. Fix $v \in V$ arbitrarily. As Appendix B shows, for fixed symmetric logit demand β of the consumer, the unique equilibrium of the pricing game played by the firms is pure and symmetric. To characterize it, suppose (b) holds and let $\sigma_i(\cdot|v) = \delta_{x_i(v)}$ for all $i \in I$. Then, taking $\pi \in (0, 1/2]$ and $-i$'s offer $x_{-i}(v) = x_{-i}$ as given, firm i maximizes profits given the demand function $\beta_i(v, x_i, x_{-i})$ as follows:

[30] If (π_1, π_2) is a solution to (3) with $\pi_1 \neq \pi_2$, the symmetry of μ implies that (π_2, π_1) is a distinct solution. Because of concavity, $\frac{1}{2}(\pi_1, \pi_2) + \frac{1}{2}(\pi_2, \pi_1)$ is then another (symmetric) solution to (3), proving the assertion.

$$\max_{x_i \geq 0} \frac{x_i \cdot \pi e^{\frac{v-x_i}{k}}}{\pi \cdot \left(e^{\frac{v-x_i}{k}} + e^{\frac{v-x_{-i}}{k}}\right) + 1 - 2\pi}$$

The first order condition is given by

$$\frac{d\Pi_i(v, x_1, x_2)}{dx_i} = \frac{\pi e^{\frac{v-x_i}{k}}}{\sum_{j \in 1,2} \pi e^{\frac{v-x_j}{k}} + 1 - 2\pi} + \frac{-x_i \cdot \pi \frac{1}{k} e^{\frac{v-x_i}{k}} (\pi e^{\frac{v-x_{-i}}{k}} + 1 - 2\pi)}{(\sum_{j \in 1,2} \pi e^{\frac{v-x_j}{k}} + 1 - 2\pi)^2} = 0$$

We can rewrite this as

$$\frac{d\Pi_i(v, x_1, x_2)}{dx_i} = \beta_i(v, x_1, x_2) \left(1 - \frac{x_i}{k} \cdot [1 - \beta_i(v, x_1, x_2)]\right) = 0. \tag{14}$$

Note that $\beta_i > 0$ for all $x_i \geq 0$ and $\lim_{x_i \to \infty} \beta_i = 0$. Therefore, $x_i \mapsto \frac{d\Pi_i(v, x_1, x_2)}{dx_i}$ crosses zero exactly once from above. It follows that $x_i \mapsto \Pi_i(v, x_i, x_{-i})$ admits a unique (interior) global maximum characterized by the first order condition. We rearrange (14) and use symmetry to see that, in equilibrium, $x_i(v) = x_{-i}(v) = x(v)$ satisfies:

$$x(v; \pi) = k \cdot \left[1 + \frac{\pi e^{\frac{v - x(v;\pi)}{k}}}{\pi e^{\frac{v - x(v;\pi)}{k}} + 1 - 2\pi}\right].$$

Define $\phi(v; \pi) = \frac{\pi e^{\frac{v - x(v;\pi)}{k}}}{\pi e^{\frac{v - x(v;\pi)}{k}} + 1 - 2\pi}$. The equilibrium firm behavior is then given by

$$x(v; \pi) = k \cdot [1 + \phi(v; \pi)],$$

where optimality requires that

$$\left(1 + e^{\phi(v;\pi)} \frac{1 - 2\pi}{\pi e^{\frac{v-k}{k}}}\right) \phi(v; \pi) = 1.$$

The above equation uniquely pins down $\phi(v; \pi)$ in $\mathbb{R}_+$: The LHS goes to 0 as $\phi \to 0$, and it goes to ∞ as $\phi \to \infty$. Furthermore, the LHS is continuously increasing in ϕ.

Lastly, note that $\phi(v; \pi) = \frac{\beta(v, x(v;\pi), x(v;\pi))}{1 - \beta(v, x(v;\pi), x(v;\pi))}$ for all $v \in V$, where $\beta_i = \beta$ because of symmetry. □

Proof of Theorem 2

Proof. We start with the proof of the if and only if statement. The necessity proof is as follows. As we argued after Lemma 2, in any competitive trading equilibrium, the sellers charge a price $x_i(v) = x(v)$ strictly above k for each $v \in V$. Now, suppose by way of contradiction that a trading equilibrium exists but $k \geq k^*$, i.e., $\mathbb{E}_\lambda \left[e^{v/k - 1}\right] \leq 1$. In equilibrium, we would have

$$\mathbb{E}_\mu \left[e^{\frac{v - x(v)}{k}}\right] < \mathbb{E}_\lambda \left[e^{v/k - 1}\right] \leq 1.$$

This is in contradiction with our hypothesis of on-path equilibrium trade. Indeed, according to Corollary 1, the consumer's trade engagement level with each firm $i \in I$ would be equal to zero. We conclude that $k < k^*$ is necessary for the existence of a competitive trading equilibrium.

We now turn to the sufficiency direction. We split the proof in two parts. First, we restrict attention to values of k for which trade occurs with probability 1. We then consider the remaining parameter values. To this end, we need to introduce some further notation. Let $\bar{k}$ be the unique solution to $\mathbb{E}_\lambda \left[e^{2-v/\bar{k}}\right] = 1$. Notice that

$$\mathbb{E}_\lambda \left[e^{2-v/k^*}\right] > \mathbb{E}_\lambda \left[e^{1-v/k^*}\right] = \mathbb{E}_\lambda \left[\frac{1}{e^{v/k^*-1}}\right] \geq \frac{1}{\mathbb{E}_\lambda \left[e^{v/k^*-1}\right]} = 1.$$

Thus, $0 < \bar{k} < k^*$.

Suppose first that $k \leq \bar{k}$. Take $x(v) = 2k$ for all $v \in V$. Observe that this configuration of prices is an equilibrium of the pricing game played by the firms when they face a symmetric logit demand with $\pi = 1/2$. At the same time, from Corollary 1, a symmetric trade engagement level $\pi = 1/2$ is consistent with this configuration of prices if and only if $\mathbb{E}_\lambda \left[e^{2-v/k}\right] \leq 1$, or equivalently, $k \leq \bar{k}$. Therefore, a symmetric sure-trade equilibrium exists. Since a symmetric sure-trade equilibrium is indeed a competitive trading equilibrium, we are done.

Now, consider the case where $k \in (\bar{k}, k^*)$. We show that if $k \in (\bar{k}, k^*)$ – or equivalently, $\mathbb{E}_\lambda \left[e^{2-v/k}\right] > 1$ and $\mathbb{E}_\lambda \left[e^{v/k-1}\right] > 1$ –, then a symmetric RVP trading equilibrium where trade occurs with probability strictly between 0 and 1 exists. To do so, define the functions $\phi = \phi(v; p)$, $x = x(v; p)$ and $F = F(p)$ as follows. For each $p \in (0, 1/2]$ and $v \in V$, let $\phi = \phi(v; p)$ be the unique solution to

$$1 = \phi \left(1 + e^{\phi} \frac{1 - 2p}{p \cdot e^{v/k-1}}\right), \tag{15}$$

let $x = x(v; p)$ be given by $x := k \cdot (1 + \phi(v; p))$, and finally let $F = F(p)$ be defined as

$$F(p) := \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-\phi}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1 - 2p)}\right]. \tag{16}$$

Since the function $F(\cdot)$ satisfies $F(p) = \frac{1}{p}\mathbb{E}[\beta_i(v, x(v; p), x(v; p))]$ for every $i \in I$, it is sufficient to show that $F(p^*) = 1$ for some $p^* \in (0, 1/2)$. (See Lemma 6 below.) We prove this by relying on the Intermediate Value Theorem, hence exploiting the continuity of $F(\cdot)$ in $p \in (0, 1/2]$. In particular, we show that there exists $0 < p_0 < p_1 < 1/2$ such that for all $p \in (0, p_0)$, we have $F(p) > 1$, and for all $p \in (p_1, 1/2)$, we have $F(p) < 1$.

Existence of $0 < p_1 < 1/2$*:* We exploit the fact that $F(\cdot)$ is continuosly differentiable. This follows from the Implicit Function Theorem that guarantees that $\phi(v; p)$ is continuously differentiable in $p \in (0, 1/2]$ for all $v \in V$.[31] Given that V is finite and $\phi(v; p) \uparrow 1$ as $p \uparrow 1/2$, for

[31]More formally, for $\phi \in (0, \infty)$, $v \in V$, and $p \in (0, 1/2 + \tau)$, let

$$G(p, \phi, v) := \phi \cdot \left(1 + e^{\phi} \cdot (1 - 2p) / \left[p \cdot e^{\frac{v-k}{k}}\right]\right) - 1.$$

Given that V is finite, one can show for a $\tau > 0$ small enough, the assumptions of the Implicit Function Theorem are satisfied by G. Thus, there exists a continuously differentiable function $\bar{\phi}(v; p)$ on $(0, 1/2 + \tau) \times V$ such that every $\varepsilon > 0$ there exists a $\bar{p}_1 \in (0, 1/2)$ such that $\phi(; pv) > 1 - \varepsilon$ for all $v \in V$ and $p \in (\bar{p}_1, 1/2)$. Fix $\varepsilon > 0$ and $\delta > 0$ small enough so that $\mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k}\right] - \delta > 1$, and let $\bar{p}_1$ be the p-threshold corresponding to ε.[32] For every $v \in V$, define

$$A(v) := \max_{p \in [\bar{p}_1, 1/2]} e^{1-v/k+\phi(v;p)} \cdot \frac{\partial}{\partial p}\phi(v;p) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}$$

where

$$D_{\max}(p) := \max_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2 > 0$$

and

$$D_{\min}(p) := \min_{v \in V} \left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2 > 0.$$

We make two observations.

Obs. 1: Each $A(v)$ is a well-defined real number since it is the maximum value of a continuous function on a compact support. Again by the finiteness of V, there exists $\bar{p}_2 \in (0, 1/2)$ such that $(1-2p) \cdot A(v) \le \delta$ for all $v \in V$ and $p \in (\bar{p}_2, 1/2)$.

Obs. 2: Since $D_{\max}(p), D_{\min}(p) \to 1$ as $p \uparrow 1/2$, we have that $D_{\max}(p)/D_{min}(p) \to 1$ as $p \uparrow 1/2$. Therefore, there exists $\bar{p}_3 \in (0, 1/2)$ such that $D_{\max}(p)/D_{min}(p) \le 1 + \delta/2$ for all $p \in (\bar{p}_3, 1/2)$.

Now, let $\bar{p} = \max\{\bar{p}_1, \bar{p}_2, \bar{p}_3\} < 1/2$. For all $p \in (\bar{p}, 1/2)$, we have:

$$\begin{aligned}
F'(p) &= \mathbb{E}_\lambda \left[\frac{2 \cdot \left(e^{1+\phi(v;p)-v/k} - 1\right) - (1-2p) \cdot e^{\phi(v;p)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(v;p)}{\left(2p + (1-2p) \cdot e^{\phi(v;p)+1-v/k}\right)^2}\right] \\
&\ge \mathbb{E}_\lambda \left[\frac{2}{D_{\max}(p)} \cdot e^{1+\phi(v;p)-v/k} - \frac{2}{D_{\min}(p)} - (1-2p) \cdot \frac{e^{\phi(v;p)+1-v/k}}{D_{\min}(p)} \cdot \frac{\partial}{\partial p}\phi(v;p)\right] \\
&= \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(v;p)-v/k} - 2 \cdot \frac{D_{\max}(p)}{D_{\min}(p)} - (1-2p) \cdot e^{\phi(v;p)+1-v/k} \cdot \frac{\partial}{\partial p}\phi(v;p) \cdot \frac{D_{\max}(p)}{D_{\min}(p)}\right] \\
&\ge \frac{1}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[2 \cdot e^{1+\phi(v;p)-v/k} - 2 \cdot \frac{D_{\max}(p)}{D_{\min}(p)} - (1-2p) \cdot A(v)\right] \\
&\ge \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{1+\phi(v;p)-v/k} - 1 - \delta\right] \\
&\ge \frac{2}{D_{\max}(p)} \cdot \mathbb{E}_\lambda \left[e^{2-\varepsilon-v/k} - 1 - \delta\right] > 0,
\end{aligned}$$

where the first inequality comes from the fact that $\frac{\partial}{\partial p}\phi(v;p) \ge 0$.[33] Since $F(1/2) = 1$ and $F'(p) > 0$ for all $p \in (0, 1/2)$ sufficiently close to $1/2$, the existence of p_1 immediately follows.

Existence of $0 < p_0 < p_1 < 1/2$*:* Given that V is finite, $\phi(v;p) \downarrow 0$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) \to 1$ as $p \downarrow 0$, for every $\varepsilon > 0$ there exists a $\underline{p} \in (0, 1/2)$ such that $\phi(v;p) < \varepsilon$ and $2p \cdot e^{\frac{v-k}{k}} \cdot e^{-\phi} + (1-2p) < 1 + \varepsilon$ for all $v \in V$ and $p \in (0, \underline{p})$. Let $\varepsilon > 0$ be small enough so that $\mathbb{E}_\lambda \left[e^{v/k-1-\varepsilon}\right] / (1+\varepsilon) > 1$. Such an $\varepsilon > 0$ exists because $\mathbb{E}_\lambda \left[e^{v/k-1}\right] > 1$. For all $p \in (0, \underline{p})$, we have:

$G(p, \bar{\phi}(v;p), v) = 0$ for all $v \in V$ and $p \in (0, 1/2 + \tau)$. As a result, $\bar{\phi}(v;p) = \phi(v;p)$ on $(0, 1/2] \times V$.

[32] Such $\varepsilon, \delta > 0$ exist because $\mathbb{E}_\lambda \left[e^{2-v/k}\right] > 1$ by assumption.

[33] See the proof of Lemma 9.

$$
\begin{aligned}
F(p) &= \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}}\cdot e^{-\phi}}{2p\cdot e^{\frac{v-k}{k}}\cdot e^{-\phi}+(1-2p)}\right]\\
&\geq \frac{\mathbb{E}_\lambda\left[e^{\frac{v-k}{k}}\cdot e^{-\phi}\right]}{1+\varepsilon}\\
&\geq \frac{\mathbb{E}_\lambda\left[e^{v/k-1-\varepsilon}\right]}{1+\varepsilon} > 1.
\end{aligned}
$$

Thus, a $p_0 \in (0, p_1)$ with the desired properties exists. This concludes the proof of existence of a trading equilibrium.

Uniqueness: Once again, we distinguish between two cases. First, suppose $k \leq \bar{k}$, or equivalently, $\mathbb{E}\left[e^{2-v/k}\right] \leq 1$. From the proof of existence, we know that a competitive sure-trade equilibrium exists. We want to show that no other symmetric trading equilibrium can exist. For each $p \in (0, 1/2]$ and $v \in V$, let $\phi = \phi(v; p)$, $x = x(v; p)$, and $F = F(p)$ be defined as above. Note that $F(1/2) = 1$. This again confirms that a competitive sure-trade equilibrium exists because $x(1/2, v) = 2k$ for every $v \in V$ and $\mathbb{E}_\lambda\left[e^{2-v/k}\right] \leq 1$ holds. To prove that no other symmetric trading equilibrium exists, it is sufficient to show that $F(p) \neq 1$ for all $p \in (0, 1/2)$. With this goal in mind, first note that $\phi(v; p)$ is strictly increasing in $p \in (0, 1/2]$ for every $v \in V$, and that $\phi(v; 1/2) = 1$. Thus, given that V is finite, when p is strictly below $1/2$, there exists $\varepsilon > 0$ small enough such that $\phi(v; p) < 1 - \varepsilon$ for all $v \in V$. Second, observe that $\mathbb{E}\left[e^{2-c-v/k}\right] < 1$ for any constant $c > 0$. Now, fix $p \in (0, 1/2)$ and its corresponding $\varepsilon > 0$. We have

$$
\begin{aligned}
F(p) &= \mathbb{E}_\lambda\left[\frac{e^{\frac{v-k}{k}}\cdot e^{-\phi}}{2p\cdot e^{\frac{v-k}{k}}\cdot e^{-\phi}+(1-2p)}\right]\\
&= \mathbb{E}_\lambda\left[\frac{1}{2p+(1-2p)\cdot e^{\phi+1-v/k}}\right]\\
&> \mathbb{E}_\lambda\left[\frac{1}{2p+(1-2p)\cdot e^{2-\varepsilon-v/k}}\right]\\
&\geq \frac{1}{2p+(1-2p)\cdot \mathbb{E}_\lambda\left[e^{2-\varepsilon-v/k}\right]} > 1.
\end{aligned}
$$

where the first strictly inequality comes from $\phi = \phi(v; p) < 1 - \varepsilon$ for all $v \in V$, the second is an application of Jensen's inequality, and the last strict inequality is implied by $\mathbb{E}_\lambda\left[e^{2-\varepsilon-v/k}\right] < 1$. Hence, $F(p) \neq 1$ for all $p < 1/2$ as required.

Now, consider the case where $k \in (\bar{k}, k^*)$. Suppose towards a contradiction that there exist $0 < p^* < p^{**} < 1/2$ such that $F(p^*) = F(p^{**}) = 1$. Define $\gamma \in (0, 1)$ implicitly by $p^{**} = \gamma p^* + (1-\gamma)1/2$. We have

$$
\begin{aligned}
F(p^{**}) &= \mathbb{E}_\lambda\left[\frac{1}{2p^{**}+(1-2p^{**})\cdot e^{\phi(v;p^{**})+1-v/k}}\right]\\
&= \mathbb{E}_\lambda\left[\frac{1}{2\gamma p^{*}+1-\gamma+\gamma(1-2p^{*})\cdot e^{\phi(v;p^{**})+1-v/k}}\right]\\
&< \mathbb{E}_\lambda\left[\frac{1}{1-\gamma+\gamma\left(2p^{*}+(1-2p^{*})\cdot e^{\phi(v;p^{*})+1-v/k}\right)}\right] \le 1.
\end{aligned}
$$

The first inequality follows from the fact that $p^{**} > p^{*}$ and that $\phi(v;p)$ is strictly increasing in $p \in (0, 1/2)$ for all $v \in V$. In order to prove the second inequality, we define

$$
g(\gamma) := \mathbb{E}_\lambda\left[\frac{1}{1-\gamma+\gamma\left(2p^{*}+(1-2p^{*})\cdot e^{\phi(v;p^{*})+1-v/k}\right)}\right].
$$

Note that $g(0) = 1$ and $g(1) = F(p^{*}) = 1$. It remains to show that $g(\gamma)$ is convex for all $\gamma \in [0,1]$. Taking the second derivative, we get

$$
g''(\gamma) = \mathbb{E}_\lambda\left[\frac{2\left(2p^{*}+(1-2p^{*})\cdot e^{\phi(v;p^{*})+1-v/k}-1\right)^2}{\left(1-\gamma+\gamma\left(2p^{*}+(1-2p^{*})\cdot e^{\phi(v;p^{*})+1-v/k}\right)\right)^3}\right] \ge 0.
$$

Thus, we reached the contradiction that $F(p^{**}) < 1$. We conclude that there is at most one $p \in (0, 1/2)$ such that $F(p) = 1$. This concludes the proof of uniqueness. □

The following Lemma was invoked during the proof of Theorem 2.

Lemma 6. *Let μ be symmetric, and β be given by (5). If $\pi = \mathbb{E}_\mu[\beta_i] \in (0, 1/2)$ for every $i \in I$, then $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \ge 1$ for every $i \in I$, and $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right)^{-1}\right] \ge 1/2$.*

Proof. For every $y > 0$ and $\gamma \in (0, 1/2)$, let

$$
g(y,\gamma) = \frac{1-2\gamma}{\gamma y+(1-2\gamma)},
$$

and

$$
h(y,\gamma) = \frac{\gamma}{\gamma+(1-2\gamma)y}.
$$

Observe that g and h are strictly decreasing and strictly convex in $y > 0$ for every $\gamma \in (0, 1/2)$. Moreover, $g(y,\gamma) = 1-2\gamma$ if and only if $y = 2$, and $h(y,\gamma) = 2\gamma$ if and only if $y = 1/2$.

From Jensen's inequality, we have

$$
2\pi = \mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu\left[h\left(\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right)^{-1},\pi\right)\right] \ge h\left(\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right)^{-1}\right],\pi\right),
$$

which implies that $\mathbb{E}_\mu\left[\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right)^{-1}\right] \ge 1/2$ because h is strictly decreasing in $y > 0$.

Similarly,

$$1-2\pi = 1-\mathbb{E}_\mu[\beta_1+\beta_2] = \mathbb{E}_\mu\left[g\left(e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}},\pi\right)\right] \geq g\left(\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right],\pi\right),$$

which implies that $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}}+e^{\frac{v-x_2}{k}}\right] \geq 2$. Since μ is symmetric, $\mathbb{E}_\mu\left[e^{\frac{v-x_1}{k}}\right] = \mathbb{E}_\mu\left[e^{\frac{v-x_2}{k}}\right]$. Therefore, $\mathbb{E}_\mu\left[e^{\frac{v-x_i}{k}}\right] \geq 1$ for every $i \in I$ as required. □

Proof of Proposition 1

Proof. Fix $k \in (0, k^*)$, and let $(\mu^M, \sigma^M, \beta^M)$ and $(\mu^C, \beta^C, \sigma^C)$ be the unique symmetric equilibrium under collusion and competition respectively associated with the cost parameter k. Set $\pi^M = \mathbb{E}_{\mu^M}[\beta_1^M + \beta_2^M]$ and $\pi^C = \mathbb{E}_{\mu^C}[\beta_i^C]$ for each $i \in I$.

If $k \leq \bar{k}$, the result is an immediate consequence of Proposition 2 and Corollary 1 of Ravid (2020). In words, while a sure-trade equilibrium cannot exist under collusion, it is the only competitive trading equilibrium outcome. Hence, $0 < \pi^M < 1 = 2\pi^C$, as required.

Now assume that $k \in (\bar{k}, k^*)$. Under collusion,[34] for every $v \in V$, each active firm plays a strategy $\sigma^M(\cdot|v) = \delta_{x^M(v)}$ such that

$$x^M(v) = k \cdot \left(1 + W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)\right), \tag{17}$$

and $W(\cdot)$ is the Lambert function.[35] Compared to the equilibrium price formula of the competition model displayed in equation (7), we note that the only difference is in the second multiplicative component, where $W\left(\frac{\pi^M}{1-\pi^M}e^{v/k-1}\right)$ is replaced by $\phi(v;\pi^C)$. As a first step, we show that if the *overall* equilibrium trade engagement level was identical in both models, the equilibrium prices under collusion would be strictly higher than the competitive ones.

Lemma 7. *For all $p \in (0, 1/2)$ and $v \in V$, we have $W\left(\frac{2p}{1-2p}e^{v/k-1}\right) > \phi(v; p)$.*

Proof. Fix $p \in (0, 1/2)$ arbitrarily. According to Lemma 2, $\phi = \phi(v; p)$ is the unique solution to equation (8) where π is replaced by p. Note that (8) is equivalent to

$$\frac{p}{1-2p}e^{v/k-1} = \phi \cdot \frac{p}{1-2p}e^{v/k-1} + \phi e^{\phi}.$$

Therefore

$$\frac{2p}{1-2p}e^{v/k-1} > \frac{p}{1-2p}e^{v/k-1} = \phi \cdot \frac{p}{1-2p}e^{v/k-1} + \phi e^{\phi} > \phi e^{\phi}.$$

Applying the Lambert's function on both sides of the above inequality yields the desired result. □

Let $W(v; 2p) := W\left(\frac{2p}{1-2p}e^{v/k-1}\right)$ for all $v \in V$ and $p \in (0, 1/2)$. Following the proof of Theorem 1 in Ravid (2020), the overall equilibrium engagement level in the collusion benchmark is given

[34] See also Proposition 2 in Ravid (2020).

[35] The Lambert's function is the inverse of the function $z \in \mathbb{R}_+ \mapsto ze^z$.

by $\pi^M = 2p^M$, where p^M the unique solution in $(0,1/2)$ to the equation:

$$G(2p) := \mathbb{E}_\lambda \left[\frac{e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)}}{2p \cdot e^{\frac{v-k}{k}} \cdot e^{-W(v;2p)} + (1-2p)} \right] = 1. \tag{18}$$

Let F be defined as in the proof of Theorem 2. We have

$$\begin{aligned}
1 &= G(2p^M) \\
&= \mathbb{E}_\lambda \left[\frac{1}{2p^M + (1-2p^M) \cdot e^{W(v;2p^M)+1-v/k}} \right] \\
&< \mathbb{E}_\lambda \left[\frac{1}{2p^M + (1-2p^M) \cdot e^{\phi(v;p^M)+1-v/k}} \right] = F(p^M),
\end{aligned}$$

where the strict inequality follows from Lemma 7. From the proof of Theorem 2, we conclude that $p^M < \pi^C$. This is equivalent to say that $\pi^M < 2\pi^C$. □

Proof of Proposition 2

Proof. Follows directly from the proof of Theorem 2. □

Proof of Corollary 2

Proof. Follows directly by combining the proofs of Lemma 1 and 2. □

Proof of Theorem 3

Preliminary analysis for the collusion benchmark. For each $k \in (0, k^*]$, let $F_k^M : [0,1) \to \mathbb{R}_+$ be defined as

$$F_k^M(p) := \mathbb{E}_\lambda \left[\frac{1}{p + (1-p) \cdot e^{W(p,v,k)+1-v/k}} \right].$$

As in the previous section, we abuse notation and write $W(p,v,k)$ for $W\left(\frac{p}{1-p} e^{v/k-1}\right)$ and let $W(\cdot)$ be the Lambert's function. We are interested in the solution $p^M(k)$ to the equation $F_k^M(p) = 1$. By the Implicit Function Theorem,[36] we know that whenever this solution exists it is continuously differentiable. In his Theorem 1, Ravid (2020) shows that $p^M(k)$ exists uniquely in $(0,1)$ whenever $k \in (0, k^*)$. The following Lemma characterizes additional properties that the solution $p^M(k)$ satisfies as k ranges in $(0, k^*]$.

Lemma 8. *We have:*

(i) $\lim_{k \uparrow k^*} p^M(k) = 0$.

(ii) $\lim_{k \uparrow k^*} \frac{\partial}{\partial k} p^M(k) = -\mathbb{E}_\lambda \left[\frac{v}{(k^*)^2} \cdot e^{v/k^*-1} \right] / \mathbb{E}_\lambda \left[\frac{2-e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}} \right]$.

Proof. *(i):* Recall from Ravid (2020) that the $F_k^M(\cdot)$ function crosses the line $y=1$ only once

[36] More precisely, one can show that there exists $\tau_k, \tau_p > 0$ small enough so that the assumptions of the Implicit Function Theorem are satisfied once the domain of $F_k^M(\cdot)$ is extended to let k range in $(0, k^* + \tau_k)$ and p range in $(-\tau_p, 1)$.

from above.[37] Therefore, it is sufficient to show that (#): for every $p \in (0,1)$, there exists $k_p \in (0,k^*)$ such that for all k strictly between k_p and k^*, $F_k^M(p) < 1$.

Since the Lambert function $W(\cdot)$ is strictly increasing $W(p,v,k)$ is strictly decreasing in k for every $p \in (0,1)$ and $v \in V$. It further satisfies $W(p,v,k) > 0$ for all $p \in (0,1)$, $v \in V$ and $k > 0$. Fix $p \in (0,1)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $W(p,v,k) > c_p$ for all $v \in V$ and $k \in (0,k^*]$. Since $\mathbb{E}_\lambda\left[e^{v/k^*-1}\right] = 1$, we have $\mathbb{E}_\lambda\left[e^{v/k^*-1-c_p}\right] < 1$. Therefore, continuity implies that there exists k_p strictly between 0 and k^* so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1$ for all $k \in (k_p,k^*)$. Fix any such k. We have:

$$\begin{aligned}
F_k^M(p) &= \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{W(p,v,k)+1-v/k}}\right] \\
&\le \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{c_p+1-v/k}}\right] \\
&\le 2p + (1-2p)\cdot \mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1.
\end{aligned}$$

Thus, (#) holds.

(ii): For each $k \in (0,k^*)$, we totally differentiate the equation $F_k^M(p(k)) = 1$ to obtain:[38]

$$\frac{\partial}{\partial k} p^M(k) = -\frac{A_M}{B_M} \tag{19}$$

where

$$A_M = \mathbb{E}_\lambda\left[\frac{v\cdot(1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}}{k^2\cdot D_M^2\cdot(1+W(p^M(k),v,k))}\right],$$

$$B_M = \mathbb{E}_\lambda\left[\frac{1}{D_M^2}\cdot\left(1 - e^{W(p^M(k),v,k)+1-v/k} + (1-p^M(k))\cdot\frac{e^{W(p^M(k),v,k)}\cdot W'\left(\frac{p^M(k)}{1-p^M(k)}e^{v/k-1}\right)}{(1-p^M(k))^2}\right)\right],$$

and

$$D_M = p^M(k) + (1-p^M(k))\cdot e^{W(p^M(k),v,k)+1-v/k}.$$

As $k \uparrow k^*$, we know from (i) that $p^M(k) \to 0$. Therefore, $A_M \to \mathbb{E}_\lambda\left[\frac{v}{(k^*)^2}\cdot e^{v/k^*-1}\right]$ and $B_M \to \mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}}\right]$.[39] This concludes the proof of Lemma 8. □

Preliminary analysis for the competition model. For each $k \in (\bar{k},k^*]$, we define $F_k^C : [0,1/2) \to \mathbb{R}_+$ as

$$F_k^C(p) := \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{\phi(v;p,k)+1-v/k}}\right],$$

where for $p > 0$, we let $\phi(v;p,k)$ be defined as the unique solution to equation (8), and we set $\phi(v;0,k) := 0$ for all $v \in V$ and $k \in (\bar{k},k^*]$. Let $p^C(k)$ be a solution to $F_k^C(p) = 1$. From Theorem 2, we know that $p^C(k)$ exists and is unique for all $k \in (\bar{k},k^*)$. Again, by the Implicit

[37] This is shown by Ravid (2020) in the proof of Theorem 1.

[38] To derive equation (22), we used the fact that $W'(x) = \frac{W(x)}{x\cdot(1+W(x))}$ for all $x > 0$.

[39] Here, we used the fact that $W'(x) = 1$ as $x \downarrow 0$.

Function theorem we know that $p^C(k)$ is continuously differentiable on $(\bar{k}, k^*)$. The next Lemma provides additional properties that $p^C(k)$ satisfies.

Lemma 9. *We have:*

(i) $\lim_{k\uparrow k^*} p^C(k) = 0$.

(ii) $\lim_{k\uparrow k^*} \frac{\partial}{\partial k} p^C(k) = -\mathbb{E}_\lambda\left[\frac{v}{(k^*)^2} \cdot e^{v/k^*-1}\right] / \mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^*})+1}{e^{2\cdot(1-v/k^*)}}\right]$.

Proof. (i): We show that (#): for every $p \in (0, 1/2)$, there exists $k_p \in (\bar{k}, k^*)$ such that for all k strictly between k_p and k^*, $F_k^C(p) < 1$. Given our proof of Theorem 2, (#) implies that for all k sufficiently close to k^*, $p^C(k) < p$, proving the statement.

From equation (8), $\phi(v; p, k)$ is strictly decreasing in k for every $p \in (0, 1/2)$ and $v \in V$, and satisfies $\phi(v; p, k) > 0$ for all $p \in (0, 1/2)$, $v \in V$ and $k > 0$. Fix $p \in (0, 1/2)$ arbitrarily. Given the finiteness of V, there exists $c_p > 0$ such that $\phi(v; p, k) > c_p$ for all $v \in V$ and $k \in (\bar{k}, k^*]$. Since $\mathbb{E}_\lambda\left[e^{v/k^*-1}\right] = 1$, we have $\mathbb{E}_\lambda\left[e^{v/k^*-1-c_p}\right] < 1$. Therefore, continuity implies that there exists k_p strictly between $\bar{k}$ and k^* so that $\mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1$ for all $k \in (k_p, k^*)$. Fix any such k. We have:

$$\begin{aligned} F_k^C(p) &= \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{\phi(v;p,k)+1-v/k}}\right] \\ &\leq \mathbb{E}_\lambda\left[\frac{1}{2p + (1-2p)\cdot e^{c_p+1-v/k}}\right] \\ &\leq 2p + (1-2p)\cdot \mathbb{E}_\lambda\left[e^{v/k-1-c_p}\right] < 1. \end{aligned}$$

Thus, (#) holds.

(ii): We first totally differentiate equation (8) to find the partial derivatives of ϕ with respect to p and k. That is, $\phi_p(v; p, k) := \frac{\partial}{\partial p}\phi(v; p, k)$ and $\phi_k(v; p, k) := \frac{\partial}{\partial k}\phi(v; p, k)$. After some algebra, one can show that

$$\phi_p(v; p, k) = \frac{1 - \phi(v; p, k)}{(1-2p)\cdot\left(p + e^{\phi(v;p,k)}\cdot(1+\phi(v; p, k))\frac{1-2p}{e^{v/k-1}}\right)} \geq 0, \tag{20}$$

and

$$\phi_k(v; p, k) = -\frac{v}{k^2}\cdot\frac{\phi(v; p, k)e^{\phi(v;p,k)}}{\frac{p}{1-2p}e^{v/k-1} + e^{\phi(v;p,k)}(1+\phi(v; p, k))} \leq 0. \tag{21}$$

Note that, as $k \uparrow k^*$ and, therefore, $p \to 0$, we have $\phi \to 0$. Therefore, $\phi_p \to e^{v/k^*-1}$ and $\phi_k \to 0$ as $k \uparrow k^*$.

Next, we totally differentiate the equation $F_k^C(p^B(k)) = 1$ with respect to $k > \bar{k}$. One can show that

$$\frac{\partial}{\partial k}p^C(k) = -\frac{A_C}{B_C} \tag{22}$$

where

$$A_C = \mathbb{E}_\lambda\left[\frac{1}{D_C^2}\cdot\left((1-2p^C(k))e^{\phi(v;p^C(k),k)+1-v/k}\cdot\left(\frac{v}{k^2}+\phi_k(v;p^C(k),k)\right)\right)\right],$$

$$B_C = \mathbb{E}_\lambda\left[\frac{1}{D_C^2}\cdot\left(2(1-e^{\phi(v;p^C(k),k)+1-v/k})+(1-2p^C(k))\cdot e^{\phi(v;p^C(k),k)+1-v/k}\cdot\phi_p(v;p^C(k),k)\right)\right],$$

and

$$D_C = 2p^C(k)+(1-2p^C(k))\cdot e^{\phi(v;p^C(k),k)+1-v/k}.$$

Letting $k \uparrow k^*$, we conclude that

$$\frac{\partial}{\partial k}p^C(k) \to -\mathbb{E}_\lambda\left[\frac{v}{(k^*)^2}\cdot e^{v/k^*-1}\right]/\mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^*})+1}{e^{2\cdot(1-v/k^*)}}\right]$$

as required. □

Concluding the proof of Theorem 3. We now use *de L'Hopital rule* to show that as $k \uparrow k^*$, the ratio $p^B(k)/p^M(k)$ is bounded above $1/2$ *strictly*. Formally:

Lemma 10. *There exists* $\Theta > 0$ *such that*

$$\lim_{k\uparrow k^*}\frac{p^C(k)}{p^M(k)} > \frac{1}{2}+\Theta.$$

Proof. Note that $\lim_{k\uparrow k^*}\frac{\partial}{\partial k}p^M(k)$ exists and is different from 0. Therefore, by *de L'Hopital rule*

$$\lim_{k\uparrow k^*}\frac{p^C(k)}{p^M(k)} = \lim_{k\uparrow k^*}\frac{\frac{\partial}{\partial k}p^C(k)}{\frac{\partial}{\partial k}p^M(k)} = \frac{\mathbb{E}_\lambda\left[\frac{2-e^{1-v/k^*}}{e^{2\cdot(1-v/k^*)}}\right]}{\mathbb{E}_\lambda\left[\frac{2(1-e^{1-v/k^*})+1}{e^{2\cdot(1-v/k^*)}}\right]} = \frac{1}{2-\frac{\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]}{2\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]-1}}.$$

Since

$$2\mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]-1 > \mathbb{E}_\lambda\left[e^{2(v/k^*-1)}\right]-1 \geq 0$$

because of Jensen inequality, the conclusion of the lemma follows. □

As the last step, note that as $k \uparrow k^*$, $p^M(k), p^C(k) \to 0$. It follows that $x_k^M(v), x_k^C(v) \to k^*$ for all $v \in V$. Now, fix $\varepsilon > 0$ so small that

$$1+2(\Theta-\varepsilon) > \frac{k^*+\varepsilon}{k^*-\varepsilon},$$

and let $\hat{k} \in (\bar{k}, k^*)$ be such that $p^C(k)/p^M(k) > 1/2+\Theta-\varepsilon$ and $x_k^m(v) \in (k^*-\varepsilon, k^*+\varepsilon)$ for all $k > \hat{k}$, $v \in V$, and $m \in \{C, M\}$. For all $k > 0$, we have that $2p^C(k)(k^*-\varepsilon) > p^M(k)(k^*+\varepsilon)$ if and only if

$$2 \cdot \frac{p^C(k)}{p^M(k)} > \frac{k^* + \varepsilon}{k^* - \varepsilon}. \tag{23}$$

Notice that (23) holds by assumption as long as $k \in (\hat{k}, k^*)$. Since by construction we have $\Pi^C(k) \geq p^C(k)(k^* - \varepsilon)$ and $p^M(k)(k + \varepsilon) \geq \Pi^M(k)$, we conclude that $2\Pi^C(k) > \Pi^M(k)$ for all $k \in (\hat{k}, k^*)$ as required.

Q.E.D.

Proof of Lemma 3

Proof. The proof of Lemma 3 relies on the following lemma.

Lemma 11. *There exists a threshold $v^* > 0$ such that $x^M(v) > x^C(v)$ if and only if $v \geq v^*$.*

Proof of Lemma 11. Let π^M be the overall equilibrium engagement level of the consumer when the firms collude, and $2\pi^C$ be the overall engagement level of the consumer in the competitive trading equilibrium. For every $v \in V$, let $W^M(v) = W\left(\frac{\pi^M}{1-\pi^M} e^{v/k-1}\right)$ and $\phi^C(v) = \phi(v; \pi^C)$ solving (8). Since $x^M(v) = k(1 + W^M(v))$ and $x^C(v) = k(1 + \phi^C(v))$, it follows that $x^M(v) > x^C(v)$ if and only if $W^M(v) > \phi^C(v)$. By the definition of the Lambert function, $W^M(v)e^{W^M(v)} = \frac{\pi^M}{1-\pi^M} e^{v/k-1}$. Moreover, $x \mapsto xe^x$ is a strictly increasing function of $x > 0$. Therefore, $W^M(v) > \phi^C(v)$ if and only if

$$\frac{\pi^M}{1 - \pi^M} e^{v/k-1} > \phi^C(v)e^{\phi^C(v)}. \tag{24}$$

From equation (8), we know that $\phi^C(v)e^{\phi^C(v)} = \frac{1-\phi^C(v)}{2} \cdot \frac{2\pi^C}{1-2\pi^C} e^{v/k-1}$. Therefore, (24) is equivalent to

$$\frac{1 - \phi^C(v)}{2} < \frac{\pi^M}{1 - \pi^M} \cdot \frac{1 - 2\pi^C}{2\pi^C}. \tag{25}$$

Because $\phi^C(v)$ is strictly increasing in v, the conclusion of the lemma follows. $\square$

Lemma 2 in Matějka and McKay (2015) shows that, in equilibrium,

$$\mathbb{E}[U^M] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right],$$

and

$$\mathbb{E}[U^C] = \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right].$$

Consider the random variables Y^C and Y^M defined by $Y^C(v) := v - x^C(v)$ and $Y^M(v) = v - x^M(v)$. Let G^C and G^M be the CDF of Y^C and Y^M respectively, and define $\omega := \mathbb{E}_\lambda[x^M(v)] - \mathbb{E}_\lambda[x^C(v)]$. By assumption, $\omega \geq 0$. Finally, denote with u_1 and u_0 the maximal and minimal element in the support of Y^C respectively. From Lemma 11, we know that $u_0 \leq Y^M \leq u_1$ with probability 1. Furthermore, one can verify that $\omega \geq 0$ together with Lemma 11 imply

$$\int_u^{\bar{u}} G^C(y)dy \leq \int_u^{\bar{u}} G^M(y)dy \quad \text{for all } u \in [u_0, u_1].$$

This means that any expected utility maximizer with an increasing and convex Bernoulli utility function $w : [u_0, u_1] \to \mathbb{R}$ would prefer the lottery Y^C over Y^M (see Theorem 4 in Meyer (1977)). Now, observe that for every $\pi \in [0, 1/2]$, we have

$$k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right] = k \cdot \mathbb{E} \left[\ln \left(2\pi \cdot e^{\frac{Y^M}{k}} + 1 - 2\pi \right) \right],$$

and

$$k \cdot \mathbb{E}_{\mu^C} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right] = k \cdot \mathbb{E} \left[\ln \left(2\pi \cdot e^{\frac{Y^C}{k}} + 1 - 2\pi \right) \right].$$

Furthermore, the function $y \in (0, +\infty) \mapsto \ln \left(2\pi \cdot e^{y/k} + 1 - 2\pi \right)$ is strictly increasing and strictly convex in $y > 0$ whenever $\pi \in (0, 1/2)$. Therefore,

$$\begin{aligned}
\mathbb{E}[U^M] &= \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right] \\
&= k \cdot \mathbb{E}_{\mu^M} \left[\ln \left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M \right) \right] \\
&= k \cdot \mathbb{E} \left[\ln \left(\pi^M \cdot e^{\frac{Y^M}{k}} + 1 - \pi^M \right) \right] \\
&\leq k \cdot \mathbb{E} \left[\ln \left(\pi^M \cdot e^{\frac{Y^C}{k}} + 1 - \pi^M \right) \right] \\
&= k \cdot \mathbb{E}_{\mu^C} \left[\ln \left(\pi^M \cdot e^{\frac{v-x}{k}} + 1 - \pi^M \right) \right] \\
&< \max_{\pi \in [0,1/2]} k \cdot \mathbb{E}_{\mu^C} \left[\ln \left(2\pi \cdot e^{\frac{v-x}{k}} + 1 - 2\pi \right) \right] = \mathbb{E}[U^C].
\end{aligned}$$

where the first inequality is implied by $\pi^M \in (0, 1)$, while the last strict inequality is implied by the fact that the overall engagement level π^M is not a best response to μ^C. We conclude that $\mathbb{E}[U^C] > \mathbb{E}[U^M]$ as required. □

Proof of Proposition 3

Proof. First, we show that the threshold $\bar{k}' \geq \bar{k}$ exists. To this goal, we make use of equation (25) introduced earlier. Specifically, the proof of Theorem 3 shows that while both π^C and π^M converge to 0 as $k \uparrow k^*$, we have

$$\lim_{k \uparrow k^*} \frac{\pi^C(k)}{\pi^M(k)} = \frac{1}{2 - \frac{\mathbb{E}_\lambda \left[e^{2(v/k^* - 1)} \right]}{2\mathbb{E}_\lambda \left[e^{2(v/k^* - 1)} \right] - 1}}.$$

Such limit is strictly less than 1 because $\mathbb{E}_\lambda \left[e^{2(v/k^* - 1)} \right] > \left(\mathbb{E}_\lambda \left[e^{v/k^* - 1} \right] \right)^2$ due to Jensen inequality. (Recall that $\mathbb{E}_\lambda \left[e^{v/k^* - 1} \right] = 1$ by definition.) Therefore, while the LHS of equation (25) converges to $\frac{1}{2}$ because $\phi^C(v) \downarrow 0$ as $k \uparrow k^*$, the RHS of (25) is converging to a limit strictly greater than 1/2. As a result, equation (25) is satisfied eventually (i.e., as k approaches k^* from below) for all $v \in V$. The existence of the threshold $\bar{k}'$ follows immediately from this observation.

We now show that $\mathbb{E}_\lambda[x^M(v)] \geq \mathbb{E}_\lambda[x^C(v)]$ for all $k \in (0, \bar{k}]$. To this goal, we begin by showing that the aggregate engagement level under collusion is weakly larger than the engagement level each competitive firm experience in the sure-trade equilibrium. Formally:

Lemma 12. *Suppose $k \leq \bar{k}$. Then, $\pi^M \geq 1/2$.*

Proof. Ravid (2020) shows that the function $F^M(\cdot)$ defined in the proof of Theorem 3 is strictly convex when $k < k^*$, and satisfies $F^M(0) > 1 > F^M(1^-)$. This implies that, if $\pi^M \in (0,1)$ is the unique solution to $F^M(\pi^M) = 1$, we have $F^M(\pi) \geq 1$ if and only if $\pi^M \geq \pi$. Therefore, it is sufficient to show that:[40]

$$\frac{1}{2}F^M(1/2) = \mathbb{E}_\lambda\left[\frac{W(1/2,v)}{W(1/2,v)+1}\right] \geq 1/2 \tag{26}$$

whenever

$$\mathbb{E}_\lambda\left[e^{1-v/k}\right] \leq 1/e, \tag{27}$$

where

$$W(\pi, v) = W\left(\frac{\pi}{1-\pi}e^{v/k-1}\right).$$

Observe that we can interpret (26) as an objective function and (27) as a constraint set on the distribution over quality levels $\lambda \in \Delta(\mathbb{R}_+)$. To simplify the problem, change variable from v to $y = e^{1-v/k}$. That is, let

$$\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}.$$

We need to show that $V^* \geq 1/2$, where

$$V^* := \inf_{F\in\mathcal{F}} \mathbb{E}_F\left[\frac{W(1/y)}{W(1/y)+1}\right] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

$y \mapsto H(y) := \frac{W(1/y)}{W(1/y)+1}$ is strictly decreasing and strictly convex in $y \geq 0$. Therefore, V^* is achieved by the degenerate distribution $F = \delta_{1/e}$. Plugging in $y = 1/e$ in $H(\cdot)$, we get $H(1/e) = 1/2$, that is (26) holds. This shows that $\pi^M \geq 1/2$, as required. $\square$

Given equations (17), (7), and the fact that $\phi^C(v) = 1$ for all $v \in V$ when competitive trade is efficient, to complete the proof of Proposition 3, it is sufficient to show that $\mathbb{E}_\lambda[W(\pi^M, v)] \geq 1$, for all $k \leq \bar{k}$. To this goal, we use the optimization approach introduced earlier once again. Formally, define

$$\mathcal{F} := \{F \in \Delta(\mathbb{R}_+) : F \text{ is finitely supported}\}.$$

Since $\pi^M \geq 1/2$ (Lemma 12), it is enough to argue that $V^{**} \geq 1$, where

$$V^{**} := \mathbb{E}_F[W(1/y)] \quad \text{subject to} \quad \mathbb{E}_F[y] \leq 1/e.$$

The function $y \mapsto G(y) := W(1/y)$ is strictly decreasing and strictly convex. Therefore, V^{**} is achieved at $F = \delta_{1/e}$. Since, $G(1/e) = 1$, we are done. $\square$

[40] Observe that $k \leq \bar{k}$ if and only if (27) holds.

Proof (Sketch) of Lemma 4

Proof. Suppose (μ, σ, β) is a competitive trading equilibrium with $N \geq 2$ firms. That such equilibrium assessment must be symmetric follows from the same arguments used for the duopoly model. From Milgrom and Roberts (1990), we know that for every $v \in V$, each firm $i \in I$ uses a pure strategy $\sigma_i(\cdot|v) = \delta_{x(v,N)}$ that solves

$$\max_{x_i \geq 0} \frac{\pi e^{\frac{v-x_i}{k}}}{\pi \left(e^{\frac{v-x_i}{k}} + \sum_{j \neq i} e^{\frac{v-x(v,N)}{k}} \right) + 1 - N\pi} \cdot x_i$$

The result follows from taking the FOC and re-arranging, i.e., following a procedure similar to the proof of Lemma 2. □

Proof of Proposition 4

Part (i) The proof of part (i) follows from a straightforward extension of the proof of Theorem 2. To show that the overall trade engagement level increases with N in the competitive trading equilibrium, a straightforward extension of the proof of Proposition 1 suffices. Here, we show how the instrumental first step is extended. *Fix $M > N \geq 2$ arbitrarily, and for every $m \in \{N, M\}$, let $\phi_m > 0$ be the unique solution to*

$$\phi_m \left(m - 1 + e^{\phi_m} \frac{1 - m\pi^m}{\pi^m e^{v/k-1}} \right), \tag{28}$$

where $\pi^m \in (0, 1/m]$. If $M\pi^M = N\pi^N$, then $\phi_N > \phi_M$.

Part (ii)

Proof. Proof. Equation (28) can be equivalently re-written as

$$\pi^m e^{v/k-1} = \phi_m(m-1)\pi^m e^{v/k-1} + (1 - m\pi^m)\phi_m e^{\phi_m}.$$

Suppose by way of contradiction that $\phi_N \leq \phi_M$. Then, $\phi_N e^{\phi_N} \leq \phi_M e^{\phi_M}$ which, given our assumption $M\pi^M = N\pi^N$, implies

$$\pi^N(1 - \phi_N(N-1)) \leq \pi^M(1 - \phi_M(M-1)).$$

The above inequality is equivalent to $\frac{M}{N}(1 - \phi_N(N-1)) \leq 1 - \phi_M(M-1)$, which implies

$$\phi_N \geq \frac{M-1}{N-1}\phi_M + (\frac{M}{N} - 1) > \phi_M,$$

a contradiction. □

Part (iii) Notice that with $N \geq 2$ firms, the maximal price that can be sustained in a symmetric equilibrium becomes $x(v, N) = k \cdot \left(1 + \frac{1}{N-1}\right) = k \cdot \frac{N}{N-1}$.[41] Therefore, a sure-trade equilibrium

[41]This price corresponds to $\pi = 1/N$ which generalizes the case $\pi = 1/2$ of the duopoly setting.

exists if and only if $k \leq \bar{k}(N)$, where $\bar{k}(N)$ is the unique solution to $\mathbb{E}_\lambda \left[e^{\frac{N}{N-1} - v/k} \right] = 1$.

Part (iv) Fix $N \geq 2$ arbitrarily and let $k \in (\bar{k}(N), k^*)$. For each $p \in [0, 1/N)$, define $F_k^{C,N}(p)$ as

$$F_k^{C,N}(p) := \mathbb{E}_\lambda \left[\frac{1}{Np + (1 - Np) \cdot e^{\phi(p,v,k,N)+1-v/k}} \right],$$

where for $p > 0$, we let $\phi(p, v, k, N)$ be defined as the unique solution to equation (9), and we set $\phi(0, v, k, N) := 0$ for all $v \in V$ and $k \in (\bar{k}(N), k^*)$. For fixed k, one can show that the consumer trade engagement level with each firm in the competitive trading equilibrium with N active firms is given by the unique solution to $F_k^{C,N}(p) = 1$. Denote with $p^{C,N}(k)$ such solution. Because as $k \uparrow k^*$, offers are converging to k^* regardless of the number of active firms, the crucial step to prove part (iv) is to show that, for $N > M \geq 2$,

$$\lim_{k \uparrow k^*} \frac{p^{C,N}(k)}{p^{C,M}(k)} > \frac{M}{N} + \Theta, \tag{29}$$

for some $\Theta > 0$. Using the same arguments as in the proof of Theorem 3

$$\lim_{k \uparrow k^*} \frac{p^{C,N}(k)}{p^{C,M}(k)} = \frac{\mathbb{E}_\lambda \left[\frac{M\left(1-e^{1-v/k^*}\right)+1}{e^{2(1-v/k^*)}} \right]}{\mathbb{E}_\lambda \left[\frac{N\left(1-e^{1-v/k^*}\right)+1}{e^{2(1-v/k^*)}} \right]},$$

which implies that (29) is satisfied.

□

B Symmetry and pure strategies in competitive trading equilibria

In this Appendix, we show that any competitive trading equilibrium must be in pure strategies and symmetric. For this, we use the finding that all best responses by the consumer have to be of the form described in Lemma 1.

More formally, suppose a competitive trading equilibrium exists, and let the assessment (μ, σ, β) be one of them. From Matějka and McKay (2015) we know that since both firms are active, we must have $\pi_i = \mathbb{E}_\mu[\beta_i] > 0$ for all $i \in I$. Fix $v \in V$ arbitrarily and let β be given by Lemma 1. For fixed $\pi \in (0, 1/2]$, the payoff function of seller $i \in I$ is:

$$u_i(x_i, x_{-i}) = \left[\frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{l\in I} \pi_l e^{\frac{v-x_l}{k}} + 1 - \pi_i - \pi_j} \right] \cdot x_i.$$

Let $w_i = \log(u_i)$ be the log-transformed payoff function of seller $i \in I$. Each w_i is twice continuously differentiable in $\mathbb{R}^2_+$ and satisfies the following conditions:

1. $\frac{\partial^2}{\partial x_i \partial x_j} w_i(x_i, x_j) = \frac{1}{k^2} \cdot \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{l\in I} \pi_l e^{\frac{v-x_l}{k}} + 1 - \pi_i - \pi_j} \cdot \frac{\pi_j \cdot e^{\frac{v-x_j}{k}}}{\sum_{l\in I} \pi_l e^{\frac{v-x_l}{k}} + 1 - \pi_i - \pi_j} \geq 0.$

2. $\frac{\partial^2}{\partial x_i \partial x_j} w_i(x_i, x_j) + \frac{\partial^2}{\partial x_i^2} w_i(x_i, x_j) = \frac{-1}{k^2} \cdot \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{l\in I} \pi_l e^{\frac{v-x_l}{k}} + 1 - \pi_i - \pi_j} \cdot \frac{1 - \pi_i - \pi_j}{\sum_{l\in I} \pi_l e^{\frac{v-x_l}{k}} + 1 - \pi_i - \pi_j} - \frac{1}{x_i^2} < 0.$

Condition 1 guarantees that the log-transformed game played by the two sellers is *smooth supermodular*. As a result, we can apply Theorem 5 in Milgrom and Roberts (1990) (henceforth MR90), which implies that the game played by the sellers admits at least one pure strategy NE. As argued in section 4 of MR90, condition 2 guarantees that such an equilibrium is unique and corresponds to the unique profile of rationalizable actions. Thus, no other equilibrium (either pure or mixed) exists. For each $v \in V$, denote such unique equilibrium by $(\hat{x}_1(v), \hat{x}_2(v))$. In Lemma 15 below, we show that if $\pi_i > \pi_j$ then $\hat{x}_i(v) > \hat{x}_j(v)$. We now use this fact to show that in case a competitive trading equilibrium exists, it must be symmetric.

Lemma 14. *Let (μ, σ, β) be a competitive trading equilibrium, i.e., $\pi_i = \mathbb{E}_\mu[\beta_i] > 0$ for every $i \in I$. Then, (μ, σ, β) is symmetric.*

The intuition behind the result is simple. Suppose that $\pi_1 > \pi_2$. Then firm 1 faces a higher demand than firm 2 at any fixed price. In equilibrium, firm 1 would therefore optimally charge a higher price than firm 2. However, this implies that the consumer should buy from firm 1 *less often* than firm 2, i.e., $\pi_1 < \pi_2$, which leads to a contradiction.

Lemmas 1, 14, and Corollary 1 taken together imply the following important corollary that we use extensively in Section 4.

Corollary 3. *(μ, σ, β) is a symmetric trading equilibrium if and only if it is a competitive trading equilibrium.*

Proof of Lemma 14

Proof. First, notice that if $\pi_1 = \pi_2$, then the profile $(u_i)_i$ (hence, $(w_i)_i$) satisfies $u_i(x_i, x_j) = u_j(x_j, x_i)$ for $i \neq j$. If $\hat{x}_1(v) \neq \hat{x}_2(v)$, then $\tilde{x}(v) = (\hat{x}_2(v), \hat{x}_1(v))$ would be another pure strategy NE, contradicting the uniqueness of $\hat{x}(v)$. This shows that, if $\pi_1 = \pi_2$, then $\hat{x}_1(v) = \hat{x}_2(v)$ for all $v \in V$, implying the symmetry of (μ, σ, β). Thus, it is sufficient to show that $\pi_1 = \pi_2 = \pi \in (0, 1/2]$ is a necessary condition for an equilibrium where both firms are active. To this goal, we first state and prove Lemma 15, which says that if $\pi_1 > \pi_2 > 0$, for each $v \in V$, the unique (pure strategy) NE between the two firms $\hat{x}(v) = (\hat{x}_1(v), \hat{x}_2(v))$ is such that $\hat{x}_1(v) > \hat{x}_2(v)$.

Lemma 15. *If $\pi_1 > \pi_2 > 0$ then $\hat{x}_1(v) > \hat{x}_2(v)$ in equilibrium.*

Proof. Consider the following two player game where the action sets are $A_i = [0, \infty)$ for every $i \in I$, and the payoffs are given by

$$w_i(x_1, x_2) = \log\left(\frac{B^i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} + C}\right) + \log(x_i), \ \ (i \in I)$$

where B^1, B^2, C are parameters such that $B^1, B^2 > 0$, and $C \geq 0$. Again, applying the same arguments of MR90, we know that this game is smooth supermodular and admits a unique NE equilibrium which is pure. Denote it by $\hat{x} = (\hat{x}_1, \hat{x}_2)$. The equilibrium must be interior and is characterized by the following system of FOC:

$$\frac{k}{\hat{x}_i} = \frac{B^{-i} \cdot e^{\frac{v-\hat{x}_{-i}}{k}} + C}{\sum_{j=1,2} B^j \cdot e^{\frac{v-\hat{x}_j}{k}} + C}, \ \ (\forall i \in I).$$

If $B^1 = B^2 = B > 0$, the unique equilibrium $\hat{x}$ must be symmetric. In what follows we show that (a) the best response function of firm $i \in I$ moves up as B^i grows, while the best response function of firm $j \neq i$ moves downwards. Moreover, we prove that (b) each best response function is upward-sloping with a slope always strictly below unity. Since such best responses cross only once, from (a) and (b) we immediately conclude that

$$B^1 > B^2 \implies \hat{x}_1 > \hat{x}_2.$$

The best response function of firm 1, say $x_1^* = x_1^*(x_2) \in [0, \infty)$, is given implicitly as the unique solution to the equation

$$k \cdot \left(\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} + C\right) = \left(B^2 \cdot e^{\frac{v-x_2}{k}} + C\right) \cdot x_1. \tag{30}$$

First, notice that since $\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} > B^2 \cdot e^{\frac{v-x_2}{k}} > 0$, we immediately infer that $x_1^*(x_2) > k$

for all $x_2 \geq 0$. Totally differentiating equation (30) for B^1 we further obtain

$$\frac{\partial x_1^*}{\partial B^1}(x_2) = \frac{e^{\frac{v-x_1^*}{k}}}{B^1 \cdot e^{\frac{v-x_1^*}{k}} + B^2 \cdot e^{\frac{v-x_2}{k}} + C}.$$

This shows that $\frac{\partial x_1^*}{\partial B^1}(x_2) > 0$ for all $x_2 \geq 0$. We now show that the opposite strict inequality obtains for the BR function $x_2^* = x_2^*(x_1)$ of firm 2 over the (relevant) domain for x_1 given by $(k, \bar{x}]$. The best response function x_2^* of firm 2 is given as the unique solution to the equation

$$k \cdot \left(\sum_{j=1,2} B^j \cdot e^{\frac{v-x_j}{k}} + C \right) = \left(B^1 \cdot e^{\frac{v-x_1}{k}} + C \right) \cdot x_2. \tag{31}$$

Totally differentiating for B^1 equation (31), we get

$$\frac{\partial x_2^*}{\partial B^1}(x_1) = \frac{e^{\frac{v-x_1}{k}}}{B^1 \cdot e^{\frac{v-x_2^*}{k}} + B^2 \cdot e^{\frac{v-x_1}{k}} + C} \cdot (k - x_2^*)$$

which is < 0 because $x_2^* > k$ always. Finally, totally differentiating equation (30) with respect to x_2 (respectively, equation (31) with respect to x_1), we get

$$\frac{\partial}{\partial x_j} x_i^*(x_j) = \frac{B^i \cdot e^{\frac{v-x_i^*}{k}}}{B^i \cdot e^{\frac{v-x_i^*}{k}} + B^j \cdot e^{\frac{v-x_j}{k}} + C} \cdot \frac{B^j \cdot e^{\frac{v-x_j}{k}}}{B^j \cdot e^{\frac{v-x_j}{k}} + C} < 1,$$

for all $i \in I$, $j \neq i$, and $x_j \geq 0$. We conclude that $B^1 > B^2$ implies that $\hat{x}_1 > \hat{x}_2$. Finally, notice that when $\pi_i = B^i > 0$ for each $i \in I$, and $C = 1 - \pi_1 - \pi_2 \geq 0$, we get back our model of Bertrand competition with a rationally inattentive consumer. Therefore, $\pi_1 > \pi_2$ implies that $\hat{x}_1(v) > \hat{x}_2(v)$ for every $v \in V$, as required. □

Suppose towards a contradiction that (μ, σ, β) is an equilibrium with $\pi_1 > \pi_2 > 0$. From Lemma 15, we know that $\hat{x}_1(v) > \hat{x}_2(v)$ for every $v \in V$. This is in contradiction with the consumer trading with firm 1 strictly more often. To see this more formally, notice that $\hat{x}_1(\cdot) > \hat{x}_2(\cdot)$ immediately implies that for each $v \in V$, we have

$$\frac{\beta_1(v, \hat{x}_1(v), \hat{x}_2(v))}{\pi_1} < \frac{\beta_2(v, \hat{x}_1(v), \hat{x}_2(v))}{\pi_2}.$$

(Recall that $\pi_1 > \pi_2 > 0$.) But then, given that V is finite and in equilibrium $\mathbb{E}_\mu[\beta_i/\pi_i] = 1$ for every $i \in I$, we would reach the conclusion that $1 = \mathbb{E}_\mu[\beta_1/\pi_1] < \mathbb{E}_\mu[\beta_2/\pi_2] = 1$, an absurd. □

C Ravid's (2020) monopoly model is equilibrium outcome-equivalent to perfect price coordination and perfect profit internalization

First, we show that a model with two firms and one manager setting two offers simultaneously to maximize joint profits is equilibrium-outcome equivalent to the monopoly model of Ravid (2020). We do so by showing that the offers *accepted* by the consumer in any equilibrium with trade of this game will be equal to

$$\hat{x}(v) = k\left[1 + W\left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} \cdot e^{v/k-1}\right)\right] \tag{32}$$

for every $v \in V$, where W denotes the Lambert's function, and each π_i is the consumer's equilibrium trade engagement level with firm $i \in I$. In other words, we will show that on-path, accepted equilibrium offers must take the *same* functional form derived by Ravid (2020) in his monopoly model when the monopolist produces a good of quality v and faces the aggregate demand

$$Q(x) = \frac{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}}}{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}} + 1 - \pi_1 - \pi_2}.$$

Since the shape of the monopolist's best response determines all the properties satisfied by the unique trading equilibrium in Ravid's (2020) model, equation (32) suffices to prove the equilibrium outcome-equivalance of the two models.

Analysis. Fix $v \in V$. Suppose each firm $i \in I$ faces a demand for its product given by

$$Q^i(x_1, x_2) := \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2}.$$

Suppose the same manager runs both firms and aims at maximizing joint profits. His payoff is given by

$$\Pi(x_1, x_2) = \sum_{i=1,2} \frac{\pi_i \cdot e^{\frac{v-x_i}{k}}}{\sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2} \cdot x_i.$$

He therefore solves (P): $\max_{x_1, x_2 \geq 0} \Pi(x_1, x_2)$. If $0 = \pi_i < \pi_j$ for some $i \in I$, the problem of the common-manager is identical to the problem faced by the monopolist in Ravid's model. The equilibrium outcome-equivalence is therefore immediate. Thus, from now on suppose that $\pi_i > 0$ for all $i \in I$. Below, we show that (P) admits a unique critical point which coincides with the global optimum. Moreover, we show that the solution to (P) is symmetric.

Let $D = D(x_1, x_2) = \sum_{j=1,2} \pi_j \cdot e^{\frac{v-x_j}{k}} + 1 - \pi_1 - \pi_2$. The first order conditions associated to problem (P) are:

$$\frac{\pi_i e^{\frac{v-x_i}{k}}}{D} - \frac{\pi_i x_i}{k \cdot D^2} \cdot \left[e^{\frac{v-x_i}{k}}\left(\pi_j e^{\frac{v-x_j}{k}} + 1 - \pi_i - \pi_j\right)\right] + \frac{\pi_j e^{\frac{v-x_j}{k}}}{D^2} x_j \cdot \frac{e^{\frac{v-x_i}{k}}}{k} = 0, \ (i \in I).$$

Multiplying both sides by $D^2 > 0$, dividing both sides by $\pi_i e^{\frac{v-x_i}{k}} > 0$ and re-arrenging, we get

$$D = \frac{\pi_j e^{\frac{v-x_j}{k}}}{k} \cdot (x_i - x_j) + \frac{x_i}{k}(1 - \pi_i - \pi_j), \ (i \in I). \tag{33}$$

Notice that for all $x_j \geq 0$, if $x_i = 0$, then the LHS of (33) is strictly greater than the RHS. This implies that any solution to (P) (if exists) has to be interior. Conversely, there exists a $\bar{x}_i > 0$ such that, for all $x_i \geq \bar{x}_i$, the RHS of (33) is strictly greater than the LHS for all $x_j \geq 0$. This means that $x_i \mapsto \Pi(x_i, x_j)$ is eventually decreasing in x_i for all $x_j \geq 0$, implying that (P) indeed admits a bounded solution.

Now, combing both equations in (33) yields

$$\frac{x_1 - x_2}{k} \cdot D = 0$$

which is true if and only if $x_1 = x_2$. Thus, any critical point of Π must lie on the 45^o-line. Therefore, problem (P) is equivalent to solve

$$\max_{x \geq 0} \frac{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}}}{(\pi_1 + \pi_2) \cdot e^{\frac{v-x}{k}} + 1 - \pi_1 - \pi_2} \cdot x \tag{34}$$

which, from the analysis in Ravid (2020), we know admits a unique critical point

$$\hat{x}(v) = k \left[1 + W \left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} \cdot e^{v/k-1} \right) \right]$$

that coincides with the global maximum. Therefore, the solution to (P) is symmetric and given by $(x_1^*, x_2^*) = (\hat{x}(v), \hat{x}(v))$ in (32).

Two comments are in order. First, given $v \in V$, the problem in (34) is equivalent to the problem faced by the monopolist in Ravid's model when the logit demand he faces is characterized by $\pi^M = \pi_1 + \pi_2$. Second, given that at the optimum $x_1^* = x_2^* = \hat{x}(v)$ for all $v \in V$, firms' offers are *a priori homogeneous*. Therefore, Lemma 1 implies that the consumer's best response depends on (π_1, π_2) only through $\pi_1 + \pi_2$. The first comment implies that Ravid's (2020) model and our two-firms-one-manager model are equilibrium outcome-equivalent. Indeed, in any equilibrium with trade, the following variables coincide across the two models: (i) Offers accepted on-path, (ii) industry's total profits, (iii) the consumer's surplus, and (iv) trading efficiency. The second comment implies that it is without loss to focus on symmetric assessments to characterize all the equilibrium outcomes of the model.[42] Indeed, the following holds.

Lemma 16. *An equilibrium with total trade engagement level $\pi^M = \pi_1 + \pi_2$ exists if and only if a symmetric equilibrium with total trade engagement level π^M exists.*

[42]To see this more clearly, fix (π_1, π_2) arbitrarily. Let $\sigma_i(\cdot|v) = \delta_{\hat{x}(v)}$ for all $v \in V$, and μ be consistent with $\sigma = (\sigma_1, \sigma_2)$. By definition, if the recommendation strategy β given in (2) characterized by (π_1, π_2) solves the consumer's problem given μ and is RVP, then (μ, σ, β) is an equilibrium. Let $\bar{\pi} = \frac{\pi_1+\pi_2}{2}$, and consider the assessment (μ, σ, β^*), where β^* is given in (5) and is characterized by $(\bar{\pi}, \bar{\pi})$. Clearly, (μ, σ, β^*) is symmetric. We show that (μ, σ, β^*) is an equilibrium as well. Since μ is symmetric, β^* is still a best response to μ. Moreover, by Corollary 1, it is RVP. On the other hand, because $\hat{x}(v)$ depends on (π_1, π_2) only through $\pi_1 + \pi_2$, each σ_i is a best response to σ_{-i} given β^*.

Relation to the collusion benchmark presented in Section 3: For fixed $v \in V$, given that Π admits a unique critical point $(\hat{x}(v), \hat{x}(v))$, we conclude that $(\hat{x}(v), \hat{x}(v))$ is also the unique NE of the game played by two independent managers that perfectly internalize each other's profits in their payoff function.[43] This is because the FOC of (P) are sufficient to characterize the managers' best responses, and any NE between the two independent managers must by a critical point of Π. Therefore, like the model of perfect price coordination discussed above, the equilibrium predictions of the collusion benchmark presented in Section 3 are equivalent to those obtained by Ravid (2020) in his monopoly analysis. In other words, the monopoly model of Ravid (2020), the model of perfect price coordination presented above, and the model of perfect profits internalization of Section 3 are all equilibrium outcome-equivalent.

The proof of Theorem 1 follows.

Proof of Theorem 1

The necessity of $k < k^*$ for the existence of any equilibrium with trade follows from the same arguments presented in the proof of Theorem 2. For the sufficiency part, focus on symmetric assessments. (This is without loss in light of Lemma 16.) Given that, for every $v \in V$, firms' equilibrium offers are given by (32), invoking Theorem 1 of Ravid (2020) suffices to prove that a trading equilibrium exists. Finally, the uniqueness of the trading equilibrium *outcome* follows from Theorem 1 of Ravid (2020), and the fact that the offers accepted on path $(\hat{x}(v))_{v \in V}$ only depend on the overall trade engagement level $\pi^M = \pi_1 + \pi_2$.

Q.E.D.

[43] Since $(\hat{x}(v), \hat{x}(v))$ is the unique NE in pure strategies, no equilibrium in mixed strategies exists.

D Obfuscation

Suppose that firms can obfuscate offers. Let us represent this by positing the existence of an ex-ante stage where firms actively interfere with the consumer's information processing cost. Specifically, we focus on a setting in the spirit of Carlin (2009): in stage 1, each firm simultaneously selects a pricing complexity $f_i \geq 0$ knowing that the profile (f_1, f_2) determines the overall information processing cost $k = \kappa(f_1, f_2) \geq 0$ the consumer bears to compare offers. In the second stage, the firms and the consumer play the pricing game described in Section 2, where the consumer's unit cost of information processing is given by k. Assume throughout that $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is unbounded, continuous and strictly increasing in both arguments, and let $k_{\min} := \kappa(0, 0) \geq 0$ be the minimal unit cost of information processing of the given industry.[44] The constant $k_{\min}$ is exogenous and industry-specific. It represents the information processing cost if the firms are as transparent as possible in marketing their offers.

Let us focus on the (pure) NE outcomes of this multi-stage game where firms anticipate a symmetric (RVP) trading equilibrium will always arise in the second stage whenever possible. For a given $0 \leq k_{\min} < k^*$, define $\mathrm{NE}^M(k_{\min})$ and $\mathrm{NE}^C(k_{\min})$ as the set of framing strategy profiles that constitute a Nash equilibrium when the firms collude or compete, respectively.[45] Finally, let $\hat{k} \in (\bar{k}, k^*)$ be the threshold identified in Theorem 3. The following Proposition holds:

Proposition 5. *Suppose that $\hat{k} \leq k_{\min} < k^*$. Then, there exists a profile $(f_1^C, f_2^C) \in NE^C(k_{\min})$ such that*

$$\Pi^M(\kappa(f_1^M, f_2^M)) < 2\Pi^C(\kappa(f_1^C, f_2^C))$$

for all $(f_1^M, f_2^M) \in NE^M(k_{\min})$.

Proposition 5 strengthens our main result: Competing firms can still generate higher expected profits than under collusion, even if strategic obfuscation is feasible. A sufficient condition for this is that the industry is complex enough. Under such a condition, strategic obfuscation cannot damage competitive firms. To see why, observe that irrespective of the market structure $m \in \{C, M\}$, any profile of price complexities (f_1, f_2) inducing a profit-maximizing information processing cost $\kappa(f_1, f_2) \in \arg\max_{k \geq k_{\min}} \Pi^m(k)$ constitutes a Nash equilibrium of the first stage. However, when $\hat{k} \leq k < k^*$, competitive firms' profits are higher than under collusion for any level of k. Therefore, when $k_{\min} \geq \hat{k}$, if competitive firms optimally coordinate their pricing complexity strategies, they generate higher profits than under collusion, irrespective of how colluding firms play in the first stage.

Proof of Proposition 5

Suppose that $\hat{k} \leq k_{\min} < k^*$. The proposition follows from Theorem 3 and the fact that

$$\Pi^M(k) < 2\Pi^C(k) \leq \max_{k' \geq k_{\min}} 2\Pi^C(k'), \quad \forall k \in [k_{\min}, k^*). \qquad \textit{Q.E.D.}$$

[44] For example, the mapping $\kappa(f_1, f_2) = k_{\min} + f_1 + f_2$ is admissible.

[45] Both $\mathrm{NE}^M(k_{\min})$ and $\mathrm{NE}^C(k_{\min})$ are non-empty sets since $k \mapsto \Pi^M(k)$ and $k \mapsto \Pi^C(k)$ are continuous functions for $k \geq k_{\min}$, and $\kappa(\cdot, \cdot)$ is continuous.

E Random marginal cost

Throughout the paper, we assumed that quality is random and marginal costs are 0. We opted for this modeling choice to simplify comparison with Ravid (2020). However, all our results go through if the quality v of the good is known ex-ante, but the (common) marginal cost of the firms is unknown.

Concretely, suppose $v \in \mathbb{R}_+$ is fixed, and let $c \sim \tilde{\lambda}$ be firms' identical realized marginal cost. The variable c is drawn according to some prior $\tilde{\lambda} \in \Delta(\mathbb{R}_+)$, which we assume to have strictly positive finite support. Finally assume that $v > c$ for all $c \in \operatorname{supp} \tilde{\lambda}$. The firms make simultaneous offers after observing the common cost parameter c. The consumer's payoff remains unchanged, while the new payoff obtained by firm i is now given by

$$\tilde{\Pi}_i^C := \beta_i(v, x_1, x_2) \cdot (x_i - c).$$

The remainder of this section explains the minor adjustments needed to carry our previous results into this new setting. The proofs follow the same steps as the proofs of the main model, with only slight adaptations in the algebra.

Since the consumer side of the market remains unchanged, Lemma 1 and Corollary 1 remain valid. Further, the arguments in Appendix B follow nearly identically. Hence, any competitive trading equilibrium is symmetric and the firms use pure strategies.

Define $\tilde{k}^*$ as the unique solution to

$$\mathbb{E}_{\tilde{\lambda}}[e^{\frac{v-c}{k}} - 1] = 1,$$

and $\tilde{\tilde{k}}$ as the unique solution to

$$\mathbb{E}_{\tilde{\lambda}}[e^{2-\frac{v-c}{k}}] = 1.$$

Theorem 1 follows as in the main text, where k^* is replaced by $\tilde{k}^*$. The colluding firms' optimal strategy is now given by

$$\tilde{x}(v) = c + k\left[1 + W\left(\frac{\pi_1 + \pi_2}{1 - \pi_1 - \pi_2} \cdot e^{\frac{v-c}{k} - 1}\right)\right].$$

With this slight modification in the firms' best response, the proof of the theorem follows steps analogous to the proof of Theorem 1 in Ravid (2020).

Similarly, the firms' competitive behavior is now given by the pure strategy

$$\tilde{x}(c) = c + k \cdot (1 + \phi(c)),$$

where $\phi(c)$ is the unique solution to

$$\left(1 + e^{\phi(c)} \cdot \frac{1 - 2\pi}{\pi e^{\frac{v-c}{k} - 1}}\right)\phi(c) = 1.$$

This can be shown by solving the pricing game between the firms for its unique Nash equilibrium.

Note that $\tilde{x}(c)$ is increasing in c. Intuitively, the larger the marginal cost, the higher the marginal revenue needs to be, given the demand the firm faces. Hence, the firm chooses a higher price, while serving less demand. An implication of this is that firms' prices are random from the point of view of the uninformed consumer.

Theorem 2 follows with the slight adaptation that a competitive trading equilibrium exists if and only if $k < \tilde{k}^*$. This also allows the results in section 4.2 to go through nearly unchanged: Fixing k, the competitive trading equilibrium features a higher trade engagement level than the collusive trading equilibrium, and the competitive trading equilibrium restores efficiency when $k \leq \tilde{\tilde{k}}$. Finally, our main result (Thereom 3) can be even strengthened: for any realized marginal cost c, competing firms generate higher profits than colluding firms if k is large enough.

F Beyond entropy cost

In this section, we generalize our model with respect to the cost function we used to represent the consumer's costly information processing. We show that the primary economic force behind our main result (Theorem 3) – i.e., the *attention effect* –[46] is robust to cost functions that are different from mutual information.

We assume that the consumer uses recommendation strategies $\beta = (\beta^1, \beta^2)$. Given a prior $\mu \in \Delta(V \times \mathbb{R}^2_+)$, the consumer pays a cost proportional to[47]

$$C(\mu, \beta) = f(\mathbb{E}_\mu[\beta^1], \mathbb{E}_\mu[\beta^2]) - \mathbb{E}_\mu[f(\beta^1, \beta^2)], \tag{35}$$

where $f : [0,1]^2 \to \mathbb{R}$ is a strictly concave function on $\mathcal{D} := \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$. Entropy costs in (1) satisfy equation (35) with f being equal to

$$f^H(p_1, p_2) = -[p_1 \log(p_1) + p_2 \log(p_2) + (1 - p_1 - p_2) \log(1 - p_1 - p_2)].$$

Throughout, we assume that f is symmetric, and four times continuously differentiable on $\mathcal{D}$. We also assume that $C(\mu, \beta)$ is overall strictly convex in β.[48]

Whenever clear from the context, we will often write $f(y)$ to denote $f(y_1, y_2)$. For all $y \in \mathcal{D}$, we will also denote with $f_i(y)$ the first derivative of f with respect to the i-th component. Similarly, $f_{ii}(y)$ denotes the second derivative with respect to the i-th component, while $f_{ij}(y)$ $(i \neq j)$ denotes the cross derivative.

The consumer's problem is

$$\max_{\beta^1(\cdot), \beta^2(\cdot)} \mathbb{E}_\mu \left[\sum_{i \in I} (v - x_i) \beta^i(v, x_1, x_2) \right] - k \cdot C(\mu, \beta) \tag{36}$$

subject to

$$\beta^i(\cdot) \geq 0, \forall i \in I, \quad \text{and} \quad \sum_{i \in I} \beta^i(\cdot) \leq 1.$$

Because pointwise (36) is a strictly concave optimization problem on a compact and convex set, a standard FOC approach suffices. The following lemma summarizes.

Lemma 17. *The strategy (β^1, β^2) solves (36) if and only if there exist functions $\lambda^i : V \times \mathbb{R}^2_+ \to \mathbb{R}_+$ $(i \in I)$, and $\theta : V \times \mathbb{R}^2_+ \to \mathbb{R}_+$ such that the following equations hold μ-almost everywhere:*

$$\frac{v - x_i}{k} + \lambda^i(v, x_1, x_2) - f_i(\mathbb{E}_\mu[\beta]) + f_i(\beta(v, x_1, x_2)) = \theta(v, x_1, x_2), \quad \forall i \in I \tag{37}$$

[46] In equilibrium, the consumer's demand expands when firms compete.

[47] Similar cost functions feature in Fosgerau et al. (2020), Ravid (2020), Yang (2019), and Flynn and Sastry (2021).

[48] As f^H is continuously differentiable only in the interior of D, the analysis in Appendix F does not incorporate our main model. We make the everywhere-differentiability assumption for tractability, but we conjecture that the arguments developed can be extended to incorporate more general costs.

$$\lambda^i(v, x_1, x_2)\beta^i(v, x_1, x_2) = 0, \quad \forall i \in I \tag{38}$$

$$\theta(v, x_1, x_2)\left[1 - \sum_{i \in I} \beta^i(v, x_1, x_2)\right] = 0. \tag{39}$$

Like in our main model, plain-vanilla optimality requires that conditions (37), (38) and (39) only need to hold μ-almost everywhere. To better discipline the best response of the consumer, we invoke the RVP refinement once again. This allows us to pin down (β^1, β^2) everywhere.

Lemma 18. *Suppose (β^1, β^2) solves (36) and is RVP. Then, equations (37), (38) and (39) are satisfied for all $(v, x_1, x_2) \in V \times \mathbb{R}^2_+$.*

Proof. Fix $\hat{v} \in V$ and $\hat{x}_1, \hat{x}_2 \geq 0$ arbitrarily, and let (μ_n) be the appropriate belief perturbation such that (i) $\mu_n \to \mu$ strongly, (ii) $\mu_n(\hat{v}, \hat{x}_1, \hat{x}_2) > 0, \forall n \in \mathbb{N}$, and β is a best response to μ_n for all $n \in \mathbb{N}$. Since $(\hat{v}, \hat{x}_1, \hat{x}_2) \in \operatorname{supp} \mu_n$, equations (37), (38) and (39) hold at $(\hat{v}, \hat{x}_1, \hat{x}_2)$. In particular,

$$\frac{\hat{v} - \hat{x}_i}{k} + \lambda^i_n(\hat{v}, \hat{x}_1, \hat{x}_2) - f_i(\mathbb{E}_{\mu_n}[\beta]) + f_i(\beta(\hat{v}, \hat{x}_1, \hat{x}_2)) = \theta_n(\hat{v}, \hat{x}_1, \hat{x}_2), \quad \forall i \in I$$

for every $n \in \mathbb{N}$. Let $\lambda^i_{(\hat{v}, \hat{x}_1, \hat{x_2})} := \liminf_n \lambda^i_n(\hat{v}, \hat{x}_1, \hat{x}_2)$ and $\theta_{(\hat{v}, \hat{x}_1, \hat{x_2})} := \liminf_n \theta_n(\hat{v}, \hat{x}_1, \hat{x}_2)$. Since $\mu_n \to \mu$ strongly and β^i is bounded for each $i \in I$, we have that $\mathbb{E}_{\mu_n}[\beta^i] \to \mathbb{E}_\mu[\beta^i]$ for every $i \in I$. Therefore,

$$\frac{\hat{v} - \hat{x}_i}{k} + \lambda^i_{(\hat{v}, \hat{x}_1, \hat{x}_2)} - f_i(\mathbb{E}_\mu[\beta]) + f_i(\beta(\hat{v}, \hat{x}_1, \hat{x}_2)) = \theta_{(\hat{v}, \hat{x}_1, \hat{x}_2)}, \quad \forall i \in I$$

Now, let multipliers $\lambda^{*,i}(\cdot)$ and $\theta^*(\cdot)$ be such that $\lambda^{*,i}(v, x_1, x_2) = \lambda^i(v, x_1, x_2)$ and $\theta^*(v, x_1, x_2) = \theta(v, x_1, x_2)$ if $(v, x_1, x_2) \in \operatorname{supp} \mu$, where λ^i and θ are the multipliers from equations (37), (38) and (39) of Lemma 17; and $\lambda^{*,i}(v, x_1, x_2) = \lambda^i_{(v, x_1, x_2)}$ and $\theta^*(v, x_1, x_2) = \theta_{(v, x_1, x_2)}$, otherwise. By construction, (β^1, β^2) and multipliers $\lambda^{*,i}(\cdot)$ and $\theta^*(\cdot)$ solve equations (37), (38) and (39) for all $(v, x_1, x_2) \in V \times \mathbb{R}^2_+$.

□

Equation (37) is a complicated condition involving the strategy (β^1, β^2) and its expectation with respect to μ. To enhance tractability and allow for easier comparative statics, we replace (37) with equations (40) and (41) displayed below. Due to strict concavity, (37) and (40) & (41) are indeed equivalent conditions.

$$\frac{v - x_i}{k} + \lambda^i(v, x_1, x_2) - \theta(v, x_1, x_2) - f_i(\pi^1, \pi^2) + f_i(\beta(v, x_1, x_2, \pi^1, \pi^2)) = 0, \quad \forall i \in I \tag{40}$$

$$\pi^i = \mathbb{E}_\mu[\beta^i], \quad \forall i \in I \tag{41}$$

Let $\pi^i := \mathbb{E}_\mu[\beta^i]$ for all $i \in I$. Since (36) is symmetric and strictly concave, it is without loss to focus on symmetric RVP recommendation strategies for the consumer whenever μ is symmetric.

The following assumption demands that the firms' pricing game is well-behaved whenever the firms face an RVP optimal symmetric demand from the consumer.

Assumption 1. *Suppose (β^1, β^2) is symmetric, solves (36) and is RVP with $\pi^i = \pi > 0$. Then,*

- *The mapping $x_i \mapsto x_i \beta^i(v, x_1, x_2, \pi^1, \pi^2)$ is strictly concave in $x_i \geq 0$ for all $v \in V$.*
- *For each $v \in V$, there exists a symmetric pure strategy equilibrium $(x^m(v), x^m(v))$ in both the pricing games under collusion ($m = M$) and under competition ($m = C$).*

The next assumption is on the derivatives of f. As Theorem 4 shows, it guarantees the existence of the attention effect within the model.

Assumption 2. *The function f satisfies*

$$0 > f_{ij} > f_{ii} \quad \text{and} \quad \frac{f_{iii} + f_{iij} + 2 f_{iij} f_{ij} / f_{ii}}{f_{ii} + f_{ij}} > -2, \quad \forall y \in \mathcal{D}. \tag{42}$$

Theorem 4. *Suppose that Assumptions 1 and 2 are satisfied. Suppose further that there exists a unique symmetric pure strategy equilibrium in both the collusion and competition model with $0 < \beta^i(v, x^m(v), x^m(v), \pi^m, \pi^m) < 1/2$ for all $v \in V$ and $m \in \{M, C\}$. Then, the equilibrium overall trade engagement level under collusion is lower than under competition, i.e., $\pi^M \leq 2\pi^C$.*

Notably, the function f^H associated with expected entropy reduction does not satisfy Assumption 2.[49] Thus, Assumption 2 is not necessary, but only sufficient for an attention effect to be present in the model.

Proof of Theorem 4

The proof consists of three steps. First, we show that given any optimal RVP symmetric demand from the consumer with $\pi^i = \pi \in (0, 1/2)$, for all $v \in V$, the equilibrium offer $x^C(v)$ increases with π. Second, we prove that for any given RVP optimal symmetric demand function from the consumer with $\pi^i = \pi > 0$, moving from the collusion equilibrium to the competitive trading equilibrium lowers prices uniformly, i.e., $x^M(v) > x^C(v)$ for all $v \in V$. Finally, we show that if we perturb any symmetric belief μ so that firms' offers are uniformly more advantageous, the consumer will best respond by increasing his overall trade engagement level. Taken together, these three claims imply that in the competitive trading equilibrium, the consumer's trade engagement level is higher than under collusion, i.e., the attention effect holds.

Lemma 19 (Step 1)**.** *Fix $v \in V$ arbitrarily, and let $x^C(v)$ be the symmetric equilibrium offer of the pricing game between competitive firms facing a symmetric RVP optimal demand with $\pi \in (0, 1/2)$ and $0 < \beta^i(v, x^C(v), x^C(v), \pi, \pi) < 1/2$. Then, $\frac{\partial}{\partial \pi} x^C(v) \geq 0$.*

Proof. Given Assumption 1, $x^C(v)$ is uniquely identified by the following firms' FOC:

$$\beta^i(v, x^C(v), x^C(v), \pi, \pi) + x^C(v) \beta^i_i(v, x^C(v), x^C(v), \pi, \pi) = 0, \tag{43}$$

[49] f^H is not differentiable at the boundary points of $\mathcal{D}$. Also, it does not satisfy the second condition of (42) everywhere in $\mathcal{D}^o$.

where, to simplify notation, we wrote β_i^i to denote $\frac{\partial}{\partial x_i}\beta^i$.[50]

We totally differentiate (43) to obtain:

$$\frac{\partial}{\partial \pi}x^C(v) \cdot \left[2\beta_i^i + \beta_j^i + x^C(v)(\beta_{ii}^i + \beta_{ij}^i)\right] = -\left(\beta_\pi^i + x^C(v)\beta_{\pi i}^i\right) \tag{44}$$

In what follows, we will show that $\left[2\beta_i^i + \beta_j^i + x^C(v)(\beta_{ii}^i + \beta_{ij}^i)\right] < 0$ and $\left(\beta_\pi^i + x^C(v)\beta_{\pi i}^i\right) > 0$, thus proving the statement.

Given that (β^1, β^2) is symmetric and interior at $(v, x^C(v), x^C(v), \pi, \pi)$ – i.e., $0 < \beta^i < 1/2$ – the Lagrangian multipliers from the consumer's FOC (40) are zero. It follows that (β^1, β^2) satisfies

$$\frac{v - x_i}{k} - f_i(\pi, \pi) + f_i(\beta^1, \beta^2) = 0, \quad \forall i \in I. \tag{45}$$

Claim 2. $2\beta_i^i + \beta_j^i + x^C(v)(\beta_{ii}^i + \beta_{ij}^i) < 0$

Proof of Claim 2. Totally differentiating (45) with respect to x_i and x_j gives:

$$\frac{1}{k} = f_{ii}\beta_i^i + f_{ij}\beta_j^i \tag{46}$$

$$0 = f_{ii}\beta_j^i + f_{ij}\beta_i^i. \tag{47}$$

Solving the resulting linear system for β_i^i and β_j^i yields:

$$\beta_i^i = \frac{1}{k}\frac{f_{ii}}{f_{ii}^2 - f_{ij}^2} \quad \text{and} \quad \beta_j^i = -\frac{1}{k}\frac{f_{ij}}{f_{ii}^2 - f_{ij}^2}.$$

Observe that, by strict concavity, $f_{ii} < 0$. Therefore, $\beta_i^i < 0$. Moreover, Assumption 2 guarantees that

$$\beta_i^i + \beta_j^i = \frac{1}{k}\frac{f_{ii} - f_{ij}}{f_{ii}^2 - f_{ij}^2} < 0. \tag{48}$$

Similarly, we totally differentiate (46) with respect to x_i and x_j to get values for β_{ii}^i and β_{ij}^i. Not surprisingly, they involve the third derivative of f. These expressions imply

$$\beta_{ii}^i + \beta_{ij}^i = -\frac{\beta_i^i + \beta_j^i}{f_{ii} + f_{ij}}\left(\beta_i^i(f_{iii} + f_{iij}) - 2\beta_j^i f_{iij}\right).$$

Since $\beta_j^i = -\beta_i^i f_{ij}/f_{ii}$, we get

$$\beta_{ii}^i + \beta_{ij}^i = -\frac{\beta_i^i + \beta_j^i}{f_{ii} + f_{ij}}\beta_i^i\left(f_{iii} + f_{iij} + 2f_{iij}f_{ij}/f_{ii}\right). \tag{49}$$

From (43), we know that $x^C(v) = -\beta^i/\beta_i^i$. Combining this fact with equations (48) and (49),

[50] Throughout, we will also write β_j^i for $\frac{\partial}{\partial x_j}\beta^i$, and β_π^i for $\frac{\partial}{\partial \pi}\beta^i$. Cross derivatives will be denoted using similar notation.

we get:

$$2\beta_i^i + \beta_j^i + x^C(v)(\beta_{ii}^i + \beta_{ij}^i) = \beta_i^i + (\beta_i^i + \beta_j^i)\left[1 + \frac{\beta^i}{f_{ii} + f_{ij}}\left(f_{iii} + f_{iij} + 2f_{iij}f_{ij}/f_{ii}\right)\right].$$

The above expression is strictly negative because $\beta^i < 1/2$ and Assumption 2 is satisfied. This completes the proof of the claim.

□

Claim 3. $\beta_\pi^i + x^C(v)\beta_{\pi i}^i > 0$

Proof of Claim 3. We totally differentiate (45) with respect to π to obtain:

$$\beta_\pi^i \cdot \left(f_{ii}(\beta^1, \beta^2) + f_{ij}(\beta^1, \beta^2)\right) = f_{ii}(\pi, \pi) + f_{ij}(\pi, \pi) \tag{50}$$

Observe that Assumption 2 together with (50) implies that $\beta_\pi^i > 0$.

We now totally differentiate equation (50) with respect to x_i to obtain an expression for $\beta_{\pi i}^i$. This expression is:

$$\beta_{\pi i}^i = \frac{-1}{f_{ii} + f_{ij}}\left[\beta_i^i(f_{iii} + f_{iij}) - 2\beta_j^i f_{iij}\right] \tag{51}$$

We conclude that

$$\beta_\pi^i + x^C(v)\beta_{\pi i}^i = \beta_\pi^i\left[1 + \frac{\beta^i}{f_{ii} + f_{ij}}\left(f_{iii} + f_{iij} + 2f_{iij}f_{ij}/f_{ii}\right)\right] > 0$$

as required.

□

This concludes the proof of Lemma 19. □

Lemma 20 (Step 2)**.** *Suppose that β is a symmetric RVP demand from the consumer with $\pi^i = \pi > 0$. For every $v \in V$, let $(x^m(v), x^m(v))$ be the symmetric (pure) NE of the pricing game between firms given β under collusion ($m = M$) and competition ($m = C$), respectively. Then, $x^C(v) < x^M(v)$ for all $v \in V$.*

Proof. Fix $v \in V$ arbitrarily. The symmetric offer $x = x^M(v)$ must satisfy the following FOC:

$$\beta^i(v, x, x, \pi, \pi) + x[\beta_i^i(v, x, x, \pi, \pi) + \beta_j^i(v, x, x, \pi, \pi)] = 0,$$

where we used the fact that $\beta_j^i = \beta_i^j$ given symmetry. Now, Assumption 2 implies that $\beta_j^i > 0$. Therefore,

$$\beta^i(v, x, x, \pi, \pi) + x\beta_i^i(v, x, x, \pi, \pi) < 0.$$

On the other hand, RVP implies that $\beta^i(v,0,0,\pi,\pi) > 0$ because $\pi > 0$.[51] Define the mapping $y \mapsto g(y) := \beta^i(v,y,y,\pi,\pi) + y\beta^i_i(v,y,y,\pi,\pi)$ for $y \geq 0$. Observe that $g(\cdot)$ is continuous and satisfies $g(0) > 0 > g(x)$. Therefore, $g(\cdot)$ must admit a zero $x^* \in (0,x)$. It is straightforward to see that $x^* = x^C(v)$, i.e., x^* is the symmetric pure strategy equilibrium between competitive firms given β. We conclude that $x^C(v) < x^M(v)$ as required.

□

Lemma 21 (Step 3)**.** *Let μ be a symmetric belief, and $\tilde{\mu}$ be a symmetric perturbation of μ such that for every $v \in V$, there exists a $c_v > 0$ with*

$$\tilde{\mu}(v,x,x) = \mu(v,x+c_v,x+c_v), \quad \forall x \geq 0.$$

Suppose that $\beta = (\beta^1,\beta^2)$ is a symmetric RVP best response to μ, and that $\tilde{\beta}$ is a RVP symmetric best response to $\tilde{\mu}$. Finally, suppose that β^i (resp., $\tilde{\beta}^i$) is in $(0,1/2)$ μ-a.e. (resp., $\tilde{\mu}$-a.e.). Then,

$$\mathbb{E}_\mu[\beta^i] = \pi^* < \tilde{\pi} = \mathbb{E}_{\tilde{\mu}}[\tilde{\beta}^i].$$

Proof. From the proof of Lemma 19 and Assumption 2, we know that $\beta^i_i + \beta^i_j < 0$. It follows that

$$\beta^i(v,x-c_v,x-c_v,\pi,\pi) > \beta^i(v,x,x,\pi,\pi), \quad \forall v \in V, \quad \forall x \geq 0, \quad \forall \pi \in [0,1/2]. \tag{52}$$

We now consider the function $g(\pi) := \mathbb{E}_\mu\left[\beta^i(v,x_1,x_2,\pi,\pi)\right] - \pi$, where β^i solves (40) everywhere. To satisfy equation (41), we need $g(\pi) = 0$. Note, that $g(0) \geq 0$. Further, by strict concavity of our objective function, there is a unique best response of the consumer. Therefore, there can be at most one π such that $g(\pi) = 0$. Since μ is symmetric, the consumer's best response has to be symmetric as well, and therefore there is a π^* such that $g(\pi^*) = 0$. Since g is continuous in π, this implies that $g(\pi) \geq 0$ for all $\pi \leq \pi^*$.

Next, consider $\tilde{g}(\pi) := \mathbb{E}_{\tilde{\mu}}\left[\beta^i(v,x_1,x_2,\pi,\pi)\right] - \pi$. From (52), we know that $\mathbb{E}_{\tilde{\mu}}\left[\beta^i(v,x_1,x_2,\pi,\pi)\right] > \mathbb{E}_\mu\left[\beta^i(v,x_1,x_2,\pi,\pi)\right]$. Thus, for all $\pi \leq \pi^*$, $\tilde{g}(\pi) > g(\pi) \geq 0$. Let $\tilde{\pi}$ be the solution to $\tilde{g}(\tilde{\pi}) = 0$, which exists since $\tilde{\mu}$ is symmetric. It must be that $\mathbb{E}_{\tilde{\mu}}[\tilde{\beta}^i] = \tilde{\pi} > \pi^* = \mathbb{E}_\mu[\beta^i]$, as required

□

To conclude the proof of Theorem 4, suppose towards a contradiction that $2\pi^C < \pi^M$. Then, competitive prices are lower than collusive ones for two reasons. By Lemma 20, prices decrease compared to collusion, when keeping the consumer's strategy fixed. By Lemma 19, prices further decrease from lowering π. However, Lemma 21 implies that with lower prices, the consumer's best reply has to feature a higher trade engagement level π, a contradiction.

Q.E.D.

[51] To see why, consider equation (40). If $\beta^i(v,0,0,\pi,\pi) = 0$, then $\lambda^i(v,0,0) \geq 0 = \theta^i(v,0,0)$. As a result, equation (40) implies $v/k - f_i(\pi) + f_i(0) \leq 0$. But this is an absurd, because Assumption 2 implies that $x \mapsto f_i(x,x)$ is a decreasing function.

F.1 Quadratic cost example

In this subsection we consider a specific example which satisfies Assumptions 1 and 2. In the spirit of Theorem 4, we show that the attention effect exists, i.e., that the trade engagement level is higher under competition than under collusion. We then show that in this example, the attention effect can dominate and lead to higher profits under competition than under collusion. This shows that our main result – namely, Theorem 3 – is not specific to entropy-based cost functions.

The cost function we consider is given by

$$f(p_1, p_2) = -\left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \theta p_1 p_2\right).$$

In order to satisfy Assumption 2, we assume $\theta \in (0, 1)$. Further, we assume that $k \geq 2\frac{v_{\max}}{1-\theta^2}$, where $v_{\max} := \max V$, to avoid existence issues and cumbersome boundary cases.

Consumer best response. We begin by finding the consumer's optimal recommendation strategy under an arbitrary belief μ, which also defines the demand the firms face. To this end, we use Lemma 18, and assume that $\beta_i(v, x_1, x_2) \in (0, \frac{1}{2})$ μ-almost surely, to find

$$\beta_i(v, x_1, x_2; \pi_i) = \frac{1}{1-\theta^2}\frac{v - x_i}{k} - \frac{\theta}{1-\theta^2}\frac{v - x_j}{k} + \pi_i.$$

Optimality requires that $\pi_i = \mathbb{E}_\mu[\beta_i(v, x_1, x_2; \pi_i)]$, and $\mathbb{E}_\mu\left[\frac{v-x_i}{k}\right] = 0$ whenever $\pi_i \in (0, \frac{1}{2})$. The assumption that $\beta_i(v, x_1, x_2) \in (0, \frac{1}{2})$ μ-almost surely has to be verified in equilibrium.

Colluding firms. Colluding firms take the consumer's strategy as given, and for every $v \in V$, solve the problem

$$\max_{x_1, x_2 \geq 0} \Pi^M(x_1, x_2) := \max_{x_1, x_2 \geq 0} x_1\beta_1(v, x_1, x_2) + x_2\beta_2(v, x_1, x_2). \tag{53}$$

For any fixed x_2, $\Pi^M(x_1, x_2)$ is concave in x_1, and vice versa. Further, note that

$$\frac{\partial \Pi^M}{\partial x_i} = \frac{1-\theta}{1-\theta^2} \cdot \frac{v}{k} - \frac{2}{1-\theta^2} \cdot \frac{x_i}{k} + \frac{2\theta}{1-\theta^2}\frac{x_j}{k} + \pi_i,$$

which is strictly positive when $x_i = 0$. Hence, any solution to (53) has to be an interior critical point, and therefore solve $\frac{\partial \Pi^M}{\partial x_i} = \frac{\partial \Pi^M}{\partial x_j} = 0$.

Re-arranging the FOC gives

$$x_i(v) = \frac{1-\theta}{2} \cdot v + \theta x_j + \frac{1-\theta^2}{2} \cdot k \cdot \pi_i, \quad \forall i \in I, j \neq i.$$

Any solution to the above system of linear equations constitutes a NE of the pricing game between colluding firms. If $\pi_i > \pi_j$, then the colluding firms would therefore choose $x_i(v) > x_j(v)$ for all $v \in V$, which contradicts optimality of $\pi_i > \pi_j$. Therefore, any equilibrium has to feature symmetric behavior by the firms and the consumer. The firm's solution to (53) is then given by

$$x(v) = \frac{1}{2} \cdot v + \frac{1+\theta}{k} \cdot k \cdot \pi,$$

where $\pi \in (0, 1/2)$ is the symmetric trade engagement level of the consumer. Note that, indeed, all offers of this form result in $\beta_i \in (0, \frac{1}{2})$.

In order to ensure that the consumer behaves optimally, we need to have $\mathbb{E}\left[\frac{v-x_i}{k}\right] = 0$ for all $i \in I$. This is satisfied in a trading equilibrium if and only if $\pi_i = \frac{1}{k} \cdot \frac{\mathbb{E}[v]}{1+\theta} =: \frac{1}{2}\pi^M$.

Competing firms. When firms compete, each (π_1, π_2) induces a pricing game between the firms. In particular, each firm solves

$$\max_{x_i \geq 0} \mathbb{E}_{\sigma_{-i}}[\Pi^C(x_1, x_2)] := \max_{x_i \geq 0} \mathbb{E}_{\sigma_{-i}}[x_i \beta_i(v, x_1, x_2)], \tag{54}$$

taking the other firm's strategy σ_{-i} as given. As in the main model, we can apply arguments from Milgrom and Roberts (1990) to determine that the unique Nash-Equilibrium of this game has to be in pure strategies. Further, since $\pi_i > \pi_j$ implies that $x_i(v) > x_j(v)$ for all $v \in V$, any competitive trading equilibrium has to be symmetric due to arguments similar to those in Appendix B. The Nash equilibrium of the pricing game is given by

$$x_i(v) = \left(1 - \frac{1}{2-\theta}\right) \cdot v + \frac{1-\theta^2}{2-\theta} \cdot k \cdot \pi.$$

As a result, we get that indeed all β_i are interior: $\beta_i \in (0, \frac{1}{2})$ μ almost-surely. It remains to find a π that makes the consumer's behavior optimal. Since $\pi \in (0, 1)$, we need $\mathbb{E}\left[\frac{v-x}{k}\right] = 0$, which is satisfied if an only if $\pi^C = \frac{1}{k} \cdot \frac{\mathbb{E}[v]}{1-\theta^2}$.

Equilibrium comparisons. In this example, one can verify that $\pi^M = (1-\theta) \cdot 2\pi^C$. In other words, the attention effect is present also for this specific cost function. In our main model, we showed that the attention effect can be large enough to create higher profits under competition than under collusion. Under quadratic costs, we get the following difference in profits:

$$2\mathbb{E}\left[\Pi^C\right] - \mathbb{E}\left[\Pi^M\right] = \frac{1}{k}\left(\frac{1}{2-\theta} - \frac{1}{2}\right)\left[\frac{2-\theta}{1-\theta^2}\mathbb{E}[v]^2 - \left(\frac{1}{2-\theta} - \frac{1}{2}\right)\frac{\mathrm{Var}(v)}{1+\theta}\right].$$

Hence, under quadratic costs, profits are higher under competition than under collusion whenever $\mathbb{E}[v]$ is large relative to the $\mathrm{Var}(v)$.

G Partial Internalization

So far, we have compared the consequences of rational inattention across two opposite market structures: Collusion, where firms perfectly internalize each others' profits, and competition, where firms are antagonists. In this section, we extend our analysis to the case of *partial and symmetric profits internalization*. Denote by $\rho \in [0, 1]$ the degree of profits' internalization, that is, for each $i \in I$, the profits of firm i are given by

$$\Pi_i^\rho := \beta_i(v, x_1, x_2) \cdot x_i + \rho \cdot (\beta_{-i}(v, x_1, x_2) \cdot x_{-i}) \,. \tag{55}$$

Collusion and competition are obtained as special cases when $\rho = 1$ and $\rho = 0$, respectively. Edgeworth (1881) and Cyert and DeGroot (1973) refer to ρ as the coefficient of *effective sympathy* between firms, while more recently Azar and Vives (2021) derives ρ from the level of portfolio diversification of common owners. Results in this section conform with the intuition developed so far: When $\rho \to 1$, the industry structure approximates collusion, and the economy displays the same feature of Ravid (2020) – trade failure and inefficiencies, while as $\rho \to 0$ we observe the implications of perfect competition discussed in section 4.2 – sure-trade and expansionary demand. Furthermore, our main result replicates in this setting: If attention costs are high enough, the firms obtain higher profits under perfect than imperfect competition.

We focus on *partially competitive trading equilibrium* in pure and symmetric strategies of both firms and the consumer,[52] and we refer to it as ρ-equilibrium for some level of profits internalization $\rho \in [0, 1]$. As for the extension on random marginal costs discussed in Appendix E, the demand side is unchanged, and Lemma 1 and Corollary 1 continue to hold. On the other hand, firms' behavior changes, adapting to the incentives of partial internalization, as shown by the following lemma.

Lemma 22. *Suppose (μ, σ, β) is a ρ-equilibrium. Then, given $v \in V$, each seller $i \in I$ plays a symmetric pure strategy $\sigma_i(\cdot|v) = \delta_{x(v)}$ given by*

$$x(v) = k \cdot (1 + \phi(v)) \tag{56}$$

where $\phi(v)$ is the unique solution to

$$\left(\frac{1-\rho}{1+\rho} + e^{\phi(v)} \cdot \frac{1-2\pi}{(1+\rho)\, \pi e^{\frac{v-k}{k}}} \right) \phi(v) = 1. \tag{57}$$

Proof. The proof is similar to the one of Lemma 2. For every $i \in I$, we need to characterize the symmetric equilibrium in pure strategy $\sigma_i(\cdot|v) = \delta_{x_i(v)}$ when $\pi_i > 0$. For every $v \in V$, $\beta_i(v, x_1, x_2)$ follows equation (5), and firm i solves the following problem

$$\max_{x_i \geq 0} \frac{\pi_i \cdot x_i \, e^{\frac{v-x_i}{k}} + \rho\, \pi_{-i} \cdot x_{-i}\, e^{\frac{v-x_{-i}}{k}}}{\pi_i e^{\frac{v-x_i}{k}} + \pi_{-i} e^{\frac{v-x_{-i}}{k}} + 1 - \pi_i - \pi_{-i}}$$

where $x_{-i}(v) = x_{-i}$ denotes firm $-i$'s offer. By taking first-order conditions and rearranging,

[52] For simplicity, we assume that the consumer randomizes uniformly between offers whenever indifferent.

we obtain

$$\frac{d\Pi_i^{\rho}(v,x_1,x_2)}{dx_i} = \beta_i(v,x_1,x_2)\left(1-\frac{x_i}{k}+\frac{1}{k}\cdot\left[x_i\,\beta_i(v,x_1,x_2)+\rho\,x_{-i}\,\beta_{-i}(v,x_1,x_2)\right]\right)=0.$$

Symmetry of firms' strategies, $x_i(v)=x_{-i}(v)=x(v)$, makes the consumer indifferent about the two option, $\beta_i=\beta_{-i}=\beta$, which implies $\pi_i=\pi_{-i}=\pi$. Hence, by imposing symmetry, we obtain

$$x(v;\pi)=k\cdot\left[1+\frac{(1+\rho)\cdot\pi e^{\frac{x(v;\pi)-v}{k}}}{(1-\rho)\cdot\pi e^{\frac{x(v;\pi)-v}{k}}+1-2\pi}\right].$$

Define $\phi(v;\pi)=\frac{(1+\rho)\cdot\pi e^{\frac{x(v;\pi)-v}{k}}}{(1-\rho)\cdot\pi e^{\frac{x(v;\pi)-v}{k}}+1-2\pi}$ to get

$$x(v;\pi)=k\cdot\left[1+\phi(v;\pi)\right].$$

Furthermore, by optimality and standard arguments, $\phi(v;\pi)$ is the unique solution to

$$\left(\frac{1-\rho}{1+\rho}+e^{\phi(v)}\cdot\frac{1-2\pi}{(1+\rho)\,\pi e^{\frac{v-k}{k}}}\right)\phi(v)=1.$$

Finally, for all $v\in V$, ϕ can be written as $\phi(v;\pi)=\frac{\beta(v,x(v;\pi),x(v;\pi))\cdot(1+\rho)}{1-\beta(v,x(v;\pi),x(v;\pi))\cdot(1+\rho)}$. □

Equation (57) is of interest, as it shows how ϕ varies with the coefficient ρ. In particular, when $\rho=0$, the equation collapses to the case of perfect competition displayed in (8); when $\rho=1$, the equation becomes

$$\left(e^{\phi(v)}\cdot\frac{1-2\pi}{2\pi e^{\frac{v-k}{k}}}\right)\phi(v)=1$$

which is equivalent to Ravid (2020). This observation is consistent with our discussion in Appendix C, where we show that the collusion and the monopoly model are outcome equivalent.

We now assess whether partial internalization affects the existence of a trading equilibrium. In light of Theorem 2 and the related discussion, we expect that intermediate levels of competition do not affect the attention cost threshold above which we do not observe trade. The following result conforms with this intuition. Define k^* as in equation (6).

Proposition 6. *For every $\rho\in[0,1]$, a ρ-equilibrium exists if and only if $k<k^*$. If a ρ-equilibrium exists, it is unique.*

Proof. The cases of $\rho\in\{0,1\}$ were previously established. The necessity direction closely follows the one of Theorem 2 and is thus omitted. For the sufficiency part, let $\bar{k}(\rho)$ be the unique solution to $\mathbb{E}_\lambda\left[e^{\frac{2}{1-\rho}-\frac{v}{k}}\right]=1$. Notice that, for every $\rho\in[0,1)$, we have that $k^*>\bar{k}(\rho)>0$ since

$$\mathbb{E}_\lambda\left[e^{\frac{2}{1-\rho}-\frac{v}{k^*}}\right]>\mathbb{E}_\rho\left[e^{1-\frac{v}{k^*}}\right]=\mathbb{E}_\lambda\left[\frac{1}{e^{\frac{v}{k^*}-1}}\right]\geq\frac{1}{\mathbb{E}_\lambda\left[e^{\frac{v}{k^*-1}}\right]}=1.$$

When $k \leq \bar{k}(\rho)$, consider firms' offers equal to $x(v) = \frac{2k}{1-\rho}$. Notably, this configuration of prices is an equilibrium of the pricing game with partial internalization when $\pi = 1/2$. By Corollary 1, $\pi = 1/2$ if and only if $\mathbb{E}_\lambda \left[e^{\frac{2}{1-\rho} - \frac{v}{k}} \right] \leq 1$, which is equivalent to $k \leq \bar{k}(\rho)$. A ρ-equilibrium exists in this case.

When $k \in (\bar{k}(\rho), k^*)$, let ϕ be the unique solution to

$$\left(\frac{1-\rho}{1+\rho} + e^{\phi} \cdot \frac{1-2\pi}{(1+\rho)\, \pi e^{\frac{v-k}{k}}} \right) \phi = 1.$$

The remainder simply adapts the proof of Theorem 2 to the ϕ defined above, and we omit it. □

Since consumer demand is attention-based, the market structure affects the overall trade engagement level: The higher the level of profit internalization, the lower the trade probability. This result extends the *attention effect* described by Proposition 1, i.e., the demand expands when firms compete, to intermediate levels of competition. We denote by π^ρ the overall trade engagement level when $\rho \in [0,1]$ is the level of partial internalization.

Proposition 7. *Let $k \in (0, k^*)$. For every $\rho', \rho'' \in [0,1]$, if $\rho' < \rho''$ then $0 < \pi^{\rho''} \leq \pi^{\rho'} \leq 1/2$.*

Proof. By inspecting the proof of Proposition 6, we can deduce that $\bar{k}(\rho'') < \bar{k}(\rho')$ for every $\rho', \rho'' \in [0,1]$ such that $\rho' < \rho''$. Thus, the statement is valid for every $k \in (0, \bar{k}(\rho'))$, since for $k \in (0, \bar{k}(\rho''))$ we have that $\pi^{\rho''} = \pi^{\rho'} = 1$ and for $k \in [\bar{k}(\rho''), \bar{k}(\rho''))$ we have that $\pi^{\rho''} < \pi^{\rho'} = 1$. We now consider the case of $k \in (\bar{k}(\rho''), k^*)$. First of all, notice that for a fixed $p \in (0, 1/2)$, $\phi^{\rho'}(p) < \phi^{\rho''}(p)$. This follows by noticing that, all things equal, the $\phi^\rho(p)$ that solves

$$\left(\frac{1-\rho}{1+\rho} + e^{\phi^\rho(p)} \cdot \frac{1-2p}{(1+\rho)\, p e^{\frac{v-k}{k}}} \right) \phi^\rho(p) = 1$$

increases as ρ increases. Let F be defined as in the proof of Theorem 2 and recall that $F(p^*) = 1$ if p^* is the equilibrium trade engagement level. For every $\rho \in [0,1]$, we denote by F^ρ the F defined by ϕ^ρ and by π^ρ the equilibrium trade engagement level under ρ. We have

$$\begin{aligned}
1 &= F^{\rho''}(\pi^{\rho''}) \\
&= \mathbb{E}_\lambda \left[\frac{1}{2\pi^{\rho''} + (1 - 2\pi^{\rho''}) \cdot e^{\phi^{\rho''}(\pi^{\rho''}) + 1 - v/k}} \right] \\
&< \mathbb{E}_\lambda \left[\frac{1}{2\pi^{\rho''} + (1 - 2\pi^{\rho''}) \cdot e^{\phi^{\rho'}(\pi^{\rho''}) + 1 - v/k}} \right] = F^{\rho'}(\pi^{\rho''}),
\end{aligned}$$

where the strict inequality follows by the argument above. From the proof of Theorem 2, we conclude that $\pi^{\rho''} < \pi^{\rho'}$. □

The following result documents the presence of sure-trade with imperfect competition. The region of attention cost where trade occurs with probability one is largest when firms perfectly compete but progressively shrinks as firms collude until disappearing under perfect collusion.

Proposition 8. *For every $\rho \in [0,1)$, a sure-trade ρ-equilibrium exists if and only if $k \leq \bar{k}(\rho)$, where $\bar{k}(\rho)$ solves*

$$\mathbb{E}\left[e^{\,2\cdot(1-\rho)^{-1} - \,v/k}\right] = 1.$$

Furthermore, $\bar{k}(\rho)$ decreases as ρ increases. Notably, $\bar{k}(0) = \bar{k}$, and $\bar{k}(1) = 0$.

Proof. The result follows from Proposition 6. □

We are finally ready to extend our main result, Theorem 3, to the case of partial internalization: For any level of imperfect competition, if the consumer is inattentive enough, firms benefit by competing more. In particular, for any two intermediate levels of competition, there exists a threshold of attention cost such that profits are higher when firms compete more for any attention costs above the given threshold. For every $\rho \in [0,1]$, denote by $\Pi^\rho(k)$ the industry profits when the degree of internalization is ρ and consumer's attention cost is k.

Proposition 9. *For every $\rho', \rho'' \in [0,1]$, if $\rho' < \rho''$ there exists $\hat{k} \in (0, k^*)$ such that $\Pi^{\rho''}(k) < \Pi^{\rho'}(k)$ for all $k > \hat{k}$.*

Proof. Part (i) of Lemma 9 continues to hold for every $\rho \in [0,1]$. We now prove the equivalent of part (ii) under partial internalization. We totally differentiate (57) with respect to p and k to find the partial derivatives of ϕ^ρ. After some calculations, we obtain

$$\phi^\rho_p(p,v,k) = \frac{\left(1 - \phi^\rho(p,v,k)\left(\frac{1-\rho}{1+\rho}\right)\right)(1+\rho)}{p(1-2p)\left(1-\rho+e^{\phi^\rho(p,v,k)}(1+\phi^\rho(p,v,k))\dfrac{1-2p}{p\cdot e^{v/k-1}}\right)} \geq 0$$

$$\phi^\rho_k(p,v,k) = -\frac{v}{k^2}\cdot\frac{\phi^\rho(p,v,k)e^{\phi^\rho(p,v,k)}}{(1-\rho)\cdot\frac{p}{1-2p}e^{v/k-1}+e^{\phi^\rho(p,v,k)}(1+\phi^\rho(p,v,k))} \leq 0$$

As $k \uparrow k^*$, $p \to 0$, and thus $\phi \to 0$. Therefore, as $k \uparrow k^*$, $\phi_p \to (1+\rho)\cdot e^{v/k^*-1}$ and $\phi_k \to 0$. Again, following the proof of Lemma 9, by total differentiating $F^\rho(p^\rho) = 1$ one can show that

$$\frac{\partial}{\partial k}p^\rho(k) = -\frac{A_\rho}{B_\rho}$$

where

$$A_\rho = \mathbb{E}_\lambda\left[\frac{1}{D^2_\rho}\cdot\left((1-2p^\rho(k))e^{\phi^\rho(p^\rho(k),v,k)+1-v/k}\cdot\left(\frac{v}{k^2}+\phi^\rho_k(p^\rho(k),v,k)\right)\right)\right],$$

$$B_\rho = \mathbb{E}_\lambda\left[\frac{1}{D^2_\rho}\cdot\left(2(1-e^{\phi^\rho(p^\rho(k),v,k)+1-v/k})+(1-2p^\rho(k))\cdot e^{\phi^\rho(p^\rho(k),v,k)+1-v/k}\cdot\phi^\rho_p(p^\rho(k),v,k)\right)\right],$$

and

$$D_\rho = 2p^\rho(k)+(1-2p^\rho(k))\cdot e^{\phi^\rho(p^\rho(k),v,k)+1-v/k}.$$

Letting $k \uparrow k^*$, we conclude that

$$\frac{\partial}{\partial k} p^{\rho}(k) \to -\mathbb{E}_{\lambda}\left[\frac{v}{(k^*)^2} \cdot e^{v/k^*-1}\right] / \mathbb{E}_{\lambda}\left[\frac{2(1-e^{1-v/k^*})+1+\rho}{e^{2\cdot(1-v/k^*)}}\right].$$

We use de L'Hopital rule to show that as $k \uparrow k^*$, the ratio $p^{\rho'}(k)/p^{\rho''}(k)$ is bounded above 1 strictly for $\rho' < \rho''$. Note that $\lim_{k\uparrow k^*} \frac{\partial}{\partial k} p^{\rho''}(k)$ exists and is different from 0. Therefore,

$$\lim_{k\uparrow k^*} \frac{p^{\rho'}(k)}{p^{\rho''}(k)} = \lim_{k\uparrow k^*} \frac{\frac{\partial}{\partial k} p^{\rho'}(k)}{\frac{\partial}{\partial k} p^{\rho''}(k)} = \frac{\mathbb{E}_{\lambda}\left[\frac{2(1-e^{1-v/k^*})+1+\rho''}{e^{2\cdot(1-v/k^*)}}\right]}{\mathbb{E}_{\lambda}\left[\frac{2(1-e^{1-v/k^*})+1+\rho'}{e^{2\cdot(1-v/k^*)}}\right]} = \frac{(3+\rho'')\cdot \mathbb{E}_{\lambda}\left[e^{2(v/k^*-1)}\right]-2}{(3+\rho')\cdot \mathbb{E}_{\lambda}\left[e^{2(v/k^*-1)}\right]-2} > 1.$$

Note that as $k \uparrow k^*$, $p^{\rho''}(k) \to 0$ and $p^{\rho'}(k) \to 0$, which implies $x_k^{\rho''}(v) \to k^*$ and $x_k^{\rho'}(v) \to k^*$ for all $v \in V$. Now, for $\Theta > 0$, fix $\varepsilon > 0$ such that

$$1 + \Theta - \varepsilon > \frac{k^* + \varepsilon}{k^* - \varepsilon},$$

and let $\hat{k} \in (\bar{k}, k^*)$ be such that $p^{\rho'}(k)/p^{\rho''}(k) > 1 + \Theta - \varepsilon$ and $x_k^{\rho}(v) \in (k^* - \varepsilon, k^* + \varepsilon)$ for all $k > \hat{k}$, $\rho \in \{\rho', \rho''\}$. For all $k > 0$, we have that $p^{\rho'}(k)(k^* - \varepsilon) > p^{\rho''}(k)(k^* + \varepsilon)$ if and only if

$$\frac{p^{\rho'}(k)}{p^{\rho''}(k)} > \frac{k^* + \varepsilon}{k^* - \varepsilon}.$$

Notice that this holds if $k \in (\hat{k}, k^*)$. Since by construction we have $\Pi^{\rho'}(k) \geq p^{\rho'}(k)(k^* - \varepsilon)$ and $p^{\rho''}(k)(k + \varepsilon) \geq \Pi^{\rho''}(k)$, we conclude that $\Pi^{\rho'}(k) > \Pi^{\rho''}(k)$ for all $k \in (\hat{k}, k^*)$ as required. $\square$

The comparative result on competition follows as a special case of Proposition 9 by fixing $\rho' = 0$. If the consumer is inattentive enough, the industry obtains higher profits under perfect than imperfect competition.

Corollary 4. *For every $\rho \in (0, 1]$, there exists $\hat{k} \in (0, k^*)$ such that $\Pi^{\rho}(k) < \Pi^{0}(k)$ for all $k > \hat{k}$.*