In [1]:
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%matplotlib notebook
import sympy as sp
```

Homework #4

These problems cover the material in Lecture 6.

Problem #1

Show that if P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Solution

If P is an orthogonal projector, then $P = P^*$.

Let $A = I - 2P$. For A to be unitary, then $A^* = A^{-1}$ therefore to show that A is unitary then we must show that $A^*A = I$ and this is as below;

$$\begin{aligned} A^*A &= (I-2P)^*(I-2P) && (1)\\ &= (I^* - 2P^*)(I-2P) && (2)\\ &= (I-2P^*)(I-2P) && (3)\\ &= I - 2P - 2P^* + 4P^*P && (4) \end{aligned}$$

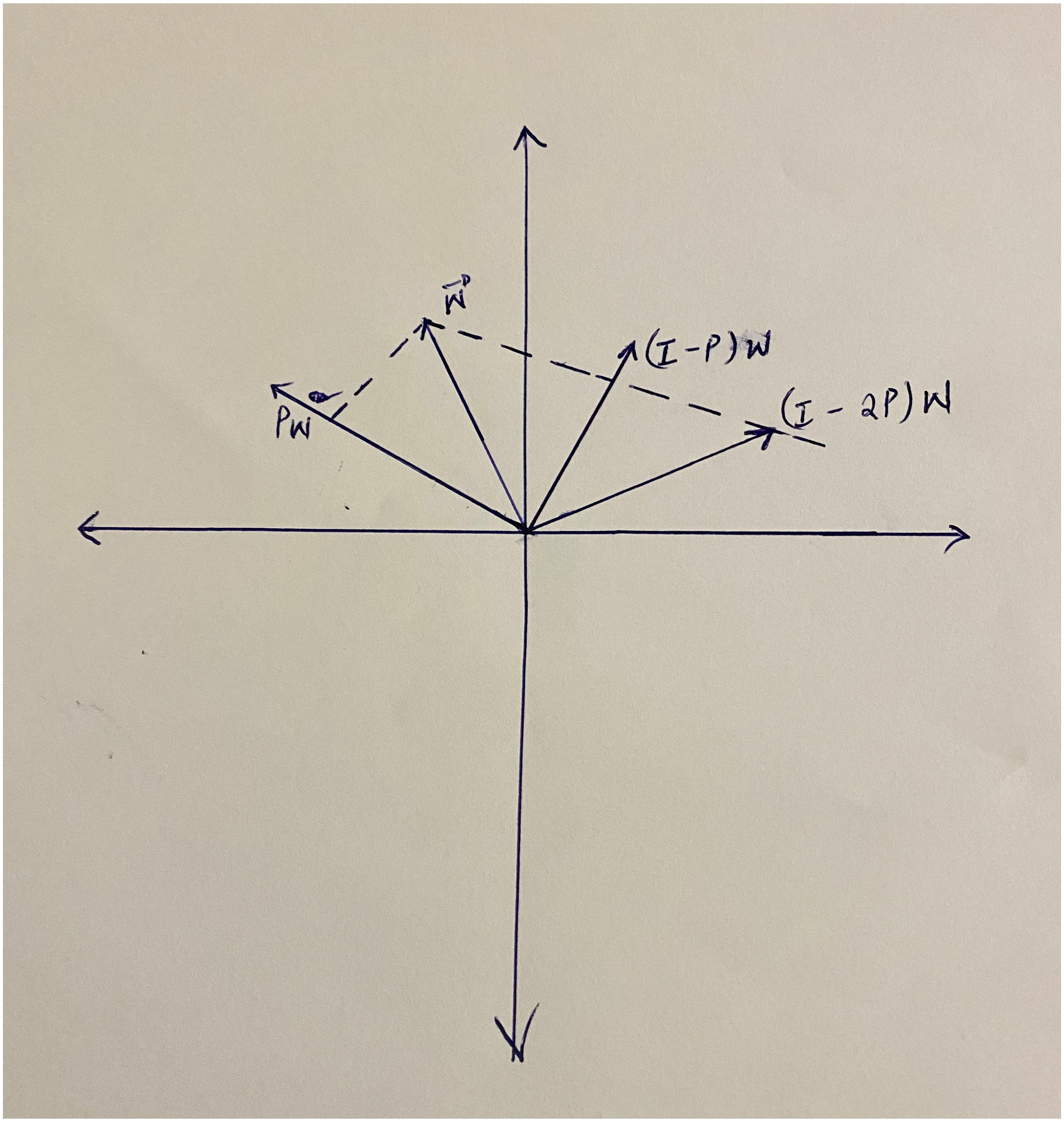
But $P = P^*$;

$$\begin{aligned} \Rightarrow A^*A &= I - 4P + 4P^2 && (5)\\ &= I - 4P + 4P && (6)\\ &= I && (7) \end{aligned}$$

Therefore $I - 2P$ is unitary since $A^*A = (I-2P)^*(I-2P) = I$.

Geometric interpretation

Suppose vector $\mathbf{w} \in range(P)$ and we have that;



$$(I - 2P)\mathbf{w} = \mathbf{w} - 2P\mathbf{w} \tag{8}$$

$$= (I - P)\mathbf{w} - P\mathbf{w} \tag{9}$$

Therefore the geometric interpretation of this is that $(I - 2P)\mathbf{w}$ is the reflection of vector $\mathbf{w}$ along the space orthogonal to $P\mathbf{w}$

Problem #2

Given $A \in \mathcal{C}^{m \times n}$ with $m \geq n$, show that A^*A is non-singular if and only if A has full rank.

Solution

Let $C = A^*A$ be a non-singular matrix. This means that $det(C) = det(A^*A) \neq 0$ implying that matrix C has full rank. The adjoint of C is then given by;

$$C^* = (A^*A)^* = A^*A \tag{10}$$
$$\Rightarrow det(C^*) = det(A^*A) \neq 0 \tag{11}$$

This then implies that C^* also has full rank.

Since A^*A is hermitian and $det(A^*A) \neq 0$, then $det(A) \neq 0$ thus A has full rank.

Therefore A^*A is non singular if and only if A has full rank.

Problem #3

The singular vectors and values of a matrix A satisfy

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \tag{12}$$

When we write this in matrix form, we obtain the SVD of A, given by

$$A = U\Sigma V^* \tag{13}$$

By convention, we order the singular values in the diagonal of Σ in decreasing order so that

$$\Sigma_{11} \geq \Sigma_{22} \geq \Sigma_{33} \geq \cdots \geq \Sigma_{rr} > 0 \tag{14}$$

where r is the rank of A. Show that it in fact doesn't matter how we arrange the singular vectors into columns of U and V and Σ, as long as we are consistent with the ordering of the vectors within U and V.

Solution

To show that it doesn't matter how we arrange the singular vectors into columns of U and V with their correponding singular values in Σ as long as we are consistent, we can use a permutation matrix say P to show this.

Assuming A is an $m \times n$ matrix with $m \geq n$ and that it has full rank that is rank(A) = r = n, P will then be a $n \times n$ matrix with exactly one entry of 1 in each row and each column and 0s elsewhere.

Let

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1r} \\ u_{21} & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mr} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{rr} \end{pmatrix} \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ \vdots & \vdots \\ v_{r1} & v_{r2} \end{pmatrix}$$

If I want to interchange Σ_{11} with Σ_{22}, then $P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

The diagonal elements Σ_{11} and Σ_{22} can be interchaged a shown below;

$$\underset{P}{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}} \underset{\Sigma}{\begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{rr} \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}} = \begin{pmatrix} \Sigma_{22} & 0 \\ 0 & \Sigma_{11} \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}$$

Since consistence is a key factor we as well have to interchage the columns of U with respect to Σ_{11} and Σ_{22} as shown below;

$$\underset{U}{\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1r} \\ u_{21} & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mr} \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}} = \begin{pmatrix} u_{12} & u_{11} & \cdots & u_{1r} \\ u_{22} & u_{21} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{m2} & u_{m1} & \cdots & u_{mr} \end{pmatrix}$$

Lastly we have to interchage the rows of V^* with respect to Σ_{11} and Σ_{22} as shown below

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \bar{v}_{11} & \bar{v}_{21} & \cdots & \bar{v}_{r1} \\ \bar{v}_{12} & \bar{v}_{22} & \cdots & \bar{v}_{r2} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{v}_{1r} & \bar{v}_{2r} & \cdots & \bar{v}_{rr} \end{pmatrix} = \begin{pmatrix} \bar{v}_{12} & \bar{v}_{22} & \cdots & \bar{v}_{r2} \\ \bar{v}_{11} & v_{21} & \cdots & \bar{v}_{r1} \\ \vdots & \vdots & \vdots & \vdots \\ \bar{v}_{1r} & \bar{v}_{2r} & \cdots & \bar{v}_{rr} \end{pmatrix}$$

P V^*

We notice then that to re-arrange the singular vectors into columns of U and V with their correponding singular values in Σ consistently A becomes;

$$A = (UP)(P\Sigma P)(PV^*) \tag{15}$$

$$= UP^2\Sigma P^2 V^* \tag{16}$$

But $P^2 = I$, $\Rightarrow A = U\Sigma V^*$. This therefore shows that it doesn't matter how we arrange the singular vectors into columns of U and V with their correponding singular values in Σ as long as we are consistent.

Problem #4

Let $P \in \mathcal{C}^{m\times m}$ be a non-zero projector. Show that $||P||_2 \geq 1$ with equality if and only if P is an orthogonal projector.

Hint: Use the definition of a project presented in the introduction to Lecture 6.

Solution

Since P is a non-zero projector, then $\|P\|_2 \neq 0$ and $P^2 = P$.

Suppose $\mathbf{v} \in range(P)$ and $\mathbf{v} = P\mathbf{x}$ for some $\mathbf{x}$, then $\mathbf{v}$lies exactly on its own shadow, and applying the projector on $\mathbf{v}$ results in;

$$P\mathbf{v} = P(P\mathbf{x}) = P^2\mathbf{x} = P\mathbf{x} = \mathbf{v} \tag{17}$$

since

$$P\mathbf{v} = \mathbf{v} \tag{18}$$

$$\Rightarrow \|P\mathbf{v}\|_2 = \|\mathbf{v}\|_2 \tag{19}$$

Using the Cauchy–Schwarz inequality, $\|P\mathbf{v}\|_2 \leq \|P\|_2\|\mathbf{v}\|_2$ the above equation becomes;

$$\|P\|_2\|\mathbf{v}\|_2 \geq \|\mathbf{v}\|_2 \tag{20}$$

$$\Rightarrow \|P\|_2 \geq 1 \tag{21}$$

If P is an orthogonal projector, then $P = P^*$.

Let $P = U\Sigma V^*$ be the SVD form of P where U and V are the left and right singular vectors respectively, $UU^* = VV^* = I$ and Σ is a diagonal matrix of singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$.

From Lecture 3, we know that $\|P\|_2 = \max\limits_{1 \leq i \leq m} \sigma_i = \sigma_1$.

Therefore $\|P\|_2 = 1$ if and only if $\sigma_1 = 1$ and $\sigma_1 = 1$ iff $\Sigma = I$.

We also know that;

$$P^2 = P \tag{22}$$

$$\Rightarrow U\Sigma V^*U\Sigma V^* = U\Sigma V^* \tag{23}$$

$$V^*U\Sigma = I \tag{24}$$

This means that $\Sigma = I$ iff $V^*U = I$ i.e. $V = U$ which also means that $\Sigma = I$ if and only iff P is an orthogonal progector.

Therefore $\|P\|_2 = 1$ if and only if P is an orthogonal progector.

Problem #5

Suppose both P_1 and P_2 are orthogonal projectors onto the subspace S. Show that $P_1 = P_2$.

Hint: Show that $\|(P_1 - P_2)\mathbf{x}\|_2^2 = 0$.

Solution

Supose $\mathbf{x} \in \mathbb{R}^m$, then If P_1 and P_2 are orthogonal projectors onto the subspace S, $P_1\mathbf{x}, P_2\mathbf{x} \subset S$ and $(I - P_1)\mathbf{x}, (I - P_2)\mathbf{x} \subset S^\perp$. Also $P_i^2 = P_i^T = P_i$ for $i = 1, 2$.

To show that $P_1 = P_2$, we must show that $\|(P_1 - P_2)\mathbf{x}\|_2^2 = 0$ and this is as below;

$$
\begin{aligned}
\|(P_1 - P_2)\mathbf{x}\|_2^2 &= (P_1\mathbf{x} - P_2\mathbf{x})^T(P_1 - P_2)\mathbf{x} &(25)\\
&= (P_1\mathbf{x})^T(P_1 - P_2)\mathbf{x} - (P_2\mathbf{x})^T(P_1 - P_2)\mathbf{x} &(26)\\
&= \mathbf{x}^T P_1(P_1 - P_2)\mathbf{x} - \mathbf{x}^T P_2(P_1 - P_2)\mathbf{x} &(27)\\
&= \mathbf{x}^T P_1(I - P_2)\mathbf{x} - \mathbf{x}^T P_2(P_1 - I)\mathbf{x} &(28)\\
&= \mathbf{x}^T P_1(I - P_2)\mathbf{x} + \mathbf{x}^T P_2(I - P_1)\mathbf{x} &(29)
\end{aligned}
$$

Since S and $S^\perp$ are complementary subspace, $P_1(I - P_2)\mathbf{x} = P_2(I - P_1)\mathbf{x} = 0$.

$$\Rightarrow \|(P_1 - P_2)\mathbf{x}\|_2^2 = 0 \qquad (30)$$

Therefore $P_1 = P_2$ and thus the projection onto S is unique.

Problem #6

Let U be the matrix of singular vectors of the matrix A. Show that $P = UU^T$ is an orthogonal projector onto $\mathrm{range}(A)$. Assume that the columns of A are linearly independent.

Solution

If we assume that the columns of matrix A are linearly independent and that $m \geq n$, then $range(A)$ is spanned by all columns of A. Thus A is full rank with $rank(A) = n$.

Let $A = U\Sigma V^T$ be the reduced SVD form of an $m \times n$ matrix A where U is an $m \times n$ matrix of orthonormal left singular vectors ,V is an $n \times n$ matrix of orthonormal right singular vectors and Σ is $n \times n$ diagonal matrix of singular values. We then have that $V^T V = I$, and $\Sigma^T = \Sigma$

If P is an orthogonal projection onto $range(A)$, then it is given by;

$$
\begin{aligned}
P &= A(A^T A)^{-1} A^T && (31)\\
&= U\Sigma V^T[(U\Sigma V^T)^T U\Sigma V^T]^{-1}(U\Sigma V^T)^T && (32)\\
&= U\Sigma V^T[V\Sigma U^T U\Sigma V^T]^{-1} V\Sigma U^T && (33)\\
&= U\Sigma V^T[V\Sigma^2 V^T]^{-1} V\Sigma U^T && (34)\\
&= U\Sigma V^T V^{-T}\Sigma^{-2} V^{-1} V\Sigma U^T && (35)\\
&= U\Sigma V^T V\Sigma^{-2} V^T V\Sigma U^T && (36)\\
&= U\Sigma\Sigma^{-2}\Sigma U^T && (37)\\
\therefore P &= UU^T && (38)
\end{aligned}
$$

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