BIE-LA1, Linear Algebra 1

6. Exercise: Eigenvalues and diagonalization

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6.1 Eigenvalues, Diagonalization and Similarity

Example 6.1. Find eigenvalues and eigenvectors of matrix $\mathbf{A} \in M_{3,3}(\mathbb{C})$ and their algebraic as well as geometric multiplicity then decide whether it is or it is not diagonalizable. If it is diagoanlizable, compute diagonal matrix D and invertible matrix S such that $A = SDS^{-1}$.

a) $\mathbf{A} = \begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}$, c) $\mathbf{A} = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$, e) $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

b) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, d) $\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$, f) $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$.

Solution. To find eigenvalues and eigenspaces of matrix means:

→ Calculate determinant $det(\mathbf{A} - \lambda\mathbf{I})$. Result is characteristic polynomial $p_A(\lambda)$.

→ Find spectrum $\sigma(\mathbf{A})$, i.e. find roots of $p_A(\lambda)$ (eigenvalues of A).

→ Find a basis of invariant subspace $\ker(\mathbf{A} - \lambda\mathbf{I})$ separately for every eigenvalue $\lambda \in \sigma(\mathbf{A})$.

To determine diagonalization of operator we just continue with:

→ Determine dimensions of every $\ker(\mathbf{A} - \lambda\mathbf{I})$ to obtain $\nu_g(\lambda)$.

→ Compare $\nu_g(\lambda)$ with $\nu_a(\lambda)$ (multiplicity of λ as a root of $p_A(\lambda)$).

→ All equal – matrix is diagonalizable, otherwise – matrix is not diagonalizable.

→ Find matrix $\mathbf{D}$ and S (diagonal matrix $\mathbf{D}$ with eigenvalues on a diagonal, and invertible matrix S with eigenvectors in columns)

a) $p_A(\lambda) = det(\mathbf{A}-\lambda\mathbf{I}) = det(\begin{pmatrix} 5-\lambda & 2 & -3 \\ 4 & 5-\lambda & -4 \\ 6 & 4 & -4-\lambda \end{pmatrix}) = -\lambda^3+6\lambda^2-11\lambda+6 = -(\lambda-1)(\lambda-2)(\lambda-3).$

There are three roots of multiplicity 1, therefore $\sigma(\mathbf{A}) = \{1, 2, 3\}$ and $\nu_a(1) = \nu_a(2) = \nu_a(3) = 1$. Eigenspaces are

- $\lambda = 1$: $V_1 = ker(A - \mathbf{I})$

$$\begin{pmatrix} 4 & 2 & -3 \\ 4 & 4 & -4 \\ 6 & 4 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \ker(A - 1I) = span(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}).$$

Thus $\mathcal{B}_1 = (\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix})$ is a basis for V_1. Then

$$\nu_g(1) = \dim(V_1) = |\mathcal{B}_1| = 1,$$

and

$$\nu_g(1) = \nu_a(1) = 1.$$

- $\lambda = 2$: $V_2 = ker(A - 2\mathbf{I})$.

$$\left(\begin{array}{ccc|c} 3 & 2 & -3 & 0 \\ 4 & 3 & -4 & 0 \\ 6 & 4 & -6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow V_2 = \ker(A - 2I) = span(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}).$$

  Hence

$$\mathcal{B}_2 = (\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})$$

  is a basis for $V_2 = ker(A - 2\mathbf{I})$. Thus

$$\nu_g(2) = \dim(V_2) = 1.$$

  Then

$$\nu_g(2) = \nu_a(2) = 1.$$

- $\lambda = 3$: $V_3 = ker(A - 3\mathbf{I})$.

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & 0 \\ 4 & 2 & -4 & 0 \\ 6 & 4 & -7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \ker(A - 3I) = span(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}).$$

  Hence

$$B_3 = (\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix})$$

  is a basis for $V_3 = ker(A - 3\mathbf{I})$. Then

$$\nu_g(3) = \dim(V_3) = 1,$$

  and

$$\nu_g(3) = \nu_a(3).$$

Since the algebraic multiplicity of each eigenvalue is the same as its geometric multiplicity, the matrix $\mathbf{A}$ is diagonal. Then

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 = (\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix})$$

is a basis of $\mathbb{C}^3$ consisting of eigenvalues of $\mathbf{A}$.

Diagonal matrix $\mathbf{D}$ and invertible matrices S, S^{-1} are:

$$\mathbf{D} = \begin{pmatrix} \overbrace{1}^{\lambda_1} & \overbrace{0}^{\lambda_2} & \overbrace{0}^{\lambda_3} \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad S = [\mathbf{I}]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} \overbrace{1}^{\lambda_1} & \overbrace{1}^{\lambda_2} & \overbrace{2}^{\lambda_3} \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} -2 & -1 & 2 \\ 2 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}.$$

b) $p_A(\lambda) = det(\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}) = det(\begin{pmatrix} 0 & \lambda & 2\lambda-\lambda^2 \\ 0 & -\lambda & \lambda \\ 1 & 1 & 1-\lambda \end{pmatrix}) = det(\begin{pmatrix} \lambda & 2\lambda-\lambda^2 \\ -\lambda & \lambda \end{pmatrix}) = \lambda^2 \begin{pmatrix} 1 & 2-\lambda \\ -1 & 1 \end{pmatrix}$
$\lambda^2(3-\lambda)$,

There is root 0 with multiplicity 2 and root 3 with multiplicity 3, therefore $\sigma(\mathbf{A}) = \{0,3\}$, $\nu_a(0) = 2$ and $\nu_a(3) = 1$.

Eigenspaces are:

- $\lambda = 0$: $V_0 = ker(A - 0\mathbf{I}) = ker(A)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \ker(A - 0I) = span(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}).$$

  Thus

$$\mathcal{B}_1 = (\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix})$$

  is a basis for V_0 and so

$$\nu_g(0) = \dim(V_0) = 2.$$

  Then $\nu_g(0) = \nu_a(0) = 2$,

- $\lambda = 3$: $V_3 = ker(\mathbf{A} - 3\mathbf{I})$.

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \ker(A - 3I) = span(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}).$$

  Thus

$$\mathcal{B}_2 = (\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$$

  is a basis for V_3. Hence

$$\nu_g(3) = \dim(V_3) = 1.$$

  Then $\nu_g(3) = \nu_a(3) = 1$,

Since the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity, matrix $\mathbf{A}$ is diagonalizable.

Therefore

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = (\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$$

is a basis of $\mathbb{C}^3$ consisting of eigenvalues of A. Then Diagonal matrix $\mathbf{D}$ and invertible matrices S, S^{-1} are:

$$\mathbf{D} = \begin{pmatrix} \overbrace{3}^{\lambda=3} & \overbrace{0}^{\lambda=0} & \overbrace{0}^{\lambda=0} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S = [\mathbf{I}]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} \overbrace{1}^{\lambda=3} & \overbrace{1}^{\lambda=0} & \overbrace{1}^{\lambda=0} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and

$$S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

c) $p_A(\lambda) = det(\begin{pmatrix} 4-\lambda & -5 & 2 \\ 5 & -7-\lambda & 3 \\ 6 & -9 & 4-\lambda \end{pmatrix}) = \lambda^2 - \lambda^3 = \lambda^2(1-\lambda),$

therefore $\sigma(\mathbf{A}) = \{0, 1\}$, $\nu_a(0) = 2$ and $\nu_a(1) = 1$. Eigenspaces are

- $\lambda = 0$: $V_0 = ker(A - 0\mathbf{I}) = ker(A)$.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 5 & -7 & 3 & 0 \\ 6 & -9 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \ker(A - 0I) = span(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}).$$

  Thus

$$\mathcal{B}_1 = (\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix})$$

  is a basis for V_0. Therefore

$$\nu_g(0) = \dim(V_0) = 1.$$

  Then $\nu_g(0) = 1 \neq \nu_a(0)$,

- $\lambda = 1$: $V_1 = ker(\mathbf{A} - \mathbf{I})$.

$$\left(\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \ker(A - 1I) = span(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}).$$

  Thus

$$\mathcal{B}_2 = (\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$$

  is a basis for V_1. Therefore

$$\nu_g(1) = \dim(V_2) = 1$$

  Hence $\nu_g(1) = \nu_a(1) = 1$.
  Since $\nu_g(0) = 1 \neq \nu_a(0)$, $\mathbf{A}$ is not diagonalizable.

and matrix $\mathbf{A}$ is NOT diagonalizable due to $\nu_g(0) = 1 \neq \nu_a(0) = 2$.

d) $p_A(\lambda) = det(\begin{pmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{pmatrix}) = -(\lambda^3 - 2\lambda^2 + \lambda) = -\lambda(\lambda - 1)^2.$

As the spectrum of $\mathbf{A}$ is the roots of its characteristic polynomial therefore $\sigma(\mathbf{A}) = \{0, 1\}$, $\nu_a(0) = 1$ and $\nu_a(1) = 2$

- $\lambda = 0 : V_0 = ker(\mathbf{A} - 0\mathbf{I}) = ker(\mathbf{A})$.

$$\ker(A - 0I) = span(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$$

thus

$$\mathcal{B}_1 = (\begin{pmatrix}1\\1\\1\end{pmatrix})$$

is a basis for V_0. Hence

$$\nu_g(0) = \dim(V_0) = 1.$$

Then $\nu_g(0) = 1 = \nu_a(0)$.

- $\lambda = 1 : V_1 = ker(A - \mathbf{I})$.

$$\ker(A - 1I) = span(\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix})$$

. Thus

$$\mathcal{B}_2 = (\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix})$$

is a basis for V_1. Hence

$$\nu_g(2) = \dim(V_1) = 2.$$

Then $\nu_g(1) = 2 = \nu_a(1)$

As algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity, matrix $\mathbf{A}$ is diagonalizable.

Therefore

$$\mathcal{B} = (\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix})$$

is a basis for $\mathbb{C}^3$ consisting of eigenvectors of A.

Diagonal matrix $\mathbf{D}$ and invertible matrices S, S^{-1} are:

$$\mathbf{D} = \begin{pmatrix} \overbrace{1}^{\lambda=1} & \overbrace{0}^{\lambda=1} & \overbrace{0}^{\lambda=0} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S = [\mathbf{I}]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \\ -1 & 3 & -1 \end{pmatrix}.$$

e) **You need to write down all details as the previous examples!** $\lambda_1 = 2$, $\nu_a(2) = 3$, $\ker(A - 2I) = \langle(1, 0, 1), (-1, 1, 0\rangle$. $\nu_g(2) = 2 \neq \nu_a(2)$ and therefore matrix $\mathbf{A}$ is not diagonalizable.

f) **It is optional.** $p_A(\lambda) = -\lambda^3 + 6\lambda^2 - 19\lambda + 22 = -(\lambda - 2)(\lambda^2 - 4\lambda + 11) = -(\lambda - 2)(\lambda - (2 + i\sqrt{7}))(\lambda - (2 - i\sqrt{7}))$.

- $\lambda = 2$: $\nu_a(2) = 1$, $\ker(A - 2I) = \langle(5, 6, -2)\rangle$,
- $\lambda = 2 + i\sqrt{7}$: $\nu_a(2 + i\sqrt{7}) = 1$, $\ker(A - (2 + i\sqrt{7})I) = \left\langle(-1, \frac{1}{2}(1 - i\sqrt{7}), 1)\right\rangle$
- $\lambda = 2 - i\sqrt{7}$: $\nu_a(2 - i\sqrt{7}) = 1$, $\ker(A - (2 - i\sqrt{7})I) = \left\langle(-1, \frac{1}{2}(1 + i\sqrt{7}), 1)\right\rangle$

and matrix $\mathbf{A}$ is diagonalizable. Diagonal matrix $\mathbf{D}$ and regular matrices $\mathbf{P}$ (find $\mathbf{P}^{-1}$) are:

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+i\sqrt{7} & 0 \\ 0 & 0 & 2-i\sqrt{7} \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 5 & -1 & -1 \\ 6 & \frac{1}{2}(1-i\sqrt{7}) & \frac{1}{2}(1+i\sqrt{7}) \\ -2 & 1 & 1 \end{pmatrix}.$$

Example 6.2. Decide whether matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{3,3}$ are similar or not:

a) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

b) $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$

c) $\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$

d) $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix},$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Solution. a) We first examine their rank and their eigenvalues.

- Since both matrix are in REF, their ranks are equal to the number of pivots. Thus $rank(A) = rank(B) = 2$.
- We have

  $$p_A(\lambda) = det(\mathbf{A} - \lambda\mathbf{I}) = det(\begin{pmatrix} \lambda & 1 & 2 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}) = \lambda^3.$$

  And

  $$p_B(\lambda) = det(\mathbf{B} - \lambda\mathbf{I}) = det(\begin{pmatrix} \lambda & 1 & 1 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}) = \lambda^3.$$

  Therefore $\sigma(\mathbf{A}) = \sigma(\mathbf{B}) = \{0\}$, and the algebraic multiplicity of $\lambda = 0$ is 3 for both matrices.
- We are now going to compare the geometric multiplicity of $\lambda = 0$.

  $$\nu_g^A(0) = \dim(ker(A)) = \text{the number of free variables in a REF of } \quad A = 1.$$

  And

  $$\nu_g^B(0) = \dim(ker(B)) = \text{the number of free variables in a REF of } \quad B = 1.$$

  Thus $\nu_g^A(0) = \nu_g^B(0) = 1$.

Their similarity has to be decided by the definition, i.e. finding of invertible matrix $\mathbf{P}$, with unknown items $a, b \dots, h, i \in \mathbb{C}$, such that $\mathbf{A} = \mathbf{PBP}^{-1}$ or equivalently $\mathbf{PA} = \mathbf{BP}$:

$$\mathbf{P} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \qquad \mathbf{PA} = \mathbf{PB} \Rightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equation $\mathbf{PA} = \mathbf{PB}$ leads to SLE

$$d = g = h = 0\,, \quad a = e = i\,, \quad 2a + b = i\,.$$

Obtained matrix $\mathbf{P}$ exists and has the form

$$\mathbf{P} = \begin{pmatrix} a & b & c \\ 0 & a & a+2b \\ 0 & 0 & a \end{pmatrix}$$

for some parameters $a, b, c \in \mathbb{C}$. In order to have invertible matrix, it is sufficient to chose $a \neq 0$.

b) By the part b) of the previous example, $\sigma(\mathbf{B}) = \{0, 3\}$, where $\lambda_1 = 0$ has algebraic multiplicity equal to 2, $\lambda_2 = 3$ has multiplicities equal to one. Moreover, $\lambda_1 = 0$ has geometric multiplicity equal to 2, $\lambda_2 = 3$ has geometric multiplicities equal to one. And matrix $\mathbf{B}$ is diagonalizable with the diagonal matrix equal to $\mathbf{A}$ here. In particular, there is an invertible matrix S such that $\mathbf{B} = S\mathbf{A}S^{-1}$. Therefore $\mathbf{A}$ and $\mathbf{B}$ are similar.

Invertible matrix $S \in M_{3,3}(\mathbb{C})$ fulfilling $\mathbf{B} = SDS^{-1}$ can be found by writing eigenvectors of $\mathbf{B}$ in columns (first corresponding to 3, then two corresponding to 0, again see b) in the previous exercise). Hence

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

c) We first compute their characteristic polynomial.

$$p_A(\lambda) = det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda^2 - 1),$$

and

$$p_B(\lambda) = det(\mathbf{B} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda^2 + 1).$$

Since $p_A(\lambda) \neq p_B(\lambda)$, matrices A and B are not similar.

d) **Optional** Similar.

6.2 Further exercises

Exercise 6.2.1. Decide, whether matrix $\mathbf{A} \in M_{4,4}(\mathbb{C})$, where

$$\text{a) } \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{b) } \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \quad \text{c) } \mathbf{A} = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$$

is similar to diagonal matrix. If yes, find invertible matrix $S \in M_{4,4}(\mathbb{C})$ and diagonal matrix $\mathbf{D} \in M_{4,4}(\mathbb{C})$, such that $\mathbf{A} = SP\mathbf{D}S^{-1}$.

Solution. You are supposed to write down all details as example 1.

a) Yes,

$$S = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

b) Yes,

$$S = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

c) No.

Exercise 6.2.2. Let $\mathbf{A}, \mathbf{B} \in M_{n,n}(F)$ are similar. Prove following statements:

a) $rank(\mathbf{A}) = rank(\mathbf{B})$;

b) $\mathbf{A}^m, \mathbf{B}^m$ are similar for arbitrary $m \in \mathbb{N}$;

c) If further $\mathbf{A}, \mathbf{B}$ are invertible, then $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ are similar.

Solution. Since A and B are similar, there exists an inevrtible matrix P such that $A = PBP^{-1}$.

a) This does not need proof in this course, but student should know this result.

b) We compute
$$A^2 = PBP^{-1}PBP = PB^2P^{-1},$$
thus A^2 and B^2 are similar. We show this claim by induction on k. Assume that $A^k = PB^kP^{-1}$, then
$$A^{k+1} = A^kA = PB^kP^{-1}PBP^{-1} = PB^kBP^{-1} = PB^{k+1}P.$$
In particular, A^{k+1} and B^{k+1} are similar. This proves that A^m and B^m are similar for every positive integer m.

c) By the properties of the inverse, we compute
$$A^{-1} = (PBP^{-1})^{-1} = (P^{-1})^{-1}B^{-1}P^{-1} = PB^{-1}P^{-1},$$
which means that A^{-1} and B^{-1} are similar.

Exercise 6.2.3. Prove that arbitrary matrix $\mathbf{A} \in M_{n,n}(\mathbb{C})$ if $0 \notin \sigma(\mathbf{A})$ then A is invertible.

Solution. The statement of the exercise is equivalent to the following one: If A is not invertible then $0 \in \sigma(A)$.

As A is not invertible, $det(A) = 0$. Thus
$$p_A(0) = det(A - 0\mathbf{I}) = det(A) = 0.$$
Therefore, 0 is a root of the characteristic polynomial. We know that the spectrum of a matrix is the set of all roots of it characteristic polynomial. Thus $0 \in \sigma(A)$, as desired.

Exercise 6.2.4. Prove that if arbitrary matrix $\mathbf{A} \in M_{n,n}(\mathbb{C})$ invertible then $0 \notin \sigma(\mathbf{A})$.

Solution. The proof is similar to the previous exercise.

Exercise 6.2.5. Let $\mathbf{A} \in M_{n,n}(\mathbb{C})$ be a diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeating according to their algebraic multiplicity). Prove that $\det \mathbf{A} = \lambda_1 \cdots \lambda_n$.

Solution. As we assume that A is diagonalizable, there is an invertible matrix S and diagonal matrix D such that $A = SDS^{-1}$. We also know that diagonal entries of D are eigenvalues of A, and determinant of a diagonal matrix is the product of all its diagonal entries (this can be seen by expansion formula.) Thus $det(D) = \lambda_1 \cdots \lambda_n$. Now by using the properties of the determinant, we compute

$$\begin{aligned} det(A) &= det(SDS^{-1}) = det(S)det(D)det(S^{-1}) \\ &= det(S)det(D)det(S)^{-1} = det(S)det(S^{-1})det(D) \\ &= det(D) = \lambda_1 \cdots \lambda_n. \end{aligned}$$

Exercise 6.2.6. Let $\mathbf{A} \in M_{n,n}(\mathbb{C})$. Prove that

a) $p_A(\lambda) = p_{A^T}(\lambda)$.

b) $\sigma(\mathbf{A}^T) = \sigma(\mathbf{A})$;

c) $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}^T$ is diagonalizable.

Solution. We recall the following properties of taking transpose:

- A matrix is invertible if and only if its transpose is invertible, moreover, their determinants are equal. Equivalently, A matrix is not invertible if and only if its transpose is not invertible.
- $(C + \alpha D)^T = C^T + \alpha D^T$ for all matrices C and D with the same size and scalar α.
- $\mathbf{I}^T = \mathbf{I}$.

  a) By using above properties of taking transpose we get

  $$\begin{aligned} p_A(\lambda) = det(A^T - \lambda\mathbf{I}) &= det(A^T - \lambda\mathbf{I}^T) = det((A - \lambda\mathbf{I})^T) \\ &= det(A = \lambda\mathbf{I}) = p_A(\lambda). \end{aligned}$$

  b) Since the spectrum of a matrix is the set of the roots of its characteristic polynomial, and A and A^T have the same characteristic polynomial by part a), we get $\sigma(A) = \sigma(A^T)$.

  c) We also have the following properties:

     i) S is invertible if and only if S^T is invertible (why?). Moreover, $(S^T)^{-1} = (S^{-1})^T$ (why?).

     ii) $(CD)^T = D^T C^T$ for squares matrices with the same size. This is true for any product of square matrices of the same size.

  Assume that A is diagonalizable, then there exist an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$, then by the above properties, we compute

  $$A^T = (S^{-1})^T D^T S^T = (S^T)^{-1} D^T S^T.$$

  Since D^T is diagonal, the above equation shows that A^T is similar to a diagonal matrix. Therefore A^T is diagonalizable.

  For the other direction, if A^T is diagoanlizable, then we have just proved that its transpose $(A^T)^T$ is diagonalizable. But we know that $A = (A^T)^T$ so A is diagonalizable.