MICHEAL BEAR, HOMEWORK 1 , SEP, 4, 2025

☐ **Problem 1** [Logic and $\mathbb{D}$]. In L0, we encountered the finite set $\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of digits in base 10. Using d as symbol to denote an element of $\mathbb{D}$, consider the predicates

$$\begin{aligned} P(d) &= \text{“}|d-5| \leq 2\text{”} \\ Q(d) &= \text{“}|d-2| \leq 5\text{”} \end{aligned}$$

where $|x|$ is absolute value. *Careful: if x is a real number, x is not a set, so the notation "$|x|$" refers to "the absolute value of x" and cannot possibly mean "the cardinality of x".* Determine the truth value of each of the following propositions.

- 1.1 [10 points] $P(3) \wedge (P(2) \vee P(1))$
- 1.2 [10 points] $\neg\big(P(3) \Rightarrow Q(3)\big)$
- 1.3 [10 points] $\exists d \in \mathbb{D}\ P(d)$
- 1.4 [10 points] $\forall d \in \mathbb{D}\ P(d)$
- 1.5 [10 points] $\forall d \in \mathbb{D}\ \big(P(d) \Rightarrow Q(d)\big)$.

solution

1)

$$\begin{aligned} &P(3) \wedge (P(2) \vee P(1)) \\ &= (|3-5| \leq 2) \wedge \Big((|2-2| \leq 2) \vee (|1-5| \leq 2)\Big) \\ &= (2 \leq 2) \wedge \Big((0 \leq 2) \vee (4 \leq 2)\Big) \\ &= (True) \wedge (True \vee False) \\ &= True \wedge (True) \\ &= True \end{aligned}$$

2)

$$\begin{aligned} \neg\big(P(3) \Rightarrow Q(3)\big) & \\ &= \neg\big((|3-5| \leq 2) \Rightarrow (|3-2| \leq 5)\big) \\ &= \neg\big((2 \leq 2) \Rightarrow (1 \leq 5)\big) \\ &= \neg\big(True \Rightarrow True\big) \\ &= \neg(True) \\ &= False \end{aligned}$$

3)

$$\exists d \in \mathbb{D} P(d)$$
$$= \exists d \in |d-5| \leq 2$$

you can also write this as:
there exists a number d in the set of digits such that $|d-5| \leq 2$
this statement is true because:
let d = 4

$$|d-5| = |4-5|$$
$$1 \leq 2$$

this makes the stament $True$

4)

$$\forall d \in \mathbb{D}\ P(d)$$
$$= \forall d \in \mathbb{D}\ |d-5| \leq 2$$

let $d = 1$

$$|1-5| = 4$$
$$4 \leq 2$$

since $4 > 2$ this statment is $False$

5)

$$\forall d \in \mathbb{D}\ \big(P(d) \Rightarrow Q(d)\big)$$
$$= \forall d \in \mathbb{D}\big(|d-5| \leq 2 \Rightarrow |d-2| \leq 5)$$

let $d = 0$

$$|0-5| \leq 2 \Rightarrow |0-2| \leq 5 = 5 \leq 2 \Rightarrow 2 \leq 5$$
$$= False \Rightarrow True$$
$$= False$$

☐ **Problem 2** [Logic and $\mathbb{N}$] In L0, we encountered the infinite set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots\}$. For each $k \in \mathbb{N}$, consider the truth set

$$A_k = \{n \in \mathbb{N} \, : \, k \leq n\}$$

To prove $a \in A_k$, you have to verify $k \leq a$. Prove each of the following propositions.

- 2.1 [10 points] $\exists \ell \in \mathbb{N} \; \ell \in A_7$
- 2.2 [10 points] $\exists \ell \in \mathbb{N} \; 7 \in A_\ell$
- 2.3 [10 points] $\neg\big(|\mathbb{N} \setminus A_{10}| = 9\big)$. *Careful: if S is a set, $|S|$ is the cardinality of S*
- 2.4 [10 points] $\forall m \in \mathbb{N} \; 2m + 1 \in A_m$
- 2.5 [10 points] $\forall m \in \mathbb{N} \; \big(m \in A_{112} \Rightarrow m \in A_{111}\big)$

solution

1)

$$\exists \ell \in \mathbb{N} \; \ell \in A_7$$

$$A_7 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

let $\ell = 2$

$$2 \in \mathbb{N} \wedge 2 \in A_7$$

2)

$$\exists \ell \in \mathbb{N} \; 7 \in A_\ell$$

let $\ell = 7$

$$7 \in \mathbb{N}$$

$$7 \leq 7 \Rightarrow 7 \in A_7$$

therefore: $7 \in \mathbb{N} \; 7 \in A_7$

3)

$$\neg\big(|\mathbb{N} \setminus A_{10}| = 9\big)$$

$$\mathbb{N} \setminus A_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$|\mathbb{N} \setminus A_{10}| = 11$$

$$\neg\big(10 = 9\big)$$

$$\neg False = True$$

4)

$$\forall m \in \mathbb{N} \; 2m + 1 \in A_m$$

$$A_m = \{1, 2, 3, ..., m\}$$

since $m + 1 > m$, $m + 1 \notin A_m$ b/c m is the largest element of A_m

5)

$$\forall m \in \mathbb{N} \; \big(m \in A_{112} \Rightarrow m \in A_{111}\big)$$

$$A_{112} = \{1, 2, 3, ..., 111, 112\}$$

$$A_{111} = \{1, 2, 3, ..., 111\}$$

since all elements of A_{111} are in A_{112} if an element is in A_{111} it implies that that same element is also in A_{112}

☐ **Bonus** [X points] Is $\forall n \in \mathbb{Z}_+ \left(\exists q \in \mathbb{Q} \left((0 < q) \wedge (q < \frac{1}{n}) \right) \right)$ true or false? Explain.