Course: Probability & Statistics

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Module 7: Continuous Probability Distributions
Lesson 2: Properties of Normal Distribution

Reading Objectives:

In this reading, you will understand the properties of normal and standard normal distributions, such as expectation and variance.

Main Reading Section:

Mean and Variance:

To compute E[X], note that

$$E[X - \mu] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x - \mu)\, e^{-(x-\mu)^2/2\sigma^2}\, dx$$

Letting *y = (x − μ)/σ* gives that

$$E[X - \mu] = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{-y^2/2}\, dy$$

But

$$\int_{-\infty}^{\infty} y\, e^{-y^2/2}\, dy = 0$$

Implies E[X − μ] = 0, or equivalently that E[X] = μ.

Using this, we now compute Var(X) as follows:

$$\mathrm{Var}(X) = E[(X - \mu)^2] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x - \mu)^2\, e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2\, y\, e^{-y^2/2}\, dy$$

After some calculations, Var(X) = σ^2

Note: A very important property of normal random variables is that if *X* is normal with mean *μ* and variance σ^2, then for any constants *a* and *b*, *b* ≠ 0, the random variable *Y* = *a*+*bX* is also a normal random variable with parameters

- $E[Y] = E[a + bX] = a + bE[X] = a + b\mu$
- $Var(Y) = Var(a + bX) = b^2 Var(X) = b^2\sigma^2$

Example 1: The power W dissipated in a resistor is proportional to the square of the voltage V. That is, $W = rV^2$

where r is a constant. If r = 3, and V can be assumed (to a very good approximation) to be a normal random variable with mean 6, and standard deviation 1, find E [W] and P{W > 120}.

Solution: $E[W] = E[3V^2] = 3E[V^2] = 3(Var[V] + E^2[V]) = 3(1 + 36) = 111$

$P\{W > 120\} = P(3V^2 > 120) = = P(V > \sqrt{40}) = 0.37275.$

Gaussian Mixture

A Gaussian Mixture is a function comprised of several Gaussians, each identified by $k \in \{1,\dots, K\}$, where K is the number of clusters of our dataset. Each Gaussian k in the mixture is comprised of the following parameters:

- A mean μ that defines its center.
- A covariance Σ that defines its width. This would be equivalent to the dimensions of an ellipsoid in a multivariate scenario.
- A mixing probability π that defines how big or small the Gaussian function will be.

The Gaussian density function is given by:

$$N[x/\mu, \Sigma] = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

where **x** represents our data points, D is the number of dimensions of each data point. μ and Σ are the mean and covariance, respectively. The mixing coefficients are themselves probabilities and must meet this condition $\sum_{1}^{K}\pi_k = 1.$

Reading Summary

In this reading, you have learned the following:

- The mean and variance of normal distribution with some properties.