Computational Math I: Assignment 3

Based on Ting Gao's solution

1. (**Exercise 8.1**) Let A be an $m \times n$ matrix. Determine the exact numbers of floating point additions, subtractions, multiplications, and divisions involved in computing the factorization $A = \hat{Q}\hat{R}$ by Algorithm 8.1.
Solution:

Additions: For $j = i + 1$ to n, we have $m - 1$ additions for each innermost loop.And for $i = 1$ to n, we have $m - 1$ additions for each outer loop. Hence, the total number of floating point additions is

$$\sum_{i=1}^{n}(m-1)(n-i) + \sum_{i=1}^{n}(m-1) = \frac{(m-1)n(n+1)}{2}.$$

Subtractions: For $j = i + 1$ to n, we have m subtractions for each innermost loop. Hence, the total number of floating point subtractions is

$$\sum_{i=1}^{n} m(n-i) = \frac{mn(n-1)}{2}.$$

Multiplications: For $j = i + 1$ to n, we have $2m$ multiplications for each innermost loop. And for $i = 1$ to n, we have m multiplications for each outer loop. Hence, the total number of floating point multiplications is

$$\sum_{i=1}^{n} 2m(n-i) + \sum_{i=1}^{n} m = mn(n-1) + mn = mn^2.$$

Divisions: For $i = 1$ to n, we have m divisions for each outer loop. Hence, the total number of floating point divisions is

$$\sum_{i=1}^{n} m = mn.$$

2. **Computer Problem:** Produce the plot of r_{jj} versus j for all three QR factorization algorithms.

3. Determine the leading order term in the operation count for Algorithm 10.3 of the textbook. Also compute the leading order term in the operation count for using Eq.(10.7) to compute Qb.

Solution:

(1)The Algorithm 10.3 involves in three operations: multiplication, addition and subtraction. The exact numbers of floating point are:

$$\text{Additions: } \sum_{k=1}^{n}(m-k) = mn - \frac{n(n+1)}{2} = mn - \frac{n^2}{2} - \frac{n}{2}.$$

$$\text{Subtractions: } \sum_{k=1}^{n}(m-k+1) = mn - \frac{n(n+1)}{2} + n = mn - \frac{n^2}{2} + \frac{n}{2}.$$

$$\text{Multiplications: } \sum_{k=1}^{n} 2(m-k+1) = 2mn - n^2 + n.$$

(2) For equation (10.7): $Q = Q_1 Q_2 \cdots Q_n$, we need to multiply b by Q_k from $k = n$ down to $k = 1$. Since for each loop, we have $m(2m-1)$ operations ($m \cdot m$ multiplications and $m \cdot (m-1)$ additions). The exact number of floating point is

$$\sum_{k=1}^{n} m(2m-1) = 2m^2 n - mn.$$

4. **Computer Problem: Exercise 11.3**

From the observation, we can see that the normal equations exhibit instability.

5. Find the condition number of the addition: $f(x) = x1 + x2$, where the vector x is the vector $(x1, x2)$. Do the same for the multiplication, $f(x) = x1 * x2$. Discuss the implication of your results.

Solution: The (relative) condition number of addition is :

$$\kappa = \frac{\|J\|_\infty \|x\|_\infty}{\|f(x)\|} = \frac{2 \cdot \max\{|x_1|, |x_2|\}}{|x_1 + x_2|}.$$

This quantity is large if $|x_1 + x_2| \approx 0$, so the problem is ill-conditioned when $x_1 \approx -x_2$.

The (relative) condition number of multiplication is:

$$\kappa = \frac{\|J\|_\infty \|x\|_\infty}{\|f(x)\|} = \frac{(|x_1| + |x_2|) \cdot \max\{|x_1|, |x_2|\}}{|x_1 x_2|}$$

For $x_1, x_2 \in \mathbb{C}$, we have $|x_1x_2| = |x_1||x_2|$. Hence, if $|x_1| > |x_2|$, we have

$$\kappa = \frac{|x_1|^2 + |x_2||x_1|}{|x_1x_2|} = \frac{|x_1|}{|x_2|} + 1;$$

If $|x_1| \le |x_2|$,we have

$$\kappa = \frac{|x_2|^2 + |x_2||x_1|}{|x_1x_2|} = \frac{|x_2|}{|x_1|} + 1.$$

Thus the problem become ill-conditioned when $|x_1| \gg |x_2|$ or $|x_1| \ll |x_2|$.