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January 18, 2022

1.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix}$$

(a)

The reduced row echelon form of matrix $\mathbf{A}$ is given by:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b)

$$\mathrm{N}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$dimN(\mathbf{A}) = 1$ and $\mathrm{N}(\mathbf{A}) \in \mathbb{R}^3$

(c)

$$\mathrm{R}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$dimR(\mathbf{A}) = 2$ and $\mathrm{R}(\mathbf{A}) \in \mathbb{R}^4$

(d)

$\mathrm{rank}(\mathbf{A}) = 2$

Since $m > n$ and $rank(\mathbf{A}) < n$ then there is a rank deffeciency.

Therefore the existence of the solution depends on $\mathbf{b}$ and if there is a solution, it is not unique.

2.

(a)

For any matrix to be symmetric, it should be equal to its transpose.

$$\Rightarrow \mathbf{A}^T\mathbf{A} = (\mathbf{A}^T\mathbf{A})^T$$

Proof:

$$(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$$

(b)

Since

$$\mathbf{y} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_m \end{pmatrix} \in \mathbb{R}^{m\times 1} \qquad \Rightarrow \quad \mathbf{y}^T = \begin{pmatrix} a_1 & a_2 & . & . & . & a_m \end{pmatrix} \in \mathbb{R}^{1\times m}$$

$$\Rightarrow \quad \mathbf{y}^T\mathbf{y} = a_1^2 + a_2^2 + ... + a_m^2 \in \mathbb{R}$$

$\therefore$ dimension of $\mathbf{y}^T\mathbf{y} = 1$

$\mathbf{y}^T\mathbf{y} = \sum_{i=1}^{m} a_i^2$

(c)

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x})$$

$$\Rightarrow \mathbf{A}\mathbf{x} = \mathbf{y}$$

(d)

We have that:

$$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x}$$

$$\Rightarrow \quad \mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = \sum_{i=1}^{m}(Ax)_i^2$$

Since $(Ax)_i^2$ for $i = 1, ..., m$ will always be greater or equal to zero, then $\sum_{i=1}^{m}(Ax)_i^2 \geq 0$

$$\therefore \mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} \geq 0$$

(e)

(i)

Given that the $rank(\mathbf{A}) = n$,

$\Rightarrow dimR(\mathbf{A}) = n$

using the rank- nullity theorem we have that;

$rank(\mathbf{A}) + dimN(\mathbf{A}) = n$

$\Rightarrow dimN(\mathbf{A}) = 0$

(ii)

No, we cannot find a nonzero vector such that $\mathbf{Ax} = \mathbf{0}$ because the dimension of the null space is zero meaning that there is no vector in the basis of the null space that satisfies that equation.

(iii)

From (ii) above we see that no nonzero vector can satisify $\mathbf{Ax} = \mathbf{0}$

Therefore if $\mathbf{x} \neq 0$, then $(\mathbf{Ax})_i^2 \neq 0$ for $i = 1, 2, ..., m$ and therefore $\sum_{i=1}^{m} (Ax)_i^2 \neq 0$

(iv)

From 2(d), we see that;

$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} \geq 0$

and from 2(e)(iii) for full rank that is $dimN(\mathbf{A}) = 0$, we see that;

$\sum_{i=1}^{m} (Ax)_i^2 \neq 0$ if $\mathbf{x} \neq 0$

Therefore for $rank(\mathbf{A}) = n$,

$\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = \sum_{i=1}^{m} (Ax)_i^2 > 0$ if $\mathbf{x} \neq 0$

meaning that $\mathbf{A}^T\mathbf{A}$ is positive definite and non singular.

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