MATH 112: Introduction to Analysis
Fall 2025 Semester
Homework 4: Due Tuesday October 07, 10:00am PST.

Assignment (2 Problems: 50 + 50 = 100 points total.)

☐ **Problem 1** [Fields] Let $(F, \oplus, \mathbb{O}, \otimes, \mathbb{I})$ be a field satisfying the 10 field axioms from L7. Prove each of the following propositions, indicating which field axioms you use in each step.

- 1.1 [10 points] $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \oplus c = y \oplus c)\ \Rightarrow x = y\Big)$.
- 1.2 [10 points] $\forall x \in F\ \mathbb{O} \otimes x = \mathbb{O}$. *Hint:* $\mathbb{O} = \mathbb{O} \oplus \mathbb{O}$.
- 1.3 [10 points] $\forall x \in F\ \forall y \in F\ \Big((x = \mathbb{O}\ \vee\ y = \mathbb{O})\ \Rightarrow\ x \otimes y = \mathbb{O}\Big)$.
- 1.4 [10 points] $\forall x \in F\ \forall y \in F\ (x \otimes y = \mathbb{O}\ \Rightarrow\ (x = \mathbb{O}\ \vee\ y = \mathbb{O}))$.
- 1.5 [10 points] $\neg\Big(\exists \sigma \in F\ \mathbb{O} \otimes \sigma = \mathbb{I}\Big)$.

1.1) $\forall x \in F\ \forall y \in F\ \forall c \in F\ \Big((x \oplus c = y \oplus c)\ \Rightarrow x = y\Big)$.
Assume $x \oplus c = y \oplus c$. By the hypothetical strategy, if $x = y$ when we make this assumption, the original statement is $True$. Since, by axiom 4, we know that all elements have $\oplus$-inverses. if we $\oplus(-c)$ to each side:

$$\begin{aligned} x \oplus c &= y \oplus c \\ x \oplus c \oplus (-c) &= y \oplus c \oplus (-c) \\ x \oplus \mathbb{O} &= y \oplus \mathbb{O} \\ x &= y. \end{aligned}$$

Since by axoim 4 any value $\oplus$ its inverse equals $\mathbb{O}$ the equation simplifies down to $x \oplus \mathbb{O} = y \oplus \mathbb{O}$. And since, by axiom 3, any value plus the identity $\mathbb{O}$ equals itself we can simplify to $x = y$. Since we have proved that when we assume $x \oplus c = y \oplus c$, $x = y$ is $True$, by the hypothetical stretegy, $(x \oplus c = y \oplus c) \Rightarrow x = y)$ is $True$ for all $x, y, c \in F$. ☐

1.2) $\forall x \in F\ \mathbb{O} \otimes x = \mathbb{O}$
let $x \in F$. If we use the Hint that $\mathbb{O} = \mathbb{O} \oplus \mathbb{O}$ and substitute $\mathbb{O}$ for $\mathbb{O} \oplus \mathbb{O}$ in the original equation:

$$\begin{aligned} \mathbb{O} \otimes x &= (\mathbb{O} \oplus \mathbb{O}) \otimes x \\ &= (\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x). \end{aligned}$$

Using axiom 9, we can take $(\mathbb{O} \oplus \mathbb{O})$ and distribute $\otimes$ over $\oplus$ to get $(\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x)$. Then we can use axiom 4 to take the inverse of $(x \oplus c)$ and $\oplus(-(\mathbb{O} \oplus x))$ to both sides:

$$\begin{aligned} (\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \oplus x)) &= (\mathbb{O} \otimes x) \oplus (\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \otimes x)) \\ \mathbb{O} &= (\mathbb{O} \otimes x) \oplus \mathbb{O} \\ &= \mathbb{O} \otimes x. \end{aligned}$$

Since a value $\oplus$ its inverse equals $\mathbb{O}$ by axiom 9, $(\mathbb{O} \otimes x) \oplus (-(\mathbb{O} \otimes x)) = \mathbb{O}$ and we can substitute one for the other on both sides. Since, by axiom 3 a value $\oplus$ the identity $\mathbb{O}$, it follows that $(\mathbb{O} \otimes x) \oplus \mathbb{O} = \mathbb{O} \otimes x$. Thus, by the slinky method, $\mathbb{O} = \mathbb{O} \otimes x$. ☐

1.3) $\forall x \in F\ \forall y \in F\ \Big((x = \mathbb{O}\ \vee\ y = \mathbb{O})\ \Rightarrow\ x \otimes y = \mathbb{O}\Big)$

First Assume $(x = \mathbb{O}\ \vee\ y = \mathbb{O})$. By the Hypothetical strategy, if we can prove $x \otimes y = \mathbb{O}$ under this assumption, the original statement is $True$. if $(x = \mathbb{O}\ \vee y = \mathbb{O})$, we can evaluate the implication by thinking of two cases: case (1) $x = \mathbb{O}$, and case (2) $y = \mathbb{O}$.

Case (1): $x = \mathbb{O}$ if we substitute our value of x into the statement $x \otimes y$:

$$\begin{aligned} x \otimes y &= \mathbb{O} \otimes y \\ &= \mathbb{O}\,. \end{aligned}$$

Since we proved that the identity, $\mathbb{O}$, $\otimes$, any value equals the identity, $\mathbb{O}$ in problem 2, it follows that

$\forall x \in F\ \forall y \in F\ (x \otimes y = \mathbb{O}\ \Rightarrow\ (x = \mathbb{O}\ \vee\ y = \mathbb{O}))$

1.4) $\neg\Big(\exists \sigma \in F\ \mathbb{O} \otimes \sigma = \mathbb{I}\Big)$

☐ **Problem 2** [Induction] Prove each of the following propositions by induction.

- 2.1 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ 2^n < n!\Big)$. *where* $n! = \prod_{k=1}^{n} k$ *and* $0! = 1$.
- 2.2 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow \frac{1}{n} < 0.112\Big)$.
- 2.3 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ 1 + 0.112n \leq (1 + 0.112)^n\Big)$
- 2.4 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}\Big)$.
- 2.5 [10 points] $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\Big)$ *for* $z \in \mathbb{C}$ *and* $z \neq 1$.

Recall from L10*: to prove* $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ P(n)\Big)$ *by induction requires two steps:*

(i) *nominate a base case* $n_0 \in \mathbb{N}$ *and prove* $P(n_0)$

(ii) *prove* $\forall n \in \mathbb{N}\ \Big((n_0 \leq n) \wedge P(n) \ \Rightarrow\ P(n+1)\Big)$ *using the hypothetical strategy from* P3.

In the inductive step (ii)*, try to format your proof of the implication via the "slinky method".*

2.1) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ 2^n < n!\Big)$. *where* $n! = \prod_{k=1}^{n} k$ *and* $0! = 1$. *proof.* we will prove this by induction. First note that the statement holds when $n = 4$.

2.2) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow \frac{1}{n} < 0.112\Big)$

2.3) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ 1 + 0.112n \leq (1 + 0.112)^n\Big)$

2.4) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ \sum_{j=1}^{n} j = \frac{n(n+1)}{2}\Big)$

2.5) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}\ \Big(n_0 \leq n \ \Rightarrow\ \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}\Big)$ *for* $z \in \mathbb{C}$ *and* $z \neq 1$.

☐ **Bonus** [X points] Construct an example of a field F with $|F| = 4$.