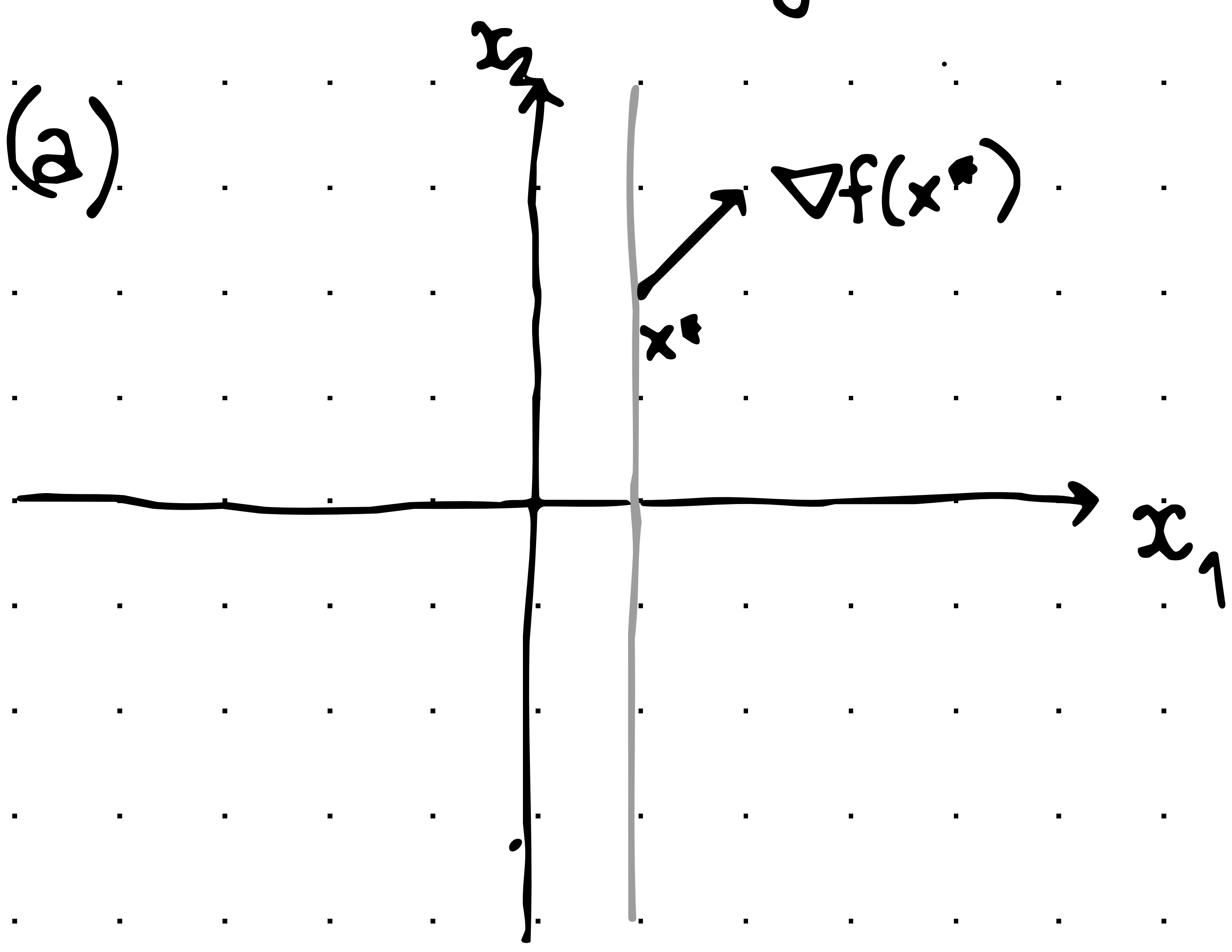
6.1 Consider the problem

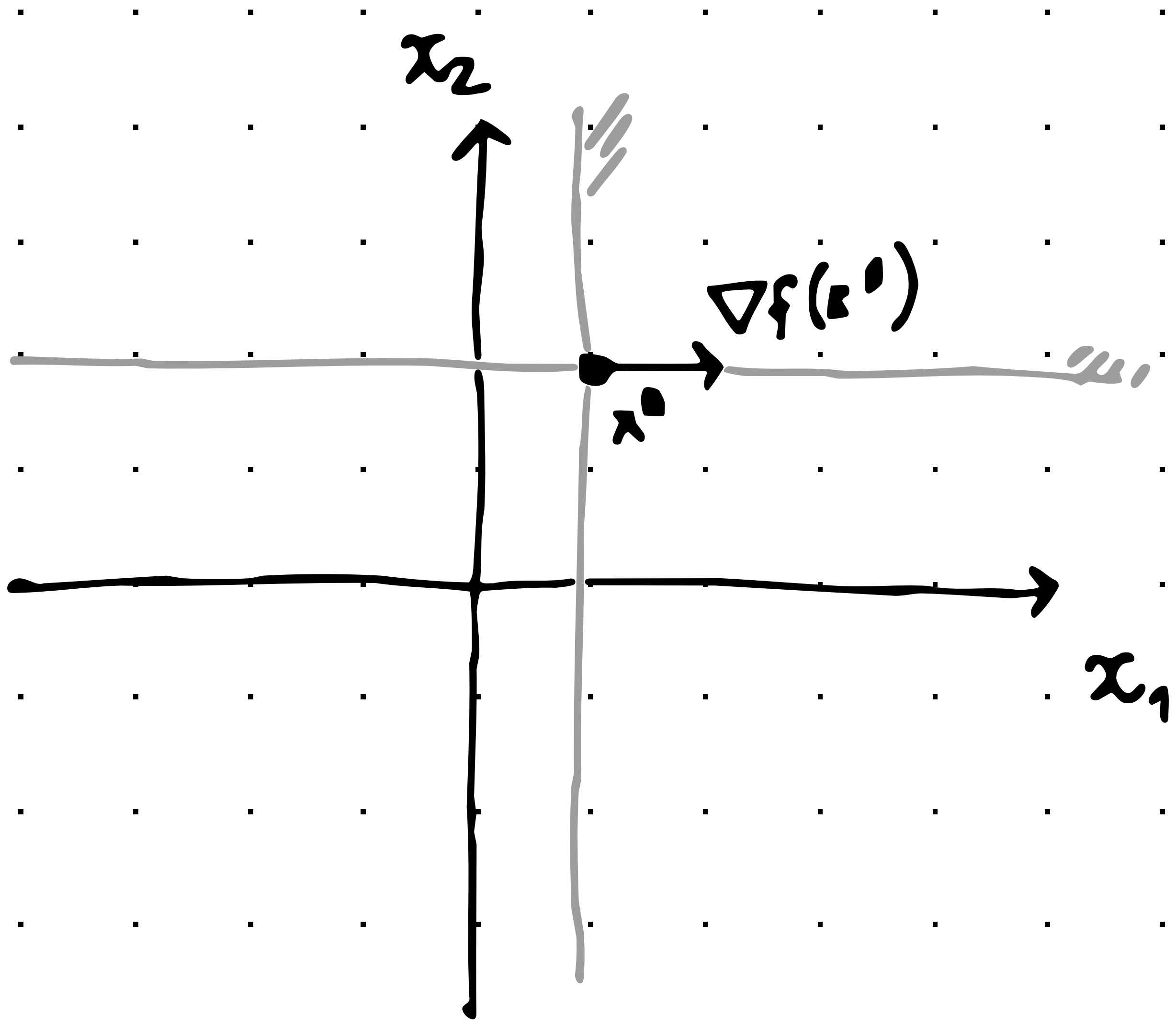
$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subj. to} && x \in \Omega \end{aligned} \qquad f \in C^2.$$

(a)



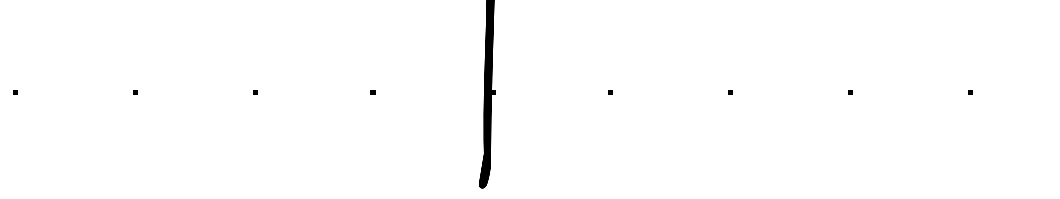
Definitely not a local minimizer.

(b)



Possibly a local minimizer.

(c)

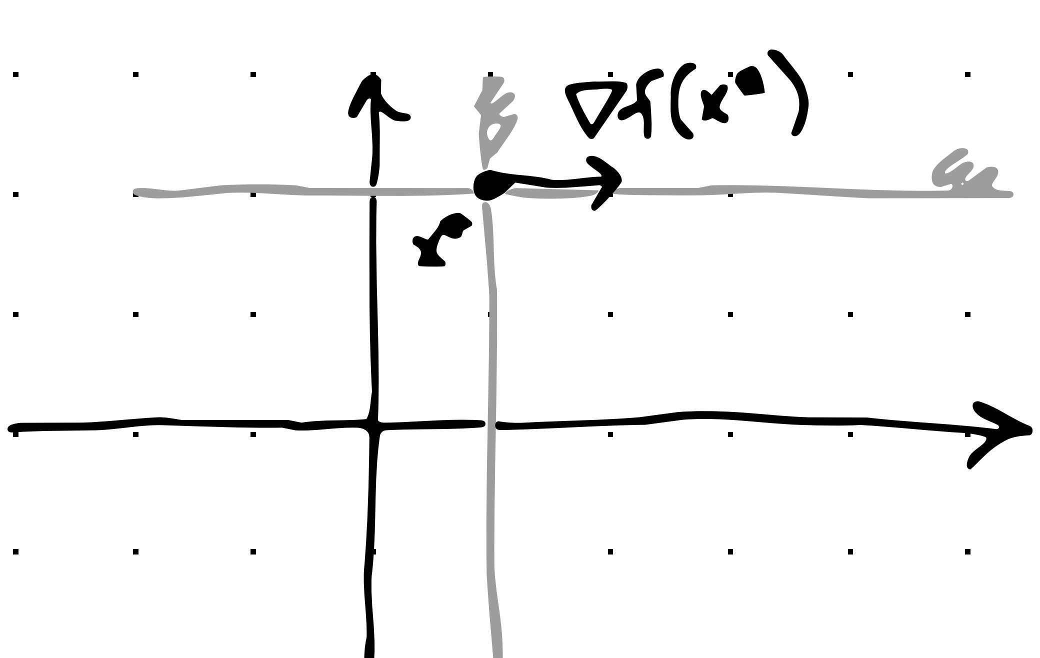


Definitely a local minimizer.

$\nabla f(x^*) = 0$

$F(x^*) = I$

(d)



$F(x^*) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Definitely not a local minimizer.

6.2 Find the minimizers and maximizers of the function

$$f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^3 - 16x_2$$

$$\frac{\partial f}{\partial x_1}(x) = x_1^2 - 4 = 0 \Rightarrow x_1^* = \pm 2$$

$$\frac{\partial f}{\partial x_2}(x) = x_2^2 - 16 = 0 \Rightarrow x_2^* = \pm 4$$

$$F(x) = \begin{pmatrix} 2x_1 & 0 \\ 0 & 2x_2 \end{pmatrix} \Rightarrow F(x^*) = \begin{pmatrix} \pm 4 & 0 \\ 0 & \pm 8 \end{pmatrix}$$

$\Rightarrow$ $x^* = (2, 4)$ global minimizer

$x^* = (-2, -4)$ global maximizer

6.8 $f(x) = x^T \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} x + x^T \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(a) $\nabla f(x) = 2\begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} x + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$F(x) = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}$$

(b) $\nabla f((1,1))^T \dfrac{\nabla f((1,1))}{|\nabla f(1,1)|} = \dfrac{(11 \quad 25)\begin{pmatrix} 11 \\ 25 \end{pmatrix}}{|(11 \quad 25)|} = \sqrt{746} = 27.313$

(c) FONC $\nabla f(x^*) = 0 \Rightarrow x^* = \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}$

SONC not satisfied.

6.9 $\quad f(x_1, x_2) = x_1 x_2 (x_1 + x_2^2)$.

(a) $\nabla f(x) = \left(x_2(x_1 + x_2^2) + x_1 x_2 \quad x_1(x_1 + x_2^2) + 2x_1 x_2^2 \right)$

$-\nabla f(x^{(0)}) = \left(-5 \quad -10 \right) \qquad x^{(0)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

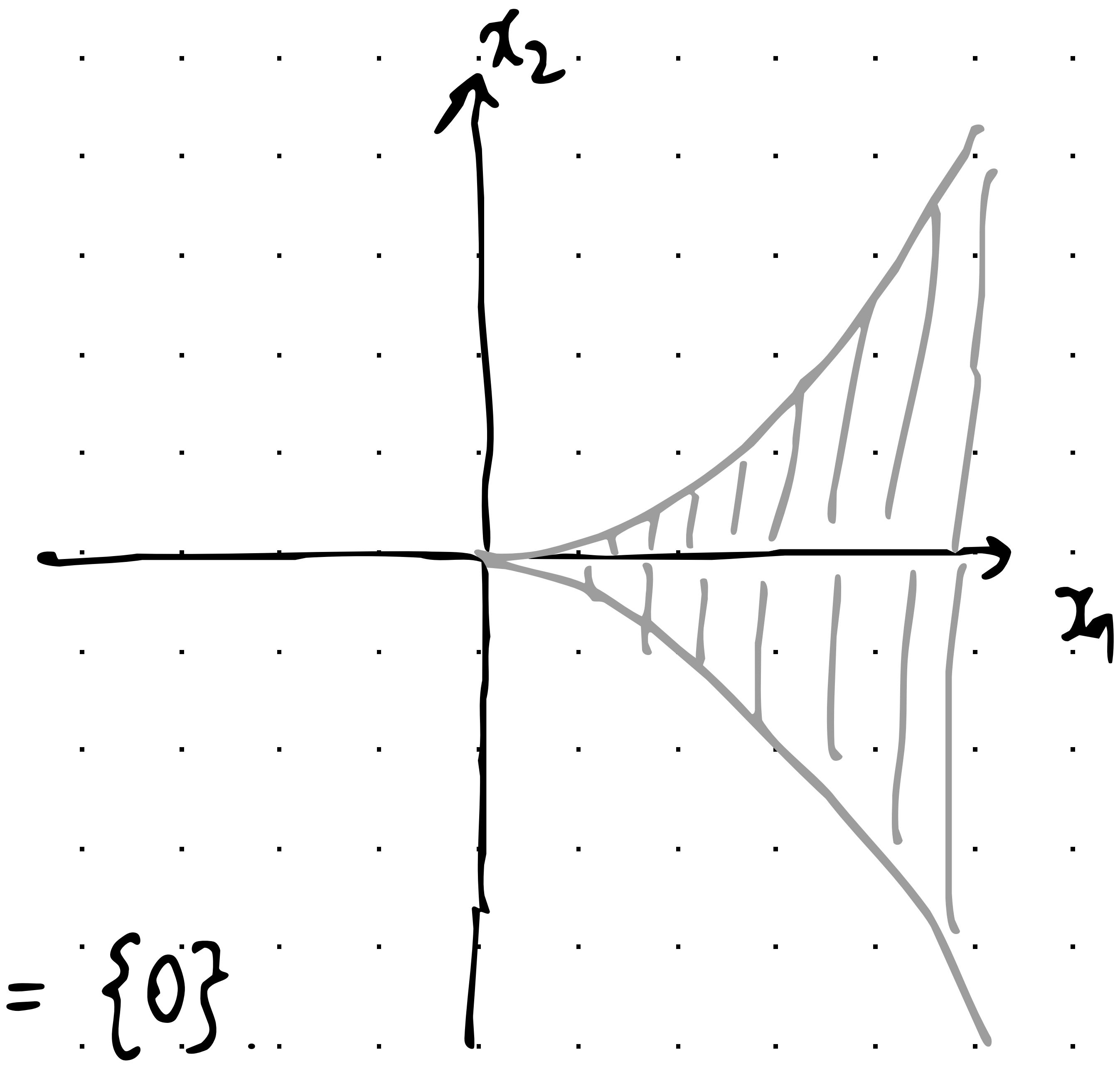
(b) $\nabla f(x^{(0)}) \cdot -\left(\frac{\nabla f(x^{(0)})}{|\nabla f(x^{(0)})|} \right) = -|\nabla f(x^{(0)})| = -\sqrt{125} = -5\sqrt{5}$.

(c) Compute $\nabla f(x^{(0)}) \cdot d/|d|$.

6.11 $\quad$ minimize $-x_2^2$

subj. to $-x_1^2 \le x_2 \le x_1^2$

$x_1 \ge 0$



(a) $x = 0$. $\quad \mathcal{F} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\nabla f(x) = (0 \quad -2x_2)$

$\nabla f(0) = (0 \quad 0)$. $\qquad \nabla f(0) \cdot \mathcal{F} = \{0\}$

FONC ✓

(b) $x = 0$ is a local maximizer (not strict)

6.26 $\quad$ minimize $f(x)$ $\qquad \nabla f(0) < 0$. (element-wise)

subj. to $x_1, x_2 \ge 0$.

Show that 0 cannot be a minimizer.

Proof. Take $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \nabla f(0) \cdot d = \frac{\partial f}{\partial x_1}(0) + \frac{\partial f}{\partial x_2}(0) < 0$

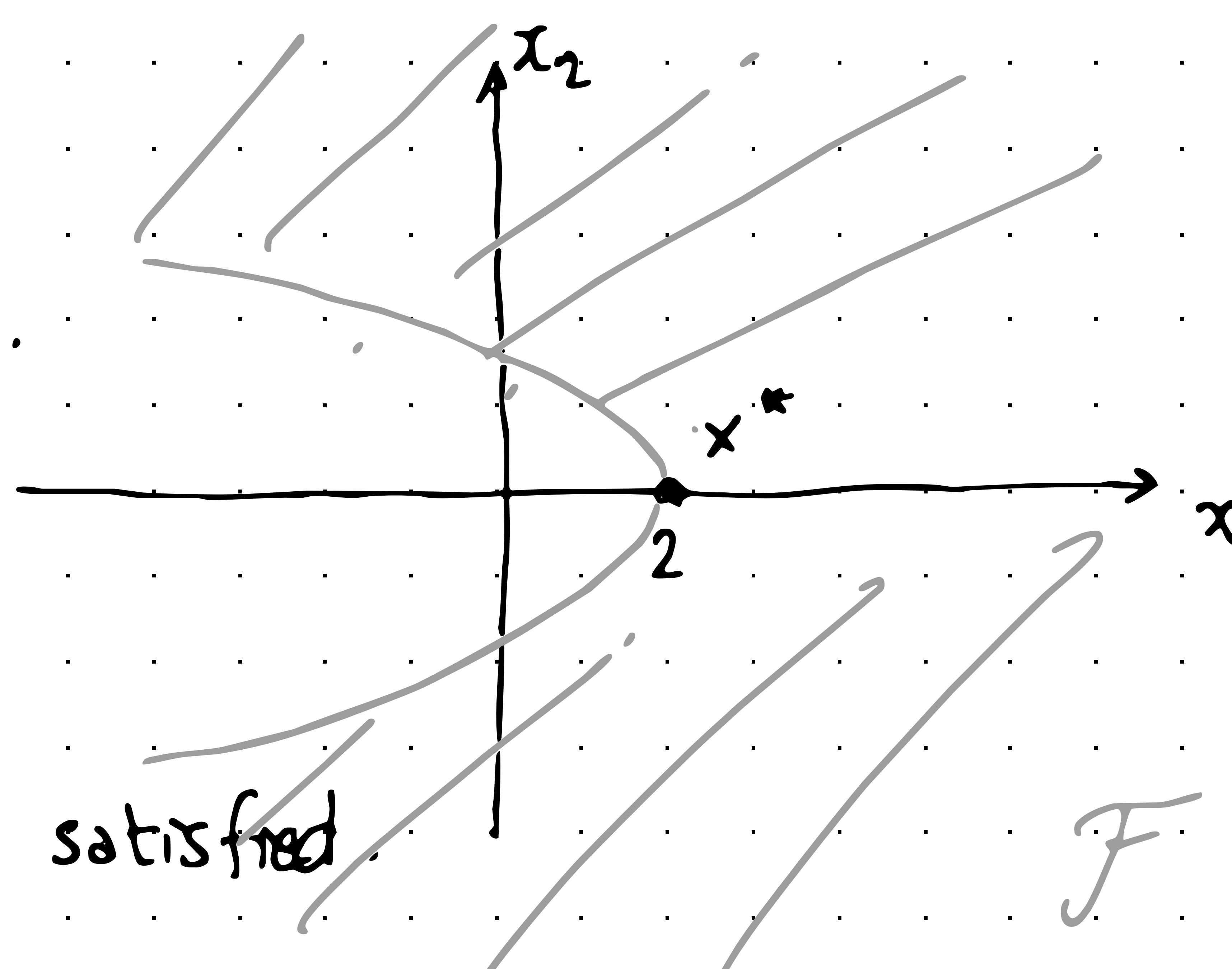
FONC is violated.

6.33 $f(x) = \frac{1}{2} x^T Q x - x^T b$, $Q = Q^T > 0$.

Show that x^* minimizes f iff x^* satisfies the FONC.

Proof. Claim: f is convex. Indeed $F(x) = Q > 0$. Done.

6.15 minimize $f(x) = 3x_1$
subject to $x \in \Omega = \{x : x_1 + x_2^2 \geq 2\}$



$\nabla f(x^*) = (3 \quad 0)$

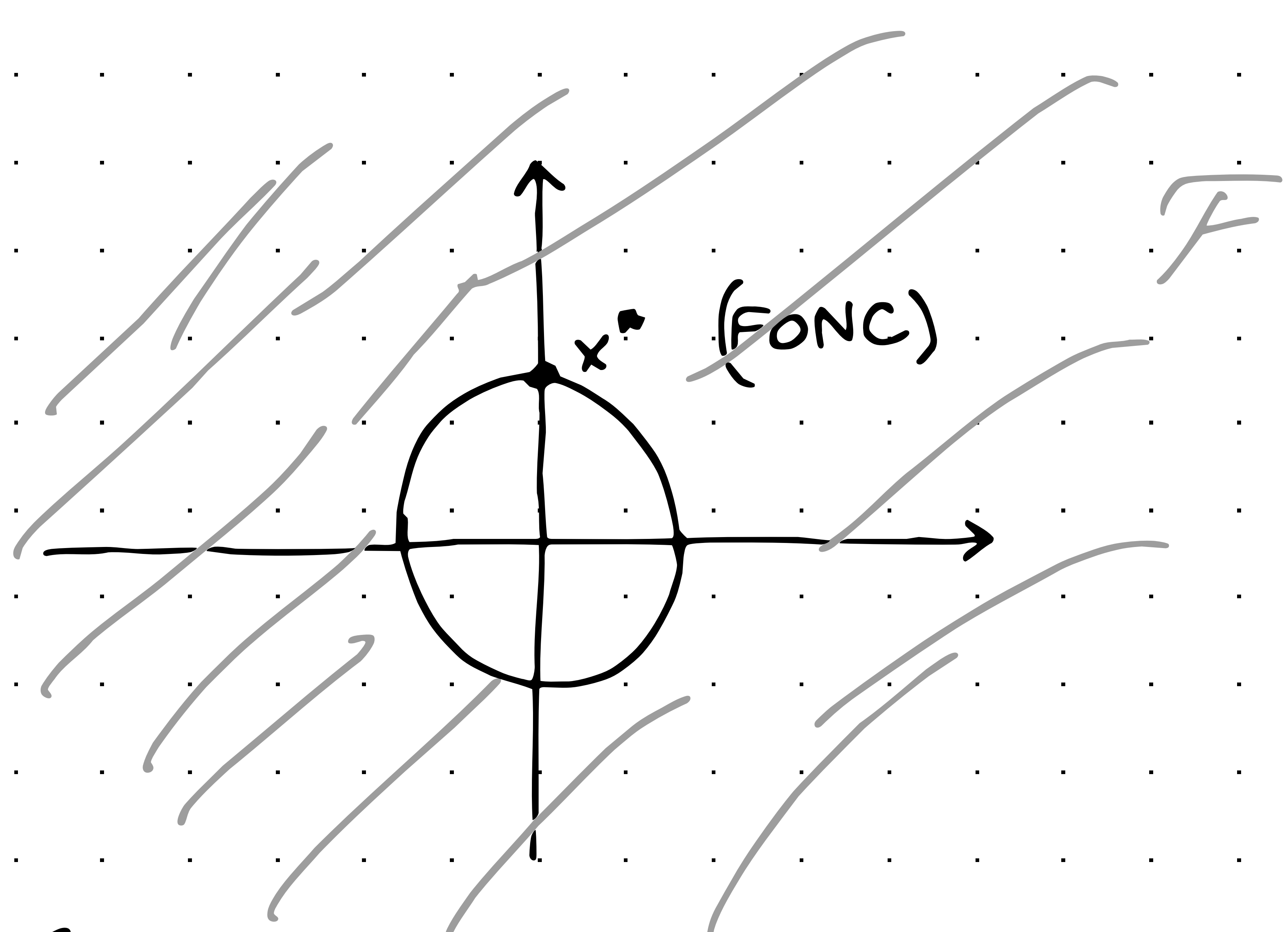
$\nabla f(x^*) d = (3 \quad 0)\begin{pmatrix} 0 \\ a \end{pmatrix} = 0$.

FONC is satisfied.

$F(x^*) = 0$. Hence SONC is satisfied.

x^* is not a local minimizer as

6.14 minimize x_2
subject to $x_1^2 + x_2^2 \geq 1$.



$\nabla f(x) = (0 \quad 1)$

$\nabla f(x) \cdot d = d_2 \geq 0$

$F(x) = 0 \Rightarrow$ SONC is satisfied.

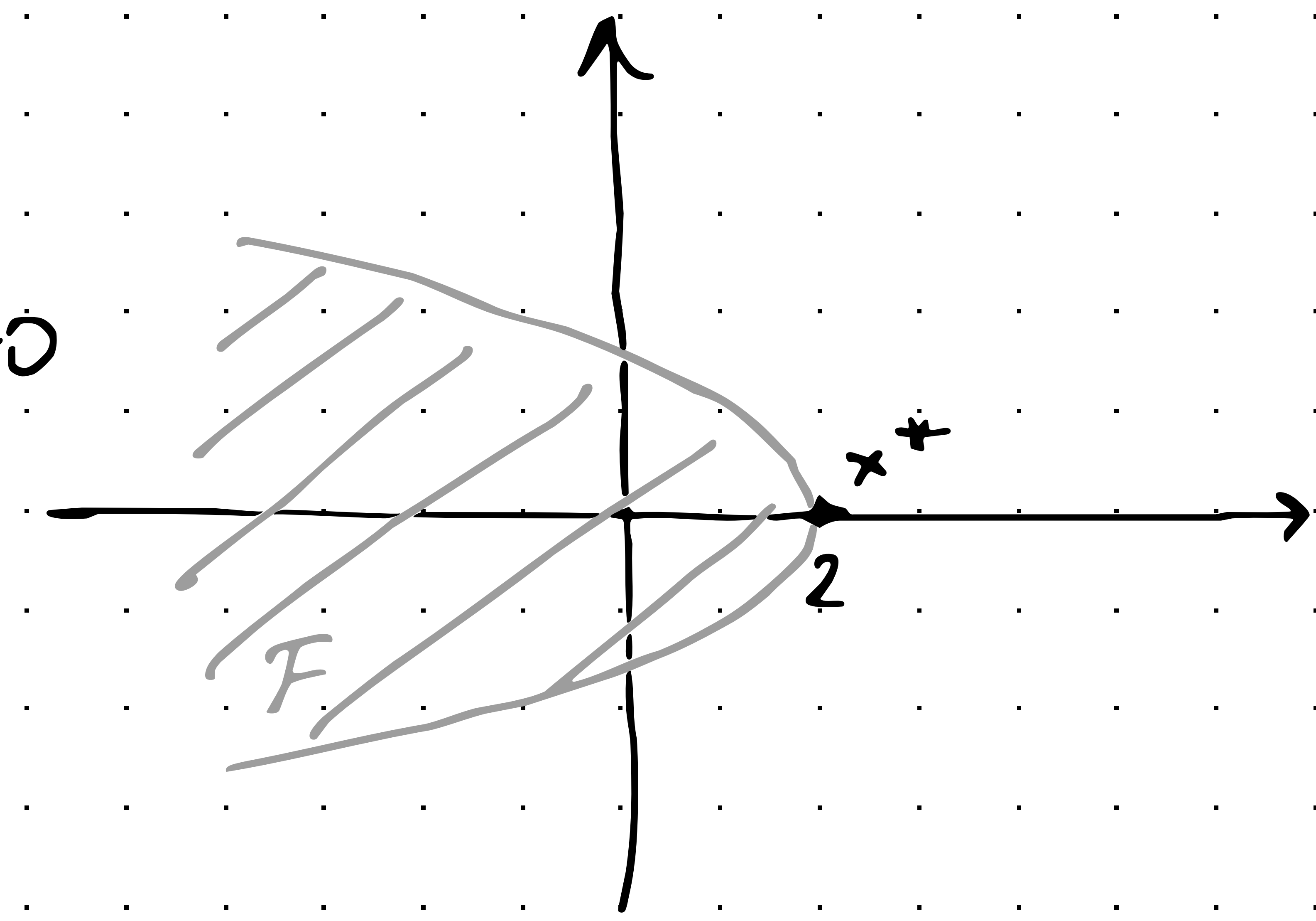
x^* is not a local minimizer.

6.13 minimize $-3x_1$ subject to $x_1 + x_2^2 \leq 2$

(a) $\nabla f(x^*) = (-3 \quad 0)$

$\nabla f(x^*) \cdot d = (-3 \quad 0)\begin{pmatrix} 0 \\ d_2 \end{pmatrix} = 0$

FONC ✓



(b) $F = 0 \Rightarrow$ SONC satisfied

(c) x^* is a local minimizer

6.12 minimize $5x_2$ subject to $x_1^2 + x_2 \geq 1$

(a) $\nabla f(x) = (0 \quad 5)$

$d = (d_1 \quad 0)$

$\nabla f(x) \cdot d = 0$ ✓

FONC satisfied

(b) SONC satisfied ✓

$F = 0$

(c) x^* is not a local minimizer