In [1]:
```
%matplotlib notebook
from numpy import *
from matplotlib.pyplot import *
from numpy.linalg import qr
from scipy.linalg import hilbert
from numpy.linalg import eig
```

In [2]:
```
def display_mat(msg,A):
    print(msg)
    display(A)
    print("")
```

Homework 10 : Conditioning and stability of linear least squares

The least squares problem

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

where $A \in \mathcal{R}^{m\times n}$, $m \ge n$ has four associated "conditioning" problems, described in the table in Theorem 18.1 of TB (page 131). These are

1. Sensitivity of $\mathbf{y} = A\mathbf{x}$ to right hand side vector $\mathbf{b}$,
2. Sensitivity of the solution $\mathbf{x}$ to right hand side vector $\mathbf{b}$,
3. Sensitivity of $\mathbf{y} = A\mathbf{x}$ to the coefficient matrix A, and
4. Sensitivity of the solution $\mathbf{x}$ to the coefficient matrix A.

Problem 1

Sensitivity of $\mathbf{y}$ to a perturbation in $\mathbf{b}$.

In TB Lecture 12, the relative condition number is defined as

$$\kappa = \sup_{\delta x}\left(\frac{||\delta f||}{||f(x)||}\Big/\frac{||\delta x||}{||x||}\right) \tag{2}$$

Problem 1(a)

Arguing directly from this definition, establish the condition number of $\mathbf{y}$ with respect to perburbations in $\mathbf{b}$ given by TB Lecture 18

$$\kappa = \frac{1}{\cos\theta} \tag{3}$$

Hint: The input "x" in this problem is $\mathbf{b}$ and the output (or model) "f" is $\mathbf{y}$. Show geometrically that the supremum is attained with $P\delta\mathbf{b} = \delta\mathbf{b}$.

Solution

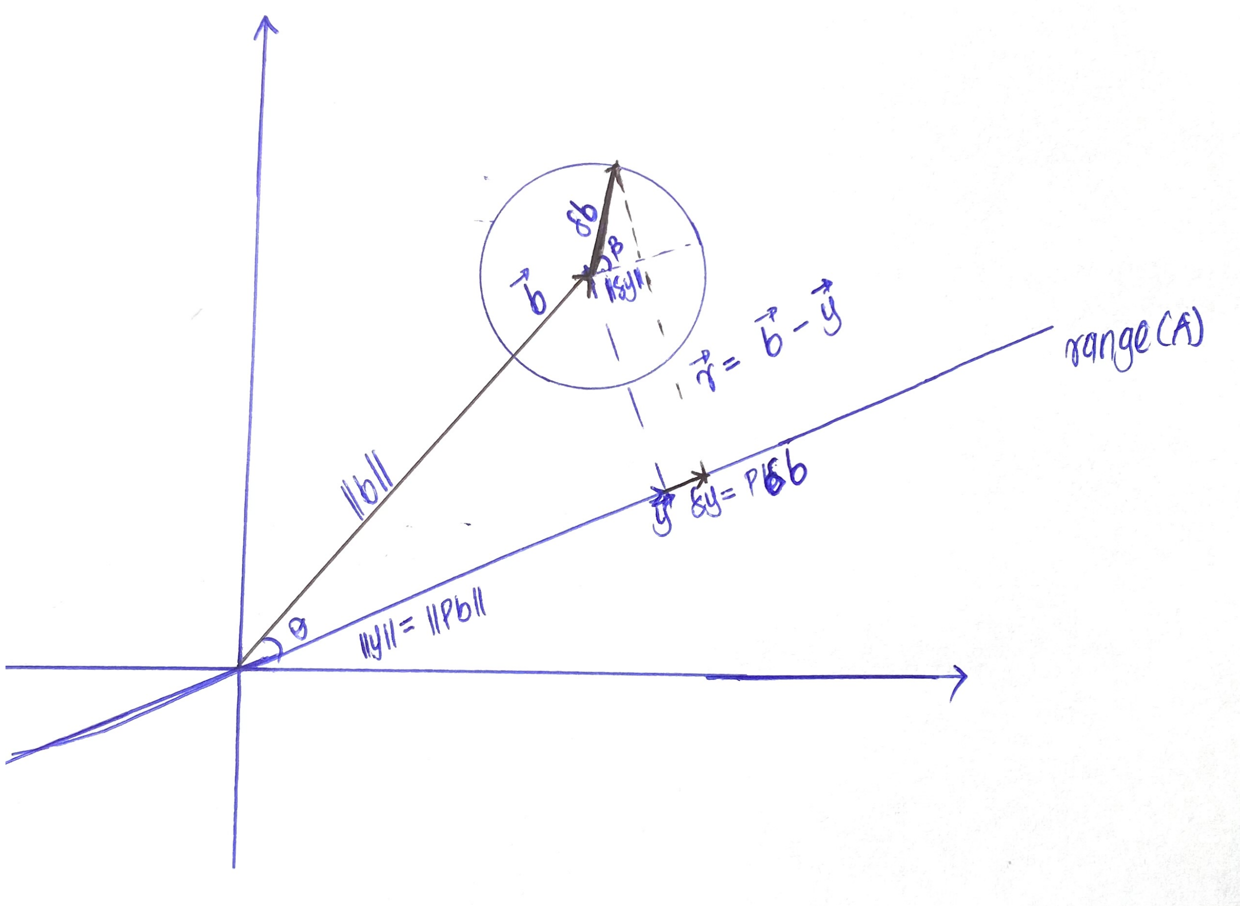
Since the relative condition number is defined as ;

$$\kappa = \sup_{\delta\mathbf{x}} \left(\frac{\|\delta\mathbf{f}\|}{\|\mathbf{f}(\mathbf{x})\|} \Big/ \frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \right), \tag{4}$$

then the relative condition number of $\mathbf{y}$ with respect to perburbations in $\mathbf{b}$ is given by;

$$\kappa = \sup_{\delta\mathbf{b}} \left(\frac{\|\delta\mathbf{y}\|}{\|\mathbf{y}\|} \Big/ \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right) \tag{5}$$

where $\mathbf{y} = A\mathbf{x} = P\mathbf{b}$ is the the projetion of $\mathbf{b}$ onto the $range(A)$, $\delta\mathbf{b}$ is a pertubation in $\mathbf{b}$ with a correponding pertubation $\delta\mathbf{y}$ in $\mathbf{y}$ (which is the projetion of $\delta\mathbf{b}$ onto the $range(A)$) . All this is illustrated geometrically in the image below;



From the above image we notice that $\cos(\theta) = \frac{\|y\|}{\|b\|}$ where θ is the angle between $\mathbf{b}$ and the $range(A)$.

The relative condition number of $\mathbf{y}$ with respect to perburbations in $\mathbf{b}$ then becomes;

$$\kappa = \sup_{\delta\mathbf{b}} \left(\frac{1}{\cos(\theta)} \times \frac{||\delta\mathbf{y}||}{||\delta\mathbf{b}||} \right)$$

$$= \sup_{\delta\mathbf{b}} \left(\frac{1}{\cos(\theta)} \times \frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||} \right)$$

$$= \frac{1}{\cos(\theta)} \times \sup_{\delta\mathbf{b}} \left(\frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||} \right)$$

But to find the $\sup_{\delta\mathbf{b}} \left(\frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||} \right)$, we must maximize the ratio $\frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||}$.

From the diagram above we see that; for $0 < \beta < \pi/2$, $||\delta\mathbf{b}||$ will always be greater than $||\delta\mathbf{y}||$ and therefore to maximize $\frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||}$, β must be equal to zero and $||\delta\mathbf{b}|| = ||P\delta\mathbf{b}||$.

This means that we obtain $\sup_{\delta\mathbf{b}} \left(\frac{||P\delta\mathbf{b}||}{||\delta\mathbf{b}||} \right)$ when $||\delta\mathbf{b}|| = ||P\delta\mathbf{b}||$ and therefore the condition number of $\mathbf{y}$ with respect to perburbations in $\mathbf{b}$ becomes;

$$\kappa = \frac{1}{\cos\theta} \tag{6}$$

Problem 1(b)

For $\theta = \pi/2$, the condition number is ∞. Illustrate what this means by considering the least squares problem

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} [\, x \,] = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{7}$$

Use the results in TB 11.11 and 11.12 (page 82) to determine the projection operator P for this problem. Then compute $\mathbf{y} = P\mathbf{b}$ and show that $P\mathbf{b} = 0$. Find a perturbation $\delta\mathbf{b}$ so that $P\delta\mathbf{b} = \delta\mathbf{b} = \delta\mathbf{y} \neq 0$. Explain what a condition number $\kappa = \infty$ might mean here. Illustrate your argument graphically.

Solution

From the above system we have that;

$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

To find the the projection operator for this problem, we use $P = AA^{\dagger}$, but

$$
\begin{aligned}
A^{\dagger} &= (A^T A) A^T \\
&= \left(\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 1 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 2 & 1 \end{bmatrix} \\
\Rightarrow P &= \frac{1}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \\
\Rightarrow \mathbf{y} = P\mathbf{b} &= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-4}{5} + \frac{4}{5} \\ \frac{-2}{5} + \frac{2}{5} \end{bmatrix} \\
\therefore P\mathbf{b} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}
\end{aligned}
$$

To find $\delta\mathbf{b}$, will let $\delta\mathbf{b} = \mathbf{b} + \mathbf{u}$ where $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $P\delta\mathbf{b} = \delta\mathbf{b}$ that is;

$$
\begin{aligned}
\Rightarrow P(\mathbf{b} + \mathbf{u}) &= (\mathbf{b} + \mathbf{u}) \\
\Rightarrow \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 + x \\ 2 + y \end{bmatrix} &= \left(\begin{bmatrix} -1 + x \\ 2 + y \end{bmatrix} \right) \\
\Rightarrow \begin{bmatrix} 4x + 2y \\ 2x + y \end{bmatrix} &= \left(\begin{bmatrix} -5 + 5x \\ 10 + 5y \end{bmatrix} \right)
\end{aligned}
$$

Equating elements in the two vectors we obtain;

$$
\begin{aligned}
x &= 2y + 5 \qquad (8) \\
-4y &= 10 - 2x \qquad (9)
\end{aligned}
$$

Subbstituting for x in equation (26), we obtain;

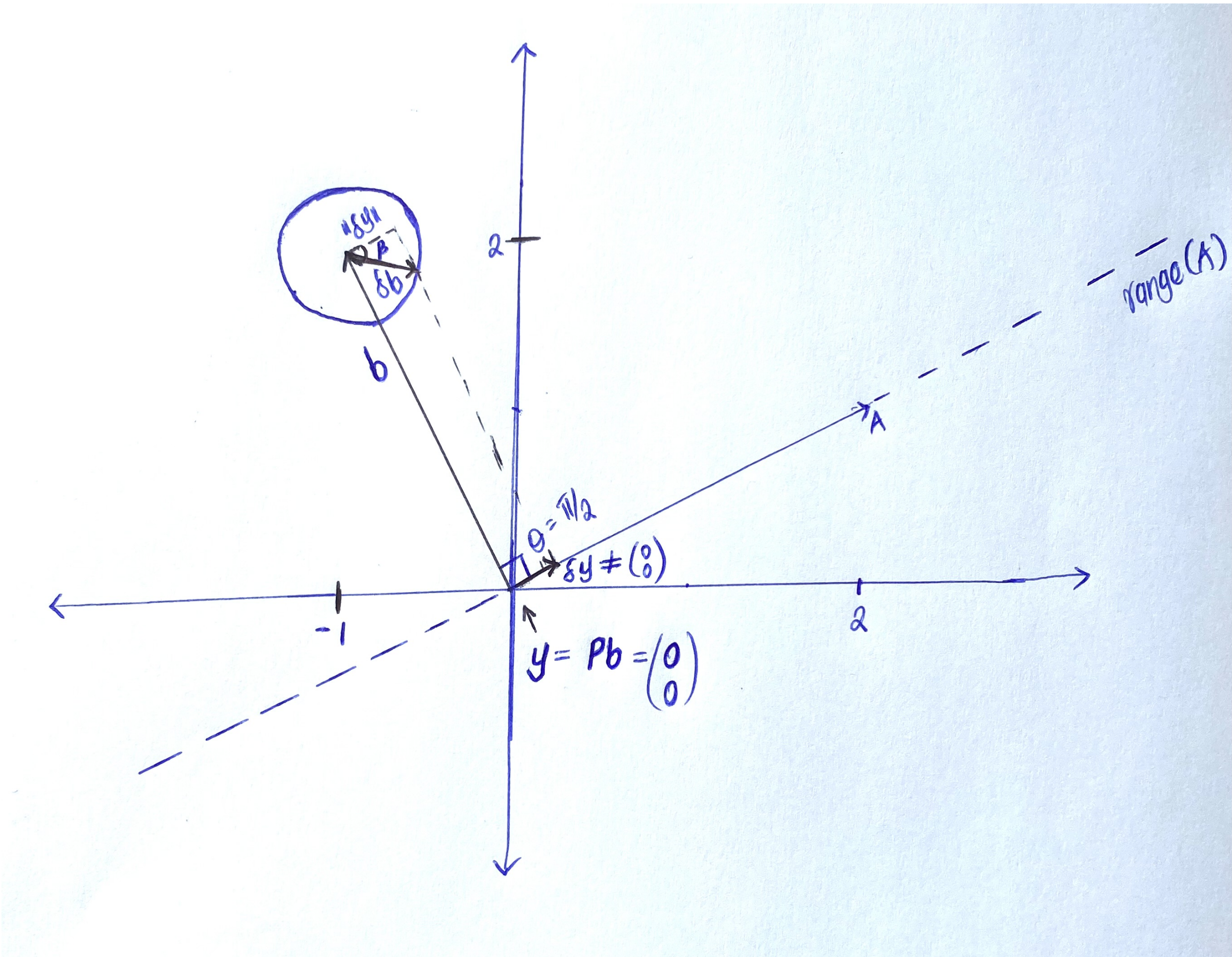
$$
\begin{aligned}
-4y &= 10 - 2(2y + 5) \qquad (10) \\
\Rightarrow 0 &= 0 \qquad (11)
\end{aligned}
$$

This means that y can take on any real number ie. $y = t \in \mathbb{R}$.

Chosing $y = t = 0$, this gives $x = 5$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\Rightarrow \delta\mathbf{b} = \begin{bmatrix} -1+5 \\ 2+0 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
$$\therefore \delta\mathbf{y} = P\delta\mathbf{b} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \delta\mathbf{b}$$

This can be geometrically illustrated as shown in the image below;



From the graph above, we see that the projection of $\mathbf{b}$ onto the $range(A)$ is a zero vector, that is $\mathbf{y} = P\mathbf{b} = \mathbf{0}$. Surprisingly, a small change in $\mathbf{b}$ say $\delta\mathbf{b}$ results into non-zero pertubation $\delta\mathbf{y}$ in the projection. This means that the residual $\mathbf{r} = \mathbf{b} - A\mathbf{x} = \mathbf{b}$ and that the change in solution is very very large.

With reference to the above ideas, we conclude that when $\theta = \pi/2$, the condition number blows up i.e. $\kappa = \infty$.

The same conclusion can be reached by using the definition of the condition number of $\mathbf{y}$ with respect to perburbations in $\mathbf{b}$;

$$\kappa = \frac{1}{cos(\theta)} = \frac{1}{cos(\pi/2)} = \infty$$

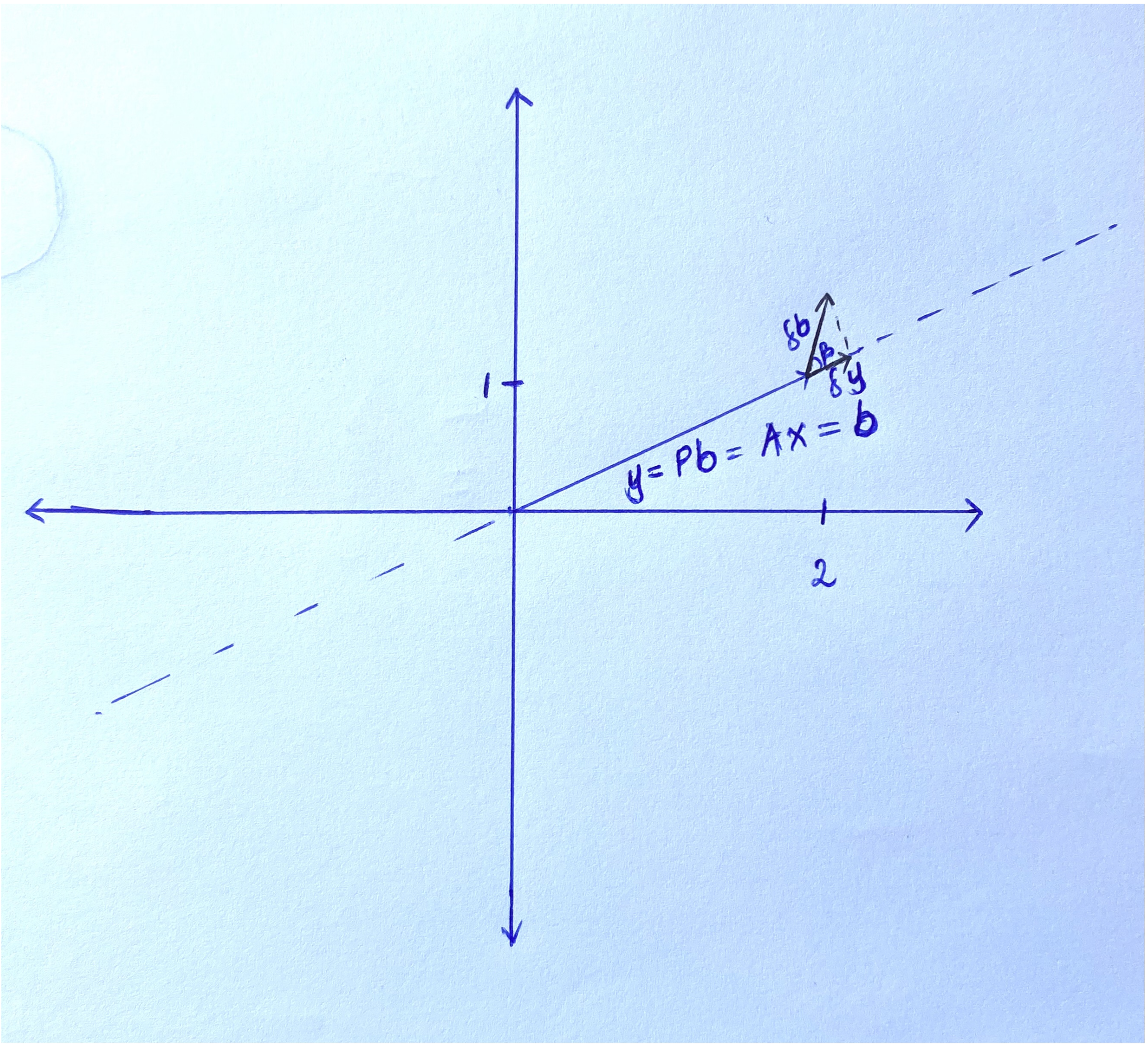
Problem 1(c)

Now consider the problem

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \left[\, x \,\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{12}$$

For this problem, show that $\kappa = 1$. What is qualitatively different about this problem than the problem in which $\kappa = \infty$?

Solution



Since in 1(a) we proved that $\kappa = \frac{1}{\cos(\theta)}$ and the angle between $\mathbf{b}$ and the $range(A)$ is zero, the condition numder for this case is then given as;

$$\kappa = \frac{1}{cos(\theta)} = \frac{1}{cos(0)} = 1$$

For this problem, we see that the projection of $\mathbf{b}$ onto the column space of A is $\mathbf{b}$ itself, that is $\mathbf{y} = P\mathbf{b} = \mathbf{b}$. This means that $\mathbf{b}$ is in the $range(A)$, the residual $\mathbf{r} = \mathbf{b} - A\mathbf{x} = \mathbf{0}$ and that the change in the solution is very very small which isn't the case when $\kappa = \infty$.

Problem 2

Problem 18.1 in TB (page 136)

2(a)

In [3]:

```
A=array([[1,1],[1,1.0001],[1,1.0001]])
b=array([[2],[0.0001],[4.0001]])
A_inv = linalg.pinv(A)
#A_inv = linalg.inv(A.T@A)@A.T
P=A@A_inv
display_mat(" A = ",A)
display_mat("Pseudo inverse of matrix A = ",A_inv)
display_mat("P = ",P)
```

```
 A =
array([[1.    , 1.    ],
       [1.    , 1.0001],
       [1.    , 1.0001]])
Pseudo inverse of matrix A =
array([[10000.99999998, -4999.99999999, -4999.99999999],
       [-9999.99999998,  4999.99999999,  4999.99999999]])
P =
array([[1.00000000e+00, 9.09494702e-13, 9.09494702e-13],
       [0.00000000e+00, 5.00000000e-01, 5.00000000e-01],
       [0.00000000e+00, 5.00000000e-01, 5.00000000e-01]])
```

2(b)

In [4]:

```
#x1 = linalg.lstsq(A,b)
#display_mat(" x = ",x1[0])
x = A_inv@b
y = P@b

display_mat(" x = ",x)
display_mat(" y = ",y)
```

```
 x =
array([[1.],
       [1.]])
 y =
array([[2.    ],
       [2.0001],
       [2.0001]])
```

2(c)

In [5]:

```
k =linalg.cond(A)
K = linalg.norm(A)*linalg.norm(A_inv)
theta = arccos(linalg.norm(y)/linalg.norm(b))
eta = (linalg.norm(A)*linalg.norm(x))/linalg.norm(y)
```

```
display_mat(" K(A) = ",k)
display_mat(" theta = ",theta)
display_mat(" eta = ",eta)
```

```
 K(A) =
42429.235416083044
 theta =
0.684702873261185
 eta =
1.000000000833278
```

2(d)

In [20]:

```
##d
K_by = 1/cos(theta)
K_bx = k/(eta*cos(theta))
K_Ay = k/(cos(theta))
K_Ax = k+ (((k**2) *tan(theta))/eta)
display_mat(" K_by = ",K_by)
display_mat(" K_bx = ",K_bx)
display_mat(" K_Ay < or = ",K_Ay)
display_mat(" K_Ax < or = ",K_Ax)
```

```
 K_by =
1.290977236078942
 K_bx =
54775.1770207547
 K_Ay < or =
54775.17706639765
 K_Ax < or =
1469883252.449082
```

2(e)

(i)

The first condition number K_{by} is attained when $\|\delta\mathbf{b}\| = \|P\delta\mathbf{b}\|$ and this occurs when $\delta\mathbf{b}$ is equal to zero except in the first $n = 2$ entries (n is the number of columns of A),

I therefore used $\delta\mathbf{b}$ as $\begin{bmatrix} 0.0001 \\ 0.0001 \\ 0 \end{bmatrix}$ and this gave $K_{by} = 1.29108$

(ii)

The condition number K_{bx} is attained when $\|A^{\dagger}\delta\mathbf{b}\|_2 = \|A^{\dagger}\|_2\|\delta\mathbf{b}\|_2$ and this occurs when $\delta\mathbf{b}$ is equal to zero except in the 2^{nd} entry.

For this I used $\delta\mathbf{b} = \begin{bmatrix} 0 \\ 10^{-8} \\ 0 \end{bmatrix}$ this gave $K_{by} = 54775.17695228298$

(iii)

The condition number K_{Ay} is attained when $\delta A = \delta p v_2^T$ where δp is the vector orthogonal to the $range(A)$ and v_2 is the second column of the right singular matrix v.

for this I used $\delta p = (I - AA^{\dagger}) \begin{bmatrix} 10^{-7} \\ 10^{-7} \\ 0 \end{bmatrix}$, $v_2^T = \begin{bmatrix} -0.70713035 & -0.70708321 \end{bmatrix}$ and obtained

$\delta A = 54775.16415665296$

Problem 3

Show that if $(\lambda, \mathbf{v})$ is an eigenvalue/eigenvector pair for matrix A, then $((\lambda - \mu)^{-1}, \mathbf{v})$ is an eigenvalue/eigenvector pair for the matrix $(A - \mu I)^{-1}$.

Why is this observation useful when using the power iteration to find an eigenvalue close to μ?

Solution

If λ is an eigenvalue of matrix A with a corresponding eigenvector $\mathbf{v}$, then we have that;

$$A\mathbf{v} = \lambda \mathbf{v} \tag{13}$$

Subtracting $\mu\mathbf{v}$ from equation() above we obtain;

$$\begin{aligned} A\mathbf{v} - \mu\mathbf{v} &= \lambda\mathbf{v} - \mu\mathbf{v} && (14) \\ (A - \mu I)\mathbf{v} &= (\lambda - \mu)\mathbf{v} && (15) \end{aligned}$$

Since $\mu \in \mathbb{R}$ is not an eigenvalue of A, then $A - \mu I$ is invertible and therefore multiplying the above equation with $(A - \mu I)^{-1}$, we get;

$$\begin{aligned} (A - \mu I)^{-1}(A - \mu I)\mathbf{v} &= (\lambda - \mu)(A - \mu I)^{-1}\mathbf{v} && (16) \\ & && (17) \\ \Rightarrow \mathbf{v} &= (\lambda - \mu)(A - \mu I)^{-1}\mathbf{v} && (18) \\ & && (19) \\ \Rightarrow (A - \mu I)^{-1}\mathbf{v} &= (\lambda - \mu)^{-1}\mathbf{v} && (20) \end{aligned}$$

Therefore from the above equation, we can conclude that $((\lambda - \mu)^{-1}, \mathbf{v})$ is an eigenvalue/eigenvector pair for the matrix $(A - \mu I)^{-1}$.

Why is this observation useful when using the power iteration to find an eigenvalue close to μ?

This observation is useful when using the power iteration to find an eigenvalue (λ_j) close to μ because the dominant eigenvalue of $(A - \mu I)^{-1}$ will be $(\lambda_j - \mu)^{-1}$ from which we can easily obtain λ_j.

Problem 4

Exercise 29.1 (Lecture 29, TB page 223). This is a five part problem that asks you to code an eigenvalue solver for a real, symmetric matrix using the shifted QR algorithm. Do your code in Python, using the Numpy `qr` algorithm where needed.

The basic steps are :

1. Reduce your matrix A to tridiagonal form. You may use the hessenberg code we wrote in class.
2. Implement the unshifted QR code (also done in class). Use the Numpy routine `qr`. Your iteration should stop when the off diagonal elements are smaller (in absolute value) than $\tau \approx 10^{-12}$.
3. Find all eigenvalues of a matrix A using the "deflation" idea described in Algorithm 28.2.
4. Introduce the Wilkinson shift, described in Lecture 29.

Notes

- Your code should work for a real, symmetrix matrix
- Your code does not have to be efficient in the sense of optimizing the cost of matrix/vector multiplies and so on.
- Apply your algorithm to the Hilbert matrix `scipy.linalg.hilbert`. The entries of the $m \times m$ Hilbert matrix are given by

$$H_{ij} = \frac{1}{i+j-1}, \qquad i,j = 1,2,\ldots m \tag{21}$$

In [7]:

```
def display_mat(msg,A):
    print(msg)
    fstr = {'float' : "{:>10.8f}".format}
    with printoptions(formatter=fstr):
        display(A)
    print("")
```

In [8]:

```
def hessenberg(A):
    m,n = A.shape
    assert m == n, "A must be square"

    H = A.copy()
    Qt = eye(m)
    for j in range(m-1):
        x = H[j+1:,j:j+1]
        I = eye(m-j-1)
        s = 1 if x[0] > 0 else -1      # sign function, with sign(0)  = 1
        v = s*linalg.norm(x,2)*I[:,0:1] + x

        vn = linalg.norm(v,2)
        if vn!=0:
            v = v/vn
```

```
            F = I - 2*(v@v.T)
            H[j+1:,j:] = F@H[j+1:,j:]
            H[0:,j+1:] = H[0:,j+1:]@F    # Apply F to the right side of H.
            break

    return H
```

In [9]:

```
def eigenvalue_QR_solver(A,kmax,method=''):
        H = hessenberg(A)
        Ak = H.copy()
        mu = 0
        e = zeros((kmax,1))
        lam = zeros((size(H,0),1))
        for k in range(kmax):
            m = size(Ak,0)
            mu = 0
            Q,R = qr(Ak - mu*eye(m))
            Ak = R@Q + mu*eye(m)
            if method == 'unshifted':
                mu=mu
                if abs(Ak[-1,-2]) < 1e-12:
                    print ("number of iterations required=",k+1)
                    lam[0:m]  = array([diag(Ak)]).T
                    break
            elif method =='Rayleigh shift':
                mu = Ak[-1,-1]
                if m==1:
                    e[0] = abs(Ak[-1,-1])
                    lam[0] = Ak[-1,-1]
                    print ("number of iterations required to find the eigenvalue
                    print("eigenvalue = {:12.4e} \n".format(Ak[-1,-1]))
                    break

                else:
                    e[1:k] = abs(Ak[-1,-2])
                    if  abs(Ak[-1,-2]) < 1e-12:
                        print ("number of iterations required to find the eigenv
                        print("eigenvalue = {:12.4e} \n".format(Ak[-1,-1]))
                        lam[1:m]= Ak[-1,-1]
                        Ak = Ak[0:m-1,0:m-1]
            elif method=='Wilkinson shift':
                if m==1:
                    print ("number of iterations required to find the eigenvalue
                    print("eigenvalue = {:12.4e} \n".format(Ak[-1,-1]))
                    lam[0]=Ak[0,0]
                    break
                else:
                    sigma = (Ak[-2,-2]-Ak[-1,-1])/2
                    if sigma !=0:
                        sn = sign(sigma)
                    else:
                        sn = -1
                    mu = Ak[m-1,m-1] - (sn*Ak[m-2,m-1]**2/(abs(sigma) + sqrt(sig
                    if abs(Ak[-1,-2]) < 1e-12:
                        print ("number of iterations required to find the eigenv
                        print("eigenvalue = {:12.4e} \n".format(Ak[-1,-1]))
                        lam[1:m]= Ak[-1,-1]
                        Ak = Ak[0:m-1,0:m-1]
```

```
        return e,sort(lam)[::-1]
```

In [10]:

```
H = hilbert(4)
```

In [11]:

```
# TRUE EIGENVALUES
eval_true1,evec_true = eig(H)
eval_true1 = sort(array(eval_true1).reshape(size(H,0),1),axis=0)
eval_true1 = sort(eval_true1)
eval_true1
```

Out[11]:

```
array([[9.67023040e-05],
       [6.73827361e-03],
       [1.69141220e-01],
       [1.50021428e+00]])
```

In [12]:

```
# USING UNSHIFTED QR
e1,lam1= eigenvalue_QR_solver(H,kmax= 20,method='unshifted')
#lam1 = sort(lam1)[::-1]
display_mat("Eigenvalues = ",lam1)
display_mat("Error = ",linalg.norm(eval_true1 - lam1))
```

```
number of iterations required= 6
Eigenvalues =
array([[0.00009670],
       [0.00673827],
       [0.16914122],
       [1.50021428]])
Error =
4.282800079194783e-12
```

In [13]:

```
#USING RAYLEIGH SHIFT
e2,lam2= eigenvalue_QR_solver(H,kmax= 20,method='Rayleigh shift')
#lam2 = sort(lam2)[::-1]
display_mat("Eigenvalues = ",lam2)
display_mat("Error = ",linalg.norm(eval_true1 - lam2))
```

```
number of iterations required to find the eigenvalue below= 6
eigenvalue =   9.6702e-05

number of iterations required to find the eigenvalue below= 8
eigenvalue =   6.7383e-03

number of iterations required to find the eigenvalue below= 13
eigenvalue =   1.6914e-01

number of iterations required to find the eigenvalue below= 14
eigenvalue =   1.5002e+00

Eigenvalues =
array([[0.00009670],
       [0.00673827],
       [0.16914122],
       [1.50021428]])
Error =
```

```
2.2303812898233666e-16
```

In [14]:

```
# USING WILKINSON SHIFT

e3,lam3= eigenvalue_QR_solver(H,kmax= 20,method='Wilkinson shift')
#lam3 = sort(lam3)[::-1]
display_mat("Eigenvalues = ",lam3)
display_mat("Error = ",linalg.norm(eval_true1 - lam3))
```

```
number of iterations required to find the eigenvalue below= 6
eigenvalue =   9.6702e-05

number of iterations required to find the eigenvalue below= 8
eigenvalue =   6.7383e-03

number of iterations required to find the eigenvalue below= 13
eigenvalue =   1.6914e-01

number of iterations required to find the eigenvalue below= 14
eigenvalue =   1.5002e+00

Eigenvalues =
array([[0.00009670],
       [0.00673827],
       [0.16914122],
       [1.50021428]])
Error =
2.2303812898233666e-16
```

Part (e)

In [15]:

```
a = arange(15,0,-1)
A = diag(a) + ones((15,15))
```

In [16]:

```
# TRUE EIGENVALUES
eval_true,evec_true = eig(A)
eval_true = sort(array(eval_true).reshape(size(A,0),1),axis=0)
eval_true = sort(eval_true)
eval_true
```

Out[16]:

```
array([[ 1.21465537],
       [ 2.25695098],
       [ 3.28777559],
       [ 4.31431185],
       [ 5.33895988],
       [ 6.36294449],
       [ 7.38709275],
       [ 8.41211207],
       [ 9.43874576],
       [10.46792166],
       [11.50098302],
       [12.54018637],
       [13.59013196],
       [14.664097  ],
       [24.22313127]])
```

In [17]:

```
# USING UNSHIFTED QR
e11,lam11= eigenvalue_QR_solver(A,kmax=1000,method='unshifted')
#lam1 = sort(lam1)[::-1]
display_mat("Eigenvalues = ",lam11)
display_mat("Error = ",linalg.norm(eval_true - lam11))
```

```
number of iterations required= 42
Eigenvalues =
array([[1.21465537],
       [2.25695098],
       [3.28777559],
       [4.31431185],
       [5.33895991],
       [6.36294484],
       [7.38709463],
       [8.41211890],
       [9.43876514],
       [10.46797027],
       [11.50111645],
       [12.54132790],
       [13.58892879],
       [14.66394811],
       [24.22313127]])
Error =
0.0016713681173739802
```

In [18]:

```
#USING RAYLEIGH SHIFT
e22,lam22= eigenvalue_QR_solver(A,kmax=1000,method='Rayleigh shift')
#lam2 = sort(lam2)[::-1]
display_mat("Eigenvalues = ",lam22)
display_mat("Error = ",linalg.norm(eval_true - lam22))
```

```
number of iterations required to find the eigenvalue below= 42
eigenvalue =   1.2147e+00

number of iterations required to find the eigenvalue below= 70
eigenvalue =   2.2570e+00

number of iterations required to find the eigenvalue below= 97
eigenvalue =   3.2878e+00

number of iterations required to find the eigenvalue below= 124
eigenvalue =   4.3143e+00

number of iterations required to find the eigenvalue below= 151
eigenvalue =   5.3390e+00

number of iterations required to find the eigenvalue below= 178
eigenvalue =   6.3629e+00

number of iterations required to find the eigenvalue below= 205
eigenvalue =   7.3871e+00

number of iterations required to find the eigenvalue below= 232
eigenvalue =   8.4121e+00

number of iterations required to find the eigenvalue below= 259
```

```
eigenvalue =   9.4387e+00

number of iterations required to find the eigenvalue below= 286
eigenvalue =   1.0468e+01

number of iterations required to find the eigenvalue below= 313
eigenvalue =   1.1501e+01

number of iterations required to find the eigenvalue below= 345
eigenvalue =   1.2540e+01

number of iterations required to find the eigenvalue below= 348
eigenvalue =   1.3590e+01

number of iterations required to find the eigenvalue below= 349
eigenvalue =   1.4664e+01

number of iterations required to find the eigenvalue below= 350
eigenvalue =   2.4223e+01

Eigenvalues =
array([[1.21465537],
       [2.25695098],
       [3.28777559],
       [4.31431185],
       [5.33895988],
       [6.36294449],
       [7.38709275],
       [8.41211207],
       [9.43874576],
       [10.46792166],
       [11.50098302],
       [12.54018637],
       [13.59013196],
       [14.66409700],
       [24.22313127]])
Error =
7.965709004822976e-14
```

In [19]:

```
# USING WILKINSON SHIFT

e33,lam33= eigenvalue_QR_solver(A,kmax=1000,method='Wilkinson shift')
#lam3 = sort(lam3)[::-1]
display_mat("Eigenvalues = ",lam33)
display_mat("Error = ",linalg.norm(eval_true - lam33))
```

```
number of iterations required to find the eigenvalue below= 42
eigenvalue =   1.2147e+00

number of iterations required to find the eigenvalue below= 70
eigenvalue =   2.2570e+00

number of iterations required to find the eigenvalue below= 97
eigenvalue =   3.2878e+00

number of iterations required to find the eigenvalue below= 124
eigenvalue =   4.3143e+00
```

```
number of iterations required to find the eigenvalue below= 151
eigenvalue =   5.3390e+00

number of iterations required to find the eigenvalue below= 178
eigenvalue =   6.3629e+00

number of iterations required to find the eigenvalue below= 205
eigenvalue =   7.3871e+00

number of iterations required to find the eigenvalue below= 232
eigenvalue =   8.4121e+00

number of iterations required to find the eigenvalue below= 259
eigenvalue =   9.4387e+00

number of iterations required to find the eigenvalue below= 286
eigenvalue =   1.0468e+01

number of iterations required to find the eigenvalue below= 313
eigenvalue =   1.1501e+01

number of iterations required to find the eigenvalue below= 345
eigenvalue =   1.2540e+01

number of iterations required to find the eigenvalue below= 348
eigenvalue =   1.3590e+01

number of iterations required to find the eigenvalue below= 349
eigenvalue =   1.4664e+01

number of iterations required to find the eigenvalue below= 350
eigenvalue =   2.4223e+01

Eigenvalues =
array([[1.21465537],
       [2.25695098],
       [3.28777559],
       [4.31431185],
       [5.33895988],
       [6.36294449],
       [7.38709275],
       [8.41211207],
       [9.43874576],
       [10.46792166],
       [11.50098302],
       [12.54018637],
       [13.59013196],
       [14.66409700],
       [24.22313127]])
Error =
7.965709004822976e-14
```