Course: Probability and Statistics

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Module 4: Elements of Probability
Lesson 1: Bayes' Formula

Reading Objectives:

In this reading, you will learn about conditional probability and Bayes' formula with examples. You will also be introduced to the concepts of independent and dependent events.

Main Reading Section:

1. Conditional Probability

We introduce one of the most important concepts in all probability theory—conditional probability. Its importance is twofold. In the first place, we are often interested in calculating probabilities when some partial information concerning the result of the experiment is available or in recalculating them in light of additional information.

The conditional probability of E given that F has occurred is given by $P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)}$. Note that this probability is well defined only when P (F) > 0, and hence, P (E|F) is defined only when P (F) > 0.

Example 1. A bin contains 5 defectives (immediately fail when put in use), 10 partially defective (fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable?

Solution: Since the transistor did not immediately fail, we know that it is not one of the 5 defectives, and so the desired probability is:

P{acceptable/not defective} = $\frac{P\{acceptable, not\ defective\}}{P\{not\ defective\}} = \frac{P\{acceptable,\}}{P\{not\ defective\}}$

where the last equality follows since the transistor will be both acceptable and not defective if it is acceptable. Hence, assuming that each of the 40 transistors is equally likely to be chosen, we obtain that

P{acceptable/not defective} = $\frac{25/40}{35/40} = \frac{5}{7}$

2. Bayes' formula

Let E and F be events. We may express E as E = (E∩F) ∪ (E∩ F^c). As E∩F and E∩ F^c are clearly mutually exclusive, we have by Axiom 3 that

P(E) = P(E∩F) + P (E∩ F^c)

= P (E/F) P(F) + P(E/F^c) P(F^c)

= P (E/F) P(F) + P(E/F^c) [1 − P(F)]

The above formula may be generalized in the following manner. Suppose that F_1, F_2, ..., F_n are mutually exclusive events such that

$$\cup_{i=1}^{n} F_i = S$$

In other words, exactly one of the events F_1, F_2, ..., F_n must occur. By writing

$$E = \cup_{i=1}^{n} E \cap F_i$$

and using the fact that the events $E \cap F_i$, i = 1,...,n are mutually exclusive, we obtain that

$$P(E) = \sum_{i=1}^{n} P(E \cap F_i)$$

$$= \sum_{i=1}^{n} P(E/F_i)\, P(F_i) \qquad (1)$$

Suppose now that E has occurred, and we are interested in determining which one of F_i also occurred. By Equation (1), we have that

$P\left(F_i/E\right) = \frac{P(E \cap F_i)}{P(E)} = \frac{P(E/F_i)P(F_i)}{\sum_{i=1}^{n} P(E/F_i)\, P(F_i)}$, this is known as Bayes' formula.

Example 2. A plane is missing, and it is presumed that it was equally likely to have gone down in any of three possible regions. Let 1 − α_i denote the probability the plane will be found upon a search of the ith region when the plane is, in fact, in that region, i = 1, 2, 3. (The constants α_i are called overlook probabilities because they represent the probability of overlooking the plane; they are generally attributable to the geographical and environmental conditions of the regions.) What is the conditional probability that the plane is in the i^{th} region, given that a search of region 1 is unsuccessful, i = 1, 2, 3?

Solution: Let R_i, i =1, 2, 3, be the event that the plane is in region i, and let E be the event that a search of region 1 is unsuccessful. From Bayes' formula, we obtain

$$P(R_1/E) = \frac{P(E \cap R_1)}{P(E)} = \frac{P(E/R_1)\,P(R_1)}{\sum_{i=1}^{3} P(E/R_1)\,P(R_1)}$$

$$= \frac{(\alpha_1)\,(1/3)}{(\alpha_1)\,(1/3) + 1(1/3) + 1(1/3)} = \frac{\alpha_1}{\alpha_1 + 2}$$

For j = 2, 3

$$P(R_j/E) = \frac{1}{\alpha_1 + 2}$$

Thus, for instance, if $\alpha_1 = .4$, then the conditional probability that the plane is in region 1 given that a search of that region did not uncover it is $\frac{1}{6}$, whereas the conditional probabilities that it is in region 2 and that it is in region 3 are both equal to $\frac{1}{2.4} = \frac{5}{12}$

3. Independent events

Definition 1: Two events, E and F, are said to be independent if P(E ∩F) = P(E)P(F). Two events, E and F, that are not independent are said to be dependent.

This definition can be extended to more than two events in a similar fashion.

Proposition 1: If E and F are independent, then so are E and F^c.

Example 3: A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, independent of other components, functions with probability p_i, i = 1,...,n, what is the probability the system functions?

Solution: Let A_i denote the event that component i functions. Then

P{system functions} = 1 − P{system does not function}

= 1 − P{all components do not function}

$= 1 - P(A_1^c\, A_2^c \,\ldots\ldots\, A_n^c)$

$$= 1 - \prod_{i=1}^{n} (1 - p_i) \quad (by\ independence)$$

Reading Summary

In this reading, you have learned the following:

- Conditional probability and Bayes’ formula with some examples.
- Independent and dependent events with examples.