In [1]:
```
%matplotlib notebook

from matplotlib.pylab import *
from numpy import *
```

Homework #7

Problem #1 (TB 11.2, page

Find the best-fit coefficients $\mathbf{c} = (c_0, c_1, c_2)$ so that

$$g(x) = c_0 e^x + c_1 \sin(x) + c_2 \Gamma(x) \text{ ''} \tag{1}$$

approximates the function $f(x) = 1/x$ as closely as possible. To find $\mathbf{c}$, find $\overline{\mathbf{c}}$ that minimizes the function

$$\mathbf{F}(c) = \|f(\mathbf{x}) - g(\mathbf{x})\|_2^2 \tag{2}$$

over an interval $x \in [a, b]$.

Task

We solve this problem by discretizing the interval $[a, b]$ using discrete nodes $\mathbf{x} = [x_0, x_1, \ldots x_{m-1}]$. We then solve the system of equations

$$c_0 e^{x_i} + c_1 \sin(x_i) + c_2 \Gamma(x_i) = f(x_i), \qquad i = 0, 1, \ldots m-1 \tag{3}$$

This will in general be an overdetermined system that can be expressed as the linear system

$$A\mathbf{c} = \mathbf{F} \tag{4}$$

where $A \in \mathbb{R}^{m \times 3}$ and $\mathbf{F} \in \mathbb{R}^{m \times 1}$.

- Solve this resulting linear system using Algorithm 11.2 in TB (page 83).
- Choose m so that the relative norm satisfies

$$\frac{\|\mathbf{f} - \mathbf{g}\|_2}{\|\mathbf{f}\|_2} \le 10^{-2} \tag{5}$$

where vectors $\mathbf{f}$ and $\mathbf{g}$ are defined as $\mathbf{f} = f(\mathbf{x})$ and $\mathbf{g} = g(\mathbf{x})$. Your value of m should not be very large.

- Display the coefficients $\mathbf{c} = (c_0, c_1, c_2)$ you obtain for each interval (a, b).
- Plot the function $f(x)$ and your approximation $g(x)$ on the same graph.

Hints

- Use `linspace` to construct discrete points $x_i, i = 0, 1, \ldots m-1$.
- Use the SciPy function `scipy.special.gamma` to get the function $\Gamma(x)$.
- Use the QR function provided below.
- For the interval (0,1), you can use an interval $(\varepsilon, 1)$, where $\varepsilon \ll 1$.
- Use the notebook "Hmwk7_exact" to compare the coefficients you get to the exact solution computed using SymPy.

In [2]:

```
def display_mat(msg,A):
    print(msg)
    fstr = {'float' : "{:>16.8f}".format}
    with printoptions(formatter=fstr):
        display(A)
    print("")
```

In [3]:

```
# Use this QR algorithm, or you may use your own from Homework #6

def QR_House(A):
    m,n = A.shape

    R = A.copy()
    Qt = eye(m)
    p = min([m,n])
    for j in range(p):
        x = R[j:m,j:j+1]
        I = eye(m-j)
        s = 1 if x[0] >= 0 else -1      # sign function, with sign(0)  = 1
        v = s*linalg.norm(x,2)*I[:,0:1] + x
        v = v/linalg.norm(v,2)
        F = I - 2*(v@v.T)
        R[j:m,j:n] = F@R[j:m,j:n]
        Qt[j:m,:] = F@Qt[j:m,:]    # Solution to Homework #6 !!!

    Qt = Qt.T
    return Qt, R
```

In [4]:

```
from scipy.special import gamma

def f(x):
    return 1/x

def g(c,x):
    return c[0]*exp(x)+c[1]*sin(x)+c[2]*gamma(x)
```

In [5]:
```
# a=1
# b=2
# m=150

# x = reshape(linspace(a,b,m),(m,1))
# V = zeros((m,3))


# V[:,0:1] = exp(x)
# V[:,1:2] = sin(x)
# V[:,2:3] = gamma(x)

# Q,R = QR_House(V)

# p   = linalg.matrix_rank(V)
# F = f(x)
# f1 = Q.T@F
# c = linalg.inv(R[0:p,0:p])@Q[:,0:p].T@F
# #c = linalg.lstsq(R,f1,rcond=None)[0]

# G = g(c,x)
```

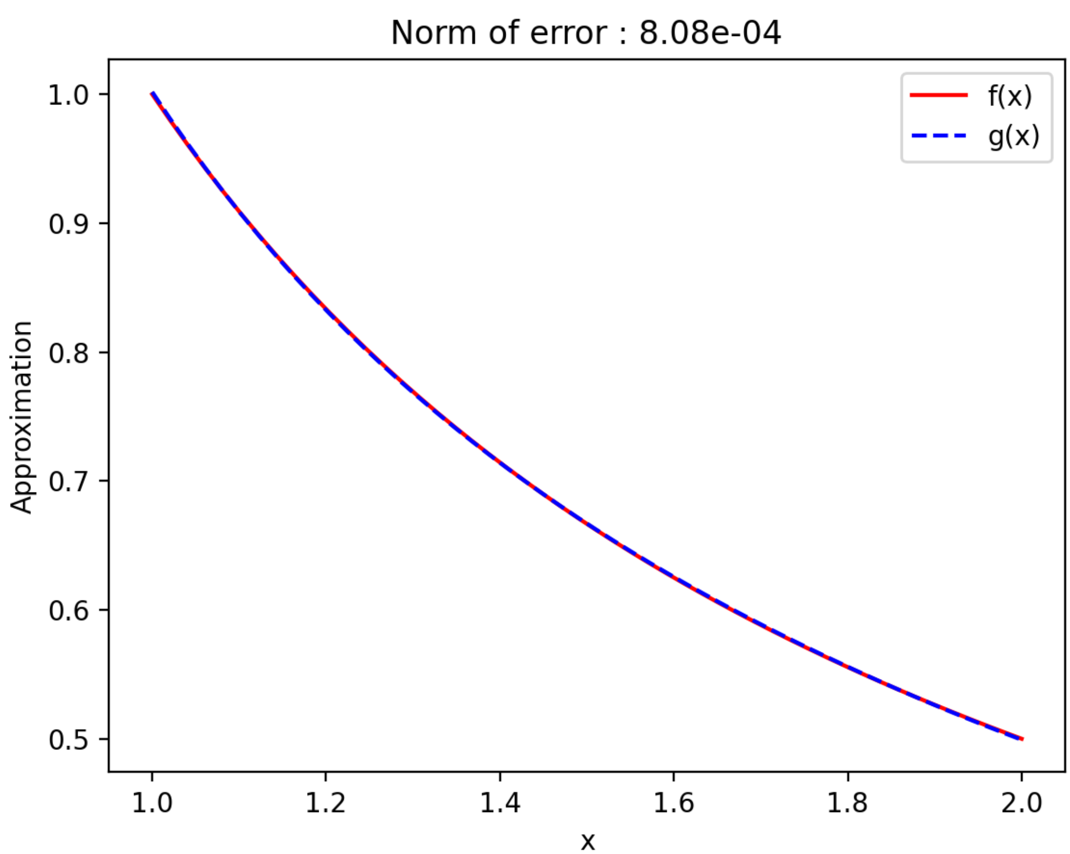
Problem 1a

Compute the coefficients **c** for the interval (a, b).

In [13]:
```
def fn(a,b,m):
    x = reshape(linspace(a,b,m),(m,1))

    V = zeros((m,3))
    for i in range(0,m):
        V[i:,0:1] = exp(x[i])
        V[i:,1:2] = sin(x[i])
        V[i:,2:3] = gamma(x[i])

    Q,R = QR_House(V)
    p   = linalg.matrix_rank(V)
    F = f(x)
    f1 = Q.T@F


    #c = linalg.inv(R[0:p,0:p])@Q[:,0:p].T@F
    c = linalg.lstsq(R,f1,rcond=None)[0]
    G = g(c,x)
    return x,c,p,F,G
a=1
b=2
m=50
x,c,p,F,G= fn(a,b,m)

c
```

Out[13]:
```
array([[-0.10774242],
       [ 0.00959252],
       [ 1.28659552]])
```

In [7]:
```
figure(1)
clf()



plot(x,F,'r',label='f(x)')
plot(x,G,'b--',label='g(x)')



rel_norm = linalg.norm(G-F,2)/ linalg.norm(F,2)

str = "Norm of error : {:.2e}".format(rel_norm)
xlabel('x')
ylabel('Approximation')
title(str)

legend()

show()
```



Problem 1b

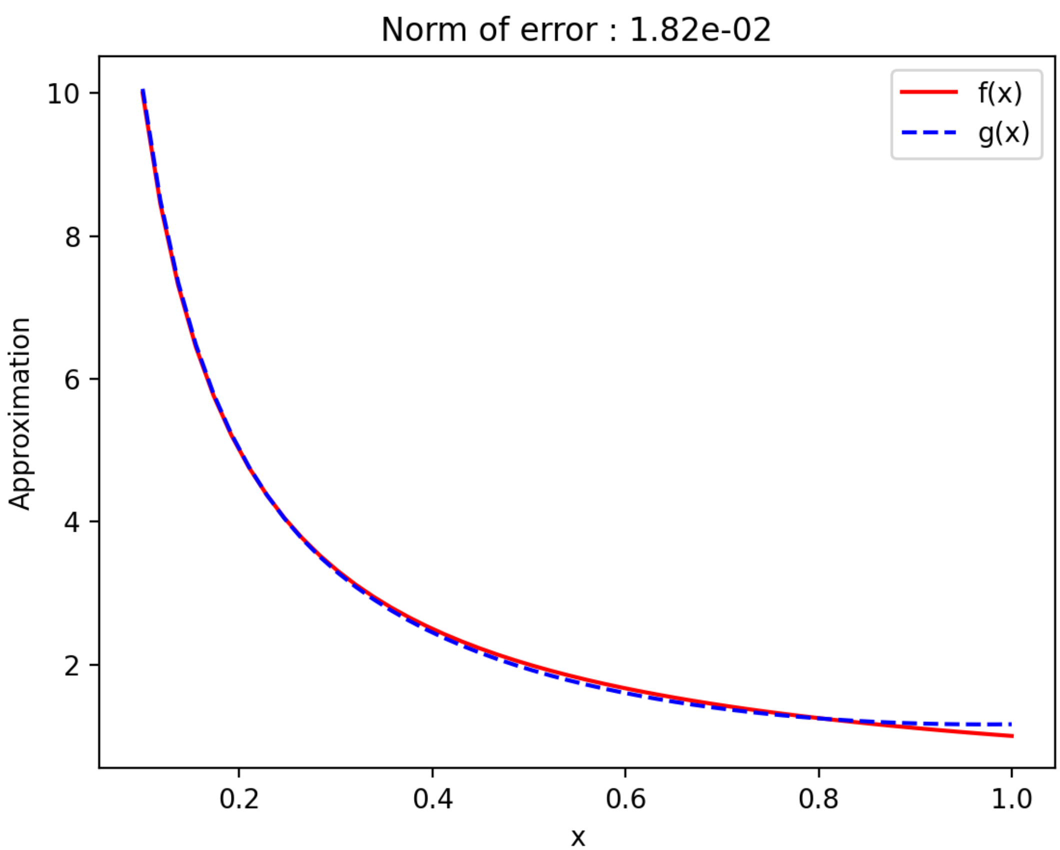
Compute the coefficients $\mathbf{c}$ for the interval $(0, 1)$.

Question :

Is there one function used to define $g(x)$ that seems to be the closest match to $f(x)$?

In [12]:

```
# Repeat the above for interval (0,1)
figure(2)
a1=0.1
b1=1
m1=50
x1,c1,p1,F1,G1= fn(a1,b1,m1)






plot(x1,F1,'r',label='f(x)')
plot(x1,G1,'b--',label='g(x)')



rel_norm = linalg.norm(G1-F1,2)/ linalg.norm(F1,2)     #  Compute relative norm
str = "Norm of error : {:.2e}".format(rel_norm)
xlabel('x')
ylabel('Approximation')
title(str)

legend()

show()
```



In [11]:
```
c1
```

Out[11]:
```
array([[ 0.63895196],
       [-1.87311681],
       [ 1.00291161]])
```

In [ ]:

In [ ]: