ELLIPTIC CURVES
Homework 1

1. Let E be an elliptic curve over an algebraically closed field k defined by a Weierstrass equation $Y^2Z = X^3 + AXZ^2 + BZ^3$. Show that every $f \in k(E)$ which has no poles except perhaps at $O = (0, 1, 0)$ can be represented by a polynomial in X/Z, Y/Z of the form $g(X/Z) + (Y/Z)h(X/Z)$, where $g, h \in k[x]$. Deduce a quick proof that if f has no poles at all, then f must be constant.

2. Let X be a smooth projective plane curve and D a divisor of degree 0 on X. Show that the vector space $\mathcal{L}(D)$ has dimension 1 if D is the divisor of a function and dimension 0 otherwise.

Note: We have seen that the latter case can occur, for instance for the divisor $P - O$ on an elliptic curve.

3. Compute the zeta function of the circle $X^2 + Y^2 = Z^2$ defined over the finite field $\mathbf{F}_q$ and show that it satisfies the functional equation. [*Hint:* Stereographic projection.]