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PROBABILITY AND STATISTICS REVIEW

QUESTION ①

Given $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $E(X) = \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

(a) Show that $E(AX) = AE(X)$

$$AX = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix}$$

$$E(AX) = \begin{pmatrix} E(a_{11}X_1 + a_{12}X_2) \\ E(a_{21}X_1 + a_{22}X_2) \end{pmatrix}$$

$$= \begin{pmatrix} E(a_{11}X_1) + E(a_{12}X_2) \\ E(a_{21}X_1) + E(a_{22}X_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}E(X_1) + a_{12}E(X_2) \\ a_{21}E(X_1) + a_{22}E(X_2) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E(X_1) \\ E(X_2) \end{pmatrix}$$

$$= AE(X)$$

(b) If

$$Y = AX = \begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\Rightarrow Y_1 = a_{11}X_1 + a_{12}X_2 \quad , \quad Y_2 = a_{21}X_1 + a_{22}X_2$$

$$\Rightarrow \text{Var}(Y_1) = \text{Var}(a_{11}X_1 + a_{12}X_2)$$

$$= \text{Var}(a_{11}X_1) + \text{Var}(a_{12}X_2) + 2\,\text{Cov}(a_{11}X_1, a_{12}X_2)$$

$$= a_{11}^2\,\text{Var}(X_1) + a_{12}^2\,\text{Var}(X_2) + 2a_{11}a_{12}\,\text{cov}(X_1, X_2)$$

$$\Rightarrow \text{Var}(Y_2) = \text{Var}(a_{21}X_1 + a_{22}X_2)$$

$$= \text{Var}(a_{21}X_1) + \text{Var}(a_{22}X_2) + 2\,\text{Cov}(a_{21}X_1, a_{22}X_2)$$

$$= a_{21}^2\,\text{Var}(X_1) + a_{22}^2\,\text{Var}(X_2) + 2a_{21}a_{22}\,\text{Cov}(X_1, X_2)$$

$$\text{Cov}(Y_1, Y_2) = \text{Cov}(a_{11}X_1 + a_{12}X_2, a_{21}X_1 + a_{22}X_2)$$

$$= E[(a_{11}X_1 + a_{12}X_2)(a_{21}X_1 + a_{22}X_2)] - E[a_{11}X_1 + a_{12}X_2] \cdot E[a_{21}X_1 + a_{22}X_2]$$

$$= E[a_{11}a_{21}X_1^2 + a_{11}a_{22}X_1X_2 + a_{12}a_{21}X_2X_1 + a_{12}a_{22}X_2^2] -$$

$$\left[(a_{11}\,E[X_1] + a_{12}\,E[X_2]) \cdot (a_{21}E[X_1] + a_{22}\,E[X_2])\right]$$

$$= a_{11}a_{21}E[X_1^2] + a_{11}a_{22}E[X_1X_2] + a_{12}a_{21}E[X_2X_1] + a_{12}a_{22}E[X_2^2] - \left(a_{11}a_{21}E^2[X_1] + a_{11}a_2E[X_1]E[X_2] - \left(a_{12}a_{21}E[X_2]E[X_1] + a_{12}a_{22}E^2[X_2]\right)\right.$$

$$= a_{11}a_{21}\left(E[X_1^2] - E^2[X_1]\right) + a_{12}a_{22}\left(E[X_2^2] - E^2[X_2]\right) + 2a_{11}a_{22}\left(E[X_1X_2] - E[X_1]E[X_2]\right)$$

$$\Rightarrow Cov(Y_1, Y_2) = a_{11}a_{21}Var(X_1) + a_{12}a_{22}Var(X_2) + 2a_{11}a_{22}Cov(X_1, X_2)$$