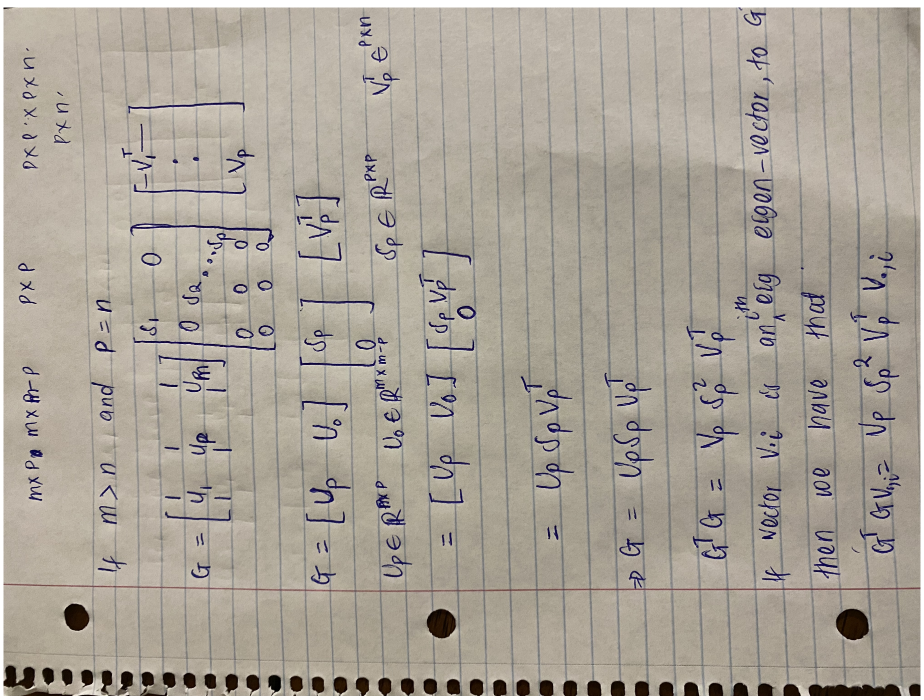
Ch3: Generalized inverse, individual activity

1 (a)

$$
\begin{aligned}
G^T G &= (USV^T)^T (USV^T) \\
&= (VS^T U^T)(USV^T) \\
&= VSU^T USV^T \\
&= VSISV^T \\
&= VS^2 V^T
\end{aligned}
$$

1 (b)



$m \times p \quad m \times m-p \qquad p \times p \qquad p \times p \cdot \times p \times n \cdot \quad p \times n$

If $m > n$ and $p = n$

$$G = \begin{bmatrix} | & | & | \\ u_1 & u_p & u_m \\ | & | & | \end{bmatrix} \begin{bmatrix} s_1 & & 0 \\ 0 & s_2 & \ldots s_p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -V_1^T- \\ \vdots \\ V_p \end{bmatrix}$$

$$G = \begin{bmatrix} U_p & U_0 \end{bmatrix} \begin{bmatrix} S_p \\ 0 \end{bmatrix} \begin{bmatrix} V_p^T \end{bmatrix}$$

$U_p \in \mathbb{R}^{p \times p}$ $\quad U_0 \in \mathbb{R}^{m \times m-p}$ $\quad S_p \in \mathbb{R}^{p \times p}$ $\quad V_p^T \in {}^{p \times n}$

$$= \begin{bmatrix} U_p & U_0 \end{bmatrix} \begin{bmatrix} S_p V_p^T \\ 0 \end{bmatrix}$$

$$= U_p S_p V_p^T$$

$$\Rightarrow G = U_p S_p V_p^T$$

$$G^T G = V_p S_p^2 V_p^T$$

If vector $V_{\cdot i}$ is an i^{th} eig eigen-vector, to G

then we have that

$$G^T G V_{\cdot, i} = V_p S_p^2 V_p^T V_{\cdot, i}$$

$m \times m$

If $m < n$ the $p = m$ and

$$\underset{m\times n}{G} = \underset{p\times p}{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_p \\ | & & | \end{bmatrix}} \begin{bmatrix} s_1 & & & 0 & 0 \\ & \ddots & & 0 & \\ & & s_p & 0 & 0 \\ 0 & & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_p^T \\ v_n^T \end{bmatrix}$$

$$= \underset{p\times p}{[U_p]} \; [\underset{p\times p}{S_p} \;\; \underset{p\times n-p}{0}] \begin{bmatrix} V_p^T \\ V_0 \end{bmatrix} \begin{matrix} p\times n \\ (n-p)\times n \end{matrix}$$

$$= [U_p] \; [S_p V_p^T]$$

$$= U_p S_p V_p^T$$

$$G^T G = V S_p^2 V_p^T$$

let V_i be the eigenvectors of $G^T G$ then

$$G^T G\, V_{\cdot,i} = V S_p^2 V_p^T V_{\cdot,i}$$

$$= (S_p^2)_i \, V_{\cdot,i}$$

In comparison with $Ax = \lambda x$, then we have that $(S_p^2)_i$ which are the singular values are the eigen vectors of $G^T G$ if $m < n$.

$$G^T G\, V_{\cdot,i} = (S_p^2)_i^2 \, V_{\cdot,i}$$

$\therefore S_p^2$ are the eigenvalues of $G^T G$

If $m > n$.

1(c)

for $m > n$ and p=n , we have;

$$\begin{aligned}
G^\dagger &= V_p S_p^{-1} U_p^T \\
&= V_p S_p^{-2} V_p^T V_p S_p U_p^T \\
\Rightarrow m_\dagger = G^\dagger d &= V_p S_p^{-2} V_p^T V_p S_p U_p^T d \\
&= V_p S_p^{-2} V_p^T G^T d
\end{aligned}$$

But

$$\begin{aligned}
(G^T G)^{-1} &= (V_p S_p U_p^T U_p S_p V_p^T)^{-1} \\
&= V_p S_p^{-2} V_p^T
\end{aligned}$$

$$\Rightarrow m_\dagger = (G^T G)^{-1} G^T d$$

Which is similar to the least squares estimates and hence $G^\dagger = (G^T G)^{-1} G^T$

2

To show that $G^\dagger$ satisfies the four properties given below, we used other properties such as; $U_p^T U_p = I_p$, $V_p^T V_p = I_p$ and $S_p S_p^{-1} = S_p^{-1} S_p = I_p$
(a)

$$\begin{aligned}
G G^\dagger G &= (U_p S_p V_p^T V_p S_p^{-1} U_p^T) U_p S_p V_p^T \\
&= (U_p S_p I_p S_p^{-1} U_p^T) U_p S_p V_p^T \\
&= (U_p S_p S_p^{-1} U_p^T) U_p S_p V_p^T \\
&= U_p I_p U_p^T U_p S_p V_p^T \\
&= U_p S_p V_p^T \\
&= G
\end{aligned}$$

(b)

$$\begin{aligned}
G^\dagger G G^\dagger &= (V_p S_p^{-1} U_p^T U_p S_p V_p^T) V_p S_p^{-1} U_p^T \\
&= (V_p S_p^{-1} I_p S_p V_p^T) V_p S_p^{-1} U_p^T \\
&= (V_p S_p^{-1} S_p V_p^T) V_p S_p^{-1} U_p^T \\
&= V_p I_p V_p^T V_p S_p^{-1} U_p^T \\
&= V_p S_p^{-1} U_p^T \\
&= G^\dagger
\end{aligned}$$

(c)

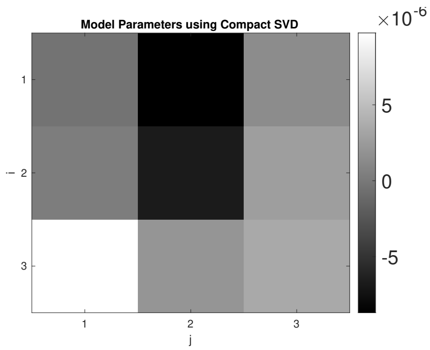
$$
\begin{aligned}
(GG^\dagger)^T &= (U_pS_pV_p^TV_pS_p^{-1}U_p^T)^T \\
&= (U_pS_pS_p^{-1}U_p^T)^T \\
&= (U_pU_p^T)^T \\
&= U_pU_p^T \\
&= U_pS_pS_p^{-1}U_p^T \\
&= U_pS_pV_p^TV_pS_p^{-1}U_p^T \\
&= (U_pS_pV_p^T)(V_pS_p^{-1}U_p^T) \\
&= GG^\dagger
\end{aligned}
$$

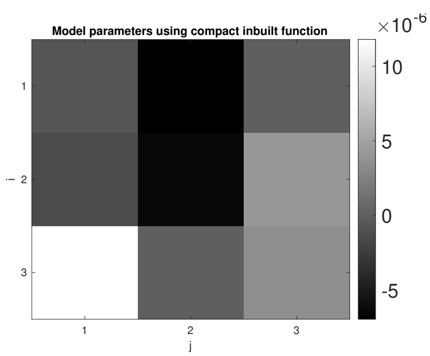
(d)

$$
\begin{aligned}
(G^\dagger G)^T &= (V_pS_p^{-1}U_p^TU_pS_pV_p^T)^T \\
&= (V_pS_p^{-1}S_pV_p^T)^T \\
&= (V_pV_p^T)^T \\
&= V_pV_p^T \\
&= V_pS_p^{-1}S_pV_p^T \\
&= V_pS_p^{-1}U_p^TU_pS_pV_p^T \\
&= (V_pS_p^{-1}U_p^T)(U_pS_pV_p^T) \\
&= G^\dagger G
\end{aligned}
$$

3.(a)

$$
m_{dagger} = 10^{-5} \times \begin{bmatrix} -0.0369 \\ -0.8697 \\ 0.1399 \\ 0.0303 \\ -0.6702 \\ 0.2732 \\ 0.9732 \\ 0.2066 \\ 0.3536 \end{bmatrix} \qquad m_{backslash} = 10^{-4} \times \begin{bmatrix} -0.0070 \\ -0.0696 \\ 0 \\ -0.0143 \\ -0.0637 \\ 0.0413 \\ 0.1180 \\ 0 \\ 0.0354 \end{bmatrix}
$$

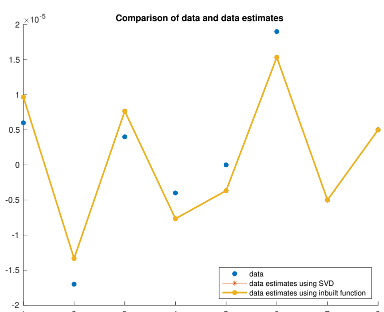




The values of the model parameters obtained in 3(a) (i) using the generalized inverse of G, with the compact SVD decomposition at positions; s_{12}, s_{15}, s_{16}, s_{17}, and s_{19}, are smaller than their corresponding values obtained using the inbuilt function in 3(a)(ii). The remaining values of model parameters in 3(a)(i) are bigger than that in 3(a)(ii).

$$t_{dagger} = 10^{-4} \times \begin{bmatrix} 0.0967 \\ -0.1333 \\ 0.0767 \\ -0.0767 \\ -0.0367 \\ 0.1533 \\ -0.0500 \\ 0.0500 \end{bmatrix} \qquad t_{backslash} = 10^{-4} \times \begin{bmatrix} 0.0967 \\ -0.1333 \\ 0.0767 \\ -0.0767 \\ -0.0367 \\ 0.1533 \\ -0.0500 \\ 0.0500 \end{bmatrix} \qquad error = 10^{-5} \times \begin{bmatrix} -0.3667 \\ -0.3667 \\ -0.3667 \\ 0.3667 \\ 0.3667 \\ 0.3667 \\ 0.0000 \\ -0.0000 \end{bmatrix}$$

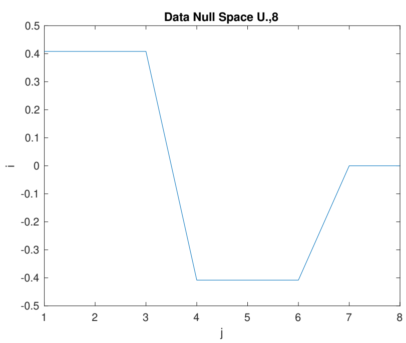
The predicted data sets obtained using model parameters in 3(a) (i) and 3(a) (ii) are the same and this is shown by the figure below. The actual data however deviates from the predicted data by a value of -0.3667 for the first three data points, and 0.3667 the next three data , and is the same for the last two values.



3(b)

dimesion of data null space= 1
The basis vector spanning the data null space is given by;

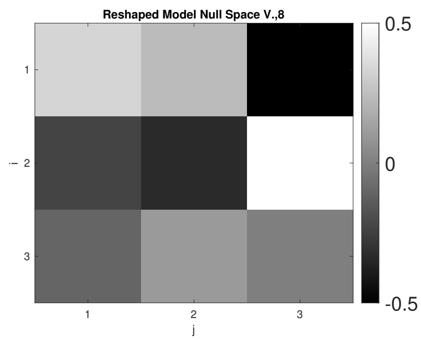
$$\begin{bmatrix} 0.4082 \\ 0.4082 \\ 0.4082 \\ -0.4082 \\ -0.4082 - 0.4082 \\ -0.0000 \\ -0.0000 \end{bmatrix}$$

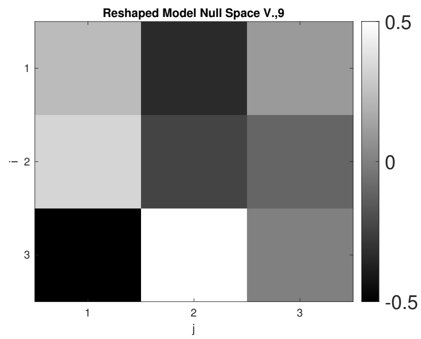


3(c)

dimension of the model null space = 2
Therefore the model null space, $N(G)$, is spanned by the two orthonormal vectors that form the 8th and 9th columns of singular vector$V$. An orthonormal basis for the model null space is;

$$\begin{bmatrix} 0.3357 & 0.2323 \\ 0.2323 & -0.3357 \\ -0.5680 & 0.1035 \\ -0.2323 & 0.3357 \\ -0.3357 & -0.2323 \\ 0.5680 & -0.1035 \\ -0.1035 & -0.5680 \\ 0.1035 & 0.5680 \\ -0.0000 & 0.0000 \end{bmatrix}$$





3(d)

Yes, it possible to have two sets of parameters that produce the same data. This is because the dimension of the model null space is two meaning that there are two non-zero vectors spanning the model null space and consquently we can always obtain new sets of model parameters that can give us the same data using; $m = m_\dagger + \alpha_1 V_8 + \alpha_2 V_9$ where V_8 and V_9 span $N(G)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

This can be seen in 3(a) above where the model paremeters obtained by generalized inverse of G, with the compact SVD decomposition different from the model parameters obtained using the backslash inbuilt function both give the same predicted data. Another set of parameters obtained using the expression above gives the same data set as shown below.
If the basis of the model null space was to be empty, then there would be no two sets of parameters that would produce the same data.

```
new_m = mdagger + 2*V(:,8)+ 3*V(:,9);
d=G*new_m
```

```
d = 8x1
1.0e+-4 *

    0.0967
```

```
   -0.1333
    0.0767
   -0.0767
   -0.0367
    0.1533
   -0.0500
    0.0500
```

3(e)
Yes,it possible to have two sets of data that produce the same model parameters. This is because the dimension of the data null space is one meaning that there is one non-zero vector spanning the data null space and consquently we can always obtain a new data set that can give us the same model parameters using; $d = d_{\dagger} + \alpha_1 U_8$ where U_8 spans $N(G^T)$ and $\alpha_1 \in \mathbb{R}$.

An example to confirm this is as shown below where the new data gives the same set of parameters as that obtained when using the actual data set.
If the basis of the data null space was to be empty, then there would be no two data sets that would give the same set of model parameters.

```
new_d = d+ 10*U(:,8);
mdagger2 = Vp* inv(Sp)*Up.'*new_d
```

```
mdagger2 = 9x1
1.0e+-5 *

   -0.0369
   -0.8697
    0.1399
    0.0303
   -0.6702
    0.2732
    0.9732
    0.2066
    0.3536
```

APPENDIX

```
t = [6e-06;-1.7e-05; 4e-06;-4e-06;0; 1.9e-05; -5e-06;5e-06];
s2=sqrt(2);
G = [1,0,0,1,0,0,1,0,0;
     0,1,0,0,1,0,0,1,0;
     0,0,1,0,0,1,0,0,1;
     1,1,1,0,0,0,0,0,0;
```

```
     0,0,0,1,1,1,0,0,0;
     0,0,0,0,0,0,1,1,1;
     s2,0,0,0,s2,0,0,0,s2;
     0,0,0,0,0,0,0,0,s2];

% Find dimensions of G
[m,n]=size(G);
% Get the singular values for the system matrix
[U,S,V] = svd(G);
 p=rank(G);


% Using the compact form
%Gdagger = pinv(G);
%mdagger = Gdagger*t
diag(S);
Vp = V(:,1:p);
Up =U(:,1:p);
Sp = S(1:p,1:p);

mdagger = Vp* inv(Sp)*Up.'*t


%Using backslash
mbackslash = G\t
tdagger = G*mdagger

tbackslash= G*mbackslash
error = t-tdagger
tt = linspace(1,8,8)

scatter(tt,t,'filled')
hold on
plot(tt,tdagger)
hold on
plot(tt,tbackslash,'*','Color','black')
hold off
legend('data','data estimates using SVD','data estimates using inbuilt function','Locatio
title('Comparison of data and data estimates')



%dimension of the data null space
m-p
```

```
U(:,p+1:m)

% Display image of null space model V.,9
%m0 = reshape(U(:,p+1),2,2)

plot(tt,U(:,p+1))
xlabel('j')

ylabel('i')
title('Data Null Space U.,8');
%dimension of the model null space
n-p

%model null space
V(:,p+1:n)

% Display null space vectors reshaped to match tomography example geometry
m01=reshape(V(:,p+1),3,3)';

m02=reshape(V(:,p+2),3,3)';
% Display image of null space model V.,8
figure(1)
clf
colormap('gray')
imagesc(m01)
caxis([-0.5 0.5]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Reshaped Model Null Space V.,8');
% Display image of null space model V.,9
figure(2)
clf
colormap('gray')
imagesc(m02)
caxis([-0.5 0.5]);
set(colorbar,'Fontsize',18);
set(gca,'xtick',[1,2,3]);
set(gca,'ytick',[1,2,3]);
xlabel('j')
ylabel('i')
title('Reshaped Model Null Space V.,9');
```