

## Homework 4

### Problem 6.3

Given  $A \in C^{m \times n}$  with  $m \geq n$ , show that  $A^*A$  is nonsingular if and only if  $A$  has full rank.

If  $A^*A$  is nonsingular, its rank will be  $n$  and it has  $n$  nonzero eigenvalues. Then from Theorem 5.4,  $A$  has  $n$  nonzero singular values. So  $A$  has full rank.

Inversely if  $A$  has full rank, the number of nonzero singular values is  $n$ . Then  $A^*A$  also has  $n$  nonzero eigenvalues and does not have zero as eigenvalue. So  $A^*A$  is nonsingular.

### Problem 6.4

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a) The orthogonal projector  $P$  onto  $\text{range}(A)$  is:

$$\begin{aligned} P &= A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \end{aligned}$$

The image under  $P$  of the vector  $(1, 2, 3)^*$  is:

$$y = Pv = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

b) The orthogonal projector  $P$  onto  $\text{range}(B)$  is:

$$\begin{aligned}
 P = B(B^*B)^{-1}B^* &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}
 \end{aligned}$$

The image under  $P$  of the vector  $(1, 2, 3)^*$  is:

$$y = Pv = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

## Problem 6.5

$P \in C^{m \times m}$  is a nonzero projector.

- First: if  $\|P\|_2 = 1$ ,  $P$  will be an orthogonal projector.

Suppose  $u \in \text{Range}(P)$ ,  $v \in \text{Null}(P)$ , with  $\|v\|_2 = 1$ . Make an orthogonal projection of  $u$  onto  $v$ , and then decompose  $u$ . We have:  $u = r + v^*uv$ ,  $r = u - v^*uv$ .

$$\text{So } \|u\|_2^2 = \|r\|_2^2 + \|v^*u\|_2^2,$$

$$Pr = P(u - v^*uv) = Pu = u.$$

$$\text{So that } \frac{\|Pr\|_2^2}{\|r\|_2^2} = \frac{\|u\|_2^2}{\|r\|_2^2} = \frac{\|r\|_2^2 + \|v^*u\|_2^2}{\|r\|_2^2} \leq 1, \text{ since } \|P\|_2 = 1.$$

Thus  $\|v^*u\|_2^2 = 0$ , this means that  $P$  is orthogonal.

- Then if  $P$  is an orthogonal projector,  $\|P\|_2 = 1$ .

Suppose  $u \in \text{Range}(P)$ ,  $v \in \text{Null}(P)$ ,  $w = u + v$ . Then

$$\frac{\|Pw\|_2^2}{\|w\|_2^2} = \frac{\|P(u+v)\|_2^2}{\|u+v\|_2^2} = \frac{\|Pu\|_2^2}{\|u\|_2^2 + \|v\|_2^2 + u^*v + v^*u} = \frac{\|u\|_2^2}{\|u\|_2^2 + \|v\|_2^2} \leq 1$$

On the other hand,  $\|w\| = \|Pw\| \leq \|P\| \|w\|$ .

Thus we can get  $\|P\|_2 = 1$ .

### Problem 7.1

a) Reduced QR factorization for A:

$$r_{11} = \|a_1\|_2 = \sqrt{2}, \quad q_1 = \frac{a_1}{r_{11}} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};$$

$$r_{12} = q_1^* a_2 = 0,$$

$$r_{22} = \|a_2 - r_{12}q_1\|_2 = 1, \quad q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} = a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Full QR factorization for A: (Assume  $a_3 = [0 \ 0 \ 1]^*$ )

$$r_{13} = q_1^* a_3 = \frac{\sqrt{2}}{2}, \quad r_{23} = q_2^* a_3 = 0,$$

$$r_{33} = \|a_3 - r_{13}q_1 - r_{23}q_2\|_2 = \frac{\sqrt{2}}{2}, \quad q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix};$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

b) Reduced QR factorization for B:

$$r_{11} = \|a_1\|_2 = \sqrt{2}, \quad q_1 = \frac{a_1}{r_{11}} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix};$$

$$r_{12} = q_1^* a_2 = \sqrt{2},$$

$$r_{22} = \|a_2 - r_{12}q_1\|_2 = \sqrt{3}, \quad q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

Full QR factorization for A: (Assume  $b_3 = [0 \ 0 \ 1]^*$ )

$$r_{13} = q_1^* b_3 = \frac{\sqrt{2}}{2}, \quad r_{23} = q_2^* b_3 = -\frac{\sqrt{3}}{3},$$

$$r_{33} = \|b_3 - r_{13}q_1 - r_{23}q_2\|_2 = \frac{\sqrt{6}}{6}, \quad q_3 = \frac{b_3 - r_{13}q_1 - r_{23}q_2}{r_{33}} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix};$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

### Problem 7.3

Give an algebraic proof of Hadamard's inequality:  $|\det A| \leq \prod_{j=1}^m \|a_j\|_2$ .

Make a QR factorization of A. We get:

$$\det A = \det(QR) = \det Q \cdot \det R = \det R = \prod_{j=1}^m r_{jj}.$$

$$a_j = \sum_{i=1}^j r_{ij} q_j, \quad \|a_j\|_2^2 = \sum_{i=1}^j \|r_{ij} q_j\|_2^2 = \sum_{i=1}^j \|r_{ij}\|_2^2 \geq \|r_{jj}\|_2^2. \quad \text{So}$$

$$\prod_{j=1}^m \|a_j\|_2^2 \geq \prod_{j=1}^m \|r_{jj}\|_2^2.$$

$$\text{Thus } |\det A| \leq \prod_{j=1}^m \|a_j\|_2.$$

For  $m=3$  case, the geometric interpretation of this result is: the volume of a parallelepiped is smaller or equal to that of a rectangular parallelepiped with the same side length. For  $m>3$  case, it follows that: the volume of a super-parallelepiped is smaller or equal to that of a super-rectangular parallelepiped with the same side length.

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