Assignment — Generative network models

```
from scipy.stats import ks_2samp
import matplotlib.pyplot as plt
import networkx as nx
from tqdm.notebook import trange, tqdm
import random
import numpy as np
```

Task 1. Watts-Strogatz model (0 points)

Implement Watts-Strogatz model (small-world model) — rewire an edge with probability `p` in a ring lattice with `n` nodes and `k` degree.

```
def watts_strogatz_graph(n, k, p):
    G = ring_lattice(n, k)
    for node in tqdm(G.nodes):
        rewire(G, node, k, p)
    return G
```

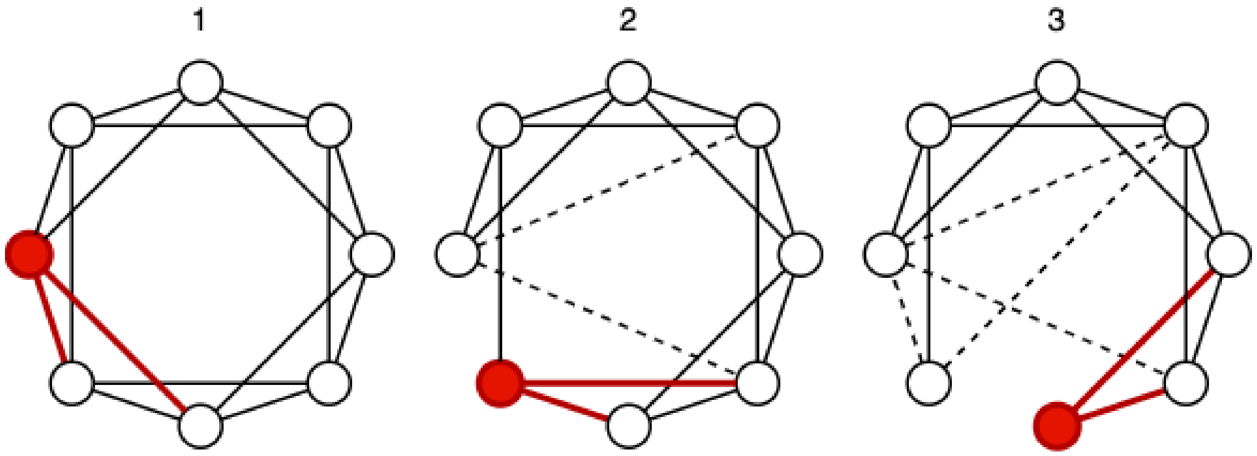
Write a function `ring_lattice` that returns a regular ring lattice with `n` nodes (0, 1, 2, ..., n-1) and `k` node degree. In a case of an odd node degree, it round it to the nearest smaller even number.

```
def ring_lattice(n, k):
    # YOUR CODE HERE
    G = nx.Graph()
    nodes = np.arange(0, n)
    G.add_nodes_from(nodes)

    for i in range(1, int(k/2 + 1)):
      G.add_edges_from(zip(nodes, (nodes + i) % n))
    return G

assert nx.degree_histogram(ring_lattice(10, 2))[2] == 10
assert nx.degree_histogram(ring_lattice(10, 3))[2] == 10
assert nx.degree_histogram(ring_lattice(10, 4))[4] == 10
```

Write a function `rewire` that takes in input a ring lattice `G`, a `node`, a model parameter `k` and probability `p`. For every right hand side neighbor *i*, the function rewires an edge (`node`, *i*) into a random edge (`node`, *j*) with probability `p` where $i \neq j \neq $ `node`.

Hints:

- *Why do we only rewire right hand side edges? We want to guarantee that only untouched in previous iterations edges will be rewound. Look at the picture — we could not move the red edges in previous iterations.*



- *To speed up the generation, do not filter nodes to random selection. If a selected node produces an existing edge or a loop, just skip it.*

```
def rewire(G, node, k, p):
    # YOUR CODE HERE
    for i in range(node+1, node+int(k/2 + 1)):
      r = np.random.rand()
      if r < p:
        target = np.random.choice(len(G.nodes))
        if G.has_edge(node, target) or target == node: continue

        G.remove_edge(node, i % len(G.nodes))
        G.add_edge(node, target)

cases = [[50, 8, 0.1],
         [1000, 10, 0.01],
         [1000, 10, 0.5],
         [1000, 10, 0.99]]
for n, k, p in cases:
    G = watts_strogatz_graph(n, k, p)
    assert nx.number_of_nodes(G) == n
    assert nx.number_of_edges(G) == int(k / 2 * n)
    degree_seq = [degree for (node, degree) in G.degree]
    nxG = nx.watts_strogatz_graph(n, k, p, 1)
    nxdegree_seq = [degree for (node, degree) in nxG.degree]
    assert ks_2samp(degree_seq, nxdegree_seq).pvalue > 0.05
```

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{"model_id":"fe71a1b60efc435c9d134b8b90e7d7b3","version_major":2,"version_minor":0}

```
/usr/local/lib/python3.11/dist-packages/scipy/stats/
_axis_nan_policy.py:531: RuntimeWarning: ks_2samp: Exact calculation
```

```
unsuccessful. Switching to method=asymp.
  res = hypotest_fun_out(*samples, **kwds)
```

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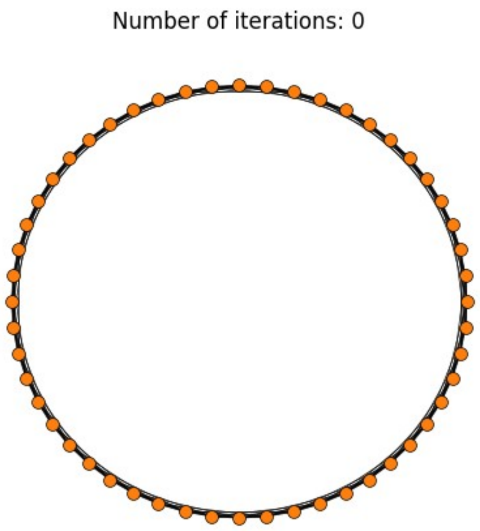
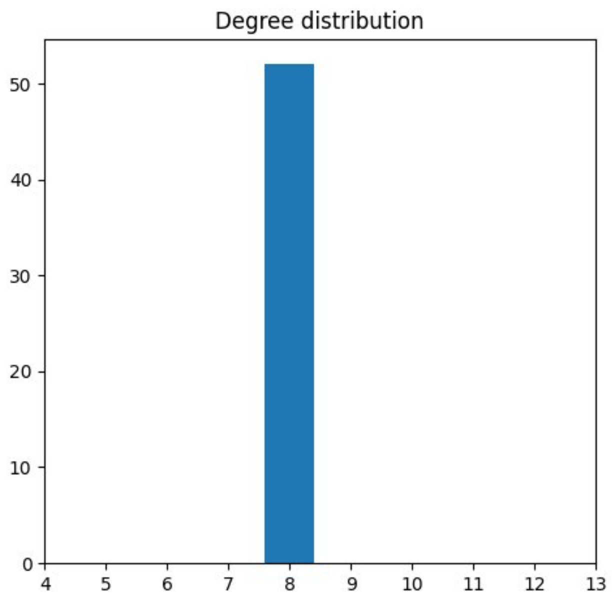
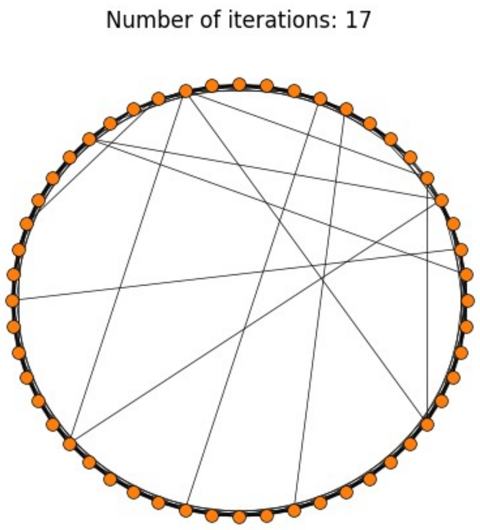
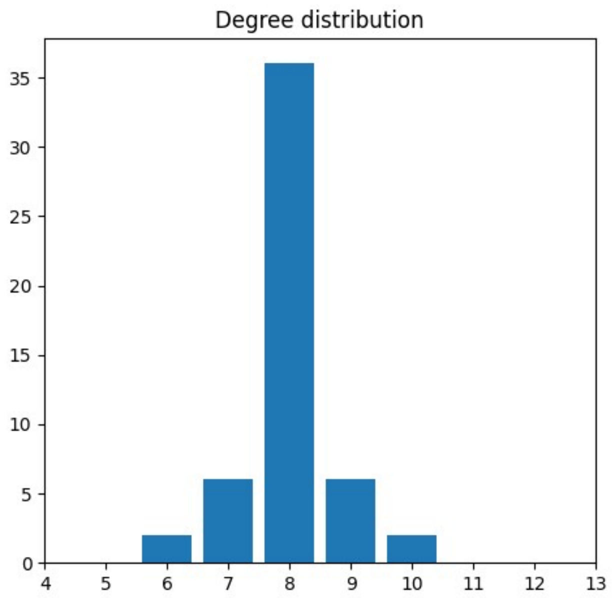
Let us draw a small-world graph in some steps of the algorithm

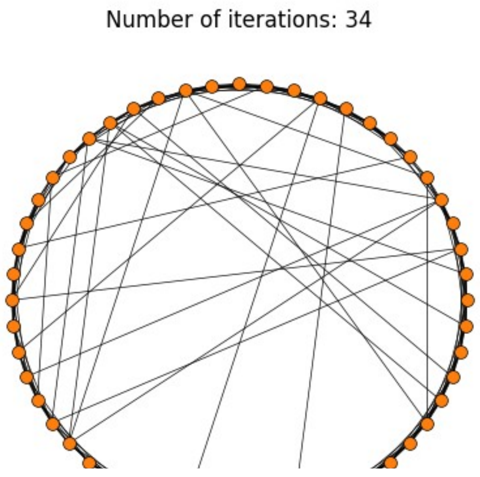
```python
n, k, p = 52, 8, 0.2
G = ring_lattice(n, k)
plt.figure(figsize=(12, 6 * 4))
i = 1
for node in G.nodes:
    if node in np.arange(0, n+1, int(n/3)):
        plt.subplot(4, 2, i)
        plt.title('Number of iterations: {}'.format(node))
        nx.draw_circular(
            G,
            node_size=50,
            width=0.5,
            linewidths=0.5,
            edgecolors='black',
            node_color='tab:orange')
        i += 1
        plt.subplot(4, 2, i)
        degree_seq = [degree for (node, degree) in G.degree]
        bins, freq = np.unique(degree_seq, return_counts=True)
        plt.bar(bins, freq)
        plt.xlim((4, 13))
        plt.title('Degree distribution')
        i += 1
    rewire(G, node, k, p)
```

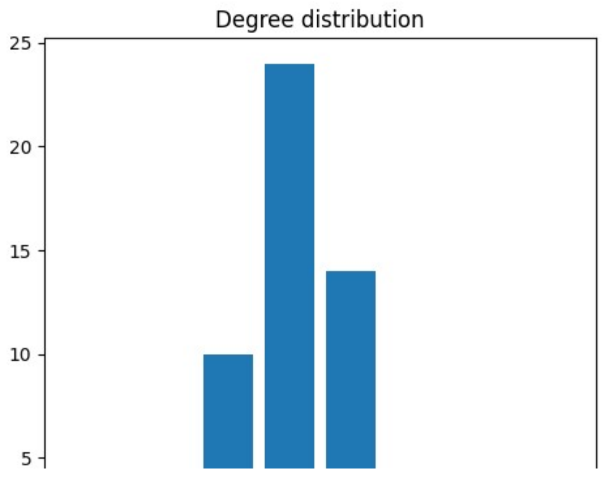

Number of iterations: 0


Degree distribution
0
10
20
30
40
50
4 5 6 7 8 9 10 11 12 13


Number of iterations: 17


Degree distribution
0
5
10
15
20
25
30
35
4 5 6 7 8 9 10 11 12 13


Number of iterations: 34


Degree distribution
5
10
15
20
25

Task 2. Average path length in Watts-Strogatz model (2 points)

Let us check that the average path length tends to theoretical value during building the small-world model.

$$\langle L \rangle = \begin{cases} N/2k, \text{if } p \to 0 \\ \log(N)/\log(k), \text{if } p \to 1 \end{cases}$$

So that we have a lower and upper limits of path lengths for $0 < p < 1$.

Write a function `smallworld_path_len` with Watts-Strogatz model parameters `n, k, p` that returns np.array of average path lengths in each step (node). The length of the array is `n`.

Hint: to calculate the average shortest path length, use `nx.average_shortest_path_length`

```
def smallworld_path_len(n, k, p):
    # YOUR CODE HERE

n, k, p = 101, 10, 0.05
lengths = smallworld_path_len(n, k, p)
step_space = np.log(np.arange(1, len(lengths) + 1))
X = np.stack([step_space, np.ones(lengths.shape[0])], axis=1)
assert lengths.shape[0] == n
assert 0.1 < -(np.linalg.pinv(X) @ np.log(lengths))[0] < 0.25

plt.figure(figsize=(10, 5))

n, k, p = 101, 12, 0.01
lengths = smallworld_path_len(n, k, p)
plt.plot(lengths, label=f'p={p}')

n, k, p = 101, 12, 0.1
lengths = smallworld_path_len(n, k, p)
plt.plot(lengths, label=f'p={p}')

n, k, p = 101, 12, 0.9
lengths = smallworld_path_len(n, k, p)
plt.plot(lengths, label=f'p={p}')

plt.xlabel('Step of the small-world building')
plt.ylabel('Average path length')
plt.grid()
plt.plot([0, 100], [n / 2 / k, n / 2 / k], '--',
         label='limit upper bound')
plt.plot([0, 100], [np.log(n) / np.log(k), np.log(n) / np.log(k)],
'--',
         label='limit lower bound')
plt.legend(loc='lower left')
plt.show()
```

Task 3. Barabasi-Albert model (0 points)

Implement Barabasi-Albert model (preferential attachment model) – a growth process where each new node connects to `m` existing nodes. The higher node degree, the higher probability of the connection. The final number of nodes is `n`.

You start from a star graph with `m + 1` nodes. In each step you create `m` edges between a new node and existing nodes. The probability of connection to the node i is

$$p(i) = \frac{k_i}{\sum k}$$

Write a function `attach` that attaches a `node` to a graph `G` through `m` edges.

Hint: Create a list with repeated nodes from a list of edges. For example, $[(1,2),(2,3),(2,4)] \rightarrow [1,2,2,3,2,4]$. *Uniformly select nodes one-by-one. Apply* `random.choice` *instead of* `np.random.choice` *to speed up the generation.*

```python
'''Do not touch the cell'''
def barabasi_albert_graph(n, m):
    G = nx.star_graph(m)
    for i in trange(1, n - m):
        attach(m + i, G, m)
    return G

def attach(node, G, m):
    # YOUR CODE HERE
    '''
    degrees = np.array([G.degree[i] for i in G.nodes])
    probs = degrees / degrees.sum()

    neigs = np.random.choice(G.nodes, p=probs, size=m, replace=False)
    for nei in neigs:
      G.add_edge(node, nei)
    '''

    edges_flatten = []
    for u, v in G.edges:
      edges_flatten.append(u)
      edges_flatten.append(v)

    neigs = set()
    while len(neigs) < m:
      nei = random.choice(edges_flatten)
      neigs.add(nei)

    for nei in neigs:
      G.add_edge(node, nei)

G = nx.star_graph(3)
attach(4, G, 3)
```

```
assert nx.number_of_edges(G) == 6

cases = [[10, 3],
         [1000, 3],
         [1000, 20]]
for n, m in cases:
    G = barabasi_albert_graph(n, m)
    degree_seq = [degree for (node, degree) in G.degree]
    nxG = nx.barabasi_albert_graph(n, m)
    nxdegree_seq = [degree for (node, degree) in nxG.degree]
    assert ks_2samp(degree_seq, nxdegree_seq).pvalue > 0.05
```

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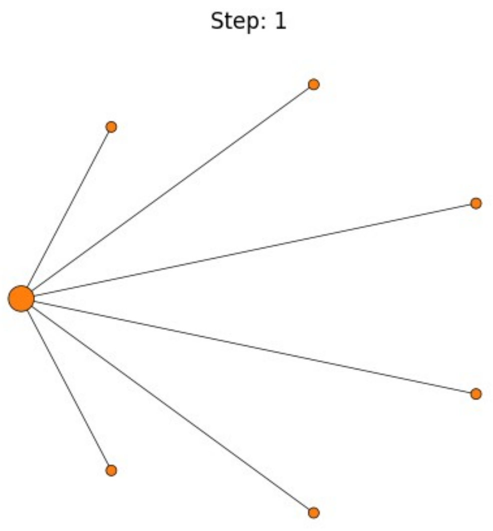
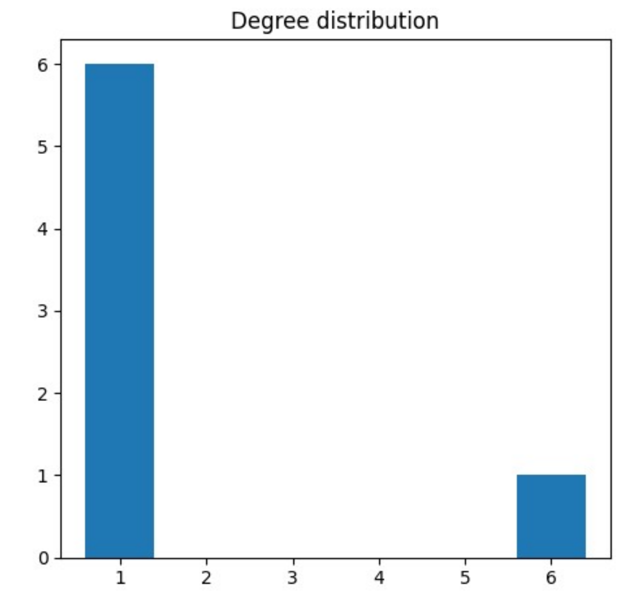
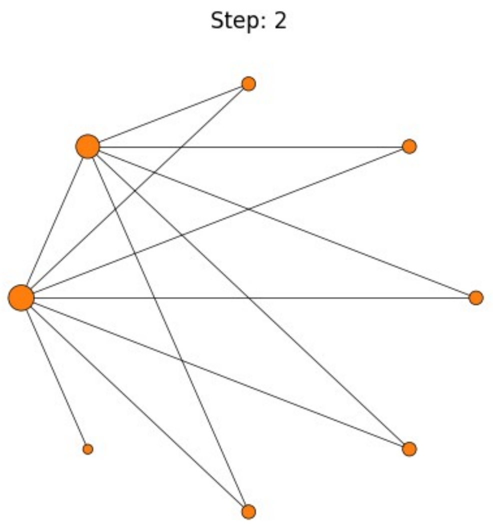
{"model_id":"4abacf4064cc466a8718c9b7c153bc76","version_major":2,"version_minor":0}

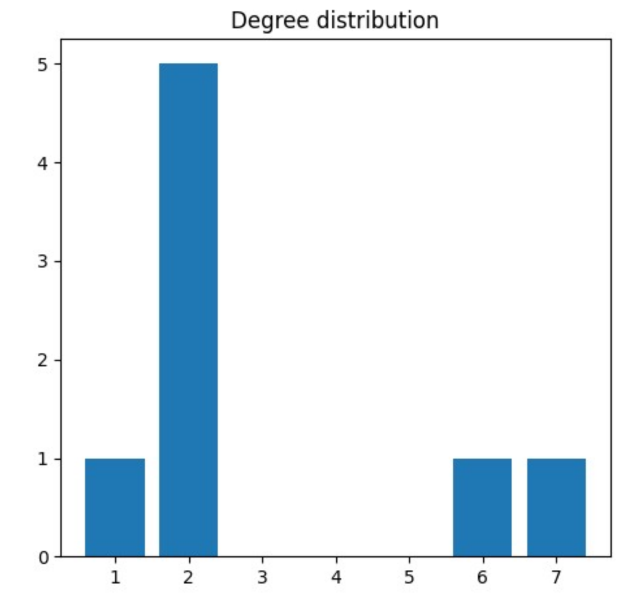
Let us see what the growth process looks like

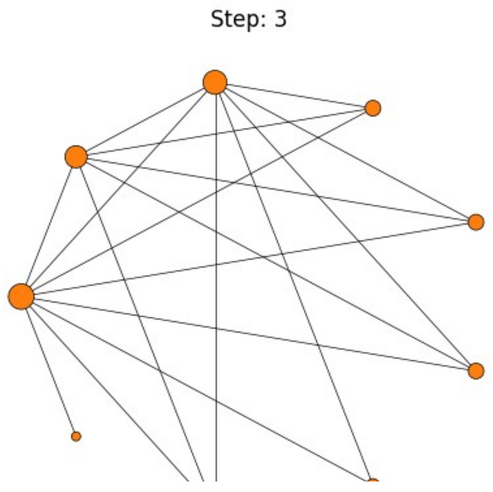
```
n, m = 1000, 6
G = nx.star_graph(m)
plt.figure(figsize=(12, 6 * 5))
j = 1
for i in range(1, n - m):
    if i in [1, 2, 3, 30, n-m-1]:
        plt.subplot(5, 2, j)
        j += 1
        sizes = np.array(list(nx.degree_centrality(G).values()))
        sizes = sizes / max(sizes) * 200
        if i <= 3:
            pos = nx.layout.shell_layout(G)
        else:
            pos = nx.layout.spring_layout(G)
        nx.draw(
            G,
            pos=pos,
            with_labels=False,
            node_size=sizes,
            width=0.5,
            linewidths=0.5,
            edgecolors='black',
            node_color='tab:orange')
        plt.title('Step: {}'.format(i))
        degree_seq = [degree for (node, degree) in G.degree]
        bins, freq = np.unique(degree_seq, return_counts=True)
        plt.subplot(5, 2, j)
        j += 1
        plt.bar(bins, freq)
        plt.title('Degree distribution')
    attach(m + i, G, m)
```

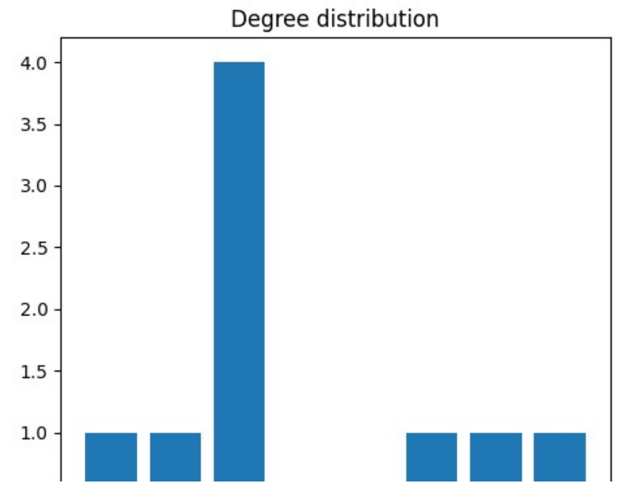

Step: 1


Degree distribution
0
1
2
3
4
5
6
1
2
3
4
5
6


Step: 2


Degree distribution
0
1
2
3
4
5
1
2
3
4
5
6
7


Step: 3


Degree distribution
1.0
1.5
2.0
2.5
3.0
3.5
4.0

Task 4. Degree distribution in Barabasi-Albert model (3 points)

Barabasi-Albert graph has a degree distribution of the form

$$P(k)=\frac{2m^2}{k^3}$$

That is Power law with $\alpha=3$ and $k_{min}=m$.

Check this fact by an experiment — generate a set of Barabasi-Albert graphs and estimate parameters of Power law using MLE

$$\alpha=1+n\left[\sum_i \log\frac{k_i}{k_{min}}\right)^{-1}$$

where the k_{min} is selected by minimal Kolmogorov-Smirnov distance between observed and theoretical distributions.

First, write a function `power_law_cdf` that takes an argument and parameters of the Power law distribution and returns the CDF.

```
def power_law_cdf(k, alpha=3.5, k_min=1):
    # Ensure k >= k_min
    if k < k_min:
        raise ValueError(f"k must be greater than or equal to k_min.
Given k: {k}, k_min: {k_min}")

    # Compute the CDF
    return 1 - (k_min / k) ** (alpha - 1)

assert power_law_cdf(2, 2, 1) == 0.5
assert power_law_cdf(10, 2, 1) == 0.9
```

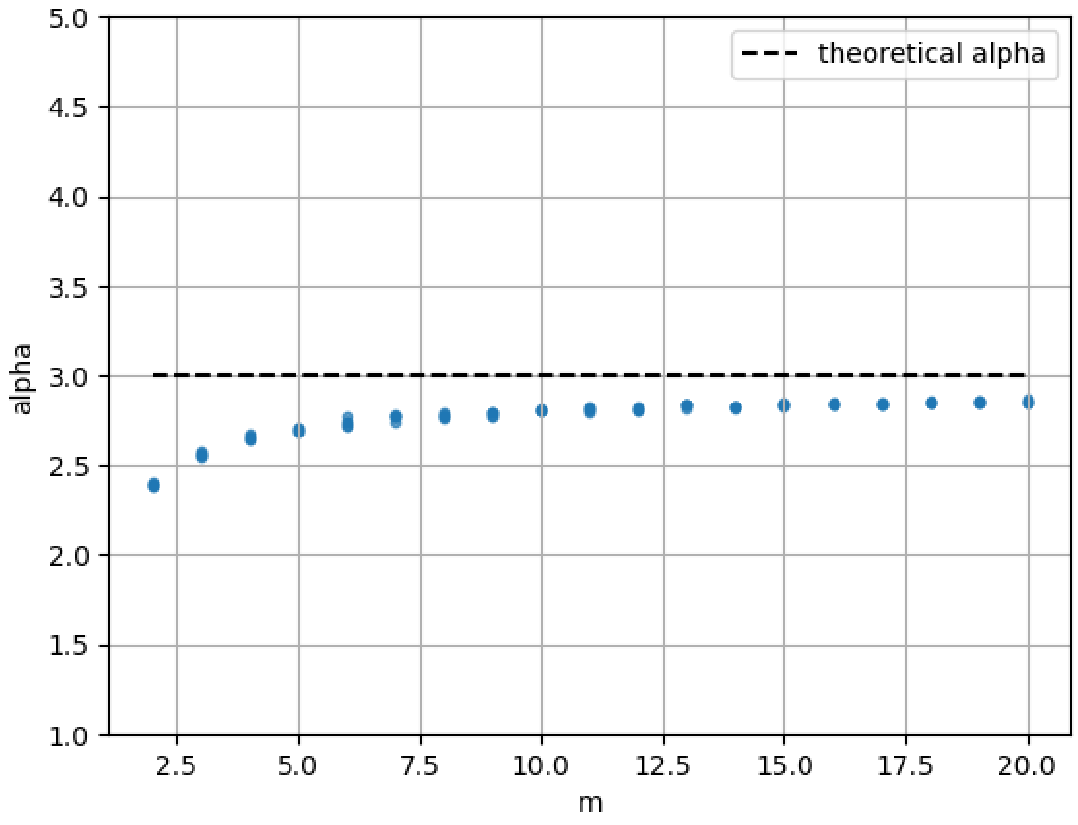
Next, write a function `mle_power_law_params` that takes a degree sequence and returns a tuple: the best α, w.r.t. MLE, the best k_{min} w.r.t. Kolmogorov-Smirnov distance

Hint: use `scipy.stats.kstest` where a theoretical CDF is a `power_law_cdf` function and `args=(alpha, k_min)`

```
import numpy as np
from scipy import stats


def mle_power_law_params(degree_sequence):
    # Sort the degree sequence in ascending order
    degree_sequence = np.array(degree_sequence)
    degree_sequence = degree_sequence[degree_sequence > 0]  # remove
zero degrees

    # Set initial values for alpha and k_min
    best_alpha = 0
    best_kmin = 0
    best_ks_stat = float('inf')  # Best KS-statistic (lower is better)

    # We will try different k_min values from min(degree_sequence) to
max(degree_sequence)
    k_min_range = np.arange(min(degree_sequence), max(degree_sequence)
+ 1)

    for k_min in k_min_range:
        # Calculate alpha using MLE formula for each k_min
        alpha = 1 + len(degree_sequence) *
np.sum(np.log(degree_sequence / k_min))**(-1)

        # Apply Kolmogorov-Smirnov test to compare the empirical and
theoretical CDF
        ks_stat, _ = stats.kstest(degree_sequence, 'powerlaw',
args=(alpha, k_min))

        # Update best parameters if the current KS-statistic is better
        if ks_stat < best_ks_stat:
            best_ks_stat = ks_stat
            best_alpha = alpha
            best_kmin = k_min

    return best_alpha, best_kmin



assert mle_power_law_params(np.array([1, 2, 3]))[0] > 0
assert mle_power_law_params(np.array([1, 2, 3]))[1] > 0
```

Write a function `estimate_power_law` that generates Barabasi-Albert graphs with `n` nodes, from `m_min` to `m_max` connections and returns a tuple of np.arrays: α and k_{min} for each graph.

```
import networkx as nx
import numpy as np

def estimate_power_law(n, m_min, m_max):
    alpha_list = []
    k_min_list = []
    for m in range(m_min, m_max + 1):
        # Generate Barabasi-Albert graph
        G = nx.barabasi_albert_graph(n, m)
        degrees = [d for _, d in G.degree()]
        # Set k_min as m
        k_min = m
        # Filter degrees >= k_min
        filtered_degrees = [k for k in degrees if k >= k_min]
        # Calculate alpha using MLE for continuous power-law
        if len(filtered_degrees) < 2:
            # Fallback in case of insufficient data (unlikely for BA
model)
            alpha = np.nan
        else:
            sum_log = np.sum(np.log(np.array(filtered_degrees) /
(k_min - 0.5)))
            alpha = 1 + len(filtered_degrees) / sum_log
        alpha_list.append(alpha)
        k_min_list.append(k_min)
    return np.array(alpha_list), np.array(k_min_list)

'''Check the Power law parameters'''
n, m_min, m_max = 500, 2, 20
alpha, k_min = estimate_power_law(n, m_min, m_max)
assert alpha.shape[0] == m_max - m_min + 1
assert 2 < alpha.mean() < 4
assert k_min[0] < k_min[-1]

n, m_min, m_max = 500, 2, 20
m_space = np.arange(m_min, m_max + 1)
for _ in range(5):
    alpha, k_min = estimate_power_law(n, m_min, m_max)
    plt.scatter(m_space, alpha, alpha=0.7, c='tab:blue', s=10)
plt.plot([2, 20], [3, 3], 'k--', label='theoretical alpha')
plt.ylim((1, 5))
plt.xlabel('m')
plt.ylabel('alpha')
plt.grid()
plt.legend()
plt.show()
```

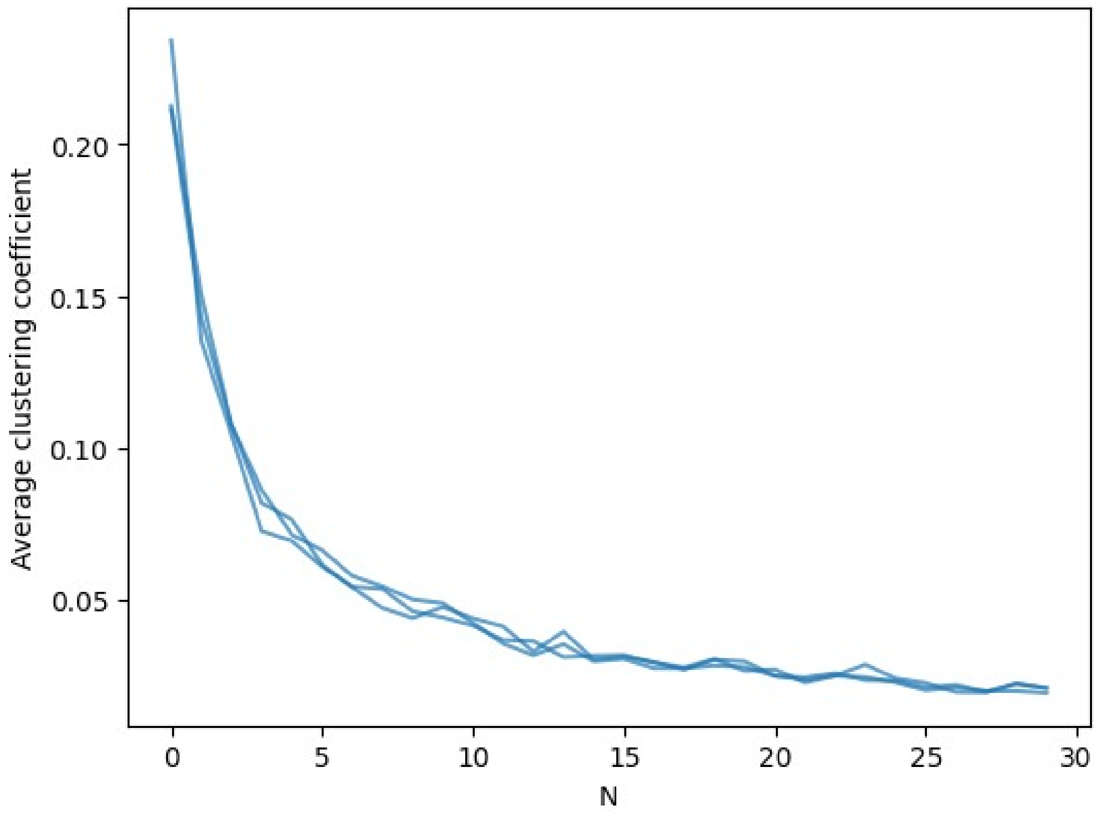


Task 5. Clustering coefficient in Barabasi-Albert model (2 points)

Measure the average clustering coefficient in function of N using Barabasi-Albert model.

Write a function `generate_clustering_coef` that takes np.array with list of `n` values for each graph and parameter `m`. The function generate Barabasi-Albert graphs and returns np.array of average clustering coefficients.

```
import numpy as np
import networkx as nx
import matplotlib.pyplot as plt

def generate_clustering_coef(n_list, m):
    # Initialize an empty list to store average clustering
coefficients
    coeffs = []

    # Iterate through each value in n_list
    for n in n_list:
        # Generate the Barabasi-Albert graph with n nodes and m edges
per new node
        G = nx.barabasi_albert_graph(n, m)

        # Calculate the average clustering coefficient for the graph
        avg_clustering = nx.average_clustering(G)

        # Append the result to the list
        coeffs.append(avg_clustering)

    # Convert the list to a numpy array and return
    return np.array(coeffs)

n_list = np.arange(100, 3100, 100)
m = 6
coeffs = generate_clustering_coef(n_list, m)
assert coeffs.shape == (30,)
X = np.log(n_list)
X = np.stack([np.ones_like(X), X], axis=1)
Y = np.log(coeffs)[:, None]
assert -0.78 < (np.linalg.pinv(X) @ Y)[1][0] < -0.63

n_list = np.arange(100, 3100, 100)
m = 6
for _ in range(3):
    coeffs = generate_clustering_coef(n_list, m)
    plt.plot(coeffs, c='tab:blue', alpha=0.7)
plt.xlabel('N')
plt.ylabel('Average clustering coefficient')
plt.show()
```



Task 6. Degree dynamics in Barabasi-Albert model (3 points)

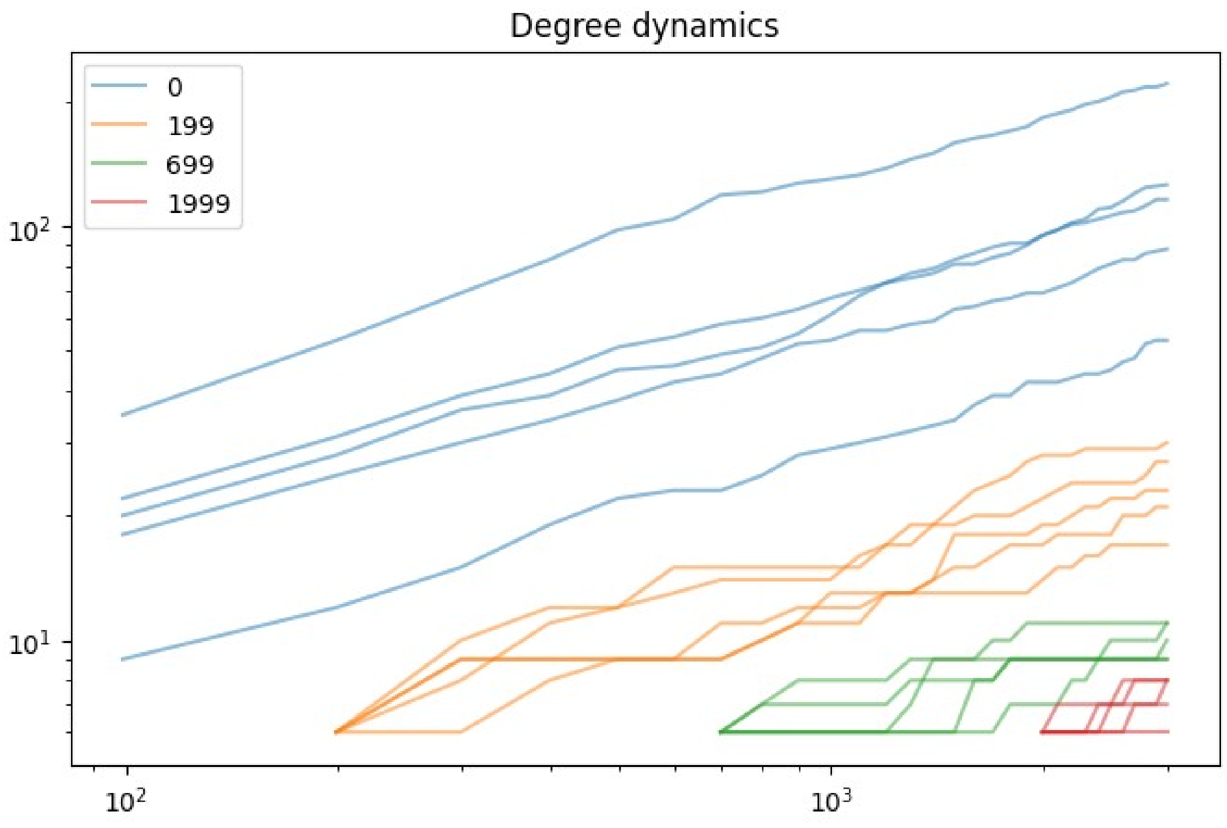
Measure the degree dynamics in Barabasi-Albert model of one of the initial nodes and of the nodes added to the network at intermediate time moments (steps of the algorithm).

Write a function `generate_degree_dynamics` that takes np.array with considered nodes, generates Barabasi-Albert graph ($n=3000$, $m=6$) and returns a np.array of the shape `(30, len(cons_nodes))` — degrees of these nodes at time moments when nodes 99, 199, 299, ..., 2999 appear. If a node does not exist yet, pass `np.nan` value.

Hint: use the `barabasi_albert_graph` *function as a template*

```python
import numpy as np
import random

def generate_degree_dynamics(cons_nodes):
    m = 6
    n = 3000
    checkpoints = [99 + 100 * i for i in range(30)]
    checkpoints_set = set(checkpoints)
    degrees = [0] * m  # Initial nodes 0-5 with degree 0
    repeated_nodes = []  # Nodes repeated according to their degree
    result = []

    for t in range(m, n):
        # Determine targets for the new node
        if t == m:
            # First new node connects to all initial nodes
            targets = list(range(m))
        else:
            # Select m distinct targets using preferential attachment
            targets_set = set()
            while len(targets_set) < m:
                node = random.choice(repeated_nodes)
                targets_set.add(node)
            targets = list(targets_set)

        # Update degrees for existing targets
        for target in targets:
            degrees[target] += 1
        # Add new node with degree m
        degrees.append(m)

        # Update the repeated_nodes list
        repeated_nodes.extend(targets)
        repeated_nodes.extend([t] * m)

        # Check if current node is a checkpoint
        if t in checkpoints_set:
            current_degrees = []
            for node in cons_nodes:
                if node >= len(degrees):
                    current_degrees.append(np.nan)
                else:
                    current_degrees.append(degrees[node])
            result.append(current_degrees)

    return np.array(result)

degree_dynamics = generate_degree_dynamics([99, 199, 699, 1999])
assert degree_dynamics.shape == (30, 4)
assert np.all(np.isnan(degree_dynamics[0]) == [False, True,  True,
True])
assert np.all(np.isnan(degree_dynamics[9]) == [False, False,  False,
True])
assert degree_dynamics[0, 0] < degree_dynamics[-1, 0]
assert degree_dynamics[1, 1] < degree_dynamics[-1, 1]
assert degree_dynamics[-1, 0] > degree_dynamics[-1, 3]

cons_nodes = [0, 199, 699, 1999]
colors = plt.cm.tab10.colors
plt.figure(figsize=(8, 5))
for _ in trange(5):
    degree_dynamics = generate_degree_dynamics(cons_nodes)
    time_space = np.arange(99, 3000, 100)
    for i in range(4):
        plt.plot(time_space, degree_dynamics[:, i], c=colors[i],
alpha=0.5)
plt.legend(cons_nodes)
plt.title('Degree dynamics')
plt.xscale('log')
plt.yscale('log')
plt.show()
```

{"model_id":"800382fd92b1490d90e591870d2f2368","version_major":2,"vers
ion_minor":0}


Degree dynamics
0
199
699
1999
10^2
10^1
10^2
10^3