MOCK FINAL EXAM

LINEAR ALGEBRA - 20221

Time: 90 minutes

Question 1 [1p] Given the linear map: $f : \mathbb{R}^2 \to \mathbb{C}$ define: $f(x,y) = (2x+y)+(2x-y)i$ and $A = \{z \in \mathbb{C} \mid z\,.\,\bar{z} = 2\ \}$. Compute $a-b$ if $f^{-1}(A) = \{(x,y) \in \mathbb{R} \mid ax^2 - by^2 = 1\}$

Question 2. [1p] Find matrix X such that: $\left(\frac{1}{2}X^T - 2E\right)^{-1} = 2\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Question 3. [2đ] Consider R^4, given vectors:

$$v_1 = (1,0,-1,0), \quad v_2 = (1,2,3,4), \quad v_3 = (-1,-2,0,1), \quad v_4 = (-2,-2,7,11)$$

a) Proof that these vectors are linear dependent and express one vector in the set as a linear combination of others

b) Let $V_1 = span\{v_1, v_2\}, \quad V_2 = span\{v_3, v_4\}$. Find basis and dimension of vector space $V_1 \cap V_2$

Question 4 [3p] Consider a linear transformation in $\mathbb{R}^3$ defined by:

$$f(2,1,-2) = (5,-3,2)$$
$$f(0,3,1) = (10,4,14)$$
$$f(4,1,0) = (11,-3,8)$$

a) Find matrix of f with respect to the standard basis E of $\mathbb{R}^3$. Find $f(0,1,1)$

b) Find a basis of $\mathbb{R}^3$ so that the matrix of f with respect to this basis is diagonal

c) **Prove**: $\exists\ v(a,b,c)$ with $a^2+b^2+c^2 \neq 0$ such that $f(v) = \theta$ (θ: null vector of $\mathbb{R}^3$)

Question 5. [2p] Vector space $\mathbb{R}^3$ with dot product $\langle x,y \rangle = x^T Ay$ $(x,y \in \mathbb{R})$ and

$$A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 6 & 1 \\ 8 & 4 & 3 \end{bmatrix}$$

a) Find the distance of two vector $v_1 = (1,2,3)$ and $v_2 = (4,-1,0)$

b) Find an orthonormal basis of subspace $V = \text{span}\{v_1, v_2\}$ using Gram-Schmidt process

c) Find a vector of V such that the distance from this vector to $w = (2,3,-1)$ is minimum

d) Which matrix A makes this dot product becomes standard ?

Question 6. [1p] Find the determinant of the following matrix A:

$$A = \begin{bmatrix} 1+\varepsilon^2 & 1 & 1 & 1 \\ 1 & 1-\varepsilon^2 & 1 & 1 \\ 1 & 1 & 1+\varepsilon^5 & 1 \\ 1 & 1 & 1 & 1-\varepsilon^5 \end{bmatrix}$$

(With $\varepsilon = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$)

Question 1.

$$\begin{aligned}
f^{-1}(A) &= \{(x,y)\in\mathbb{R}^2 \mid f(x,y)\in A\}\\
&= \{(x,y)\in\mathbb{R}^2 \mid (2x+y)+(2x-y)i\in A\}\\
&= \{(x,y)\in\mathbb{R}^2 \mid |(2x+y)+(2x-y)i\,|^2 = 2\}\\
&= \{(x,y)\in\mathbb{R}^2 \mid (2x+y)^2+(2x-y)^2 = 2\}\\
&= \{(x,y)\in\mathbb{R}^2 \mid 4x^2+y^2 = 1\}
\end{aligned}$$

So, $a=4$ and $b=-1$ and $a-b=5$

Question 2.

$$\left(\frac{1}{2}X^T-2E\right)^{-1} = 2\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix} = \begin{pmatrix}2 & 2\\ 4 & 6\end{pmatrix}$$

$$\Rightarrow \frac{1}{2}X^T-2E = \begin{pmatrix}2 & 2\\ 4 & 6\end{pmatrix}^{-1} = \begin{pmatrix}\frac{3}{2} & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{pmatrix}$$

$$\Rightarrow \frac{1}{2}X^T = \begin{pmatrix}\frac{7}{2} & \frac{3}{2}\\ 1 & \frac{5}{2}\end{pmatrix}$$

$$\Rightarrow X^T = \begin{pmatrix}7 & 3\\ 2 & 5\end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix}7 & 2\\ 3 & 5\end{pmatrix}$$

Question 3.

a) Consider the linear combination

$$x_1\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+x_3\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}+x_4\begin{bmatrix}-2\\-2\\7\\11\end{bmatrix}=0$$

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with variables x_1,x_2,x_3,x_4 $\boxed{\text{and } (x_1,x_2,x_3,x_4)\neq(0,0,0,0)}$

$\in \mathbb{R}$

$$\iff \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -2 \\ -1 & 3 & 0 & 7 \\ 0 & 4 & 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We have:

$$[A|0] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ -1 & 3 & 0 & 7 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_2 \to R_2]{R_3+R_1\to R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 4 & -1 & 5 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_1-R_2\to R_1 \\ R_3-4R_2\to R_3 \\ R_4-4R_2\to R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & 5 & 15 & 0 \end{array}\right] \xrightarrow[\frac{1}{5}R_4 \to R_4]{\frac{1}{3}R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_2+R_3\to R_2 \\ R_4-R_3\to R_4}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the general solution is given by

$$x_1 = x_4$$
$$x_2 = -2x_4$$
$$x_3 = -3x_4$$

where x_4 is a free variable.

Suppose $x_4 = 1$ then we have nonzero solution that

$$x_1 = 1,\ x_2 = -2,\ x_3 = -3,\ x_4 = 1$$

$\Rightarrow$ $\boxed{\text{(x1,x2,x3,x4) khác (0,0,0,0)}}$

Therefore, the set is linearly dependent.

Substituting these values, we have:

$$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}-2\begin{bmatrix}1\\2\\3\\4\end{bmatrix}-3\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}+\begin{bmatrix}-2\\-2\\7\\11\end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix}-2\\-2\\7\\11\end{bmatrix}=-\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+2\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+3\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix} \quad \text{(linear combination)}$$

b) Assume $v \in V_1 \cap V_2$. Then

$\Rightarrow \begin{cases} v \in V_1 \rightarrow v = a(1,0,-1,0)+b(1,2,3,4) \\ v \in V_2 \rightarrow - - - - - \end{cases}$

$$v = a(1,0,-1,0)+b(1,2,3,4) = c(-1,-2,0,1)+(-2,-2,7,11)$$

$$\Longleftrightarrow a\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}+b\begin{bmatrix}1\\2\\3\\4\end{bmatrix}-c\begin{bmatrix}-1\\-2\\0\\1\end{bmatrix}-d\begin{bmatrix}-2\\-2\\7\\11\end{bmatrix}=0$$

We have:

$$[A|0]=\left[\begin{array}{cccc|c}1&1&1&2&0\\0&2&2&2&0\\-1&3&0&-7&0\\0&4&-1&-11&0\end{array}\right]\xrightarrow[\frac{1}{2}R_2\to R_2]{R_3+R_1\to R_3}\left[\begin{array}{cccc|c}1&1&1&2&0\\0&1&1&1&0\\0&4&1&-5&0\\0&4&-1&-11&0\end{array}\right]$$

$$\xrightarrow[\substack{R_3-4R_2\to R_3\\R_4-4R_2\to R_4}]{R_1-R_2\to R_1}\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&1&1&0\\0&0&-3&-9&0\\0&0&-5&-15&0\end{array}\right]\xrightarrow[-\frac{1}{5}R_4\to R_4]{-\frac{1}{3}R_3\to R_3}\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&1&1&0\\0&0&1&3&0\\0&0&1&3&0\end{array}\right]$$

$$\xrightarrow[R_4-R_3\to R_4]{R_2-R_3\to R_2}\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&-2&0\\0&0&1&3&0\\0&0&0&0&0\end{array}\right]$$

Thus, the general solution is given by

$$a = -t$$
$$b = 2t$$
$$c = -3t$$
$$d = t \quad \text{t is any scalar}$$

We have:

$$v = a(1,0,-1,0) + b(1,2,3,4) = -t(1,0,-1,0) + 2t(1,2,3,4) = t(1,4,7,8)$$

So ~~the~~ one basis of $V_1 \cap V_2 = \{(1,4,7,8)\} \Rightarrow dim(V_1 \cap V_2) = 1$

Question 4.

a) Standard basis of $\mathbb{R}^3$: $E = \{(1,0,0),(0,1,0),(0,0,1)\}$
Call $v_1 = (2,1,-2)$, $v_2 = (0,3,1)$, $v_3 = (4,1,0)$ and A: matrix of f with respect to E. We have:

$$[f(v_1) \quad f(v_2) \quad f(v_3)]_E = A[v_1 \quad v_2 \quad v_3]_E$$

$$\Rightarrow \begin{bmatrix} 5 & 10 & 11 \\ -3 & 4 & -3 \\ 2 & 14 & 8 \end{bmatrix} = A \begin{bmatrix} 2 & 0 & 4 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}^{-1} . \begin{bmatrix} 5 & 10 & 11 \\ -3 & 4 & -3 \\ 2 & 14 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

b) $det(A-\lambda I)=\begin{vmatrix}2-\lambda & 3 & 1\\ -1 & 1-\lambda & 1\\ 1 & 4 & 2-\lambda\end{vmatrix}=-\lambda^3+5\lambda^2-6\lambda=0\Leftrightarrow\left[\begin{array}{l}\lambda=3\\ \lambda=2\\ \lambda=0\end{array}\right.$

+) $\lambda=3$: Solve $(A-3I)(x)_E=0$

$$\Rightarrow\begin{bmatrix}-1 & 3 & 1\\ -1 & -2 & 1\\ 1 & 4 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow x=(\mathrm{t},0,\mathrm{t})\Rightarrow$ Eigen vector $(1,0,1)$

+) $\lambda=2$: Solve $(A-2I)(x)_E=0$

$$\Rightarrow\begin{bmatrix}0 & 3 & 1\\ -1 & -1 & 1\\ 1 & 4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow x=(12\mathrm{t},-3\mathrm{t},\mathrm{t})\Rightarrow$ Eigen vector $(12,-3,1)$

+) $\lambda=0$: Solve $(A-0I)(x)_E=0$

$$\Rightarrow\begin{bmatrix}2 & 3 & 1\\ -1 & 1 & 1\\ 1 & 4 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow x=(2\mathrm{t},-3\mathrm{t},5\mathrm{t})\Rightarrow$ Eigen vector $(2,-3,5)$

So, matrix of f with respect to basis $D=\{(1,0,1),(12,-3,1),(2,-3,5)\}$ is diagonal

c) **Option 1**: $f(v)=\theta\Rightarrow A[v]_E=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}2 & 3 & 1\\ -1 & 1 & 1\\ 1 & 4 & 2\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$Det(A)=0\Rightarrow r(\overline{A})=r(A)<3$

$\Rightarrow$ System has infinite solution $\Rightarrow$ q.e.d

Option 2: $Det(A)=0\Rightarrow dim(Imf)<3\Rightarrow dim(kerf)>0\Rightarrow$ q.e.d

Question 5.

- We have:

  $v_1-v_2=(-3,3,3)$

  $d(v_1,v_2)=\|v_1-v_2\|=\sqrt{\langle v_1-v_2,v_1-v_2\rangle}=\sqrt{(v_1-v_2)\cdot A\cdot(v_1-v_2)^T}=\sqrt{90}$

- v_1, v_2 are independent and $B = \{v_1, v_2\}$ is a basis of V.
  We orthogonalize using Gram-Schmidt method:
  $$u_1 = v_1, \qquad e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{14}}(1,2,3$$
  $$u_2 = v_2 - proj_{u_1}(v_2) = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u2} u_2 = \left(\frac{27}{7}, \frac{-9}{7}, \frac{-3}{7}\right), \qquad e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{91}}(9,-3,$$
  Hence, one orthonormal basis of subspace V is $B' = \{u_1, u_2\}$.

- We can find one vector that is the projection of w to V.
  $$\begin{aligned} proj_V(w) &= w - \langle w, u_1 \rangle u_1 - \langle w, u_2 \rangle u_2 \\ &= w - \frac{5}{14} u_1 - \frac{10}{39} u_2 \\ &= \left(\frac{17}{26}, \frac{34}{13}, -\frac{51}{26}\right) \end{aligned}$$
  Hence, one vector that we can find is $\left(\frac{17}{26}, \frac{34}{13}, -\frac{51}{26}\right)$

- This dot product becomes standard when $A = I_3$.

Question 6.

- We have: $\det(A) =$

$$\begin{vmatrix} 1+\varepsilon^2 & 1 & 1 & 1 \\ 1 & 1-\varepsilon^2 & 1 & 1 \\ 1 & 1 & 1+\varepsilon^5 & 1 \\ 1 & 1 & 1 & 1-\varepsilon^5 \end{vmatrix} \xrightarrow[R_3-R_4\to R_3]{R_1-R_2\to R_1} \begin{vmatrix} \varepsilon^2 & \varepsilon^2 & 0 & 0 \\ 1 & 1-\varepsilon^2 & 1 & 1 \\ 0 & 0 & \varepsilon^5 & \varepsilon^5 \\ 1 & 1 & 1 & 1-\varepsilon^5 \end{vmatrix}$$

$$\xrightarrow[C_3-C_4\to C_3]{C_1-C_2\to C_1} \begin{vmatrix} 0 & \varepsilon^2 & 0 & 0 \\ \varepsilon^2 & 1-\varepsilon^2 & 0 & 1 \\ 0 & 0 & 0 & \varepsilon^5 \\ 0 & 1 & \varepsilon^5 & 1-\varepsilon^5 \end{vmatrix}$$

- Hence,

$$\det(A) = -\varepsilon^2 \begin{vmatrix} \varepsilon^2 & 0 & 1 \\ 0 & 0 & \varepsilon^5 \\ 0 & \varepsilon^5 & 1-\varepsilon^5 \end{vmatrix} = (-\varepsilon^2)\times(-\varepsilon^5)\begin{vmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^5 \end{vmatrix} = \varepsilon^{14}$$

$$\Rightarrow \det(A) = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^{14} = \cos(4\pi) + i\sin(4\pi) = 1$$