MICHEAL BEAR, MIDTERM 1, 9/24/25

Assignment (2 problems $50 + 50 = 100$ points total)

☐ **Problem 1.** [Addition and Multiplication of Intergers as Functions]
For any integer $k \in \mathbb{Z}$,define two functions $\alpha_k : \mathbb{Z} \to \mathbb{Z}$ by the formulas

$$\alpha_k(m) = k + m$$
$$\gamma_k(m) = k \cdot m$$

where $+$ and $\cdot$ are addition and multiplication in $\mathbb{Z}$, respectively, For example, if we choose $k = 2$ and $m = 6$, then $\alpha_2(6) = 2 + 6 = 8$ and $\gamma_2(6) = 2 \cdot 6 = 12$, whereas if we just choose $k = 2$ and allow m to vary, then α_2 and γ_2 are functions with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$. The subscript k is not an input to the fucntion. The subscript k labels diffrent functions!

1.1 [10 points] Find the set of all $k \in \mathbb{Z}$ for which α_k is a bijection. Give proofs.

1.2 [10 points] Find the set of all $k \in \mathbb{Z}$ for which γ_k is a bijection. Give proofs.

1.3 [10 points] is $\alpha_{111} \circ \gamma_{112} = \gamma_{112} \circ \alpha 111$ true or false? Prove it.

1.4 [10 points] is $\forall m \in \mathbb{Z}\, \exists k \in \mathbb{Z}\, |\alpha_k(m) + \alpha_k(m)| \leq$ true or false? Prove it.

1.5 [10 points] is $\forall m \in \mathbb{Z}\, \exists k \in \mathbb{Z}\, |\gamma(m) + \gamma(m)| \leq$ true or false? Prove it.

Hint: draw the arrow aiagram for α_k and γ_k for $k \in \{-1, -1, 0, 1, 2\}$ to gain intuition, then for each, see if you can rpove if it is a bijecttion or not. is α_1 a bijection or not? Can you prove it? What about α_0? γ_0? Answer these before tackling Problems 1.1 and 1.2.

Caution: in the case $k = 2$. the function $\gamma_2 : \mathbb{Z} \to \mathbb{Z}$ in this problem is not the same as $\Delta : \mathbb{N} \to E$ from L4. Why? Although γ_2 and Δ both multiply their input by 2, they aren't the same functions since they don't have the same domain nor the same codomain.

solution

1.1

1.2

1.3

1.4

1.5

☐ **Problem 2.** [The Field with Two Elements] Let $F = \{a, b\}$ be the finite set with $a \neq b$ and $|f| = 2$. Consider the two binary operations $\otimes$ and $\oplus$ on F defined by the formulas

$$\begin{aligned} a \oplus a &= a & \qquad a \otimes a &= a \\ a \oplus b &= b & \qquad a \otimes b &= a \\ b \oplus a &= b & \qquad b \otimes a &= a \\ b \oplus b &= a & \qquad b \otimes b &= b \end{aligned}$$

The first binary operation $\otimes$ takes two inputs $x \in F$ and $x \in F$ and $y \in F$ and returns a single output $x \otimes y \in F$ according to the table on the left. Similarly, the second binary operation $\otimes$ takes two inputs $x \in F$ and returns a single output $x \otimes y \in F$ acording to the table on the right. These tules enable calculations such as

$$(a \oplus (b \oplus b)) \otimes b = (a \oplus a) \otimes b = a \otimes b = a$$

Determine the truth value of each propsition below. Give proofs and explain your reasoning.

2.1 [10 points] $\forall x \in F \Big((x \otimes x = x) \Rightarrow (x = a) \Big)$

2.2 [10 points] $\exists w \in F \ \forall x \in F x \oplus w = x$

2.3 [10 points] $\forall x \in F \ \exists \ y \in F \ x \otimes y = b$

2.4 [10 points] $\exists g \in F \ \forall \ x \in x \otimes g = x$

2.5 [10 points] $\forall x \in F \ \exists y \in F \ x \oplus y = a$

solution

1.1

1.2

1.3

1.4

1.5