On Political Spectra *

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(Preliminary – please do not circulate)

Abstract

A political spectrum is a chart that places political opinions in a geometric space to illustrate their positions in relation to one another. We argue that its most essential feature is geometric representation of political disagreement: opinions which disagree more are positioned more separately in a spectrum. Building upon this insight, we formally define political spectra within a general social choice framework in which political opinions correspond to preferences. Specifically, we model a political spectrum as a mapping from preferences to a Euclidean space that geometrically represents disagreement in the preferences. We show that the model can accommodate a broad range of applications, and establish a number of existential and structural properties of political spectra thus modeled. We also find that a key parameter — the dimensionality of a spectrum — reflects the underlying ideological diversity within the society, and moreover, it is directly connected to the possibility for the society to elect a widely supported collective choice — the formal statement of this result subsumes findings in Grandmont (1978) and Nakamura (1979).

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I. Introduction

Political spectra are widely in practice, especially by media, educators and anlaysts of politics, not only in that they offer a highly visual, easily understandable description of the underlying political landscape which would otherwise be too abstract and complicated to grasp, but also because they can serve as a useful frame of reference that facilitates political discussion and analysis. The best known spectrum is possibly the linear spectrum where political opinions are positioned on a horizontal scale which captures some version of the Leftism-Rightism orientation. Another popular spectrum (Heywood (1992)) is more complex: opinions are positioned in a bi-axial coordinate system where one axis captures attitude towards individual liberty and the other captures attitude towards economic equality.

Despite many specific examples in practice, there seems to be a lack of a unified, theoretical understanding of political spectra in general. What, after all, is a political spectrum? Of course, it is possible to offer a loose summary that generalizes these examples: a political spectrum is a chart that places political opinions in a geometric space to illustrate their positions in relation to one another. However, this summary seems too loose to satisfy sharper and deeper inquiries. What is the essential function of a plitical spectrum? What are the components and structure that are necessary for that function? What are the principles that make a political spectrum functional? These questions motivate our paper, in which we will try to answer them and incorporate our answers in a model of political spectra that offers a unified framework within which one can analyze a spectrum or construct one for a political environment. We will also show that analyzing a spectrum based on the model allows us to derive interesting results about its underlying political environment.

A. The nature of political spectra

A political spectrum is a chart, and a chart is, essentially, a model: a visual representation that reflects the fundamental structure of what is represented, in our case a political environment consisting of various opinions. But what is the fundamental structure of the environment that is reflected by a spectrum? — In our view, it is the disagreement relation between political opinions.

Specifically, we argue that a political spectrum is essentially a model that uses geometric separation to reflect disagreement between political opinions. Roughly speaking, if two opinions are positioned more distantly on a spectrum, this should suggest that there is a larger disagreement between these two opinions. Indeed, what makes a political spectrum useful is that it translates an abstract relation — political disagreement — into a more concerete notion — geometric separation, so that the conceptually complex and murky political landscape of various opinions is more clearly illustrated in terms of an easy-to-grasp geometric configuration of dots or regions. In this sense, a political spectrum is similar to the graph of a function that translates algebraic relation into geometric relation, and in doing so renders a more straightforward picture of what the function "looks like".

What, then, are the necessary components of a political spectrum to serve the purpose discussed above? If we examine some of the popular spectra in use, it seems that they all have a coordinate system with one or more axes, where each axis cooresponds to a key aspect that the underlying political environment concerns. For instance, in the linear spectrum, the single axis corresponds to the Leftism vs. Rightism orientation, whereas in the bi-axial spectrum, one axis corresponds to individual liberty and the vertical axis to attitude economic equality.

In spite of its prevalence, We wish to argue that a coordinate system is not conceptually necessary for a political spectrum. This is because, after all, the disagreement relation between political opinions is not a cardinal relation, i.e. there is not an a priori scale that measures by how much two opinions disagree, and hence the scales in a coordinate system do not capture anything that is intrinsic to political disagreement. And if we remove the coordinate system from a spectrum, the underlying political landscape, i.e. how are opinions positioned in relation to each other, is still represented without compromise. The reason most political spectra have a coordinate system is because it is instrumental to their construction and interpretation. For instance, if there is a known way to respectively rate individual attitudes towards liberty and equality on a scale, say $0-10$, then constructing a spectrum is no more than placing opinions on the $[0,10]^2$ system, and interepreting the relation between two opinions is no more than reading off their respective coordinates.

Since a coordinate system is not essential to a political spectrum, the most general spectrum model should not entail it. Moreover, for certain political environments it can be difficult to come up with a suitable coordinate system. Indeed, to do that one would have to distill the key political aspects about the environment, and then translate various political opinions into coordinates to determine their positions. As such, a substantial extent of a priori understanding about the environment is called for to find a coordinate system, and if the environment is new, alien, or unfit for any known paradigm, then finding a suitable coordinate system would be in question. Thus, a spectrum model free of a coordinate system is not only conceptually more general, but practically also has a broader range of application.

Now we set out to describe our approach.

B. Political opinions as preferences

First, we model political opinions, and we model them as prefrences. This is because observational data about political opinions, which are either actual political choices, or responses to questionnaires about hypothetical political choices or general ideological stances, are, or can be easily translated to, preferences: over candidates, parties, policies, ideologies, or simply answers to survey questions. Thus, we are taking a revealed preference approach in modeling opinions. In addition to its empirical basis, this approach also has two theoretical advantages. First, since preferences are ordinal, cardinal benchmarks such as those afforded by a coordinate system is no longer necessary — just as we have argued above. Second, existing ideas and techniques for analyzing preferences will come handy for our study.

One may contend that revealed preferences do not always faithfully reflect the underlying opinions. For example, different ideologies can lead to the same preference relation over presidential candidates. We admit this, but this is a practical issue about the data not fully capturing the ideological variation underneath, and any spectrum model, as long as it has to base on data, would suffer from it regardless of how opinions are modeled. On the other hand, if a spectrum is taken as a theoretical model, then conceptually it is possible to construct a sufficiently detailed survey that fully identifies each opinion. In principle, our approach is no different from modeling individual needs and tastes for commodities as preferences.

As political spectra are about disagreement between opinions, we also need to formalize disagreement when opinions are modeled as preferences. Given two preference relations R and R', we define their *agreement set* $A(R, R')$ as the set of binary comparisons they agree on, i.e. $A(R, R') = \{(x, y) : xRy \text{ and } xR'y\}$. Naturally, a larger agreement set indicates a lower degree of disagreement between two opinions, since they often prefer the same alternative in a binary comparison. Agreement sets also help us determine the ideological relation between multiple opinions. Specifically, given three opinions R, R' and R'', we say that (R, R') *have less disagreement than* (R, R'') (or simply $R' \lhd_R R''$) if $A(R, R'') \subset A(R, R')$. Here, R serves as a reference point, and R' is interpreted as ideologically closer to R than R'' is, as (R, R') agree on more binary comparisons than (R, R''). Fixing the reference point R, the disagreement relation $\lhd_R$ clearly induces an ordering on all opinions which implies their relative ideological distance to R. The ordering is not necessarily complete because set inclusion is not a complete relation, but since our notion of ideological distance is quite innocuous, it seems that any other such notion would have to subsume ours.

C. Geometric representation of disagreement

We have argued above that a political spectrum is essentially a mapping f that projects opinions to a Euclidean space by which political disagreement is geometrically represented, i.e. if (R, R') have less disagreement than (R, R''), then $(f(R), f(R'))$ is geometrically less separated than $(f(R), f(R''))$. The most natural notion of geometric separateness is the Euclidean distance, that is, the shorter the distance, the less separateness there is. However, the Euclidean distance is not quite suitable for our purpose, the main reason being that the Euclidean distance to any reference point induces a complete ordering, whereas the ideological distance to a reference opinion R implied by $\triangleleft_R$ is not necessarily complete. This would cause a number of problems that compromise the use of a spectrum — we will elaborate on this point in Discussion.

We wish to come up with a notion of geometric separateness that resembles the mathematical structure of ideological distance, and that can is suggestive of not only separateness between points, but also that between regions, i.e. sets of points, because many spectra are like maps where an opinion occupies a region rather than just a point, especially when there are finitely many opinions. Our idea is formalized as follows: given three regions r, r', r'', (r, r) are said to *have less separation* than (r, r'') (or simply $r' \blacktriangleleft_r r''$) if there exist three points $s \in r, s' \in r', s'' \in r''$ *on the same line* such that $|s - s'| \leq |s - s''|$. To interpret, here r serves as a reference region from which r' is less separated than r'', because it is possible to find a straight route that starts from a point (s) in r and goes through a point (s') in r' before reaching some point (s'') in r''. Fixing the refrence region r, the separation relation $\blacktriangleleft^r$ induces an ordering on all regions in the space. This ordering is not necessarily complete, because not any two regions can be linked by a straight ray emitted from within r. Just like the disagreement relation, the separation relation is so innocuous that any other reasonable notion of geometric separateness, including the Euclidean distance, would have to subsume it.

D. Political spectra and their properties

Now we are ready to formally define a political spectrum in our framework: it comes down to partioning a Euclidean space into regions (which could be single points) and assigning each region r with an opinion $f(r)$, such that such that separateness between regions reflects disagreement between the underlying opinions, i.e. if $r' \blacktriangleleft_r r''$, then $f(r') \triangleleft_{f(r)} f(r'')$.

We show that a spectrum must exist for any political environment, i.e. any set of opinions. Moreover, it has a simplistic structure: the space is partitioned by a number of hyperplanes, and each region is thus a convex polytope. On the other hand, the dimensionality of the spectrum, i.e. how many dimensions there need to be for an Euclidean space to accommodate the spectrum, varies with the underlying environment, and more than three dimensions may be necessary. Although a high-dimensional spectrum may be difficult to visualize, its orderly mathematical structure can still help us better understand the underlying political environment, much like a high-dimensional "graph" of a function is helpful to the understanding of the underlying algebraic relation. We will focus on the analysis of spectrum dimensionality a little down below.

We show that for a wide range of political settings — precisely, those where opinions are rationalizable by utility functions that are linear in the agents' types, including the classical Hotelling-Downs model — it is straightforward to construct a political spectrum directly on the type space, thus giving theoretical foundation to the common practice of referring to the type space, such as the Hotelling "beach", as a political spectrum.

E. Spectrum dimensionality and its implications

We have brought up spectrum dimensionality. In addition to its practical importance that concerns the possibility of visualization, this parameter is also of a theoretical importance. Specifically, spectrum dimensionality is indicative of the diversity of opinions in the underlying political environment.

We show that a more diverse environment, i.e. has more opinions in the set-inclusion sense, calls for a (weakly) higher-dimensional spectrum. In particular, a consensual environment — where there is only one opinion — can be represented by a zero-dimensional spectrum, i.e. a single point, whereas if opinions in the environment satisfy the single-crossing condition — a well known restriction on preference variation — then they can be represented by a one-dimensional spectrum, that is, a classical linear spectrum. Is there an upper bound on spectrum dimensionality? We show that there indeed is, and that bound is $n-1$, where n is the number of alternatives over which the opinions are about. This upper bound is reached by the logically most diverse environment, which is the one that has every possible opinion. In addition, for settings discussed above that are based on utility functions linear in types, the dimensionality of the type is the spectrum dimensionality, because the type space doubles as a spectrum.

We proceed to unveil a deeper and more striking connection between spectrum dimensionality and the underlying environment when it comes to social choice. We show that the higher the spectrum dimensionality, the more difficult it is for the society to make a social choice that enjoys a broad popularity. More precisely, we define the *popularity* q of a social alternative x as the minimal fractional support it would get against another alternative, i.e. when pitched against any other alternative y, x is preferred by at least a fraction q of the population. We show that for an environment whose spectrum dimensionality is d, the popularity of the most popular alternative can be as low as $1/(1+d)$, but not lower. To interpret, the more diverse opinions are, resulting in a higher spectrum dimensionality, the lower the popularity of even the most popular social choice. This finding seems to confirm the intuition that diversity in political opinions compromises the possibility of reaching a broadly popular social agreement.

In fact, the above finding follows from a more general result that connects spectrum dimensionality to the existence of Condorcet voting cycles. In particular, we show that if the environment has a spectrum dimensionality of d, then there can be a $q-$Condorcet voting cycle, i.e. voting cycles that arise from pairwise ballots with winning threshold q, if and only if $q \leq 1/(1+d)$. For various settings, this result subsumes what is respectively found in Grandmont (1978) (in which $d = 1$), Nakamura (1979) (in which $d = n - 1$), and Greenberg (1979) (in which d is the same as the type space dimensionality).

Moreover, the above exercise is our attempt to demonstrate that the spectrum model is not only useful in illustrating a political landscape, but can also be used as an auxiliary model based on which analysis of the political landscape may be easier, thanks to some useful additional structures of our spectrum model. For example, proving this finding is made possible by an geometric convexity property of our model but is absent if one analyzes a discrete set of preferences directly. It is our hope that the spectrum model can make it easier to derive other social choice and mechanism design results.

F. Literature

Within the political economy literature, most studies that involve political spectra take them as given, and they usually come with an a priori coordinate. The entire literature on the Hotelling-Downs spatial model of elections (surveyed by Duggan (2005)) have this feature. Our paper complement this literature by giving a theoretical foundation to these spectra, anlaysing their essential properties, and providing an additional finding on social choice that subsumes some of the existing ones.

Academic works on political spectra per se mostly take the conventional coordinate system structure for granted and focuse on what the most appropriate coordinate system should be for specific political environments, e.g. whether the linear spectrum is adequate, what political orientations the axes should stand for to best represent ideologies in the society, etc. For examples, see Ferguson (1941), Eysenck (1975) Lester (1994) and Petrik (2010). Our paper discusses political spectra on a more general and abstract level and develops an approach that allows one to consider political spectra without knowledge or assumptions about a coordinate system.

Our notion of disagreement between preferences is motivated by Sprumont (1993), which in turn is inspired by a condition termed as *intermediateness* first studied in Grandmont (1978), although Grandmont (1978) essentially defines intermediateness on a continuous type space instead of a preference domain as Sprumont (1993) and we do.

Our analysis also contributes to the literature on Condorcet voting cycles. Most papers in this literature focus on simple-majority cycles, i.e. $q = 1/2$, and they are wonderfully surveyed in Monjardet (2009). For more recent progress, see Saari (2014), which discusses general cycles, and Puppe and Slinko (2019). Our contribution is that we identify novel and often easily verifiable conditions under which general Concorcet voting cycles are absent, namely those of spectrum dimensionality. As mentioned above, our finding also subsumes a number of existing results as special cases, including Grandmont (1978) and Nakamura (1979).

II. Model

Political environment

As motivated in the Introduction, we model individual political opinions as preferences. Specifically, there is a set X of relevant political alternatives, and each individual opinion is a rational preference relation R[1] over X. In most of our analysis we assume R is strict and X is finite,[2] though in a few places their relaxations are discussed.

We use xRy to denote that alternative x is preferred to alternative y under R, and $\overline{D}$ to denote the set of all strict rational preference relations over X. The set of opinions present in a specific political environment is described by its *domain*, i.e. the set of preference relations $D \subset \overline{D}$ that represent the opinions. In general, we use $\mathcal{V}(X, D)$ to denote a political environment thus modeled.

To interpret, each alternative in X can be an actual political alternative, e.g.. a presidential candidate, a particular parliament composition, a policy, etc., and the environment thus captures that of an actual social choice situation. In this case, a political opinion is a ranking of the available political alternatives. Another interpretation is that each altrnative in X represents an ideology or a system of values, and a political opinion is thus a preference over ideologies. One interesting case is that each $x \in X$ represents a specific response, i.e. a set of answers, to a questionnaire, something that is often used to determine one's political view. With this interpretation, our analysis will shed light on how to construct a political spectrum directly from responses to a questionnaire.[3]

[1]We will discuss the case of incomplete preferences.

[2]These assumptions are mostly made for expository simplicity. Relaxing them do not significantly change the analysis.

[3]Of course, if the only observational datum we have from an individual is his actual response to a

In the Introduction we have laid out the roadmap towards modeling political spectra as mappings that geometrically represents political disagreement in the underlying environment. Here we explain in more detail.

Political Disagreement

Given political opinions are modeled as preference relations, to determine the extent of disagreement them, it is logical to examine what *binary* comparisons they agree and disagree upon, since, ultimately, preference relations are binary relations and only concern binary comparisons. To that end, we define the *agreement set* between two opinions (R, R') as

$$A(R, R') := \{(x, y) \in X^2 : xRy \text{ and } xR'y\}.$$

Agreement sets give us a natural basis to develop a notion of ideological distance, i.e. what means for there to be "more" or "less" disagreement between opinions. This notion is formalized as follows.

Definition. (*Disagreement relation*) Given three opinions R, R', R'', (R, R') are said to *have less disagreement than* (R, R'') *do* if $A(R, R') \subset A(R, R'')$.

The disagreement relation is formally a triadic relation, which we denote as $R' \lhd_R R''$ Here, R serves as a reference point, and R' is interpreted as ideologically closer to R than R'' is, as (R, R') agree on more binary comparisons than (R, R'').

Fixing the reference point R, $\lhd_R$ becomes a binary relation on all opinions that implies their relative ideological distance to R. Clearly $\lhd_R$ is transitive but not necessarily complete because set inclusion is transitive but not always complete.[4] Thus it is not always possible to say which one of two given opinions disagrees more with R.

We are of the view that the our disagreement relation entails possibly the weakest conceptual requirement when it comes to comparing disagreement between opinions, or preference relations in general. Thus, any other reasonable notion of disagreement would imply ours, and hence imply the results obtained based on this notion.

Our notion is closely related to the notion of *intermediateness* proposed by Grandmont (1978). Indeed, $R' \lhd^R R''$ if and only if, using the terminology of Grandmont (1978), R' is intermediate between R and R''. Intermediateness is also known to be closely related to the the *single-crossing condition* which will also have implications to our analysis. Let us first define the condition.

questionnaire, then we cannot logically retrieve his entire preference over all possible responses.

[4]Technically, $\lhd^R$ induces a partial order on $\overline{D}$ known as the weak Bruhat order. In fact, with the weak Bruhat order, $\overline{D}$ becomes a lattice (Ziegler (1993)).

Definition. *(Single-crossing)* A domain D satisfies the *single-crossing* condition if opinions in D can be ordered as $R_1, ..., R_{|D|}$ such that for every $x, y \in X$ there is some $n \in \{1, ..., |D|\}$ given which xR_iy if and only if $i < n$.

It turns out that single-crossing is closely connected to the completeness of $\triangleleft_R$:

Lemma 1. *D satisfies the single-crossing condition if and only if there exists some $R \in D$ such that $\triangleleft^R$ is complete on D.*

Geometric Separateness

As a political spectrum geometrically reflects political disagreement, we need a notion of geometric separateness that is both intuitive and mathematically well-behaved. The most intuitive notion of geometric separateness is, of course, the Euclidean distance. However, as we will show in Discussion, a number of shorcomings render it less mathematically well-behaved for our purpose. In short, it is because the algebraic structure implied by the Euclidean distance is too dissimilar to that implied by the disagreement relation, which the former is meant to structurally reflect. Instead, we adopt the following notion.

Definition. (*Separation relation*) Given three subsets (which we usually call *regions*) $r, r', r'' \subset \mathbb{R}^d$, (r, r') are said to *have less separation than* (r, r'') *do* if there exist three points $s \in r$, $s' \in r'$ and $s'' \in r''$ such that $s'' - s = \lambda(s' - s)$ for some $\lambda > 1$.

To interpret, r serves as a reference region from which r' is less separated than r''. This holds when there there are three points respectively from the three regions such that they lie on the same line, and traveling from the point in r towards that in r'' i.e. the region more separated from r, we must first encounter the point from r', i.e. the region less separated from r. When the three regions are single points or convex and mutually disjoint sets, then the separation relation clearly coincides with our intuition about geometric separateness as a spatial relation. This is less so if the regions are arbitrary sets. Indeed, it is possible that a region is mroe separated from itself than a different regions is. However, it turns out that in our model of political spectra which will come soon, opinions will be be represented by convex and mutually disjoint sets, and hence geometric intuition applies.

The separation relation is formally parallel to the disagreement relation, as both are triadic relations with respect to a reference point, and both are meant to capture some sense of relative distance. It is partly this apparent resemblence that motivate us to use the separation relation to substantiate geometric separateness in order to best reflect political disagreement implied by the disagreement relation. We will see soon that, when applied to a political spectrum, both relations have a structural resemblence as well.

Political spectrum

Finally it is time to define the political spectrum. In short, a political spectrum is a configuration that assigns regions in a Euclidean space with political opinions, so that geometric separateness between regions reflects political disagreement between the underlying opinions. Formally:

Definition. (*Political spectrum*) A *political spectrum* that represents political environment $\mathcal{V}(X, D)$ is a tuple (S, P, f) such that

- $S \in \mathbb{R}^d$ is a convex Euclidean subspace;
- P is a partition of S;
- $f : P \to \overline{D}$ is a one-to-one mapping such that $D \subset f(P)$ and for any $r, r', r'' \in P$, if $r' \blacktriangleleft^r r''$, then $f(r') \triangleleft^{f(r)} f(r'')$.

To interpret, the spectrum is fit into a Euclidean subspace S, which is partitioned into a number of regions according to P, where each region r is assigned with an opinion by f. For this spectrum to be an appropriate description of the political environment $\mathcal{V}(X, D)$, we require that every opinion present in the environment be represented, i.e. $D \subset f(P)$, and that geometric separateness reflects political disagreement, i.e. $r' \blacktriangleleft^r r'' \to f(r') \triangleleft^{f(r)} f(r'')$.

Let us discuss other details embedded this definition. First, we require the entire subspace S to be convex, because (1) almost every political spectrum in use seems to have this property, (2) every point in the space can be interpreted, and (3) we can get rid of some over- and underrepresentation issues, see Discussion.[5] Second, we do not require $f(P)$ to be exactly equal to D, meaning the spectrum can include opinions other than what are present in the relevant environment. The foremost reasons for this leniency is technical: as we will discuss in more detail in the Dicussion, it guarantees the existence of a spectrum. Also, from a practical point, although allowing opinions not present in a given environment makes the spectrum more complex than necessary, the spectrum nonetheless does its fundamental job, which is to illustrate how opinions present in the society stand against each other, as this is not interfered by the presence of irrelevant opinions. Third, we only require that the separateness relation between regions implies the disagreement relation between the underlying opinions, but not vice versa. Thus, technically f need not be an isomorphism, and it is possible that (R, R') have less disagreement than (R, R'') but the regions representing these opinions

[5]Technically, our analysis will go through as long as S is connected.

do not have the corresponding separation relation.[6] We will discuss implications of the alternative formulation that requires isomorphism in Discussion section.

Here we collect some useful properties of a political spectrum that also help us envision its geometric structure.

Proposition 1. *Suppose (S, P, f) is a political spectrum that represents $\mathcal{V}(X, D)$.*

1. *Any distinct $r, r' \in P$ are separated by a hyperplane that does not intersect the interior of any $\hat{r} \in P$.*

2. *For any $r, r', r'' \in P$ the following conditions are equivalent: (a) $r' \blacktriangleleft^r r''$; (b) $r' \blacktriangleleft^{r''} r$; (c) $co[r \cup r''] \cap r'$ is nonempty, whereas $co[r \cup r'] \cap r''$ and $co[r'' \cup r'] \cap r$ are both empty.*

Part 1 of the lemma presents quite a sharp picture of what a spectrum looks like: a partitioning of a Euclidean space by hyperplanes. This immediately implies that each region is a polytope, and hence is convex. Since regions are convex and mutually disjoint, the separateness relation between them becomes a reasonable notion of actual geometric separateness. In fact, when regions are cut out by hyperplanes, $\triangleleft_r$ becomes antisymmetric, i.e. if r'' is more separated from r then r' is then the reverse cannot be true. This property is also in line with what our notion of geometric separateness would suggest. Note that Part 1 of the lemma can also be easily applied to tell which spectrum configurations, in particular with regard to those used in practice, do not technically satisfy our model.

To interpret Part 2, we can think of region r' as the "middle ground" between two extremes r and r''. The equivalence of (a) and (b) thus says that from the standpoint of either extreme, the middle ground is closer than the other extreme, whereas (c) gives an alternative interpretation of what makes r' geometrically in the middle: only r' overlaps with the convex hull of the other two regions. This finding further confirms the geometric intuition behind the separateness relation.

III. Political Spectra and the Envrionemnts they Represent

In this section we explore properties of our model of political spectra, and what implications we can draw from them about the underlying environments.

The foremost question is, of course, whether every environment can be represtend by some spectrum. This we answer affirmatively.

Proposition 2. *Every political environment $\mathcal{V}(X, D)$ can be represented by a political spectrum.*

[6]On the other hand, it is easy to verify that $R' \triangleleft^R R''$ implies $f^{-1}(R'') \not\blacktriangleleft^{f^{-1}(R)} f^{-1}(R')$. Thus $\blacktriangleleft^r$ on P is more sparse than $\blacktriangleleft^{f(r)}$ on $f(P)$.

The proof of the proposition relies on an observation that is worth noting in itself:

Lemma 2. *If political spectrum* (S, P, f) *represents* $\mathcal{V}(X, D)$*, then* (S, P, f) *represents any* $\mathcal{V}(X, D')$ *where* $D' \subset D$*.*

Put simply, a spectrum competent in representing $\mathcal{V}(X, D)$ must also be competent in representing any environment that is politically less diverse than $\mathcal{V}(X, D)$. Thus, if there exists a spectrum that represents $\mathcal{V}(X, \overline{D})$, i.e. the most politically diverse environment, then that spectrum represents any environment with the same set of alternatives. The proof of Proposition 2 is based on an explicit construction of such a catch-all spectrum.

When it comes to practicality, however, the catch-all spectrum we construct may be more than adequate to the extent of being too complex. First, obviously, it contains irrelevant opinions, those that are absent in the environment of interest. Second, more importantly, that spectrum could have too many dimensions. Indeed, its dimension is $|X| - 1$ — this will become an important step for our later analysis, so it is worth rephrasing it in terms of a lemma:

Lemma 3. *Any* $\mathcal{V}(X, D)$ *can be represented by a political spectrum* (S, P, f) *where* $S \subset \mathbb{R}^{|X|-1}$*.*

Seeing Lemma 3, one is concerned whether all $|X| - 1$ dimensions are necessary to represent an environment of modest political diversity. This concern is particularly relevant if $|X|$ is large, i.e. there are many alternatives, because an $|X| - 1$ dimensional spectrum is difficult to grasp conceptually, let alone impossible to plot visually. It is thus logical to ask what is the least necessary dimensionality for there to eixst a spectrum representation of a given environment. Formally we consider the following concept.

Definition. *(Least spectrum dimensionality)* The *least spectrum dimensionality (LSD)* of $\mathcal{V}(X, D)$ is the minimal $d \in \mathbb{N}$ such that there exists a political spectrum (S, P, f), where $S \subset \mathbb{R}^d$, that represents $\mathcal{V}(X, D)$.

We use $d(X, D)$ to denote the LSD of $\mathcal{V}(X, D)$. Lemma 2 immediate implies that LSD is increasing in the political diversity of the underlying environment.

Proposition 3. *If* $D \subset D'$*, then* $d(X, D) \leq d(X, D')$*.*

Proposition 3 suggests that LSD can potentially serve as a useful index for political diversity in the environment. That a politically more diverse environment calls for a more complex, i.e. higher-dimensional, spectrum to represent agrees with intuition and practice. Indeed, one of the main motivations for practitioners to go beyond the linear spectrum is that the diversity and richness in ideologies are not longer sufficiently captured by the mere one-dimensional model. Now we explore the implications of LSD about the underlying environment and its spectrum representation.

First, we observe that environments with the lowest LSD, zeros, are exactly those that are politically least diverse, where a single opinion prevails.

Proposition 4. *$d(X, D) = 0$ if and only if D is a singleton.*

That LSD equals zero means the spectrum is essentially just a point, which is, obviously, sufficient to represent just one opinion.

Then we observe that environments that are one tier above in terms of diversity (LSD $= 1$), i.e. those representable by a linear spectrum, are exactly those that satisfy the single-crossing condition.

Proposition 5. *$d(X, D) = 1$ if and only if $|D| > 1$ and D satisfies the single-crossing condition.*

Proposition 5 is not surprising given Lemma 1, as the latter reveals that single-crossing is essentially the possibility of linearly ordering opinions by disagreement. A related linearity interpretation of the single-crossing condition is given by Puppe and Slinko (2019), who use graph-theorical graphs to represent domains and show that single-crossing domains are isomorphic to chains, i.e. linear graphs.

Societies of higher LSD cannot be as cleanly characterized. But the following observation provides an angle for analyzing them.

Proposition 6. *Suppose (S, P, f) is a political spectrum representing $\mathcal{V}(X, D)$. If $S' \subset S$ is a convex subset of S and P' and f' are the respective restrictions of P and f to S', then (S', P', f') is a political spectrum representing an environment $\mathcal{V}(X, D')$ where $D' \subset D$.*

A useful implication of Proposition 6 is that a lower-dimensional section of a spectrum is itself a spectrum. In particular, Proposition 5 implies that a line arbitarily drawn across a given spectrum represents a collection of opinions that can be completely ordered in disagreement. For example, given a two-dimensional spectrum on which it is reasonable to impose an auxiliary coordinate system where the $x-$axis represents stance on economic freedom and the $y-$axis on social liberty, then any vertical section of the spectrum is also a linear spectrum that represents opinions that agree on economic issues but diverge on social ones. Thus, from a theoretical pespective, the condition LSD$= d$ can be interepted as a generalization of the single-crossing condition to higher dimensions: that the political disagreement within society can be fundamentally decomposed to d dimensions/aspects. More importantly, in the case a a priori coordinate system is absent, studying the configuration of the spectrum may shed light on the main political orientations across the opinions. In other words, a coordinate system, along with the main political aspects that underpin it, may emerge from close examination of a spectrum. For instance, after opinions about a Presidential election are properly arranged in a two-dimensional spectrum, upon studying these opinions vis-à-vis each other on the spectrum, one may find that the two main dimensions that explain the divergence in opinions are national security and tax rates.

As said above, determining the LSD of an arbitrary environment is not straightforward, unless we impose additional structure on the setting — such cases will be analyzed in the following section. On the other hand, we are able to determine the LSD of $(X, \overline{D})$, i.e. the most politically diverse environment.

Proposition 7. *$d(X, \overline{D}) = |X| - 1$.*

This proposition also confirms that the LSD upper bound established in Lemma 3 is tight. It is worth noting that there exist other environments for which all the $|X|-1$ dimensions are needed for spectrum representation. The fact that the maximal LSD grows with the number of alternatives is also in line with interpreting LSD as an index for political diversity, as multitude in alternatives is a prerequisite for multitude in tastes.

IV. Political Spectra for Linear Environments

In this section we discuss how to explicitly construct a suitable political spectrum for a class of widely used social choice environments, namely those where opinions can be rationalized by linear utility functions. Opinons in these environments are usually modeled as *types* each of which summarize a full preference relation. Specifically, each agent has a type $t \subset \mathbb{R}^k$ that determines her utility function $u(\cdot, t)$ regarding the alternatives in X. The environment is formulated in terms of the set of alternatives X, the set of all possible types (*type space*) $T \subset \mathbb{R}^k$, and a collection of utility functions $u : X \times T \to \mathbb{R}$. We denote such type-based formulation of an environment as $\mathcal{T}(X, T, u)$. We say that $\mathcal{T}(X, T, u)$ is *linear* if $u(x, t)$ is linear in t for every $x \in X$.

It would be useful to see some examples first.

Example 1. *(Multi-attribute candidates)* A hiring comittee considers a list X of job candidates. The candidates have gone through a vetting process that evaluates each of them regarding $k+1$ attributes, e.g. intellectuality, maturity, etc., and consequently every candidate $x \in X$ receives $k+1$ scores $v_1(x), ..., v_{k+1}(x) \in \mathbb{R}$, one for each attribute.

Committee members compute weighted average scores to rank candidates, but different members may use different sets of weights. Specifically, the overall evaluation for candidate x by a committee member who uses weights $t = (t_1, ..., t_k)$ (and $t_{k+1} = 1 - \sum_{i=1}^{k} t_i$) is

$$u(x,t) = \sum_{i=1}^{k} t_i v_i(x) + [1 - \sum_{i=1}^{k} t_i] v_{k+1}(x)$$

which is linear in t.

Example 2. *(Subjective beliefs)* A board considers a set X of risky projects the returns of which depend on an unknown state which can take any value from $\{1, ..., k+1\}$. If state $i \in \{1, ..., k+1\}$ obtains, project x generates a return of $v_i(x)$.

Board members evaluate the projects by expected return, but their subjective beliefs about the uncertainty differ. Specifically, the value of project x to a board member whose subjective belief is given by $t = (t_1, ..., t_k)$ (and $t_{k+1} = 1 - \sum_{i=1}^{k} t_i$) , where t_i is the probability he assigns to state i, is

$$u(x,t) = \sum_{i=1}^{k} t_i v_i(x) + [1 - \sum_{i=1}^{k} t_i] v_{k+1}(x)$$

which is linear in t.

Example 3. *(Single-peaked preferences)*[7] There is a presidential election that invovles k political issues, e.g. tax rate, LGBT rights, etc. Each candidate $x \in X$ proposes a platform $v(x) = (v_1(x), ..., v_k(x))$ where $v_i(x) \in [0,1]$ stands for her stance on the ith issue.

Each voter's preference is single-peaked at his "ideal" platform $t = (t_1, ..., t_k)$, i.e. candidate x is preferred to candidate y by this voter if $|v(x) - t| < |v(y) - t|$. It turns out that this preference can be rationalized by the utility function

$$u(x,t) = -\sum_{i=1}^{k} v_i^2(x) + 2 \sum_{i=1}^{k} t_i v_i(x)$$

which is linear in t.[8]

These examples suggest that linear environments may arise in many political-economic

[7]This example generalizes the leading examples in Tullock (1967) and Grandmont (1978). In effect, it is a generalization of the Hotelling-Downs political competition model (Hotelling (1929); Downs (1957)) to a higher-dimensional policy space.

[8]To see that $u(x,t)$ rationalizes the single-peaked preference with peak t, note that $u(\cdot, t)$ is a monotone transformation of $\hat{u}(\cdot, t) = -v(x) \cdot t$

situations, and it would be useful to construct political spectra that represent them. Moreover, we will show that such a spectrum can provide additional insights about the underlying environment.

The first step towards a political spectrum is to reformulate the environment so that opinions are preferenes instead of types. This translation is straightforward:

Definition. $\mathcal{V}(X, D)$ is said to be *induced* by $\mathcal{T}(X, T, u)$ if D is the set of all preference relations rationalizable by $u(\cdot, t)$ for some $t \in T$.

For exposition, we will focus on the case where the type space T is a convex set.[9] The spectrum construction is based on a very simple idea, that is, viewing the type space T itself — a convex Euclidean set — as the geometric space in which the spectrum is set, and group types into regions by the implied (ordinal) preference relation on X. It turns out that the separation relation between these regions exactly reflects the disagreement relation between the implied preference relations. Formally:

Proposition 8. *Suppose $\mathcal{V}(X, D)$ is induced by the linear environment $\mathcal{T}(X, T, u)$ where the type space $T \subset \mathbb{R}^k$ is convex. If construction (S, P, f) satisfies the following:*

- *$S = T$;*
- *for any $r \in P$, $t, t' \in r$ if and only if $u(\cdot, t)$ and $u(\cdot, t')$ induce the same preference relation which we denote as $R(r)$;*
- *$f(r) = R(r)$,*

then (S, P, f) is a political spectrum that represents $\mathcal{V}(X, D)$.

Proposition 8 makes it very easy to construct a spectrum for a linear environment. In addition, this spectrum has the advantage that each point has an immediate interpretation: it stands for a type. In fact, an important observation (due to Grandmont (1978)) that paves the way towards Proposition 8 is that, for a linear environment, the separation relation between types (viewed as singleton convex sets) preserves the disagreement relation between the implied preference relations.

A particularly interesting case is when each type represents a unique preference relation on X.[10] This, for example, captures the standard Hotelling-Downs model (Example 3) where for each policy platform in the platform space $[0, 1]^k$ is proposed by a presidential candidate $x \in X$. In this case, the type space itself can be viewed as a political spectrum, where each

[9] If T is not convex, then in the analysis below we can simply replace T with a convex superset of it.

[10] In this case X would have to be infinite, but our analysis so far does not logically rely on finiteness.

point stands for a unique opinion. Sometimes the type space in the Hotelling-Downs model is directly referred to as a political spectrum. Our theory therefore endows this practice with precise content and formal justification.

In addition to easy construction and interpretation, the type space-founded spectrum has some other advantages over alternative spectra. First, every preference relation in the spectrum is also present in the society, so this spectrum is in a sense the most economical possible.

Proposition 8 also generates a useful theoretical implication on the LSD of a linear environment:

Corollary 1. *If $\mathcal{V}(X, D)$ is induced by the linear environment $\mathcal{T}(X, T, u)$ where the type space $T \subset \mathbb{R}^k$ is convex, then $d(X, D) \leq k$.*

Note that this upperbound k, which is the dimensionality of the type space, can be much lower than $|X| - 1$ which is the uniform upperbound resulting from Proposition 7. For instance, the necessary spectrum dimensionality does exceed the number of evaluated attributes in Example 1, the number of states in Example 2, or the number of relevant policy issues in Example 3. This observation also relates back to using LSD as an index for diversity in the opinions. Indeed, in all three examples, k in a way stands for the perceived level of complexity of the relevant social choice problem and hence allows for divergence across opinions.

A related question is when would all k dimensions be necessary for there to exist a spectrum representing a linear environment. If the answer is positive, then the spectrum constructed in Proposition 8 is not only economical in that it admits no redundant preference relations, but also economical in that it admits no redundant dimensions. We will leave this discussion in Discussion.

V. Political Spectrum and Social Choice

The primary purpose of a spectrum is to manifest a political landscape that would otherwise be osbure. Although a spectrum representation of a political environment becomes difficult to visualize if there are more than three dimensions, with a spectrum it would still be easier to understand the political landscape than without, much like how thinking about the "graph" of a many-variable function makes the function easier to understand.

In addition to this primary, illustrative, purpose, a spectrum representation, albeit high-dimensional, can potentially afford us additional, geometry-based mathematical tools to better analyse the underlying environment. In this section we make one attempt to explore this potential. In particular, we carry out an exercise that reveals an interesting connection between the LSD of an environment and the possibility for the there to be a social common ground.

Let us elaborate. Imagine that the environment is of the social choice nature, that is, agents want to make a collective choice from the alternatives in X. If there is an alternative x that is widely supported by the population against any other alternative, then we would think that there is a decent social common ground in x, as any opposition against x in favor of another alternative would be weak. We will show that the existence of such an widely supported alternative is related to the LSD.

Now let us be a little more formal. Given an alternative $x \in X$ and a preference profile $\mathcal{R} = (R_1, ..., R_n)$ which describes what opinions inviduals have in a society, define[11]

$$q(x|\mathcal{R}) := \max\{q \in [0,1] : \ |\{i : xR_i y\}| \geq qn \quad \forall y \in X, y \neq x\}.$$

To put in words, $q(x|\mathcal{R})$ reflects how much social support x has against its *strongest* opponent, when opinions are given by the preference profile $\mathcal{R}$ — we might as well call $q(x|\mathcal{R})$ the *popularity* of x under $\mathcal{R}$. Of particular significance is the popularity of the most popular alternative, i.e. $\overline{q}(\mathcal{R}) := \max_{x \in X} q(x|\mathcal{R})$, as it gives us a sense of how firm the firmest social common ground, or the "greatest common denominator", that the society as a whole may agree on. Indeed, if $\overline{q}(\mathcal{R})$ is high, then there is an alternative that has an unambiguously strong support from the society against any other alternative. Not only would that alternative have a better chance of becoming the social choice under any procedure that reasonably respects democracy, but also, when it is chosen, it would stand steadily as the ground of dissent is thin. On the other hand, if $\overline{q}(\mathcal{R})$ is low, then every alternative could face significant challenge from some other alternative, and even if an alternative survives the social choice procedure and gets elected, it is more of a tenuous compromise with a weak legitimacy, and can be easily undermined by organized opposition.

Clearly, the value of $\overline{q}(\mathcal{R})$ can be arbitarily close to 100% regardless of the environment $\mathcal{V}(X, D)$, as long as a large fraction of the agents share the same top choice. What is more interesting is how *low* this popularity can reach. It turns out that this is bounded from above by a function of the LSD.

Theorem 2. *Given an environment $\mathcal{V}(X, D)$, for any $n \in \mathbb{N}$ and preference profile $\mathcal{R} = (R_1, ..., R_n) \in D^n$,*

$$\overline{q}(\mathcal{R}) \geq \frac{1}{1 + d(X, D)}.$$

[11] Here for simplicity we assume there are finitely many agents in the society. The definition can obviously be generalized to the case where the agents are distributed on a continuum. Our subsequent analysis will not change to this generalization.

Thus, roughly speaking, if an environment has a low LSD, i.e. representable by a low-dimensional spectrum, then any society that is an instance of this environment, regardless of how opinions are distributed, has a firm social common ground (in the alternative with the highest popularity). On the other hand, if a society has a high LSD, then it is possible that even the firmest social common ground is rather fragile, because what we establish in the proof is that the lower bound $1/(1 + d(X, D))$ is tight for certain distribution of opinions. For instance, if the society calls for a 3D spectrum to represent, then it is possible that every alternative in the society may face an opposition of no less than three quarters $(1 - 1/(1 + 3))$ of the population in favor of another alternative. Recall that LSD can be viewed as an index for political diversity. Thus, Theorem 4 suggests that the less diversity there is in the opinions, the firmer the social common ground is, or in other words, the easier it is for the society to have a widely supported social choice. This implication certainly agrees with common sense.

Theorem 4 is also closely related to the presence of Condorcet voting cycles. Recall that, given a preference profile $\mathcal{R}$, a Condorcet voting cycle with voting threshold q[12] is present if there is a sequence of alternatives $x_1, ..., x_k$ where $x_1 = x_k$ such that for each $i = 1, ..., k-1$, no less than a fraction q of the people prefer x_i to x_{i+1}. It is clear that if a Condorcet cycle with voting threshold q is absent under $\mathcal{R}$, then $\overline{q}(\mathcal{R})$ is at least q.[13] What we actually establish in the proof is that, as long as the voting threshold does not go below $1/(1 + d(X, D))$, a Condorcet voting cycle with threshold q is absent for certain.

It is worth noting that Theorem 4 (or more precisely the stronger statement we establish by its proof) in fact subsumes a number of classical results on Condorcet voting cycles. For instance, it implies that if the society satisfies the single-crossing condition, i.e. its LSD is 1, then a voting cycle with threshold $q = 1/2$ does not exist. This is precisely the now well-understood result that pairwise majority voting cycles do not exist in single-crossing domains, see for example Grandmont (1978). Also, since the LSD of a society does not go above $|X| - 1$ by Proposition 7, Theorem 4 implies that a voting cycle with threshold lower than $1/|X|$ does not exist, which is precisely Nakamura (1979).

A. Special Case: Linear Domains

Although theoretically insightful, the notion of intermediate type spaces might be difficult to handle in applications and the dimensionality of a preference domain might be hard to tell in general. For this reason, we introduce the following notion of linear domains, which

[12] Much of the literature on Condorcet voting cycles focuses on $q = 1/2$, i.e. pairwise majority voting, but it is straightforward to generalize the concept to any other threshold.

[13] On the other hand, a voting cylce with threshold $q > \overline{q}(\mathcal{R})$ could exist as long as it does not involve the alternative with the highest popularity.

is easier to handle in applications since they are represented by utility functions of simple functional forms and their dimensionality is straightforward to verify.

Let each candidate x be associated with the common value $v(x)$ for all voters and the vector of characteristics $\kappa(x) \in \mathbb{R}^n$ that are valued differently by voters. The preference relation of a voter with the preference parameter $\theta \in \Theta \subset \mathbb{R}^n$ is represented by the utility function $u(x;\theta) := v(x) + \theta \cdot \kappa(x)$. The *linear domain* induced by (v, κ, Θ), denoted as $LDR(v, \kappa, \Theta)$, is the set of preference relations that are represented by $u(x;\theta)$ for some $\theta \in \Theta$.

Proposition 9. *The dimension of the linear domain $LDR(v, \kappa, \Theta)$ is no greater than the dimension of the perpendicular projection of Θ onto the affine space spanned by $\kappa(X)$. [Question: Is the dimension of the projection of A onto B the same as the dimension of the projection of B onto A? If so, is there a name for that?]*

Proof. For each $\theta \in \Theta$, let $P\theta$ be the perpendicular projection of θ onto the affine space spanned by $\kappa(X)$. Let m be the dimension of $P\Theta$ and we want to show that the dimension of $LDR(v, \kappa, \Theta)$ is no greater than m.

Arbitrarily take $\zeta \in P\Theta$ and let $(e^1, e^2, \ldots, e^m)$ be a basis of the linear space $P\Theta - \zeta$. For each $\theta \in \Theta$, we can represent $P\theta$ as $\zeta + \sum_{j=1}^{m} t^j(\theta) e^j$, where $t(\theta) \in \mathbb{R}^m$ is the coordinates of $P\theta$. Arbitrarily take $x^0 \in X$. By definition of perpendicular projection, we have $(\theta - P\theta) \cdot (\kappa(x) - \kappa(x^0)) = 0$ for all $x \in X$ and $\theta \in \Theta$. Therefore, we have

$$\begin{aligned} u(x;\theta) &= v(x) + \theta \cdot \kappa(x) \\ &= v(x) + \theta \cdot \kappa(x^0) - P\theta \cdot \kappa(x^0) + (\zeta + \sum_{j=1}^{m} t^j(\theta) e^j) \cdot \kappa(x) \\ &= v'(x) + w(\theta) + t(\theta) \cdot e\kappa(x) \end{aligned}$$

where $v(x) := v(x) + \zeta \cdot \kappa(x)$, $w(\theta) := \theta \cdot \kappa(x^0) - P\theta \cdot \kappa(x^0)$, and $e\kappa(x)$ is a vector in $\mathbb{R}^m$ whose j-th coordinate is $e^j \cdot \kappa(x)$. Let the intermediate type space $T \subset \mathbb{R}^m$ be the convex hull of $t(\Theta)$, and let $r(t)$ be the preference relation represented by $\hat{u}(x;t) := v'(x) + t \cdot e\kappa(x)$. Because every preference relation in the linear domain induced by (v, κ, Θ) is represented by $u(x;\theta)$ for some $\theta \in \Theta$, it is also represented by $\hat{u}(x;t(\theta))$ and is therefore contained in $r(T)$. In addition, if the inequality $\hat{u}(x', t) > \hat{u}(x, t)$ holds for $t = t'$ and t'', it also holds for every convex combination of t' and t'' by linearity. Therefore, by Definition **??**, the dimension of $LDR(v, \kappa, \Theta)$ is no greater than m. $\square$

Corollary 3. *The dimension of the linear domain $LDR(v, \kappa, \Theta)$ is no greater than the dimension of Θ or $\kappa(X)$.*

If preferences are induced by the interaction of more voter characteristics and candidate characteristics, the linear preference domain will contain more diverse preferences and its dimension will potentially increases.

Proposition 10. *The dimension of the linear domain $LDR(v, \kappa, \mathbb{R}^n)$ is exactly* $\dim \kappa(X)$.

Proof. Let $d := \dim \kappa(X)$. By Corollary 3, the dimension of the linear domain induced by $(v, \kappa, \mathbb{R}^n)$ is no greater than d. So it is sufficient to show that its dimension is greater than $d-1$. By Lemma 4, it is suffcient to find a ranking wheel configuration of degree $d+1$ in $LDR(v, \kappa, \mathbb{R}^n)$.

Because $\dim \kappa(X) = d$, there exists a set of $d+1$ candidates $\{x^k\}_{k=0}^d$ s.t. $\{\kappa(x^k)\}_{k=0}^d$ are affinely independent. Then we can show that for each preference relation $\succsim$ on $\{x^k\}_{k=0}^d$, we can find $\theta \in \mathbb{R}^N$ s.t. the utility function $u(x;\theta)$ restricted to $\{x^k\}_{k=0}^d$ represents $\succsim$. To see this, first we find a vector $u = (u^k)_{k=0}^d$ that represents $\succsim$, i.e. $u^{k'} \geq u^k$ iff $x^{k'} \succsim x^k$. Because $\{\kappa(x^k) - \kappa(x^0)\}_{k=1}^d$ are linearly independent, we can find $\theta \in \mathbb{R}^N$ s.t.

$$[\kappa(x^k) - \kappa(x^0)] \cdot \theta + v(x^k) - v(x^0) = u^k - u^0$$

for each $k = 1, 2, \ldots, d$. By construction, the difference $u(x^k;\theta) - u^k$ does not depend on k, and so $u(x;\theta)$ restricted to $\{x^k\}_{k=0}^d$ also represents $\succsim$.

For each $k = 0, 1, \ldots, d$, define $\succsim^l$ on $\{x^k\}_{k=0}^d$ s.t. $x^k \succ^l x^{k+1}$ for all $k, l \in \{0, 1, 2, \ldots, d\}$ with $k \neq l$. Find $\theta^l \in \mathbb{R}^n$ s.t. $u(x;\theta^l)$ restricted to $\{x^k\}_{k=0}^d$ represents $\succsim^l$. Let $R^l \in LDR(v, \kappa, \mathbb{R}^n)$ be the preference relation on X that is represented by $u(x;\theta^l)$. Then $\{R^l\}_{l=0}^d$ and $\{x^k\}_{k=0}^d$ form a ranking wheel configuration of degree $d+1$ in $LDR(v, \kappa, \mathbb{R}^n)$, which completes the proof. $\square$

It might be tempting to think that every domain of dimension d can be represented by a d-dimensional linear domain, but it is not true even with $d = 1$.

Note that a linear domain may fail to be multi-dimensional single-peaked in the sense of Greenberg (1978) because v may fail to be concave. Therefore our nonempty core result complements the result of Greenberg (1978) that the $1/(d+1)$-core is nonempty with a d-dimensional single-peaked domain.

B. Application to Savage Models

Another application of linear domains is the Savage model. Let there be $n+1$ states. A belief over the states is $\theta \in \mathbb{R}^n_+$ with $\sum_{j=1}^n \theta_j \leq 1$, where θ_j is the belief on the state j and $\theta_0 := 1 - \sum_{j=1}^n \theta_j$ is the belief on the state 0. Voters share the same posterior utility function $u_j(x)$ in state j but have different subjective beliefs over the states. The preference relation of a voter with belief θ is represented by the utility funtion

$$\begin{aligned} u(x;\theta) &= \sum_{j=0}^{n} \theta_j u_j(x) \\ &= (1 - \sum_{j=1}^{n} \theta_j) u_0(x) + \sum_{j=1}^{n} \theta_j u_j(x) \\ &= u_0(x) + \theta \cdot [u(x) - u_0(x)], \end{aligned}$$

where $u(x) - u_0(x) \in \mathbb{R}^n$ is the vector whose j-th coordinate is $u_j(x) - u_0(x)$. Therefore, the domain induced by the Savage model satisfies the definition of a linear domain with $v(x) = u_0(x)$ and $\kappa(x) = u(x) - u_0(x)$. By Proposition 9, the preference domain induced by the Savage model is no greater than the dimension of the perpendicular projection of Θ onto the affine space spanned by $(u - u_0)(X)$. Corollary 3 implies that the dimension is no greater than $\dim(u - u_0)(X)$ or $\dim \Theta$, and an apparent upper bound for the dimension is n.

Proofs

Sarri (2014) studies the cyclic patterns contained in preference domains, which he labeled as ranking wheel configurations.

Definition. A ranking wheel configuration of degree n contained in a preference domain D consists of a set of n preference relations $\{R^l\}_{l=1}^n$ in D and a set of n candidates $\{x^k\}_{k=1}^n$ in X s.t. $x^k P^l x^{k+1}$ for all $k, l \in \{1, 2, \ldots, n\}$ with $k \neq l$. (When $k = n$, we let the notation $k + 1$ be identified with 1.)

Lemma 4. *A d-dimensional preference domain cannot contain a ranking wheel configuration of degree $d + 2$.*

Proof. Because the domain D has dimension d, by definition, there exists an intermediate type space, i.e. a convex set $T \subset \mathbb{R}^d$, and a function $r : T \to \overline{D}$ s.t. $r(T) \supset D$ and for any $t, t', t'' \in \mathbb{R}^d$ with t'' being a convex combination of t and t', we have $x' \hat{r}(t'') x$ whenever $x' \hat{r}(t) x$ and $x' \hat{r}(t') x$.

Suppose, to the contrary, that D contains a ranking wheel configuration of degree $d+2$, i.e. there exist $R^1, R^2, \ldots, R^{d+2} \in D$ and $x^1, x^2, \ldots, x^{d+2} \in X$ s.t. $x^k P^l x^{k+1}$ for all $k, l \in \{1, 2, \ldots, d+2\}$ with $k \neq l$. For each $k = 1, 2, \ldots, d+2$, we let $T^k := \{t \in T : x^k \hat{r}(t) x^{k+1}\}$. Note that T^k is convex set in $\mathbb{R}^d$ because, by definition of intermediate type spaces, strict preferences are preserved under convex combination. Besides, every $d + 1$ of the T^k's have nonempty intersection. To see this, because $r(T) \supset D$, for each k we can find a type $t^k \in T$

with $r(t^k) = R^k$. By definition of the ranking wheel configuration, we have $t^k \in \bigcap_{k' \neq k} T^{k'}$. By Helly's theorem ($d+2$ convex sets in $\mathbb{R}^d$ must have nonempty intersection if every $d+1$ of them have nonempty intersection), we have $\bigcap_{k=1}^{d+2} T^k \neq \emptyset$. Take some $t^* \in \bigcap_{k=1}^{d+2} T^k$, and we have $y^k \hat{r}(t^*) y^{k+1}$ for all $k = 1, 2, \ldots, d+2$. This contradicts to the transitivity of $r(t^*)$. $\square$

Let I be a set of finitely many voters, and each voter i has a preference relation R_i in the domain D. A *supermajority voting rule* is described by $q \in [0,1]$ that specifies the fraction of voters that can enforce a change. Under this voting rule, we say that candidate x' dominates x' if the fraction of voters who strictly prefer x' to x is greater than q, i.e.

$$\frac{|\{i \in I : x' P_i x\}|}{|I|} > q,$$

where P_i stands for the strict version of R_i. The *q-core*, or C_q, is the set of candidates undominated by any other candidate under this voting rule. As a special case, when $q = 1/2$ and the total number of voters is odd, the supermajority voting is the familiar simple majority voting. When $q = 1$, the supermajority voting becomes the unanimity voting.

When the q-core is nonempty, a candidate in the q-core, by definition, is supported by at least a fraction $1-q$ of the voters against any other candidate. In other words, we can find a candidate that is socially agreeable to some extend. However, the q-core may be empty in general. A well-known example by Condorcet shows that the 1/2-core can be empty with as few as three candidates and three voters: Voter 1 strictly prefers candidate x to y to z, voter 2 strictly prefers y to z to x, and voter 3 strictly prefers z to x to y. Then we see that x dominates y, y dominates z, z dominates x, and so there is no undominated candidate.

It is interesting to know when the q-core is nonempty, i.e. a socially agreeable candidate can be found. A set of finitely many candidates $\{x^1, x^2, \ldots, x^N\}$ is called a *q-cycle* if x^1 dominates x^2, x^2 dominates x^3, ..., and x^N dominates x^1 under the supermajority voting rule q. To obtain a nonempty core, the primary difficulty is to rule out social preference cycles by limiting the richness of the preference domain D. This is exactly what our main result is about.

We show that if the preference domain D has dimension d, there exists no $1/(d+1)$-cycle. With some additional regularity conditions, the $d/(d+1)$-core is always nonempty. This result is summarized by the following theorem.

Theorem 4. *If the preference domain D has dimension d, there exists no $1/(d+1)$-cycle under any preference profile $\{R_i\}_{i \in I} \in D^I$. If, in addition, the set X of candidates is a compact topological space and the preference relations in D are upper semi-continuous, then the $1/(d+1)$-core is nonempty under every preference profile in $\{R_i\}_{i \in I} \in D^I$.*

Proof. If D has dimension d, by Lemma 4, it contains no ranking wheel configuration of degree $d+2$. By Sarri (2014), Theorem 4, there exists no $1/(d+1)$-cycle under any preference profile $\{R_i\}_{i \in I} \in D^I$.

Now we show the nonempty core result with the additional assumption of compact X and upper semi-continuous preferences.

Take any preference profile $\{R_i\}_{i\in I} \in D^I$. For each candidate $x \in X$, let $ND(x)$ be the set of candidates that are not dominated by x under the supermajority voting rule $q = 1/(d+1)$. Then the $1/(d+1)$-core can be represented as $\bigcap_{x \in X} ND(x)$.

We claim that $ND(x)$ is a closed set for each $x \in X$. To see this, we have

$$\begin{aligned} ND(x) &= \{x' \in X : |\{i \in I : x P_i x'\}| \leq |I|/(d+1)\} \\ &= \{x' \in X : |\{i \in I : x' R_i x\}| \geq |I| \cdot d/(d+1)\} \\ &= \bigcup_{J \subset I : |J| \geq |I| \cdot d/(d+1)} \bigcap_{i \in J} \{x' \in X : x' R_i x\}, \end{aligned}$$

where $q = 1/(d+1)$. By upper semi-continuity, the set $\{x' \in X : x' R_i x\}$ is closed, and so is $ND(x)$.

Moreover, for any finitely many candidates x^1, x^2, ..., and x^N, the intersection $\bigcap_{n=1}^N ND(x^n)$ is nonempty since no $1/(d+1)$-cycle implies that there exist some x^k undominated by any $x^{k'}$. Then by compactness of X, the infinite intersection $\bigcap_{x \in X} ND(x)$ is also nonempty. This completes the proof of the nonempty core. □

As a special case, when $d = 1$, the notion of d-dimensional single-crossing domain reduces to the standard notion of single-crossing, and the our nonempty core result reduces to the well-known fact that the simple majority voting always has a winner under a single-crossing preference domain. Besides, when there are finitely many candidates, because the universal domain $\overline{D}$ has dimension $N-1$, and so our theorem implies that the $(N-1)/N$-core is nonempty. This result was found by Nakamura (1978).

Proposition 11 (Nakamura, 1978)**.** *If the set X of candidates is finite, the $(N-1)/N$-core is nonempty under any preference profile.*

The next proposition provides a partial converse to Theorem 4.

Theorem 5. *Given $N \geq d+1$ and $q < d/(d+1)$, there exists a preference domain D of dimension d s.t. the q-rule is empty under some preference profile $\{R_i\}_{i\in I} \in D^I$ if $|I|$ is sufficiently large or $d+1$ divides $|I|$.*

Proof. Take $d+1$ candidates x^0, x^1, ..., and x^d and construct the preference domain $D = \{R^l\}_{l=0}^d$ as follows. For each $l = 0, 1, \ldots, d$, we construct R^l by letting $x^k P^l x^{k+1}$ for all $k \neq l$,

$x^k P^l x$ for all $x \in X \backslash \{x^0, x^1, \ldots, x^d\}$, and $x R^l x'$ and $x' R^l x$ for all $x, x' \in X \backslash \{x^0, x^1, \ldots, x^d\}$. We can verify that the domain D has dimension d.

If $d+1$ divides $|I|$, we construct the preference profile by letting each preference relation R^l possessed by a fraction of $1/(d+1)$ of voters. Then it is straightforward to verify that the q-core is empty since each x^k is dominated by x^{k-1}.

If $d+1$ does not divide $|I|$, let $|I| = m(d+1) + r$, where $1 \leq r \leq d$. We construct the preference profile by letting each preference relation R^l possessed by n or $n+1$ voters. When $m > q(d+1)^{-1}[d/(d+1) - q]^{-1}$, again we can verify that the q-core is empty since each x^k is dominated by x^{k-1}. $\square$

References

Downs, Anthony (1957), "An economic theory of political action in a democracy." *Journal of Political Economy*, 65, 135–150.

Duggan, John (2005), "A survey of equilibrium analysis in spatial models of elections."

Eysenck, H. J. (1975), "The structure of social attitudes." *British Journal of Social and Clinical Psychology*, 14, 323–331.

Ferguson, Leonard W. (1941), "The stability of the primary social attitudes: I. religionism and humanitarianism." *The Journal of Psychology*, 12, 283–288.

Grandmont, Jean-Michel (1978), "Intermediate preferences and the majority rule." *Econometrica*, 46, 317–330.

Greenberg, Joseph (1979), "Consistent majority rules over compact sets of alternatives." *Econometrica*, 47, 627–36.

Heywood, Andrew (1992), *Political ideologies : an introduction*. St. Martin's Press, New York.

Hotelling, Harold (1929), "Stability in competition." *The Economic Journal*, 39, 41–57.

Lester, J.C. (1994), "The evolution of the political compass (and why libertarianism is not right-wing)." *Journal of Social and Evolutionary Systems*, 17, 231–241, URL `https://www.sciencedirect.com/science/article/pii/1061736194900116`.

Monjardet, Bernard (2009), *Acyclic Domains of Linear Orders: A Survey*, 139–160. Springer Berlin Heidelberg, Berlin, Heidelberg.

Nakamura, K. (1979), "The vetoers in a simple game with ordinal preferences." *International Journal of Game Theory*, 8, 55–61.

Petrik, Andreas (2010), "Core concept "political compass". how kitschelt's model of liberal, socialist, libertarian and conservative orientations can fill the ideology gap in civic education." *Journal of Social Science Education*, 4.

Puppe, Clemens and Arkadii Slinko (2019), "Condorcet domains, median graphs and the single-crossing property." *Economic Theory*, 67, 285–318.

Saari, Donald G. (2014), "Unifying voting theory from Nakamura's to Greenberg's theorems." *Mathematical Social Sciences*, 69, 1 – 11, URL `http://www.sciencedirect.com/science/article/pii/S016548961400002X`.

Sprumont, Yves (1993), "Intermediate preferences and rawlsian arbitration rules." *Social Choice and Welfare*, 10, 1–15.

Tullock, Gordon (1967), "The general irrelevance of the general impossibility theorem." *The Quarterly Journal of Economics*, 81, 256–270.

Ziegler, Günter M (1993), "Higher Bruhat orders and cyclic hyperplane arrangements." *Topology*, 32, 259–279.