Course: Probability & Statistics

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Module 7: Continuous Probability Distributions
Lesson 1: Normal Distribution

Reading Objectives:

In this reading, you will understand the normal and standard normal distributions with some examples.

Main Reading Section:

Definition 1: A random variable is said to be **normally distributed** with parameters μ and σ^2, and we write X ∼ N (μ, σ^2) if its density is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

The normal density *f(x)* is a bell-shaped curve that is symmetric about μ and attains its maximum value of $\frac{1}{\sqrt{2\pi}\,\sigma} \approx 0.399/\sigma$ at x = μ.

Definition 2: It follows from the foregoing that if X ∼ N (μ, σ^2), then Z = $\frac{X-\mu}{\sigma}$ is a normal random variable with mean 0 and variance 1. Such a random variable Z is said to have a standard, or unit, normal distribution. Let Φ(·) denote its distribution function. That is

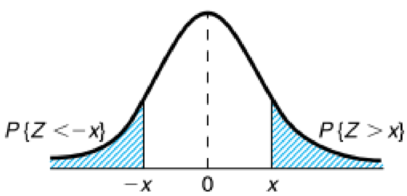
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \quad -\infty < x < \infty$$

This result that Z = (X − μ)/σ has a standard normal distribution when X is normal with parameters μ and σ^2 is quite important, for it enables us to write all probability statements about X in terms of probabilities for Z.

Note: For instance, to obtain P{X<b}, we note that X will be less than b if and only if $\frac{X-\mu}{\sigma}$ is less than $\frac{b-\mu}{\sigma}$, and so

$$P\{X < b\} = P\left\{\frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right\} = \Phi\left(\frac{b-\mu}{\sigma}\right)$$

It remains for us to compute $\Phi(x)$



We can also obtain $\Phi(-x)$ from the table using the symmetry (about 0) of the standard normal probability density function. That is, for x > 0, if Z represents a standard normal random variable, then

$$\Phi(-x) = P\{Z < -x\}$$

$$= P\{Z > x\} = 1 - \Phi(x)$$

Example 1. If X is a normal random variable with mean μ = 3 and variance $\sigma^2 = 16$, find

a) P{ X < 11}

b) P{X > -1}

c) P{2 < X < 7}

Solution:

a) $P\{X < 11\} = P\left\{\frac{X-3}{4} < \frac{11-3}{4}\right\} = \Phi(2) = 0.9772$

b) $P\{X > -1\} = P\left\{\frac{X-3}{4} > \frac{-1-3}{4}\right\} = P\{Z > -1\} = P\{Z < 1\} = 0.8413$

c) $P\{2 < X < 7\} = P\left\{\frac{2-3}{4} < \frac{X-3}{4} < \frac{7-3}{4}\right\} = \Phi(1) - \Phi(-1/4) = \Phi(1) - (1 - \Phi(-1/4))$

$= 0.8413 + 0.5987 - 1 = 0.4400$

Example 2: Students of a class were given an aptitude test. Their marks were found to be normally distributed, with mean 60 and a standard deviation 5. What percentage of students scored more than 60 marks?

Solution: X = 60, μ = 60, and σ = 5

$$Z = \frac{X-\mu}{\sigma} = \frac{60-60}{5} = 0$$

If x > 60, then Z > 0

The area lying to the right of z is 0.5.

The percentage of students getting more than 6 marks is 50%.

Example 3: The mean inside diameter of a sample of 200 washers produced by a machine is 0.502, and the standard deviation is 0.005 cm. The purpose for which these washers are intended is to allow a maximum tolerance in the diameter of 0.496 to 0.508 cm; otherwise, the washers are considered defective. Determine the percentage of defective washers the machine produces, assuming the diameters are normally distributed.

Solution: $Z_1 = \frac{X-\mu}{\sigma} = \frac{0.496-0.502}{0.005} = -1.2$

$Z_2 = \frac{X-\mu}{\sigma} = \frac{0.508-0.502}{0.005} = 1.2$

Area for non-defective washers = Area between Z = -1.2 and Z = 1.2

= 2. Area between Z=0 and Z = 1.2

= 2(0.3849) = 0.7698 = 76.98%

Percentage of defective washers = 100-76.98 = 23.02%

Reading Summary

In this reading, you have learned the following:

- Normal and standard normal distribution function with its probability density function
- Some examples of the normal curve and its area