MATH 112: Introduction to Analysis
Fall 2025 Semester
Homework 4: Due Tuesday October 07, 10:00am PST.

Assignment (2 Problems: 50 + 50 = 100 points total.)

☐ **Problem 1** [Fields] Let $(F, \oplus, \mathbb{O}, \otimes, \mathbb{I})$ be a field satisfying the 10 field axioms from L7. Prove each of the following propositions, indicating which field axioms you use in each step.

- 1.1 [10 points] $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \oplus c = y \oplus c)\ \Rightarrow x = y\Big)$.
- 1.2 [10 points] $\forall x \in F\ \mathbb{O} \otimes x = \mathbb{O}$. *Hint:* $\mathbb{O} = \mathbb{O} \oplus \mathbb{O}$.
- 1.3 [10 points] $\forall x \in F\ \forall y \in F\ \Big((x = \mathbb{O}\ \vee\ y = \mathbb{O})\ \Rightarrow\ x \otimes y = \mathbb{O}\Big)$.
- 1.4 [10 points] $\forall x \in F\ \forall y \in F\ (x \otimes y = \mathbb{O}\ \Rightarrow\ (x = \mathbb{O}\ \vee\ y = \mathbb{O}))$.
- 1.5 [10 points] $\neg\Big(\exists \sigma \in F\ \mathbb{O} \otimes \sigma = \mathbb{I}\Big)$.

1.1) $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \oplus c = y \oplus c)\ \Rightarrow x = y\Big)$.
Assume $x \oplus c = y \oplus c$. By the hypothetical strategy, if $x = y$ when we make this assumption, the original statement is $True$. Since, by axiom 4, we know that all elements have $\oplus$-inverses. if we $\oplus(-c)$ to each side:

$$\begin{aligned} x \oplus c &= y \oplus c \\ x \oplus c \oplus (-c) &= y \oplus c \oplus (-c) \\ x \oplus \mathbb{O} &= y \oplus \mathbb{O} \\ x &= y. \end{aligned}$$

Since by axoim 4 any value $\oplus$ its inverse equals $\mathbb{O}$ the equation simplifies down to $x \oplus \mathbb{O} = y \oplus \mathbb{O}$. And since, by axiom 3, any value plus the identity $\mathbb{O}$ equals itself we can simplify to $x = y$. Since we have proved that when we assume $x \oplus c = y \oplus c$, $x = y$ is $True$, by the hypothetical stretegy, $(x \oplus c = y \oplus c) \Rightarrow x = y)$ is $True$ for all $x, y, c \in F$. □

1.2) $\forall x \in F\ \mathbb{O} \otimes x = \mathbb{O}$
let $x \in F$. If we use the Hint that $\mathbb{O} = \mathbb{O} \oplus \mathbb{O}$ and substitute $\mathbb{O}$ for $\mathbb{O} \oplus \mathbb{O}$ in the original equation:

$$\begin{aligned} \mathbb{O} \otimes x &= (\mathbb{O} \oplus \mathbb{O}) \otimes x \\ &= (\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x). \end{aligned}$$

Using axiom 9, we can take $(\mathbb{O} \oplus \mathbb{O})$ and distribute $\otimes$ over $\oplus$ to get $(\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x)$. Then we can use axiom 4 to take the inverse of $(x \oplus c)$ and $\oplus(-(\mathbb{O} \oplus x))$ to both sides:

$$\begin{aligned} (\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \oplus x)) &= (\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \otimes x)) \\ \mathbb{O} &= (\mathbb{O} \otimes x) \oplus \mathbb{O} \\ &= \mathbb{O} \otimes x. \end{aligned}$$

Since a value $\oplus$ its inverse equals $\mathbb{O}$ by axiom 9, $(\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \otimes x)) = \mathbb{O}$ and we can substitute one for the other on both sides. Since, by axiom 3 a value $\oplus$ the identity $\mathbb{O}$, it follows that $(\mathbb{O} \otimes x) \oplus \mathbb{O} = \mathbb{O} \otimes x$. Thus, by the slinky method, $\mathbb{O} = \mathbb{O} \otimes x$. □

1.3) $\forall x \in F\ \forall y \in F\ \Big((x = \mathbb{O}\ \vee\ y = \mathbb{O})\ \Rightarrow\ x \otimes y = \mathbb{O}\Big)$

First Assume $(x = \mathbb{O}\ \vee\ y = \mathbb{O})$. By the Hypothetical strategy, if we can prove $x \otimes y = \mathbb{O}$ under this assumption, the original statement is $True$. if $(x = \mathbb{O} \vee y = \mathbb{O})$, we can evaluate the implication by thinking of two cases: case (1) $x = \mathbb{O}$, and case (2) $y = \mathbb{O}$.

Case (1): $x = \mathbb{O}$ if we substitute our value of x into the statement $x \otimes y$:

$$\begin{aligned} x \otimes y &= \mathbb{O} \otimes y \\ &= \mathbb{O}\,. \end{aligned}$$

Since we proved that the identity, $\mathbb{O}$, $\otimes$, any value equals the identity, $\mathbb{O}$ in problem 2, it follows that $\mathbb{O} \otimes y = \mathbb{O}$. Thus, by the sliky method, $x \otimes y = \mathbb{O}$ when $x = \mathbb{O}$.

Case (2): $y = \mathbb{O}$ if we substitute our value of x into the statement $x \otimes y$:

$$\begin{aligned} x \otimes y &= x \otimes \mathbb{O} \\ &= \mathbb{O} \otimes x \\ &= \mathbb{O}\,. \end{aligned}$$

First use axiom 2, commmutativity, it follows that: $x \otimes \mathbb{O} = \mathbb{O} \otimes x$. Since we proved that the identity, $\mathbb{O}$, $\otimes$, any value equals the identity, $\mathbb{O}$ in problem 2, it follows that $\mathbb{O} \otimes x = \mathbb{O}$. Thus, by the sliky method, $x \otimes y = \mathbb{O}$ when $y = \mathbb{O}$. Since $x \otimes y = \mathbb{O}$ when $x = \mathbb{O}$ or when $y = \mathbb{O}$, the original statement holds true. $\square$

1.4) $\forall x \in F\ \forall y \in F\ (x \otimes y = \mathbb{O}\ \Rightarrow\ (x = \mathbb{O}\ \vee\ y = \mathbb{O}))$

let $x, y \in F$. Assume $x \otimes y = \mathbb{O}$, Assume $x \neq 0$. Since, by problem 2, $\mathbb{O} = x \otimes \mathbb{O}$ we can substitute one for the other in the eqation. Next, by axiom 8, all $x \neq \mathbb{O}$ have $\otimes$-inverses. We can $\otimes\, x^{-1}$ to each side:

$$\begin{aligned} (x \otimes y) &= \mathbb{O} \\ x^{-1} \otimes (x \otimes y) &= x^{-1} \otimes \mathbb{O} \\ (x^{-1} \otimes x) \otimes y &= \mathbb{O} \\ \mathbb{I} \otimes y &= \\ y &= . \end{aligned}$$

By axiom 5, $\otimes$ is associative, it follows that $x^{-1} \otimes (x \otimes y) = (x^{-1} \otimes x) \otimes y$. by axiom 8 a value $\otimes$ its inverse is equal to $\mathbb{I}$, so $x^{-1} \otimes x = \mathbb{I}$. Thus, by the slinky method, $y = \mathbb{O}$. Acccording to the Hypothetical strategy if we prove that statement is true, it is sufficient to prove that, assuming $x \otimes y = \mathbb{O}$, then $(x = \mathbb{O} \vee y = \mathbb{O})$ is true. since we have proved that $(x \otimes y = \mathbb{O}) \Rightarrow y = \mathbb{O}$, then $(x = \mathbb{O} \vee y = \mathbb{O})$ regardless of the truth value of $x = \mathbb{O}$. Thus the original statement holds true. $\square$

1.5) $\neg\Big(\exists\sigma \in F\ \mathbb{O} \otimes \sigma = \mathbb{I}\Big)$

We attempt a proof by contradiction. First, assume the negation of the original statement:

Assume $\exists\sigma \in F\, \mathbb{O} \otimes \sigma = \mathbb{I}$, Evaluate the statement:

$$\begin{aligned} \mathbb{I} &= \mathbb{O} \otimes \sigma \\ &= \mathbb{O}\,. \end{aligned}$$

By the slinky method, $\mathbb{I} = \mathbb{O}$, which is a contradiction. By axiom 10, $\mathbb{O} \neq \mathbb{I}$. Therefore the original statement is $True$. $\square$

☐ **Problem 2** [Induction] Prove each of the following propositions by induction.

- 2.1 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ 2^n < n!\Big)$. *where* $n! = \prod_{k=1}^{n} k$ *and* $0! = 1$.
- 2.2 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow \frac{1}{n} < 0.112\Big)$.
- 2.3 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ 1 + 0.112n \le (1+0.112)^n\Big)$
- 2.4 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}\Big)$.
- 2.5 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\Big)$ *for* $z \in \mathbb{C}$ *and* $z \ne 1$.

Recall from L10*: to prove* $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ P(n)\Big)$ *by induction requires two steps:*
(i) *nominate a base case* $n_0 \in \mathbb{N}$ *and prove* $P(n_0)$
(ii) *prove* $\forall n \in \mathbb{N}\ \Big((n_0 \le n) \land P(n)\ \Rightarrow\ P(n+1)\Big)$ *using the hypothetical strategy from* P3*.*
In the inductive step (ii)*, try to format your proof of the implication via the "slinky method".*

2.1) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \le n\ \Rightarrow\ 2^n < n!\Big)$. *where* $n! = \prod_{k=1}^{n} k$ *and* $0! = 1.//$ *proof.* we will prove this by induction. First note that the statement holds when $n = 4$.

$$\begin{aligned} 2^4 &= 16 \\ &< 4! \\ &= 24. \end{aligned}$$

since $16 < 24$ the statment holds true.
By the principle of mathmatical induction, if we show that if the base case is true then, $\forall n \in \mathbb{N}\ (4 \le n \lor (2^n < n!) \Rightarrow 2^{n+1} < (n+1)!)$.
First note that $4 \le n \Rightarrow 2 < n+1$. Assum $4 \le n$. By the hypothertical stretegy, if we prove that $2 < n+1$ is true under this assumption, it is suficent to prove that $4 \le n \Rightarrow 2 < n+1$.

$$\begin{aligned} 2 &< 5 \\ &= 4+1 \\ &\le n+1 \end{aligned}$$

we know that $2 < 5$. and 5 can be rewritten as $4+1$ since we have assumed $4 \le n$ it follows that $4+1 \le n+1$. thus, By the slinky method: $2 < n+1$ Simmilarly using the hypothetical strategy, assume $4 \le n \land (2^n < n!)$.

$$\begin{aligned} 2^{n+1} &= 2^n \cdot 2 \\ &< 2^n(n+1) \\ &< n!(n+1) \\ &= (n+1)! \end{aligned}$$

You can rewrite 2^{n+1} as $2^n \cdot 2$ by rules of exponents. Since we have previously proved that $2 < n+1$ when $4 \le n$, it follows that $2^n \cdot 2 < 2^n(n+1)$ when $4 \le n$. Since, by our assumption that $2^n < n!$, it follows that $2^n(n+1) < n!(n+1)$. Lastly, we know that $n!(n+1) = (n+1)!$ by rules of algebra and factorials. thus, By the Slinky method, the original statment is $True$ via induction $\square$

2.2) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \left(n_0 \leq n\ \Rightarrow \frac{1}{n} < 0.112\right)$
proof. Choose $n_0 = 9$, We'll show

$$\forall n \in \mathbb{N}(4 \leq n \Rightarrow \frac{1}{n} < 0.112)$$

By the principal of Mathmatical induction, if we show the statement holds for $n = 9$.
let $n = 9$:

$$\frac{1}{9} < 0.112.$$

Since $\frac{1}{9} < 0.112$ is true, the statement holds for $n = 9$.
Next we will show that:

$$\forall n \in \mathbb{N}\left(9 \leq n \wedge \frac{1}{n} < 0.112 \Rightarrow \frac{1}{n+1} < 0.112\right)$$

Assume $9 \leq n \wedge \frac{1}{2} < 0.112$.

$$\begin{aligned} \frac{1}{n+1} &< \frac{1}{n} \\ &\leq 0.112 \end{aligned}$$

We know that $n+1 > n$, because since we have chosen $n_0 = 9$ then $n > 0$. Using this, it follows that $\frac{1}{n+1} < \frac{1}{n}$ since the larger a denominator is, the smaller the fraction is. Since we have assumed $\frac{1}{n} \leq 0.112$, we can just evaluate it as given. By the slinky method, we have: $\frac{1}{n+1} < 0.112$. Thus we have proven the original statement true by induction. $\square$

2.3) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ 1 + 0.112n \leq (1+0.112)^n\Big)$

proof. Choose $n_0 = 1$. We'll show:

$$\forall n \in \mathbb{N}\left(1 \leq n \Rightarrow 1 + 0.112n \leq (1+0.112)^n\right)$$

By the principal of mathmatical induction, to prove this we will first show this statment holds for $n = 1$

let $n = 1$

$$\begin{aligned} 1 + 0.0112n &= 1 + 0.0112 \cdot 1 \\ &= 1.112 \\ &= (1+0.112)^1 \\ &= (1+0.112)^n \end{aligned}$$

First plug substitute n for 1. this expression evaluates to 1.112. you could also write 1.112 as $1 + 0.112)^1$. This is the same thing as $(1 + 0.112)^n$ if you substituted 1 for n. thus $1 + 0.0112n \leq (1 + 0.112)^n$ when $n = 1$, by the slinky method.

Next we will show that:

$$\forall n \in \mathbb{N}\left(1 \leq n \land 1 + 0.112n \leq (1+0.112)^n \Rightarrow 1 + 0.112(n+1) \leq (1+0.112)^{n+1}\right)$$

First, we will prove that $0.112 \leq (0.112)^2 n$. Let $n \in \mathbb{N}$, Assume $1 < N$:

$$\begin{aligned} 0.112 &< 1(0.0112)^2 \\ &\leq n(0.0112)^2 \\ &= (0.112)^2 n. \end{aligned}$$

We know that $0.112 < 1 \cdot (0.0112)^2$. Using our assumption that $1 < N$, it follows that $1(0.0112)^2 \leq n(0.0112)^2$. The last statement just moves around the n term. thus, by the slinky method, $0.112 < (0.0112)^2 n$.

Assume $1 \leq n \land 1 + 0.112n \leq (1 + 0.112)^n$

$$\begin{aligned} 1 + 0.112(n+1) &= 1 + (0.112)n + (0.112) \\ &< 1 + (0.112)n + (0.112)^2 n \\ &= 1 + (0.112)n + (0.112)n)(0.112) \\ &= 1 + (0.112)n(1 + 0.112) \\ &< (1+0.112)^n \cdot (1+0.112) \\ &= (1+0.112)^{n+1} \end{aligned}$$

First, $1 + 0.112(n + 1)$ can be reweitten by distributing 0.112 with $(n + 1)$ to get $1 + (0.112)n + (0.112)$. We already proved that $(0.112) < (0.112)^2 n$ above, and it follow that $1 + (0.112)n + (0.112) < 1 + (0.112)n + (0.112)^2 n$. we can rearange $1 + (0.112)n + (0.112)^2 n$ to get $1 + (0.112)n + (0.112)n)(0.112)$. You can then factor $(1 + 0.112)$ out to get $1 + (0.112)n(1 + 0.112)$. Using our asumption that $1 + 0.112n < (1 + 0.112)^n$, it follows that $1 + (0.112)n(1 + 0.112) < (1 + 0.112)^n \cdot (1 + 0.112)$. this can be further rewritten as $(1 + 0.112)^{n+1}$ by power rules of algebra. Thus, By the slinky method, The original statement is $True$. $\square$

2.4) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N} \left(n_0 \leq n \ \Rightarrow \ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}\right)$

proof. Choose $n_0 = 1$. We'll show:

$$\forall n \in \mathbb{N} \left(n_0 \leq n \ \Rightarrow \ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}\right)$$

By the principal of mathmatical induction, to prove this we will first show this statment holds for $n = 1$
let $n = 1$

$$\begin{aligned}
\sum_{j=1}^{n} j &= \sum_{j=1}^{1} j \\
&= 1 \\
&= \frac{2}{2} \\
&= \frac{1(1+1)}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

Thus the statement holds when $n = 1$

$$\forall n \in \mathbb{N} \left(n_0 \leq n \ \wedge \ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \Rightarrow z\right)$$

Assume $\leq n \ \wedge \ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

$$\begin{aligned}
\sum_{j=1}^{n+1} j &= \sum_{j=1}^{n} +(n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Thus, the original statement is true by induction.

2.5) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N} \left(n_0 \leq n \ \Rightarrow \ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\right)$ *for* $z \in \mathbb{C}$ *and* $z \neq 1$.

proof. Choose $n_0 = 0$. We'll show:

$$\forall n \in \mathbb{N} \left(n_0 \leq n \ \Rightarrow \ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\right)$$

By the principal of mathmatical induction, to prove this we will first show this statment holds for $n = 0$
let $n = 0$

$$\begin{aligned} \sum_{k=0}^{n} z^k &= z^0 \\ &= 1 \\ &= \frac{1-z^1}{1-z} \\ &= \frac{1-z^{n+1}}{1-z}. \end{aligned}$$

Thus, the statement holds, for $n = 0$.

$$\forall n \in \mathbb{N} \left(n_0 \leq n \ \wedge \ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z} \Rightarrow \sum_{k=0}^{n+1} z^k = \frac{1-z^{n+2}}{1-z}\right)$$

Assume $n_0 \leq n \ \wedge \ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$

$$\begin{aligned} \sum_{k=0}^{n} z^{n+1} &= \sum_{k=0}^{n} z^k + z^{n+1} \\ &= \frac{1 - z^{n+1} + (1-z)(z^{n+1}}{1-z} \\ &= \frac{a - z^{n+1} + z^{n+1} - z^{n+2}}{1-z} \\ &= \frac{1-z^{n+2}}{1-z}. \end{aligned}$$

Thus, the original statment is True, by induction.

☐ **Bonus** [X points] Construct an example of a field F with $|F| = 4$.