MICHEAL BEAR, HW3, DATE

☐ **Problem 1** (Real Functions)**.** Define $D \subseteq \mathbb{R}$ by $D = \mathbb{R}\setminus\{-1\} = \{x \in \mathbb{R} \ : \ x \neq -1\}$. Consider the real function $f : D \to \mathbb{R}$ defined by the formula

$$f(x) = \frac{x-2}{x+1}.$$

1.1 [10 points] Prove that f is injective *Hint: proceed from the definition in* L4

1.2 [10 points] Prove that $1 \notin \text{range}\,(f)$. *Hint: use "proof by contradiction" from* P5

1.3 [10 points] Prove that $\text{range}\,(f) = \mathbb{R} \setminus \{1\}$. *Hint: proceed from the definition in* L4

1.4 [10 points] Is $\forall x \in D\ \exists u \in \mathbb{R}\ \ f(x) \leq u$ true or false? Give a proof. *Hint:* P4 *& Desmos*

1.5 [10 points] Is $\exists u \in \mathbb{R}\ \forall x \in D\ \ f(x) \leq u$ true or false? Give a proof. *Hint:* P4 *& Desmos*

solution

1.1) a function $f : X \to Y$ is injective if $\forall a \in X\ \forall b \in X\ (f(a) = f(b) \Rightarrow a = b)$

Consider $a \in X$ and $b \in X$. Then $f(a) = \frac{a-2}{a+1}$ and $f(b) = \frac{b-2}{b+1}$.
Suppose $f(a) = f(b)$, it follows that $\frac{a-2}{a+1} = \frac{b-2}{b+1}$. if you solve for a you get:

$$\begin{aligned}
\frac{a-2}{a+1} &= \frac{b-2}{b+1} \\
(a-2)\cdot(b+1) &= (b-2)\cdot(a+1) \\
ab + a - 2b - 2 &= ab + b - 2a - 2 \\
ab - ab + a + 2a &= b + 2b - 2 + 2 \\
3a &= 3b \\
a &= b
\end{aligned}$$

Since the equality simplifies to a = b the statment $\big(f(a) = f(b) \Rightarrow a = b\big)$ is true, because $True \Rightarrow True$. ☐

1.2) The $range(f) = \{y \in Y : \exists\, x \in X \; y = f(x)\}$.
We attempt a proof by contradiction:

First assume $1 \in range\ (f)$. consider $y \in Y$ let $y = 1$ (since we are looking for the $x \in X$ where the statement$y = f(x)$ is True). Then evaluate $f(x) = 1$:

$$\begin{aligned} f(x) &= \frac{x-2}{x+1} \\ 1 \cdot (x+1) &= (x-2) \\ x+1 &= x-2 \\ x-x+1 &= -2 \\ 1 &= -2 \end{aligned}$$

Since $1 \neq -2$ and there are no solutions to this equation, we have reached a contradiciton. Thus, by proof of contradiciton $1 \notin range\ (f)$ $\square$

1.3) We already know that $1 \notin range(f)$ from the problem above. we can split this problem into two cases: (1) the case where $x \in D$ for all $r \in \mathbb{R}\, r < 1$, and (2) the case where $x \in D$ for all $r \in \mathbb{R}\, r > 1$.
Case (1): let $x \in D$. Assume $r \in R$ such that $r < 1$ Then solve for x:

$$\begin{aligned} f(x) = \frac{x-2}{x+1} &= r \\ (x-2) &= r \cdot (x+1) \\ x-2 &= rx + r \\ -2-1 &= rx - x + r \\ -3 - r &= x(r-1) \\ \frac{-(r+3)}{r-1} &= x \end{aligned}$$

Since the equation is true when $r < 1$, the statement: for all $r \in \mathbb{R}$ where $r < 1$, $r \in range(f)$ holds true. Case(2): let $x \in D$. Assume $r \in R$ such that $r > 1$ then sove for x:

$$\begin{aligned} f(x) = \frac{x-2}{x+1} &= r \\ x-2 &= rx + 1 \\ -3 - r &= x(r-1) \\ \frac{-(r+3)}{r-1} &= x \end{aligned}$$

since the equation holds true when $r > 1$, the statement: for all $r \in \mathbb{R}$ where $r > 1$, $r \in range(f)$ holds true. since both cases are true, the whole proposition holds true
$\square$

1.4) $\forall x \in D\ \exists\ u \in\ \ R\ f(x) \leq u$

Let $x \in D$. Nominate $u \in \mathbb{R}$ in terms of x by setting $u = x$ then:

$$\begin{aligned} f(x) = \frac{x-2}{x+1} &\leq u \\ \frac{x-2}{x+1} &\leq x \\ (x-2) &\leq x \cdot (x+1) \\ x-2 &\leq x^2+1 \\ -2-1 &\leq x^2-x \\ -3 &\leq x(x-1). \end{aligned}$$

There are two solutions to $-3 \leq x(x-1)$:

$$-3 \leq x$$

and

$$\begin{aligned} -3 &\leq x-1 \\ -4 &\leq x. \end{aligned}$$

Since both are acceptable x values the statement: $f(x) \leq u$ holds true $\square$

1.5) $\exists u \in \mathbb{R}\ \ \forall x \in D\ f(x) \leq u$

let $u \in \mathbb{R}$ Assume $x \in D$ such that $f(x) \leq u$. Solve for x.

$$\begin{aligned} f(x) = \frac{x-2}{x+1} &\leq u \\ (x-2) &\leq u \cdot (x+1) \\ x-2 &\leq ux+1 \\ -2-1 &\leq ux-x \\ -3 &\leq -x(u-1) \\ \frac{-3}{u-1} &\leq -x \\ \frac{3}{u-1} &> x \end{aligned}$$

since the equation can reverse the statement holds true $\square$

☐ **Problem 2.** [Compositions, Implications, and Counterexamples]
Let X, Y, and Z be three sets (possibly infinite)
and let $f : X \to Y$ and $g : Y \to Z$ be two functions
since codomain$(F) = Y =$ domain(g), $g \circ f : X \to Z$ is a well defined function
prove that each given implication below is false by providing an explicit counterexample

2.1 [10 points] if f is constant and g is bijective then $g \circ f$ is surjective

2.2 [10 points] if $|X| \leq |Z|$ then $g \circ f$ is injective

2.3 [10 points] if $g \circ f$ is bijective then $|X| = |Y|$

solution

2.1) A function $f : X \to Y$ is surjective if $\forall y \in Y\ \exists x \in X\ y = f(x)$.

Consider $y \in Y$, let $x \in X$, let$z \in Z$. The function $g \circ f$ can be rewritten as: $g(f(x))$. Since f is constant no matter what x you put in f, you will always get back out the same y. we can re write our function as $g(y)$. Since there is only one input $y \in Y$ if |y| > 1 we can't hit all $z \in Z$ of our codomain. Since g is bijective each input will have exactly one output. This shows that the proposition is False. ☐

2.2) Just because the domain is less than the codomain does not mean that it will only go to one output.

2.3) Let $X = \{1, 2, 3, 4\}$ and let {1,2,3,4,5,6,7,8}. $f(x) = x+2$. A function being injective only guaranties the $range(f)$ has the same cardinality of X.

☐ **Problem 3.** [Compositions, Surjectivity, and Injectivity]
Let X,Y, and Z be three sets (possibly infinite)
and let $f\ :\ X \to Y$ and $g\ : Y \to Z$ be two functions.

3.1 [10 points] Prove that if f is surjective and g is surgective then $g \circ f$ is surjective.

3.2 [10 points] prove that if f is injective and g is injective then $g \circ f$ is injective

solution

3.1 a function $f : X \to Y$ is surjective if $\forall y \in Y\ \exists x \in X\ y = f(x)$

Consider $y \in Y$, let $x \in X$. The composition $(g \circ f)(x)$ can be re written as: $g(f(x))$. Since $f(x)$ is surjective, we know that for all $y \in Y$ there exists an $x \in X$ such that $y = f(x)$. it follows that we can then substitute $f(x)$ for the $y \in Y$ that corisponds with the x. That then gives us $g(y)$. since we know g(y) is surjective. the whole function is surjective ☐

3.2 a function $f : X \to Y$ is injective if $\forall a \in X\ \forall b \in X\ (f(a) = f(b) \Rightarrow a = b)$

Consider $a \in X$, let $b \in x$. The composition $g \circ f$ can be rewritten as $f(g(x))$. since $g(x)$ is injective we know that for all $a \in X$ and for all $b \in X$, $f(a) = f(b)$ implies $a = b$. It follows that if g is surjective, then $g(f(a)) = g(f(b))$ this implies that $a = b$. thus $g \circ f$ is injective ☐