MICHEAL BEAR, MIDTERM 1, 9/24/25

Assignment (2 problems $50 + 50 = 100$ points total)

☐ **Problem 1.** [Addition and Multiplication of Intergers as Functions]
For any integer $k \in \mathbb{Z}$,define two functions $\alpha_k : \mathbb{Z} \to \mathbb{Z}$ by the formulas

$$\alpha_k(m) = k + m$$
$$\gamma_k(m) = k \cdot m$$

where $+$ and $\cdot$ are addition and multiplication in $\mathbb{Z}$, respectively, For example, if we choose $k = 2$ and $m = 6$, then $\alpha_2(6) = 2 + 6 = 8$ and $\gamma_2(6) = 2 \cdot 6 = 12$, whereas if we just choose $k = 2$ and allow m to vary, then α_2 and γ_2 are functions with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$. The subscript k is not an input to the fucntion. The subscript k labels diffrent functions!

1.1 [10 points] Find the set of all $k \in \mathbb{Z}$ for which α_k is a bijection. Give proofs.

1.2 [10 points] Find the set of all $k \in \mathbb{Z}$ for which γ_k is a bijection. Give proofs.

1.3 [10 points] is $\alpha_{111} \circ \gamma_{112} = \gamma_{112} \circ \alpha 111$ true or false? Prove it.

1.4 [10 points] is $\forall m \in \mathbb{Z}\, \exists k \in \mathbb{Z}\, |\alpha_k(m) + \alpha_k(m)| \leq 112$ true or false? Prove it.

1.5 [10 points] is $\forall m \in \mathbb{Z}\, \exists k \in \mathbb{Z}\, |\gamma(m) + \gamma(m)| \leq 112$ true or false? Prove it.

Hint: draw the arrow aiagram for α_k and γ_k for $k \in \{-2, -1, 0, 1, 2\}$ to gain intuition, then for each, see if you can rpove if it is a bijecttion or not. is α_1 a bijection or not? Can you prove it? What about α_0? γ_0? Answer these before tackling Problems 1.1 and 1.2.

Caution: in the case $k = 2$. the function $\gamma_2 : \mathbb{Z} \to \mathbb{Z}$ in this problem is not the same as $\Delta : \mathbb{N} \to E$ from L4. Why? Although γ_2 and Δ both multiply their input by 2, they aren't the same functions since they don't have the same domain nor the same codomain.

solution

1.1 Find the set of all $k \in \mathbb{Z}$ for which α_k is a bijection.
α_k is bijective for $k = \{..., -2, -1, 0, 1, 2, ...\} = \mathbb{Z}$

Proof: $\alpha_k(m) = k + m$.
First prove α_k is surjective: $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ \ y = f(x)$.
Consider $y \in \mathbb{Z}$, let $x \in \mathbb{Z}$, $x = -k + y$. plug x into f:

$$\begin{aligned} f(-k+y) &= k - k + y \\ &= y \end{aligned}$$

Since that satisfies the propsition, α_k is surjective for all $y \in \mathbb{Z}$.
Next prove α_k is injective: $\forall b \in \mathbb{Z}\ \forall b \in \mathbb{Z}\,((f(a) = f(b)) \Rightarrow (a = b))$. Consider $a, b \in \mathbb{Z}$, assume $f(a) = f(b)$.

$$\begin{aligned} f(a) &= f(b) \\ k + a &= k + b \\ a &= b \end{aligned}$$

Since this satisfies the origonal proposition because $True \Rightarrow True = True$ by the truth table. Thus, α_k is injective for all $a, b \in \mathbb{Z}$. Since α_k is Surjective and injective, it is bijective for all $x \in \mathbb{Z}$. $\square$

1.2 Find the set of all $k \in \mathbb{Z}$ for which γ_k is a bijection.
γ_k is bijective for $k = \{... - 3, -2, -1, 1, 2, 3\} = \{\mathbb{Z} \mid |k| \geq 1\}$.
proof: $\gamma_k = m \cdot k$
First, Surjectivity. Prove $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ \ y = f(x)$.
let $y = k + 1$, $x = \frac{k+1}{k}$

$$\begin{aligned} f\Big(\frac{k+1}{k}\Big) &= \frac{k+1}{k} \cdot k \\ &= k + 1 \end{aligned}$$

since $\frac{k+1}{k}$ only works when $|k| \geq 1$, it follows that γ_k is surjective for all $y, x \in \mathbb{Z}$ when $|k| \geq 1$.
Then, Injectivity: $\forall b \in \mathbb{Z}\ \forall b \in \mathbb{Z}\,((f(a) = f(b)) \Rightarrow (a = b))$
Consider $a, b \in \mathbb{Z}$, assume $f(a) = f(B)$.

$$\begin{aligned} f(a) &= f(b) \\ a \cdot k &= b \cdot k \\ a &= b. \end{aligned}$$

This statment is true when $k \neq 0$. Thus γ_k is injective when $|k| > 0$.
Since γ_k is surjective when $|k| \geq 1$ and γ_k is injective when $|k| > 0$, it follows that γ_k is bijective when $|k| \geq 1$ $\square$

1.3 is $\alpha_{111} \circ \gamma_{112} = \gamma_{112} \circ \alpha 111$ true or false?
$\boxed{False}$ consider $m \in \mathbb{Z}$, assume $\alpha_{111} \circ \gamma_{112} = \gamma_{112} \circ \alpha 111$

$$\begin{aligned} \alpha_{111}(\gamma_{112}(m)) &= \gamma_{112}(\alpha_{111}(m)) \\ \alpha 111(112 \cdot m) &= \gamma_{112}(111 + m) \\ 111 + 112m &= 112(111 + m) \\ 111 + 112m &\neq 112 \cdot 111 + 112m \end{aligned}$$

Since this is not equal, the two compositions are also not equal and the whole equation is $False$. $\square$

1.4 is $\forall m \in \mathbb{Z} \, \exists k \in \mathbb{Z} \, |\alpha_k(m) + \alpha_k(m)| \leq 112$ true or false?
$\boxed{False}$ Let $m = 1$, consider $k \in \mathbb{Z}$.Evaluate the statement for our value of m.

$$\begin{aligned} |k + 1 - k + 1| &\leq 112 \\ |2| &\leq 112. \end{aligned}$$

This statement is false when $m = 1$. Therefore this statment does not hold for all $m \in \mathbb{Z}$. it follows that the statement $\forall m \in \mathbb{Z} \, \exists k \in \mathbb{Z} \, |\alpha_k(m) + \alpha_k(m)| \leq 112$ is $False$. $\square$

1.5 is $\forall m \in \mathbb{Z} \, \exists k \in \mathbb{Z} \, |\gamma(m) + \gamma(m)| \leq 112$ true or false?
$\boxed{False}$ consider $m, k \in \mathbb{Z}$. Evaluate the statement for our values of m and k

$$\begin{aligned} |k \cdot m + -k \cdot m| &\leq 112 \\ |0| &\leq 112 \end{aligned}$$

since the equation evaluates to 0 for all $k, m \in \mathbb{Z}$. it follows that the statment $\forall m \in \mathbb{Z} \, \exists k \in \mathbb{Z} \, |\gamma(m) + \gamma(m)| \leq 112$ is $False$. $\square$

☐ **Problem 2.** [The Field with Two Elements] Let $F = \{a, b\}$ be the finite set with $a \neq b$ and $|f| = 2$. Consider the two binary operations $\otimes$ and $\oplus$ on F defined by the formulas

$$\begin{aligned} a \oplus a &= a & \qquad a \otimes a &= a \\ a \oplus b &= b & \qquad a \otimes b &= a \\ b \oplus a &= b & \qquad b \otimes a &= a \\ b \oplus b &= a & \qquad b \otimes b &= b \end{aligned}$$

The first binary operation $\otimes$ takes two inputs $x \in F$ and $x \in F$ and $y \in F$ and returns a single output $x \otimes y \in F$ according to the table on the left. Similarly, the second binary operation $\otimes$ takes two inputs $x \in F$ and returns a single output $x \otimes y \in F$ acording to the table on the right. These tules enable calculations such as

$$(a \oplus (b \oplus b)) \otimes b = (a \oplus a) \otimes b = a \otimes b = a$$

Determine the truth value of each propsition below. Give proofs and explain your reasoning.

2.1 [10 points] $\forall x \in F\Big((x \otimes x = x) \Rightarrow (x = a)\Big)$

2.2 [10 points] $\exists w \in F \ \forall x \in F x \oplus w = x$

2.3 [10 points] $\forall x \in F \ \exists \ y \in F \ x \otimes y = b$

2.4 [10 points] $\exists g \in F \ \forall \ x \in F \otimes g = x$

2.5 [10 points] $\forall x \in F \ \exists y \in F \ x \oplus y = a$

solution

2.1 $\forall x \in F\Big((x \otimes x = x) \Rightarrow (x = a)\Big)$

$\boxed{\text{False}}$: let $x = b$, if we plug x into the equation given we get:

$$x \otimes x = x$$
$$b \otimes b = b.$$

The equation $b \otimes b = b$ is true according to the formulas given. However, the statement $\Big((b \otimes b = b) \Rightarrow (b = a)\Big)$ is $False$. This is because $True \Rightarrow False$ is $False$ by the truth table of implies. Since the statment does not hold for all $x \in F$, the proposition $\forall x \in F\Big((x \otimes x = x) \Rightarrow (x = a)\Big)$ is false. ☐

2.2 $\exists w \in F\ \forall x \in F\ x \oplus w = x$

$\boxed{\text{True}}$: Nominate $w = a$. We can slip all $x \in F$ into the the case (1) where $x = a$ and case (2) where $x = b$.

Case 1: Let $x = a$. If we plug in our values of x and w into the equation we get:

$$x \oplus w = x$$
$$a \oplus a = a$$

by the given formulas. Since this statment is true, the propositon holds when $x = a$.

Case 2: Let $x = b$. If we plug in our values of x and w into the equation we get:

$$x \oplus w = x$$
$$b \oplus a = b$$

by the given formulas. Since this statement is true, the proposition holds when $w = a$.

Since, when $x = a$, the proposition $x \oplus w = x$ holds for both $x = a$ and $x = b$–and a, b are the only elements of F, this proposition holds for all $x \in F$. Thus:
$\exists w \in F\ \forall x \in F\ x \oplus w = x$ is true $\square$

2.3 $\forall x \in F\ \exists\ y \in F\ x \otimes y = b$

$\boxed{\text{False}}$: Consider $x = a$, let $y = a$. if we plug in x and y we get:

$$x \otimes y = b$$
$$a \otimes a = a.$$

By the equations given, $a \otimes a = a$ and $a \otimes a! = b$ as it would need to be for the proposition to be true, thus the statement does not hold when $y = a$.
Now consider $x = a$, let $y = b$. If we plug in x and y we get:

$$x \otimes y = b$$
$$a \otimes b = a.$$

By the equations given $a \otimes b = a$ and $a \otimes b! = b$ as it would need to be for the proposition to be true, thus the statement does not hold when $y = b$.

Since the statement does not hold when $y = a$ and it does not hold when $y = b$, and a and b are all elements of F, there are no values of y that make the proposition true when $x = a$.

Since the proposition is always false when $x = a$ it follows that the proposition is not true for all $g \in F$, thus the statement $\forall x \in F\ \exists\ y \in F\ x \otimes y = b$ is $False$. $\square$

2.4 $\exists g \in F \; \forall x \in F \; x \otimes g = x$

$\boxed{\text{True}}$: consider $g = a$. Since there are only two elements of F, in order to prove the statement is true for all elements $x \in F$ we can consider two cases: case (1) where $x = a$ and case (2) where $x = b$.

Case 1: $x = a$. plug in our values of x and g

$$x \otimes g = x$$
$$a \otimes a = a.$$

since $a \otimes a = a$ in our given formulas, the statments holds true for $x = a$.

Case 2: $x = b$. plug in the values of x and g:

$$x \otimes g = x$$
$$b \otimes a = a.$$

Since $b \otimes a = a$ in our given formulas, then the statement holds true for $x = b$.

Since the statement holds for $x = a$ and $x = b$ and a and b are all the elements of F. $\exists g \in F \; \forall x \in F \; x \otimes g = x$ is $True$. $\square$

2.5 $\forall x \in F \; \exists y \in F \; x \oplus y = a$

$\boxed{\text{True}}$: Since there are only two elements of F we can evaluate all $x \in F$ by splitting it into two cases: case (1) where $x = a$ and case (2) where $x = b$. // Case 1: consider $x = a$,let $y = a$. Plug in our values of x and y:

$$x \oplus y = a$$
$$a \oplus a = a$$

This is true by the given formulas. It follows that, when $x = a$, the statment holds true. Case 2: consider $x = b$, let $y = b$. Plug in our values of x and y:

$$x \oplus y = a$$
$$b \oplus b = a.$$

This is true by the given formulas. Thus, the statement holds when $x = b$.
Since a, and b are all the values of F, all values of x have been evalutated.
Furthermore, since the proposition holds for all members of F, the statement:
$\forall x \in F \; \exists y \in F \; x \oplus y = a$ is $True$. $\square$