Course: Probability & Statistics

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Module 6: Discrete Probability Distributions
Lesson 1: Binomial Distribution

Reading Objectives:

In this reading, you will understand the binomial probability distribution with its corresponding probabilities, expectation, and its properties.

Main Reading Section:

Definition 1: Suppose now that *n* independent trials, each of which results in a "success" with probability *p* and in a "failure" with probability 1 − *p*, are to be performed. If X represents the number of successes that occur in the *n* trials, then X is said to be a binomial random variable with parameters *(n, p).*

The probability mass function of a binomial random variable with parameters *n* and *p* is given by:

$$P\{X = i\} = \binom{n}{i} p^{i} (1 - p)^{n-i}, \quad i = 0,1,2,\ldots,n$$

Where $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$ is the number of different groups of *i* objects that can be chosen from a set of *n* objects.

Properties: Since a binomial random variable X, with parameters *n* and *p*, represents the number of successes in *n* independent trials, each having success probability *p*, we can represent X as follows:

$$X = \sum_{i=1}^{n} X_i \text{ where } X_i = \{1, \quad if\ the\ ith\ trial\ is\ a\ success\ 0, \qquad otherwise$$

Because the X_i, $i = 0,1,2,\ldots,n$ are independent Bernoulli random variables, we have that

$$E[X_i] = P\{X_i = 1\} = p$$

$$\mathrm{Var}(X_i) = E[X_i^2] - p^2 = p(1 - p)$$

Mean and Variance:

It is now an easy matter to compute the mean and variance of X:

Mean: $E[X] = \sum_{i=1}^{n} E[X_i] = np$

Variance: $Var[X] = \sum_{i=1}^{n} Var(X_i) = np(1 - p)$ (since the X_i are independent)

Recurrence Relation:

Suppose that X is binomial with parameters *(n,p)*. The key to computing its distribution function

$$P\{X \le i\} = \sum_{k=0}^{i} \binom{n}{k} p^k (1-p)^{n-k}, \qquad i = 0,1,\ldots n$$

The relation between $P\{X = k + 1\}$ and $P\{X = k\}$ is

$$P\{X = k + 1\} = \frac{p}{1-p} \frac{n-k}{k+1} P\{X = k\}$$

Example 1. It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

Solution: If X is the number of defective disks in a package, then assuming that customers always take advantage of the guarantee, it follows that X is a binomial random variable with parameters (10, 0.01). Hence, the probability that a package will have to be replaced is

$$\begin{aligned} P\{X > 1\} &= 1 - P\{X = 0\} - P\{X = 1\} \\ &= 1 - \binom{10}{0}(0.01)^0 (0.99)^{10} - \binom{10}{1}(0.01)^1 (0.99)^9 \\ &\approx 0.005 \end{aligned}$$

Because each package will, independently, have to be replaced with probability 0.005, it follows from the law of large numbers that in the long run 0.5 percent of the packages will have to be replaced.

It follows from the foregoing that the number of packages that will be returned by a buyer of three packages is a binomial random variable with parameters $n = 3$ and $p = 0.005$. Therefore, the probability that exactly one of the three packages will be returned is

$$\binom{3}{1}(0.005)(0.995)^2 = 0.015$$

Example 2: Let X be a binomial random variable with $E[X] = 7$ and $\text{Var}(X) = 2.1$

Find $P\{X = 4\}$ and $P\{X > 12\}$

Solution: Since $E(X) = np = 7$, $\text{Var}(X) = np(1 - p) = 2.1$
It follows that $p = 0.7$, $n = 10$. Hence,
$P\{X = 4\} = \binom{10}{4}(0.7)^4\,(0.3)^6 = 0.036$
$P\{X > 12\} = 0$

Reading Summary

In this reading, you have learned the following:

- Binomial distribution function with its probability mass function
- Expectation of the binomial distribution function with some examples