15.1

maximize $2x_1 + x_2$
subject to $0 \le x_1 \le 2$
$x_1 + x_2 \le 3$
$x_1 + 2x_2 \le 5$
$x_2 \ge 0.$

minimize $-2x_1 - x_2$
subject to $x_1 + x_3 = 2$
$x_1 + x_2 + x_4 = 3$
$x_1 + 2x_2 + x_5 = 5.$
$x_1, x_2 \ge 0$

15.2

minimize x_2
subject to $x_{k+1} = ax_k + bu_k$
$x_0 = 1$
$|u_i| \le 1.$

min x_2
s.t. $x_1 - ax_0 - bu_0 = 0$
$x_2 - ax_1 - bu_1 = 0$
$u_0 \le 1$
$u_0 \ge -1$
$u_1 \le 1$
$u_1 \ge -1$

15.8

min $p_1 + \dots + p_n$
s.t. $g_{i,1} p_1 + \dots + g_{im} p_n \ge P, \quad i = 1, \dots, m$
$p_1, \dots, p_n \ge 0$

or

min $\sum_i p_i$
s.t. $Gp \ge P\mathbf{1}$; $\quad G = [g_{ij}]$
$p \ge 0$

17.6

Primal

(a) min $c^T x$
s.t. $Ax \le b$

Dual

max $y^T b$
s.t. $y^T A = c^T$
$y \le 0$

If we define $\lambda = -y$
dual becomes

minimize $\lambda^T b$
subj. to $\lambda^T A = -c^T$
$\lambda \ge 0$

(b) $\underbrace{\exists y \geq 0 \text{ s.t. } y^T A = -c^T}_{}$

Dual: minimize 0
subj. to $y^T A = -c^T$
$y \geq 0$

y is optimal feasible for the dual.

Hence, primal has optimal value 0.
$x = 0$ satisfies this optimal value and is feasible.

17.8 minimize $\sum_{i=1}^{n} x_i$
subject to $a^T x = 1$, $0 < a_1 < a_2 < \cdots < a_n$.
$x \geq 0$

(a) Dual: maximize λ
subject to $\lambda a \leq 1 \iff \lambda \leq \min\{1/a_i\}_{i=1}^{n} = 1/a_n$.

(b) $\sum x_i = 1/a_n \Rightarrow x_n = 1/a_n$ and $x_1 = \ldots = x_{n-1} = 0$.

17.13 minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Show if $\exists \lambda, \mu$ s.t. $A^T \lambda + \mu = c$, $\mu^T x = 0$, $\mu \geq 0$ $\Rightarrow$ x is opt. feasible, λ opt. feasible dual

Complementary slackness says that if
$(y^T A - c^T) x = 0 \Rightarrow x, y$ are optimal

Set $\mu = y^T A - c^T$ so that $\mu^T x = 0$ and $\mu \geq 0$.

17.15 minimize $x_1 + x_2$
subject to $x_1 + 2x_2 \geq 3$
$2x_1 + x_2 \geq 3$
$x_1, x_2 \geq 0$

maximize $3(y_1 + y_2)$
subject to $y_1 + 2y_2 \leq 1$
$2y_1 + y_2 \leq 1$
$y_1, y_2 \geq 0$

$y^T = c_B^T B^{-1} = (1/3 \;\; 1/3) \Rightarrow 3(1/3 + 1/3) = 2 = 1 + 1$.

17.17 Show that $\exists x$ s.t. $Ax \geq b$ and $x \geq 0$ iff $\forall y$ with $A^T y \leq 0$ and $y \geq 0$, we have $y^T b \leq 0$.

$$\begin{array}{lll} \text{minimize} & 0^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \quad \overset{\text{dualize}}{\Longrightarrow} \quad \begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & y^T A \leq 0 \\ & y \geq 0 \end{array}$$

17.5

Primal

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 \\ \text{subject to} & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + x_5 = 3 \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{minimize} & 2y_1 + 7y_2 + 3y_3 \\ \text{subject to} & -2y_1 - y_2 + y_3 \geq 1 \\ & y_1 + 2y_2 \geq 2 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \\ & y_3 \geq 0 \end{array}$$

(b) $x^* = [3 \quad 5 \quad 3 \quad 0 \quad 0]^T$. What is y^*?

$$B = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ 1 & -1/2 & 3/2 \end{bmatrix} ; \quad c_B^T = [1 \quad 2 \quad 0]$$

$$y^{*T} = c_B^T B^{-1} = [0 \quad 1 \quad 2].$$

Optimal values: $c^T x^* = 13 = y^{*T} b = 7 \cdot 1 + 3 \cdot 2$.