MICHEAL BEAR, HW2, 9/13/25

☐ **Problem 1** [Cartesian Products and Solution Sets] In our preliminary L0, we encountered the *digit set* $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ commonly used to express numbers in base 10. Alternatively, the *bit set* $\mathbb{B} = \{0, 1\}$ from L3 can be used to express numbers in base 2. Note: $|\mathbb{B}| = 2$. The set of *bit strings of length* 3 is the set of 3-tuples as defined in L3:

$$\begin{aligned}\mathbb{B}^3 &= \{0,1\}^3 \\ &= \Big\{(b_1, b_2, b_3) \ : \ b_1 \in \{0,1\} \wedge b_2 \in \{0,1\} \wedge b_3 \in \{0,1\}\Big\} \\ &= \Big\{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\Big\}\end{aligned}$$

with $|\mathbb{B}^3| = 2^3 = 8$. For each set below (i) present it explicitly and (ii) find its cardinality.

- 1.1 [5 points] $W = \{(b_1, b_2, b_3) \in \mathbb{B}^3 \ : \ b_1 = 0\}$
- 1.2 [5 points] $Y = \{(b_1, b_2, b_3) \in \mathbb{B}^3 \ : \ b_1 + b_2 + b_3 = 2\}$
- 1.3 [5 points] $W \cap Y$
- 1.4 [5 points] $(W \cup Y) \cap W$
- 1.5 [5 points] $(W \cap Y) \cup W$

solution

1. i) the set of bit strings for which the first element is 0 is:
$$\boxed{W = \big\{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\big\}}$$
   ii) $|W| = 4$

2. i) the set of bit strings for which b1, b2, and b3, add up to 2:
$$\boxed{Y = \{(0,1,1), (1,0,1), (1,1,0)\}}$$
   ii) $|Y| = 3$

3. i) the set of bit strings in the intersection of W and Y, or the set containing the bit strings that exisit in both the set W and the set Y:
$$\boxed{W \cap Y = (0,1,1)}$$
   ii) $\boxed{|W \cap Y| = 1}$

4. i) the set of bitstrings in the union of W and Y is:
$W \cup Y = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,1),(1,1,0)\}$
the set of bit strings in $(W \cup Y) \cap W$ is the set of bit strings in $W \cup Y$ and in W:
$$(W \cup Y) \cap W = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}$$
since this containes all values of W and only values of W, this is equal to W
$$\boxed{(W \cup Y) \cap W = W}$$
ii) $\boxed{|(W \cup Y) \cap W| = 4}$

5. i) as shown above:
$$W \cap Y = \{(0,1,1\}$$
the set of bit strings in $(W \cap Y)$ and/or in W:
$$(W \cap Y) \cup W = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}$$
$$\boxed{(W \cap Y) \cup W = W}$$
ii) $\boxed{|(W \cap Y) \cup W| = 4}$

☐ **Problem 2** [Nested Quantifiers] Recall $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $\mathbb{N}$ from L0. Determine the truth value of each proposition. Careful: just like we used 3 different variables x_1, x_2, x_3 in our discussion of ordered triples in L3, here n and n_0 are 2 different variables.

- 2.1 [5 points] $\forall d \in \mathbb{D}\, \exists m \in \mathbb{D}\ \ (|d - m| < 5)$
- 2.2 [5 points] $\exists m \in \mathbb{D}\, \forall d \in \mathbb{D}\ \ (|d - m| < 5)$
- 2.3 [5 points] $\forall n_0 \in \mathbb{N}\, \exists n \in \mathbb{N}\, (n_0 \leq n)$
- 2.4 [5 points] $\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n_0 \leq n)$
- 2.5 [5 points] $\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n_0 \leq n \ \Rightarrow\ 4 < n)$

Solution

1) $\forall d \in \mathbb{D}\, \exists m \in \mathbb{D}(|d - m < 5|)$
   for all d in $\mathbb{D}$ there exists an m in $\mathbb{D}$ such that $|d - m < 5|$
   for all d $\leq$ 5, if m = 1, then $|d - m| \leq 5$ is $\boxed{true}$
   for all d $>$ 5, if m = 5, then then $|d - m| \leq 5$ is $\boxed{true}$

2) $\exists m \in \mathbb{D}\, \forall d \in \mathbb{D}\ \ (|d - m| < 5)$
   there exists an m in $\mathbb{D}$ for all d in $\mathbb{D}$ such that $(|d - m| < 5)$
   if m = 1 for all d $\leq$ 5, then $|d - m| \leq 5$ i s $\boxed{true}$
   if m = 5 for all d $>$ 5, then then $|d - m| \leq 5$ is $\boxed{true}$

3) $\forall n_0 \in \mathbb{N}\, \exists n \in \mathbb{N}\, (n_0 \leq n)$
   for all n_0 in $\mathbb{N}$ there exists an n in $\mathbb{N}$ such that $(n_0 \leq n)$ this statment is true because you can pick n to be equal to n_0 $\boxed{true}$

4) $\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n_0 \leq n)$
   there exists n_0 in $\mathbb{N}$ for all n in $\mathbb{N}$ such that $(n_0 \leq n)$
   let $n_0 = n$
   $n_0 \leq n$ is $\boxed{true}$

5) $\exists n_0 \in \mathbb{N}\, \forall n \in \mathbb{N}\, (n_0 \leq n \ \Rightarrow\ 4 < n)$
   there exists n_0 in $\mathbb{N}$ for all n in $\mathbb{N}$ such that $(n_0 \leq n \ \Rightarrow\ 4 < n)$ this is not true because if $n_0 = 5$ and $n = 5$ $n_0 \leq n$ is true but $4 < n$ is false. there being an $n_0 \leq n$ does not imply that $4 < n$ $\boxed{true}$

☐ **Problem 3** [Functions]. Recall the infinite set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and the infinite set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ both introduced in L0. Recall also from L0 in our review of *divisibility* that a natural number $n \in \mathbb{N}$ is *even* or *odd* if the existential proposition "$\exists k \in \mathbb{N}\ n = 2k$" is true or false, respectively. For example, if we take $n = 0$, we can conclude that 0 is even since $\exists k \in \mathbb{N}\ 0 = 2k$ is true. The piecewise-formula

$$\lambda(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

defines a function $\lambda : \mathbb{N} \to \mathbb{Z}$ as in L4 with domain $\mathbb{N}$ and codomain $\mathbb{Z}$. In L4, I called this the *interleaving function*. Make sure you know why $\lambda(5) = -3$, then fill in the table below:

$\lambda(n)$	0	-1	1	-2	2	-3	3	-4	4	-5	5	-6	6	-7	7	-8	8
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- 3.1 [10 points] Prove that $n \in \mathbb{N}$ is even if and only if $\lambda(n)$ is non-negative.
- 3.2 [10 points] Prove that $n \in \mathbb{N}$ is odd if and only if $\lambda(n)$ is negative.
- 3.3 [15 points] Prove that $\lambda : \mathbb{N} \to \mathbb{Z}$ is surjective.
- 3.4 [15 points] Prove that $\lambda : \mathbb{N} \to \mathbb{Z}$ is injective.

Hint 1: Reread the discussion in L4 *of this "interleaving function" $\lambda : \mathbb{N} \to \mathbb{Z}$.*
Hint 2: To prove $P \Leftrightarrow Q$, you must prove $P \Rightarrow Q$ and also prove $Q \Rightarrow P$.

solution

1) let n be some even number. then:$\lambda(n) = \frac{n}{2}$.
since n is even $\lambda(n)$ is a whole number and it isnt negative.
then let $\lambda(n)$ be a non negative number since $\lambda(n)$ is a piece wise function it must be using the $\frac{n}{2}$ since its the only non negative option
and that part of the peicewise fxn only happens when n is even
☐

2) let n be some odd number. then:$\lambda(n) = -\frac{n+1}{2}$.
since n is odd $\lambda(n)$ is a whole number and it is negative.
then let $\lambda(n)$ be a negative number since $\lambda(n)$ is a piece wise function it must be using the $-\frac{n+1}{2}$ since its the only negative option
and that part of the peicewise fxn only happens when n is odd ☐

3) a function $f : X \to Y$ is surjective if $\forall y \in Y \; \exists x \in X \; y = f(x)$
let n be an even number, then $\lambda(x)$ = $\frac{n}{2}$ since it divides by 2 it increments by 1 as n increases by 2. $\frac{n}{2} = \{0, 1, 2, 3...\}$ let n be an odd number. then then $\lambda(x)$ = $-\frac{n+1}{2}$ since it divides by 2 it decrements by 1 as n increases by 2. $-\frac{n+1}{2} = \{-1, -2, -3...\}$ if we take the union of the two sets (which is itself the piecewise funciton $\lambda(x)$ it goes to all elements of the integers since every x in the natural numbers eventually go to every y in the integers, $\lambda(x)$ is surjective $\square$

4) a function $f : X \to Y$ is injective if $\forall a \in X \; \forall b \in X \; (f(a) = f(b) \Rightarrow a = b)$
let a be even.
$(\frac{a}{2} = \lambda(b))$ if they are equal to eachother then b must also be even
$(\frac{a}{2} = \frac{b}{2})$ if you multiply both sides by two you get:

$$a = b$$

let a be odd.
$(-\frac{a+1}{2} = \lambda(b))$ if they are equal to eachother then b must also be odd
$(-\frac{a+1}{2} = -\frac{b+1}{2})$ if you multiply both sides by two you get:

$$a + 1 = b + 1$$

subtract both sides by 1:

$$a = b$$

$\square$

☐ **Bonus** [X points] For any non-empty set S, finite or infinite, the *power set* $\mathcal{P}(S)$ is

$$\mathcal{P}(S) = \{A \ : \ A \subseteq S\}$$

the set of all subsets of S. $f : S \to \mathcal{P}(S)$.

solution