MATH 568

Linear Algebra review, individual activity

1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix}$$

   (a) Transform the matrix into reduced row echelon form (RREF). You may use computer software or an online calculator to do this.

   (b) Find the null space of $\mathbf{A}$, i.e. $N(\mathbf{A})$, and determine the dimension of it. Is $N(\mathbf{A}) \in \mathbb{R}^3$ or $\in \mathbb{R}^4$?

   (c) Find the column space of $\mathbf{A}$, i.e. $R(\mathbf{A})$, and determine the dimension of it. Is $R(\mathbf{A}) \in \mathbb{R}^3$ or $\in \mathbb{R}^4$?

   (d) Find the rank of $\mathbf{A}$. Discuss if it is possible for there to be a solution of $\mathbf{Ax} = \mathbf{b}$, and it is possible, if the solution will be unique.

2. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. The following questions relate to the notion that we estimate solutions of $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

   (a) Show that $\mathbf{A}^T\mathbf{A}$ is symmetric.

   (b) If $\mathbf{y} \in \mathbb{R}^m$ identify the dimension of $\mathbf{y}^T\mathbf{y}$ and re-write the product of vectors using sums, i.e. using Σ notation.

   (c) Use matrix algebra to find a $\mathbf{y}$ so that you can express $\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x}$ as $\mathbf{y}^T\mathbf{y}$.

   (d) Use your result in 2c. with sums (i.e. Σ notation from 2b. rather than vector multiplication) to show that $\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} \geq 0$, which means that $\mathbf{A}^T\mathbf{A}$ is positive semi-definite.

   (e) Let $\text{rank}(\mathbf{A}) = n$.

      i. Find $\dim R(A)$ and $\dim N(A)$.

      ii. Can you find a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$? Justify your answer.

      iii. Use 2(e)ii. to justify the statement: $\displaystyle\sum_{i=1}^{m} (\mathbf{Ax})_i^2 \neq 0$ if $\mathbf{x} \neq 0$.

      iv. Use 2d and 2(e)iii to show that $\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} > 0$ if $\mathbf{x} \neq 0$. This means that $\mathbf{A}^T\mathbf{A}$ is positive definite and thus is nonsingular.