MATH 112: Introduction to Analysis
Fall 2024 Semester
Homework 5: Due Tuesday October 14, 10:00am PST

Assignment (2 Problems: 50 + 50 = 100 points total.)

☐ **Problem 1** [Ordered Fields and Averages] Let $(F, \preccurlyeq, \oplus, \mathbb{O}, \otimes, \mathbb{I})$ be an ordered field, i.e. a set F that is both a field $(F, \oplus, \mathbb{O}, \otimes, \mathbb{I})$ satisfying the 10 field axioms from L7 and an ordered set $(F, \preccurlyeq)$ with a compatible total order $\preccurlyeq$ on F satisfying the 4 ordered field axioms from L11. Prove each of the following propositions. Indicate which axioms you use in each step.

- 1.1 [10 points] $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \prec y\ \wedge\ c \prec \mathbb{O})\ \Rightarrow\ c \otimes x \succ c \otimes y\Big)$
  Note: recall from L9 *that "$a \succ b$" means the same thing as "$b \prec a$".*
- 1.2 [10 points] $\forall x \in F\ (x \neq \mathbb{O}\ \Rightarrow\ \mathbb{O} \prec x^2)$. *Recall from* L8*: for all $x \in F$, $x^2 = x \otimes x$.*
- 1.3 [10 points] $-\mathbb{I} \prec \mathbb{O} \prec \mathbb{I}$. *Recall from* L7*: $-x$ is the additive inverse of $x \in F$.*
- 1.4 [10 points] $\exists g \in F\ (\mathbb{I} \oplus \mathbb{I}) \otimes g = \mathbb{I}$.
- 1.5 [10 points] $\forall x \in F\ \forall y \in F\ \Big((x \prec y) \Rightarrow (x \prec g \otimes (x \oplus y) \prec y)\Big)$ where the element $g \in F$ is the one you proved exists in the previous Problem 1.4.

1.1 $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \prec y\ \wedge\ c \prec \mathbb{O})\ \Rightarrow\ c \otimes x \succ c \otimes y\Big)$

Assume $x \prec y \wedge c \prec \mathbb{O}$. By the hypothetical strategy, if we can prove that $c \otimes x \succ c \otimes y$, under this assumption, then the original statement is true. Start from our assumption.

$$
\begin{aligned}
&\Leftrightarrow & x &\prec y \\
&\Leftrightarrow & (-c) \otimes x &\prec (-c) \otimes y \\
&\Leftrightarrow & (c \otimes x) \oplus ((-c) \otimes x) &\prec (c \otimes x) \oplus ((-c) \otimes y) \\
&\Leftrightarrow & (c \oplus (-c)) \otimes x &\prec (c \otimes x) \oplus ((-c) \otimes y) \\
&\Leftrightarrow & \mathbb{O} \otimes x &\prec (c \otimes x) \oplus ((-c) \otimes y) \\
&\Leftrightarrow & \mathbb{O} &\prec (c \otimes c) \oplus ((-c) \otimes y) \\
&\Leftrightarrow & \mathbb{O} \oplus (c \otimes y) &\prec (c \otimes x) \oplus ((-c) \otimes y)) \oplus (c \otimes y) \\
&\Leftrightarrow & \mathbb{O} \oplus (c \otimes y) &\prec (c \otimes x) \oplus (y \otimes ((-c) \oplus c)) \\
&\Leftrightarrow & \mathbb{O} \oplus (c \otimes y) &\prec (c \otimes x) \oplus (y \otimes \mathbb{O}) \\
&\Leftrightarrow & \mathbb{O} \oplus (c \otimes y) &\prec (c \otimes x) \oplus \mathbb{O} \\
&\Leftrightarrow & c \otimes y &\prec c \otimes x \\
&\Leftrightarrow & c \otimes x &\succ c \otimes y
\end{aligned}
$$

Using Axiom 14 we know that we can $\otimes$ any value to both sides, as long as it is greater than $\mathbb{O}$. Since we have asusmed $c \prec \mathbb{O}$, by axiom 4, $(-c) \succ \mathbb{O}$ since $c \oplus (-c) = \mathbb{O}$. Next use axiom 14, which tells us we can $\oplus$ any element to both sides without affecting $\prec$ adding $(c \otimes x)$. Use Axiom 9, to distribute $\otimes$ over $\oplus$, on the left side which gets us $c \oplus (-c) \otimes x)$. Then use axiom 4 to get that $(c \oplus (-c)) = \mathbb{O}$. After this, use Hw 4, problem 1.2 to get $\mathbb{O} \otimes x = \mathbb{O}$. Use axiom 13 to $\oplus(c \otimes y)$ to both sides. Use axiom 1 to group $(c \otimes x) \oplus ((-c) \otimes y) \oplus (x \otimes y)$ together, then use axiom 9 to distribute $\otimes$ over

$\oplus$, to get: $y \otimes ((-c) \oplus c)$. Use axiom 4, to get $(-c) \oplus c) = \mathbb{O}$. Use Hw 4, problem 1.2 to get $y \otimes \mathbb{O} = \mathbb{O}$. Use axiom 3, that any value plus $\mathbb{O}$ equals itself to get: $(c \otimes x)$. Lastly use what we learned in L8, that you can flip $\prec$ to $\succ$ if you switch the terms sides as well.

1.2 $\forall x \in F\ (x \neq \mathbb{O} \ \Rightarrow\ \mathbb{O} \prec x^2)$.

EXPLANATION

Assume $x \neq \mathbb{O}$, means either $x < \mathbb{O}$ or $x > \mathbb{O}$.

Case 1: $x < \mathbb{O}$

$$\begin{aligned}
\Leftrightarrow && x \prec \mathbb{O} \\
\Leftrightarrow && x \otimes (-x) \prec \mathbb{O} \otimes (-x) \\
\Leftrightarrow && x \otimes (-x) \prec \mathbb{O} \\
\Leftrightarrow && (x \otimes x) \oplus (x \otimes (-x)) \prec (x \otimes x) \oplus \mathbb{O} \\
\Leftrightarrow && x \otimes (x \oplus (-x)) \prec (x \otimes x) \oplus \mathbb{O} \\
\Leftrightarrow && x \otimes \mathbb{O} \prec (x \otimes x) \oplus \mathbb{O} \\
\Leftrightarrow && \mathbb{O} \prec (x \otimes x) \oplus \mathbb{O} \\
\Leftrightarrow && \mathbb{O} \prec (x \otimes x) \\
\Leftrightarrow && \mathbb{O} \prec x^2
\end{aligned}$$

EXPLANATION.

Case 2: $x > \mathbb{O}$

$$\begin{aligned}
&& x \prec \mathbb{O} \\
\Leftrightarrow && x \otimes x \prec \mathbb{O} \otimes x \\
\Leftrightarrow && x \otimes x \prec \mathbb{O} \\
\Leftrightarrow && x^2\ \mathbb{O}
\end{aligned}$$

EXPLANATION

Since this statement _____, holds for both possible cases and these the only possible cases that statisgy $x \neq \mathbb{O}$, the orignal statment is True.

1.3 $-\mathbb{I} \prec \mathbb{O} \prec \mathbb{I}$.

EXPLANATION

by axiom 10, we have: $\mathbb{O} \neq \mathbb{I}$

$$\begin{aligned}
&& \mathbb{O} \neq \mathbb{I} \\
&& \mathbb{O} \prec \mathbb{I}^2 \\
\Leftrightarrow && \mathbb{O} \prec \mathbb{I} \otimes \mathbb{I} \\
\Leftrightarrow && \mathbb{O} \prec \mathbb{I}
\end{aligned}$$

We now know that $\mathbb{O} \prec \mathbb{I}$, use to show $-\mathbb{I} \prec \mathbb{O}$.

$$\begin{aligned}
&& \mathbb{O} \prec \mathbb{I} \\
\Leftrightarrow && (-\mathbb{I}) \oplus \mathbb{O} \prec (-\mathbb{I}) \oplus \mathbb{I} \\
\Leftrightarrow && (-\mathbb{I}) \oplus \mathbb{O} \prec \mathbb{O} \\
\Leftrightarrow && -\mathbb{I} \prec \mathbb{O}
\end{aligned}$$

1.4 $\exists g \in F\ (\mathbb{I} \oplus \mathbb{I}) \otimes g = \mathbb{I}$.

Let $g = (\mathbb{I} \oplus \mathbb{I})^{-1}$. Substitute g into the equation.

$$\begin{aligned}(\mathbb{I} \oplus \mathbb{I}) \otimes g &= (\mathbb{I} \otimes \mathbb{I}) \otimes (\mathbb{I} \oplus \mathbb{I})^{-1} \\ &= \mathbb{I}\end{aligned}$$

EXPLANATION

1.5 $\forall x \in F\ \forall y \in F\ \Big((x \prec y) \Rightarrow (x \prec g \otimes (x \oplus y) \prec y)\Big)$

Assume $x \prec y$, let $g = (\mathbb{I} \oplus \mathbb{I})^{-1}$.

First show: $x \prec y \Rightarrow x \prec g \otimes (x \oplus y)$

$$\begin{aligned}
& & x &\prec y \\
\Leftrightarrow & & x \oplus x &\prec x \oplus y \\
\Leftrightarrow & & (x \otimes \mathbb{I}) \oplus (x \otimes \mathbb{I}) &\prec x \oplus y \\
\Leftrightarrow & & x \otimes (\mathbb{I} \oplus]II) &\prec x \oplus y \\
\Leftrightarrow & & (\mathbb{I} \oplus \mathbb{I}) \otimes x &\prec x \oplus y \\
\Leftrightarrow & & (\mathbb{I} \oplus \mathbb{I}) \otimes x &\prec (\mathbb{I} \oplus \mathbb{I})^{-1}]x(x \oplus y) \\
\Leftrightarrow & & \mathbb{I} \otimes x &\prec (\mathbb{I} \otimes \mathbb{I})^{-1} \otimes (x \otimes Y) \\
\Leftrightarrow & & x &\prec (\mathbb{I} \otimes \mathbb{I})^{-1} \otimes (x \otimes y) \\
\Leftrightarrow & & x &\prec g \otimes (x \otimes y)
\end{aligned}$$

EXPLANATION

Next show: $x \prec y \Rightarrow g \otimes (x \otimes y)$

$$\begin{aligned}
& & x &\prec y \\
\Leftrightarrow & & x \oplus y &\prec y \oplus y \\
\Leftrightarrow & & x \oplus y &\prec (y \otimes \mathbb{I}) \oplus (y \otimes \mathbb{I}) \\
\Leftrightarrow & & x \oplus y &\prec y \otimes (\mathbb{I} \oplus \mathbb{I}) \\
\Leftrightarrow & & (x \oplus y) \otimes (\mathbb{I} \oplus \mathbb{I})^{-1} &\prec y \otimes (\mathbb{I} \oplus \mathbb{I}) \otimes (\mathbb{I} \otimes \mathbb{I})^{-1} \\
\Leftrightarrow & & (x \oplus y) \otimes (\mathbb{I} \oplus \mathbb{I})^{-1} &\prec y \otimes ((\mathbb{I} \oplus \mathbb{I}) \otimes (\mathbb{I} \otimes \mathbb{I})^{-1}) \\
\Leftrightarrow & & (x \oplus y) \otimes (\mathbb{I} \oplus \mathbb{I})^{-1} &\prec y \otimes \mathbb{I} \\
\Leftrightarrow & & (x \oplus y) \otimes (\mathbb{I} \oplus \mathbb{I})^{-1} &\prec y
\end{aligned}$$

EXPLANATION

MORE EXPLANATION Since we have $a \prec b \wedge b \prec c$. by axiom 11, this implies we have $a \prec c$. furthermore we have $a \prec b \prec c$.

☐ **Problem 2** [Optimal Bounds of Intervals in the Continuum] Recall from L9 that in any totally ordered set $(S, \preceq)$, if $A \subseteq S$ is a subset, we say that $u_\star = \sup S$ is an optimal upper bound of A in S if (i) $u_\star$ is an upper bound for A in $(S, \preceq)$ and (ii) any upper bound u of A satisfies $u_\star \preceq u$. In L9, we saw that special subsets of S are the closed and open intervals

$$[a,b]_S := \Big\{ x \in S \ : \ a \preceq x \preceq b \Big\}$$

$$(a,b)_S := \Big\{ x \in S \ : \ a \prec x \prec b \Big\}.$$

- 2.1 [25 points] Prove that $\sup[111, 112]_{\mathbb{R}} = 112$.
- 2.2 [25 points] Prove that $\sup(111, 112)_{\mathbb{R}} = 112$.

2.1 Prove that $\sup[111, 112]_{\mathbb{R}} = 112$.
We are given, in l9 and in the begining of this problem, that:

$$[a,b]_S := \Big\{ x \in S \ : \ a \preceq x \preceq b \Big\}.$$

Thus:

$$[112, 112]_{\mathbb{R}} := \Big\{ x \in \mathbb{R} : \ 112 \preceq x \preceq 112 \Big\}.$$

To prove that $\sup[111, 112]_{\mathbb{R}} = 112$, we will first show that 112 is an upper bound for $[111, 112]_{\mathbb{R}}$. Recall that, from L9, the definition: an upper bound for A in S is an element $u \in S$ so $\forall \alpha \in A \ \alpha \leq u$. To prove that 112 is an upper bound, it is sufficent to show that $\forall x \in \mathbb{R} \ \ x \preceq 112$.

2.2 Prove that $\sup(111, 112)_{\mathbb{R}} = 112$.
We are given, in l9 and in the begining of this problem, that:

$$(a,b)_S := \Big\{ x \in S \ : \ a \prec x \prec b \Big\}.$$

Thus:

$$(112, 112)_{\mathbb{R}} := \Big\{ x \in \mathbb{R} : \ 111 \prec x \prec 112 \Big\}.$$

To prove that $\sup[111, 112]_{\mathbb{R}} = 112$, we will first show that 112 is an upper bound for $[111, 112]_{\mathbb{R}}$. Recall that, from L9, the definition: an upper bound for A in S is an element $u \in S$ so $\forall \alpha \in A \ \alpha \leq u$. To prove that 112 is an upper bound, it is sufficent to show that $\forall x \in \mathbb{R} \ \ x \preceq 112$.

☐ **Bonus** [X points] Despite our proofs in L6 that the set of rational numbers $\mathbb{Q}$ is countably infinite and the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountably infinite, prove that
(i) between any two rational numbers there is an irrational number
(ii) between any two irrational numbers there is a rational number.