

# FEniCS Course

## Lecture 9: Incompressible Navier–Stokes

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### *Contributors*

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# The incompressible Navier–Stokes equations

$$\begin{aligned}\rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) &= f && \text{in } \Omega \times (0, T] \\ \nabla \cdot u &= 0 && \text{in } \Omega \times (0, T] \\ u &= g_D && \text{on } \Gamma_D \times (0, T] \\ \sigma \cdot n &= g_N && \text{on } \Gamma_N \times (0, T] \\ u(\cdot, 0) &= u_0 && \text{in } \Omega\end{aligned}$$

- $u$  is the fluid velocity and  $p$  is the pressure
- $\rho$  is the fluid density
- $\sigma(u, p) = 2\mu\epsilon(u) - pI$  is the Cauchy stress tensor
- $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$  is the strain rate tensor
- $f$  is a given body force per unit volume
- $g_D$  is a given boundary velocity
- $g_N$  is a given boundary traction
- $u_0$  is a given initial velocity

## Mixed variational formulation of Navier–Stokes

Multiply the **momentum equation** by a test function  $v$  and integrate by parts:

$$\int_{\Omega} \rho(\dot{u} + u \cdot \nabla u) \cdot v \, dx + \int_{\Omega} \sigma(u, p) : \epsilon(v) \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_N \cdot v \, ds$$

Short-hand notation:  $\langle \cdot, \cdot \rangle$  is  $L^2$ -inner product

$$\langle \rho \dot{u}, v \rangle + \langle \rho u \cdot \nabla u, v \rangle + \langle \sigma(u, p), \epsilon(v) \rangle = \langle f, v \rangle + \langle g_N, v \rangle_{\Gamma_N}$$

Multiply the **continuity equation** by a test function  $q$  and sum up: find  $(u, p) \in V$  such that

$$\langle \rho \dot{u}, v \rangle + \langle \rho u \cdot \nabla u, v \rangle + \langle \sigma(u, p), \epsilon(v) \rangle + \langle \nabla \cdot u, q \rangle = \langle f, v \rangle + \langle g_N, v \rangle_{\Gamma_N}$$

for all  $(v, q) \in \hat{V}$

# Discrete mixed variational form of Navier–Stokes

Time-discretization leads to a *saddle-point* problem on each time step:

$$\begin{bmatrix} M + \Delta t A + \Delta t N(U) & \Delta t B \\ \Delta t B^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- Efficient solution of the saddle-point problem relies on the efficiency of special-purpose preconditioners (Uzawa iteration, Schur complement preconditioners, ...)
- We will use another approach (simpler and often more efficient)

# A splitting scheme for Navier–Stokes

Use a Crank-Nicolson approximation with explicit convection for the time-discretization of the momentum equation:

$$\rho D_t u^n + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-1/2}) = f^{n-1/2}$$

**Step 1:** Compute the *tentative velocity*  $u^\star$  using the approximation

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, \color{red}{p^{n-3/2}}) = f^{n-1/2}$$

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$$D_t u = (u^n - u^{n-1})/k_n \text{ and } k_n = t_n - t_{n-1}$$

## A splitting scheme for N-S (contd.)

Subtract the equation for the tentative velocity from the equation for the *real* velocity:

$$\rho(D_t u^n - D_t u^\star) - \nabla \cdot \sigma(0, p^{n-1/2} - p^{n-3/2}) = 0$$

Expanding the  $D_t$ , using the equations, and rearranging give

$$\rho u^n = \rho u^\star - k_n \nabla(p^{n-1/2} - p^{n-3/2}) \quad (1)$$

Take the divergence and set  $\nabla \cdot u^n = 0$ :

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star \quad (2)$$

- **Step 2:** Compute  $p^{n-1/2}$  by solving the Poisson problem (2)
- **Step 3:** Compute  $u^n$  by solving the projection problem (1)

*NB: Which boundary conditions for the Poisson problem (2)?*

## Boundary conditions

- For outflow boundary conditions, corresponding to so-called “do-nothing” boundary conditions for the Laplacian formulation, we take  $\partial_n u = 0$ :

$$\begin{aligned}\sigma(u, p) \cdot n &= (2\mu\epsilon(u) - pI) \cdot n = \mu\nabla u \cdot n + \mu(\nabla u)^\top \cdot n - pn \\ &= \mu\nabla u \cdot n - pn \approx \mu\nabla u^{n-1/2} \cdot n - p^{n-3/2}n\end{aligned}$$

- Boundary conditions for the pressure Poisson problem:

$$\partial_n \dot{p} = 0$$

on the pressure Neumann boundary

# Incremental pressure correction scheme (IPCS)

- ① Compute the tentative velocity  $u^\star$  by

$$\begin{aligned} & \langle \rho D_t^n u^\star, v \rangle + \langle \rho u^{n-1} \cdot \nabla u^{n-1}, v \rangle + \langle \sigma(u^{n-\frac{1}{2}}, p^{n-3/2}), \epsilon(v) \rangle \\ & - \langle \mu \nabla u^{n-\frac{1}{2}} \cdot n, v \rangle_{\partial\Omega} + \langle p^{n-3/2} n, v \rangle_{\partial\Omega} = \langle f^{n-1/2}, v \rangle \end{aligned}$$

- ② Compute the corrected pressure  $p^{n-1/2}$  by

$$k_n \langle \nabla p^{n-1/2}, \nabla q \rangle = k_n \langle \nabla p^{n-3/2}, \nabla q \rangle - \langle \rho \nabla \cdot u^\star, q \rangle$$

- ③ Compute the corrected velocity  $u^n$  by

$$\langle \rho u^n, v \rangle = \langle \rho u^\star, v \rangle - k_n \langle \nabla(p^{n-1/2} - p^{n-3/2}), v \rangle$$

# Useful FEniCS tools (I)

Note grad vs.  $\nabla$ :

*Python code*

```
dot(grad(u), u)  
dot(u, nabla_grad(u))
```

Defining operators:

*Python code*

```
def sigma(u, p):  
  
    return 2.0*mu*sym(grad(u))-p*Identity(len(u))
```

The facet normal  $n$ :

*Python code*

```
n = FacetNormal(mesh)
```

## Useful FEniCS tools (II)

Assembling matrices and vectors:  
*Python code*

```
A = assemble(a)
b = assemble(L)
```

Solving linear systems:  
*Python code*

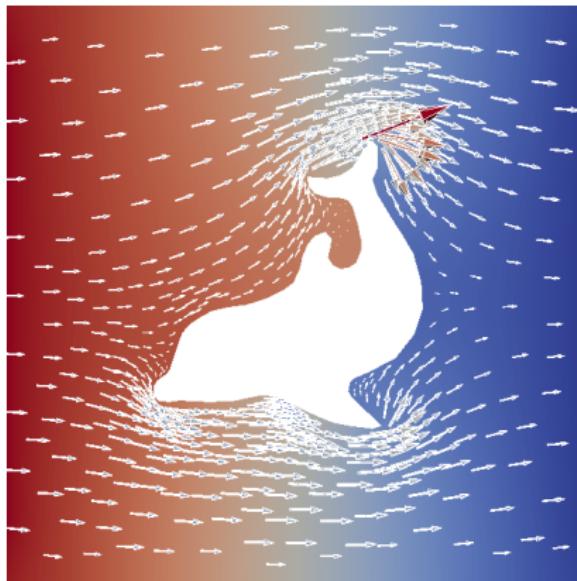
```
solve(A, x, b)
solve(A, x, b, "gmres", "ilu")
solve(A, x, b, "cg", "amg")
```

Extracting left- and right-hand sides:  
*Python code*

```
F = <complicated expression>
a = lhs(F)
L = rhs(F)
```

## *The FEniCS challenge!*

Solve the incompressible Navier–Stokes equations for the flow of water around a dolphin. The water is initially at rest and the flow is driven by a pressure gradient.



## *The FEniCS challenge!*

- Use the mesh `dolfin_channel.xml.gz`.
- Compute the solution on the time interval  $[0, 0.1]$  with time steps of size  $k = 0.0005$
- Set  $p = 1$  kPa at the inflow and  $p = 0$  at the outflow
- The density of water is  $\rho = 1000$  kg/m<sup>3</sup> and the viscosity is  $\mu = 0.001002$  kg/(m · s)
- To check your answer, compute the average velocity in the  $x$ -direction:

$$\bar{u}_x = \frac{1}{|\Omega|} \int_{\Omega} u \cdot (1, 0) \, dx$$

*The student(s) who first produce the right answer will be rewarded with an exclusive FEniCS surprise!*